Lecture 8

Fourier series representation of periodic signals

(Analysis)

(Ref: Chapter 3 O&W)

I. Fourier Series: Decomposition of Continuous-Time Periodic Signals into Complex Sinusoids

II. Fourier Series Examples

III. The Convergence Question

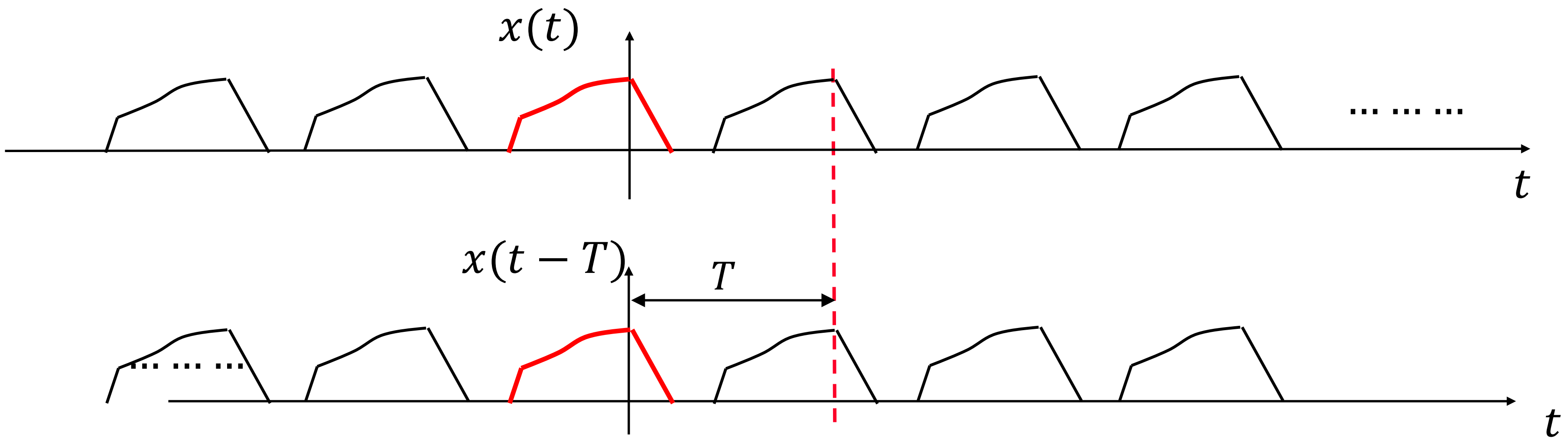
IV. Conjugate Symmetry and FS as Real Sinusoids

I. Fourier Series decomposition of CT Periodic Signals

- We begin with ***periodic*** CT signals.
- CT signal $x(t)$ is T-periodic if $x(t) = x(t - T)\ \forall t$.
- Reciprocal of T gives the ***fundamental frequency***:

$$\omega_0 = \frac{2\pi}{T} \text{ or } f_0 = \frac{1}{T}.$$

- Below is an example of a T-periodic CT signal.



Continuous-Time Fourier Series (CTFS)

- Joseph Fourier (1768-1830), a French mathematician, proposed that "*any*" *T-periodic CT signal can be decomposed into, or synthesized as, a superposition of a discrete set of T-periodic complex sinusoids*:

$$\underbrace{x(t)}_{T\text{-periodic CT signal}} = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\frac{2\pi}{T}t}}_{\substack{\text{Complex sinusoid} \\ \text{at frequency} = k\omega_0}} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

The Synthesis Equation

- The complex sinusoid at frequency $k\omega_0$ is called the k-th ***harmonic*** of the signal. The k-th harmonic is hence $\frac{T}{k}$-periodic. Thus all harmonics are T-periodic and so is their weighted sum.
- The weights a_k's are called the ***Fourier series (FS) coefficients***.

The DC Term

- The 0-th harmonic is the unity constant: $\boxed{e^{j0\frac{2\pi}{T}t} = 1}$
- The 0-th harmonic is also called the ***DC*** term (***DC* = *Direct Current***, an EE jargon for the average value of a signal), because all other harmonics have a zero average value:

all $e^{jk\frac{2\pi}{T}t}$ with $k \neq 0$ integrate to zero over one period of duration T:

$$e^{jk\frac{2\pi}{T}t} = \cos\left(k\frac{2\pi}{T}t\right) + j\sin\left(k\frac{2\pi}{T}t\right)$$

Both are T-periodic sinusoids

Therefore, $$\boxed{\int_0^T e^{jk\frac{2\pi}{T}t}\ dt = \begin{cases} 0 & k \neq 0 \\ T & k = 0 \end{cases}}$$

- All harmonics have a constant power of 1: $\boxed{\left|e^{jk\frac{2\pi}{T}t}\right|^2 = 1}$

How to compute the Fourier Coefficients?

- The FS coefficients a_k are determined by the ***analysis equation*** below:

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$$

The Analysis Equation

Means integrating over any contiguous interval of width T; i.e., over one period.

Since both terms in the integrand are T-periodic, integrating over any contiguous interval T produces the same result.

- We recognize that the analysis equation is a normalized ***inner product*** that computes a ***projection coefficient*** that specifies how much of $e^{jk\omega_0 t}$ there is in $x(t)$.

Recall from last lecture that the inner product for two signals integral/sum of conjugate product:

$$\langle x(t), g(t)\rangle = \int x(t) g^*(t)\, dt$$

Conjugating a complex sinusoid = negating its frequency

$$\langle x(t), e^{jk\omega_0 t}\rangle = \int_0^T x(t)(e^{jk\omega_0 t})^*\, dt = \int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

And projection coefficient of $x(t)$ on $g(t)$ is:

$$\frac{\int x(t) g^*(t) dt}{\int |g(t)|^2 dt}$$

normalization

AKW

Orthogonality of Harmonics

- To prove the analysis equation, we observe that the inner product of the m-th and k-th harmonics is zero unless $m = k$ because:

$$\left\langle e^{jm\omega_0 t}, e^{jk\omega_0 t} \right\rangle = \int_0^T e^{jm\omega_0 t}\left(e^{jk\omega_0 t}\right)^* dt = \int_0^T e^{j(m-k)\omega_0 t} dt = \begin{cases} 0 & m \neq k \\ T & m = k \end{cases}$$

Conjugate multiplying the m-th and k-th harmonics creates the $(m-k)$-th harmonic, which integrates to 0 over a contiguous interval of T unless $m-k$=0

- This is known as the ***orthogonality of the harmonics***: the inner product of two different harmonics is zero and each harmonic's self inner-product is T.

$$\langle \phi_m(t), \phi_k(t) \rangle = 0, \qquad m \neq k$$

$$\langle \phi_k(t), \phi_k(t) \rangle = T$$

self inner-product,
energy over 1 period

- Hence, if $x(t)$ is a weighted sum of harmonics, its inner product with the k-th harmonic will tell us how much of the k-th harmonic is in $x(t)$:

$$\overbrace{\int_0^T x(t)\, e^{-jk\omega_0 t} dt}^{\text{Inner product}} = \int_0^T \left(\sum_{m=-\infty}^{\infty} a_m\, e^{jm\omega_0 t} \right) e^{-jk\omega_0 t} dt = \overbrace{\sum_{m=-\infty}^{\infty} a_m \int_0^T e^{j(m-k)\omega_0 t}\, dt}^{\text{Sum of individual inner products}} = a_k T$$

Replace $x(t)$ by the synthesis sum, using m as variable for summation

Only the $m = k$ term integrates to T.
All other terms integrate to 0

Observations:

- Inner product is linear. Inner product with a sum equals sum of inner products.
- In the sum of inner products, all the terms are zero except the $m = k$ term.

Fundamental Principle: To find how much of some basic component ϕ there is in x, do an inner product of x with ϕ !

II. Fourier Series Examples

In some cases the FS coefficients can be obtained by inspection without using the analysis equation.

- **Example** What are the FS coefficients for $x(t) = \cos\omega_0 t$?

From Euler's relation, $\cos\omega_0 t = \mathrm{Re}\{e^{j\omega_0 t}\} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

This means in the FS sum $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$ there are only two non-zero terms:

$a_1 = a_{-1} = \frac{1}{2}$; and $a_k = 0$ for $k \neq \pm 1$

II. Fourier Series Examples

Example 3.3 What are the FS coefficients for $x(t) = \sin\omega_0 t$?

From Euler's relation, $\sin\omega_0 t = \mathrm{Im}\{e^{j\omega_0 t}\} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

This means in the FS sum $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$ there are only two non-zero terms:

$a_1 = \frac{1}{2j}$; $a_{-1} = -\frac{1}{2j}$; and $a_k = 0$ for $k \neq \pm 1$

$\frac{1}{j} = -j$;

$a_1 = \frac{-j}{2}$; $|a_1| = \frac{1}{2}$; $\sphericalangle a_1 = -\frac{\pi}{2}$

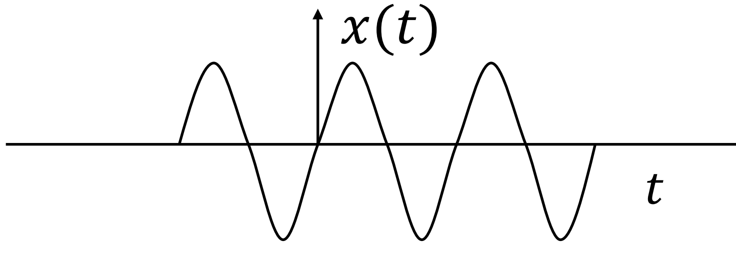
$a_{-1} = \frac{j}{2}$; $|a_{-1}| = \frac{1}{2}$; $\sphericalangle a_{-1} = \frac{\pi}{2}$

Or in polar form:

$a_1 = \frac{1}{2} e^{-j\pi/2}$; $a_{-1} = \frac{1}{2} e^{j\pi/2}$;

Example 3.3 (cont.)

- The FS coefficients tell that $x(t)$ is simply the sum of 2 complex sinusoids with the given magnitudes and phases. The FS coefficients provide a ***frequency domain representation*** of the signal.

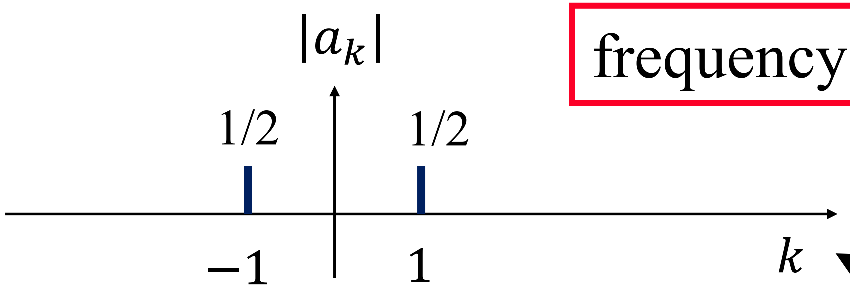


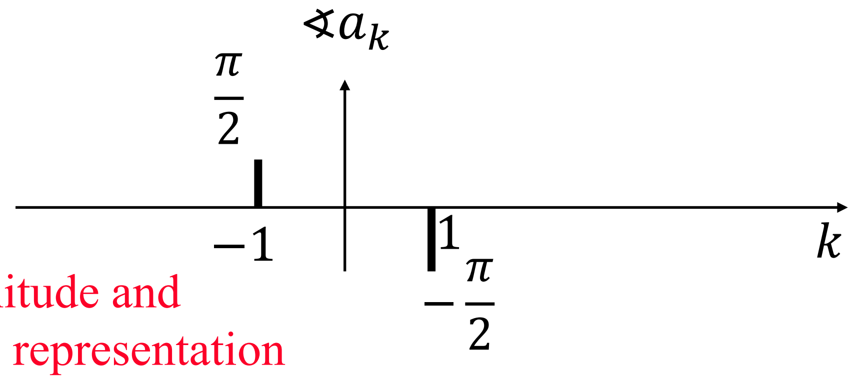
time domain

$$x(t) = \sin\omega_0 t$$

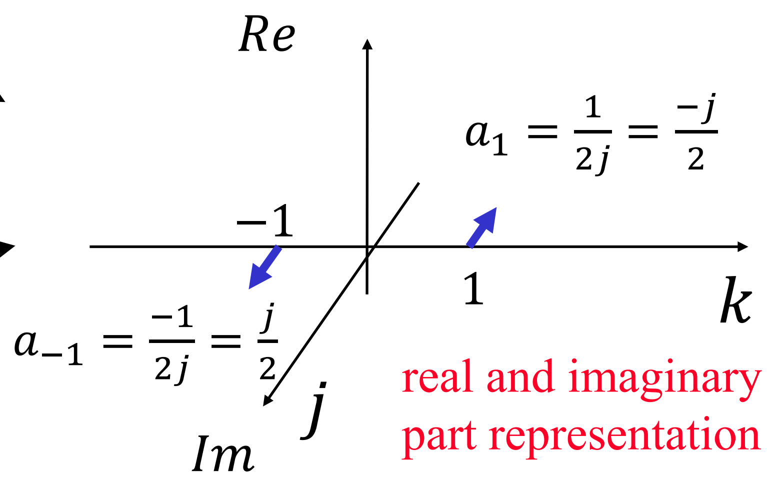
frequency domain

$$x(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$





Magnitude and phase representation



real and imaginary part representation

Exercise 1

$$x_1(t) = \overset{a_0}{1} + \overset{\omega_0 = 2}{\sin(2t)} + 2.5\overset{3\omega_0}{\cos(6t)}$$

- What is the fundamental frequency? $\omega_0 = 2;$ $f_0 = \frac{\omega_0}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$
- What is the period T? $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$
- What is its DC? $a_0 = 1$
- What are its Fourier coefficients

$$a_0 = 1;$$

Example 3.3: $\sin(2t) = \frac{-j}{2}e^{j2t} + \frac{j}{2}e^{-j2t} \Rightarrow$ $a_1 = \frac{-j}{2};$ $a_{-1} = \frac{j}{2};$

$\cos(6t) = \frac{1}{2}e^{j6t} + \frac{1}{2}e^{-j6t} \Rightarrow$ $a_3 = \frac{2.5}{2};$ $a_{-3} = \frac{2.5}{2};$ $a_k = 0$ all other k's

Exercise 2

$$x_2(t) = \sin(\overset{2\omega_0}{2t}) - 2.5\cos(\overset{5\omega_0}{5t})$$

- What is the fundamental frequency? $\omega_0 = 1$ which is the common divisor of 2 and 5
- What is the period T? $T = \dfrac{2\pi}{\omega_0} = 2\pi$
- What is its DC? $= 0$ because there is no constant term
- What are its Fourier coefficients?

$a_2 = \frac{-j}{2}$; $\quad a_{-2} = \frac{j}{2}$;

$a_5 = \frac{-2.5}{2}$; $\quad a_{-5} = \frac{-2.5}{2}$; $\quad a_k = 0$ all other k's

FS Examples

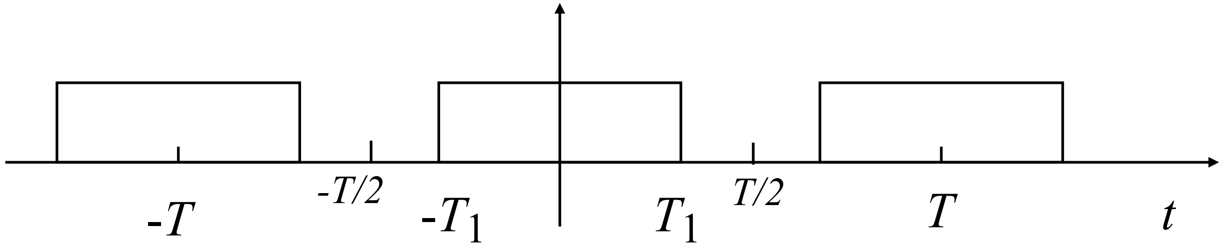
In other cases we apply the analysis integral to find the FS coefficients.

- **Example 3.5** Periodic square/rectangular wave, defined as:

$x(t)$ is T-periodic

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

One of the ways to specify a periodic signal:
1. Say that it is T-periodic
2. Specify what it is within one period



- Period = T ; i.e., $x(t-T) = x(t)$
- Fundamental frequency = $\omega_0 = 2\pi/T$

- To find the FS coefficients, we apply the integral: $a_k = \frac{1}{T}\int_T x(t)\, \underbrace{e^{-jk\omega_0 t}}_{\phi_k^*(t)} dt$

For $k = 0$, $a_0 = \frac{1}{T}\int_T x(t) \underbrace{dt}_{\phi_0(t) = 1} = \frac{1}{T}\int_{-T_1}^{T_1} x(t)\, dt = \frac{2T_1}{T}$ which is the average, or *DC value* of the signal

Since $x(t) = 1$ only in the interval $(-T_1, T_1)$

For $k \neq 0$, $a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \frac{1}{T}\frac{1}{(-jk\omega_0)} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{2}{(k\omega_0 T)}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$

half difference/j = imaginary part $= \sin(k\omega_0 T_1)$

$$= \frac{2}{(k\omega_0 T)}\sin(k\omega_0 T_1) = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$\omega_0 = \frac{2\pi}{T}$

$\boxed{\textbf{Eq (3.44)}}$

- a_k is close to zero when argument of sine function is near $m\pi$;

  i.e., $k\omega_0 T_1 = k\frac{2\pi}{T}T_1 \cong m\pi$, or $k \cong \frac{mT}{2T_1}$ $m = \pm 1, \pm 2, \pm 3$

<u>This is about the most complex integration you ever need to do in ELEC2100!</u>

FS coefficients of the periodic square/rectangular wave are the sampled values of a ***Sinc function*** (function in the form of $\frac{\sin x}{x}$; normalized sinc function defined as $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$).

$\omega_0 = \frac{2\pi}{T}$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{1}{k\pi}\sin\left(k\pi\frac{2T_1}{T}\right)$$

$$= \alpha\frac{\sin(\pi\alpha k)}{\pi\alpha k}$$

where $\alpha = \frac{2T_1}{T}$ is the *duty cycle* of the wave $= a_0$

$$= \alpha\,\mathrm{sinc}(\alpha k)$$

Figure 3.7 shows the a_k's for various values of $\frac{T_1}{T}$.

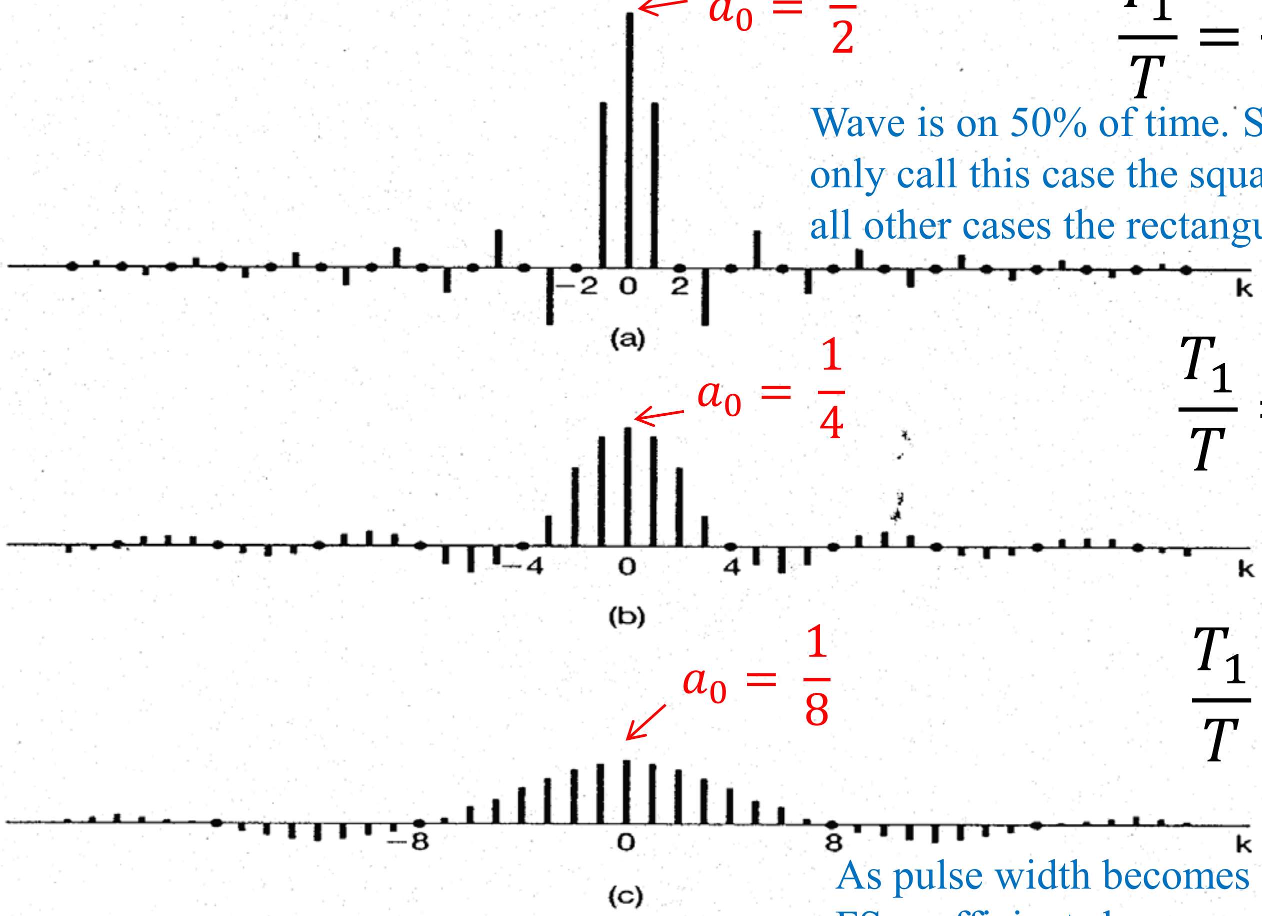
We will see the sinc function very often going forward.

$a_0 = \frac{1}{2}$ $\qquad \frac{T_1}{T} = \frac{1}{4}$

Wave is on 50% of time. Some people only call this case the square wave and all other cases the rectangular wave.

$a_0 = \frac{1}{4}$ $\qquad \frac{T_1}{T} = \frac{1}{8}$

$a_0 = \frac{1}{8}$ $\qquad \frac{T_1}{T} = \frac{1}{16}$



As pulse width becomes narrower, the FS coefficients become more spread out

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

III. The Convergence Question

- The great mathematicians Laplace and Lagrange were skeptical of Fourier series. Lagrange pointed out that as sinusoids are differentiable everywhere, we should not be able to add them together to reproduce the edges in the square wave

- We observe that the coefficients a_k in Eq.(3.44) has a generally decreasing trend with k. So, to test the validity of Fourier series we can consider the following ***truncated synthesis sum***, where we add up the lowest frequency $2N + 1$ harmonics alone :

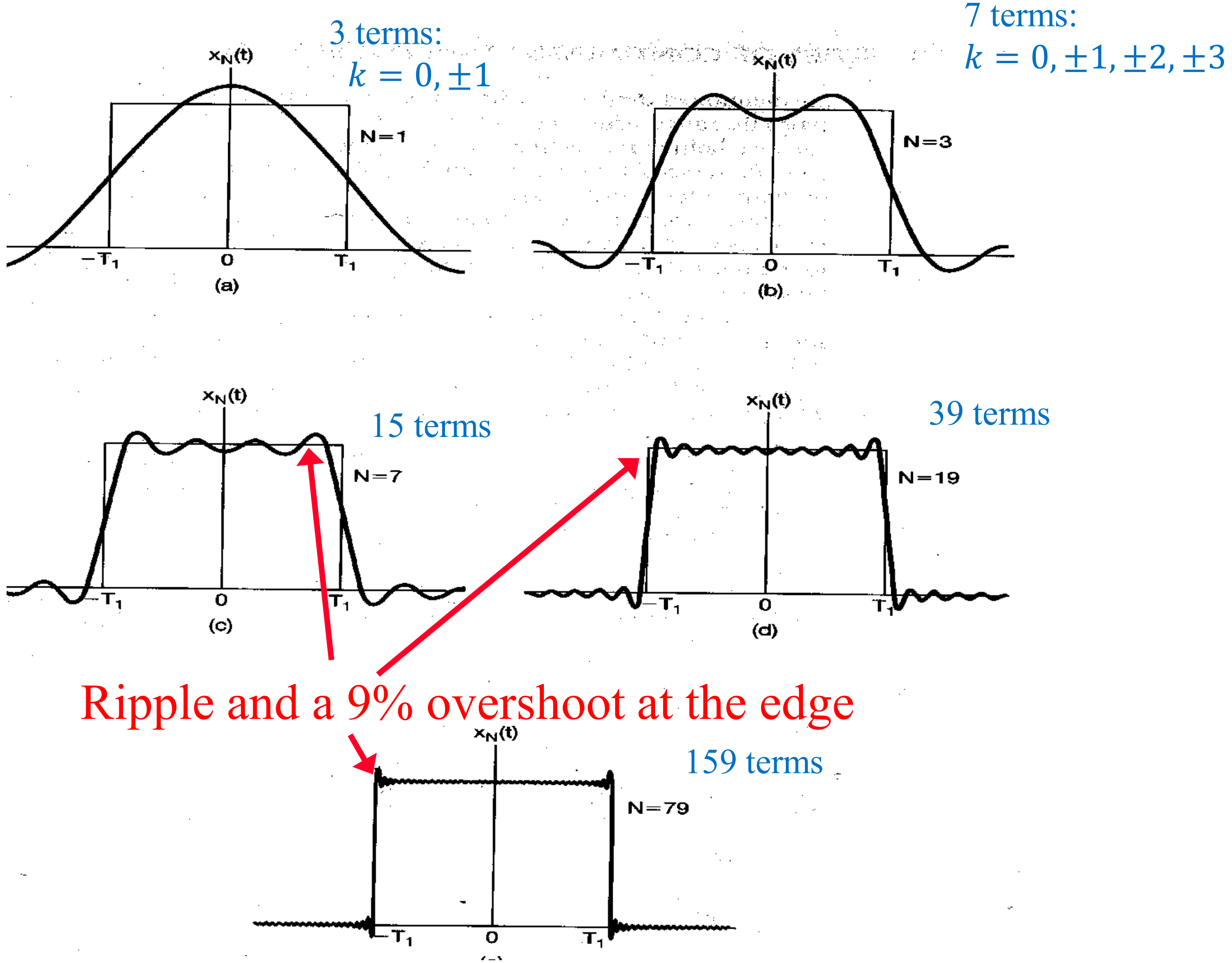
$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\frac{2\pi}{T}t}$$ *A truncated synthesis sum*

- We use the a_k as determined by Eq. (3.44) and check what happens to $x_N(t)$ when we add more terms by increasing N.

- Next slide shows **Fig. 3.9** of text which plots $x_N(t)$ over one period for different N's.

Gibbs Phenomenon

Figure 3.9



- As we increase N, we do get a closer and closer approximation to $x(t)$. But we see also that there is an overshoot of 9% at the edges that does not go away no matter how large we make N.
- This overshoot was first observed by *Willard Gibbs* (a professor at Yale and the first PhD in the USA) and is known as the ***Gibbs phenomenon***.
- But when N is large, the overshoot dies away quickly
- Gibbs phenomenon occurs because, as pointed out by Lagrange, sinusoids are differentiable everywhere, so they cannot be added up to reproduce the edges in the square wave exactly.

Convergence in Energy Sense

- So, what do we mean when we say that a periodic CT signal can be represented by its Fourier series?

- Let $x_N(t)$ be the truncated F.S. representation of a periodic signal $x(t)$,
  and let $e_N(t) = x(t) - x_N(t)$ be the difference between $x(t)$ and $x_N(t)$.
  the error signal

We say that the FS *converges* if the energy in the error signal goes to zero when N goes to infinity. That is:

$$\lim_{N\to\infty} \int_0^T |e_N(t)|^2 \, dt = 0$$

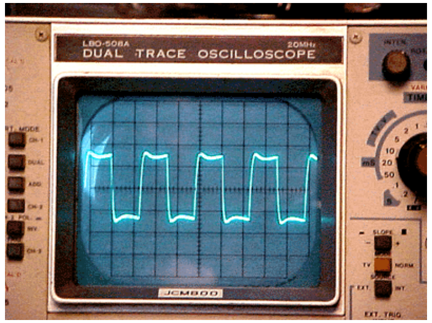
When there is no energy in the error signal, there is no meaningful difference between $x(t)$ and its FS representation, and we can say they are equal:

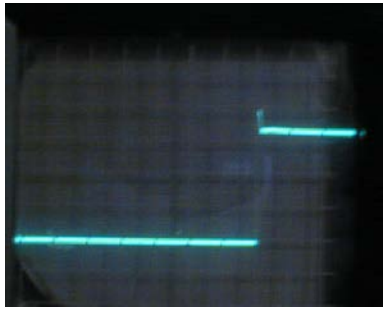
$$x(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t}$$

For the periodic square wave, for example, as we add up more and more sinusoids, the overshoot dies off so quickly that the energy in the error becomes closer and closer to zero!

Convergence Classes

- As given in Section 3.4 of the reference text, the Fourier series converges mathematically if the CT periodic signal satisfies either one of the two conditions below:
  1. Finite energy over one period
  2. Satisfies the three Dirichlet conditions: absolute integrable over one period, bounded variations, bounded discontinuities
- All practical signals satisfy the conditions above. Those signals whose Fourier series do not converge are not practically meaningful in engineering.
- Further, signals that you see in the lab do not contain discontinuities. Discontinuity is a mathematically idealization that we used to approximate real signals.





Take an oscilloscope and look at a square wave generated by any circuitry. Zoom in close enough and you will see that the edges are really not discontinuous. They either have smooth edges or contain overshoots very much like what we see in the Gibbs phenomenon

For Reference Only – Comparison with Taylor Series Expansion

- Fourier series decomposition is also called Fourier series expansion.
- Another series expansion we mentioned earlier is the Taylor series expansion:

$$x(t) = \sum_{k=0}^{\infty} \frac{x^{(k)}(a)}{k!}(t-a)^k$$

where $x(t)$ is decomposed into a weighted sum of power functions $(t-a)^k$. The coefficients and given by the k-derivative of $x(t)$ at fixed point a divided by $k!$.

- Euler's relation is discovered by applying the Taylor series expansion to $e^{j\theta}$. Your calculator actually evaluates many functions using Taylor series expansion.
- Both Taylor series and Fourier series convert a function $x(t)$ into an alternate description by a discrete set of coefficients.
- Convergence of Taylor series requires $x(t)$ to be infinitely differentiable - *analytic*.
- Fourier series is applicable to $x(t)$ that is periodic and satisfies the convergence classes described in the previous slides.

IV. Conjugate Symmetry and FS as Real Sinusoids

We stated in the last lecture that we can also decompose real signals into real sinusoids.

- Recall again the FS synthesis sum: $x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\frac{2\pi}{T}t}}_{\phi_k(t)}$ ----(1)
- Conjugating both sides of (1), we have :

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* \underbrace{e^{-jk\frac{2\pi}{T}t}}_{\phi_k^*(t)=\phi_{-k}(t)} \overset{\text{let } k' = -k}{=} \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\frac{2\pi}{T}t} \quad \text{----(2)}$$

Conjugate of sum = sum of conjugates
Conjugate of product = product of conjugates

- If $x(t)$ is real, then $x(t) = x^*(t)$ meaning that (1) and (2) are equal.
- Then, comparing terms in the two sum, we observe that we must have $a_k = a^*{}_{-k}$.
- This means *the Fourier coefficients for a real signal must be* ***conjugate symmetric***.

- Therefore the a_k and a_{-k} terms in the synthesis equation is a conjugate pair and combining them gives a real sinusoid:

Conjugate if $x(t)$ is real

$$a_k e^{jk\frac{2\pi}{T}t} + a_{-k} e^{-jk\frac{2\pi}{T}t} = 2Re\left\{a_k e^{jk\frac{2\pi}{T}t}\right\} = 2|a_k|\cos\left(k\frac{2\pi}{T}t + \measuredangle a_k\right)$$

conjugate

Sum of a conjugate pair is 2x real part

real part of complex sinusoid is cosine. Multiply by a_k means scaling by magnitude and shift by phase of a_k

- Hence, real periodic signals with period T can alternatively be represented by a set of real cosine functions with arbitrary phases:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2|a_k|\cos(k\omega_0 t + \measuredangle a_k)$$

- Using the identity $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$, we can also express Fourier series as a sum of cosine and sine functions:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2|a_k|\cos(\measuredangle a_k)\cos(k\omega_0 t) - \sum_{k=1}^{\infty} 2|a_k|\sin(\measuredangle a_k)\sin(k\omega_0 t)$$

Self-Test

It is known that $x(t)$ is a real periodic signal with period T,

and that $a_0 = 2$, $a_1 = j$, $a_3 = 1 + j$, and $a_k = 0$ for all other $k>0$

- Write $x(t)$ as a sum of real sinusoids:

Solution to Self-Test

It is known that $x(t)$ is a real periodic signal with period T, $x(t)$ real $\Rightarrow$ $a_k = a_{-k}^*$

and that $a_0 = 2$, $a_1 = j$, $a_3 = 1 + j$, and $a_k = 0$ for all other $k > 0$

$a_{-1} = -j$ $\quad a_{-3} = 1 - j$

- Write $x(t)$ as a sum of real sinusoids: If you can, just write down the answer from inspection:

$$x(t) = \overset{a_0}{2} + \underbrace{2}_{2\times|a_1|} \cos\left(\omega_0 t + \underbrace{\frac{\pi}{2}}_{\measuredangle a_1}\right) + \underbrace{2\times\sqrt{2}}_{2\times|a_3|} \cos\left(3\omega_0 t + \underbrace{\frac{\pi}{4}}_{\measuredangle a_3}\right)$$

Otherwise, you can start from the individual complex sinusoids:

$$x(t) = \overset{a_0}{2} + \underbrace{\overset{a_1}{j} e^{j\omega_0 t} - \overset{a_{-1}}{j} e^{-j\omega_0 t}}_{\text{conjugate}} + \underbrace{\overset{a_3}{(1+j)} e^{j3\omega_0 t} + \overset{a_{-3}}{(1-j)} e^{-j3\omega_0 t}}_{\text{conjugate}}$$

Subtracting conjugate pair gives $j \times 2j\sin(\omega_0 t) = -2\sin(\omega_0 t)$
$= -2\cos\left(\omega_0 t - \frac{\pi}{2}\right) = 2\cos\left(\omega_0 t + \frac{\pi}{2}\right)$

Adding conjugate pair gives $2 \times \sqrt{2}\cos\left(3\omega_0 t + \frac{\pi}{4}\right)$