Lecture 10

Discrete-Time Fourier Series (DTFS)

(Analysis)

(Ref: Chapter 3 O&W)

I. DTFS – DT Signals as sum of Complex Sinusoids

II. Periodicity and Symmetry in DTFS

III. DTFS Examples

I. DTFS – DT Signals as sum of Complex Sinusoids

DTFS is to decompose an N-periodic DT signal into a weighted sum of the N distinct DT harmonic complex sinusoids with period N:

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = 0,1,2 \ldots N-1$$

$\frac{2\pi}{N}$ is fundamental frequency ω_0

There are only N distinct harmonics

- In CTFS, we have potentially an infinite number of harmonics – the frequency in CT signal can be arbitrarily high, and $-\infty \leq k \leq \infty$
- In DTFS: we have a ***finite*** sum of N harmonics only!

  Recall:

  - *There are only N distinct complex sinusoids with period N.*
  - $\phi_k[n] \equiv \phi_{k+N}[n] \equiv \phi_{k+mN}[n]$

Self Test:

1. How many distinct DT complex sinusoids there are that is periodic with period $N = 9$? Please list them.

9

$\omega_0 = \frac{2\pi}{9}$

$e^{jk\frac{2\pi}{9}n}, \quad k = 0, 1, 2, \ldots 8$

2. Illustrate $e^{j2\times\frac{2\pi}{9}n}$ graphically.

$\omega_0 = \frac{4\pi}{9}$

3. Illustrate $e^{j11\times\frac{2\pi}{9}n}$ and $e^{j(-7)\times\frac{2\pi}{9}n}$ graphically.

Same as 2.

Solution to Self Test:

1. How many distinct DT complex sinusoids there are that is periodic with period $N = 9$? Please list them.

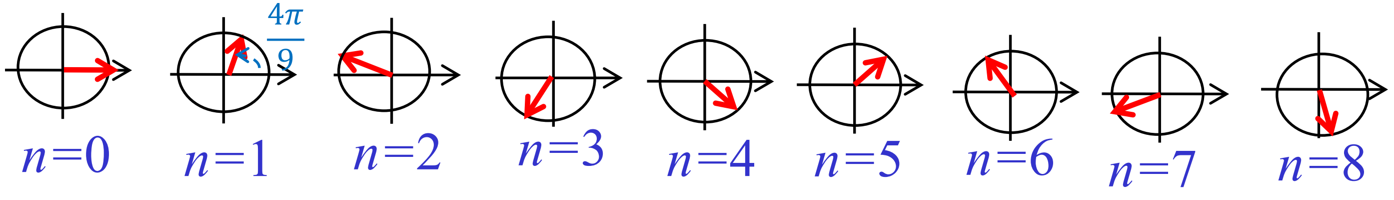
9 $\qquad e^{jk\frac{2\pi}{9}n};\ k = 0,1,2,3 \ldots .8$

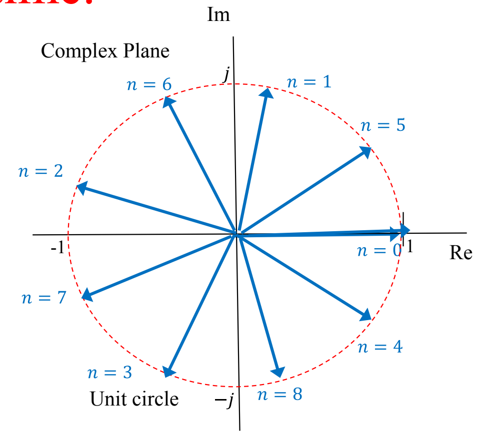
or $k = -4, -3, -2, \ldots 4$ $\quad$ or $k = -3, -2, -1, \ldots 5$

2. Illustrate $e^{j2\times\frac{2\pi}{9}n}$ graphically.

It is the 2nd-harmonic; $\quad$ frequency $k\omega_0 = 2 \times \frac{2\pi}{9}$: phase change is $\frac{4\pi}{9}$ per unit time.

$e^{j2\times\frac{2\pi}{9}n}$





3. Illustrate $e^{j11\times\frac{2\pi}{9}n}$ and $e^{j(-7)\times\frac{2\pi}{9}n}$ graphically.

Both are the same as $e^{j2\times\frac{2\pi}{9}n}$

The DTFS Synthesis and Analysis Equations

$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}$

This sum is an inner product $< x[n], \phi_k[n] >$

inner product!

synthesis

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Eq (3.94)

Synthesis: $x[n]$ as a weighted sum of harmonic complex sinusoids

analysis

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Eq (3.95)

Analysis: to determine how much of each harmonic is in $x[n]$

Decomposition!

- The ***synthesis equation*** states that $x[n]$ is a weighted sum of N complex sinusoids.
- The ***analysis equation*** calculates the ***FS coefficients*** which are the weights for the harmonics.

  It is an inner product with normalization: $\frac{1}{N} < x[n], e^{jk\left(\frac{2\pi}{N}\right)n} >$

  get back a_k

- The normalization is $\frac{1}{N}$ because N is the self-inner product of any harmonic:

$$< e^{jk\left(\frac{2\pi}{N}\right)n}, e^{jk\left(\frac{2\pi}{N}\right)n} > = N$$

$$< \phi_k[n], \phi_k[n] > = \sum_{n=0}^{N-1} |\phi_k[n]|^2 = N$$

1

Periodicity of DTFS Coefficients

- We observe from Eq (3.95) that the DTFS coefficients a_k can be regarded as N-periodic in k because:

$< x[n], \phi_{k+N}[n] >$ $\phi_{k+N}[n]$ and $\phi_k[n]$ are the same

$a_{k+N} = a_k$

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(k+N)\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} = a_k$$

-combination!

- Therefore the terms in the synthesis sum Eq (3.94) are N-periodic, and we can shift the summation window to sum over any N contiguous terms:

synthesis

Eq (3.94)

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Both N-periodic in k

means summing over any N contiguous terms in k

e.g.: 1 to N, 2 to $N+1$, etc.

- The terms in the analysis equation Eq (3.95) are obviously N-periodic in n. So we can also sum over any N contiguous terms:

analysis

Eq (3.95)

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

sum over any N contiguous terms in n

Both $x[n]$ and $\phi_k[n]$ are N-periodic in n.

Reference

For future classes in DSP, we need to get used to expressing the ranges of summation:

If N even (e.g., $N = 8$):

$$\sum_{k=0}^{N-1}, \quad \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}, \quad \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \qquad \sum_{k=0}^{7} \quad \sum_{k=-4}^{3} \quad \sum_{k=-3}^{4}$$

If N odd (e.g., $N = 9$):

$$\sum_{k=0}^{N-1}, \quad \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} \qquad \sum_{k=0}^{8} \quad \sum_{k=-4}^{4}$$

Proof of the Synthesis/Analysis Pair for DTFS

- Validity of the synthesis/analysis equations pair is again because of the ***orthogonality of harmonics*** – the inner product of two different harmonics is zero.

- To show two different harmonics are orthogonal, we first show that the self-sum of any non-DC harmonic over one period is zero:

Self-sum of a harmonic

$$\sum_{n=<N>} e^{jk\frac{2\pi}{N}n} = \begin{cases} 0 & k \neq 0 \text{ (or } mN) \\ N & k = 0 \text{ (or } mN) \end{cases}$$

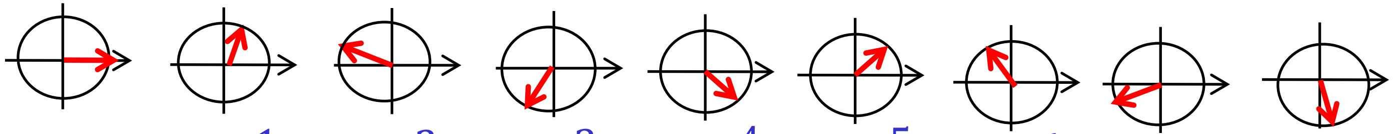
$k = 0, \pm N, \pm 2N \ldots.$ is the DC.

$\phi_k[n] \equiv \phi_{k+mN}[n]$

Self-Sum of a Harmonic

For example, for $k = 2$, $N = 9$, how do we show that $\sum_{n=0}^{8} e^{j2\times\frac{2\pi}{9}n} = 0$?

$e^{j2\times\frac{2\pi}{9}n}$



Recall the result of a *finite geometric sum*: $$\sum_{n=0}^{M} \alpha^n = \frac{1-\alpha^{M+1}}{1-\alpha}$$

$M = N - 1$ $\qquad$ $M + 1 = N$

Thus, $$\sum_{n=0}^{N-1} e^{jk\left(\frac{2\pi}{N}\right)n} = \frac{1-e^{jk\left(\frac{2\pi}{N}\right)N}}{1-e^{jk\left(\frac{2\pi}{N}\right)}}$$

$\alpha = e^{jk\left(\frac{2\pi}{N}\right)}$ $\qquad$ α

The numerator is always 0 because $e^{jk\left(\frac{2\pi}{N}\right)N} = e^{jk2\pi} = 1$, so the sum equals 0,

unless $k \overset{\text{DC}}{=} mN$ so that the denominator is also 0 because $e^{jk\left(\frac{2\pi}{N}\right)} = e^{jm2\pi} = 1$, in which case we recognize the self-sum equals N because every term in the sum is 1.

Orthogonality of Harmonics

Again, let $\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}$ represents the k-harmonic

- The inner product of the r-harmonic and k-harmonic is the self-sum of the $(r-k)$-harmonic:

conjugation

$$< \phi_r\ [n], \phi_k[n] > \quad = \sum_{n=0}^{N-1} \phi_r[n]{\phi_k}^*[n] \quad = \sum_{n=0}^{N-1} e^{jr\left(\frac{2\pi}{N}\right)n} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{n=0}^{N-1} e^{j(r-k)\left(\frac{2\pi}{N}\right)n} \quad = \quad \begin{cases} 0 & r-k \neq 0 \text{ (or } mN) \\ N & r-k = 0 \text{ (or } mN) \end{cases}$$

self-sum of $(r-k)$-harmonic

Equals 0 unless $r-k$ represents the DC.

- Hence, the inner product of two different harmonics is zero – two different harmonics are orthogonal.

Proof of the Synthesis/Analysis Pair for DTFS

Now we can consider the summation in Eq. (3.95) again:

replace $x[n]$ by its synthesis sum Eq.(3.94)

$$\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{r=<N>} a_r \underbrace{e^{jr\left(\frac{2\pi}{N}\right)n}}_{\phi_r[n]} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Reverse order of summation

$$= \sum_{r=<N>} a_r \underbrace{\sum_{n=<N>} e^{jr\left(\frac{2\pi}{N}\right)n} e^{-jk\left(\frac{2\pi}{N}\right)n}}_{\text{Inner product of two harmonics: } < \phi_r[n], \phi_k[n] >} = \sum_{r=<N>} a_r \underbrace{\sum_{n=<N>} e^{j(r-k)\left(\frac{2\pi}{N}\right)n}}_{\substack{= \text{self-sum of } (r-k)\text{-harmonic,} \\ \text{which is 0 unless } r = k + mN}} = Na_k$$

Hence, we have shown that Eq. (3.95) is true if Eq. (3.94) is true – we have proven the validity of the synthesis/analysis pair.

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Chapter 3 - Discrete-Time Fourier Series (DTFS)

I. DTFS – DT Signals as sum of Complex Sinusoids

II. Periodicity and Symmetry in DTFS

III. DTFS Examples

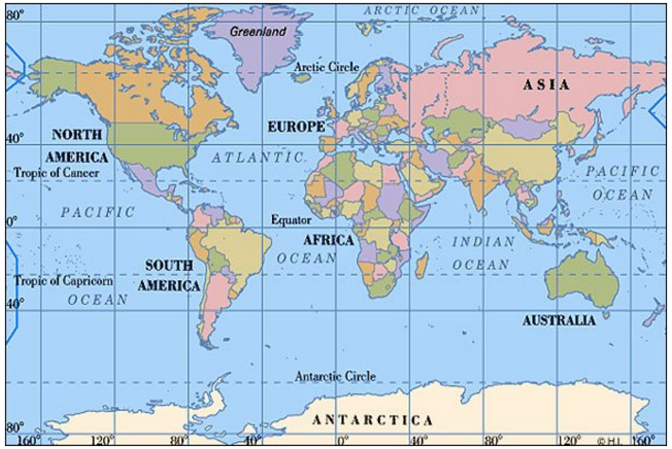
II. Periodicity and Symmetry in DTFS

Let us consider again why we treat the DTFS coefficients as N-periodic; i.e., $a_k = a_{k+N}$

- If there are only N distinct harmonics, why don't we limit k to say $\{0, 1, 2, \ldots, N-1\}$?

The reason is that k is circular and the mathematical notation is simpler if we allow k to be any arbitrary integer. It is the same reason that we allow an angle θ, which is circular, to be of arbitrary value.

Consider the world globe and the world map below:

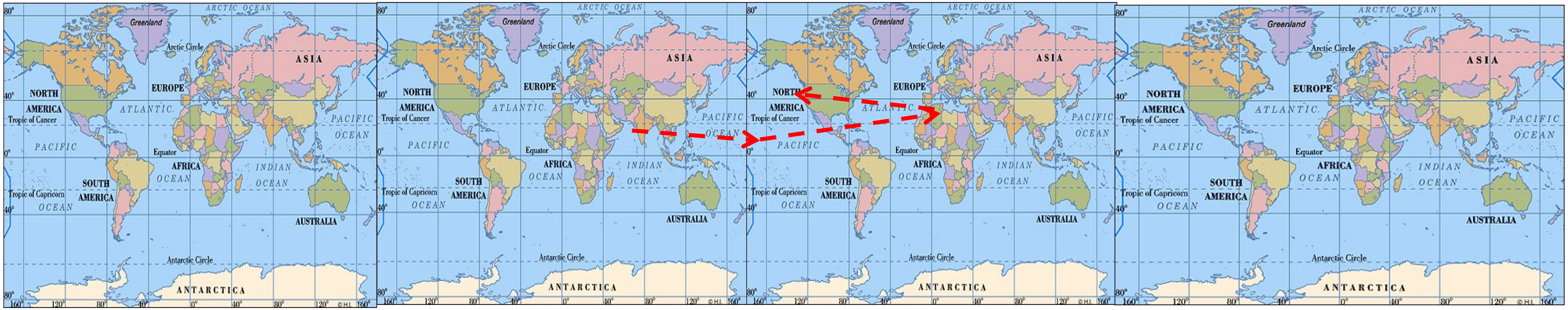


The world is finite but circular –a *circular domain*.

The map on the left can be very misleading if you do not know that the world is circular. But drawing a circular globe on paper is cumbersome.

Showing a Circular Domain as Periodic

Sailors on Christopher Columbus' ship would find it more comforting to see the following "***unwrapped***" map of the world, where we show a *circular domain* as periodic:



Longitude on the earth is over a finite range (180° west to 180° east), but longitude is circular, and it is much more convenient to allow the longitude to take on any value.

We will be doing a lot of *frequency shifting* in the processing of DT signals. It is much more convenient to allow the frequency ω or the harmonic number k to assume any value. We just need to remember that ω and $\omega + 2\pi$ are equivalent, and k and $k + N$ are equivalent.

Symmetry in the DTFS Synthesis and Analysis Equations

- Note that the synthesis equation and the analysis equation in DTFS are highly symmetrical – they are the same except for a sign change and a scaling constant $1/N$.

Synthesis
$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Analysis
$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$x[0] = \sum_{k=<N>} a_k \qquad \phi_k[0] = 1\ \forall k$$

$$x[1] = \sum_{k=<N>} a_k\, e^{jk\left(\frac{2\pi}{N}\right)} \qquad \phi_k[1]$$

$\dots$

$$x[N-1] = \sum_{k=<N>} a_k\, e^{jk\left(\frac{2\pi}{N}\right)(N-1)} \qquad \phi_k[N-1]$$

Same except for minus sign in exponent and scaling by $1/N$

$$a_0 = \frac{1}{N}\sum_{n=<N>} x[n] \qquad \phi_0^*[n] = 1$$

$$a_1 = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j\left(\frac{2\pi}{N}\right)n} \qquad \phi_1^*[n]$$

$\dots$

$$a_{N-1} = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j(N-1)\left(\frac{2\pi}{N}\right)n} \qquad \phi_{N-1}^*[n]$$

- This symmetry makes sense when we regard periodic/finite duration DT signals as vectors and view DTFS as a *change in coordinates*, which is a rotation and a scaling. To change back to the original coordinate, we simply reverse the rotation and scaling.

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As in the CT case, sometimes we can easily recognize the FS coefficients without using the analysis equation.

Example 3.11 List the FS coefficients of:

$$x[n] = 1 + 3\cos\left(\frac{2\pi}{N}n\right) + \sin\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$a_0 = 1$, $a_1 = \frac{3}{2} + \frac{1}{2j}$ $a_{-1} = \frac{3}{2} - \frac{1}{2j}$ $a_2 = \frac{j}{2}$ $a_{-2} = \frac{-j}{2}$

We can solve this problem by expressing $x[n]$ in terms of complex sinusoids:

$$x[n] = 1 + \frac{3}{2}e^{j\frac{2\pi}{N}n} + \frac{3}{2}e^{-j\frac{2\pi}{N}n} - \frac{j}{2}e^{j\frac{2\pi}{N}n} + \frac{j}{2}e^{-j\frac{2\pi}{N}n} + \frac{e^{j\frac{\pi}{2}}}{2}e^{j\frac{4\pi}{N}n} + \frac{e^{-j\frac{\pi}{2}}}{2}e^{-j\frac{4\pi}{N}n}$$

$a_0 = 1$

$a_1 = \frac{3}{2} - \frac{j}{2}$

$a_{-1} = a_1^* = \frac{3}{2} + \frac{j}{2}$

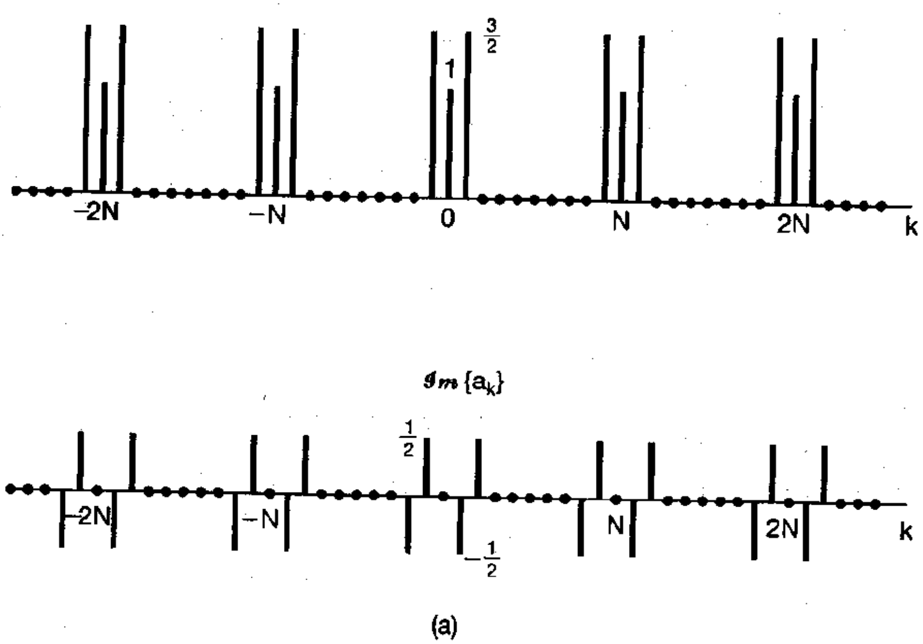
$a_2 = \frac{e^{j\frac{\pi}{2}}}{2} = \frac{j}{2}$

$a_{-2} = \frac{e^{-j\frac{\pi}{2}}}{2} = \frac{-j}{2}$

Example 3.11 - (cont.)

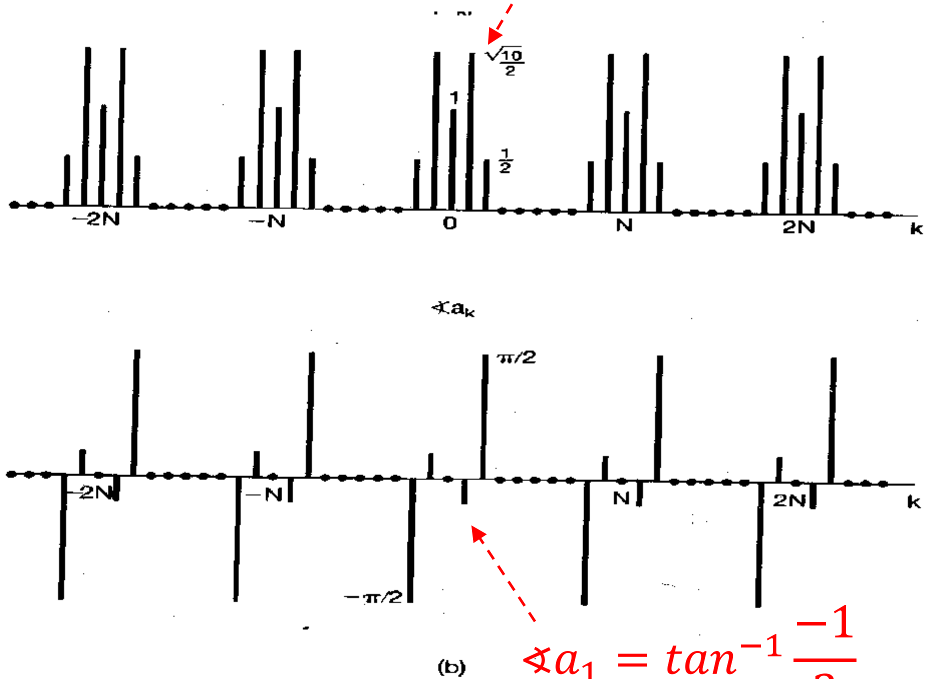
- We show below the FS Coefficients in rectangular form (real and imaginary parts), and in polar form (magnitude and phase).

<u>Rectangular Form</u>



<u>Exponential Form</u>

$|a_k|$ $\quad |a_1| = \left|\frac{3}{2} - \frac{j}{2}\right| = \frac{\sqrt{10}}{2}$

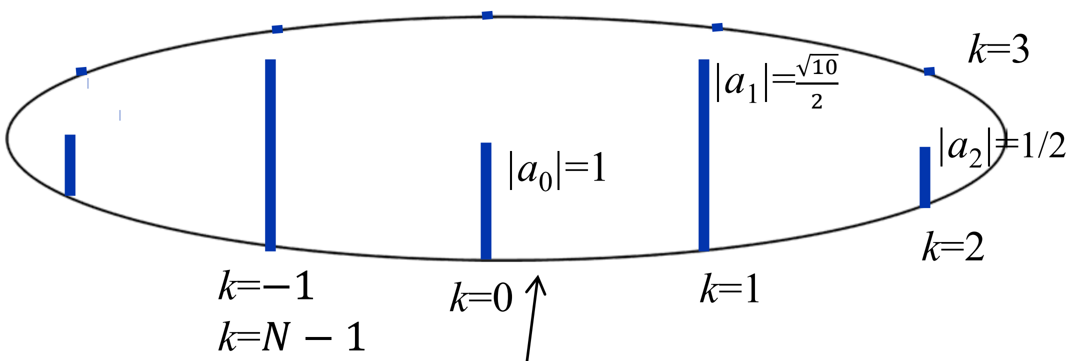


$\sphericalangle a_1 = tan^{-1}\frac{-1}{3}$

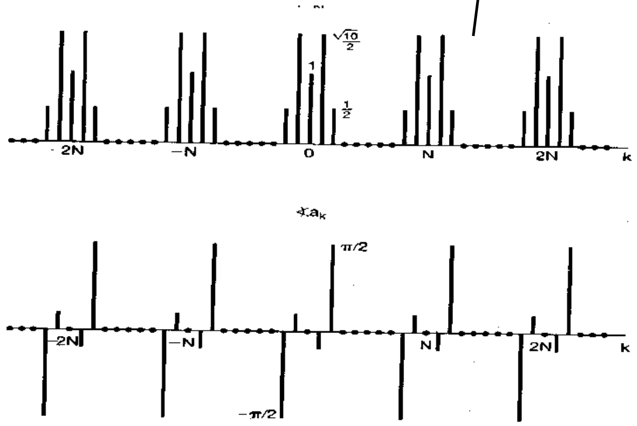
The FS coefficients are N-periodic as an unwrapped representation of a circular domain.

Circular Domain

The FS coefficients of a DT periodic signal are actually on a circular domain:



The periodic representation is simply a more convenient representation of the circular domain above.



(b)

Another Example

$k = \pm 2$ $\qquad$ $k = \pm 3$

List the F.S. coefficients of $x[n] = 3 + 5\cos\left(\frac{4\pi}{9}n + \frac{\pi}{8}\right) + \sin\left(\frac{6\pi}{9}n\right)$ $\qquad \omega_0 = \frac{2\pi}{9}$

$$a_0 = 3$$

$$|a_2| = \frac{5}{2} \qquad \sphericalangle a_2 = \frac{\pi}{8} \qquad |a_3| = \frac{1}{2} \qquad \sphericalangle a_3 = -\frac{\pi}{2}$$

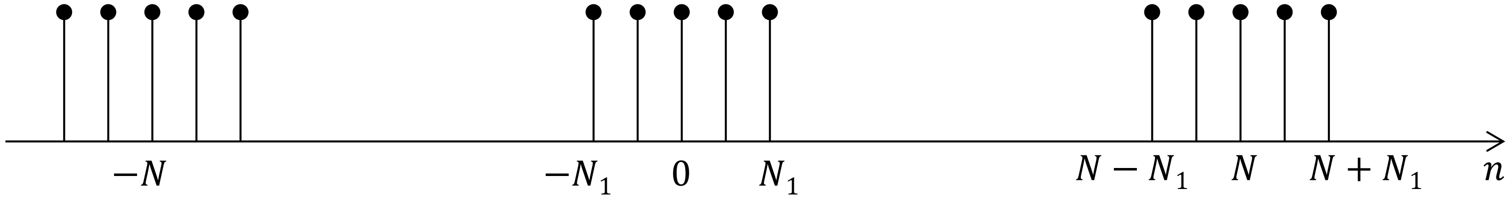
$$a_2 = \frac{5}{2}e^{j\frac{\pi}{8}} \qquad a_{-2} = \frac{5}{2}e^{-j\frac{\pi}{8}}$$

$$a_3 = \frac{1}{2}e^{-j\frac{\pi}{2}} = \frac{-j}{2} \qquad a_{-3} = \frac{1}{2}e^{j\frac{\pi}{2}} = \frac{j}{2}$$

Example 3.12 – DTFS for a DT rectangular wave

Sometimes finding the FS coefficients requires some algebraic manipulation:

Consider the DT rectangular wave: N-periodic



Summing over an interval that covers $-N_1$ to N_1:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$x[n] = 1$

Choose interval of summation $<N>$ that includes $-N_1$ to N_1

Letting $m = n + N_1$:

$n = N_1 \rightarrow m = 2N_1$

$n = m - N_1$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{e^{jk\left(\frac{2\pi}{N}\right)N_1}}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m} = \frac{e^{jk(2\pi/N)N_1}}{N}\left[\frac{1-e^{-jk(2\pi/N)(2N_1+1)}}{1-e^{-jk(2\pi/N)}}\right]$$

$n = -N_1 \rightarrow m = 0$

Finite geometric sum

$$\sum_{m=0}^{M} \alpha^m = \frac{1-\alpha^{M+1}}{1-\alpha}$$

- **Rectangular wave (cont.)**

From previous slide: $a_k = \frac{e^{jk\left(\frac{2\pi}{N}\right)N_1}}{N}\left[\frac{1-e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1-e^{-jk\left(\frac{2\pi}{N}\right)}}\right]$

An algebraic manipulation used in many optics and wave problems is to extract phase from two complex numbers of same magnitude to obtain a conjugate pair. With this, we can re-express a_k as

$$1 - e^{-j\theta} = e^{-\frac{j\theta}{2}}\left(e^{\frac{j\theta}{2}} - e^{-\frac{j\theta}{2}}\right)$$

Given a sum or difference of two complex numbers with same magnitude, we extract the average of their phases to create a conjugate pair

$$a_k = \frac{1}{N}\frac{e^{jk\left(\frac{2\pi}{N}\right)N_1}e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)}}{e^{-jk\left(\frac{2\pi}{2N}\right)}}\frac{\left[e^{jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} - e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)}\right]}{\left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)}\right]}$$

$$= \frac{1}{N}\frac{\sin\left(\left(\frac{\pi}{N}\right)(2N_1+1)k\right)}{\sin\left(\frac{\pi}{N}k\right)}$$

We note that this expression is N-periodic in k.

The above is the DT equivalence of the sinc function.

Note that when $k = 0, \pm N, \pm 2N \ldots\ldots$, both the denominator and numerator go to zero and we need to take limit. We easily obtain for these cases:

$$a_k\big|_{k=0,\pm N,\pm 2N,\ldots} = \frac{2N_1+1}{N}$$

Example 3.12 – (cont.)

We plot a_k for different N's while keeping $2N_1 + 1$ fixed at 5

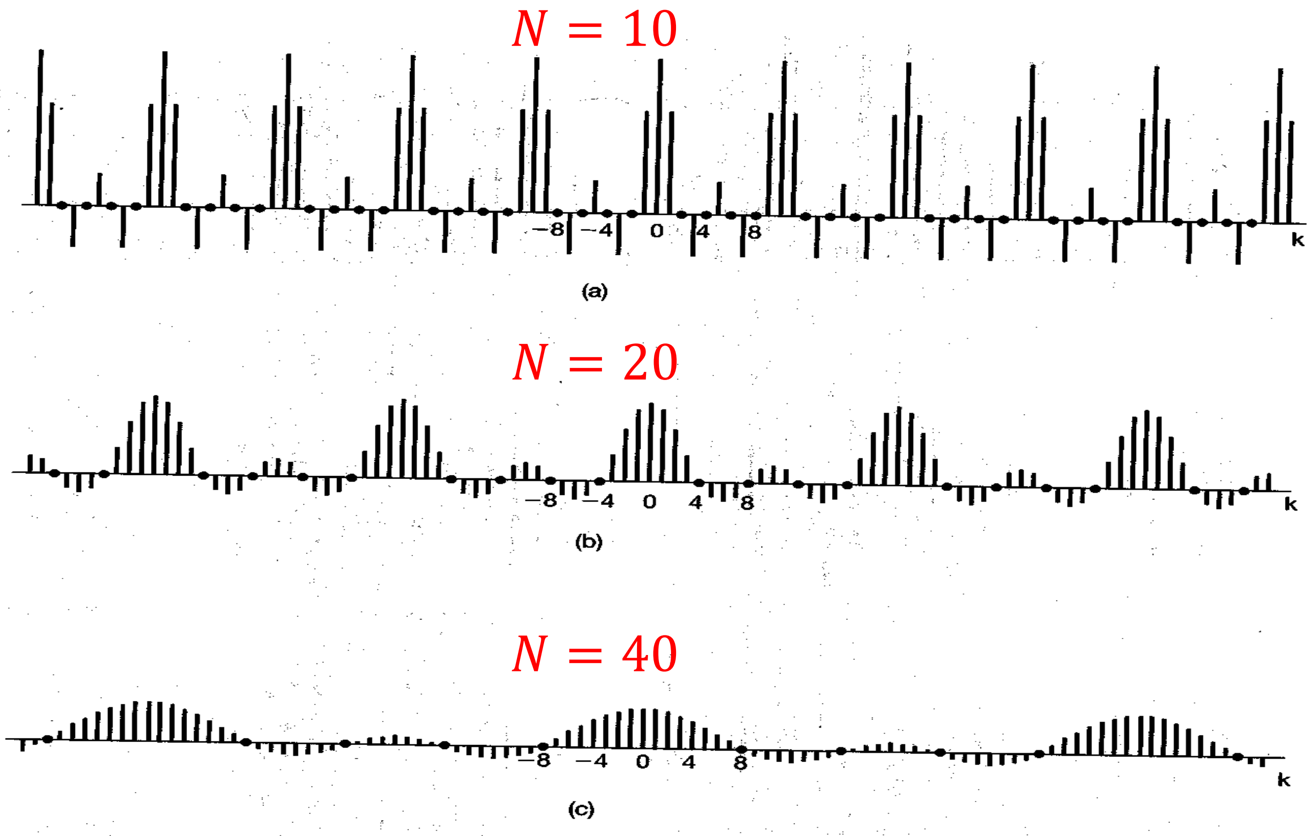
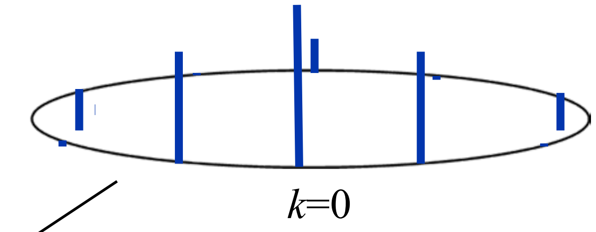
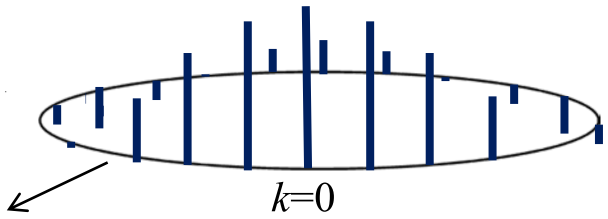


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.



N=10, hence 10 FS coefficients



N=20, hence 20 FS coefficients

Convergence for DTFS

- Unlike CTFS, there is no convergence issue in DTFS because DTFS is simply a change of coordinate (basis) in an N-vector space.
- For the **truncated synthesis equation** for the DT periodic rectangular wave, when N terms are included, $x[n]$ is exactly reproduced. There is no convergence issue.

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

include $2M + 1$ terms only
In the sum instead of $< N >$ terms

Rectangular wave is exactly reproduced when all N terms are included in the synthesis

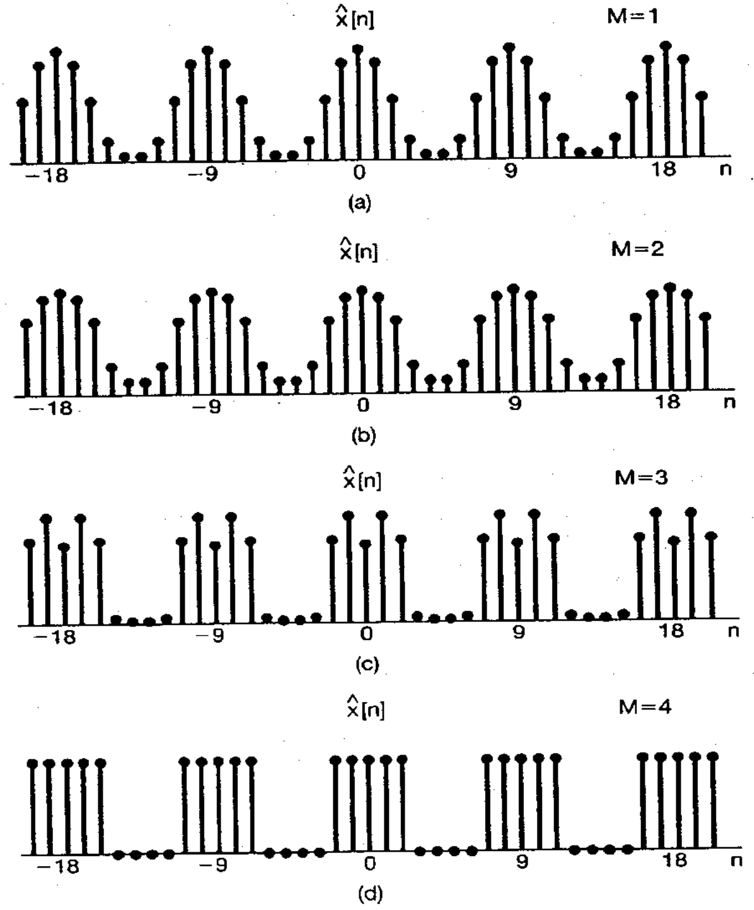


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

No Gibb's phenomena – there is no discontinuity in DT signals

For Reference Only

For Reference Only - Correspondence between CTFS and DTFS

Note also that there is a close correspondence between CT and DT Fourier Series:

CT

Periodicity | **Harmonics** | **Orthogonality of Harmonics**

$$x(t) = x(t+T)$$

T-periodic

$$\phi_k(t) = e^{jk\left(\frac{2\pi}{T}\right)t} \quad \omega_0 = \frac{2\pi}{T}$$

$$\int_0^T \phi_k(t)\phi_l^*(t)\,dt = \begin{cases} T & k = l \\ 0 & k \neq l \end{cases}$$

Inner product: $< \phi_k(t), \phi_l(t) >$

$\phi_k(t)\phi_k^*(t) = |\phi_k(t)|^2 = 1$

Self inner-product $= T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,\phi_k(t)$$

Synthesis — An infinite weighted sum

$$a_k = \frac{1}{T}\int_0^T x(t)\phi_k^*(t)\,dt = \frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$$

Analysis

DT

$$x[n] = x[n+N]$$

N-periodic

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n} \quad \omega_0 = \frac{2\pi}{N}$$

$$\sum_{n=0}^{N-1} \phi_k[n]\phi_l^*[n] = \begin{cases} N & k = l + mN \\ 0 & k \neq l + mN \end{cases}$$

l and $l + mN$ are the same harmonic

$$x[n] = \sum_{k=0}^{N-1} a_k\,\phi_k[n]$$

Synthesis — Finite sum; only N distinct harmonics

$$\underbrace{a_k}_{\text{Projection coefficient}} = \frac{1}{N} \underbrace{\sum_{n=<N>} x[n]\,\phi_k^*[n]}_{\text{Inner product: } < x[n], \phi_k[n] >} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Self-inner product: $< \phi_k[n], \phi_k[n] >$

Analysis

AKW

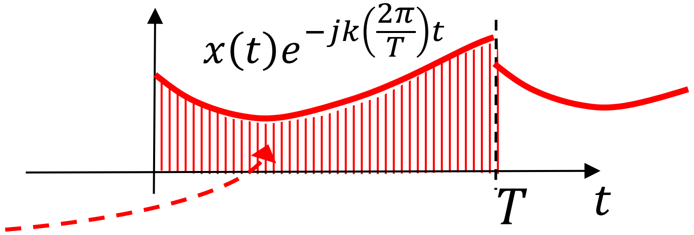
For Reference Only - DTFS as Approximation of CTFS

- We can also view the DTFS coefficients as a staircase approximation result of the CTFS analysis integral:

CT

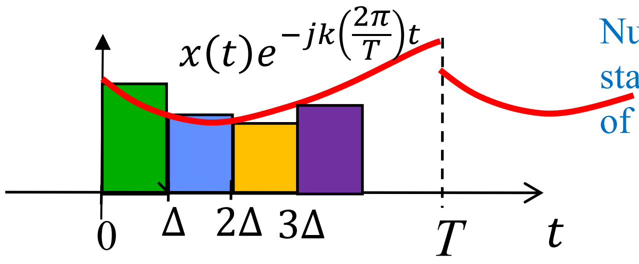
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}; \quad a_k = \underbrace{\frac{1}{T}\int_0^T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt}_{\text{CTFS analysis integral}}$$

T-periodic



Now if you ask a computer to compute the CTFS coefficients a_k, the computer can only perform a numerical integration by sampling the integrand and sum:

$$\frac{1}{T}\underbrace{\sum_{n=0}^{N-1} x(n\Delta)e^{-jk\left(\frac{2\pi}{T}\right)n\Delta}\Delta}_{\text{Numerical integration}}$$



Numerical integration = staircase approximation of integral

where we assume there are $N = \frac{T}{\Delta}$ samples. Hence $\frac{\Delta}{T} = \frac{1}{N}$ and the numerical integral is a DTFS analysis sum:

$$\underbrace{\tilde{a}_{k\ DTFS} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}}_{\text{DTFS Analysis sum}} \quad \text{where } x[n] = x(n\Delta);$$

Now in DT the computer can only produce N distinct coefficients, whereas $x(t)$ may originally have an infinite number of coefficients. Some information is potentially lost.