Noise!

Ch7.1: Baseband Communications & Noise

Information source and input transducer → Source Coding → Channel Coding → Modulator → Channel

- Questions to be answered:

white gaussian noise!

  - **System Model:** AWGN Channel
  - **White Gaussian Noise:** A Random Process
  - **Suboptimal Receiver:** Integrate-and-Dump
  - **Performance Evaluation:** How Good is the Receiver?

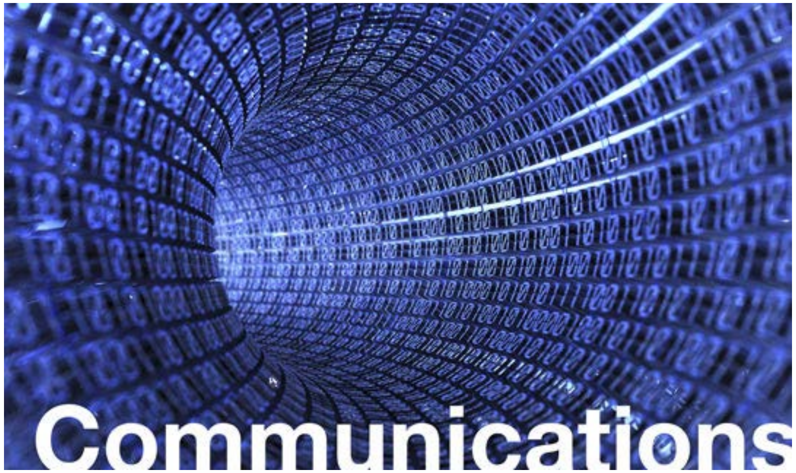
Pe

Ch7.1: Binary Communications

Channel → Demodulator (Matched Filter) → Channel Decoding → Source Decoding → Information sink and output transducer

Ch7.1: Baseband Communications & Noise

- **System Model**
- White Gaussian Noise
- Suboptimal Receiver
- Performance Evaluation
  - BER
  - A General Case



Digital Communications

- In **digital communication systems**, the **signals take discrete values** to represent binary **data**.

- For example, **-A, A** can be used to **represent logical levels, 0 and 1**, respectively.

- The above modulation signaling is called **binary** digital communications since **there are only two bits**.

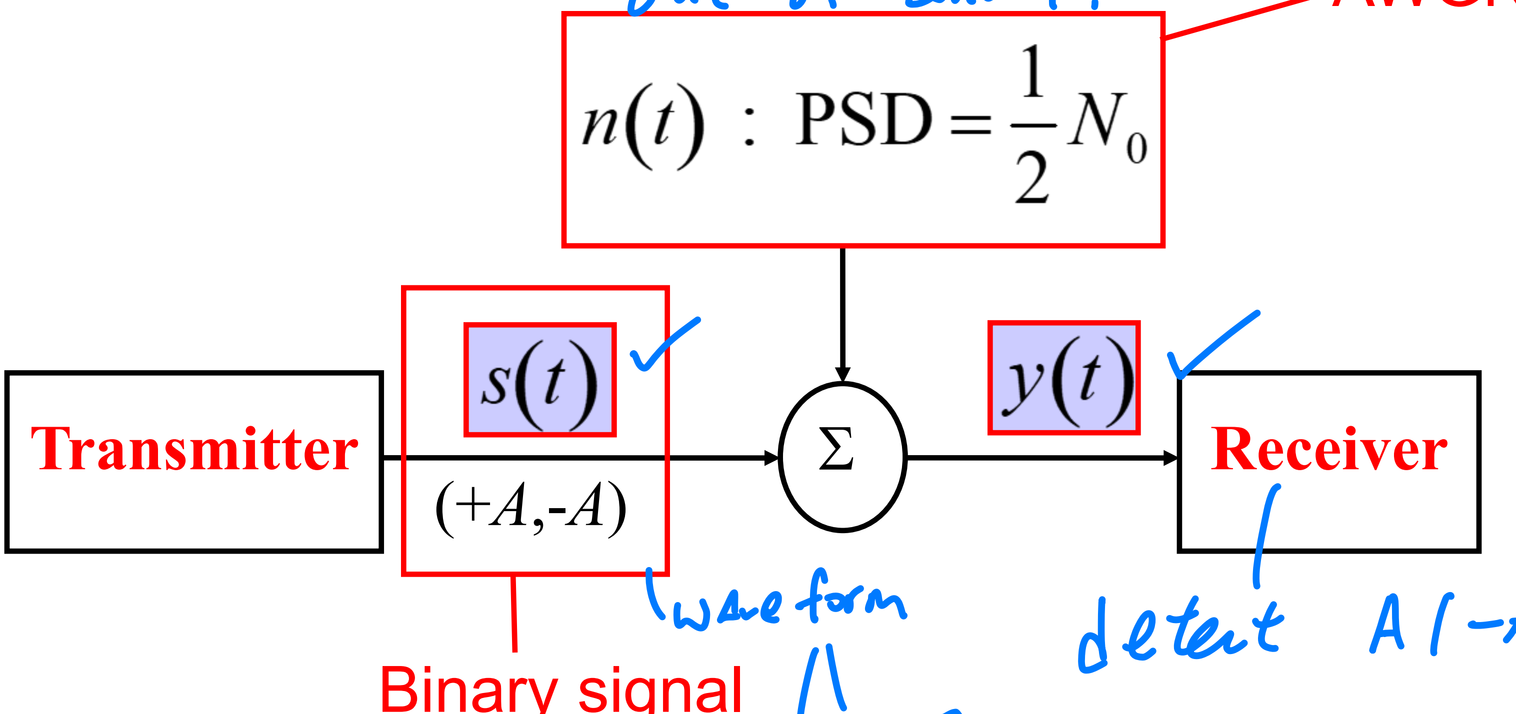
Baseband Digital Data Transmission

Goal. Minimize Pe

- The system model for **baseband digital data transmission** is

wire ..

out of control!



waveform

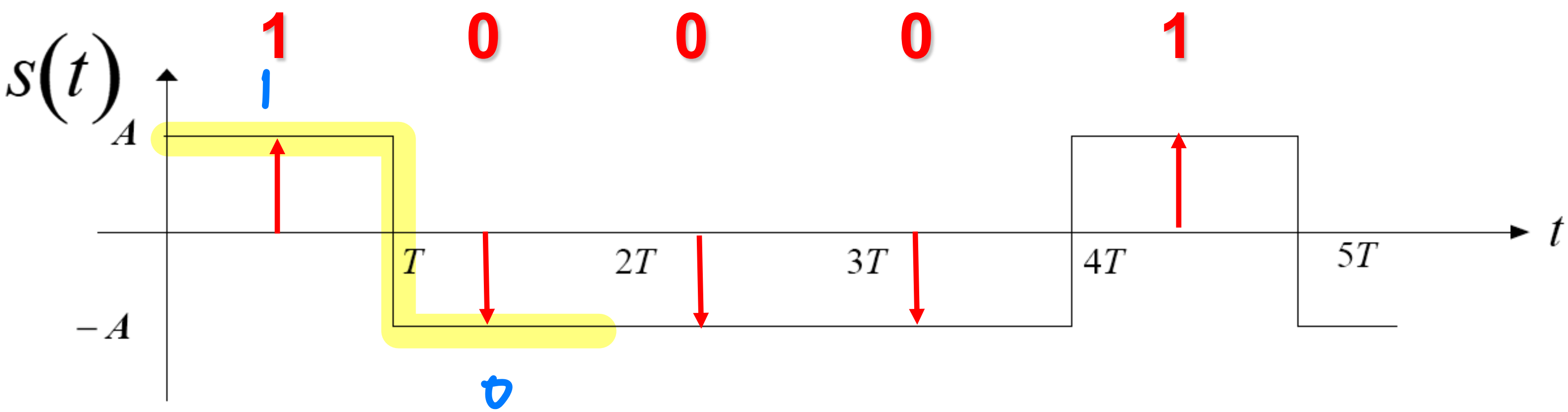
detect A/-A!

0 1 不同!!

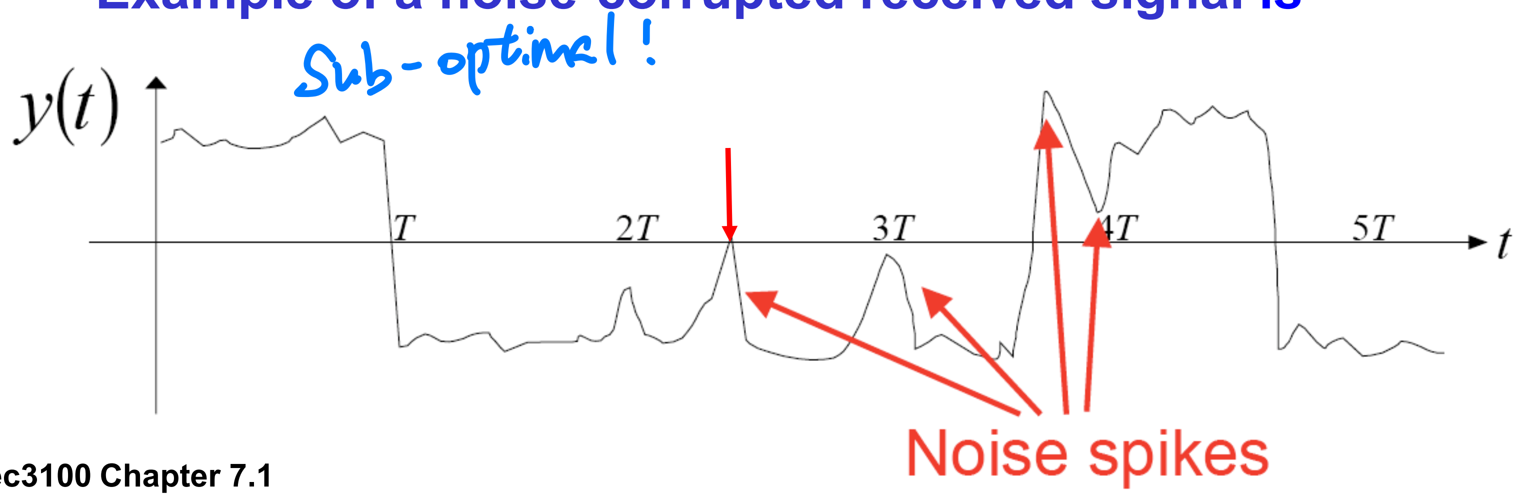
retanguler pulse one/zero!

Baseband Digital Data Transmission

- Example of a digital signal and transmitted waveform is



- Example of a noise-corrupted received signal is



Ch7.1: Baseband Communications & Noise

- System Model
- **White Gaussian Noise**
- Suboptimal Receiver
- Performance Evaluation
  - BER
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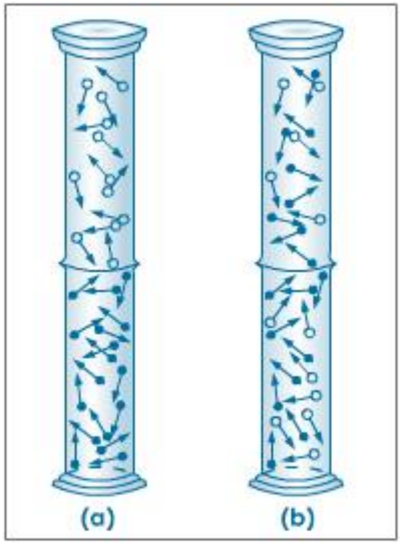
Types of Noise

- In this course, we assume **internal noise** sources are much more significant than external ones and deal with these only.

- Noise internal to a communications system arises as a result of random motion of charge carriers within the devices (transistors, resistors, diodes, etc) composing the system.

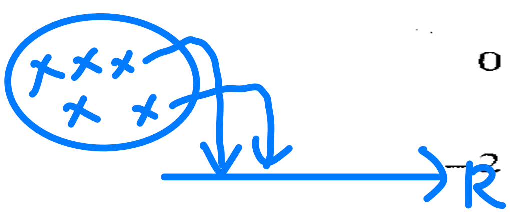
- Internal noise is classified as ***thermal***, **shot**, **flicker** or others.



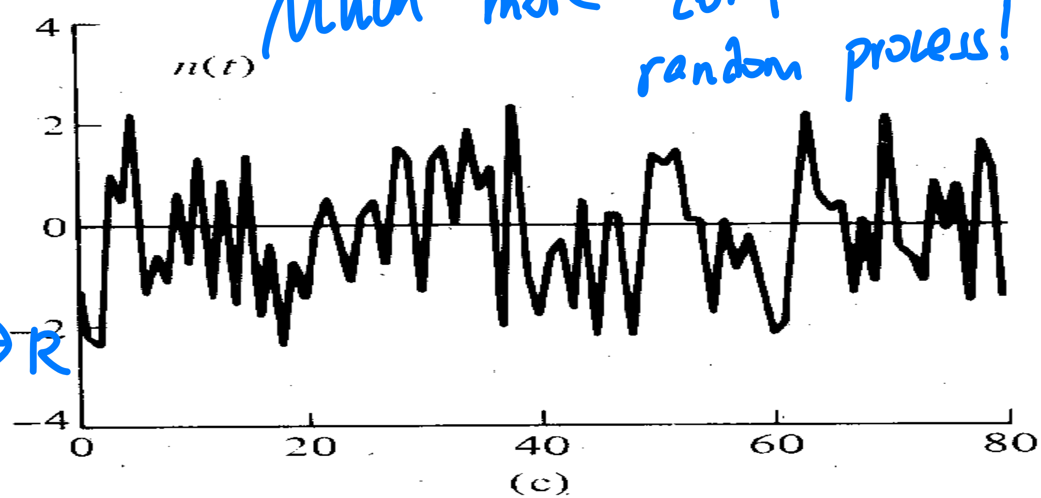
Thermal Noise

- **Thermal noise** arises from the random motion of charges in conducting medium (such as resistors) and is the **most significant noise source** we need to consider in ELEC3100.
- Essentially, by connecting a very sensitive oscilloscope across a resistor we can observe thermal noise.
- A **typical noise output** would look like:

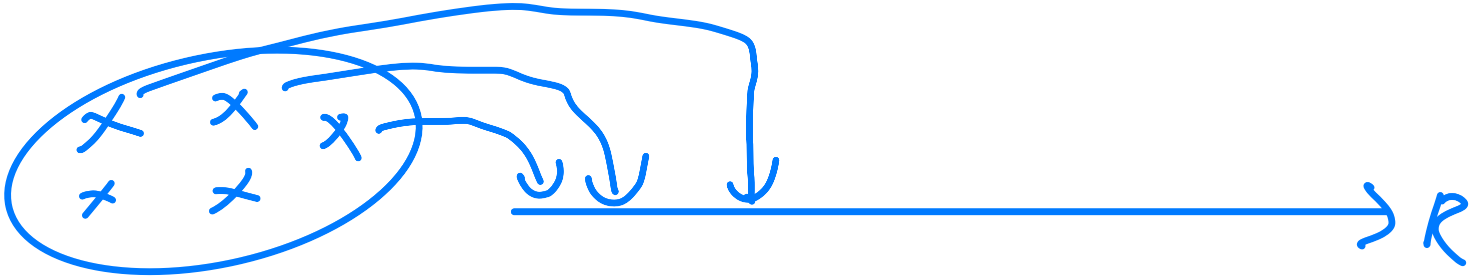
Random variable!



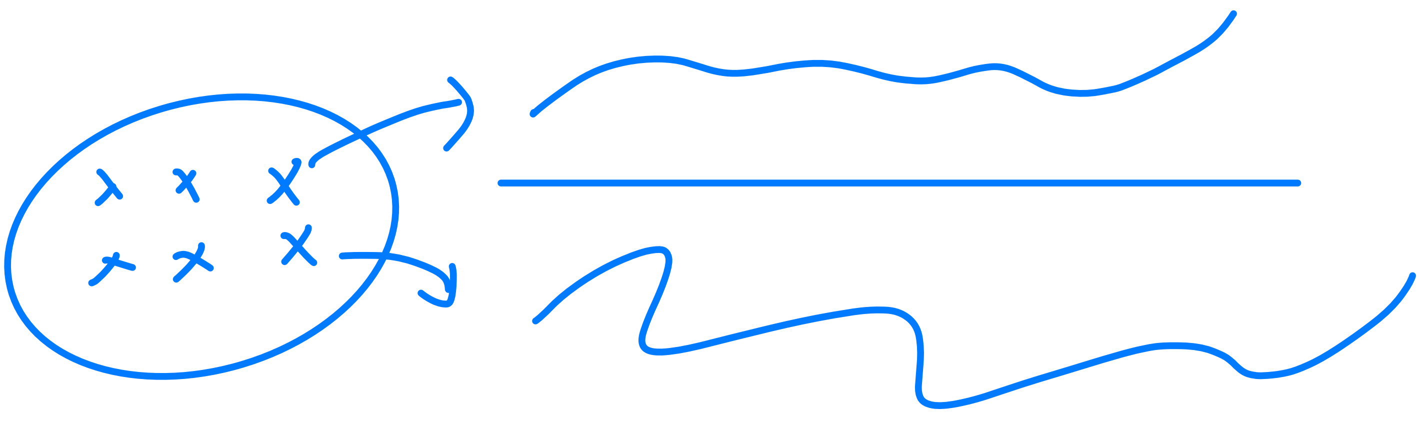
Much more complicated random process!



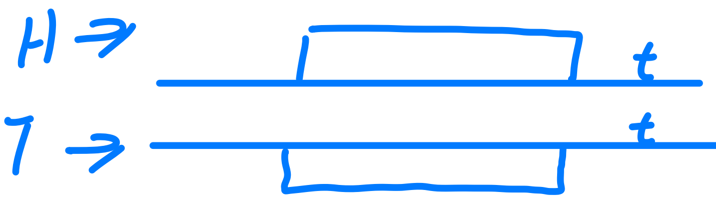
Random Variable



Random Process (RP) (noise is waveform)



e.g. $S = \{H, T\}$ RP



↳ in this course RV

Nyquist Theorem

- The signal is **completely random** and **cannot be predicted.** It has an **average value** of **zero** but has associated with it a certain **mean square voltage**.
- The mean square voltage associated with the thermal noise can be found from the **Nyquist Theorem**:

$$\overline{v_n^2} = 4kTBR$$

Another!

不用的！

where

T = Temperature of the resistor in Kelvin $\boxed{[\mathrm{K}] = [^\circ\ \mathrm{C}] + 273.15}$

k = Boltzmann's constant 1.38 x 10^{-23} joule/K

B = Bandwidth of interest

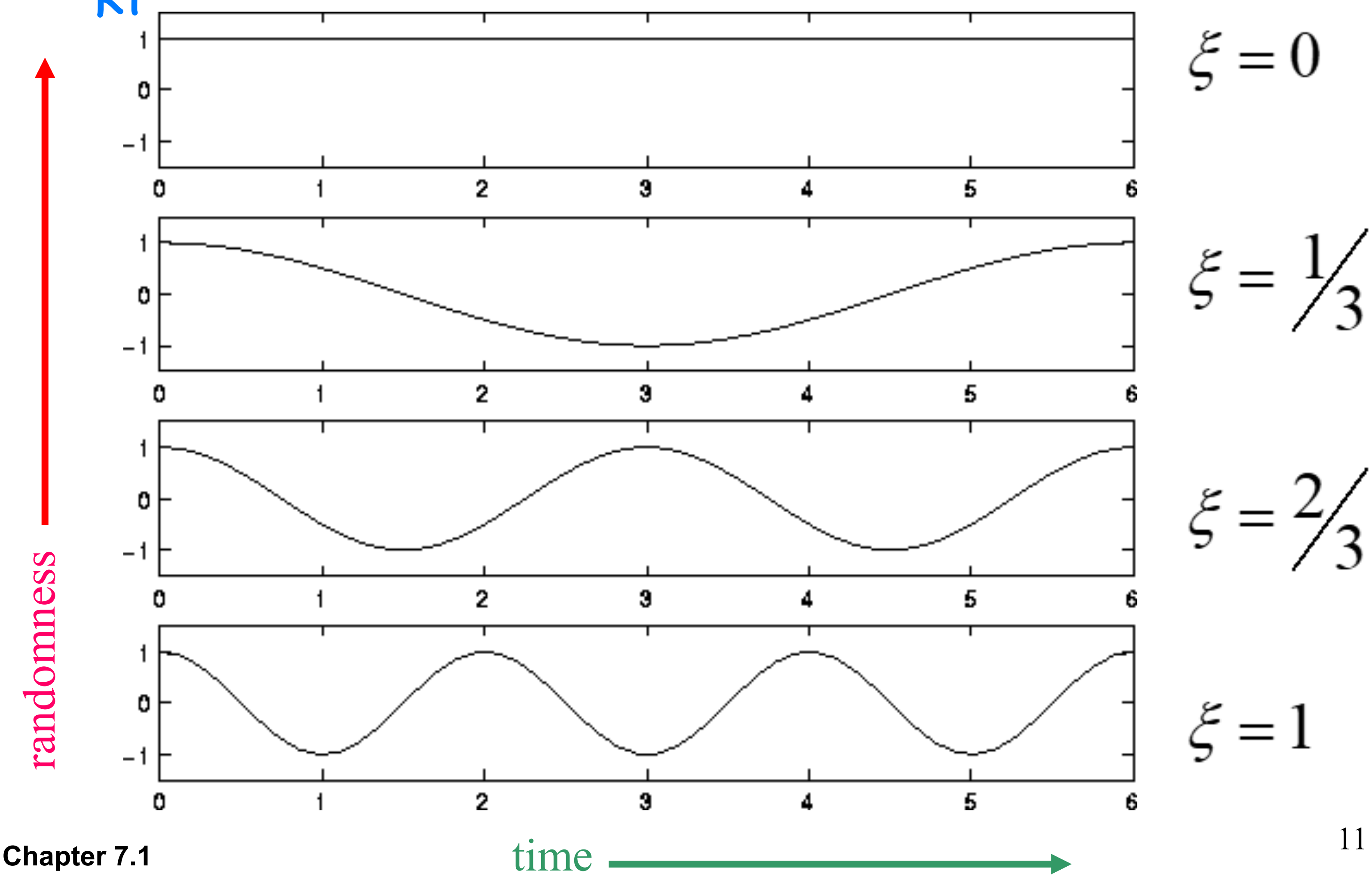
R = Resistance

Noise can be modeled as a Random Process

- **Definition:** A **random process** maps a probability space S to a **set of functions**, $X(t,\xi)$.
- It assigns to every outcome $\xi \in S$ a **time** function $X(t,\xi)$ for $t \in I$ where I is a discrete or continuous index set.
- If I is discrete (e.g. integer valued), $X(n,\xi)$ is a discrete-time random process.
- If I is continuous, $X(t,\xi)$ is a continuous-time random process.
- For a fixed t, $X(t,\xi)$ is a random variable.
- Basically, **we can understand a random process as a sequence of random variables.**

Random Process: Example

- Suppose that ξ is selected *at random* from S = [0,1] and consider $\underbrace{X(t,\xi)}_{\text{RP}} = \cos(\xi\pi t)$ for $t \in \Re$



Characterization of A RP: Mean and Variance Functions

functions!

- **Mean**

$$m_X(t) = E[X(t)] = \int x f_{X(t)}(x)dx$$

- **Variance**

$$\mathrm{Var}[X(t)] = E\left[\left(X(t) - m_X(t)\right)^2\right] = \int (x - m_X(t))^2 f_{X(t)}(x)dx$$

$$= E[X(t)^2] - m_X(t)^2$$

- Note that the mean and variance may be functions of time. However, since the randomness comes from "ξ", we treat "t" as a constant in the above calculations.

Characterization of A RP: Autocorrelation and Autocovariance

- **Autocorrelation**

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int\int xyf_{X(t_1),X(t_2)}(x,y)dxdy$$

- **Autocovariance**

$$\begin{aligned} C_X(t_1,t_2) &= E[(X(t_1)-m_X(t_1))(X(t_2)-m_X(t_2))] \\ &= R_X(t_1,t_2)-m_X(t_1)m_X(t_2) \end{aligned}$$

- **Correlation coefficient**

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}}$$

Wide Sense Stationary

- **Definition:** A process ① $X(t)$ is *wide sense stationary* (WSS) if and only if its mean is constant and its ② autocorrelation function $R_X(t_1, t_2)$ (or autocovariance function) depends only upon the time difference $t_1 - t_2$. $= \Delta$

$$m_X(t) = m \quad \text{for all } t \qquad C_X(t_1, t_2) = C_X(t_1 - t_2) \text{ for all } t_1, t_2$$

$C_X(\Delta)$ same for time difference!

- Suppose $X(t)$ is WSS with autocorrelation $R_X(t)$ where $t = t_1 - t_2$.
  - $R_X(0)$ is the *average power* of the process, $E[X(t)^2]$. $= R_X(0)$
  - $R_X(\tau)$ is an even function of τ.
  - $|R_X(\tau)| \le R_X(0)$

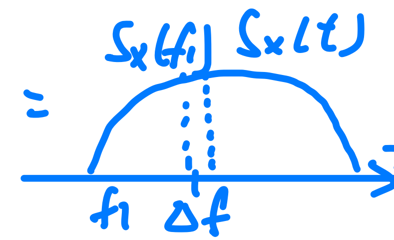
    (the autocorrelation function is maximum at the origin)

Wiener-Khinchine Theorem

- Wiener-Kinchine theorem states that the **autocorrelation function** and the **power spectral density** of a stationary random process are Fourier transform pairs.

$$R_X(\tau) \leftrightarrow \underline{\underline{S_X(f)}}$$

$S_x(f) =$



- Sample functions $x(t, \xi_i)$ of stationary random processes are power signals.

$P([f_1, f_1+\Delta]) \approx S_x(f_1)\Delta$

- The power spectral density of a stationary random process x(t) is defined as

$$S_X(f) = \lim_{T \to \infty} \frac{E[|X_T(f, \xi_i)|^2]}{T}$$

where $X_T(f, \xi_i) = \int_{-T/2}^{T/2} x(t, \xi_i)\, e^{-j2\pi f t} dt$

Gaussian Process

The additive noise in a communications system can be modeled as a Gaussian process (by the central limit theorem)

- at a particular time *t*, the noise signal amplitude will be Gaussian distributed.

- we further assume that the Gaussian process is stationary and has zero mean:

$$\mu(t_i) = E\left[X(t_i)\right] = 0 \qquad i = 1,2,\ldots,n$$

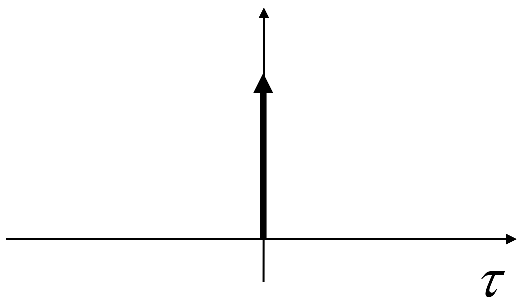
and its autocorrelation is

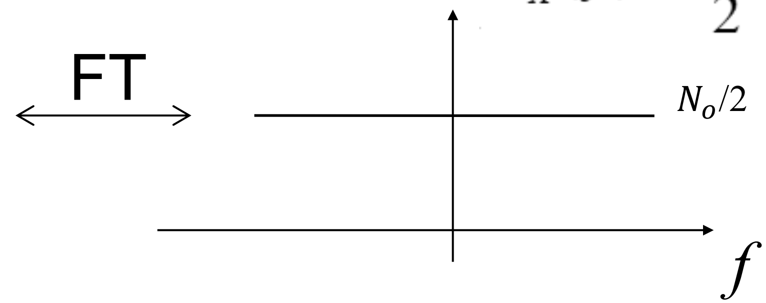
$$R_X(\tau) = E\left[X(t_i)X(t_i+\tau)\right] = \frac{N_o}{2}\delta(\tau) \quad i = 1,2,\ldots,n$$

White Noise

If the power spectrum density is “white”, that is, all frequency components have equal power.

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau) \qquad\qquad S_X(f) = \frac{N_0}{2}$$ = PSD?

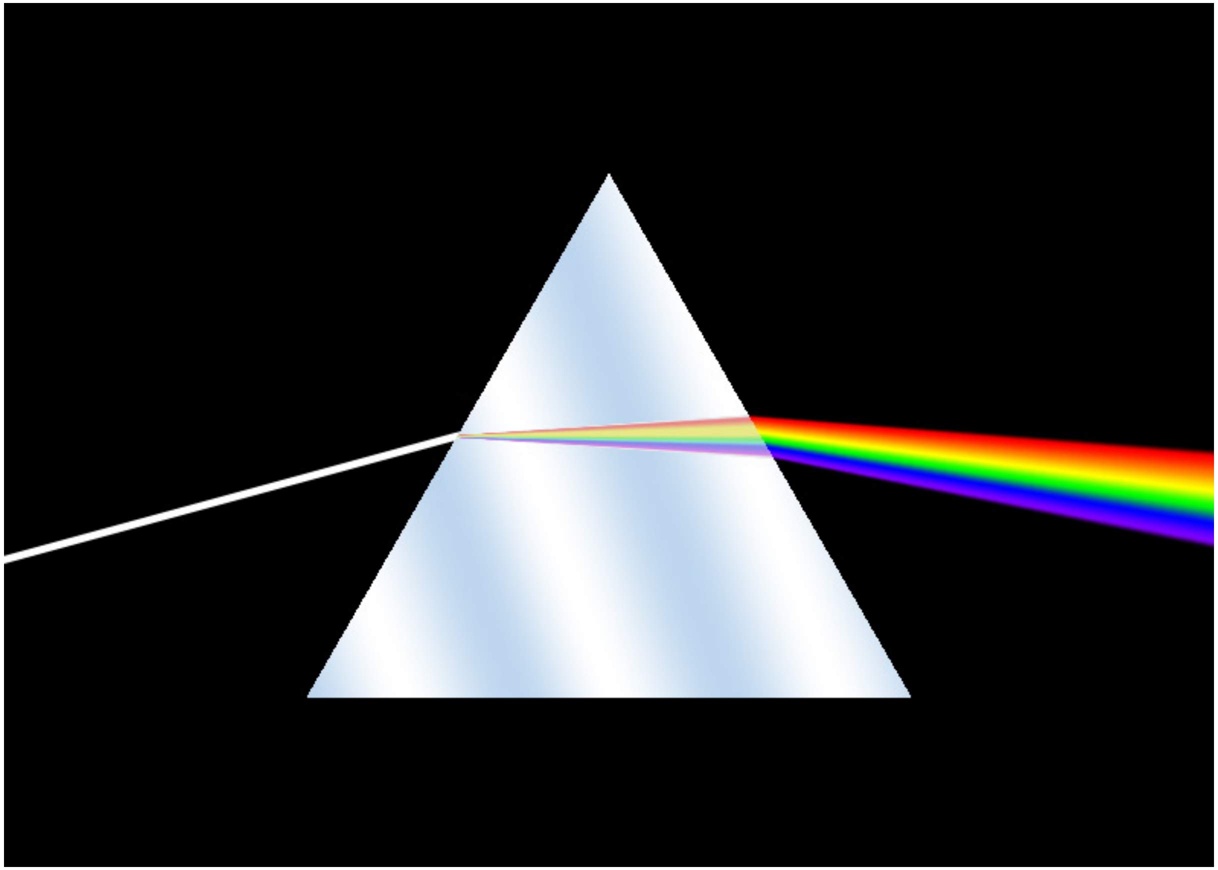




$R_X(\tau)$ is zero except for τ = 0 implies:

- $R_X(t_i, t_j) = E[X(t_i)X(t_j)]=0$ for $i \neq j$
  - ➢ $X(t_i)$ and $X(t_j)$ are uncorrelated
- $R_X(0) = \infty = \overline{X^2(t)} = \sigma_X^2$
  - ➢ Noise has infinite power

Why White noise?



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- System Model
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- **Suboptimal Receiver**
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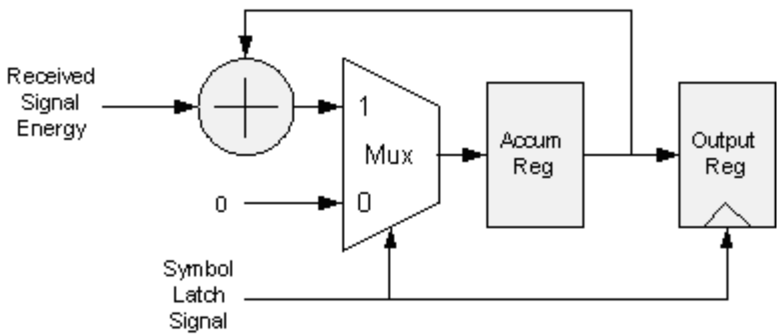
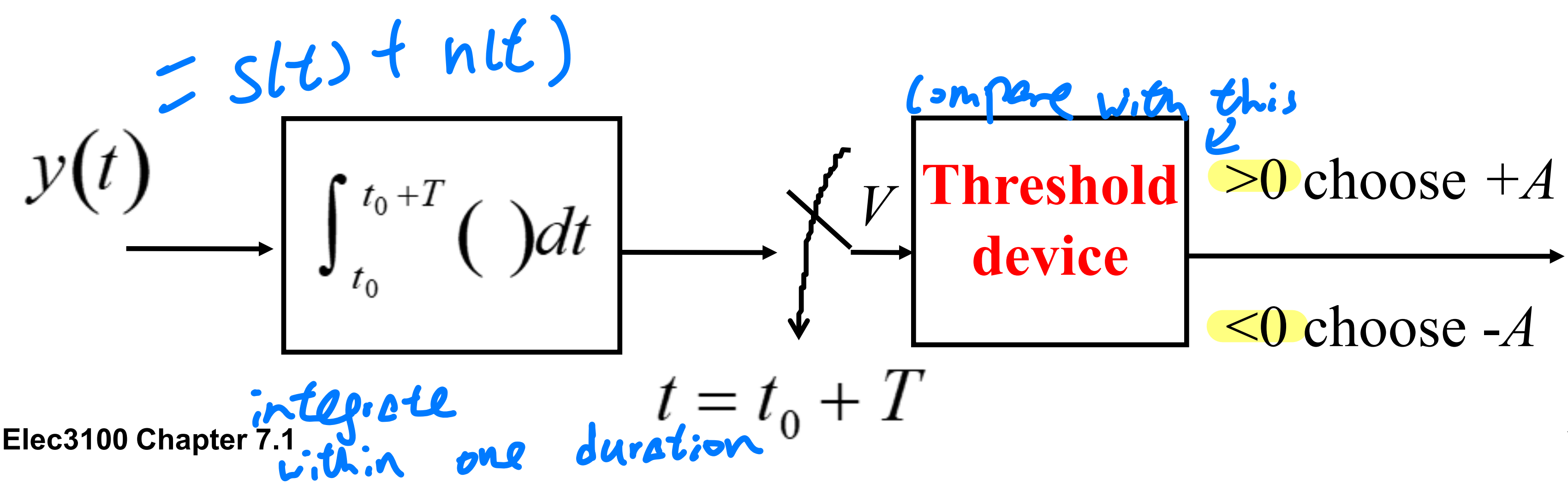


Receiver Structure

- One of the **significant differences** between analog and digital communications systems is that for digital systems, the **probability of error** is used as a **measure of performance** where as in analog systems SNR is used.

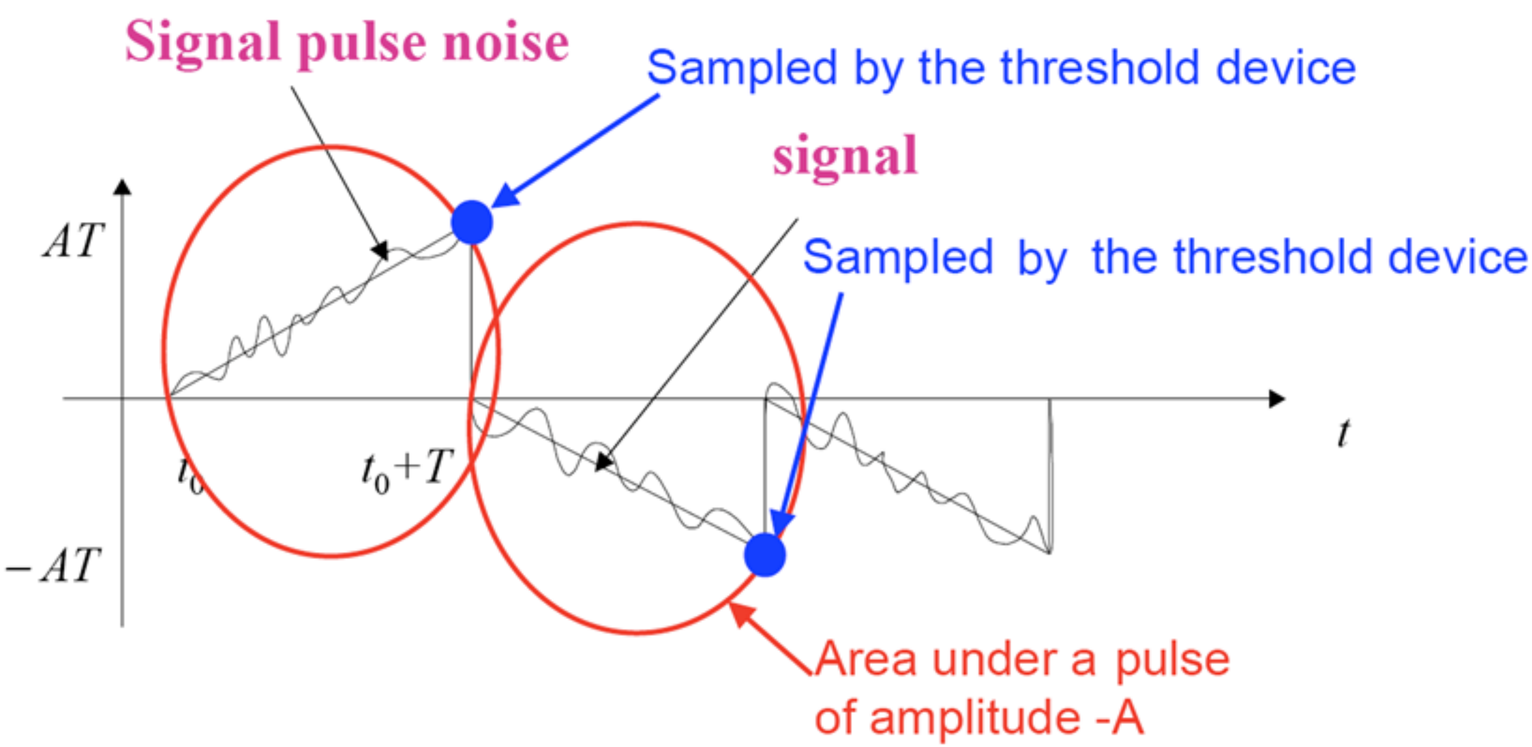
- We have modeled the AWGN. The next question is how to build a receiver to obtain a good performance.

- A possible receiver structure (integrate-and-dump) for detecting the digital transmitted signals is shown below



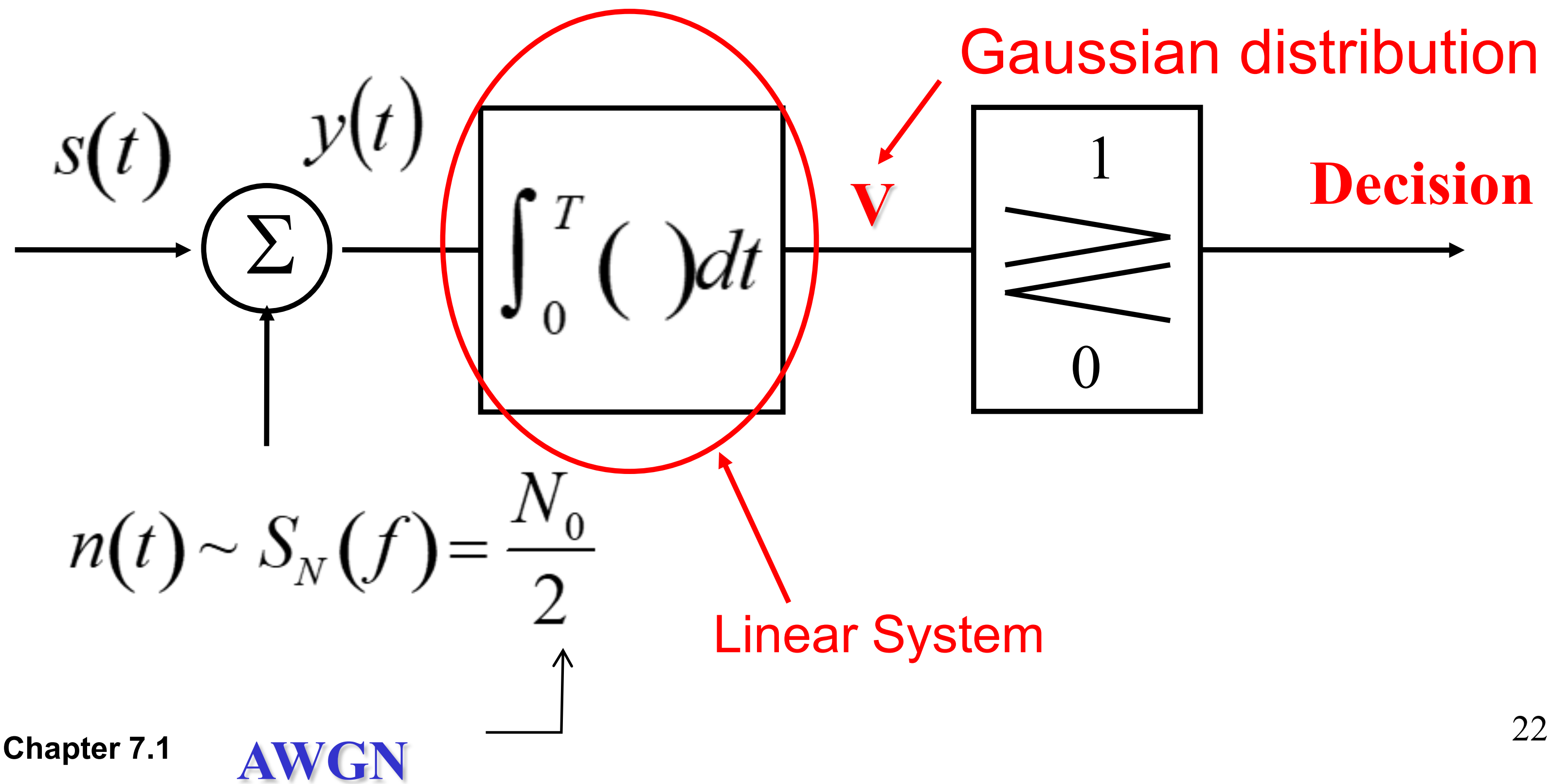
Integrate-and-Dump

- Not necessarily optimum in all situations.
- The **integrator** averages out the noise received so that the output waveform will look like



Integrate-and-Dump

- We know that the noise at the input to the receiver is AWGN.
- We can expect the output from the integrator to have a Gaussian noise distribution.
- Putting these ideas into a mathematical framework we get



Integrate-and-Dump

$$s(t) = \begin{cases} A & 0 \le t < T & \text{if "1" transmitted} \\ -A & 0 \le t < T & \text{if "0" transmitted} \end{cases}$$

signal part is constant.

$$V = \int_0^T [s(t) + n(t)]dt$$

$$= \begin{cases} AT + N & \text{if "1" is sent} \\ -AT + N & \text{if "0" is sent} \end{cases}$$

Random variable

N is Gaussian distributed

with

$$\boxed{N = \int_0^T n(t)dt}$$

Linear combination of Gaussian

Ch7.1: Baseband Communications & Noise

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Key Figure of Merit

- The **probability of receiving a bit in error** for digital systems is an important **measure of performance**.

- **Digital communications relies heavily on these error calculations - VIP**

Problem

Total prob. thm!

Assume the decision threshold is set at 0

$$P_e = P(0 \text{ received} \mid 1 \text{ sent})P(1 \text{ sent}) + P(1 \text{ received} \mid 0)P(0 \text{ sent})$$

$$P_e = P(\boxed{V < 0} \mid 1)P(1) + P(\boxed{V > 0} \mid 0)P(0)$$

bit 1 bit 0

V is the integrated signal within T: time interval!

$$P_e = P(E \mid 1)P(1) + P(E \mid 0)P(0)$$

Total probability theorem

Gaussian!

Conditional Error Probability

Prior Probability

Error Probability Computation

- **To compute** P_e, **we need to compute**

  $P(E|0)$ and $P(E|1)$.

- **We have**

$$V = \int_0^T [s(t) + n(t)]dt$$

$$= \begin{cases} AT + N & \text{if "1" is sent} \\ -AT + N & \text{if "0" is sent} \end{cases}$$

- ***V* is Gaussian distributed with variance** σ^2 .

Conditional Distribution

- The **key** to estimating the error probability is to **find out the distribution of the received signal.**
- **This in turn relies on the distribution of the noise.**
- The **noise mean** can be calculated as

$$E[N] = E\left[\int_0^T n(t)dt\right] = \int_0^T \underbrace{E[n(t)]}_{0} dt = 0$$

mean is 0!

AWGN noise!

Noise Variance

E(0)   Just E(N²)

$$Var[N] = E[N^2] = E\left[\left\{\int_0^T n(t)\,dt\right\}^2\right]$$

E(N) = 0

Autocorrelation

$$= \int_0^T \int_0^T \boxed{E[n(t)n(v)]}\,dt\,dv$$

AWGN

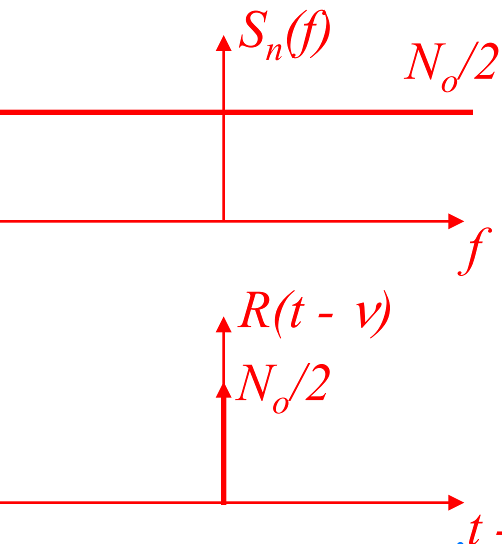
$$= \int_0^T \int_0^T \boxed{R_n(t-v)}\,dt\,dv$$

AWGN

$$= \int_0^T \int_0^T \boxed{\frac{N_0}{2}\delta(t-v)}\,dt\,dv$$

$$= \int_0^T \boxed{\frac{N_0}{2}}\,dv$$

$\frac{N_0 T}{2} = \sigma^2$



find the distribution of V

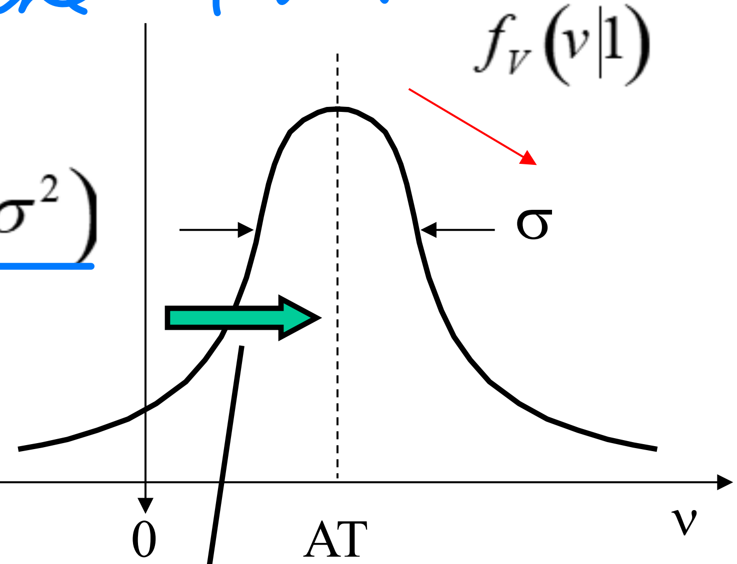
$$= \frac{N_0 T}{2} \equiv \sigma^2$$

Conditional Probabilities

Consider the RV V!

Mean = AT

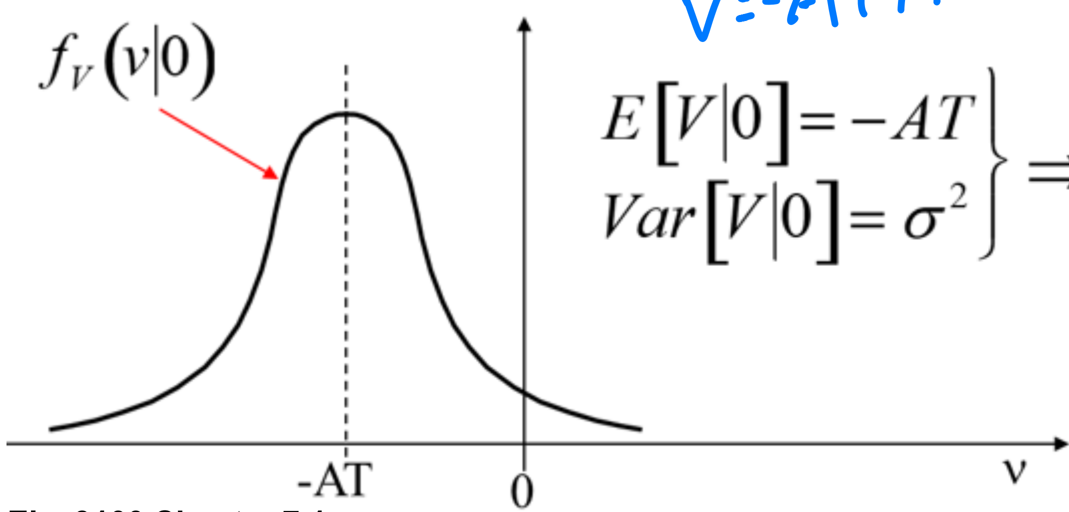
$$\left.\begin{aligned} E[V|1] &= +AT \\ Var[V|1] &= \sigma^2 = \frac{N_0 T}{2} \end{aligned}\right\} \Rightarrow f_V(v|1) \sim \underline{N\left(AT, \sigma^2\right)}$$



Shifted positive due to
The positive pulse +A

V = -AT + N

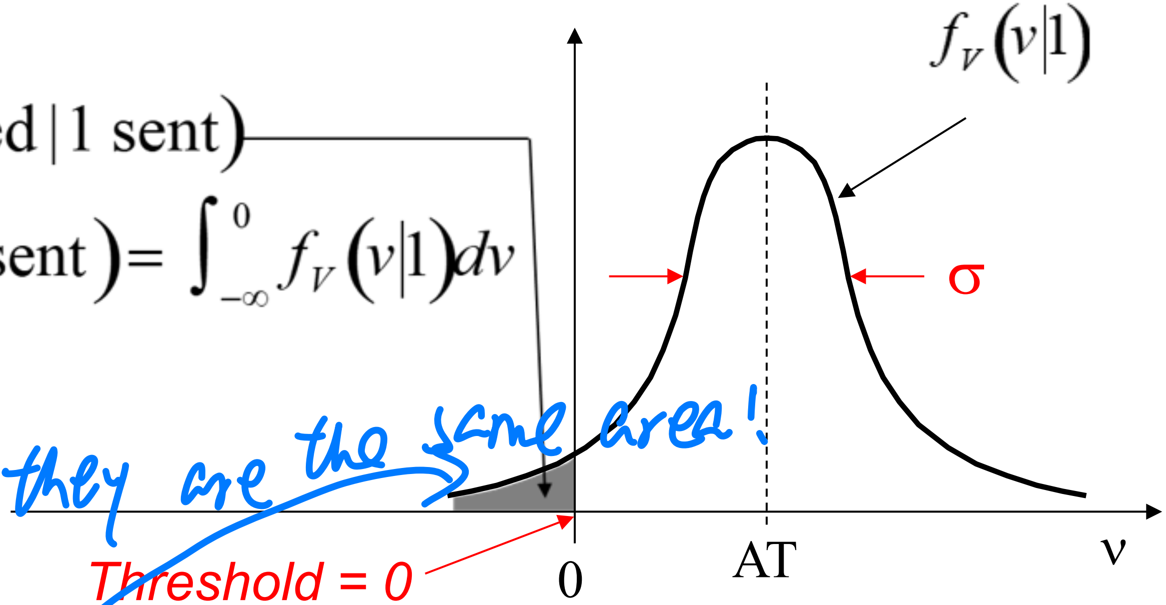
$$\left.\begin{aligned} E[V|0] &= -AT \\ Var[V|0] &= \sigma^2 \end{aligned}\right\} \Rightarrow f_V(v|0) \sim N\left(-AT, \sigma^2\right)$$



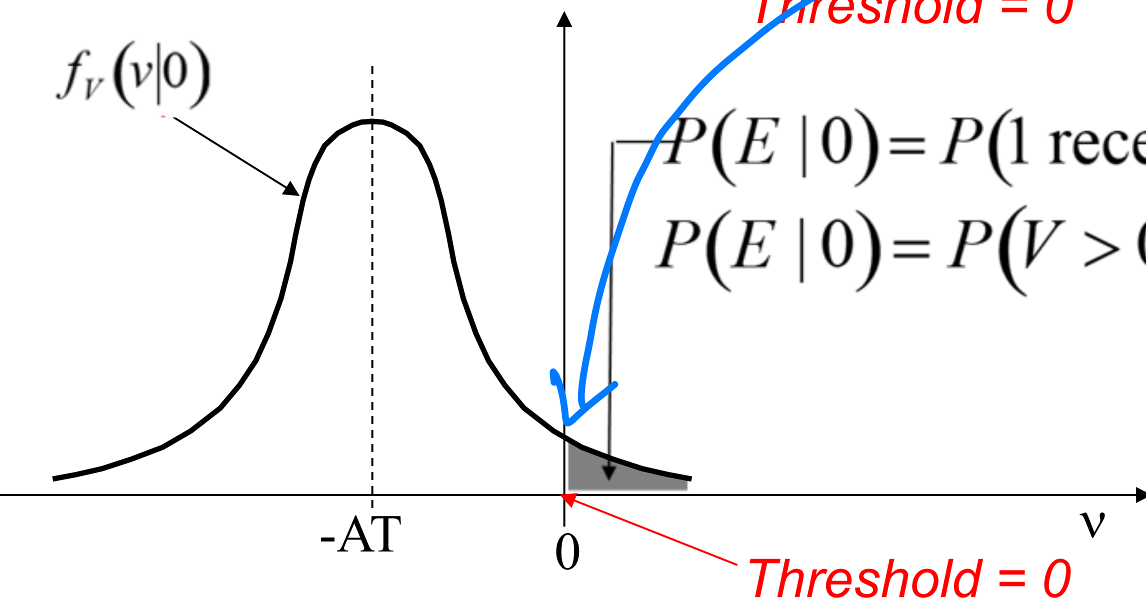
Conditional Probabilities

$$P(E\,|\,1) = P(0 \text{ received}\,|\,1 \text{ sent})$$

$$P(E\,|\,1) = P(V < 0 \,|\, 1 \text{ sent}) = \int_{-\infty}^{0} f_V(v|1)\,dv$$



they are the same area!



$$P(E\,|\,0) = P(1 \text{ received}\,|\,0 \text{ sent})$$

$$P(E\,|\,0) = P(V > 0 \,|\, 0 \text{ sent}) = \int_{0}^{\infty} f_V(v|0)\,dv$$

Conditional Error Probability

- Thus, we know that the **output noise from the integrator** **will have** the following **Gaussian distribution**

$$\therefore N \sim N\left(0,\sigma^2\right)$$

$$\Rightarrow f_n(n) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{n^2}{2\sigma^2}\right)$$

- Now,

$$P(E|0) = P(V > 0|0 \text{ sent}) = \int_0^{\infty} f_V(v|0)dv$$

$$P(E|1) = P(V < 0|1 \text{ sent}) = \int_{-\infty}^{0} f_V(v|1)dv$$

First: write down

$$v: \begin{cases} AT + N & \text{if 1 is sent} \\ -AT + N & \text{if 0 is sent} \end{cases}$$

$N \sim N(0, \sigma^2)$ $\quad \sigma^2 = \frac{N_0 T}{2}$

(wss noise)

$v|1 \sim N(AT, \sigma^2)$

$v|0 \sim N(-AT, \sigma^2)$

$P_e = P(E|1)P(1) + P(E|0)P(0)$

$= P(v<0|1)P(1) + P(v>0|0)P(0)$

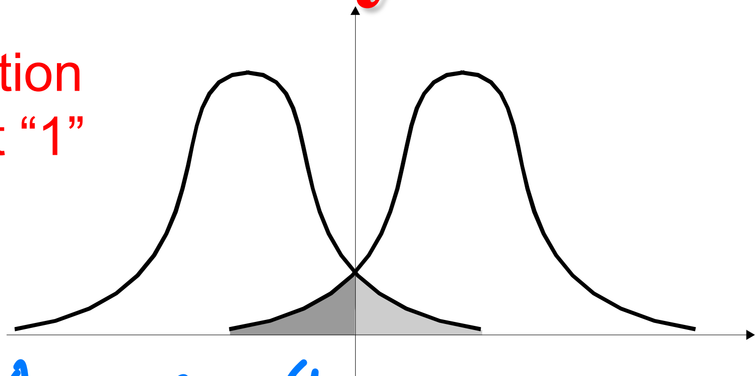
As $P(v<0|1) = Q\left(\frac{AT}{\sigma}\right)$

Total Error Probability

- **Assumptions:**

1. Symmetry,

Noise distribution is the same at "1" and "0"



$$P(E|0) \equiv P(E|1)$$

Assume they are the same

(After conditioning!!!)

2. "0" and "1" are **equally likely**, **then**

Eq likely

$$P(0) \equiv P(1) = \frac{1}{2} \quad \Rightarrow \qquad P_e = P(E|0) = P(E|1)$$

- We therefore have

$$P_e \boxed{= P(E|1)P(1) + P(E|0)P(0)} = \frac{1}{2}\left[P(E|0) + P(E|0)\right] = P(E|0)$$

$$P_e = P(E|0) = \int_0^\infty f_V(v|0)\,dv = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(v+AT)^2}{2\sigma^2}}\,dv$$

Evaluation Relies on Q-Function

- Let $x = \dfrac{v + AT}{\sigma} \Rightarrow dx = \dfrac{dv}{\sigma}$ VIP Transformation

$$P_e = \int_{\frac{AT}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \triangleq Q\left[\frac{AT}{\sigma}\right]$$

- where

$$\boxed{Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt}$$

Q(.) function

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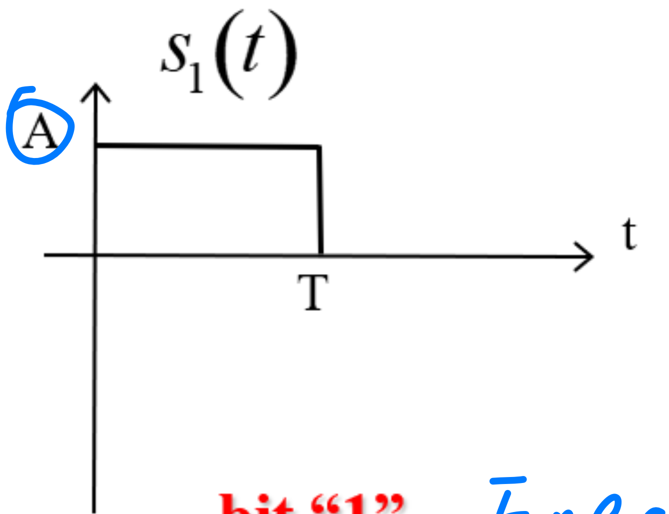
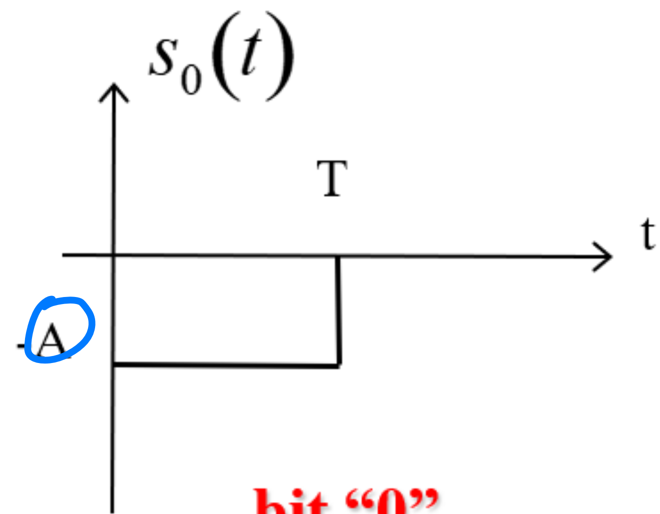
A General Case

- Next, note that one can represent $s(t)$ as $s_0(t)$ (“0” sent) or $s_1(t)$ (“1” sent).



bit “1”

Energy!



bit “0”

$$E_1 \triangleq \int_0^T s_1^2(t)dt = A^2T \qquad E_0 \triangleq \int_0^T s_0^2(t)dt = A^2T$$

Energy

In this case, they have the same Energy!

A General Result

$$\therefore E_b \triangleq \text{Energy/bit} = E_0 P(\text{"0" sent}) + E_1 P(\text{"1" sent})$$

$$= \frac{1}{2}(E_0 + E_1)$$

$$\textit{But} \qquad E_1 = E_0 = \underline{A^2 T.}$$

- We **can use** these **energy calculations** to **find a more general result** for the error probabilities. That is,

$$P_e = Q\left[\frac{AT}{\sigma}\right] = Q\left[\sqrt{\frac{A^2T^2}{\sigma^2}}\right]$$

where

$$\sigma^2 = \frac{N_0 T}{2}$$

SNR vs. Eb/No

$E_b = A^2 T$

$\sigma^2 = \frac{N_0 T}{2}$

- Therefore, $P_e = Q\left[\sqrt{\frac{A^2T^2}{\sigma^2}}\right] = Q\left[\sqrt{\frac{A^2T}{\sigma^2 / T}}\right]$

$$\equiv Q\left[\boxed{\sqrt{\frac{2E_b}{N_0}}}\right]$$

Energy per bit!

key quantity

$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]$$

Signal Energy to Noise Power Spectral Density Ratio

Relation with SNR?

where $\operatorname{erfc}(u) = 1 - \operatorname{erf}(u)$

with $\operatorname{erf}(u) \triangleq \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2}\, dt$

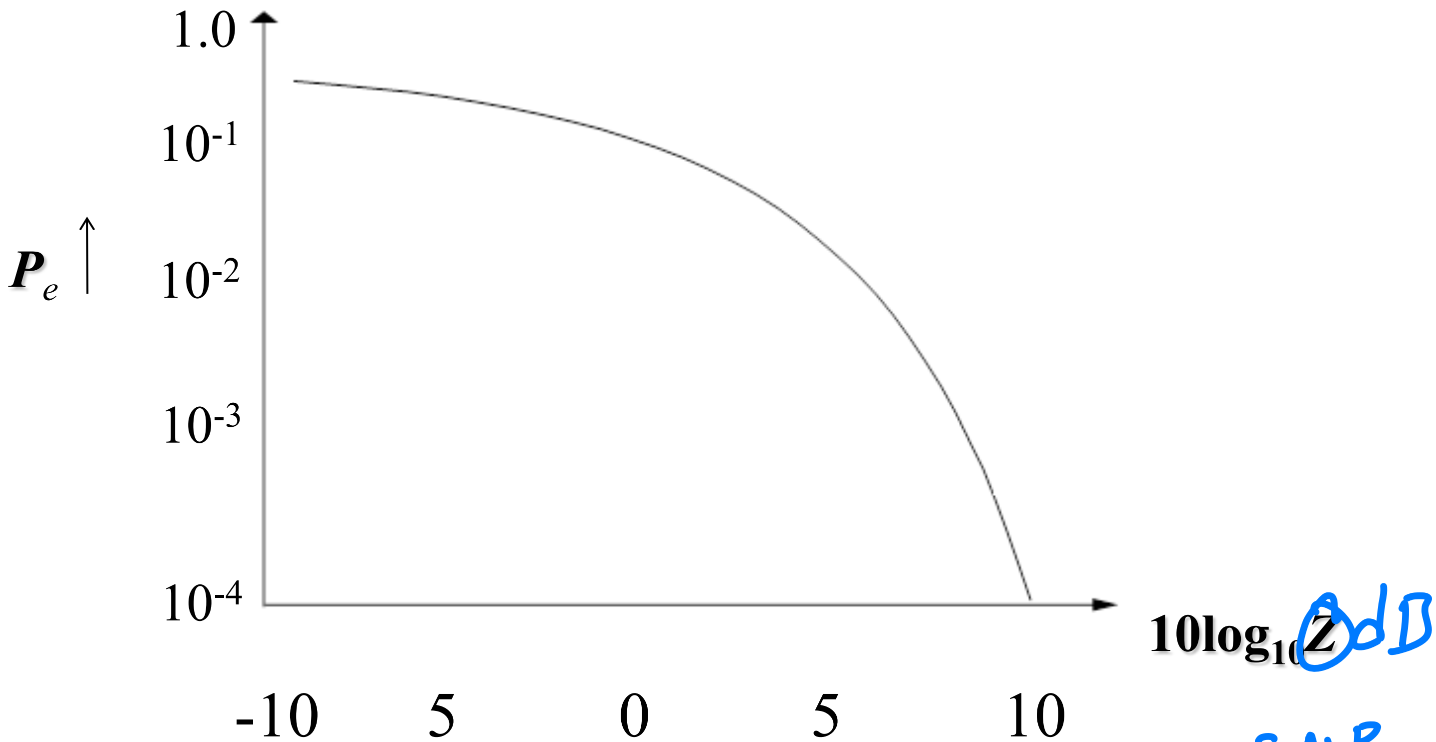
Error Function

relationship between erfc and Q function

Complementary Error Function

Bit Error Rate (BER)

- A graph of P_e for **baseband signaling** is



where $P_e = Q\left[\sqrt{2Z}\right] = \frac{1}{2}\mathrm{erfc}\left[\sqrt{Z}\right]$ $\quad Z = \boxed{\frac{E_b}{N_0}}$

SNR,

design by us!

Example 7.1

A baseband digital Tx system sends ±A valued rectangular pulses through a channel at a rate of 1Mbps with amplitude 1V when the noise PSD is 10^{-7} W/Hz.

$1 \sim T$ $\quad T = \frac{1}{1M}$ $\quad A \sim 1$ $\quad = \frac{N_0}{2}$

$Q\sqrt{2 \times 10^{-6}/(2\times10^{-7})}$

Answer: $Q\left(\sqrt{2E_b/N_0}\right) = Q\left(\sqrt{2A^2T/N_0}\right)$

$$T = 1/1000000 = 10^{-6}$$

$$\Rightarrow Q\left(\sqrt{2\times10^{-6}/(2\times10^{-7})}\right) = Q(\sqrt{10}) = Q(3.16)$$

$$Q(u) \approx \frac{e^{-u^2/2}}{u\sqrt{2\pi}}$$

$$Q(3.16) \approx 0.00085$$

Probability of Error, P_e

Example 7.2

Digital data is to be transmitted through a baseband system with $N_0 = 10^{-7}$ W/Hz and the received signal amplitude $A = 20$mV.

$P_e = Q\left(\sqrt{2E_b/N_0}\right) = Q(2.828)$

(a) If 1000 bits per second (bps) are transmitted what is the error probability? **Ans.** $\boldsymbol{P_e = 2.58 \cdot 10^{-3}}$.

(b) If 10000 bps are transmitted, to what value must A be adjusted in order to attain the same error probability as in part a)?

Ans. $\boldsymbol{A = 63.2 \cdot 10^{-3}}$ V = 63.2mV.

$\downarrow P_e = Q\sqrt{\frac{2A^2}{N_0/T\downarrow}}$ $\qquad 2A^2\left(\frac{1}{10000}\right)/10^{-7} = 8$

$A = 63.2$mV

$\Downarrow$

Then A need to increase!

BER vs. Data Rate

- In the last example, for a fixed amplitude, error probability increases as bit rate increases.
- This can be understood by looking at the error probability expression

$$P_e = Q\left[\sqrt{\frac{A^2T^2}{\sigma^2}}\right] = Q\left[\sqrt{\boxed{\frac{A^2T}{\sigma^2/T}}}\right] = Q\left[\sqrt{\boxed{\frac{2E_b}{N_o}}}\right]$$

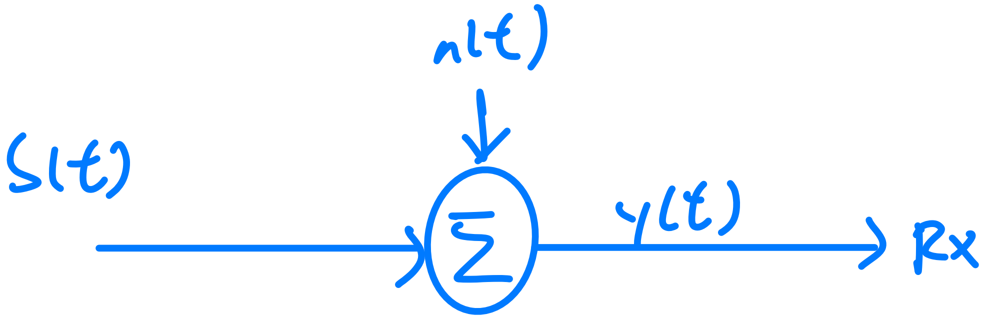
- Increase in bit rate $\rightarrow$ smaller bit period T

  $\rightarrow$ higher noise power

  $\rightarrow$ lower signal energy

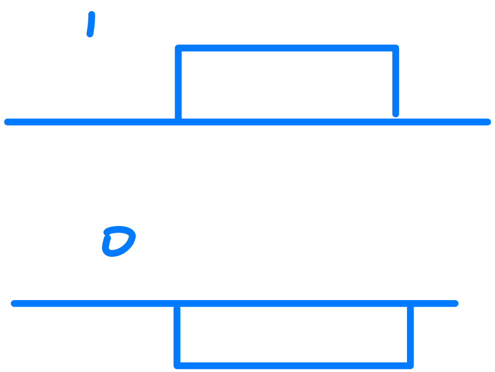
$$P_e = Q\left[\sqrt{\frac{2A^2}{\boxed{N_o/T}}}\right]$$

Are they happening at the same time?



So far

s(t)



n(t): AWGN

Rx : integrate-and-dump

Next : In general

① Optimal receiver

② Digital modulation

③ Channel model

④ Multiplexing!

design s(t)

extension of Rx

Allow multiple users!