Lecture 4

Basic Characterization of Systems
(Language – Keywords)
(Ref: Chapter 1 O&W)

I. Example CT and DT Systems

II. Basic Characterization of Systems

III. Time Invariance

IV. Linearity and LTI Systems

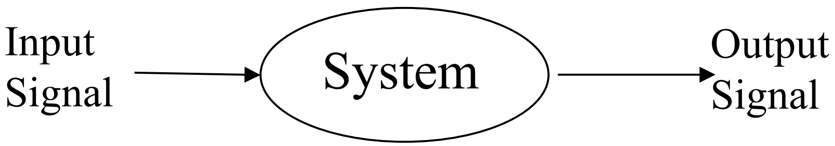
I. Example CT and DT Systems

What are Systems?

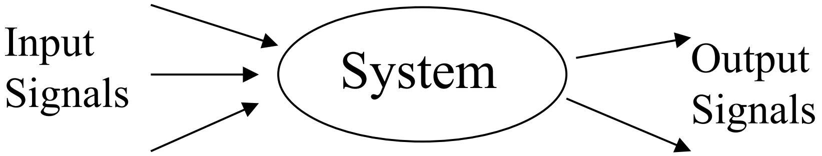
- System is anything that ***takes input signals and produces output signals***.

- In ELEC2100, we will focus on ***Single-Input Single-Output systems***

Single-Input Single-Output Systems

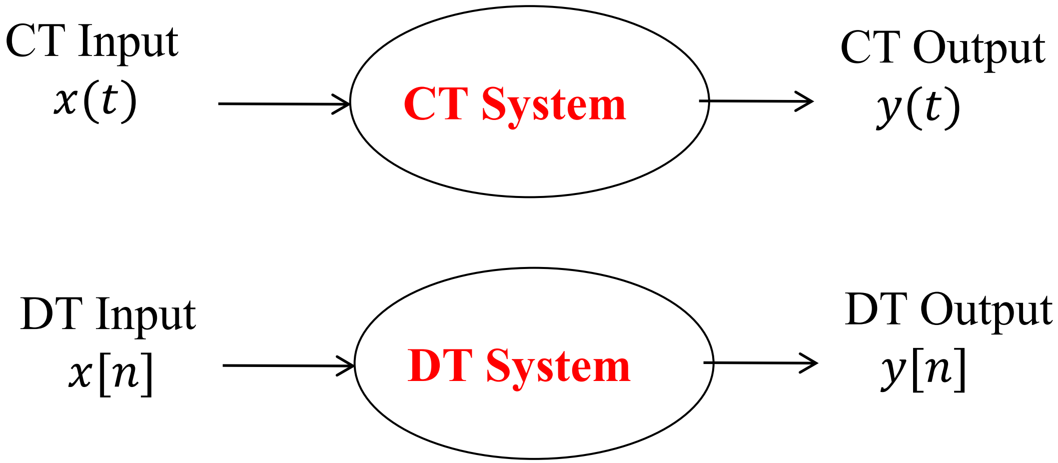


Multiple-Input Multiple-Output Systems



CT and DT System

- A system that processes CT signals is a Continuous-Time (CT) System.
- A system that processes DT signals is a Discrete-Time (DT) System.



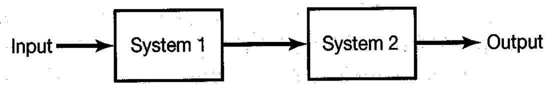
- Many of the concepts/properties that we will discuss apply to both CT and DT signals/systems.

Example Systems that we will talk about

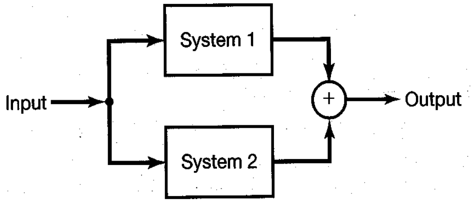
1. An RC circuit
2. A bank saving account
3. Automobile suspension system
4. Couponed bond investment
5. JPEG/MPEG for image/video compression
6. Modulation and demodulation in communications
7. Telephone subscriber loop
8. Multipath radio communication channel and OFDM for Mobile Communications and WiFi
9. Dispersive optical fiber communication
10. Anti-aliasing filtering for digital telephony, digital camera, and thumbnail generation
11. Surface Acoustic Wave (SAW) filter for conventional TV tuning
12. Spectrum analyzer
13. Stabilizing an inverted pendulum

Building Complicated Systems using Simple systems

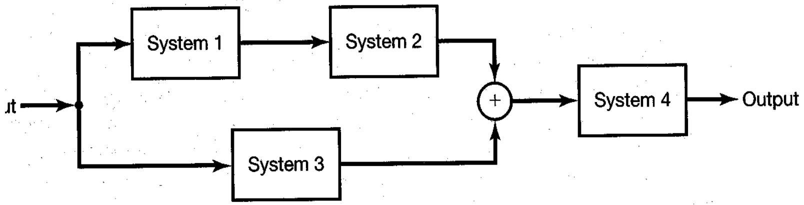
- A common theme in engineering is to build complicated system by interconnecting simpler systems together in a systematic way.
- Interconnections can be:
  a) Serial/cascade
  b) Parallel
  c) Serial-Parallel Combination
  d) Feedback
- So, to understand and build complex systems, we need to first understand some basic elementary systems (proto-systems)



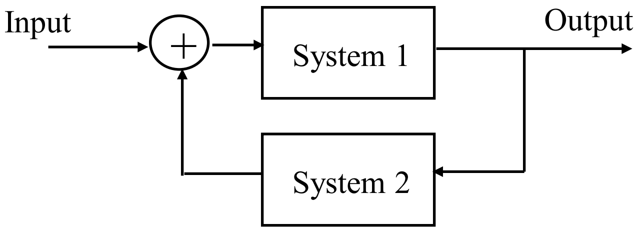
(a) Serial



(b) Parallel



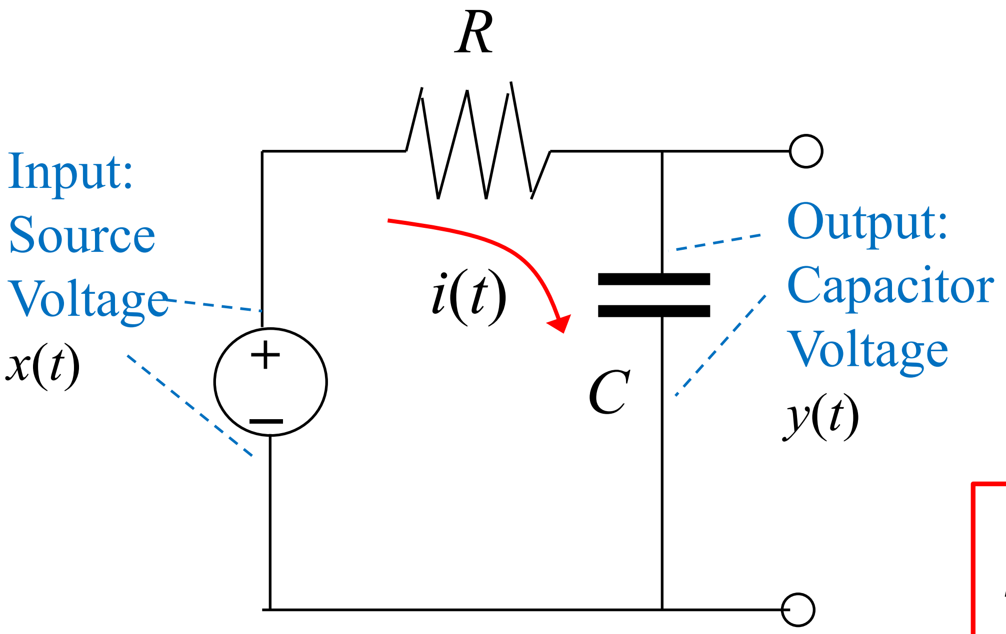
(c) Combination



(d) Feedback

Example 1: An RC circuit (CT)

- For the RC circuit below, let the source voltage be input $x(t)$, and the voltage across the capacitor be output $y(t)$.
- The input-output relationship is represented by a 1st order ***differential equation***:



$$\underbrace{x(t)}_{\text{Source Voltage}} = \underbrace{i(t)R}_{\text{Voltage across resistor}} + \underbrace{y(t)}_{\text{Voltage across capacitor}}$$

$$i(t) = C\,\frac{dy(t)}{dt}$$

Current is equal to capacitance times rate of voltage increase across capacitor

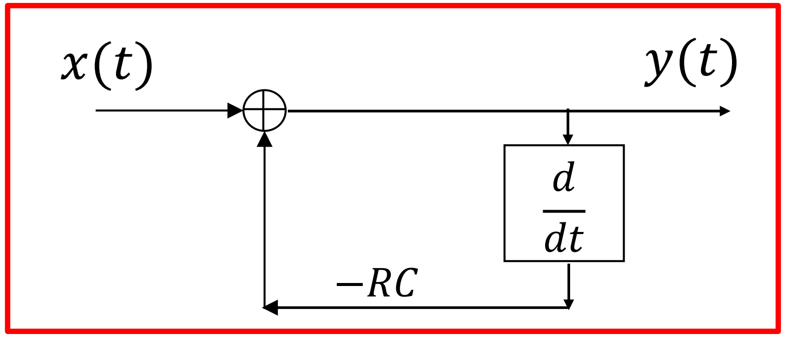
$$\boxed{x(t) = RC\,\frac{dy(t)}{dt} + y(t)}$$

Input-output related by a 1st-order differential equation.

Example 1 (Continued): RC circuit as a Feedback System

- Rearranging terms, we can view the RC circuit as a feedback system in which the output is differentiated, multiplied by a gain $(-RC)$, and fed back to add to the input.

$$y(t) = x(t) - RC\frac{dy(t)}{dt}$$



- The RC circuit is a proto-system from which we can build arbitrarily complex systems.

Example 2: Balance in a bank account (DT)

Assume that a bank account bears an interest of 0.2% per month. The balance at the end of each month can be modeled as a DT system described by the ***difference equation*** below:

$$y[n] = 1.002y[n-1] + x[n]$$

Output: balance at end of month n

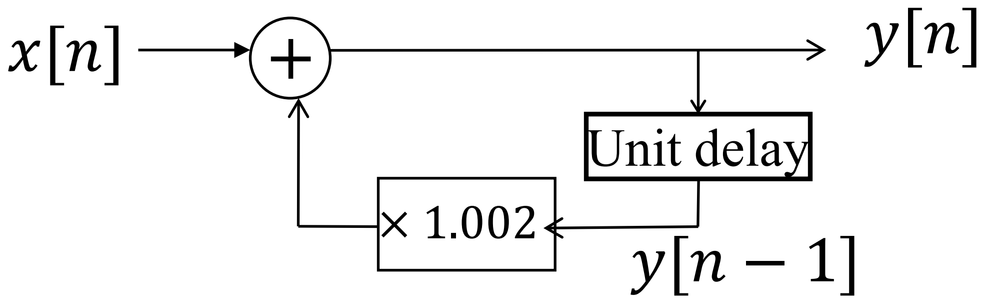
growth factor

Balance of previous month

Input: net deposit in month n

Putting all the y's on one side, we have: $\Rightarrow y[n] - 1.002y[n-1] = x[n]$

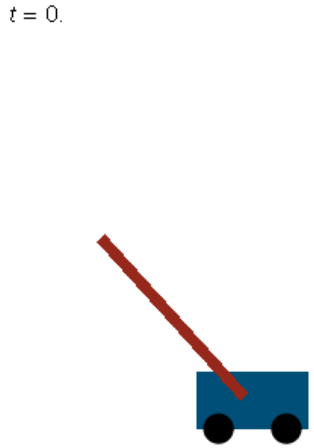
The system can be represented by the block diagram below:



The DT difference equation is the counterpart to the CT differential equation and forms the basis of many digital signal processing systems.

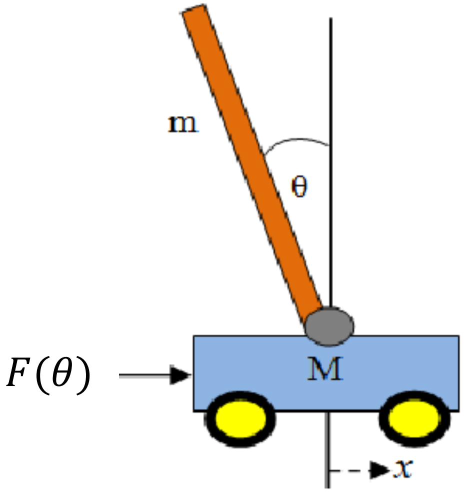
Example 3: Control System – An Inverted Pendulum

- Stabilization of a Segway or a hoverboard can be understood from the study of the inverted pendulum



- We can build a sensor to sense the angle θ. Then we apply a force $F(\theta)$ that is a function of θ to keep the system upright.

- What kind of function $F(\theta)$ do we need? Which of these will work?
  - Proportional feedback: $F(\theta) = K_1\theta$
  - Derivative feedback: $F(\theta) = K_2\frac{d\theta}{dt}$
  - Combination? $F(\theta) = K_1\theta + K_2\frac{d\theta}{dt}$



II. Basic Characterization of Systems

First, we define 4 basic characterizations of systems:

1. *Memoryless*

- If the output of a system at time t (or n) depends only on the input signal at time t (or n) then we say the system is ***memoryless***.
  - A memoryless system is *instantaneous*
- Otherwise, the system *has memory*.

- **Examples** (classify the followings as memoryless or not)

$y(t) = 5x(t)$ ✓ amplifier with gain of 5

$y[n] = 5x[n-1]$ ☒ delayed by 1 and amplified with gain of 5

$y(t) = 5x(t+1)$ ☒ time-advanced and amplified

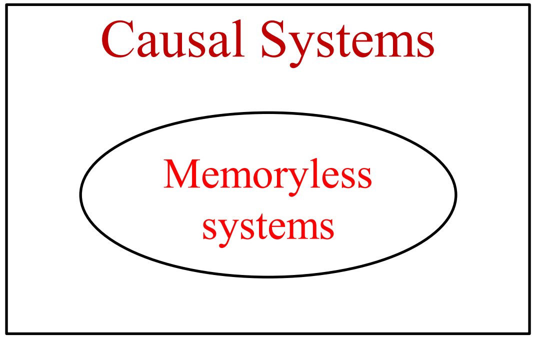
$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ ☒ first integral of x; $y(t) = \int x(t)$; $y = \int x$, $y(t) = x^{(-1)}(t)$

$y[n] = x[n]\cos(\omega(n+5))$ ✓ This is memoryless! It is amplification with time-dependent gain.

2. Causality 因果关係

- A system is ***causal*** if the output at any time does not depend on *future* input - ***not anticipative***.

- Memoryless systems are causal. Causal systems may or may not be memoryless.



- We expect that real-time physical systems with time as the independent variable should be causal. But in scenarios such as processing of stored data, or in image processing where the independent variable is “space” rather than “time”, there is no need for systems to be causal.

- **Examples** - causal or not?

$$y(t) = x(t+5)$$

time advance by 5; depends on future input⇒ not causal/non-causal

$$y[n] = 5x[n-3]$$

time delay by 3 with gain of 5 ⇒ causal

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

y is first sum of x ; adds up all past and present values of x ⇒ causal

$$y(t) = \int_{-\infty}^{t+5} x(\tau)d\tau$$

Integrate x from $-\infty$ to 5 time units into the future ⇒ not causal

$$y(t) = x(t)\cos(t+5)$$

Amplifier with time-dependent gain; memoryless and causal

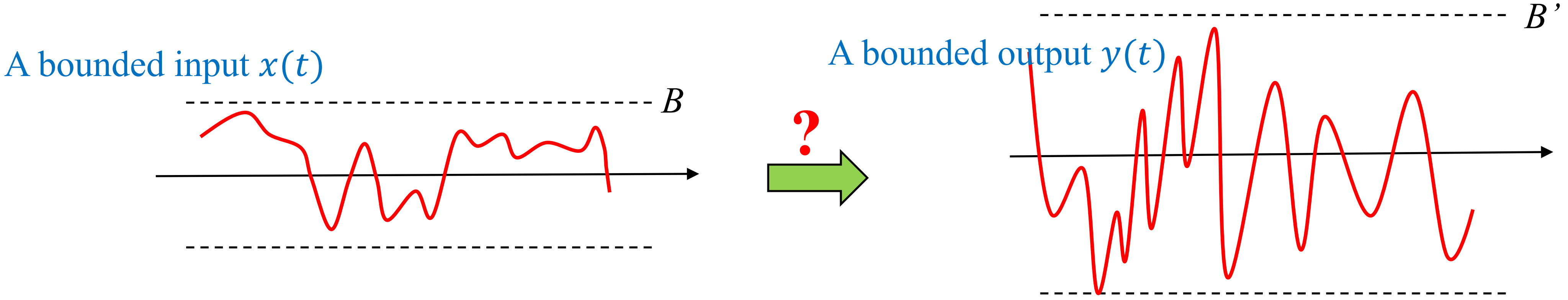
3. *Stability*

- There can be different definitions for stability.
- We will focus on BIBO (Bounded-Input Bounded Output) stability which is whether a system keeps signals bounded. A signal is bounded if there is a finite constant *B* such that the magnitude of the signal is never greater than *B* for all *t*.

- A system is ***BIBO stable*** if bounded inputs always lead to bounded outputs:

Means "there is"

$$\boxed{\begin{array}{l}\text{If } \exists B < \infty \text{ such that } |x(t)| \le B \ \forall t, \\ \text{then } \exists B' < \infty,\ |y(t)| \le B' \ \forall t.\end{array}}$$



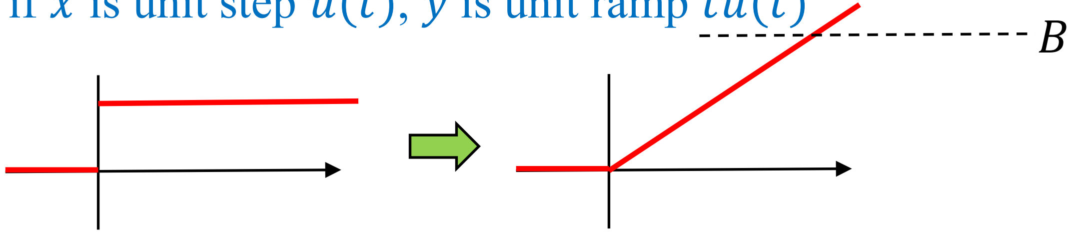
Examples BIBO Stable?

1. $y(t) = 5x(t)$ ✓

Amplification by 5; if x bounded by B; y bounded by $5B$

2. $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ $\boxed{\times}$

Counter example: if x is unit step $u(t)$, y is unit ramp $tu(t)$



3. Example 2 of systems (bank balance) $y[n] - 1.002y[n-1] = x[n]$ $\boxed{\times}$

Counter example: if x is unit impulse $\delta[n]$, y grows exponentially as $1.002^n u(t)$

4. $y(t) + 2\frac{dy(t)}{dt} - 3\frac{d^2y(t)}{dt^2} = x(t)$ describing a causal physical system $\boxed{\times}$

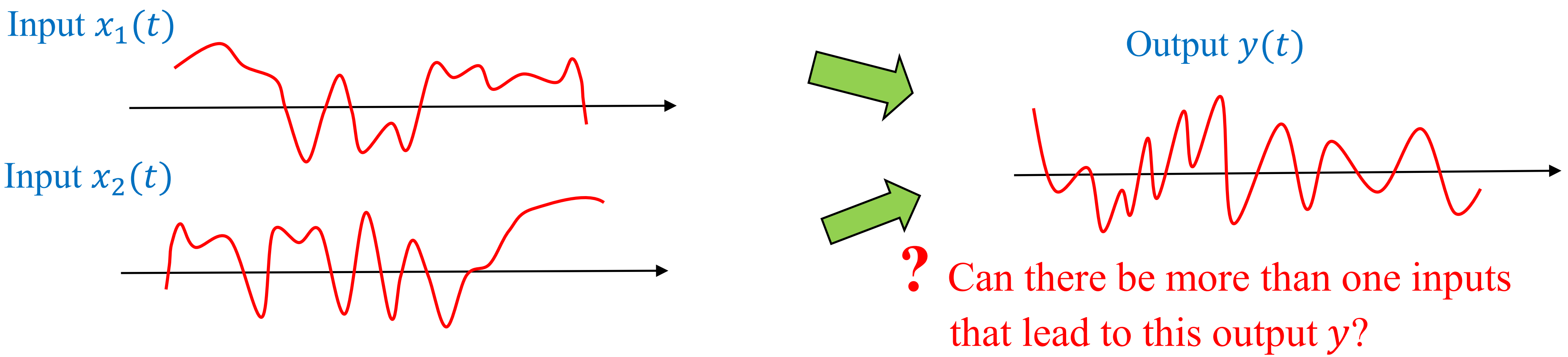
How do we know this system is not stable?
We need the deduction in ELEC2100

$\frac{2 \pm \sqrt{4+12}}{2(-3)} = \frac{2 \pm 4}{-6} = -1$ or $\frac{1}{3}$

4. *Invertibility*

- A system is said to be ***invertible*** if distinct inputs lead to distinct outputs; otherwise it is ***non-invertible***.

That is, the input-output mapping is one-to-one. For one specific output y (a function defined over all time), there is only one corresponding input x.



- For example, can we recover the original speech signal with a room filled with echoes?
- But in a practical systems there are other important considerations. (Delay? Is the inversion causal – not needing future observations? Is the inversion stable or sensitive to errors?)
- We will not discuss invertibility in detail in ELEC2100.

III. Time Invariance

5. *Time-invariance*

A system is ***time-invariant*** (TI) if it obeys *shifted input-shifted output*: if input $x(t)$ leads to output $y(t)$, then input $x(t - t_0)$ leads to output $y(t - t_0)$.

We expect time-invariance if the behavior of a system does not change over time.

Example 1.14. Consider a CT system defined by

$$y(t) = \sin[x(t)]$$

This operator itself does not depend on time

Is it TI?

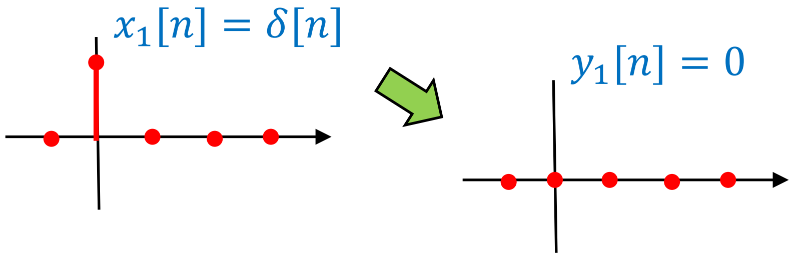
Yes, because what the system does to the input does not depend on time

- Any system that performs a time−dependent operation on the input is in general not TI .
- Examples: - Raising input to the n-th power: $y[n] = x^n[n]$
  - Multiplying by a function of time: $y(t) = x(t)\cos(\omega t)$; $y[n] = g[n]x[n]$

Example 1.15.

Multiply by a function of time $g[n] = n$

$$y[n] = nx[n]$$

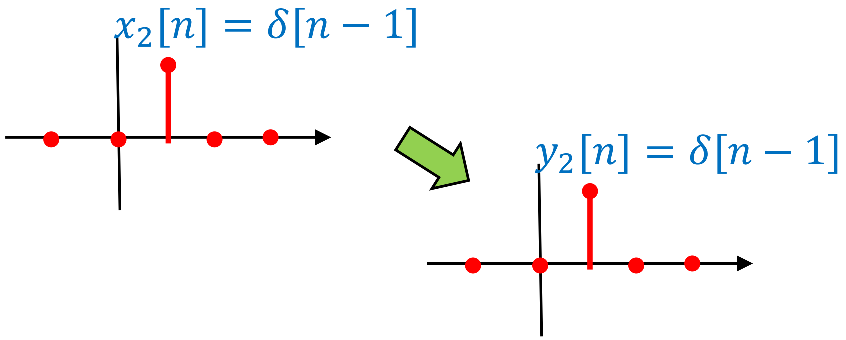


Consider the input $x_1[n] = \delta[n]$

The output is $y_1[n] = 0 \quad \forall n$

Now consider a shifted input:

$$x_2[n] = x_1[n-1] = \delta[n-1] = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$



The output is: $y_2[n] = \delta[n-1] \neq y_1[n-1]$

hence the system violates TI

- Systems that have a specific time reference/origin is not TI

Example 1.16 Time Compression

$$y(t) = x(2t)$$

- Time-compression is not TI because the operation is defined around a specific time origin, meaning that not all time instances are the same. We can verify by the example in Fig 1.47.

- Similarly, these systems are also not TI:

$y(t) = x(-t)$ Time reversal, not TI

$y(t) = Ev\{x(t)\}$ Taking even part, not TI

$y(t) = Od\{x(t)\}$ Taking odd part, not TI

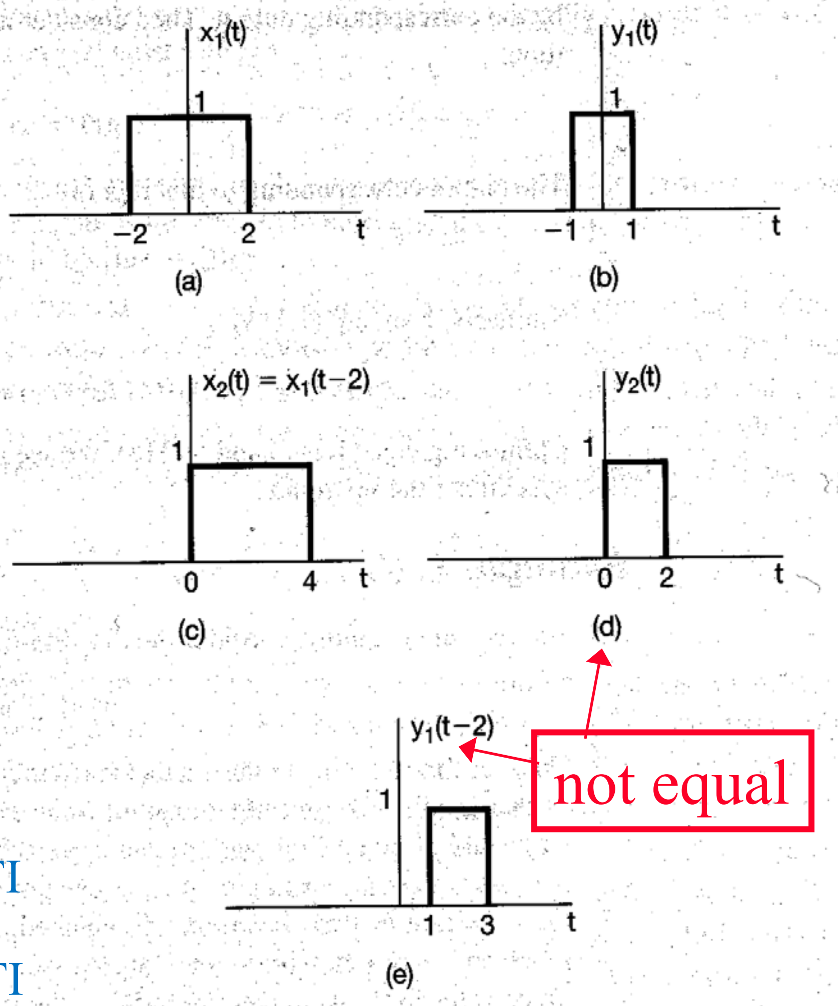


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the

$x_2(t) = x_1(t-2)$ but
$y_2(t) \neq y_1(t-2)$
Hence system not TI!

Time-invariance - Derivatives and Integrals

- A system that takes the derivative of the input is TI: $y(t) = \dfrac{dx(t)}{dt}$

  Derivative is the slope of $x(t)$. If we shift $x(t)$ in time, obviously its slope is shifted correspondingly.

- Taking the k-th derivative or integral is also TI: $y(t) = x^{(k)}(t)$

  where the superscript (k) means the k-th derivative, with $x^{(-1)}(t)$ meaning the first integral:

$$x^{(-1)}(t) = \int x(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- Be careful, however, that if we integrate the input over a defined time window (definite integral), then the operation is not TI unless the window moves with time

$$y(t) = \int_{t-3}^{t} x(\tau)d\tau, \quad y(t) = \int_{t-3}^{t+2} x(\tau)d\tau \qquad \boxed{\text{TI!}}$$

$$y(t) = \int_{-3}^{1} x(\tau)d\tau, \qquad y(t) = \int_{0}^{t} x(\tau)d\tau \qquad \boxed{\text{Not TI!}}$$

IV. Linearity and LTI systems

6. *Linearity*

A system is said to be ***linear*** if it satisfies the ***superposition*** property.

Superposition Property: If the input is a *superposition* (= *weighted sum* = *linear combination*) of individual inputs, the output will be the same weighted sum of the individual outputs:

CT: if $x_1(t) \rightarrow y_1(t)$

and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

where " $x \rightarrow y$ " means x produces the output y

Input that is a weighted sum will produce an output that is the same weighted sum of the individual outputs.

DT: if $x_1[n] \rightarrow y_1[n]$ and $x_2[n] \rightarrow y_2[n]$

then $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$

Additivity and Homogeneity

- The superposition property is sometimes decomposed into two sub-properties:

  i. The ***additive*** property:

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

Simply the superposition property with $a = b = 1$

  ii. The ***homogeneity*** (or ***scaling***) property:

$$ax_1(t) \rightarrow ay_1(t)$$

Simply the superposition property with $b = 0$ and an arbitrary a.

- We can easily see that the additive property and the homogeneity properties together implies superposition.

- Because of homogeneity, linear system must satisfy *zero-input/zero-output* – if input is 0, output must be 0

Linearity and Decomposition

- Linearity can be easily generalized to the superposition of multiple inputs.

$$\text{if } x_k(t) \to y_k(t); \quad k = 1,2 \ldots K$$

$$\text{Then, for a linear system:} \quad \sum_{k=1}^{K} a_k x_k(t) \to \sum_{k=1}^{K} a_k y_k(t)$$

- The superposition property for linear systems forms the basis of all of the *analysis* that we do in this course. With linearity, we can *decompose* a signal into a sum of individual components, determine how a system would response to them, and add the individual responses together at the end. Nonlinear systems cannot be analyzed via decomposition!

- **Example 1.17** $y(t) = tx(t)$

You can easily verify that this is linear.

This is a system where the output is the input multiplied by a time-dependent gain:

$$y(t) = g(t)x(t).$$

If input is $ax_1(t) + bx_2(t)$, then the output is:

$$g(t) \times \{ax_1(t) + bx_2(t)\} = ag(t)x_1(t) + bg(t)x_2(t)$$

which is the same weighted sum of the individual outputs!

- **Example 1.18** $y(t) = x^2(t)$

Squaring is obviously not linear

We can apply the linearity checking procedure:

If $x_3(t) = ax_1(t) + bx_2(t)$, does $y_3(t) = ay_1(t) + by_2(t)$??

Defining $x_1(t)$, $x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

and

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Clearly, we can specify $x_1(t)$, $x_2(t)$, a, and b such that $y_3(t)$ is not the same as $ay_1(t) + by_2(t)$. For example, if $x_1(t) = 1$, $x_2(t) = 0$, $a = 2$, and $b = 0$, then $y_3(t) = (2x_1(t))^2 = 4$, but $2y_1(t) = 2(x_1(t))^2 = 2$. We conclude that the system S is not linear.

Example 1.20

Does not satisfy zero-in/zero-out

Consider the system

$$y[n] = 2x[n] + 3. \tag{1.132}$$

This system is not linear, as can be verified in several ways. For example, the system violates the additivity property: If $x_1[n] = 2$ and $x_2[n] = 3$, then

$$x_1[n] \to y_1[n] = 2x_1[n] + 3 = 7, \tag{1.133}$$

$$x_2[n] \to y_2[n] = 2x_2[n] + 3 = 9. \tag{1.134}$$

However, the response to $x_3[n] = x_1[n] + x_2[n]$ is

$$y_3[n] = 2[x_1[n] + x_2[n]] + 3 = 13, \tag{1.135}$$

which does not equal $y_1[n] + y_2[n] = 16$. Alternatively, since $y[n] = 3$ if $x[n] = 0$, we see that the system violates the "zero-in/zero-out" property of linear systems given in eq. (1.125).

$y[n] = 2(ax_1[n] + bx_2[n]) + 3$

$a y_1[n] + b y_2[n]$
$= 2ax_1[n] + 3a$
$+ 2x_2[n] + 2b + 3$
$\neq y[n]$
$\Rightarrow$ not linear!

Although this system is not linear, it is *incrementally linear*, meaning that the difference in the output depends linearly on the difference in the input. Such systems, such as *linear differential equations with initial conditions*, can be addressed using techniques for addressing linear systems.

How about a system described by a differential equation as shown below?

$$y(t) + 0.5\frac{d^2y(t)}{dt} = t^2x(t) - \frac{dx(t)}{dt}$$

Implicit relation: Output not expressed as explicit function of input

It is **Linear** since: differentiation is linear and the relation depends linearly the derivatives of x and y (derivative multiplied by a constant or a time dependent function).

If $x_1(t)$ and $y_1(t)$ is an input-output pair and $x_2(t)$ and $y_2(t)$ is another input-output pair, it is obvious that $ax_1(t) + bx_2(t)$ and $ay_1(t) + by_2(t)$ will also be an input-output pair.

The following is not linear because of squaring of one of the terms:

$$y^2(t) + 0.5\frac{d^2y(t)}{dt} = t^2x(t) - \frac{dx(t)}{dt}$$

Linear time-invariant (LTI) systems

ELEC2100 will focus on the study of *Linear Time-Invariant* (LTI) systems, for which both linearity and time-invariance hold:

- **Linearity:** ***Superposition holds***

$$x(t) = \sum a_i x_i(t) \quad \rightarrow \quad y(t) = \sum a_i y_i(t)$$

- **Time Invariance:** ***Shifted input Shifted Output***

$$x(t) \rightarrow y(t) \Rightarrow x(t - t_0) \rightarrow y(t - t_0)$$

- LTI systems are thoroughly understood and *analyzable*. Many physical systems are LTI or can be *approximated* as LTI. Often we want to keep man-made systems LTI.

In Chapter 2 we will begin the study of LTI systems.

Self Test – Continue

- For each system below, determine if it is memoryless, causal, linear, time-invariant, and stable

	Memoryless	Causal	Linear	Time-Invariant	Stable
$y_1[n] = e^{x_1[n-1]}$	NO $x[n-1]$ is past input	YES $x[n-1]$ is past not future	NO Exponentiation is not linear	YES The exponentiation operation has no dependency on time	YES If x is bounded. e^x is bounded
$y_2[n] = x_2[n] + x_2[-n-1]$	NO Involves time reversal	NO time reversal; e.g.: $y[-5] = x[-5] + x[4]$	YES Time reversal and time shifting is linear	NO Involves time reversal which is not TI	YES If x is bounded by B, y is bounded by 2B
$y_3(t) = \int_0^t x_3(\tau)\,d\tau$	NO Integration means adding x from different times	NO* The fixed lower limit means when t<0, the interval of integration involves future times	YES Integration is linear	NO Integral with defined lower limit	NO Consider if input is unit step, output is unit ramp

AKW