1. Intro.

Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Common FT Pairs

$x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

$x^*(t) \leftrightarrow X^*(-j\omega)$

$x(at) \leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a})$

$x(t) * y(t) \leftrightarrow X(j\omega) Y(j\omega)$

$x(t) y(t) \leftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$

$\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$\sin(\omega_0 t) \leftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

$x(t) = \begin{cases} 1 & |t|<T_1 \\ 0 & |t|>T_1 \end{cases} \leftrightarrow \frac{2\sin\omega T_1}{\omega}$

$\frac{\sin Wt}{\pi t} \longleftrightarrow X(j\omega) = \begin{cases} 1 & |\omega|<W \\ 0 & |\omega|>W \end{cases}$

$\delta(t) \leftrightarrow 1 \qquad u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$

$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$

$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$

$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n}$

$(-jt)^n x(t) \longleftrightarrow \frac{d^n}{d\omega^n} X(j\omega)$

$x'(t) \longleftrightarrow j\omega X(j\omega)$

$\Pi(t/T) \longleftrightarrow T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$

$\Lambda(t/T) \longleftrightarrow T\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$

$\int_{-\infty}^{t} x(t)dt \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Wireless Channel

- attenuation
- Fading
- noise
- Time variation
- Filtering

FER: $\frac{\#\text{ of frame corrupted}}{\#\text{ of frame transmitted}}$

3. Signal Space:

$$E_s = \int |x(t)|^2 dt = \int |H(f)|^2 df = \|\vec{x}\|^2$$

Gram-Schmidt

$\phi_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|}$

$v_2 = \vec{x}_2 - \langle\vec{\phi}_1, \vec{x}_2\rangle \vec{\phi}_1, \quad \phi_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$

$v_3 = \vec{x}_3 - \langle\vec{\phi}_1, \vec{x}_3\rangle\vec{\phi}_1 - \langle\vec{\phi}_2, \vec{x}_3\rangle\vec{\phi}_2$

$v_m = \vec{x}_m - \sum_{i=1}^{m-1} \langle\vec{\phi}_i, \vec{x}_m\rangle \vec{\phi}_i$

$\phi_m = \frac{\vec{v}_m}{\|\vec{v}_m\|}$

☆ Product to Sum formula!

$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$

$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

$\cos(\alpha)\sin(\beta) = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$

2. Digital-mod

$n(t)$: AWGN, PSD $= \frac{N_0}{2}$

$E(n) = 0, \quad E(n^2) = \frac{N_0 T}{2}$

$f_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right)$

$P_e = P(E|1)P(1) + P(E|0)P(0)$

$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2} dt$

$\quad = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \forall x \geq 3$

Optimal threshold:

$V_T = \frac{S_{01}+S_{00}}{2} + \frac{\sigma^2}{S_{01}-S_{00}} \ln\frac{P(0)}{P(1)} = \frac{E_1 - E_0}{2}$

$P_e = P(0)\,Q\left(\frac{V_T - S_{00}}{\sigma}\right) + P(1)\,Q\left(\frac{S_{01}-V_T}{\sigma}\right)$

- Optimal filters

Assume $P(0) = P(1) = 0.5$

$P_e = Q\left(\sqrt{\frac{(S_{01}-S_{00})^2}{4\sigma^2}}\right) = Q\left(\sqrt{\frac{E_g}{2N_0}}\right)$

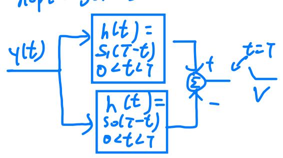
Define

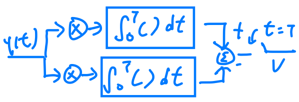
$g(t) = s_1(t) - s_0(t)$

$E_g = \int_0^T g(t)^2 dt$

Optimal filter

$h_{opt} = g(T-t)$





Antipodal signaling

$s_1(t) = A$

$s_0(t) = -A$

$P_e = Q\sqrt{\frac{2A^2T}{N_0}} = Q\left(\sqrt{\frac{E_b}{2N_0}}\right)$

Non-return to Zero

$s_1(t) = A$

$s_0(t) = 0$

$P_e = Q\left(\sqrt{\frac{A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

Amplitude shift Keying (ASK)

$s_1(t) = A\cos(\omega_c t + \theta_c)$

$s_0(t) = 0$

$P_e = Q\left(\sqrt{\frac{A^2T}{4N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

Phase Shift Keying (BPSK)

$s_0(t) = A\cos(\omega_c t + \theta_c)$

$s_1(t) = -A\cos(\omega_c t + \theta_c)$

$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

Frequency Shift Keying (BFSK)

$s_0(t) = A\cos(\omega_1 t + \theta_c)$

$s_1(t) = A\cos(\omega_2 t + \theta_c)$

$f_2 > f_1, \quad \Delta f = f_2 - f_1, \quad \Delta f = \frac{n}{T}$

Sum to product formula

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

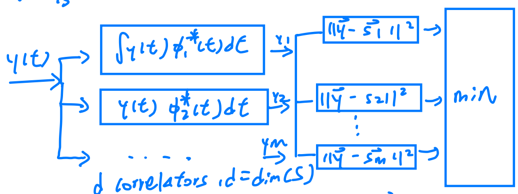
4. General M-ary mod. demod

Symbol Rate: $\frac{1}{T_s}$

Bit Rate: $\frac{1}{T_s} \times \frac{\text{bits}}{\text{symbol}} = (\log_2 M)\frac{1}{T_s}$

Transmittion BW

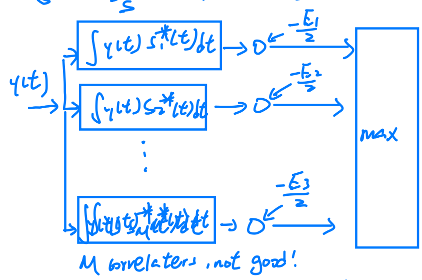
$W = \frac{1}{T_s}(1+\alpha)$



$\hat{s} = \arg\min \|\vec{y} - \vec{s}\|^2, \; s \in \{\vec{s}_1, \cdots \vec{s}_M\}$

Equivalently

$\hat{s} = \underset{s}{\arg\max}\left[\langle\vec{y}, \vec{s}\rangle - \frac{1}{2}\|\vec{s}\|^2\right]$



noise vector: AWGN, PSD $= \frac{N_0}{2}$

$E(\vec{n}) = 0, \quad \sigma^2 = \frac{N_0}{2}$

$P_e(s_i) = 1 - \int_{R_i} p(\vec{y}|\vec{s}_i) dy$

$P_e = 1 - \sum_{j=1}^{M} \int_{R_j} p(s_j) p(y|s_j) dy$

Optimal detection: $s^* = \underset{s \in \{s_1 \cdots s_N\}}{\arg\max}\, p(s)p(y|s)$

Maximal likelihood detection: If $p(s) = \frac{1}{M}$

$s^* = \underset{s \in \{s_1 \cdots s_N\}}{\arg\max}\, p(y|s)$

When s_i is sent, error occurs

$\|\vec{y} - \vec{s}_i\|^2 \geq \|\vec{y} - \vec{s}_j\|^2 \Leftrightarrow \int_0^{T_s}[s_j(t) - s_i(t)]n(t)dt$

$\int_0^{T_s}[s_j(t) - s_i(t)]n(t)dt \quad -\frac{1}{2}\int_0^{T_s}[s_j(t) - s_i(t)]^2 dt$

$\mu = 0$

$\sigma_z^2 = \frac{N_0}{2}\int_0^{T_s}(s_j - s_i)^2 dt$

$P(e|s_i) \leq \sum_{j \neq i} Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right)$

$P_e \approx \frac{1}{M} Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$

6. MFSK - Error analysis

Let $x = \frac{b_1 - E - a}{\sqrt{N_0 E/2}}$, $\quad b_1 = \begin{cases} E + k_1 & \text{if } k=1 \\ n_k & \text{otherwise} \end{cases}$

$P(\text{correct}|n_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[1 - Q\left(x + \sqrt{\frac{2E}{N_0}}\right)\right]^{M-1} e^{-\frac{x^2}{2}} dx$

$k = \log_2 M, \quad E_s = kE_b$

Particular symbol error $= \frac{P_{eM}}{M-1}$

Prob. n bit errors $= \binom{k}{n}\frac{P_{eM}}{M-1}$

Average $= \sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{M-1}$, on k bits

$P_b = P_e = \frac{1}{k}\sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{M-1} = \frac{M}{2(M-1)} P_{eM}$

$P_{UB}(e|s_j) = (M-1)\,Q\left(\sqrt{\frac{E_s}{N_0}}\right) = P_{UB}(e)$

$\Rightarrow P_b = \frac{M}{2}\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

If $M\uparrow$, $P_{UB} \gg P_e$, exact!

$P_{UB}(e) = (M-1)\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

☆ $\rightarrow \infty$

$E_s = kE_b, \quad k = \log_2 M$

7. MQAM - Error Analysis

MPSK : $s_k(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left[w_c t + \frac{2\pi(k-1)}{M}\right]$

$w_c = \frac{n \cdot 2\pi}{T_s}$

dimension = 2 , $\varphi_1(t) = \sqrt{\frac{2}{T_s}} \cos(w_c t)$

$\varphi_2(t) = \sqrt{\frac{2}{T_s}} \sin(w_c t)$

$P_{eM} = \frac{1}{\pi} \int_0^{\pi(1-\frac{1}{M})} \exp\left[\frac{(E_s/N_0)\sin^2(\pi/M)}{\sin^2\phi}\right] d\phi$

$P_M \approx 2Q\left(\sin\frac{\pi}{M}\sqrt{\frac{2E_s}{N_0}}\right)$

Grey Coding

Adjacent symbol

only one bit changes

$P_b \approx \frac{P_{eM}}{\log_2 M}$

MQAM modulation

$s_k(t) = a_k\cos(w_c t) + b_k\cos(w_c t)$, $k = \{1, 2, \cdots M\}$

$a_k, b_k \in \{\pm\alpha, \pm 3\alpha, \cdots, \pm(\sqrt{M}-1)\alpha\}$, $M = \{4, 16, 64, 256\}$

$R_b = \frac{1}{T_s}\log_2 M$, $E_s = \frac{2}{\sqrt{M}}(2\alpha^2 + 2(3\alpha)^2 + \cdots)$

for 16-QAM , $E_s = 10\alpha^2$

Average Transmit Power: $P_s = \frac{E_s}{T_s}$

$P_{correct} = (1 - P_{\sqrt{M}})^2$

$P_{\sqrt{M}} = 2\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$

$P_M = 1 - (1-P_{\sqrt{M}})^2 \approx 2P_{\sqrt{M}} = 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$

gain $= \frac{3/(M-1)}{2\sin^2(\pi/M)}$, if $M > 4$ then better

MQAM union bound

$P_e \approx 4Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right) \le 4Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$

MFSK, $M\uparrow$, bandwidth $\uparrow$, $P_b\downarrow$

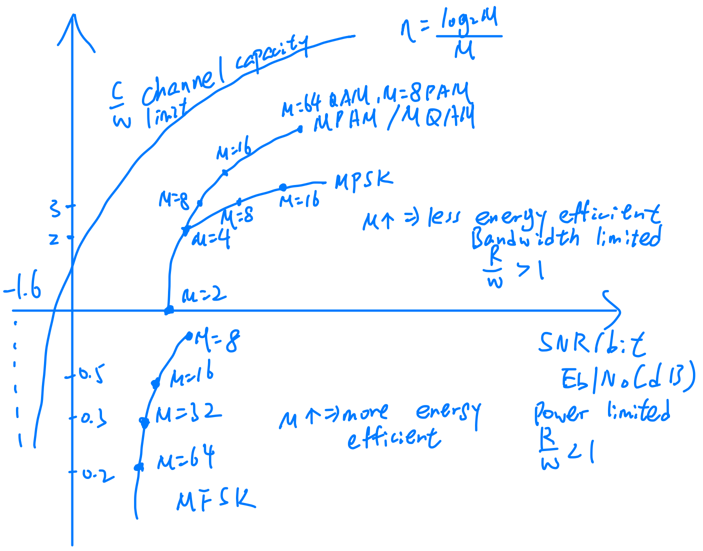
MPSK, $M\uparrow$, bandwidth $\downarrow$, $P_b\uparrow$

Shannon Limit

$R = B\log_2(1+SNR)$

R: bit rate

B: bandwidth



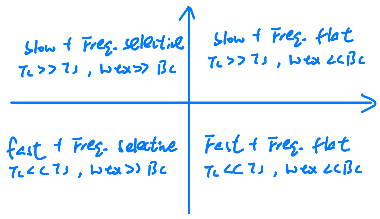
$h(t, \tau) = \sum_i \alpha_i(t)\delta(\tau - \tau_i)$

$H(t, f) = \sum_i \alpha_i(t) e^{-j2\pi f\tau_i}$

$\tau_{RMS} = \sqrt{\overline{\tau^2} - \bar{\tau}^2}$

$\bar{\tau} = \frac{\sum P_i \tau_i}{\sum P_i}$ $\overline{\tau^2} = \frac{\sum P_i \tau_i^2}{\sum P_i}$

$B_c = \frac{1}{k\sigma_\tau}$



# of resolvable multipaths

$\left\lceil\frac{T_s}{\sigma_\tau}\right\rceil = \left\lceil\frac{W_{tx}}{B_c}\right\rceil$

Joint Gaussian distribution

$f(\alpha_I, \alpha_Q) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\alpha_I^2 + \alpha_Q^2}{2\sigma^2}\right)$

$f(r, \theta) = \frac{r}{2\pi\sigma^2}\exp\left(\frac{r^2}{2\sigma^2}\right)$

$f(\theta) = \frac{1}{2\pi}$ $\theta \in [0, 2\pi)$

$f(r) = \frac{r}{\sigma^2}e^{-r^2/2\sigma^2}$, $E(r) = \sigma\sqrt{\frac{\pi}{2}}$

Power distribution : exp. distribution

$f(p) = \frac{1}{p_0}e^{-p/p_0}$, $p_0 \ge 0$

$p_0 = 2\sigma^2$: Mean power

Time-Varying fading

$S_\alpha(f) \propto \frac{1}{\sqrt{f_D^2 - f^2}}$

$R_\alpha(\tau) \propto J_0(2\pi f_D \tau)$

$f_D = \frac{v}{c}f_c$

$f(\theta) = f_c + f_D\cos\theta$

$S_\alpha(f) = \frac{K}{\sqrt{f_D^2 - f^2}}$ $|f| \le f_D$

$f_d = \frac{v}{\lambda}$, $T_c \approx \frac{9}{16\pi f_D}$

Random CDMA

$G_P = \frac{T_m}{T_c} = \frac{B_c}{B_m}$

B_m: message bandwidth

B_c: signal bandwidth
chip

G_P larger, better!

Diversity Techniques

- Selection combining

  $y_{sel} = y_{d^*}$, $d^* = \underset{d \in \{1, \cdots D\}}{\arg\max} |\alpha_d|^2$

  select most reliable one!

- Equal Gain combining

  $y_{EGC} = \sum_d \tilde{\alpha}_d y_d$,

  $\tilde{\alpha}_d = \exp(-j\alpha_d)$

- Maximal ratio combining (the best)

  $y_{MRC} = \sum_d \alpha_d^* y_d$

$\forall$ cases, $SER \approx \frac{1}{SNR^D}$