Lecture 13

Properties of Fourier Transform

(Deduction)

(Chapter 4 O&W)

I. Linearity, Time Shifting, and Conjugate Symmetry
II. Differentiation and Integration
III. Time and Frequency Scaling, Duality, Parseval's Theorem
IV. Convolution and Multiplication
V. Example Frequency Responses

I. Linearity, Time Shifting and Conjugate Symmetry

1. Linearity

FT is linear. Weighted sum of signals in time domain leads to weighted sum of spectrums in frequency domain.

$$\text{If } x(t) \xleftrightarrow{FT} X(j\omega), \text{ and } \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$\boxed{g(t) = ax(t) + by(t) \xleftrightarrow{FT} G(j\omega) = aX(j\omega) + bY(j\omega)}$$

FT and IFT are integrals. As integral they are linear. For FT:

$$\Im\{g(t)\} = \int \underbrace{(ax(t) + by(t))}_{g(t)} e^{-j\omega t} dt = \underbrace{a \int x(t) e^{-j\omega t} dt + b \int y(t) e^{-j\omega t} dt}_{\text{Weighted sum of individual FTs}}$$

2. Time shifting

Time shift a signal by $t_0 \Leftrightarrow$ phase shift of transform by ωt_0: a ***linear phase shift*** (proportional to frequency).

Delay:
Time shift by $-t_0$

Multiply by $e^{-j\omega_0 t}$
= linear phase shift

$$x(t - t_0) \overset{FT}{\longleftrightarrow} \underline{e^{-j\omega t_0}} X(j\omega)$$

✓ bigger frequency, bigger the phase shift

only phase shift, no change in frequency!

Table 4.1
4.3.2

If we view $x(t)$ as a superposition of complex sinusoids, when we shift $x(t)$, we shift all the complex sinusoids. Time shifting the complex sinusoid $e^{j\omega t}$ by $-t_0$ is equivalent to multiplication by $e^{-j\omega t_0}$, a phase shift that equals frequency times the time shift.

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Synthesis integral: $x(t)$ as superposition of complex sinusoids

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}\,d\omega$$

Shifting $x(t)$ = shifting all complex sinusoids

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [e^{-j\omega t_0}X(j\omega)]e^{j\omega t}\,d\omega$$

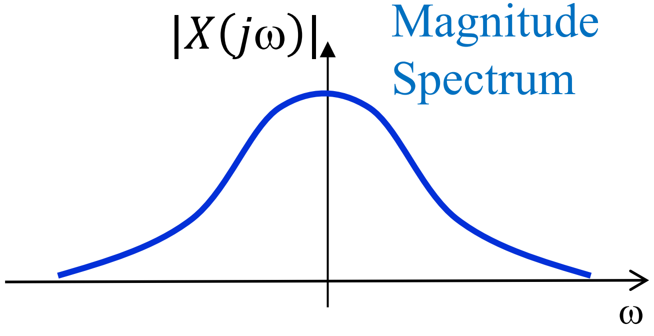
$e^{j\omega(t-t_0)} = e^{-j\omega t_0}e^{j\omega t}$

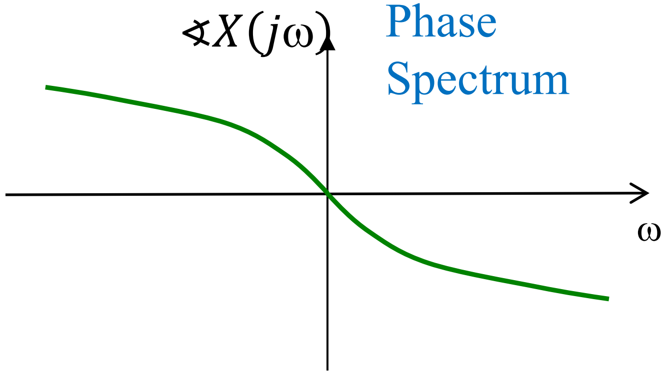
Shifting $e^{j\omega t}$ by $-t_0$ = phase shift by $-\omega t_0$

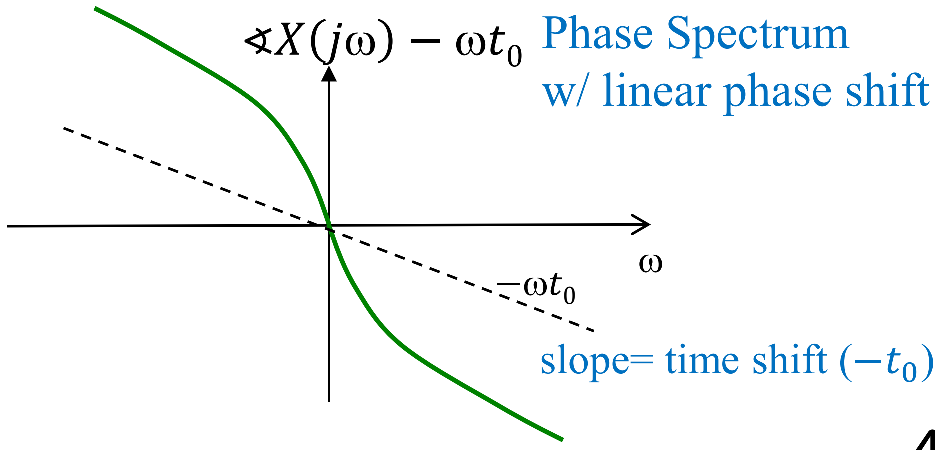
$$\Im\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j(\sphericalangle X(j\omega)-\omega t_0)}$$

Magnitude of spectrum is unchanged

Linear phase shift of spectrum



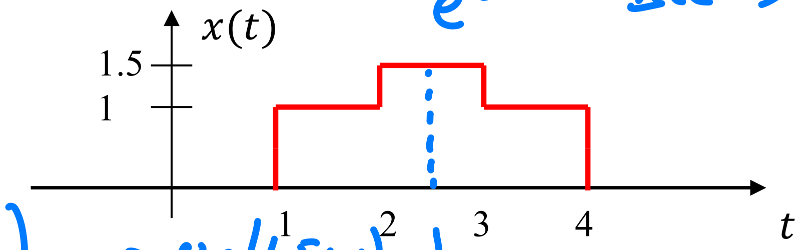




Example – Linearity & Time Shifting

$e^{-j\omega \cdot 2.5} X(e^{j\omega})$

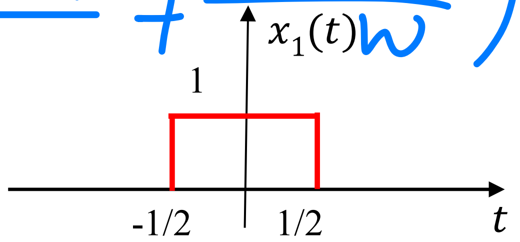
Example 4.9: Find FT of $x(t)$ shown

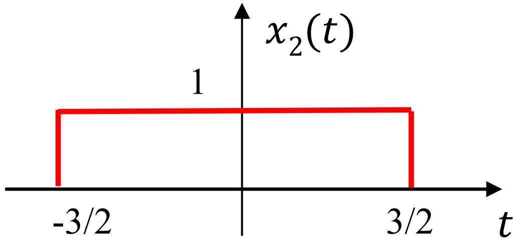


We regard $x(t)$ as sum of two shifted window/pulse signals:

$e^{-j\omega \cdot 2.5}\left(\frac{\sin(\frac{\omega}{2})}{\omega} + \frac{2\sin(1.5\omega)}{\omega}\right)$

$$x(t) = \frac{1}{2} x_1(t - 2.5) + x_2(t - 2.5)$$





From Table 4.2, $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \overset{FT}{\longleftrightarrow} 2\frac{\sin\omega T_1}{\omega}$

FT of the two pulse signals are: $X_1(j\omega) = 2\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}$; $X_2(j\omega) = 2\frac{\sin\left(\frac{3\omega}{2}\right)}{\omega}$

Therefore:

$$X(j\omega) = \underbrace{e^{-j5\omega/2}}_{\text{Linear phase shift term for time delay of 2.5}} \left[\frac{\sin\left(\frac{\omega}{2}\right)}{\omega} + \frac{2\sin\left(\frac{3\omega}{2}\right)}{\omega}\right]$$

3. Conjugation and conjugate symmetry:

We state two sub-properties:

First, $\boxed{\text{A. Conjugation of time signal} \Leftrightarrow \text{Conjugation and frequency reversal of spectrum}}$

Conjugation frequency reversal

A. $$\boxed{x^*(t) \overset{FT}{\longleftrightarrow} X^*(-j\omega)}$$

Table 4.1, 4.3.3

Proof

$$\Im\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt = \Bigg[\underbrace{\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}\,dt}_{X(-j\omega)}\Bigg]^* = X^*(-j\omega)$$

Second,

$$\boxed{\textbf{B.}\ \text{Time reversal} \Leftrightarrow \text{frequency reversal}}$$

Table 4.1, 4.3.5 **B.**

$$\boxed{x(-t) \xleftrightarrow{FT} X(-j\omega)}$$

Proof (by algebraic manipulation)

let $t' = -t$

Associate minus sign with ω

$$\Im\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}\,dt = -\int_{\infty}^{-\infty} x(t')e^{-j(-\omega)t'}\,dt' = \int_{-\infty}^{\infty} x(t')e^{-j(-\omega)t'}\,dt' = X(-j\omega)$$

Cancel negative sign by reversing lower and upper limits

$t \to \infty \Rightarrow t' \to -\infty$

$dt = -dt'$

$t \to -\infty \Rightarrow t' \to \infty$

Conjugation and conjugate symmetry -

From sub-property A, $\Im\{x^*(t)\} = X^*(-j\omega)$, we can conclude:

1. If $x(t)$ is **real**, which means $x^*(t) = x(t)$

Then its FT is **conjugate symmetric**: $X^*(-j\omega) = X(j\omega)$

FT FT

Conjugate Symmetry means:

$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$

→ the real part of the FT is even

and

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$

→ the imaginary part of FT is odd

or alternatively:

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

→ the magnitude spectrum is even

→ the phase spectrum is odd

Now, adding sub-property B, $(x(-t) \overset{FT}{\leftrightarrow} X(-j\omega))$, we conclude:

2. If $x(t)$ is real and even, then $X(j\omega)$ is real and even

$x(t)$ real

$x(t) = x^*(t) \Rightarrow$

$X(j\omega)$ conjugate symmetric

$X(j\omega) = X^*(-j\omega)$

$x(t)$ even

$x(t) = x(-t) \Rightarrow$

$X(j\omega)$ even

$X(j\omega) = X(-j\omega)$

$\Rightarrow$ $\boxed{X(j\omega) \text{ even and real}}$

3. If $x(t)$ is real and odd, $X(j\omega)$ is purely imaginary and odd

$x(t)$ real

$x(t) = x^*(t) \Rightarrow$

$X(j\omega)$ conjugate symmetric

$X(j\omega) = X^*(-j\omega)$

$x(t)$ odd

$x(t) = -x(-t) \Rightarrow$

$X(j\omega)$ odd

$X(j\omega) = -X(-j\omega)$

$\Rightarrow$ $X(j\omega)$ purely imaginary and odd

<u>4.</u> For a real signal, its *even* part leads to the *real and even part* of its FT and its *odd* part leads to the *imaginary and odd* part of its FT.

- Recall that $x(t)$ can be <u>decomposed</u> into even and odd parts

$$\underbrace{x(t)}_{\text{real}} = \underbrace{x_{even}(t)}_{\text{real \& even part}} + \underbrace{x_{odd}(t)}_{\text{real \& odd part}}$$

Then,

$$\Im\{x(t)\} = \underbrace{\Im\{x_{even}(t)\}}_{\text{real \& even}} + \underbrace{\Im\{x_{odd}(t)\}}_{\text{Purely imaginary \& odd}}$$

Table 4.1
4.3.3

Do some decomposition

II. Differentiation and Integration

4. Differentiation and integration

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Table 4.1
4.3.4

Proof View $x(t)$ as a superposition of complex sinusoids. Differentiating $x(t)$ means differentiating all of the individual complex sinusoids. Differentiating $e^{j\omega t}$ gives $j\omega\ e^{j\omega t}$.

$$\underbrace{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega}_{\text{Express } x(t) \text{ as synthesis integral - superposition of complex sinusoids}} \quad \xRightarrow{\text{Differentiate both sides with respect to } t} \quad \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}\,d\omega$$

FT of $\frac{dx(t)}{dt}$

A synthesis integral

Lower & upper limits are constant

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t}$$

Differentiation in time corresponds to multiplication by $j\omega$ in frequency, which is multiplication by frequency plus a 90^{o} phase shift. Differentiation amplifies high frequency.

Integration in time means division by $j\omega$ in frequency, but there is an extra *DC adjustment* term.

DC adjustment term, which
is an impulse at $\omega = 0$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \overbrace{\pi X(0)\delta(\omega)} = X(j\omega)\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$$

$X(0)\delta(\omega) = X(j\omega)\delta(\omega)$

zero frequency!

Formal proof for the extra term is cumbersome, but is related to the ***total value theorem***:

等重的?<

$$\int_{-\infty}^{\infty} x(\tau)d\tau = X(0)$$

total value of $x(t)$ = value of transform at zero frequency

Since $y(t)$ is the integral of $x(t)$, $y(t)$ becomes a constant as $t \to \infty$: $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \int_{-\infty}^{t} x(\tau)d\tau = X(0)$.

The transform of the constant leads to an impulse at $\omega = 0$.

A relation related to the total value theorem is: $x(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(j\omega)d\omega$

Signal at zero time is the total value of spectrum (with scaling of $1/2\pi$)

Integration Example

- **Example 4.11** Find $X(j\omega)$ the FT of $x(t) = u(t)$, the unit step.

Recall the unit step is the integral of the impulse $x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$,

and the FT of the impulse is the constant 1: $\delta(t) \xleftrightarrow{FT} G(j\omega) = 1$

Hence, from the integration property, FT of the unit step signal is:

$$u(t) \xleftrightarrow{FT} \overbrace{\frac{1}{j\omega}}^{G(j\omega)=1} + \pi\overbrace{\delta(\omega)}^{G(0)=1} \qquad \textbf{(4.33)}$$

Table 4.2

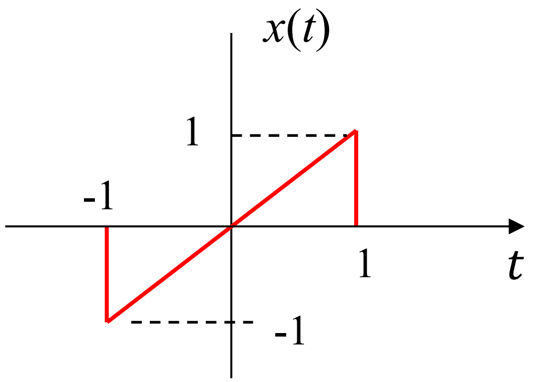
Checking our result, we consider $\delta(t)$ as the derivative of $u(t)$. Then the FT for $u(t)$ suggests that the FT of $\delta(t)$ is:

$$\mathfrak{J}\{\delta(t)\} = j\omega\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) = 1 + \underbrace{j\omega\pi\delta(\omega)}_{} = 1$$

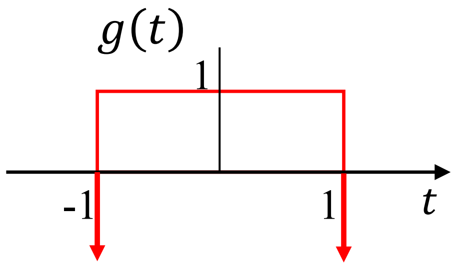
What is this product at $\omega = 0$?
It makes no difference.

- **Example 4.12:**

Find FT of $x(t)$ as shown to the right.



Consider $x(t)$'s derivative: $g(t) = \frac{d}{dt}x(t)$.



$g(t)$ can be viewed as the sum of a window and two shifted impulses. Its FT can be easily found:

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}; \quad G(0) = 2 - 1 - 1 = 0$$

DC adjustment term is 0

Using the integration property:

$$\boxed{\begin{aligned} X(j\omega) &= \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega}\left\{\left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}\right\} \\ &= \left(\frac{2\sin\omega}{j\omega^2}\right) - \frac{2\cos\omega}{j\omega} \end{aligned}}$$

III. Time & Frequency Scaling, Duality, Parseval's Theorem

5. Time and frequency scaling

Scaling of signal in time ⇔ Reciprocal scaling of Fourier transform in frequency with normalization

$$x(at) \stackrel{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Table 4.1
4.3.5

Examples:

$$x(2t) \stackrel{FT}{\longleftrightarrow} \frac{1}{2} X\left(\frac{j\omega}{2}\right)$$

$$x(-2t) \stackrel{FT}{\longleftrightarrow} \frac{1}{2} X\left(\frac{-j\omega}{2}\right)$$

Proof:

$$\mathfrak{I}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$$

$t = \frac{\tau}{a}$ $dt = \frac{d\tau}{a}$

let $\tau = at$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}}\, d\tau$$

Sign reversal if $a < 0$ because of reversal in limits of integration

If $a < 0$, when $t \to \infty$, $\tau \to -\infty$, and when $t \to -\infty$, $\tau \to \infty$,

The normalization constant

For example, let $y(t) = x(-2t)$. We observe that the total value of $y(t)$ is ½ of the total value of $x(t)$. Time reversal does not affect the total value but time scaling does.

Generally, if $y(t) = x(at)$, then $\int_{-\infty}^{\infty} y(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} x(t)dt$.

The above means $Y(0) = \frac{1}{|a|}X(0)$, which is why we have the normalization constant $\frac{1}{|a|}$

Duality

$\boxed{\text{Symmetry in the FT integral and the IFT integral leads to duality}}$

6. Duality

if $$x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$$

then $$\frac{1}{2\pi}X(jt) \overset{FT}{\longleftrightarrow} x(-\omega)$$

<u>**Ignore the *j***</u>
The only variable is ω or t!

- We have seen that the FT of a window signal is the sinc function

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

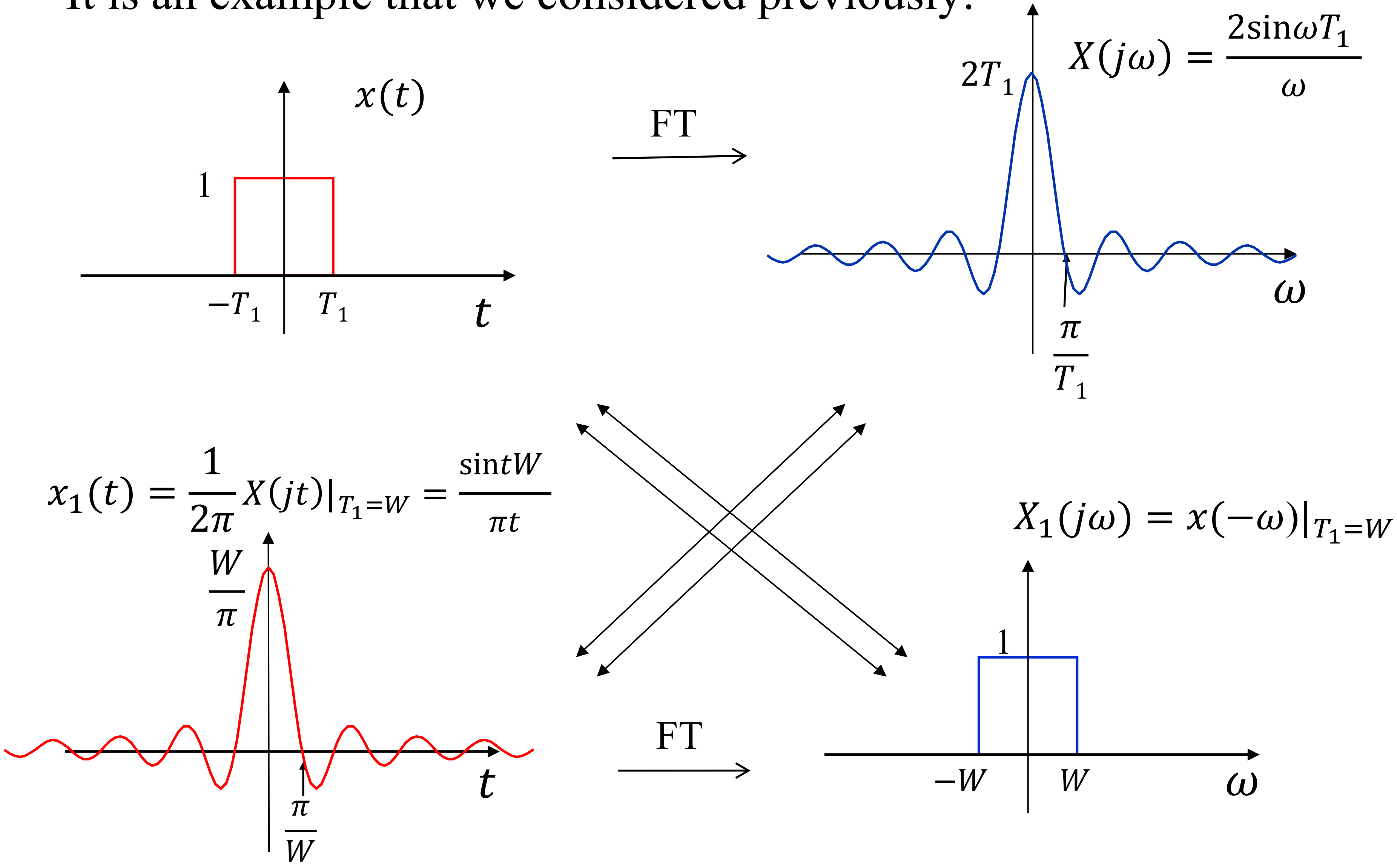
Replace ω by t
& divide by 2π;
T_1 replaced by W

Replace t by $-\omega$; here $x(-\omega) = x(\omega)$

Therefore

$$x_1(t) = \frac{\sin tW}{\pi t} \overset{FT}{\longleftrightarrow} X_1(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

- It is an example that we considered previously:



Duality Applied to Other Properties

a. Differentiation in frequency → Multiply by $-jt$ in time

Analysis integral for $X(j\omega)$

$$\frac{dX(j\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Only the $e^{-j\omega t}$ term in integrand is dependent on ω; $\frac{de^{-j\omega t}}{d\omega} = -jte^{-j\omega t}$

$$= \int_{-\infty}^{\infty} \underbrace{-jtx(t)}_{\text{IFT of } \frac{dX(j\omega)}{d\omega}}e^{-j\omega t}\,dt$$

$$\Rightarrow\ -jtx(t) \overset{FT}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

Table 4.1
4.3.6

Duality Applied to Other Properties

b. Shifting in frequency → Multiply by complex sinusoid in time

$$e^{j\omega_0 t}x(t) \overset{FT}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Table 4.1
4.3.6

This is an important property. Multiplying any complex sinusoid by $e^{j\omega_0 t}$ increases the frequency of the complex sinusoid by ω_0. Multiplying a signal $x(t)$ by $e^{j\omega_0 t}$ increases the frequency of all the frequency components of $x(t)$ by ω_0, therefore shifting the spectrum to the right by ω_0 .

Parseval's Relation

For periodic signals, the Parseval's relation considers the power of signals. For aperiodic signals, the Parsevals' relation considers the energy.

7. Parseval's relation:

Total energy can be computed by integrating $|x(t)|^2$, power of the time signal over all time or by integrating the energy per unit frequency $|X(j\omega)|^2/2\pi$ over all frequencies

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

Table 4.1
4.3.7

- $|X(j\omega)|^2$ is called the ***energy-density spectrum*** of $x(t)$

-The energy in signal $x(t)$ = the area under the energy-density spectrum

Proof (Skip - For reference only)

x(f)

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\overbrace{\cancel{\frac{1}{2\pi}}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega}^{x(t)}\right]^* dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right] dt$$

Change order of integration

$$= \int_{-\infty}^{\infty} X^*(j\omega)\left[\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

Again, the proof is based on the fact that complex sinusoids at different frequencies are orthogonal – their inner products are zero.

IV. Convolution and Multiplication

8. Convolution

Convolution of two signals in time ⇔ multiplication of the two FTs in frequency.

$$\boxed{y(t) = x(t) * h(t) \quad \xleftrightarrow{FT} \quad Y(j\omega) = X(j\omega)H(j\omega)}$$

Table 4.1
4.4

- $y(t) = x(t) * h(t)$ means we can view $y(t)$ as the output of an LTI system with $x(t)$ as input and $h(t)$ as impulse response:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t)\end{array}} \longrightarrow y(t) = x(t) * h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)H(j\omega)}_{Y(j\omega)}e^{j\omega t}d\omega$$

$e^{j\omega t}$ is eigenfunction

- Now if we view $x(t)$ as a superposition of complex sinusoids, each complex sinusoid is scaled by the frequency response $H(j\omega)$, so the spectrum for the output is product of the spectrum of the input and $H(j\omega)$.

Proof (Skip – For reference only):
We can prove the convolution property through a more formal algebraic derivation as follow:

Substituting $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ into the FT analytic integral, we have:

$$Y(j\omega) = \Im\{y(t)\} = \int_{-\infty}^{\infty} \Bigg[\overbrace{\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}^{y(t)\,=\,x(t)\,*\,h(t)} \Bigg] e^{-j\omega t}\, dt$$

Change order of integration

$$= \int_{-\infty}^{\infty} x(\tau) \Bigg[\underbrace{\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t} dt}_{} \Bigg] d\tau$$

$\Im\{h(t-\tau)\} = e^{-j\omega\tau} H(j\omega)$ time shift → phase shift

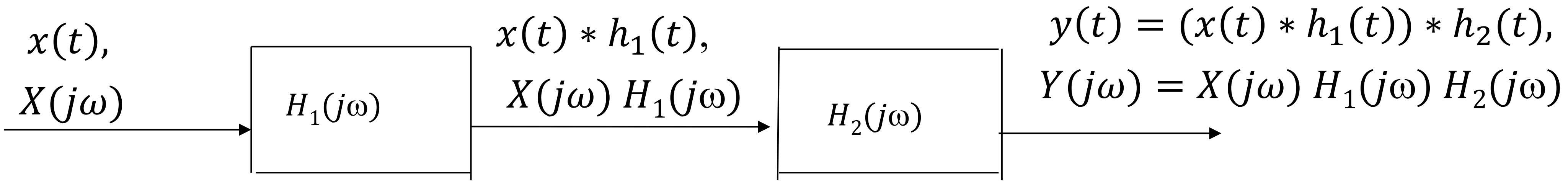
$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(j\omega)\, d\tau$$

$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau \qquad = X(j\omega) H(j\omega)$$

Associative and Commutative Properties Revisited

- Since convolution in time is simply multiplication in frequency domain, the commutative and associative properties of convolution must be true!

Multiple systems in cascade just means multiple multiplications of Fourier transforms.
Multiplication is commutative and associative.



Multiplication Property

Based on duality: Multiplication in time domain ⇔ convolution in frequency domain

9. Multiplication property

$$g(t) = x(t)y(t) \quad \overset{FT}{\longleftrightarrow} \quad G(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\gamma) Y\big(j(\omega - \gamma)\big) d\gamma$$

γ is a dummy variable for integration

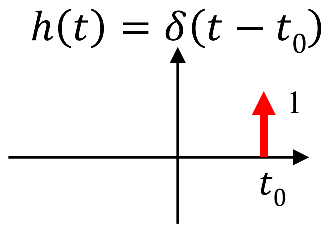
The convolution integral may look somewhat unfamiliar only because of our insistence to include j in the function arguments:

We will come back to the multiplication property next lecture when we talk about frequency mixing and modulation.

V. Example Frequency Responses

Example 1: Delay System

Recall that *frequency response* of an LTI system is the *Fourier transform of its impulse response*.
We conclude this lecture by looking at several example frequency responses.



- **Example 4.15** Consider a CT LTI system with $h(t) = \delta(t - t_0)$.

It is a delay system as we know all along: $y(t) = x(t) * \delta(t - t_0) = x(t - t_0)$

FT

From the time shifting property of FT: $Y(j\omega) = e^{-j\omega t_0} X(j\omega)$

This means the frequency response is $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = e^{-j\omega t_0}$

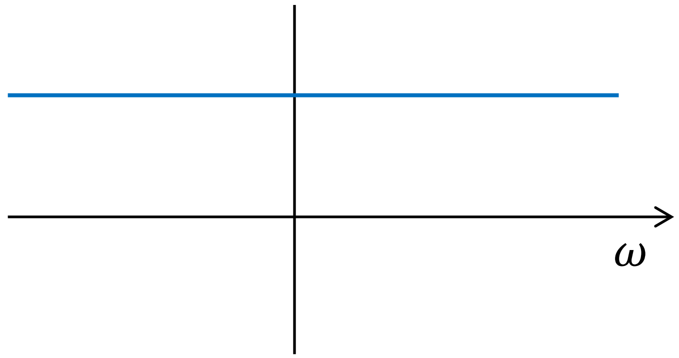
Or taking the FT of $h(t) = \delta(t - t_0)$ directly, $H(j\omega) = \mathfrak{J}\{\delta(t - t_0)\} = e^{-j\omega t_0} \mathfrak{J}\{\delta(t)\} = e^{-j\omega t_0}$

Example 1: Delay System

In frequency domain, a time delay system simply multiplies the input spectrum by $e^{-j\omega t_0}$, which is a ***linear phase shift*** with respect to ω:

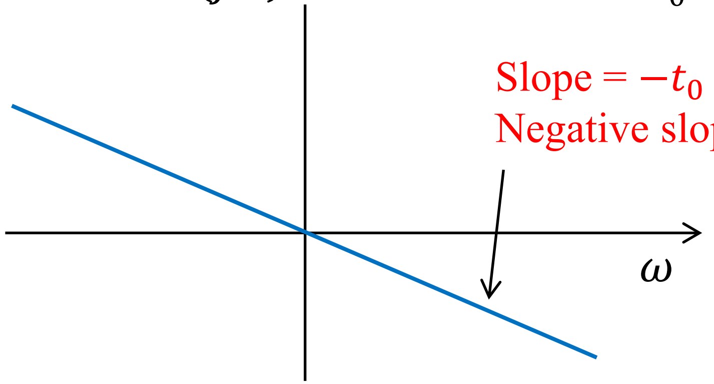
Magnitude response = 1 for all frequencies

$$|H(j\omega)| = \left|e^{-j\omega t_0}\right| = 1$$



Phase response linear in frequency

$$\measuredangle H(j\omega) = \measuredangle e^{-j\omega t_0} = -\omega t_0$$



Slope = $-t_0$ gives the time shift.
Negative slope means delay.

Example 2: Differentiator & Integrator

- **Example 4.16:** $y(t) = \dfrac{dx(t)}{dt}$

From the differentiation property of FT:

$$Y(j\omega) = j\omega X(\omega)$$

which means the frequency response of a differentiator is: $H(j\omega) = j\omega$

Hence, magnitude response is $|H(j\omega)| = \omega$, and phase response is $\measuredangle H(j\omega) = \frac{\pi}{2}$.

If input is $\cos(\omega_1 t)$, output is $|H(j\omega_1)|\cos(\omega_1 t + \measuredangle H(j\omega_1)) = \omega_1 \cos\left(\omega_1 t + \frac{\pi}{2}\right) = -\omega_1 \sin(\omega_1 t)$

- **Example 4.17:** $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

From integration property of FT,

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \underbrace{\pi X(j\omega)\delta(\omega)}_{\text{DC Adjustment}} = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) X(j\omega)$$

This means the frequency response of an integrator is $H(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$

which is the FT of the unit step $u(t)$.

We can represent differentiation and integration by multiplication in frequency domain:

$\boxed{j\omega}$ $\boxed{\frac{d}{dt}}$

Block diagrams
for a differentiator

$\boxed{\frac{1}{j\omega}}$ $\boxed{\int}$

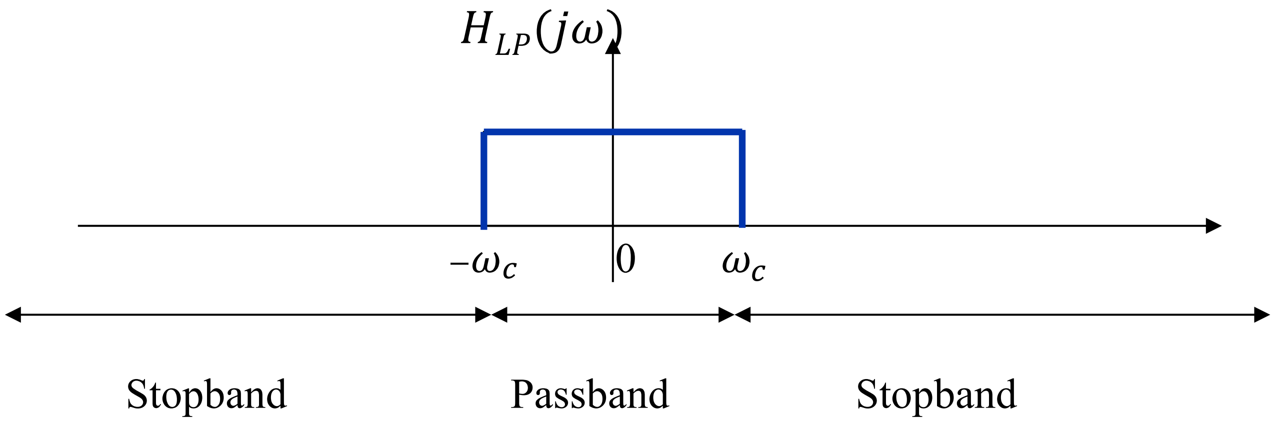
Block diagrams
for an integrator (we
ignore the $\pi\delta(\omega)$)

Frequency Response Example 3: Ideal Low Pass Filter (ILP)

- **Example 4.18:** We introduced the *Ideal Low Pass Filter* (ILP) earlier. The frequency response is a window function:

$$H_{LP}(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

Cut-off frequency

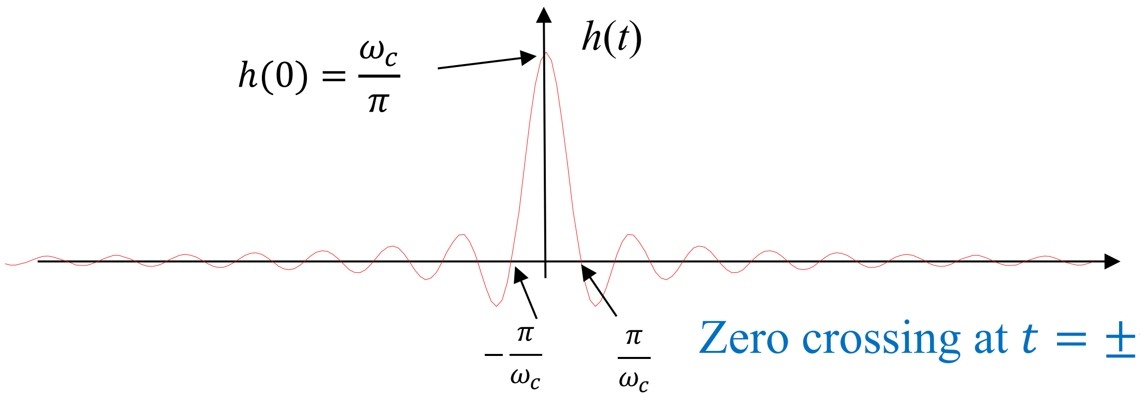


The ILP passes all frequencies below the cut-off frequency (*passband*) unchanged and reject all frequencies above (*stopband*).

We will need the ILP in many application examples that we will look at soon.

- **Example 4.18 (cont.):**

From **Example 4.5**, the corresponding impulse response is: $h(t) = \dfrac{\sin\omega_c t}{\pi t}$



Zero crossing at $t = \pm m\frac{\pi}{\omega_c}$, $m \geq 1$

The impulse response of the ILP means that it is not achievable:

– $h(t)$ is not causal. It is 2-sided so no amount of delay can make it causal. So $h(t)$ is *not feasible* to be implemented exactly.

So how do we build practical low pass filters?

– We will look at a couple of simple possible options

– Filter design involves many trade-offs that are topics in future courses