# Lecture 3

## The Impulse and the Complex Exponential
## (Language – Definition & Math)

I. DT Unit Step $u[n]$ and Impulse Signal $\delta[n]$

II. CT Impulse Signal $\delta(t)$

III. CT Complex Exponential $e^{st}$ and Complex Sinusoid $e^{j\omega t}$

IV. DT Complex Exponential $z^n$ and Complex Sinusoid $e^{j\omega n}$

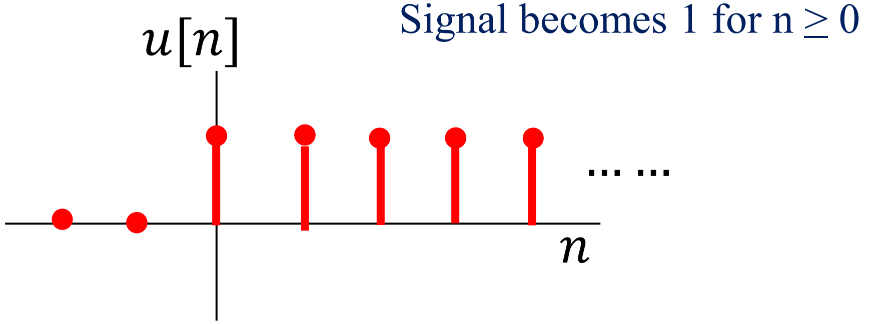
(Ref: Chapter 1, O&W)

# I. DT Unit Step $u[n]$ & Impulse $\delta[n]$

As in the CT case, the **DT *unit step*** represents a signal that is turned on at time 0

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

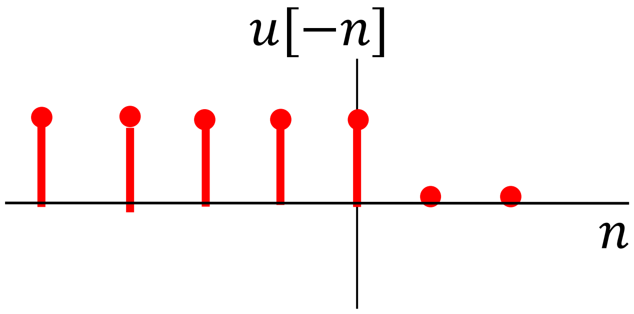
Unit step




$u[-n]$ is the time-reversal of $u[n]$

$$u[-n] = \begin{cases} 1, & n \leq 0 \\ 0 & n > 0 \end{cases}$$

Time Reversed Unit step



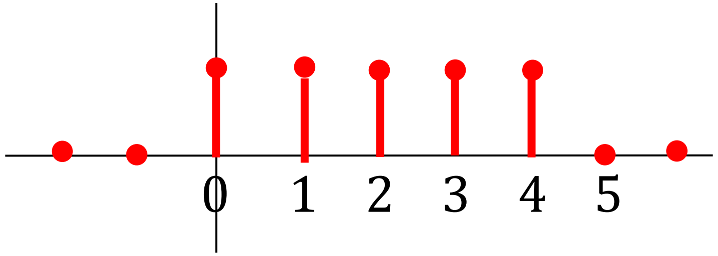
# DT Window Signal

As in CT, we can express a window as difference of two step signals

$$\boxed{u[n]-u[n-N]=\begin{cases}0, & n<0\,;n\geq N\\ 1, & 0\leq n\leq N-1\end{cases}}$$

Example:

$u[n]-u[n-5]$




Note that while in the CT case $u(t)-u(t-T)$ is a window from 0 to $T$, $u[n]-u[n-N]$ is a DT window from 0 to $N-1$.

# DT Unit Impulse

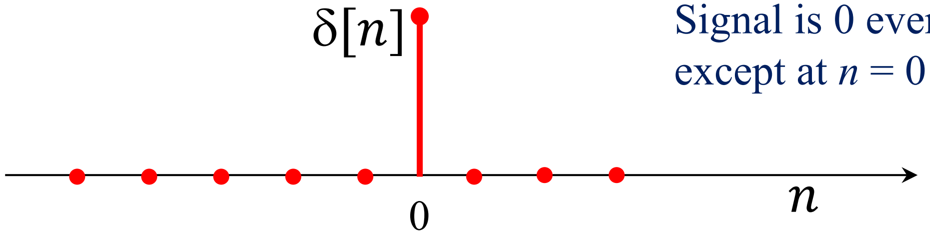
The DT *unit impulse* or *unit sample* signal, denoted $\delta[n]$, is the signal that is equal to 1 at $n = 0$ and is 0 everywhere else.

$= u[n] - u[n-1]$

$\delta[n]$ is also called the *delta function*.

Unit impulse/ Unit sample/Delta function

$$\boxed{\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}}$$

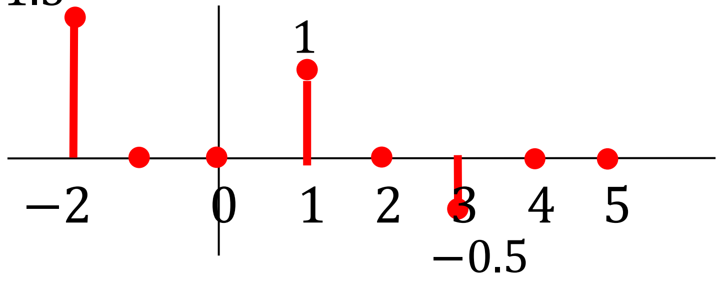


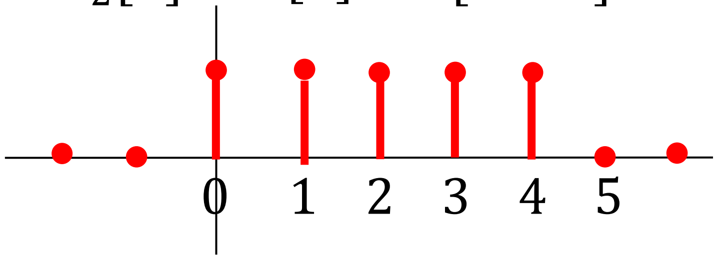

Signal is 0 everywhere except at $n = 0$

# Any Signal as Sum of Impulses

- $\delta[n]$ is the most basic DT signal. We can express any DT signal as a weighted sum of shifted $\delta[n]$ signals:

For example, given the signals $x_1$ and $x_2$ below:




We can express them as:

$$x_1[n] = 1.5\delta[n+2] + \delta[n-1] - 0.5\delta[n-3]$$

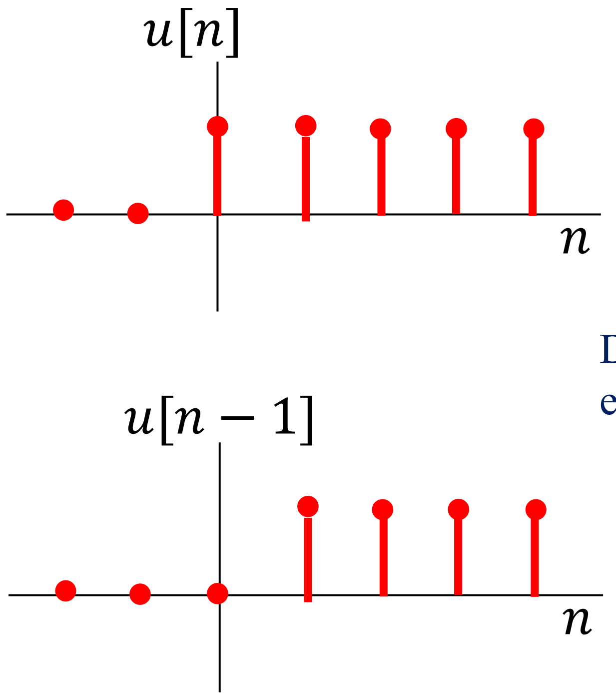
$$x_2[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] = \sum_{k=0}^{4} \delta[n-k]$$

# $\delta[n]$ as First Difference of $u[n]$

In DT, the unit impulse $\delta[n]$ is a window of one unit sample.

$\delta[n]$ is the difference of the unit step and a unit step delayed by 1. It is called the *first difference* of $u[n]$:

$$\delta[n] = u[n] - u[n-1]$$

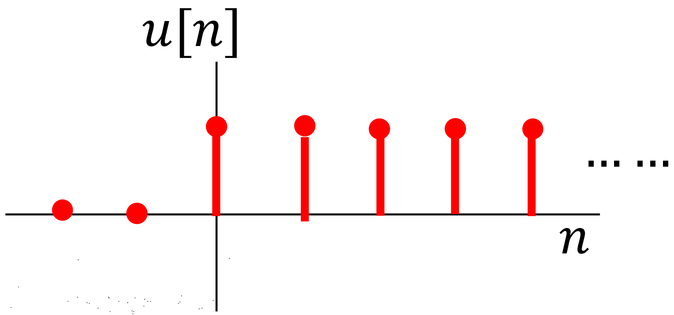



Difference is 0 everywhere except at $n = 0$

# $u[n]$ as First Sum of $\delta[n]$

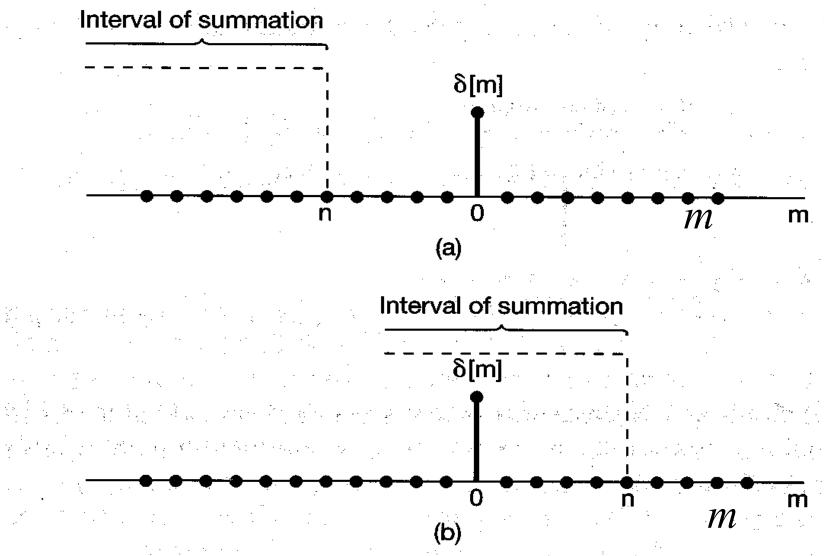
The DT unit step can be expressed as a sum function of the DT unit impulse in two ways:

Unit step




1 $$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

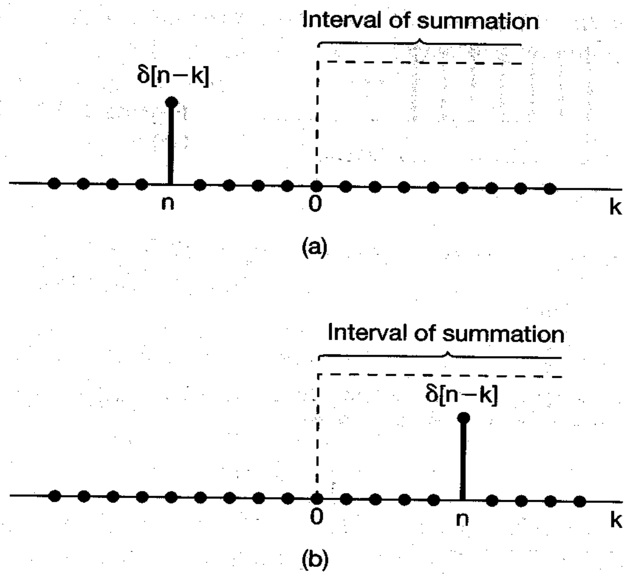
2 $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$




$\boxed{n<0}$

$\boxed{n \geq 0}$

Figure 1.3
eq. (1.66): (




$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \cdots = \sum_{k=0}^{\infty} \delta[n-k]$$

Figure 1.31 Re
eq. (1.67): (a) $n <$

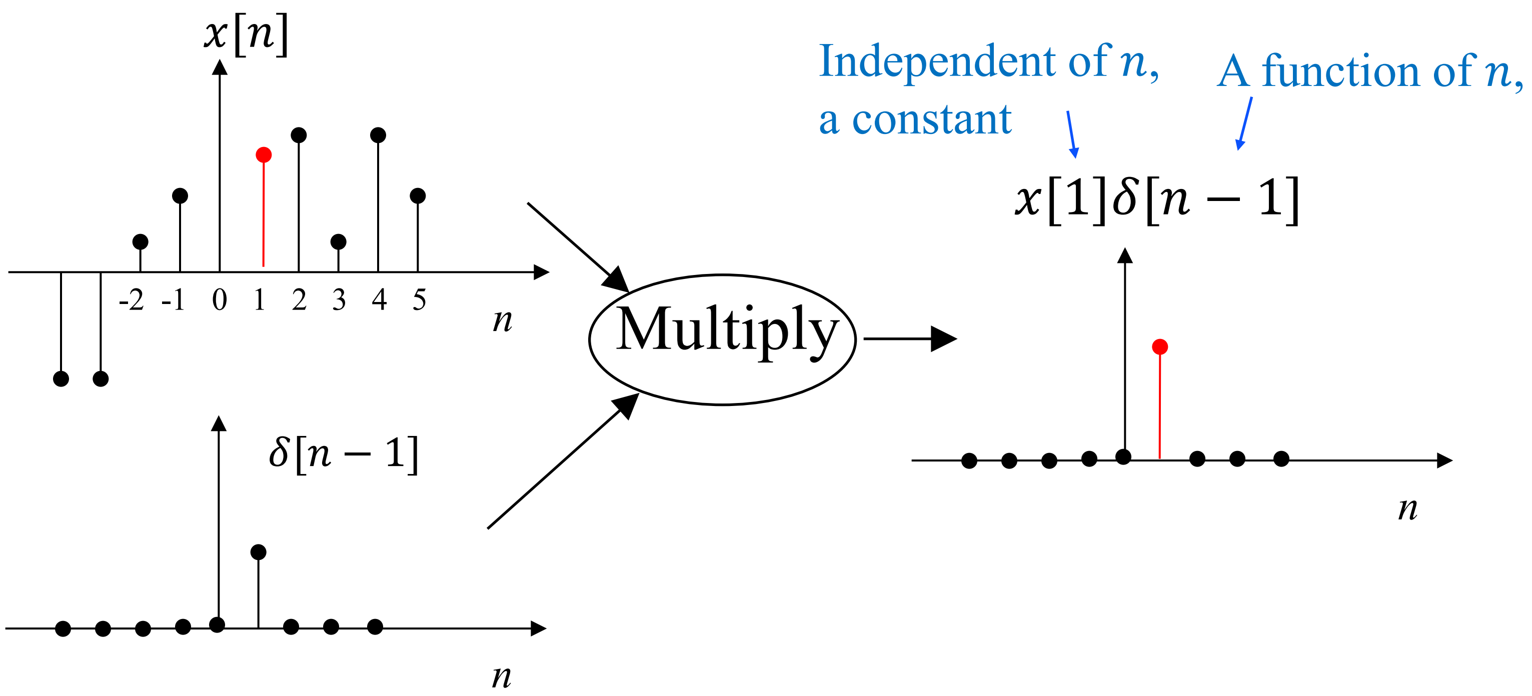
# The Sampling Property of $\delta[n]$

When we multiply any signal $x[n]$ with $\delta[n]$ , only one *sampled* value of $x[n]$ matters.

The **sampling property**:

$$x[n]\,\delta[n] \;=\; x[0]\,\delta[n]$$

$$x[n]\,\delta[n-n_0] \;=\; x[n_0]\,\delta[n-n_0]$$



# II. The CT Impulse Signal $\delta(t)$

The CT impulse signal, $\delta(t)$, is more complicated to explain.

We define it as the signal whose integral gives the unit step:

$$\boxed{u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad (1.71)}$$

- The definition suggests that $\delta(t)$ is the derivative of $u(t)$, i.e.,

$$\boxed{\delta(t) = \frac{du(t)}{dt} \qquad (1.72)}$$

- Equation (1.72) is problematic to mathematicians because $u(t)$ is discontinuous and not differentiable at $t = 0$. *But* as engineers and for simplicity, we accept (1.72) as is.

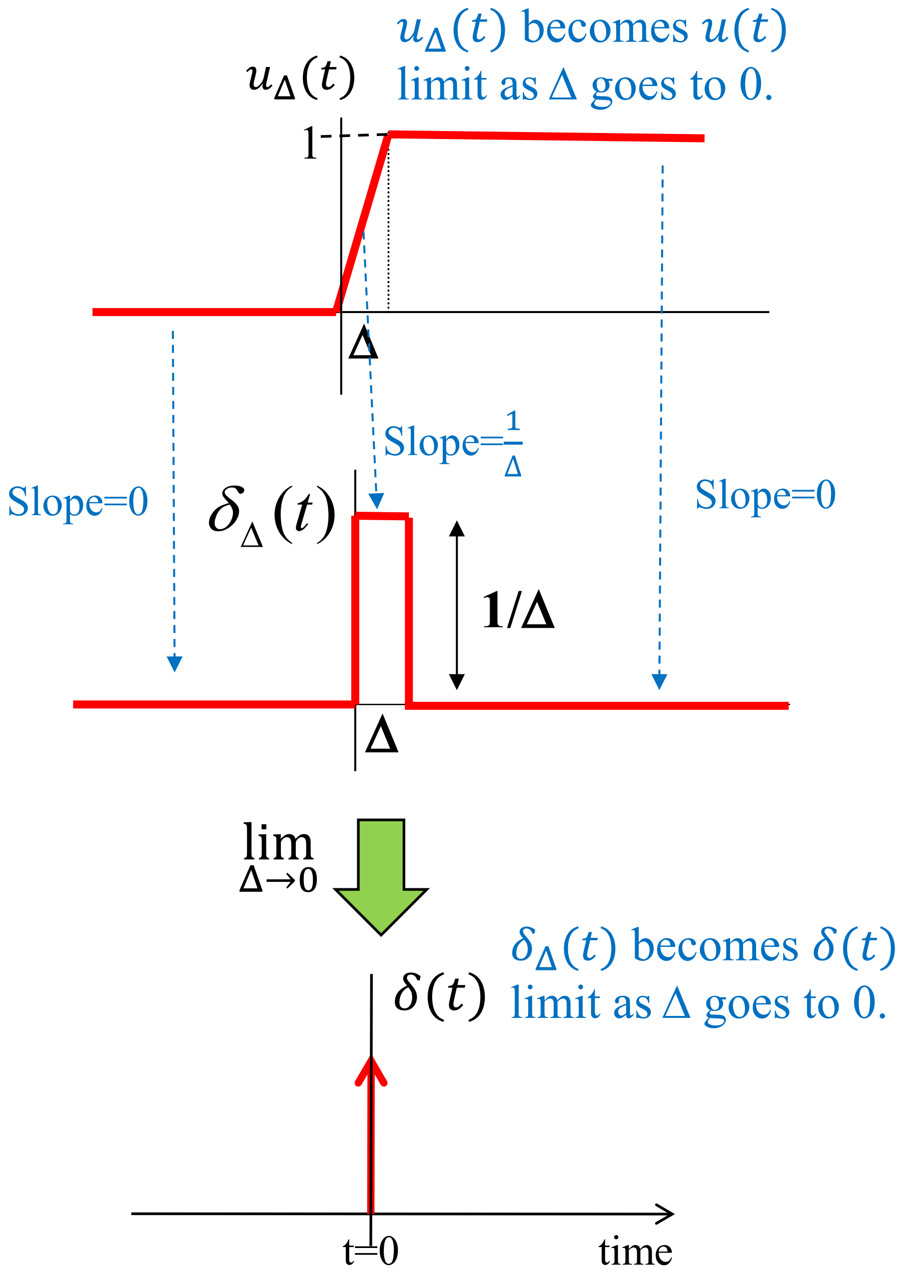
# The Unit Impulse as limit of a Tall and Narrow Pulse

We often interpret $\delta(t)$ as a very tall and very narrow pulse by the following argument:

- Consider an "approximate" step $u_\Delta(t)$ that takes a small time interval of Δ to jump from 0 to 1:

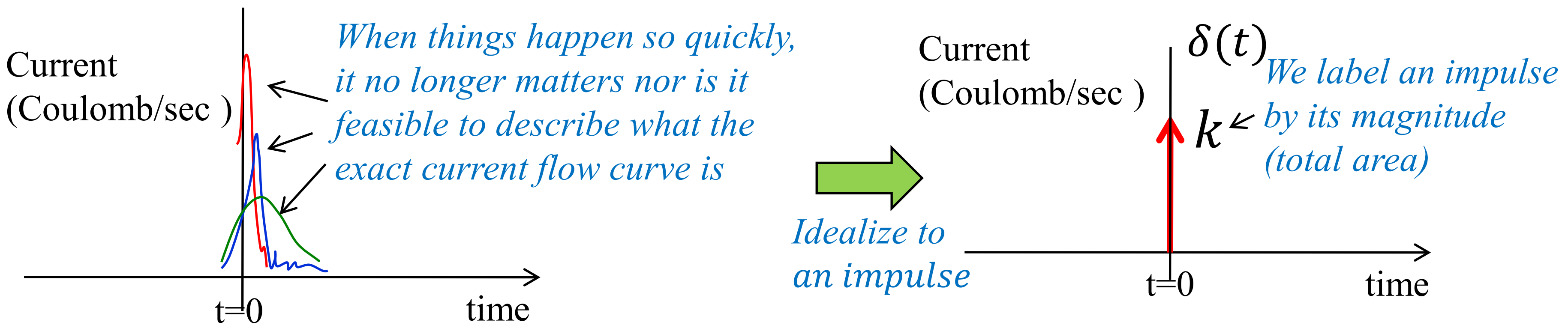
- $\delta_\Delta(t) = \frac{d}{dt}u_\Delta(t)$ , the derivative of the approximate step, is now a very narrow and very tall pulse with an area of 1 under it.

- When Δ goes to 0, the pulse becomes infinitely tall and infinitesimally narrow; i.e., $\lim_{\Delta\to 0}\delta_\Delta(t) = \delta(t)$. We represent $\delta(t)$ graphically as an arrow.



# Impulse as an Idealization

- The impulse signal is a ***mathematical idealization*** of something that causes a quick change.
- For example, integral of current gives voltage change across a capacitor. Imagine there is a lightning strike. It is ***infeasible*** and also ***meaningless*** to try to describe the current flow exactly over this short interval of time.




- A unit impulse causes one unit of charges to be dumped into the capacitor "instantaneously". An impulse of area/magnitude $k$ causes $k$ units of charges to be dumped into the capacitor.

# Mathematical Properties of the Impulse Signal

**1. Zero everywhere else, infinite at $t = 0$:**

$$\delta(t) = 0 \quad \forall t \neq 0$$

$$\lim_{t \to 0} \delta(t) = +\infty$$

**2. Unit total area:**

The total area under the $\delta(t)$ function is 1:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

**3. Integral is the unit step:**

$u(t)$ is the first integral of $\delta(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Alternate notations for first integral:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

$$y(t) = \int x(t)$$

$$y(t) = x^{(-1)}(t)$$

4. **Scaled impulse** $k\delta(t)$ has total area $k$.

Clearly $$\int_{-\infty}^{\infty} k\delta(t)dt = k\int_{-\infty}^{\infty} \delta(t)dt = k$$

We label a scaled impulse by its area as shown previously:

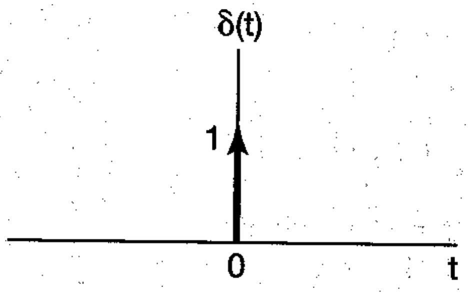



**Figure 1.35** Continuous-time unit impulse.

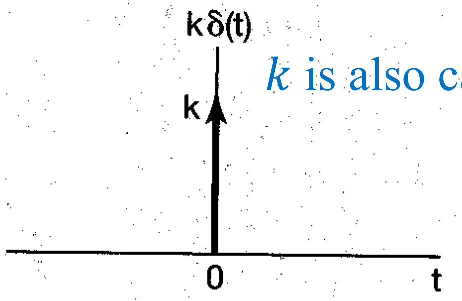



**Figure 1.36** Scaled impulse.

$k$ is also called the magnitude of the impulse

Integrating a scaled impulse $k\delta(t)$ gives $ku(t)$:

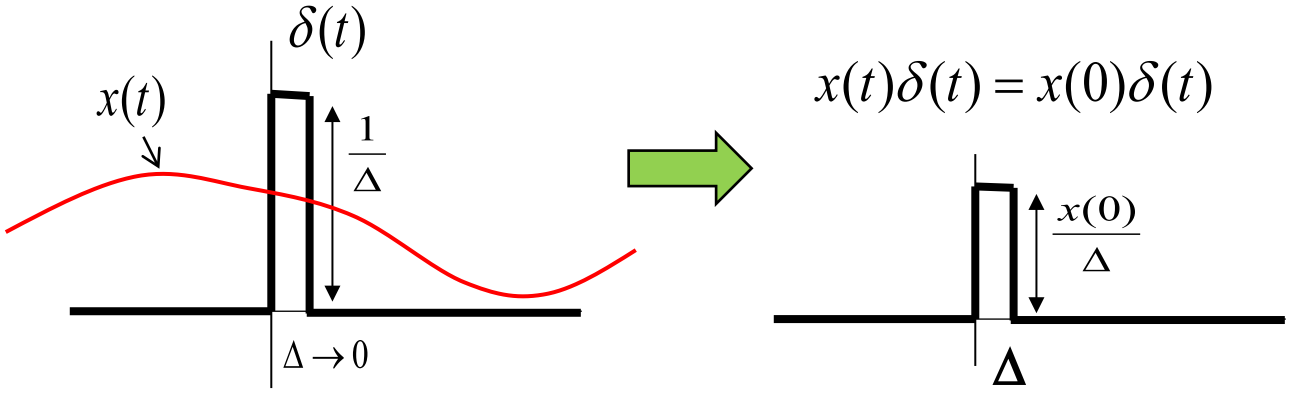
$$\int_{-\infty}^{t} k\delta(\tau)d\tau = ku(t)$$

## 5. **Sampling property**:

Multiplying any signal $x(t)$ with an unit impulse results in a scaled impulse with magnitude $x(0)$ .

$$\boxed{x(t)\delta(t)= x(0)\delta(t)}$$

The value of $x(t)$ anywhere except at $t = 0$ is irrelevant as visualized below:



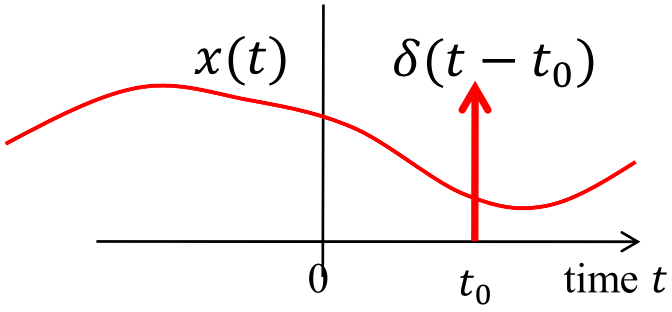
# Multiplying by a Shifted Impulse

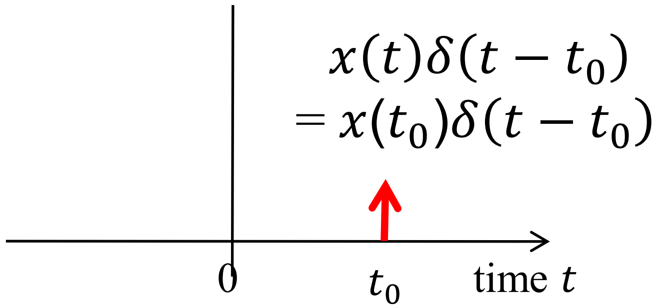
- As in DT, if we multiply any signal $x(t)$ with a shifted impulse at $t_0$, the result is the shifted impulse scaled with magnitude $x(t_0)$:

$$\boxed{x(t)\underbrace{\delta(t-t_0)}_{\text{shifted impulse}} = x(t_0)\delta(t-t_0)}$$

- $\delta(t-t_0)$ is a shifted impulse. It is non-zero only at $t = t_0$.

- If we multiply it with any $x(t)$, only the sampled value $x(t_0)$ scales the magnitude of the impulse



## 6. **Sifting property**:

Multiplying any signal with a shifted impulse at $t_0$ and integrating over all time extracts the value of the signal at $t_0$.

$$\boxed{\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)}$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt \overset{\text{Sampling property}}{=} \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)\,dt = x(t_0)\overbrace{\int_{-\infty}^{\infty} \delta(t-t_0)\,dt}^{\text{Integrates to 1}} = x(t_0)$$

## 7. **Infinite energy**:

$\delta(t)$ is the most important conceptual tool in this course, but it contains infinite energy delivered over a zero interval of time. Therefore $\delta(t)$ is an idealization that ***cannot physically exist***!

$$\text{Total Energy} = \int_{-\infty}^{\infty} |\delta(t)|^2 dt = \lim_{\Delta \to 0} \int_0^{\Delta} \left|\frac{1}{\Delta}\right|^2 dt = \lim_{\Delta \to 0} \frac{1}{\Delta} \to \infty$$

# Summary – Key Properties of $\delta(t)$ and $\delta[n]$

Integral/first sum is unit step: $\boxed{u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau}$ $\boxed{u[n] = \sum_{m=-\infty}^{n} \delta[m]}$

Sampling: $\boxed{x(t)\delta(t) = x(0)\delta(t)}$ $\boxed{x[n]\delta[n] = x[0]\delta[n]}$

$\boxed{x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)}$ $\boxed{x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]}$

Sifting: $\boxed{\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)}$ $\boxed{\sum_{-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]}$

# III. CT Complex Exponential $e^{st}$ and Complex Sinusoid $e^{j\omega t}$

- The ***Complex Exponential*** is an exponential function of time where the growth constant is generalized to be complex:

complex

$$\boxed{x(t) = e^{st}}$$

- $s$ is called the ***complex frequency***.
- Let $s = \sigma + j\omega$, where $\sigma = Re\{s\}$ and $\omega = Im\{s\}$

then:

$$e^{st} = e^{(\sigma + j\omega)t}$$

$$= e^{\sigma t} e^{j\omega t}$$

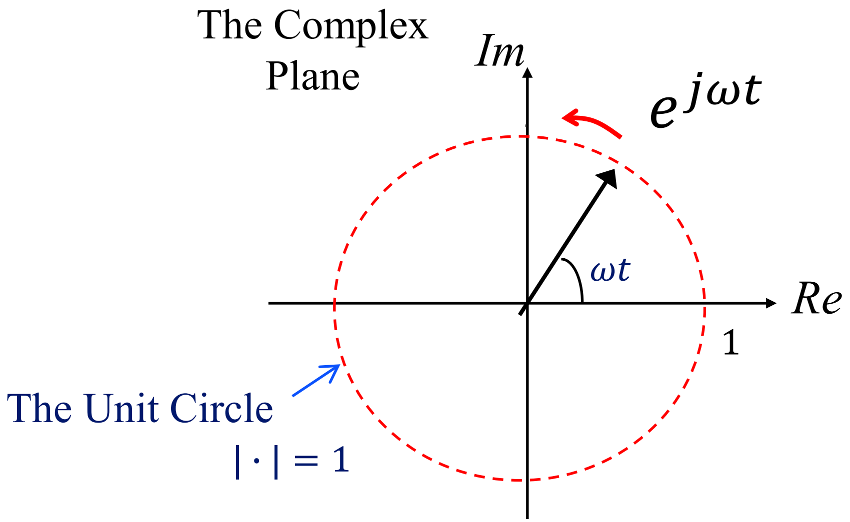
Exponent is real

Exponent is purely imaginary

- $e^{\sigma t}$ is a real exponential
- $e^{j\omega t}$ is what we call a ***complex sinusoid***
- Hence, a complex exponential is the product of a real exponential and a complex sinusoid.

# Complex Sinusoid $e^{j\omega t}$ as a Revolution

- We can conceptualize $e^{j\omega t}$ as a ***revolution*** around the "unit circle" on the complex plane:
  - Magnitude of $e^{j\omega t}$ is always 1: $|e^{j\omega t}| = 1$ ;
  - Angle increases linearly with time: $\measuredangle e^{j\omega t} = \omega t$

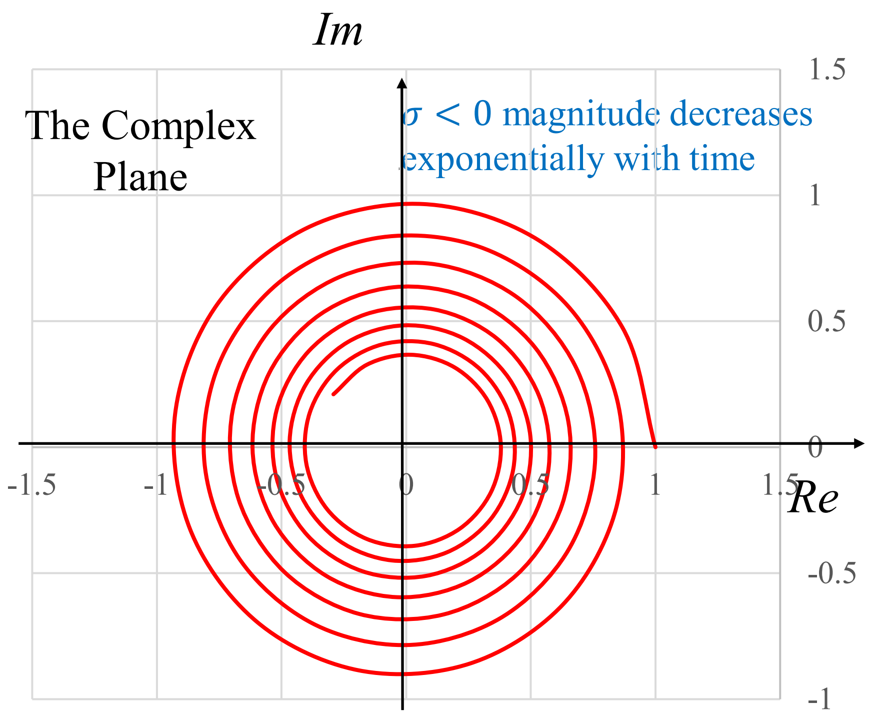


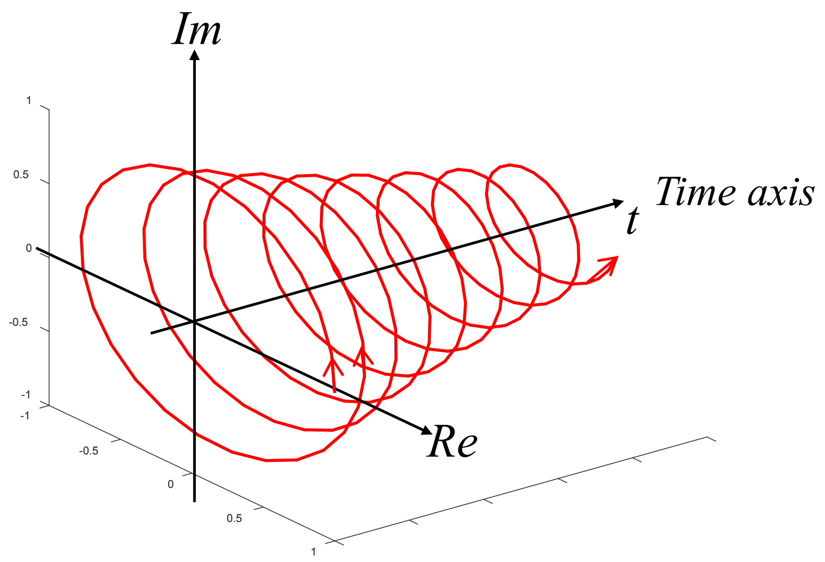

- The period of revolution is $T = 2\pi/\omega$.
- The projection of $e^{j\omega t}$ on the real axis or imaginary axis (i.e., the real and imaginary part) will give us a sinusoidal oscillation:

$$e^{j\omega t} = \underbrace{\cos \omega t}_{\text{Real part}} + j \underbrace{\sin \omega t}_{\text{Imaginary part}}$$

# Back to the General Complex Exponential

- Now back to the complex exponential $e^{st}$ where $\mathrm{Re}\{s\} = \sigma \neq 0.$
- $e^{st} = e^{\sigma t} e^{j\omega t}$ means the signal both grows/decays and revolves with time:
  - Magnitude $|e^{st}| = e^{\sigma t}$ grows ($\sigma > 0$) or decays ($\sigma < 0$) exponentially with time
  - Phase $\measuredangle e^{st} = \omega t$ which increases linearly with time
- We can visualize $e^{st}$ as a spiral on the complex plane



# Complex Exponentials to Represent Damped Oscillations

- The real and imaginary parts of $e^{st}$ are sinusoidal oscillations with an exponential envelop:

$$e^{st} = e^{\sigma t} e^{j\omega t}$$

Recall: Complex exponential = Real exponential × complex sinusoid

Real part Imaginary part

$$= \underbrace{e^{\sigma t} \cos\omega t}_{\text{Real part}} + j\underbrace{e^{\sigma t}\sin\omega t}_{\text{Imaginary part}}$$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

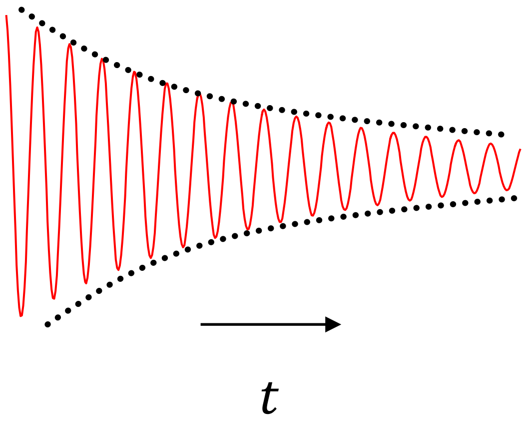
- Signals in the form $e^{\sigma t}\cos\omega t$ or $e^{\sigma t}\sin\omega t$ (product of an exponential with a sinusoid) are called ***damped oscillations***.

# Decaying and Growing Damped Oscillations

- For $Re\{e^{st}\} = e^{\sigma t}\cos\omega t$, the *complex frequency s* concisely specifies the damped oscillation:
  - $\sigma$, the real part of $s$ specifies how fast the oscillation amplitude grows or decays.
  - $\omega$, the imaginary part of $s$ specifies the frequency of oscillation.

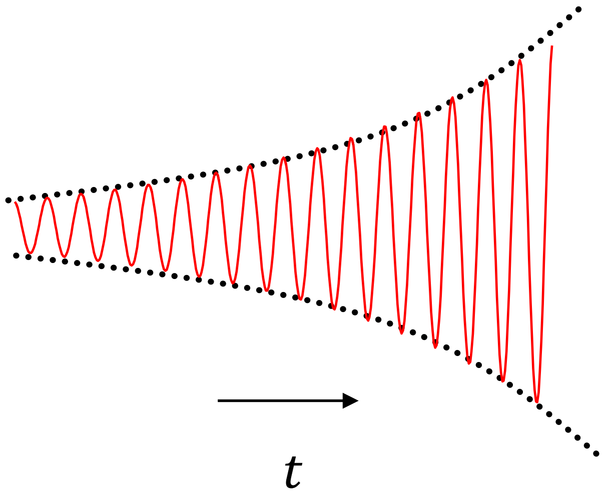
$\sigma < 0$

decaying damped oscillation




$\sigma > 0$

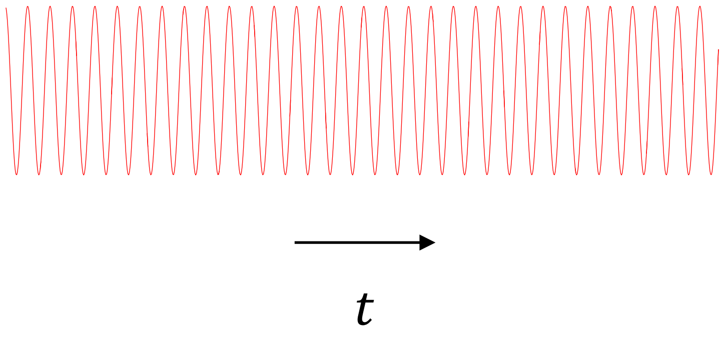
growing damped oscillation




$\sigma = 0$

$e^{\sigma t} = e^{0 \cdot t} = 1$

oscillation at constant amplitude




Period of oscillation is $\frac{2\pi}{\omega}$.

## Damped Oscillations as Half-Sum of Conjugate Complex Exponential Pair

- We can also treat damped oscillation as half-sum of a conjugate complex exponential pair:

$$e^{\sigma t}\cos\omega t = Re\{e^{st}\} = \frac{1}{2}e^{st}+\frac{1}{2}(e^{st})^{*} = \frac{1}{2}e^{st}+\frac{1}{2}e^{s^{*}t}$$
$$= \frac{1}{2}e^{\sigma t}e^{j\omega t}+\frac{1}{2}e^{\sigma t}e^{-j\omega t}$$

- We notice that conjugating a complex exponential simply means negating $\omega$, its oscillation frequency.

- Likewise, we can represent real sinusoid by complex sinusoid as:

$$\cos\omega t = Re\{e^{j\omega t}\} \qquad \text{or} \qquad \cos\omega t = \frac{1}{2}e^{j\omega t}+\frac{1}{2}e^{-j\omega t}$$

# Advantages of Using the Complex Exponential and Complex Sinusoid

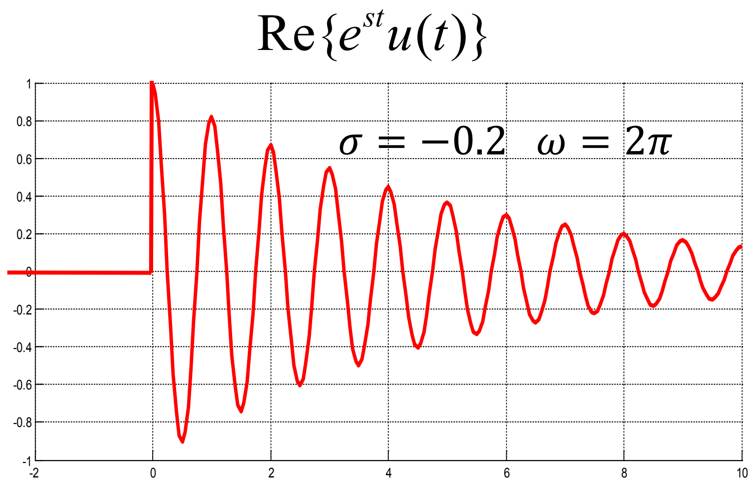
From now on, we will often use and $e^{j\omega t}$ and $e^{st}$ to represent oscillations and damped oscillations.

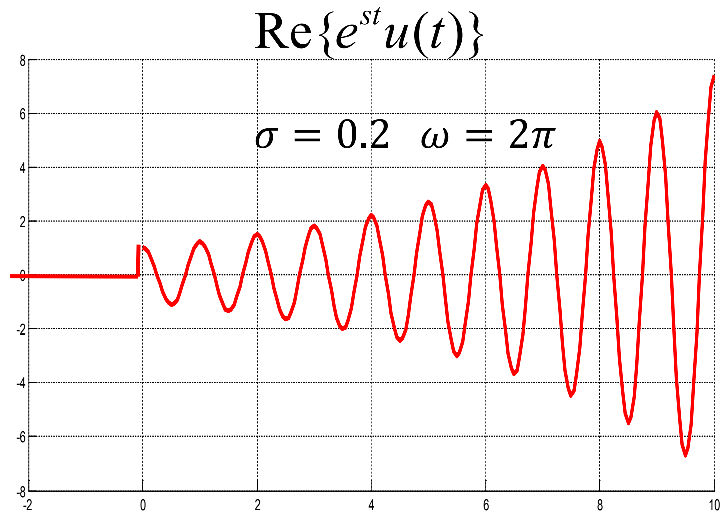
<u>Advantages</u>:

1. Unifies the sinusoid, exponential, and damped oscillation into one general class of signals, by just using a different $s$.
2. Enables concise representation of phase changes, differentiation, and time shift in algebraic forms. ⇒ all turned into multiplication by a complex constant!
3. Allows us to solve many problems much more efficiently.
4. By thinking of a signal or system as a function of the complex frequency $s$, we can have new powerful ways to understand them.

# One-Sided Complex Exponential

- The complex exponential $e^{st}$ with $Re\{s\} = \sigma \neq 0$ ***may not*** represent a realistic signal because nothing grows or decays exponentially forever from $t = -\infty$ to $t = \infty$ .
- $e^{st}u(t)$, the product of the complex exponential with the unit step, is a ***right-sided*** complex exponential. It represents a damped oscillation that starts growing or decaying at $t = 0$.

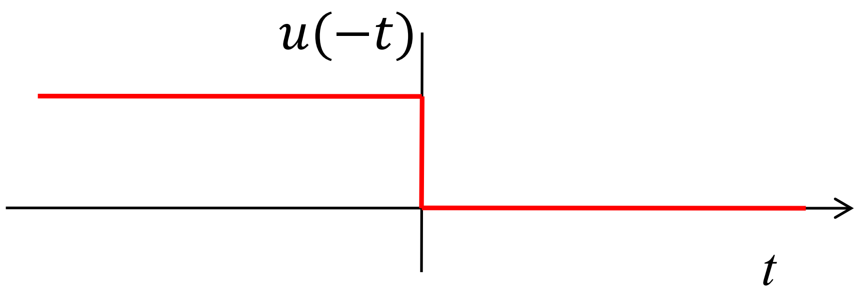


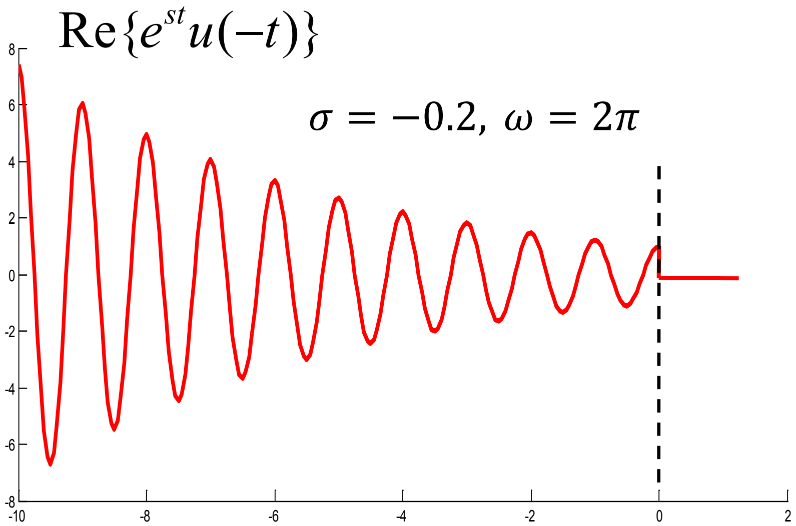

- $e^{st}u(t)$ for $\sigma > 0$ is a valid representation until the signal has grown beyond the dynamic range of the system.

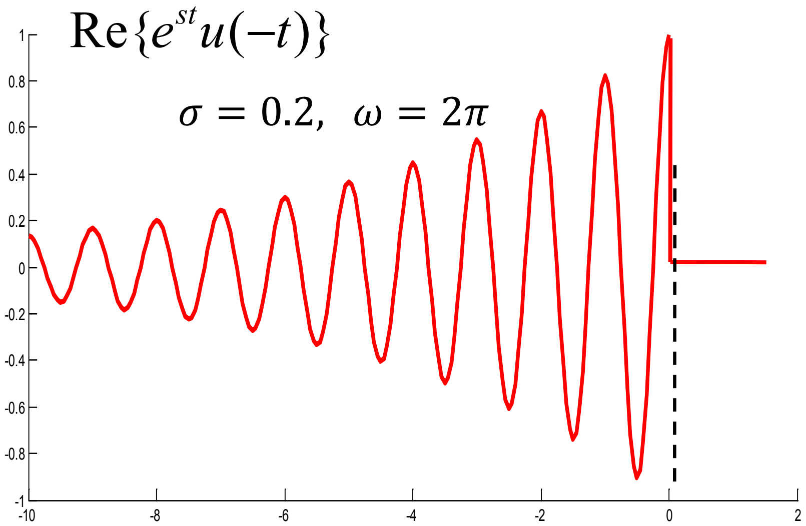
# Left-Sided Complex Exponential

$u(-t)$ is the time-reversed version of $u(t)$




Multiplying $e^{st}$ by $u(-t)$ gives a ***left-sided*** complex exponential $e^{st}u(-t)$ :



# IV. DT Complex Exponential & Complex Sinusoid

- The CT complex exponential $e^{st}$ becomes $e^{sn}$ in DT.

- But often we will express $e^{sn}$ in geometric form as $z^n$ where $z = e^s = e^{\sigma + j\omega} = e^{\sigma} e^{j\omega}$.

- Now, $z^n = |z|^n e^{j\omega n}$ is the product of a DT real exponential with a DT complex sinusoid.

DT real exponential DT complex sinusoid

$|z| = e^{\sigma}$ tells the magnitude change per unit time

$\measuredangle z = \omega$ tells the phase change per unit time

# CT and DT Complex Frequency

- $s$ and $z$ are both called ***complex frequency***, for CT and DT case respectively. $z = e^{s}$
- $s$ and $z$ concisely describe the rate at which the signal grows or decays and oscillates.

CT: $s = \mathrm{Re}\{s\} + j\,\mathrm{Im}\{s\}$

*rate of growth/decay* ($Re\{s\} = 0$ means no growth/decay)

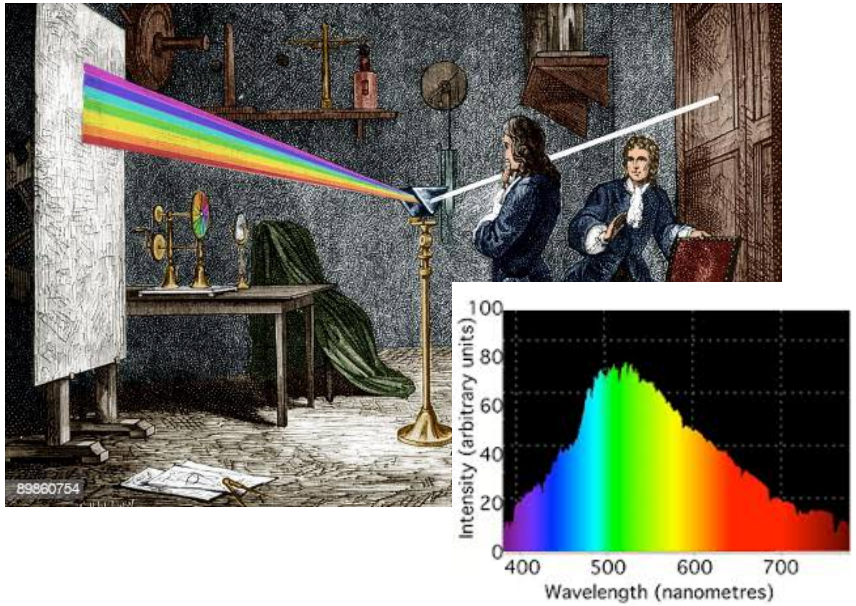
*frequency of oscillation* ($Im\{s\} = 0$ *means no oscillation)*

DT: $z = |z|e^{j\measuredangle z}$

*rate of growth/decay* ($|z| = 1$ means no growth/decay)

*frequency of oscillation* ($\measuredangle z = 0$ means no oscillation)

- In the future, we will transform signals and systems from functions of time to functions of these complex frequencies $s$ and $z$. This is called spectral analysis and will give us new ways of understanding signals and systems



# Damped Oscillations from DT Complex Exponentials

- Like $e^{st}$ in CT, the real and imaginary parts of $z^n$ are *damped oscillations*, sinusoidal signal with an exponential/geometric envelop:

$$z^n = |z|^n e^{j\omega n} = |z|^n \cos\omega n + j|z|^n \sin\omega n$$

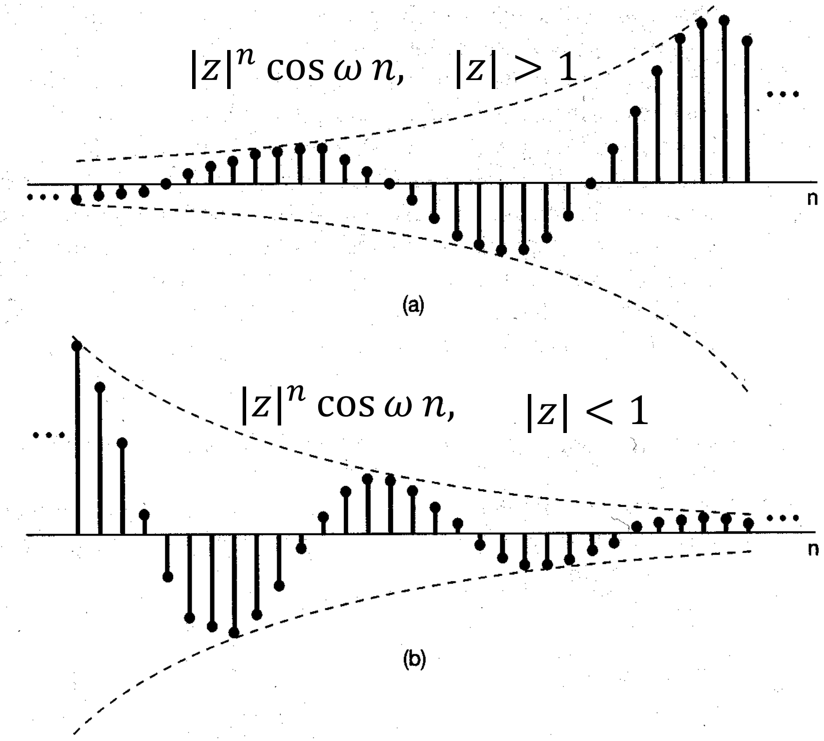



**Figure 1.26** (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

- The imaginary part is the same as the real part except for a phase shift
- $|z|$ gives the rate of growth/decay, and $\omega = \measuredangle z$ gives the frequency of oscillation.

# One-sided DT Damped Oscillations

As in CT case, we can use unit step to define right-sided or left-sided damped oscillations:

<u>Right-sided DT Damped Oscillations:</u> $Re\{z^n u[n]\}$

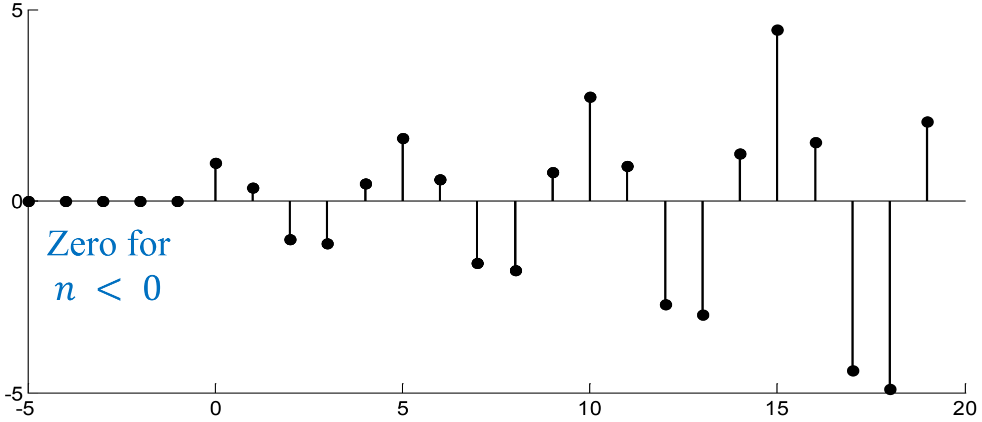
$\measuredangle z = 0.4\pi$

10 time units to grow by a factor of $e$

5 time units to oscillate once

$$z = e^{0.1+j0.4\pi} = 1.105e^{j0.4\pi}$$

$|z| = e^{0.1} > 1 \approx 1.105$

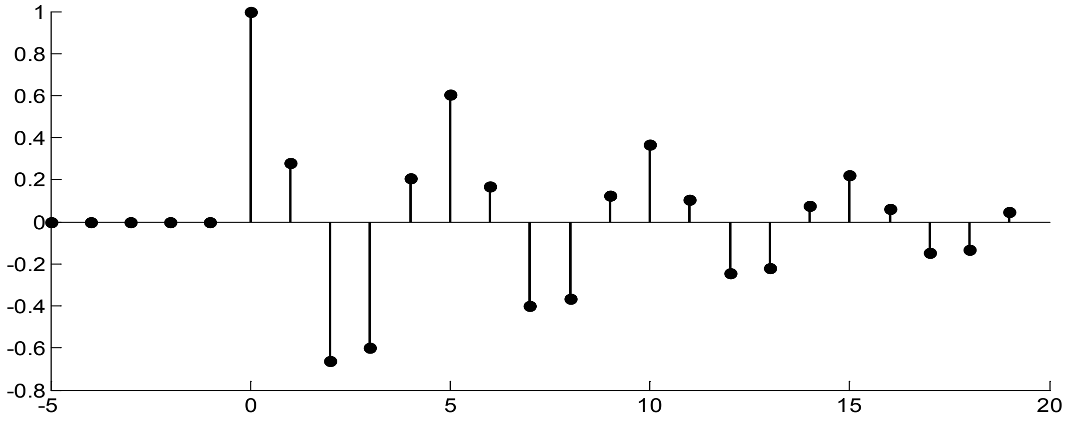



10 time units to decay by a factor of $e$

5 time units to oscillate once

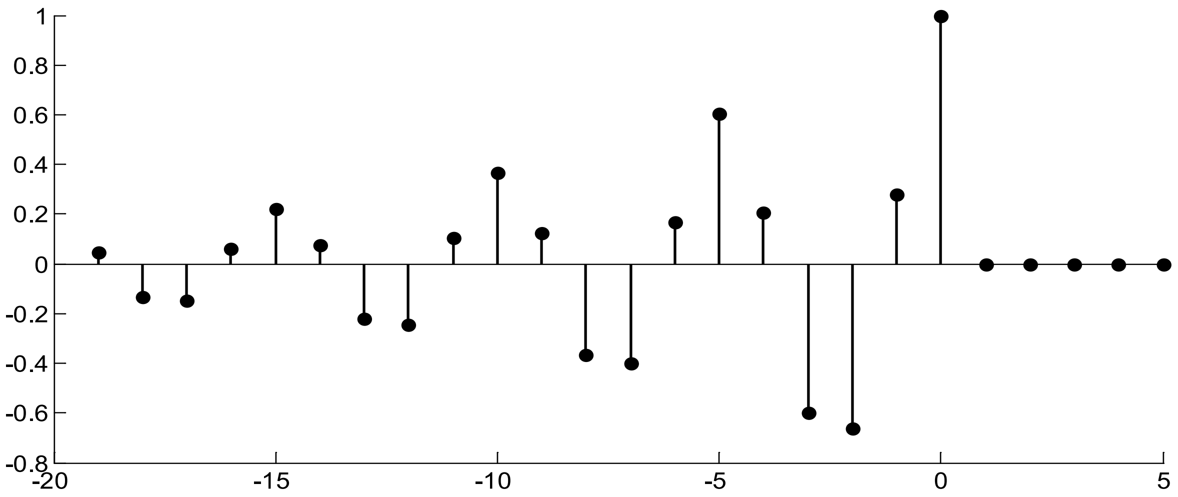
$$z = e^{-0.1+j0.4\pi} = 1.105e^{j0.4\pi}$$

$|z| = e^{-0.1} < 1 \approx 0.905$

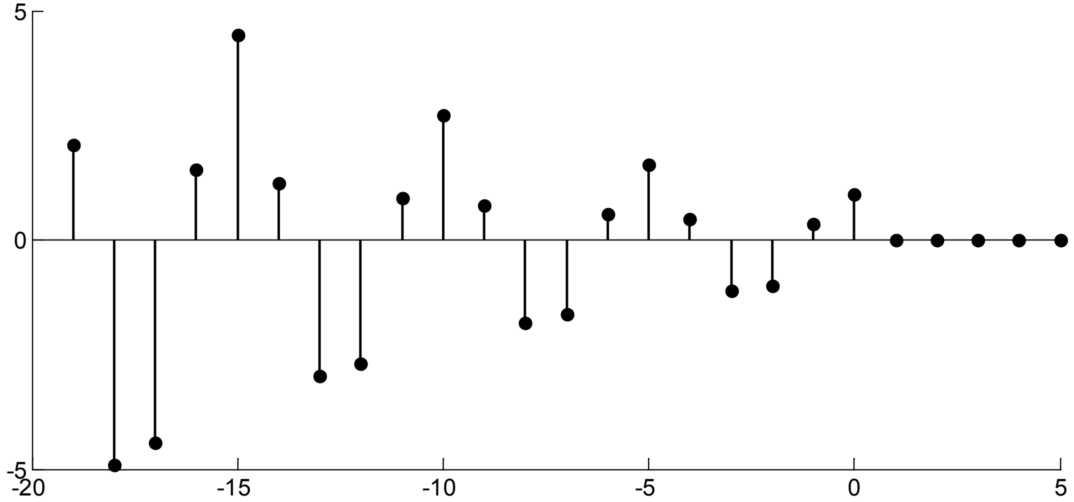


$z^n u[-n]$ <u>Left-sided DT Complex Exponential</u>

$z = e^{0.1+j0.4\pi} = 1.105e^{j0.4\pi}$




$z = e^{-0.1+j0.4\pi} = 0.905e^{j0.4\pi}$



# DT Complex Sinusoid $e^{j\omega n}$

- When $|z| = 1$ (equivalent to $\sigma = Re\{s\} = 0$), $z^n$ becomes the DT complex sinusoid $e^{j\omega n}$, which again is a revolution around the unit circle.

  The DT ***complex sinusoid*** is

$$x[n] = z^n\big|_{|z|=1} = e^{j\omega n} = \cos\omega n + j\,\sin\omega n$$

DT complex exponential

Means "given"

Sum of a real and an imaginary DT sinusoid

- DT complex sinusoids (or real sinusoids) have two important special properties that distinguish them from their CT counterparts:

# DT Sinusoids – Fact 1: Periodicity

**1.** DT sinusoid is periodic only if its ordinary frequency $f$ is a rational number, or its angular frequency $\omega$ is a rational number times 2π.)

- If a DT sinusoid $\cos(\omega n)$ is periodic, it is unchanged after time shift by some integer $N$:

$$\cos(\omega(n+N)) = \cos(\omega n + \omega N) = \cos(\omega n) \;\; \forall n$$

This means the phase change $\omega N$ must be an integer multiple of $2\pi$;

i.e., $\omega N = m2\pi$ for some integer $m$, or $\omega = m2\pi/N$

Hence $\omega = \frac{m}{N} \times 2\pi$ , a rational number $\frac{m}{N}$ multiplied by $2\pi$ .

Or, the ordinary frequency $f = m/N$ is rational.

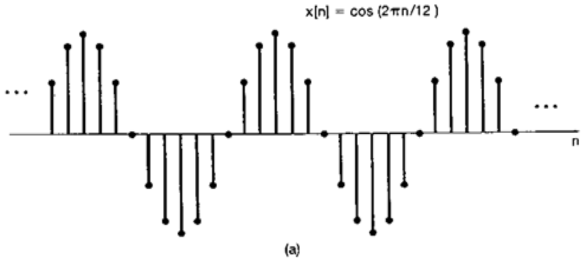
# Examples

Which of the three DT sinusoids below is(are) periodic?

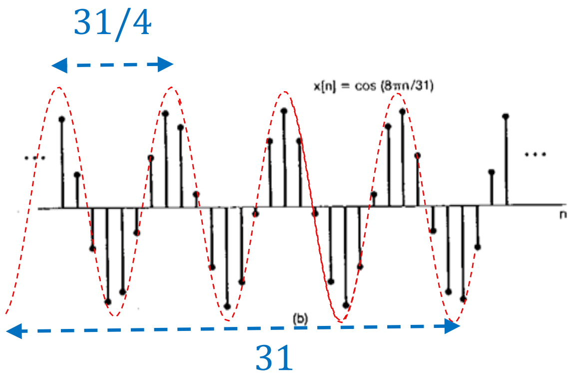
A. $\cos\left(\frac{2\pi n}{12}\right)$

B. $\cos\left(\frac{8\pi n}{31}\right)$

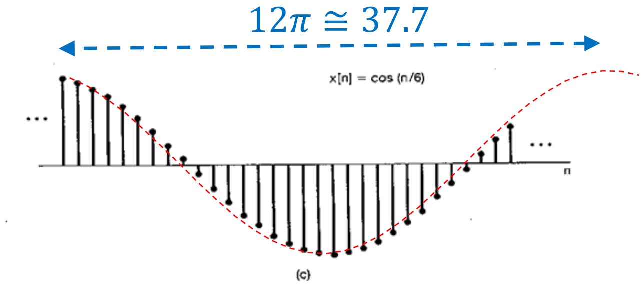
C. $\cos\left(\frac{n}{6}\right)$




Periodic?
Period =




Periodic?
Period =




Periodic?
Period =

$\omega = \frac{2\pi}{12}$, and $f = \frac{1}{12}$ is rational
Signal is periodic with period = 12

$\omega = \frac{8\pi}{31}$, and $f = \frac{4}{31}$ and is rational

Signal is periodic with period = 31
Envelop goes through 4 cycles in 31 time units

$\omega = \frac{1}{6}$ and $f = \frac{1}{12\pi}$ is not rational
Signal is not periodic

Envelop has a period of $12\pi \cong 37.7$

# DT Sinusoid – Fact 2: Periodicity of DT Frequency

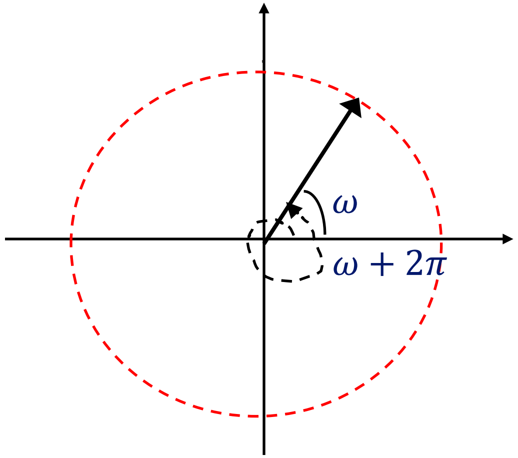
**2.** DT frequency is periodic (Very Important!)

In DT:

$\omega$ and $\omega + k2\pi$ refer to the same angular frequency!

$f$ and $f + k$ refer to the same ordinary frequency!

Angular frequency $\omega$ is the phase change per unit time. For a DT signal we observe the phase change only at discrete time instants, so there is no difference between a phase change of $\omega$ and phase change of $\omega + k2\pi$.




$$\cos((\omega + k2\pi)n) = \cos(\omega n + \cancelto{0}{kn2\pi}) = \cos(\omega n)$$

E.g.: 0.9 Hz, 1.9 Hz, 2.9 Hz, $-0.1$ Hz , $-1.1$ Hz are all the same frequency!

E.g. $\cos(0.9 \times 2\pi n) \equiv \cos(-1.1 \times 2\pi n)$

(In addition, because cosine is an even function, $-0.9$ Hz, 0.1 Hz, 1.1 Hz, 2.1 Hz , etc., are also the same frequency! We will come back to this later.)

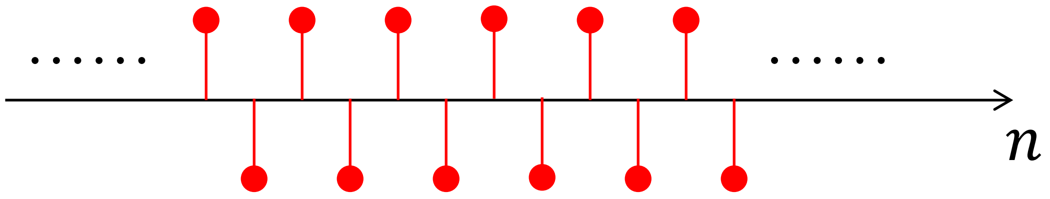
E.g. $\cos(0.9 \times 2\pi n) \equiv \cos(1.1 \times 2\pi n)$

# Maximum Frequency of DT Signals

The fastest rate at which a DT signal can oscillate is at an ordinary frequency of $f = ½$, or angular frequency of $\omega = \pi$.

The following DT oscillation, with period $N$=2, represents the DT oscillation at the highest frequency: (What if $N$=1?)

$$\cos \pi n = (-1)^n$$




But note that we have not specified what the "time unit" is. If the time unit is 1 ns (DT signal generated at 1 GHz), then an $f = ½$ actually represents a frequency of 0.5 GHz.

<u>**Examples:**</u>

- Traditional digital telephone networks sample at 8 KHz $\Longrightarrow$ maximum frequency in speech signal transmitted is 4 KHz.
- Compact Disc music sample at 44.1 KHz $\Longrightarrow$ maximum frequency in CD music is 22.05 KHz.

# Example DT Complex Sinusoid

Sketch $e^{j\omega n}$ for $\omega = \frac{4\pi}{9}$