Lecture 11

Properties of Discrete-Time Fourier Series (DTFS)

(Deduction & Language)

(Ref: Chapter 3 O&W)

I. Properties of DTFS: (Deduction)

- Frequency Shifting
- Multiplication

II. More Properties of DTFS:

- Periodic Convolution
- First Difference
- Parseval's Theorem

III. LTI Systems as Filters

I. Properties of DTFS

- Most properties of DTFS are similar to those of CTFS
  - Linearity, time-shifting, time reversal, conjugate symmetry, etc
- Here, we examine several DTFS properties which manifest some differences because the signal is DT:
  - Frequency shifting
  - Multiplication
  - Periodic convolution
  - First Difference
  - Parseval's theorem
- Recall the DTFS synthesis and analysis equation:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Properties of DTFS – 1. Frequency Shifting

<u>1. Frequency Shifting</u>: Multiply by the M harmonic $\Leftrightarrow$ Shift harmonics by M

$$x[n] \overset{FS}{\longleftrightarrow} a_k \quad \Longrightarrow \quad y[n] = \underbrace{e^{jM\left(\frac{2\pi}{N}\right)n}}_{\phi_M[n]} x[n] \overset{FS}{\longleftrightarrow} b_k = a_{k-M}$$

Table 3.2

Shift to right by M

filtering !

Proof :

$$b_k = \frac{1}{N}\sum_{n=0}^{N-1} \overbrace{e^{jM\left(\frac{2\pi}{N}\right)n} x[n]}^{y[n]} e^{-jk\left(\frac{2\pi}{N}\right)n} = \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j(k-M)\left(\frac{2\pi}{N}\right)n}}_{\langle x[n], \phi_{k-M}[n]\rangle} = a_{k-M}$$

$e^{j\frac{2\pi n}{2\cdot 2\cdot 20}\cdot 20} = e^{j\frac{2\pi n}{20}\cdot 5}$

Example 1: Consider the periodic square wave of example 3.12 with $N = 20$ and $2N_1 + 1 = 5$. Multiply this square wave by $e^{j\frac{\pi n}{2}}$ and find the DTFS of the result.

$b_k = a_{k-5}$

$M = 5$

$N_1 = 2$

$$e^{j\frac{\pi n}{2}} = e^{j5\times\frac{2\pi n}{20}} \Rightarrow M = 5$$

$e^{j\frac{\pi n}{2}}$ is the 5-th harmonic

Therefore the harmonics are shifted by 5

$$b_k = a_{k-5} = \frac{1}{N}\frac{\sin\left(\frac{2\pi(k-M)\left(N_1+\frac{1}{2}\right)}{N}\right)}{\sin\left(\frac{\pi(k-M)}{N}\right)} = \frac{1}{20}\frac{\sin\left(\frac{5\pi(k-5)}{20}\right)}{\sin\left(\frac{\pi(k-5)}{20}\right)}$$

Example 2: For the same example, multiply the square wave by $e^{j25\times\frac{2\pi}{20}n}$ and find the DTFS of the result.

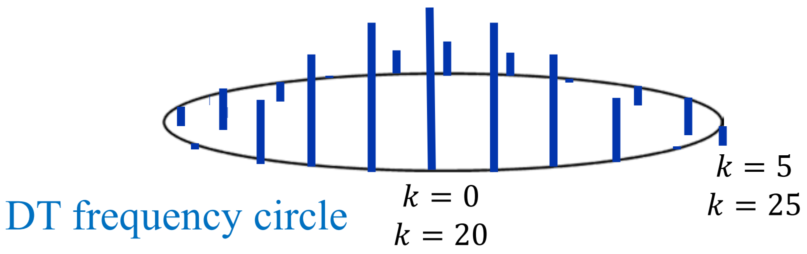
$e^{j5\times\frac{2\pi}{20}n}$ = Same!

We notice that for N=20, the 25-th harmonic and the 5-th harmonic are the same harmonic:

$$\phi_{25}[n] = e^{j(5+20)\times\frac{2\pi}{20}n} \equiv \phi_5[n] = e^{j5\times\frac{2\pi}{N}n}$$

Therefore our answer is the same as in the previous slide!

Another way to look at this, for N=20, the FS coefficients are periodic with period 20 (since there are 20 FS coefficients). So shifting the coefficients by $M = 5$ or by $M = 25$ are equivalent.



Properties of DTFS – 2. Multiplication

<u>2. Multiplication</u>: Multiply two DT periodic signals ⇔ ***<u>periodic convolution</u>*** of their DTFS coefficients

Only do one period!

m n → m+n-1

$$g[n] = x[n]y[n] \overset{FS}{\longleftrightarrow} c_k = \sum_{l=<N>} a_l\, b_{k-l} = \sum_{l=m}^{m+N-1} a_l b_{k-l}$$

Eq.(3.108)
Table 3.2

All N-periodic

N contiguous terms

N terms in sum

Recall that for CTFS, multiplying two signals leads to *convolution* of their FS:

$$x(t)y(t) \overset{FS}{\leftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

?

In *periodic convolution*, we <u>sum over only one period</u>.

Whenever we apply convolution to ***two signals that are <u>both N/T-periodic</u>***, we must apply a *periodic convolution*. If we apply the *regular convolution*, the result will "blow up" unless it is zero – either way it is not very useful.

Proof :

- Again, when we multiply two complex sinusoids, we add their frequencies.
- But now we have only a finite number of harmonics in $x[n]$ and $y[n]$:

$g[n]$ = product of two synthesis sums

Product of two sums = double sum of individual products

$$g[n] = x[n]y[n] = \left(\sum_{k=0}^{N-1} a_k\, e^{jk\frac{2\pi}{N}n}\right)\left(\sum_{k=0}^{N-1} b_k\, e^{jk\frac{2\pi}{N}n}\right) \quad = \sum_{l=0}^{N-1}\sum_{m=0}^{N-1} a_l b_m\, e^{j(l+m)\frac{2\pi}{N}n}$$

Substitute variable: $k = l + m$

$$= \sum_{l=0}^{N-1}\sum_{k=l}^{l+N-1} a_l b_{\underbrace{k-l}_{k-l=m}}\, e^{j\overbrace{k}^{l+m}\frac{2\pi}{N}n}$$

Now consider the inner sum. Both b_{k-l} and $e^{jk\frac{2\pi}{N}n}$ are N-periodic in k. Therefore we can replace the range of the inner summation to obtain:

$$g[n] = \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} a_l b_{k-l}\, e^{jk\frac{2\pi}{N}n}$$

Reverse the order of summation, and the FS coefficients become apparent: $g[n] = \sum_{k=0}^{N-1}\underbrace{\sum_{l=0}^{N-1} a_l b_{k-l}}_{c_k}\, e^{jk\frac{2\pi}{N}n}$

Example: Modified version of Exercise 3.30 in text. Let $x[n]$, $y[n]$ be 6-periodic:

$[0, \frac{1}{2}, 1, \frac{1}{2}, 0, 0]$

$$x[n] = 1 + \cos\left(\frac{2\pi}{6}n\right) \qquad y[n] = \sin\left(\frac{4\pi}{6}n\right)$$

$[\frac{-1}{2j}, 0, 0, 0, \frac{1}{2j}, 0]$

$a_0 = 1$

$a_1 = \frac{1}{2}$ $\quad a_{-1} = \frac{1}{2}$

$a_2 = \frac{1}{2j}$ $\quad a_{-2} = \frac{-1}{2j}$

(a) Determine FS coefficients of $x[n]$ $\qquad a_0 = 1; \quad a_1 = \frac{1}{2}; \quad a_{-1} = \frac{1}{2}; \quad \frac{1}{2}e^{j\left(\frac{2\pi}{6}\right)n} + \frac{1}{2}e^{-j\left(\frac{2\pi}{6}\right)n} = \cos\left(\frac{2\pi}{6}n\right)$

(b) Determine FS coefficients of $y[n]$ $\qquad b_2 = \frac{-j}{2}; \quad b_{-2} = \frac{j}{2}; \quad \frac{-j}{2}e^{j\left(\frac{4\pi}{6}\right)n} + \frac{j}{2}e^{-j\left(\frac{4\pi}{6}\right)n} = \sin\left(\frac{4\pi}{6}n\right)$

(c) Determine for FS coefficients of $g[n] = x[n]y[n]$

$$c_k = \sum_{l=<N>} a_l\, b_{k-l} = \sum_{l=<N>} b_l\, a_{k-l}$$

We can easily show that periodic convolution is also commutative. Let's enumerate over b's because there are only 2 non-zero b's

$$c_0 = b_2 \times a_{-2} + b_{-2} \times a_2 = 0$$

$$c_1 = b_2 \times a_{-1} + b_{-2} \times a_3 = \frac{-j}{2} \times \frac{1}{2} = \frac{-j}{4}$$

$$c_2 = b_2 \times a_0 + b_{-2} \times a_4 = \frac{-j}{2} \times 1 = \frac{-j}{2}$$

$$c_3 = b_2 \times a_1 + b_{-2} \times a_5 = \frac{-j}{2} \times \frac{1}{2} + \frac{j}{2} \times \frac{1}{2} = 0$$

$c_4 = ?$ $\qquad c_4 = c_{-2} = c_2^* = \frac{j}{2}$

$c_5 = ?$ $\qquad c_5 = c_{-1} = c_1^* = \frac{j}{4}$

We have computed the coefficients for 6 distinct harmonics and we are done!

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II. More Properties of DTFS: (Deduction)
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- Parseval's Theorem

III. LTI Systems as Filters

II. More Properties of DTFS

- The ***periodic convolution*** of two DT N-periodic signals is defined as:

$$x[n] \circledast y[n] = \sum_{r=0}^{N-1} x[r]y[n-r] = \sum_{r=<N>} x[r]y[n-r]$$

Since terms in the sum are N-periodic

The symbol $\circledast$ is used to denote periodic convolution.

- For a regular convolution, we have an infinite sum: $x[n] * y[n] = \sum_{r=-\infty}^{\infty} x[r]y[n-r]$

Since $x[n]$ and $y[n]$ are both periodic, $x[r]\ y[n-r]$ is periodic in r, and the infinite sum in the regular convolution will "blow up" (unless it is 0).

For two periodic signals, we must apply the periodic convolution. You will learn that the periodic convolution is a very important tool if you take a course in DSP (Digital Signal Processing).

The Periodic Convolution Property

<u>3. Periodic Convolution:</u> Periodic Convolution in time domain ⇔ multiplication of Fourier Coefficients

$$g[n] = \sum_{r=<N>} x[r]y[n-r] \overset{FS}{\longleftrightarrow} c_k = N a_k b_k$$

Table 3.2
Example 3.15

In DSP, this property allows us to convert convolution to multiplications, and makes many signal processing functions hundreds of times more efficient.

Proof (Skip) We **skip** this proof because it is the same as the proof for the multiplication property due to the symmetry of the DTFS analysis and synthesis equations

$g[n]$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{r=0}^{N-1} x[r]y[n-r]e^{-jk(2\pi/N)n}$$

separate n into r and $n-r$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{r=0}^{N-1} x[r]e^{-jk(2\pi/N)r}y[n-r]e^{-jk(2\pi/N)(n-r)}$$

let $m = n - r$

$$= \frac{1}{N}\sum_{m=-r}^{N-1-r}\sum_{r=0}^{N-1} x[r]e^{-jk(2\pi/N)r}y[m]e^{-jk(2\pi/N)(m)}$$

adjust lower and upper limits of summation

$< x[n], \phi_k[n] >$

$$= \frac{1}{N}\sum_{m=-r}^{N-1-r} Na_k y[m]e^{-jk(2\pi/N)(m)} = Na_k b_k$$

More Properties of DTFS – 4. First Difference

First difference in DT is the counterpart to differentiation in CT.

4. First difference:

$$x[n] - x[n-1] \overset{FS}{\leftrightarrow} a_k\left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right)$$

The proof is trivial from the time-shift property:

$$x[n-1] \overset{FS}{\leftrightarrow} a_k e^{-jk(2\pi/N)}$$

Time delay by 1 Phase delay by angular frequency

More Properties of DTFS – 5. Parseval's Theorem

5. Parseval's Theorem: Power of signal is sum of powers of individual harmonics

$$\frac{1}{N}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2$$

This is again due to the orthogonality of harmonics.

Proof

Prove it as self-test.

Summary - Properties of DTFS

1. **Frequency shifting** $e^{jM(2\pi/N)n}x[n] \overset{FS}{\longleftrightarrow} a_{k-M}$

2. **Multiplication** $x[n]y[n] \overset{FS}{\longleftrightarrow} \sum_{<N>} a_n b_{k-n}$ Periodic convolution

3. **Periodic convolution** $\sum_{r=0}^{N-1} x[r]y[n-r] \overset{FS}{\longleftrightarrow} Na_k b_k$ Multiplication

4. **First difference** $x[n]-x[n-1] \overset{FS}{\longleftrightarrow} a_k(1-e^{-jk(2\pi/N)})$

5. **Parseval's Theorem** $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

- Our discussion on the Fourier decomposition of periodic CT and DT signals is now complete, and we will move on to discuss how we can determine the output of LTI systems using the concept of ***Filtering***

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Chapter 3 – Properties of Discrete-Time Fourier Series (DTFS)

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III. LTI Systems as Filters (Language)

III. LTI System as Filter

- For an LTI system, the output y to input x can be computed in two ways :

  - **Method 1 (time-domain)**: We use convolution

$$y(t) = x(t) * h(t)$$

$$y[n] = x[n] * h[n]$$

$h(t)$, $h[n]$: impulse response

- Alternatively, we can go to *frequency domain* and specify an LTI system by its *frequency response* :

CT:

$$H(j\omega) = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt;$$

Fourier transform: special case of the Laplace transform

DT:

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Discrete-time Fourier transform (DTFT): special case of the z-transform $H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$

Determination of Output using Frequency Response

Method 2 (frequency-domain): By regarding $x(t)$, $x[n]$ as a sum of complex sinusoids, we can easily specify the output knowing that the LTI system will simply scale each complex sinusoid by a constant. For the case of periodic inputs:

$\phi_k(t)$: eigenfunction

Frequency response $\times\, \phi_k(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \dashrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H\left(jk\frac{2\pi}{T}\right) e^{jk\left(\frac{2\pi}{T}\right)t}$$

$\phi_k[n]$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \dashrightarrow y[n] = \sum_{k=0}^{N-1} a_k H\left(e^{jk\frac{2\pi}{N}}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

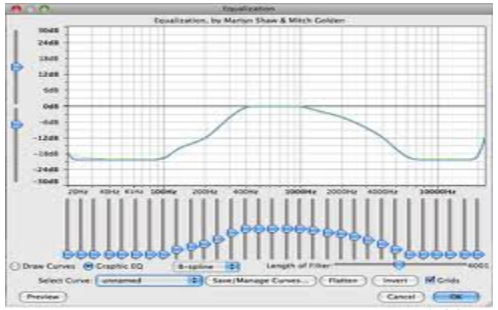
$\phi_k[n]$

A finite sum for the DT case

- When the input to an LTI system is periodic, the output is also clearly periodic.
- Furthermore, if $\{a_k\}$ are the FS coefficients for the input, then $\{a_k H(jk\omega_0)\}$ (CT) or $\{a_k H(e^{jk\omega_0})\}$ (DT) will be FS coefficients for the output .

LTI Systems as Frequency-Shaping Filters

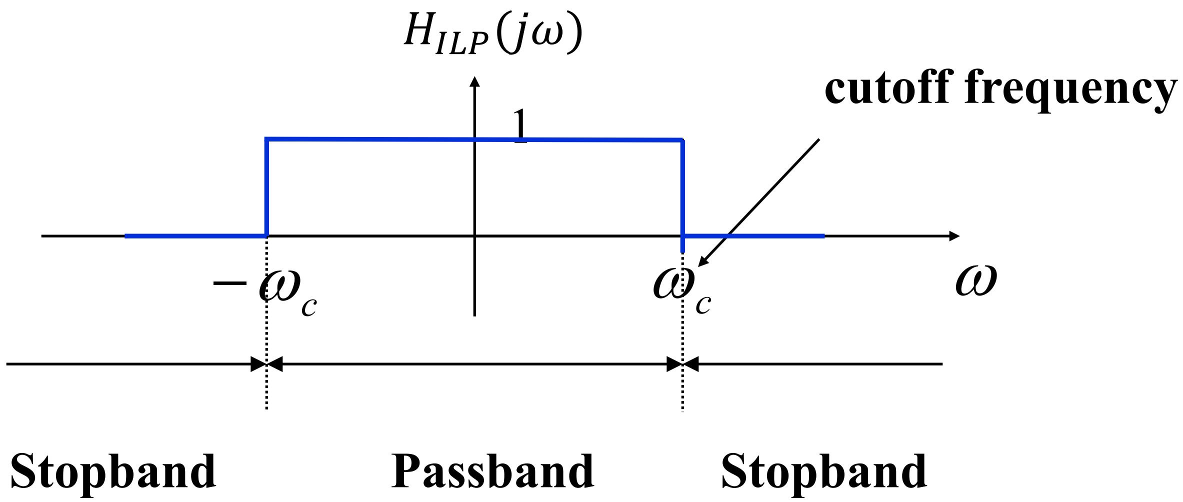
- Hence, ***all LTI systems can be regarded Frequency-Shaping filters*** that change the relative amplitudes of different frequency components in signals.
- Example: Shown below is an equalizer that an audiophile may have at home. Its function is to "equalize" the uneven response that your speakers may have to different frequencies.



Different graphics showing how equalizer shape amplitude of different frequency bands

Basic Types of Filters

- In the future, our discussion on LTI systems as filters will be based on several *idealized* basic filter types.
- Shown below is the frequency response of a CT ***ideal low-pass filter (ILPF)*** .
- An ILPF rejects all frequency components above a cutoff frequency (*stopband*) and passes all frequency components below the cutoff unchanged (*passband*).



$$H_{ILP}(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

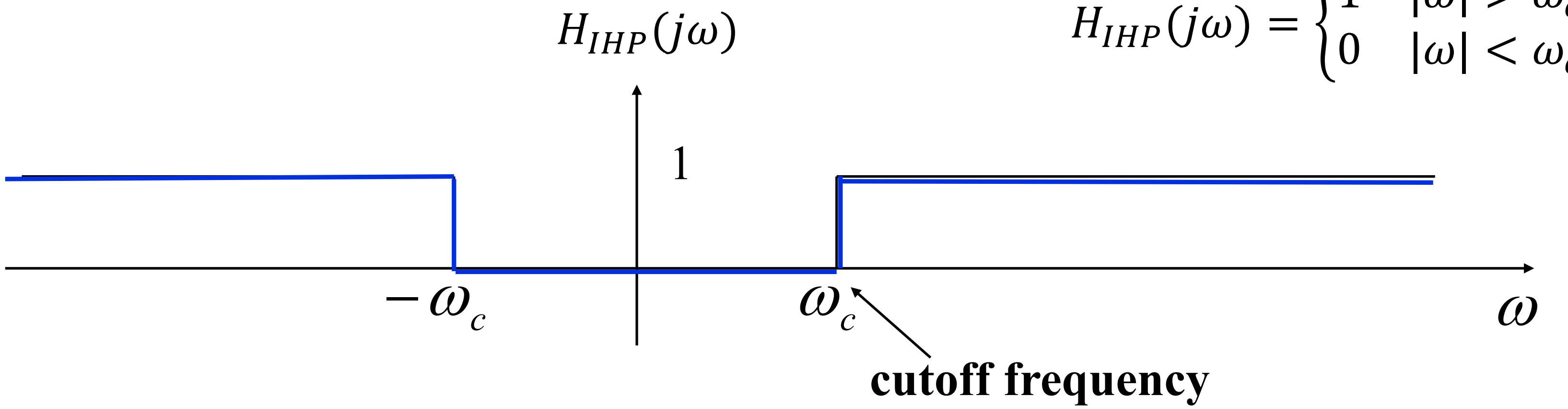
(3.140)

low pass!

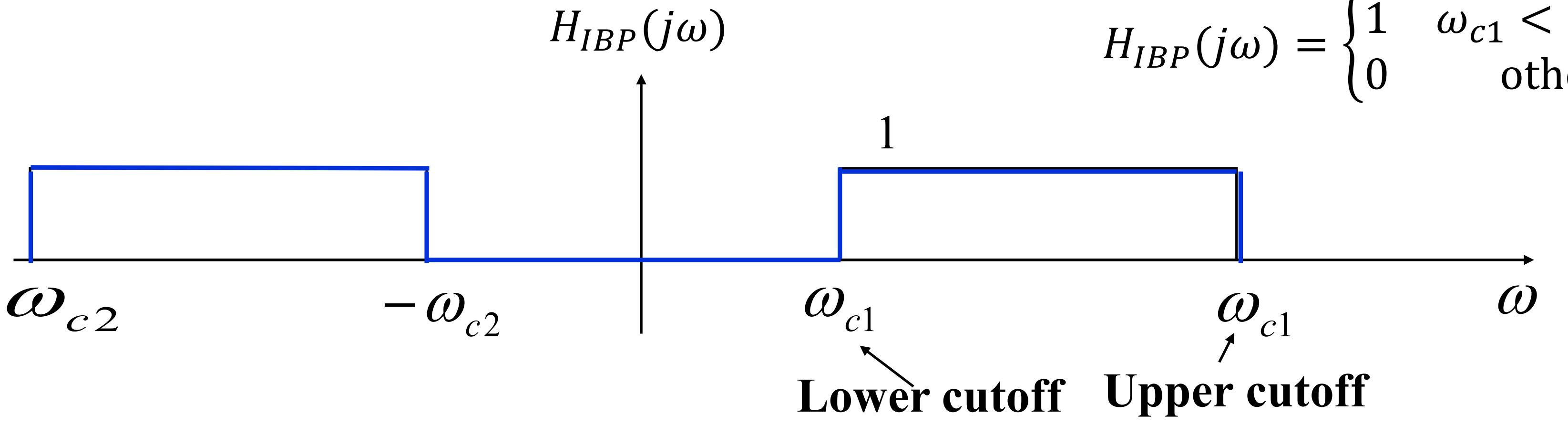
Apply Gaussian filter!

- Below are the frequency response of the CT ***Ideal High Pass Filter*** and ***Ideal Bandpass Filter*** :

$$H_{IHP}(j\omega) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & |\omega| < \omega_c \end{cases}$$



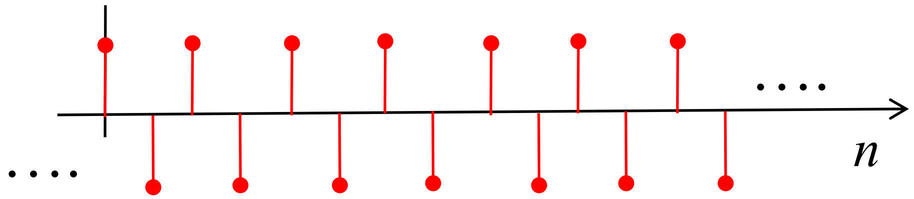
$$H_{IBP}(j\omega) = \begin{cases} 1 & \omega_{c1} < |\omega| < \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$



Discrete Time Frequency Filters

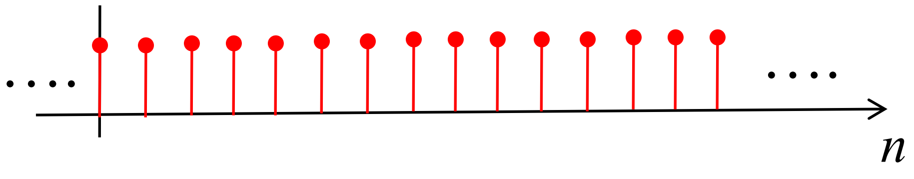
- Recall that in DT, frequency is circular: ω and $\omega + 2\pi$ are the same frequency.
- Therefore, the frequency response of a DT filter is 2π-periodic in frequency. Alternatively, the frequency response of the filter over a frequency range of 2π, say from $-\pi$ to π, fully specifies the filter!
- Recall also that π is the highest frequency for DT signals. For DT signals, frequency around π, 3π, 5π, etc., is high frequency. Frequency around 0, 2π, 4π, etc., is low frequency.

$$\cos(\pi n) = \cos(3\pi n) = \cos(5\pi n) \ldots .$$



Signal of high frequency

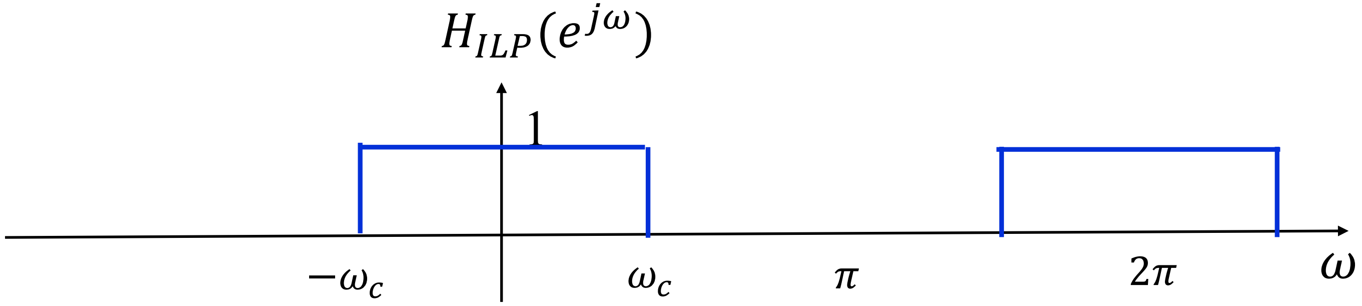
$$\cos(0n) = \cos(2\pi n) = \cos(4\pi n) \ldots .$$



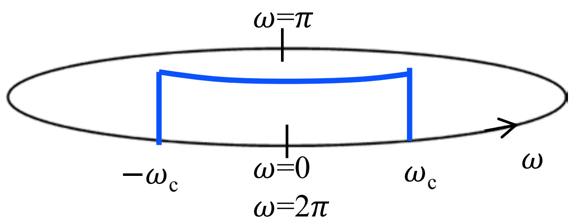
Signal of low frequency

Filtering – Discrete Time

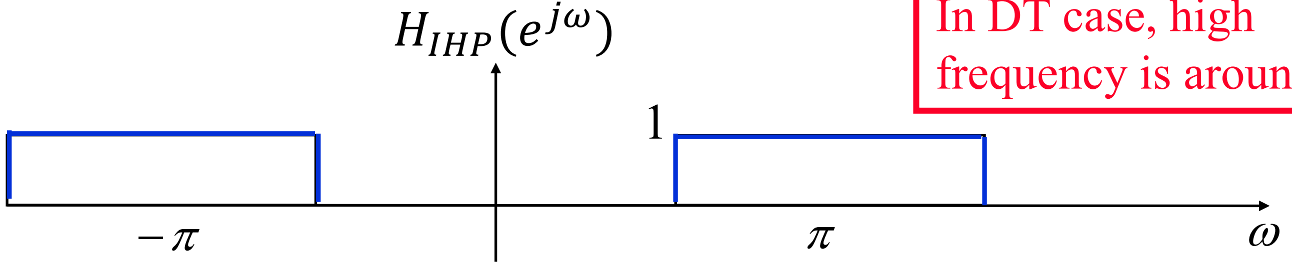
- DT Ideal lowpass filter



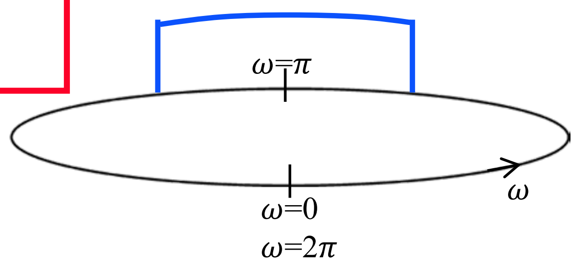
DT frequency is actually circular



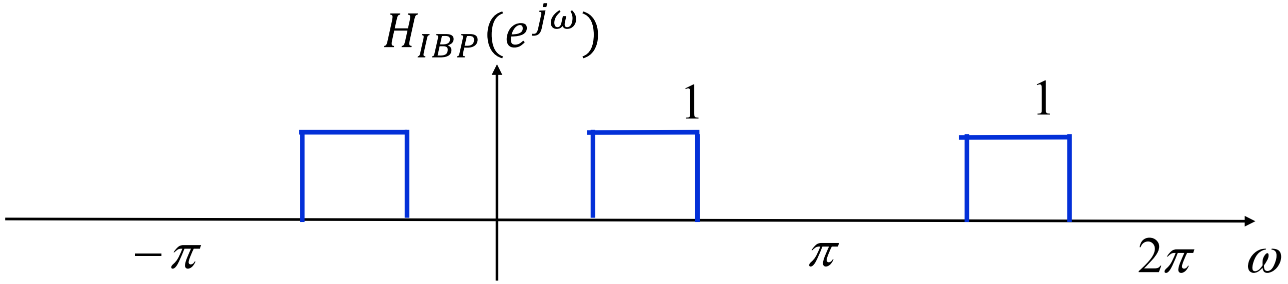
- DT Ideal highpass filter

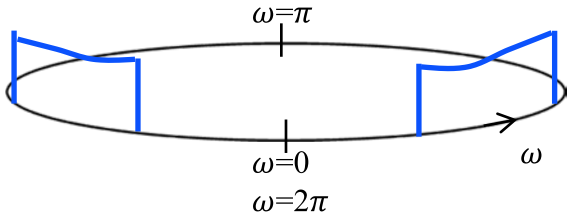


In DT case, high frequency is around π!



- DT Ideal bandpass filter





Characterization, Design and Implementation of Filters

- In the next chapter you will see how we can characterize a communication link such as the telephone channel as a filter.
- You will also see how communications and signal processing systems need to implement lowpass, bandpass, and highpass filters to perform different functions.
- How do we implement filters? How do we design the filters that we need? You will see some examples going forward.
- Ideal lowpass, bandpass, and highpass filters are not achievable, but in the real world we try to get as close to them as we can.