Ch3: Frequency Analysis for DT Signals & Systems

Information source and input transducer → Source Coding → Channel Coding → Modulator

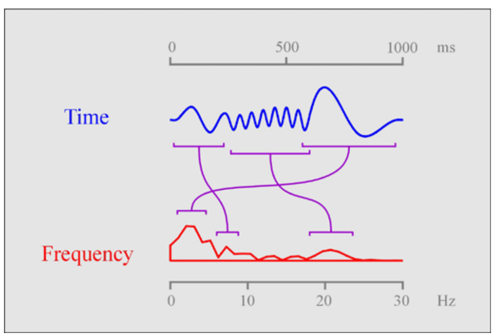
↓

Channel

↓

Demodulator (Matched Filter) → Channel Decoding → Source Decoding → Information sink and output transducer

Ch3.1: DTFT



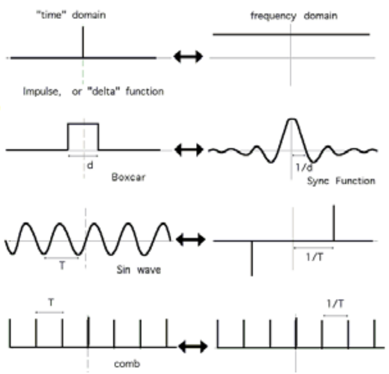
- **Questions to be answered:**
  - **Time Domain vs. Frequency Domain**
  - **Discrete-Time Fourier Transform:** Frequency analysis for DT signals.
  - **Band-Limited DT Signals:** How to determine the bandwidth of DT signals?
  - **Frequency Response:** Frequency analysis for DT systems.

Reading: Mitra (Ch3.1 – 3.3, 3.5, 4.8)

Ch3.1: DTFT for DT Signals & Systems

- **Discrete-Time Fourier Transform**
- Band-Limited DT Signals
- The Frequency Response

Frequency Analysis for CT Signals



- Frequency analysis for continuous-time periodic signals: **Fourier Series**

$$x(t) = \sum_{k=-\infty}^{\infty} \boxed{X_k} e^{2\pi jkt/T_0}, \qquad X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-2\pi jkt/T_0}dt.$$

- Frequency analysis for continuous-time aperiodic signals: **Fourier Transform**

Complex Amplitude

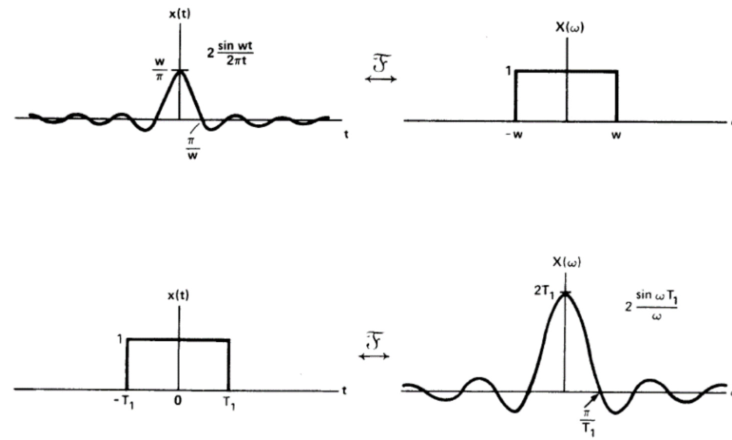
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{X(j\omega)} e^{j\omega t}d\omega$$

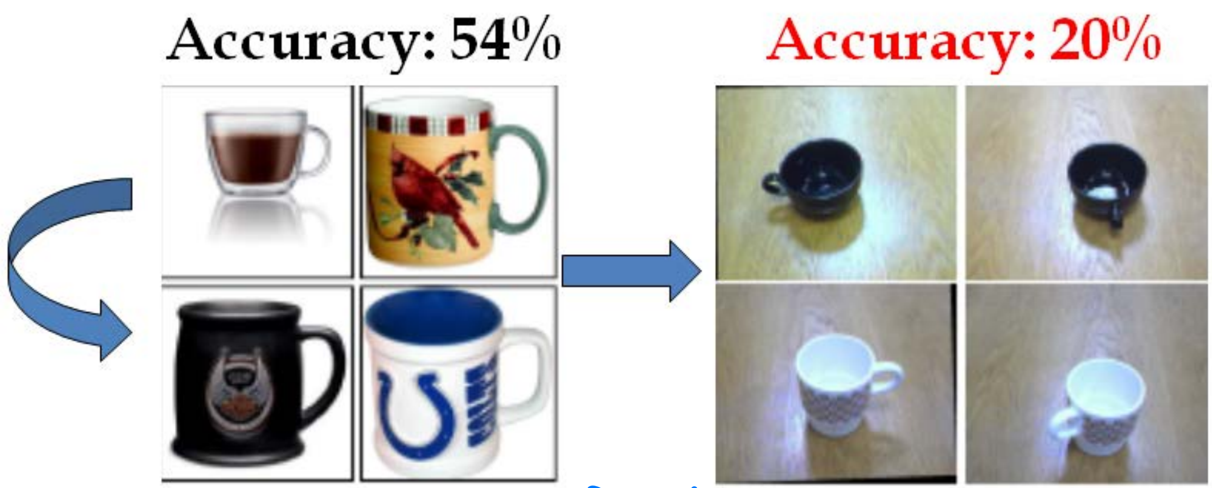
- What happens if we have discrete-time signals?

Discrete in time domain → Continuous in freq.
→ Discrete in frequency domain

How to do it in discrete-time?

- What is time-domain?
- What is frequency-domain?
- Are we talking about the same thing?
- What is the relationship between CTFT, CTFS, and DTFT?

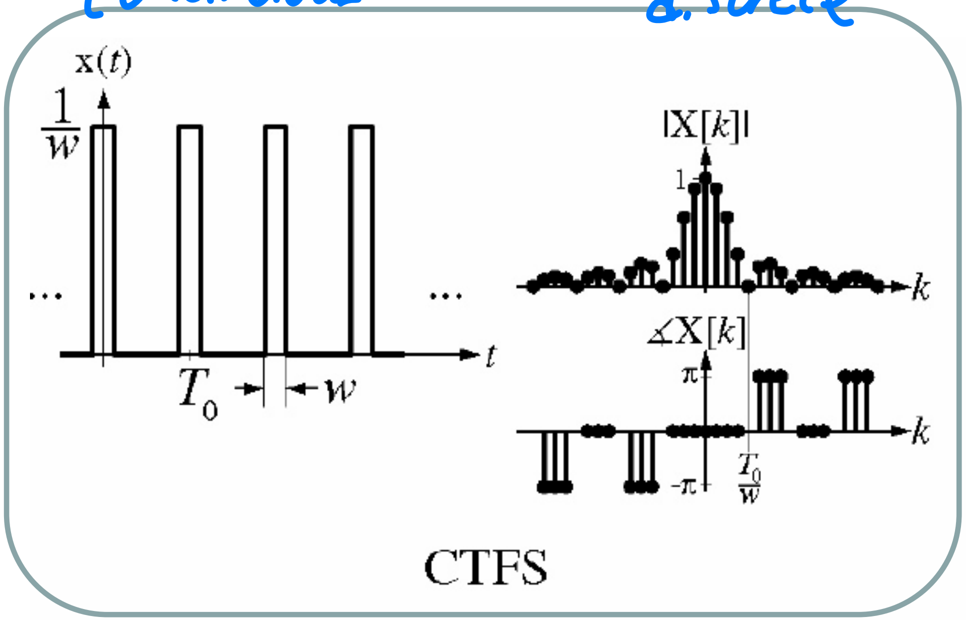




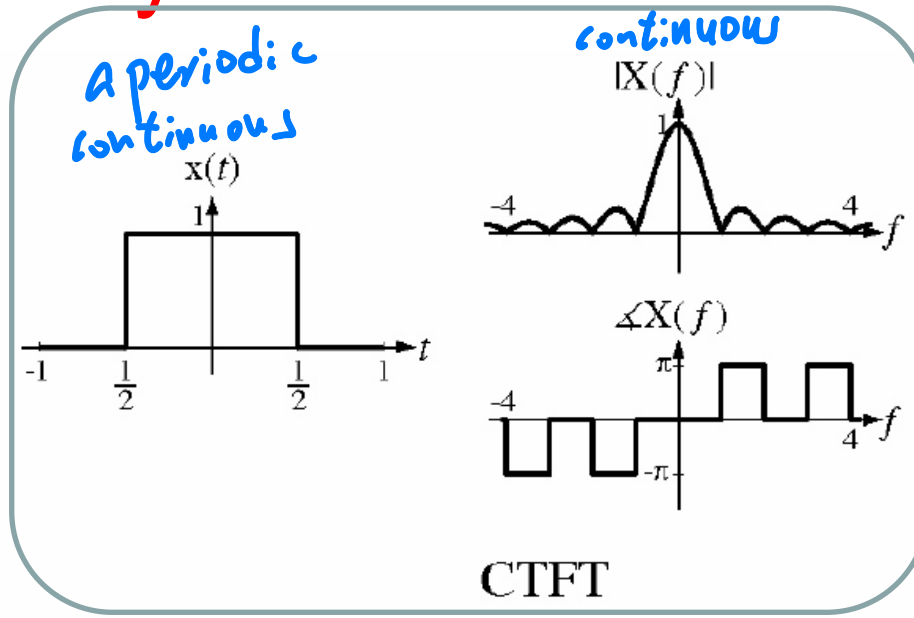
process the transform signal

Fourier Analysis

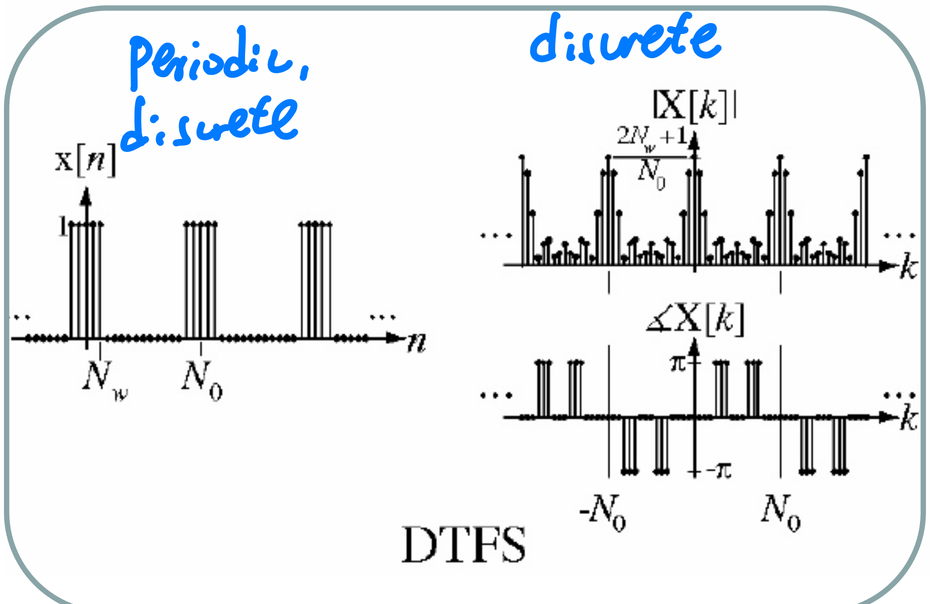
Periodic
continuous



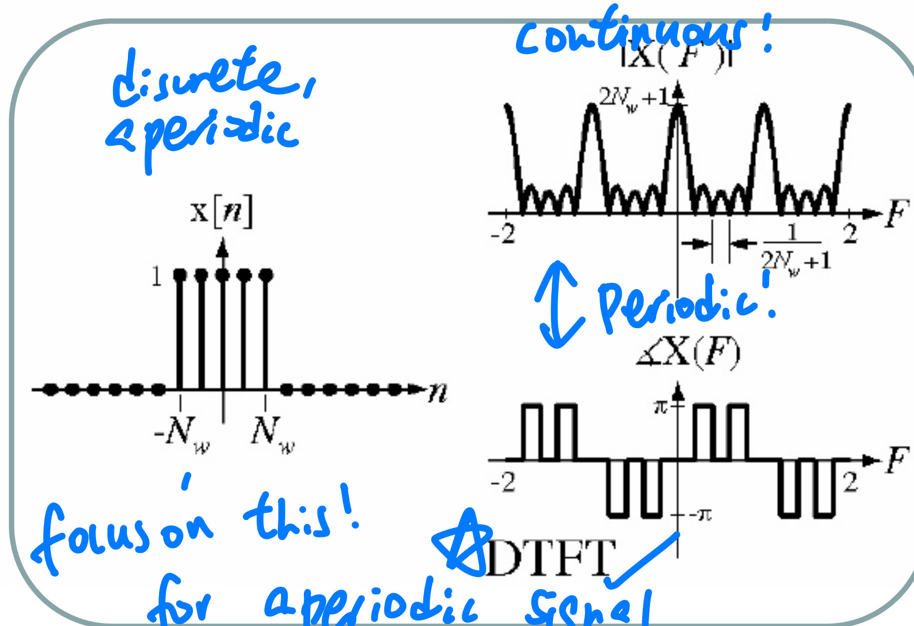
CTFS



CTFT



DTFS



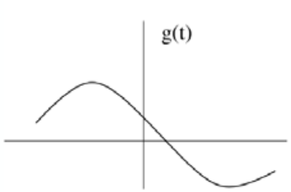
DTFT

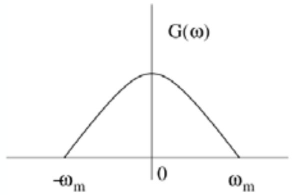
Frequency Domain Representation of Signals Why?

- why we are interested to represent signals in frequency domain?
- Application 1 - Efficient Storage of Signals
  - Most physical signals have long timespan in time domain. (e.g. waveform of a song may last for 6 minutes)
  - However, most of the physical signals are 'band-limited" ~ only have finite frequency span in frequency domain (e.g. bandwidth of music ~ 20kHz)
  - One needs to store significant frequency components —> compression of signals

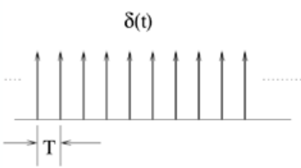
- Application 2: Signal Processing
  - e.g. Nyquist Sampling of signal g(t)

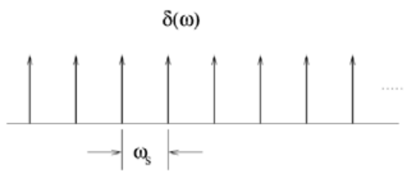
Let $g(t)$ be a bandlimited signal whose bandwidth is f_m





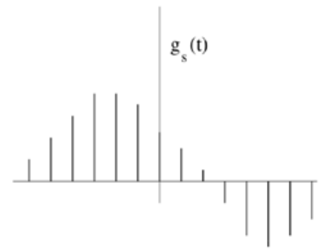
$\delta_T(t)$ is the sampling signal with $f_s = 1/T > 2f_m$.

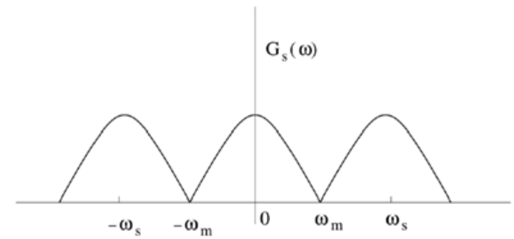




sampled signal $g_s(t) = g(t)\delta_T(t)$

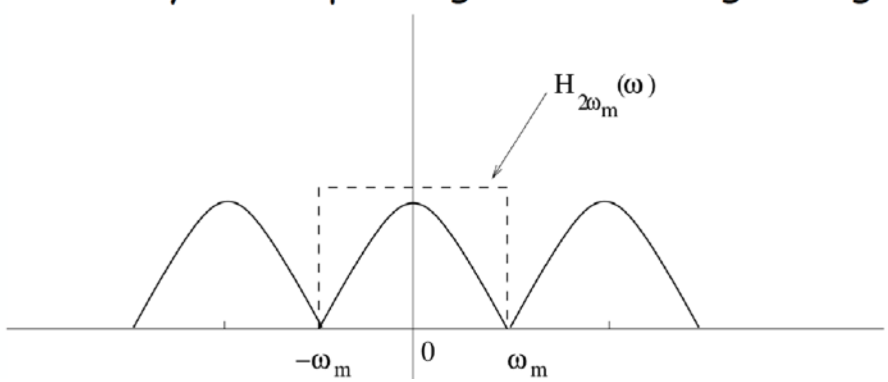
$$\begin{aligned}\mathcal{F}(g_s(t)) &= \mathcal{F}\left[g(t)\delta_T(t)\right] \\ &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega - n\omega_0)\end{aligned}$$





No overlapping (aliasing) of G(f) if $f_s > f_m$

Recovery of sampled signal to the original signal g(t)

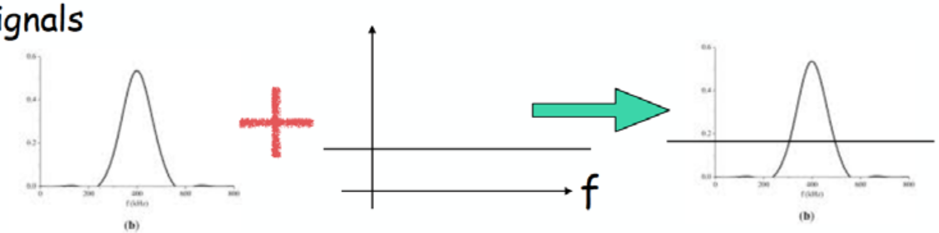


Application 3: Telecommunication (noise filtering)

- Signals are usually contaminated with noise (thermal noise).

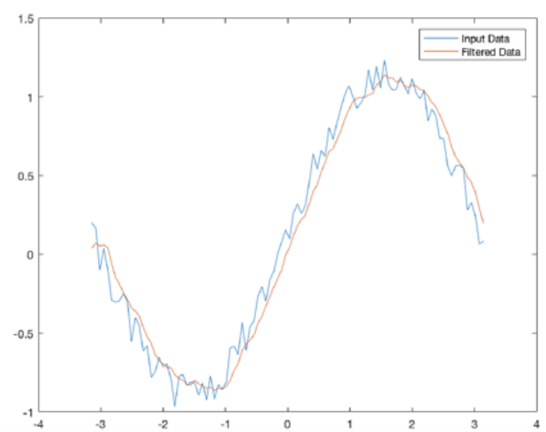
maybe different frequency → must do filtering in frequency domain

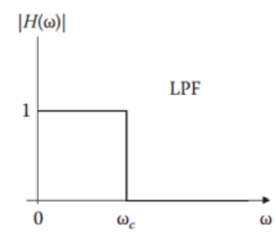
- In time domain, noise added into signals cannot be separated
- Yet, in frequency domain, signal and noise occupies different frequency spectrum ==> low pass filtering to "clean up" noisy signals



A **filter** is a device designed to pass signals with desired frequencies and block or attenuate others.

1. A *lowpass filter* (LPF) passes low frequencies and rejects high frequencies.
2. A *highpass filter* (HPF) passes high frequencies and rejects low frequencies.
3. A *bandpass filter* (BPF) passes frequencies within a frequency band and blocks or attenuates frequencies outside the band.
4. A *bandstop filter* (BSF) passes frequencies outside the frequency band and stops or attenuates frequencies within the band.





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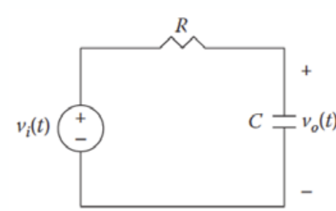

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DTFT : $x[n] \longrightarrow X(e^{j\omega})$ ← want this also discrete!

Discrete-Time Fourier Transform

- The **discrete-time Fourier transform (DTFT)** $X(e^{j\omega})$ of a sequence $x[n]$ is given by

Just like Fourier Series

$$\underbrace{X\left(e^{j\omega}\right)}_{\text{continuous}} = \sum_{n=-\infty}^{\infty} \underline{x[n]}e^{-j\omega n}$$

just sum up!

- $X(e^{j\omega})$ can be expressed as $X\left(e^{j\omega}\right) = |X\left(e^{j\omega}\right)|e^{-j\theta(\omega)}$ where $|X\left(e^{j\omega}\right)|$ is called the **magnitude function** (spectra) and $\theta(\omega) = \arg\{X(e^{j\omega})\}$ is called the **phase function** (spectra).

Graph → must show magnitude and phase

Periodic!

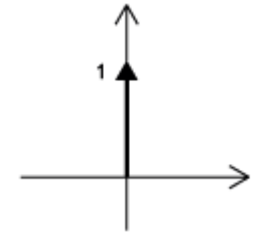
DTFT Example

- **Example:** The DTFT of the unit sample sequence $\delta[n]$ is given by

only one term is non zero!

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1$$

$e^{-jw\cdot 0}$



- **Example:** Consider the causal sequence $x[n] = a^n\mu[n]$, $|a| < 1$. Its DTFT is given by

不可depends on 未来!

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \overbrace{a^n\mu[n]}^{x[n]}e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

since $\left|ae^{-j\omega}\right| < 1$.

$|\alpha|$

$= |a|\,|e^{-jw}|$

$\downarrow = 1$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

Discrete Time Fourier Transform

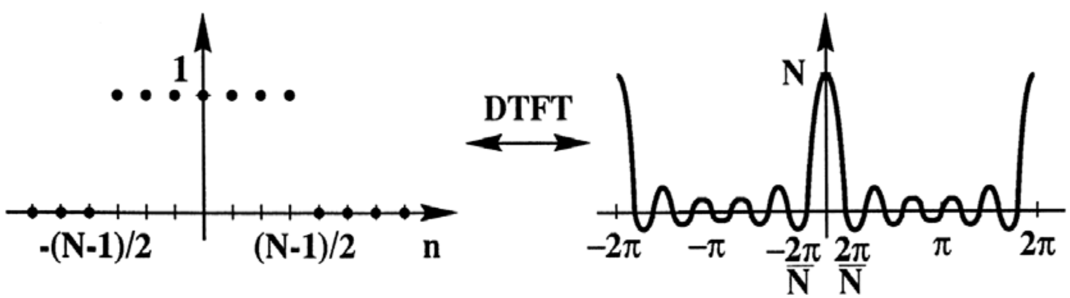
- In general, $X(e^{j\omega})$ is a complex function of ω.
- $X(e^{j\omega})$ is also a periodic function of ω with a period 2π.
- **Proof :**

Same!

$$X\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\underbrace{e^{-j2\pi kn}}_{=1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X\left(e^{j\omega}\right)$$

- Where does the periodic property come from?



$X(e^{j\omega}) \to x[n]$

Inverse Discrete-Time Fourier Transform

- Since $X(e^{j\omega})$ is a periodic function ω with a period 2π,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Discrete

Continuous + Periodic → take Fourier series

represents the <u>**Fourier series**</u> representation of a periodic function.

- As a result, the **Fourier coefficients** $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

$$\underline{x[n]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

original signal

- The proof is similar to that for CT Fourier series.

Convergence Condition

- An infinite series of the form $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ may or may not converge.

- Let

change ∞ to K

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Then, for **convergence** of $X(e^{j\omega})$, we need

$$\lim_{K\to\infty} |X_K(e^{j\omega})| < \infty$$

- Can we get a **sufficient condition** for convergence? Bounded!

- Consider the absolutely summable sequence $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$.

Absolutely summable $\rightarrow$ sufficient condition

Convergence Condition

- For an absolutely summable sequence $\underline{\sum_{n=-\infty}^{\infty}|x[n]| < \infty}$, we can obtain $\Downarrow$

$$\lim_{K\to\infty}\left|\underline{X_K}\left(e^{j\omega}\right)\right| = \lim_{K\to\infty}\underline{\left|\sum_{n=-K}^{K} x[n]e^{-j\omega n}\right|} \leq \boxed{\sum_{n=-\infty}^{\infty}|x[n]|} < \infty$$

for all values of ω.

$|a+b| \leq |a| + |b|$

$\Rightarrow$ finite energy! ✓

- Thus, the **absolute summability** is a sufficient condition for the existence of DTFT. $\Rightarrow X(e^{j\omega})$ finite

DTFT exists

- Since $\sum_{n=-\infty}^{\infty}|x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty}|x[n]|\right)^2$, an absolutely summable sequence has a finite energy.

absolute summable $\not\Leftarrow \Rightarrow$ DTFT exists

$\not\Uparrow \Rightarrow$ finite energy

Convergence Condition

- However, a finite-energy sequence is not necessarily absolutely summable. Converse is not correct!

- Example - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

$\sum |\frac{1}{n}| = \infty$

- But, $x[n]$ is not absolutely summable

Commonly Used DTFT Pairs

$$\delta[n] \stackrel{DTFT}{\Longleftrightarrow} 1$$

$$\delta[n-n_0] \stackrel{DTFT}{\Longleftrightarrow} e^{-j\omega n_0}$$

$$1 \stackrel{DTFT}{\Longleftrightarrow} 2\pi\,\tilde{\delta}(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega + 2\pi k)$$

$$e^{j\omega_0 n} \stackrel{DTFT}{\Longleftrightarrow} 2\pi\,\tilde{\delta}(\omega-\omega_0) = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_0 + 2\pi k)$$

$$u[n] \stackrel{DTFT}{\Longleftrightarrow} \frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)$$

$$a^n u[n] \quad (|a|<1) \stackrel{DTFT}{\Longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

$$(n+1)a^n u[n] \stackrel{DTFT}{\Longleftrightarrow} \frac{1}{(1-ae^{-j\omega})^2}$$

$$\sin(\omega_0 n + \phi) \stackrel{DTFT}{\Longleftrightarrow} \frac{j}{2}\left[e^{-j\phi}\tilde{\delta}(\omega+\omega_0+2\pi k) - e^{+j\phi}\tilde{\delta}(\omega-\omega_0+2\pi k)\right]$$

$$\cos(\omega_0 n + \phi) \stackrel{DTFT}{\Longleftrightarrow} \frac{1}{2}\left[e^{-j\phi}\tilde{\delta}(\omega+\omega_0+2\pi k) + e^{+j\phi}\tilde{\delta}(\omega-\omega_0+2\pi k)\right]$$

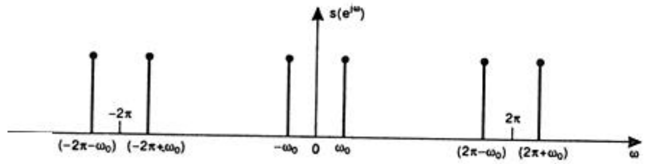
$$\frac{\sin(\omega_c n)}{n} = \omega_c\,\mathrm{sinc}(\omega_c n) \stackrel{DTFT}{\Longleftrightarrow} \tilde{\mathrm{rect}}\left(\frac{\omega}{\omega_c}\right) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

$$\text{Window}: \quad \mathrm{rect}\left(\frac{n}{M}\right) = \begin{cases} 1 & |n| \leqslant M \\ 0 & \text{otherwise} \end{cases} \stackrel{DTFT}{\Longleftrightarrow} \frac{\sin[\omega(M+\frac{1}{2})]}{\sin(\omega/2)}$$

DTFT Example

- **Example:** Determine the DTFT of the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$



- **Solution:** Its DTFT is given by

$$\underline{X(e^{j\omega})} = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

Periodic!

where $\delta(\omega)$ is an impulse function of ω and $-\pi \leq \omega_o \leq \pi$. The function is called a periodic impulse train.

To verify that $X(e^{j\omega})$ is indeed the DTFT of $x[n]$, we compute the inverse DTFT with

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)}_{X(e^{j\omega})} e^{j\omega n} d\omega$$

only consider one term!

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_o)\, e^{j\omega n} d\omega = e^{j\omega_o n}$$

$$= \int \delta(\omega-\omega_0) f(\omega) d\omega$$
$$= f(\omega_0)$$

DTFT Properties

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

DTFT Properties

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

DTFT Properties: Example

- **Example:** Determine the DTFT of the complex exponential sequence

$$z[n] = x[n]y[n] = x[n]e^{j\omega_o n}$$

- **Solution:** Let $X(e^{j\omega})$ denote the DTFT of $x[n]$. The DTFT of $y[n] = e^{j\omega_o n}$ is given by $Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$.

It thus follows from the **modulation theorem**,

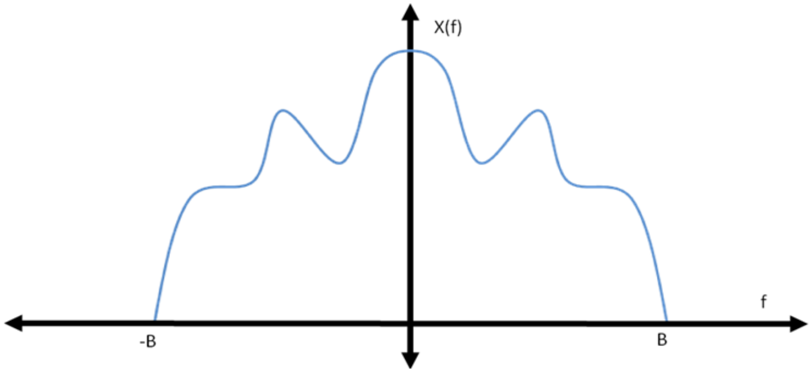
$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \theta - \omega_o + 2\pi k)\,d\theta$$

$$= \int_{-\pi}^{\pi} X(e^{j\theta})\,\delta(\omega - \theta - \omega_o)d\theta = X\left(e^{j(\omega-\omega_o)}\right)$$

Ch3.1: DTFT for DT Signals & Systems

- Discrete-Time Fourier Transform
- **Band-Limited DT Signals** Just look at frequency domain
- The Frequency Response



Band-limited Discrete-time Signals

- Since $X(e^{j\omega})$ is a periodic function ω with a period 2π, a full-band signal has a spectrum occupying the frequency range $-\pi \leq \omega \leq \pi$.
- A band-limited discrete-time signal has a spectrum that is limited to a portion of the range $-\pi \leq \omega \leq \pi$.
- An ideal band-limited signal has a spectrum that is zero outside a frequency range $0 < \omega_a \leq |\omega| \leq \omega_b < \pi$

$$X(e^{j\omega}) = \begin{cases} 0, 0 \leq |\omega| \leq \omega_a \\ 0, \omega_b \leq |\omega| \leq \pi \end{cases}$$

- **Question:** Can we generate an ideal band-limited discrete-time signal in practice? **Why?**

No! Only define some approximate bandwidth

Bandwidth of Discrete-time Signals

- A **lowpass** discrete-time real signal has a spectrum occupying the frequency range $0 < |\omega| \leq \omega_p < \pi$ and has a bandwidth of ω_p.

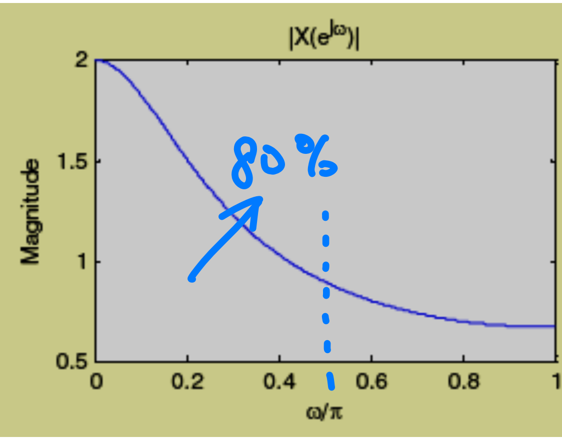
- A **highpass** discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_p \leq |\omega| < \pi$ and has a bandwidth of $\pi - \omega_p$.

- A **bandpass** discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_L \leq |\omega| \leq \omega_H < \pi$ and has a bandwidth $\omega_H - \omega_L$.

Band-limited Discrete-time Signals

- **Example:** Consider the sequence $x[n] = (0.5)^n \mu[n]$.
- Its DTFT and magnitude spectrum are shown below.

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



- It can be shown that 80% of the energy of this lowpass signal is contained in the frequency range $0 \le |\omega| \le 0.5081\pi$. Thus, we can define the 80% bandwidth to be 0.5081π radians.

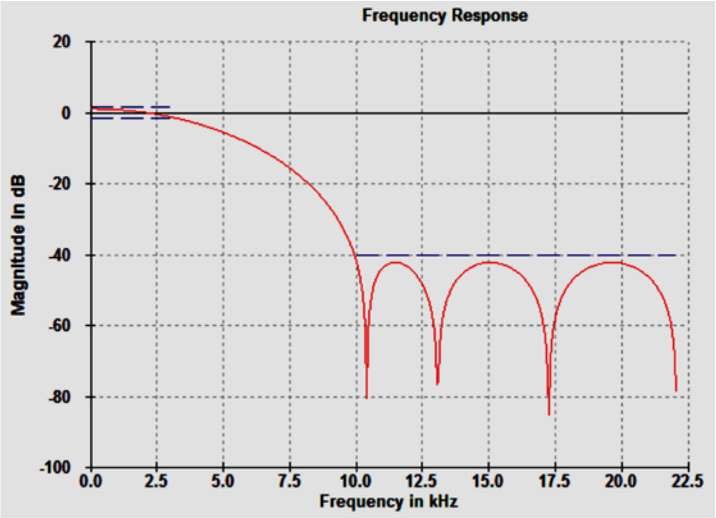
Ch3.1: DTFT for DT Signals & Systems

- Discrete-Time Fourier Transform

- Band-Limited DT Signals

Want discrete frequency

- **The Frequency Response**



The Frequency Response

- Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies. interest in specific freq.
- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition Property.
- An important property of an LTI system is that for certain types of input signals, called **eigen functions**, the output signal is the input signal multiplied by a complex constant.

Same waveform!

$$g(t) \xrightarrow{LTI} \alpha \cdot g(t)$$

$$f(t) = \sum g_i(t) \longrightarrow \sum \alpha_i g_i(t) \quad \text{easy to analyze!}$$

Eigen Response

- Consider a LTI system with impulse response $h[n]$. The output of this system to an input $x[n]$ is given by the convolution sum with $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$.
- For the input $x[n] = e^{j\omega n}$, we can obtain the output as

def.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\underbrace{e^{j\omega(n-k)}}_{x[n-k]}$$

DTFT of impulse response, $H(e^{j\omega})$

indep. on k!

$$= \sum_{k=-\infty}^{\infty} h[k]\boxed{e^{-j\omega k}}\boxed{e^{j\omega n}} = \boxed{H(e^{j\omega})}e^{j\omega n}$$

contains k!

- The relation $y[n] = H(e^{j\omega})e^{j\omega n}$ implies that for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$.

just DTFT of impulse!

The Frequency Response

- Thus, $e^{j\omega n}$ is an eigen-function of the system.

  → know the system

- $H(e^{j\omega})$ is called the frequency response of the LTI DT systems, which provides a frequency-domain description of the system.

- $H(e^{j\omega})$ is precisely the DTFT of the impulse response $h[n]$ of the LTI system.

  $|H(e^{j\omega})|$ $\qquad$ $\angle H(e^{j\omega})$

- What is the **magnitude response** and **phase response**?

- Can you explain LTI in the frequency domain?