Lecture 19

Differential Equations as LTI Systems (Analysis)

I. Linear Constant Coefficient Differential Equations as LTI Systems

II. From Frequency Response to Impulse Response

III. 2^{nd}-order Systems

I. Linear Constant-Coefficient Differential Equations

- A differential equation is an equation that relates a function (say function y) to its derivatives.
- An ***Ordinary Differential Equation*** (ODE) is a differentiation equation where the function y has only one independent variable – say t:
- We can express an ODE in the following general form:

$$f\left(y(t), \frac{dy(t)}{dt}, \frac{d^2y(t)}{dt^2}, \ldots, \frac{d^ky(t)}{dt^k}, t\right) = 0 \quad \text{where } f(\cdot) \text{ is any function}$$

Examples of ODE:

1. The logistic equation in chaos theory: $$\frac{dy(t)}{dt} = y(t)(1 - y(t))$$

2. An ODE with an input function $x(t)$: $$\frac{dy(t)}{dt} = x(t)\ln(y(t))$$

LCCDE

- An extremely important class of ODE is the ***Linear Constant-Coefficient Differential Equation*** (LCCDE), which equates a weighted sum of the 0 to N-th derivatives of the output to a weighted sum of the 0 to M-th derivatives of the input:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where usually $N > M$ and is the order of the system.

- LCCDEs model many physical systems.

- As explained in Chapter 1, LCCDEs are LTI. Therefore, they are easily analyzable and provide a framework for designing man-made systems. For the rest of this course, we will learn to characterize and build systems as LCCDEs.

Frequency Response of LCCDE

- The frequency response of an LCCDE as an LTI system can be easily determined by taking Fourier transform of both sides of the equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \underset{\text{Taking FT of both sides}}{\Rightarrow} \quad \Im\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \Im\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\underset{\text{Linearity of FT}}{\Rightarrow} \quad \sum_{k=0}^{N} a_k\, \Im\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k\, \Im\left\{\frac{d^k x(t)}{dt^k}\right\}$$

Independent of k

$$\Im\left\{\frac{d^k y(t)}{dt^k}\right\} = (j\omega)^k Y(j\omega)$$

$$\Rightarrow \sum_{k=0}^{N} a_k\, (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k\, (j\omega)^k X(j\omega) \quad \Rightarrow Y(j\omega) = \underbrace{\frac{\sum_{k=0}^{M} b_k\, (j\omega)^k}{\sum_{k=0}^{N} a_k\, (j\omega)^k}}_{H(j\omega)} X(j\omega)$$

- This means $H(j\omega)$ is in the form of the ratio of a numerator polynomial and a denominator polynomial in $j\omega$. Such a frequency response is said to be ***rational***, or of ***rational form***.

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{N(j\omega)}{D(j\omega)}}$$

Numerator Polynomial

Denominator Polynomial

Impulse Response from Rational Frequency Response

From the frequency response, we can determine the impulse response:

Example 4.25: $$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$b_1 = 1$ $b_0 = 2$

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

$a_2 = 1$ $a_1 = 4$ $a_0 = 3$

Factorizing $D(j\omega)$ followed by *partial fraction expansion*, we obtain:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{1/2}{(j\omega + 1)} + \frac{1/2}{(j\omega + 3)}$$

$(j\omega)^2 + 4(j\omega) + 3 = (j\omega + 1)(j\omega + 3)$

From Table 4.2, we find that the corresponding ***causal*** impulse response is a sum of two right-sided exponentials:

$$e^{-at}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{(j\omega + a)}$$

$a = 1$ $a = 3$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Example – Output for a given Input

Using the frequency response, we can determine the output for any input in frequency domain:

Example 4.25a– Assume now that the input is

$$x(t) = e^{\overset{a=4}{-4t}}u(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{\underset{a=4}{j\omega + 4}}$$

In frequency domain, we just multiply the input spectrums by the frequency response:

$$Y(j\omega) = H(j\omega)X(j\omega) = \overset{H(j\omega)}{\frac{j\omega + 2}{(j\omega+1)(j\omega+3)}}\overset{X(j\omega)}{\frac{1}{(j\omega+4)}}$$

Partial fraction expansion

$$= \frac{1/6}{(j\omega+1)} + \frac{1/2}{(j\omega+3)} - \frac{2/3}{(j\omega+4)}$$

$$\Big\downarrow \text{IFT}$$

And we can do inverse transform to obtain time domain solution:

$$y(t) = \left[\underset{}{\overset{a=1}{\frac{1}{6}e^{-t}}} + \overset{a=3}{\frac{1}{2}e^{-3t}} - \overset{a=4}{\frac{2}{3}e^{-4t}}\right]u(t)$$

Partial Fraction Expansion – Finding the Residues

The partial fraction expansion of a rational Fourier transform involves:

1) Finding the roots of the denominator polynomial, and

2) For each partial fraction term, evaluating the numerator constant, called the ***residue***.

residues

$$H(j\omega)=\frac{N(j\omega)}{D(j\omega)}=\frac{N(j\omega)}{(j\omega-\alpha_1)(j\omega-\alpha_2)(j\omega-\alpha_3)}=\frac{c_1}{(j\omega-\alpha_1)}+\frac{c_2}{(j\omega-\alpha_2)}+\frac{c_3}{(j\omega-\alpha_3)}$$

The following trick, illustrated for the case where the denominator polynomial has three distinct roots, will enable you to find the residue quickly:

$$\boxed{c_1=H(j\omega)\,(j\omega-\alpha_1)\big|_{j\omega=\alpha_1};\;\; c_2=H(j\omega)\,(j\omega-\alpha_2)\big|_{j\omega=\alpha_2};\;\; c_3=H(j\omega)\,(j\omega-\alpha_3)\big|_{j\omega=\alpha_3}}$$

$$H(j\omega)\,(j\omega-\alpha_1)\big|_{j\omega=\alpha_1}$$

$$=\left(c_1+\frac{c_2\cancelto{0}{(j\omega-\alpha_1)}}{(j\omega-\alpha_2)}+\frac{c_3\cancelto{0}{(j\omega-\alpha_1)}}{(j\omega-\alpha_3)}\right)\Bigg|_{j\omega=\alpha_1}=c_1$$

- **Notes** - If the denominator contains higher order roots, we can apply the trick to find as many residues as we can, and solve for the remaining ones.

<u>**Example**</u>:

$$H(j\omega) = \frac{N(j\omega)}{(j\omega-\alpha_1)(j\omega-\alpha_2)^2} = \frac{c_1}{(j\omega-\alpha_1)} + \frac{c_2}{(j\omega-\alpha_2)^2} + \frac{c_3}{(j\omega-\alpha_2)}$$

$$c_1 = H(j\omega)\,(j\omega-\alpha_1)\big|_{j\omega=\alpha_1}; \quad c_2 = H(j\omega)(j\omega-\alpha_2)^2\big|_{j\omega=\alpha_2}$$

Then we solve for c_3 after we have determined c_1 and c_2.

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II. From Frequency Response to Impulse Response

III. 2^{nd}-order Systems

II. From Frequency Response to Impulse Response

- From Example 4.25/4.25a, we have seen three equivalent ways of specifying the differential equation as LTI system:

1. By Differentiation Equation

$a_2 = 1$ $a_1 = 4$ $a_0 = 3$ $b_1 = 1$ $b_0 = 2$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$[(j\omega)^2 + 4(j\omega) + 3]Y(j\omega) = (j\omega + 2)X(j\omega)$$

2. By Frequency Response

$b_1 = 1$ $b_0 = 2$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

$a_2 = 1$ $a_1 = 4$ $a_0 = 3$

$$H(j\omega) = \frac{1/2}{(j\omega + 1)} + \frac{1/2}{(j\omega + 3)}$$

$a = 1$ $a = 3$

3. By Impulse Response

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Rational Fourier Transform in three Standard Forms

Also, a rational frequency response can be expressed in three standard forms:

1. The Polynomial Form

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{\sum_{k=0}^{M} b_k\,(j\omega)^k}{\sum_{k=0}^{N} a_k\,(j\omega)^k} = \frac{b_{N-1}(j\omega)^{N-1} + \cdots + b_1(j\omega) + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

higher order!

where $M = N - 1$.

because want $H(j\omega) \to 0$ when super high frequency!

This form allows us to write the differential equation. Usually, we assume the order of $D(j\omega)$ is higher than that of $N(j\omega)$ so that $H(j\omega) \to 0$ as $\omega \to \infty$ (systems are usually bandlimited)

2. The Factored Form

Where both $N(j\omega)$ and $D(j\omega)$ are factorized into a product:

$$\boxed{H(j\omega) = \frac{b_{N-1}\prod_{i=1}^{N-1}(j\omega - \beta_i)}{\prod_{k=1}^{N}(j\omega - \alpha_k)}}$$

Roots: value of $j\omega$ s.t. the polynomial evaluates to 0

This form allows us to plot the magnitude and phase response as function of ω.

$$|H(j\omega)| = \frac{|b_{N-1}|\prod_{i=1}^{N-1}|j\omega - \beta_i|}{\prod_{k=1}^{N}|j\omega - \alpha_k|}$$

We can control the magnitude and phase response of a system by controlling the locations of the roots.

$$\sphericalangle H(j\omega) = \sphericalangle b_{N-1} + \sum_{i=1}^{N-1}\sphericalangle(j\omega - \beta_i) - \sum_{k=1}^{N-1}\sphericalangle(j\omega - \alpha_k)$$

This form also says that we can implement an Nth-order system as a cascade (product) of 1st-order systems

3. The Partial Fraction Form

$$H(j\omega) = \sum_{k=1}^{N} \frac{c_k}{j\omega - \alpha_k}$$

Residues

Roots

This form allows us to find the inverse transform as sum of one-sided exponentials.

From Table 4.2 , the FT of a one-sided exponential is:

$$e^{-at}u(t);\ \ Re\{a\} > 0 \overset{FT}{\longleftrightarrow} \frac{1}{(j\omega + a)}$$

absolute integrable iff $Re\{a\} > 0$

By linearity (and assuming $h(t)$ is causal; more later), the impulse response is a sum of one-sided exponentials:

$$H(j\omega) = \sum_{k=1}^{N} \frac{c_k}{j\omega - \alpha_k} \Rightarrow h(t) = \sum_{k=1}^{N} c_k\, e^{\alpha_k t} u(t)$$

This form also says that we can implement an Nth-order system as a sum of 1st-order systems

Understanding Rational System by Roots of the Denominator Polynomial

$h(t)$ is completely specified by the roots α_k's and residues c_k's: $h(t) = \sum_{k=1}^{N} c_k\, e^{\alpha_k t} u(t)$ ✓

The roots α_k specify the exponential constants and tell us everything about the characteristics of the system.

- For a real system, $h(t)$ is real. Hence, for any complex root α_k, there must be another conjugate root $\alpha_{k'} = \alpha_k^*$ with residue $c_{k'} = c_k^*$, so that:

$$c_k e^{\alpha_k t} u(t) + c_k^* e^{\alpha_k^* t} u(t) = 2\mathrm{Re}\{c_k e^{\alpha_k t} u(t)\} = 2|c_k|\,\{e^{\mathrm{Re}\{\alpha_k\} t} \cos(\mathrm{Im}\{\alpha_k\} t + \measuredangle c_k)\}$$

The k-th term and k'-th term form a conjugate pair

Real part of a causal complex exponential multiplied by a complex number c_k

= A damped oscillation scaled by $|c_k|$ and shifted in phase by $\measuredangle c_k$

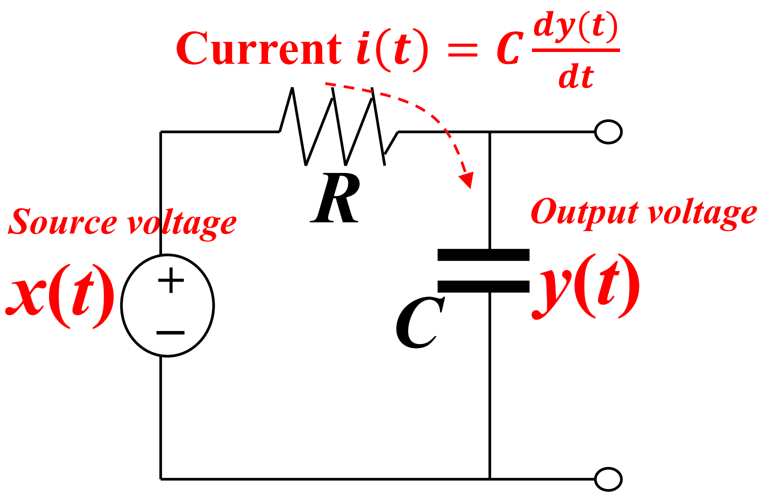
1. This means complex roots must occur in conjugate pairs, and if there is any complex root, the impulse response is <u>*oscillatory*</u>.

2. The roots α_k's are the exponential constants of the right-sided exponentials. Hence, if any root has a *non-negative real part*, the corresponding exponential grows with time and the system is *unstable*.

We can understand and implement any real Nth-order system as a product or sum of 1st and 2nd order real systems.

1^{st}-Order System as Non-Ideal Low-Pass Filter

We have seen in Chapter 1 that a simple RC circuit produces a 1^{st}-order LCCDE



$$x(t) = \overset{a_1}{RC}\frac{dy(t)}{dt} + y(t)$$

$X(j\omega) = [RC(j\omega) + 1]\, Y(j\omega)$

$H(j\omega) = \frac{1}{RC(j\omega)+1}$

Taking transform of both sides $\Rightarrow X(j\omega) = RCj\omega Y(j\omega) + Y(j\omega)$ $= \frac{1}{RC}\frac{1}{j\omega + \frac{1}{RC}}$

Or by inspection, frequency response is:

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{RCj\omega + 1} = \frac{\overset{\text{Residue}}{1/RC}}{j\omega + 1/RC}$$

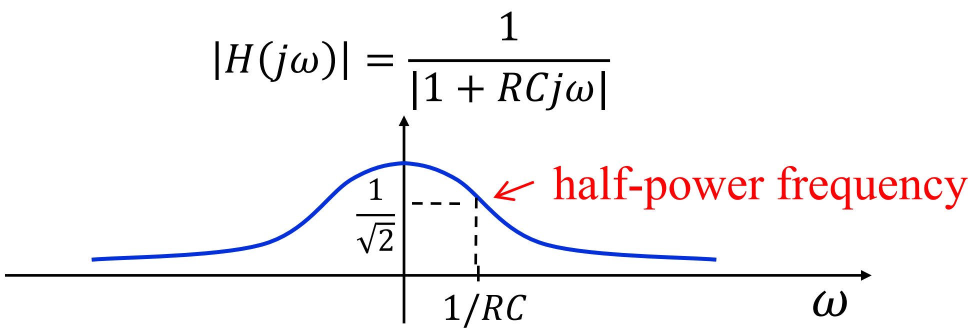
$a_1 = RC$ $\qquad \alpha_1 = -\frac{1}{RC}$

$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$

- From Example 4.1 or Table 4.2, the causal impulse response is a right-sided exponential:

$$h(t) = \mathfrak{I}^{-1}\{H(j\omega)\} = \overset{\text{residue}}{\frac{1}{RC}}\, e^{-\overset{\alpha_1}{\frac{1}{RC}}t}u(t)$$

- The magnitude response, $|H(j\omega)| = \frac{1}{|RCj\omega+1|}$ decreases with ω:



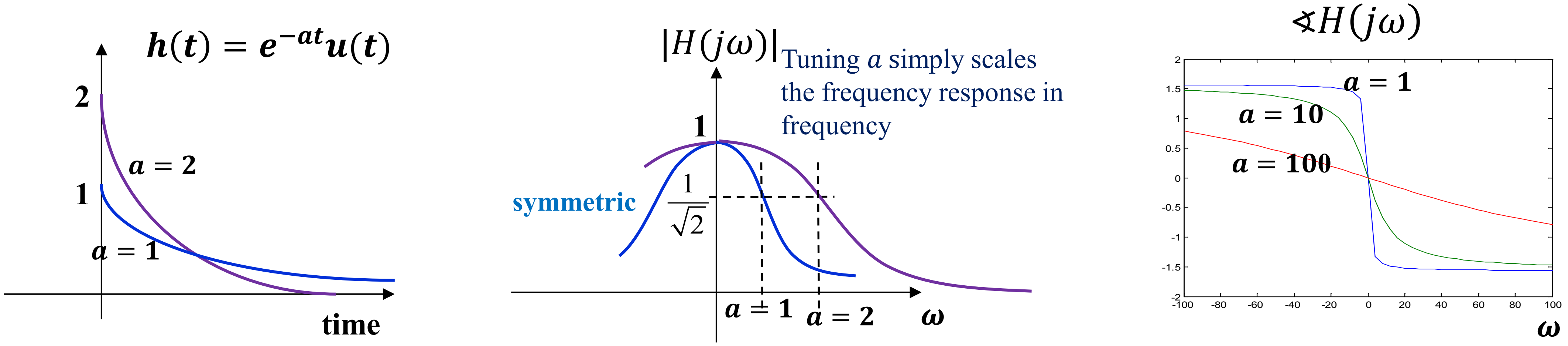
$|H(0)| = 1$

$$|H(j\omega)|_{\omega=\frac{1}{RC}} = \frac{1}{|1+j|} = \frac{1}{\sqrt{2}}$$

- *We observe that all (real) 1st order systems are a* ***non-ideal low-pass filter***.
- For the real 1-st order system, $a = |\alpha_1| = \frac{1}{RC}$ determines the half-power cut-off frequency of the filter since $|H(ja)| = 1/\sqrt{2}$.

- The 1-st order filter does not provide a very sharp cut-off between pass-band and stop-band. The only parameter that one can tune is a.
- Tuning a simply scales the frequency response in frequency but does nothing to change the shape of the filter.

Impulse, Magnitude, and Phase Response for different $a = 1/RC$



- To make filter with sharper cut-off we need higher order (later).

Lecture 19

Differential Equations as LTI Systems

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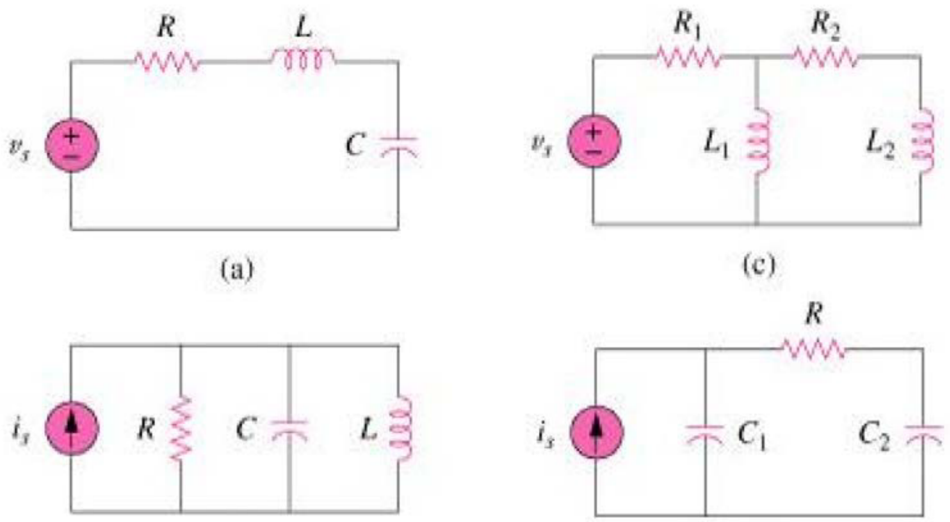
III. 2^{nd}-order Systems

III. 2nd-Order Systems

- An RC circuit gives a 1st-order system. Circuits with resistors and two energy storage elements give a second order system.

only decay!



← exchange energy

overshoot!

- Using many 1st- and 2nd- order circuits, we can build high-order CT low-pass and band-pass filters.

Frequency and Impulse Response of 2nd-order Systems

- For a 2nd order LCCDE, we can normalize the leading coefficient a_2 to 1 for convenience:

$$y^{(2)}(t) + a_1 y^{(1)}(t) + a_0 y(t) = b_1 x^{(1)}(t) + b_0 x(t)$$

Leading coefficient, a_2, normalized to 1

- For a real system, all coefficients a_0, a_1, b_0, b_1 are real.
- Frequency response is rational:

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{b_1(j\omega)+b_0}{(j\omega)^2+a_1(j\omega)+a_0}$$

We can find roots of $D(j\omega)$ and express $H(j\omega)$ in partial fraction form:

Residues

$$H(j\omega) = \frac{b_1(j\omega)+b_0}{(j\omega-\alpha_1)(j\omega-\alpha_2)} = \frac{c_1}{(j\omega-\alpha_1)} + \frac{c_2}{(j\omega-\alpha_2)}$$

Roots

poles!

Which gives causal impulse response as: $h(t) = c_1 e^{\alpha_1 t} u(t) + c_2 e^{\alpha_2 t} u(t)$

Roots of the Denominator Polynomial

- The roots of $D(j\omega) - \alpha_1, \alpha_2$ are the exponential constants in the impulse response and completely determine the characteristics of the response.

- The roots α_1, α_2 are given by the quadratic formula:

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_0}}{2}$$

the "radical"

$$\overset{1}{\cancel{A}}x^2 + \overset{a_1}{\cancel{B}}x + \overset{a_0}{\cancel{C}} = 0$$

$$\Rightarrow \text{Roots} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

- We note that α_1, α_2 are complex if ${a_1}^2 < 4a_0$, or $|a_1| < 2\sqrt{a_0}$

Radical is imaginary if the term inside the square root is negative

- We note also that: 1. $\alpha_1 \alpha_2 = a_0$;

  and 2. $\alpha_1 + \alpha_2 = -a_1$

Because $(j\omega - \alpha_1)(j\omega - \alpha_2) = (j\omega)^2 - \overbrace{(\alpha_1 + \alpha_2)}^{a_1} j\omega + \overbrace{\alpha_1 \alpha_2}^{a_0}$

System Characterization by Roots 1 – Oscillatory Response

1. **<u>Oscillatory Response</u>** - *<u>System is oscillatory if roots of denominator polynomial are complex.</u>*

- If α_1, α_2 are real,

  then $h(t) = c_1 e^{\alpha_1 t} u(t) + c_2 e^{\alpha_2 t} u(t)$ is a sum of two real one-sided exponentials.

- For a real second order system, if α_1, α_2 are complex, they must be a conjugate pair and so are

  c_1, c_2 , and

  the impulse response is a damped oscillation:

Real part of complex exponential is a damped oscillation

$$h(t) = c_1 e^{\alpha_1 t} u(t) + \; c_1^* e^{\alpha_1^* t} u(t) \; = 2Re\{c_1 e^{\alpha_1 t} u(t)\} \; = 2|c_1| e^{\sigma t} \cos(\omega t + \sphericalangle c_1)\, u(t)$$

where $\sigma = \mathrm{Re}\{\alpha_1\}$ and $\omega = \mathrm{Im}\{\alpha_1\}$

Systems Characterization by Roots 2 - Stability

2. Stability $h(t) = c_1 e^{\alpha_1 t} u(t) + c_2 e^{\alpha_2 t} u(t)$

$(s + \sigma j)$ $(s - \sigma j)$

For $h(t)$ to be stable, both α_1, α_2 must have negative real parts. This means *with a_2 normalized to 1, all coefficients in the denominator must be positive* $(a_0, a_1 > 0)$.

This is because, with leading coefficient a_2 normalized to 1:

- Recall $\alpha_1\alpha_2 = a_0$. If $\alpha_1\alpha_2 = a_0 < 0$, it implies α_1, α_2 must be real with one being positive and one negative. α_1, α_2 cannot be complex because if they are, they must be a conjugate pair and their product cannot be <0.
- If $a_0 = 0$, at least one root is equal to 0, meaning $h(t)$ contains a unit step.
- Recall $\alpha_1 + \alpha_2 = -a_1$. If $a_1 \le 0$, it means $\alpha_1 + \alpha_2 = -a_1 \ge 0$. Then:

  growing!

  If α_1, α_2 are complex, then $\alpha_1 = \alpha_2{}^*$, implying that Re$\{\alpha_1\}$=Re$\{\alpha_2\}$= $\frac{\alpha_1 + \alpha_2}{2} \ge 0$

  If α_1, α_2 are real, at least one of them is non-negative

Characterization by Natural Frequency and Zeta Parameter

We often express a 2nd order system using two alternative parameters :

i. The *natural frequency* $\omega_n = \sqrt{a_0}$, which provides a scaling in frequency.

ii. A *zeta parameter* $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$ which conveniently shows if the roots are complex.

$a_1 = 2\zeta\omega_n$

Re-express $D(j\omega)$ in terms of ω_n and ζ as:

$$D(j\omega) = (j\omega)^2 + a_1(j\omega) + a_0 = (j\omega)^2 + \overset{a_1}{2\zeta\omega_n}(j\omega) + \overset{a_0}{\omega_n{}^2}$$

Then:

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_0}}{2} = \frac{\overset{a_1}{2\zeta\omega_n} \pm \sqrt{(\overset{a_1}{2\zeta\omega_n})^2 - \overset{a_0}{4\omega_n{}^2}}}{2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

scaling

$|\zeta| \geq 1 \Rightarrow \alpha_1, \alpha_2$ real

$|\zeta| < 1 \Rightarrow \alpha_1, \alpha_2$ complex

While ω_n provides a scaling in frequency, ζ makes explicit whether ${a_1}^2 - 4a_0 \geq 0$ and it also fully characterizes the system. (Recall $\omega_n = \sqrt{a_0}$, $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$, $\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$)

$\nearrow$ decay / constant $\Rightarrow$ unstable!

3a. If $\zeta \leq 0$, system is unstable (since it means $a_1 \leq 0$)

3b. If $|\zeta| < 1$, roots have imaginary part and system is *oscillatory* (if $|\zeta| < 1$ means ${a_1}^2 - 4a_0 < 0$)

– decay very slowl

3c. If $\zeta \gg 1$, one root is nearly $-2\zeta\,\omega_n$ but the other root is only slightly less than 0. Therefore the impulse response decays very slowly. The system is *over-damped*.

$\zeta \gg 1$ implies $\sqrt{\zeta^2 - 1} = \zeta - \varepsilon$ where $\varepsilon = 0^+$ <----- *Means just slightly larger than 0*

Hence, $\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) = \omega_n(-2\zeta + \varepsilon), \omega_n(-\varepsilon)$ Close to 0; decay very slowly

3d. If $\zeta \cong 1$, both roots are near $-\omega_n$, and system impulse response decays at the fastest rate possible. System is *critically damped* – desirable in a suspension system.

3e. If $\zeta \to 0^+$, system is *under-damped*. $|H(j\omega)|$ may become very large around ω_n as we will show in a few slides.

Example

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_0}}{2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \qquad \omega_n = \sqrt{a_0};\ \zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$$

- These systems are not stable:

$$y^{(2)}(t) + 2y^{(1)}(t) - y(t) = 5x^{(1)}(t) + 3x(t)$$

For stable systems, coefficients on output side must be all positive after a_2 is normalized to 1

$$y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 5x^{(1)}(t) + 3x(t)$$

$$-y^{(2)}(t) + 2y^{(1)}(t) + y(t) = 5x^{(1)}(t) + 3x(t)$$

More Examples

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_0}}{2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \quad \omega_n = \sqrt{a_0};\ \zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$$

$\zeta = \frac{a_1}{2\omega_n}$

- Oscillatory:

$\omega_n = \sqrt{4} = 2$ ✓ $\quad \zeta = \frac{3}{2\times 2} = \frac{3}{4} \Rightarrow |\zeta| < 1 = \frac{3}{4}$

$\Rightarrow$ no real roots!

$$y^{(2)}(t) + 3y^{(1)}(t) + 4y(t) = 5x^{(1)}(t) + 3x(t)$$

decay very fast!

one term $e^{-\zeta t}$

- Critically damped:

$\omega_n = \sqrt{1} = 1 \quad \zeta = \frac{2}{2\times 1} = 1$?

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = -2x^{(1)}(t) + x(t)$$

decay very slow!!

$\alpha_1, \alpha_2 = 0.1(-5 \pm \sqrt{24})$

$= 0.5 \pm 0.1 \times 5$

$\approx -.1 + \zeta, -\zeta$

- Over-damped:

no oscillation!

$\omega_n = \sqrt{0.01} = 0.1 \quad \zeta = \frac{1}{2\times 0.1} = 5 \Rightarrow \zeta \gg 1$

$$y^{(2)}(t) + y^{(1)}(t) + 0.01y(t) = 5x^{(1)}(t) + 3x(t)$$

Underdamping Example

$a_1 = 2$

$H(j\omega) = \dfrac{20j\omega + 7}{(j\omega)^2 + 2j\omega + 100}$

a_0

$\zeta \to 0$

Dangerous!

- An under-damped system: $\omega_n = \sqrt{100} = 10;\quad \zeta = \dfrac{a_1}{2\omega_n} = \dfrac{2}{2 \times 10} = 0.1 \to 0^+$

$$y^{(2)}(t) + 2y^{(1)}(t) + 100y(t) = 20x^{(1)}(t) + 7x(t)$$

≈ 0

System resonates at large amplitude if stimulated at its natural frequency of 10 rad/sec.

Magnitude response at $\omega = 10$ is:

$\sqrt{200^2 + 7^2} \cong 200$

$$|H(j10)| = \left|\frac{20(j10)+7}{\cancel{(j10)^2}+2(j10)+\cancel{100}}\right| \cong \frac{200}{20} = 10$$

no damping at all!

Input = $\cos(10t)$ → Output = $10\cos(10t + \measuredangle H(j10))$

get amplified by 10 times!!!

In the next slide, we show video of the infamous 1940 collapse of the Tacoma Bridge because of excitation by wind at the natural frequency

http://www.youtube.com/watch?v=3XE5qU0c5qU

1940 Collapse of Tacoma Narrow Bridge

under-damped

Resonance at natural frequency led to the infamous collapse of the Tacoma Bridge