ELEC2100: Signals and Systems

Lecture 12

Continuous-time Fourier Transform

(Analysis)

(Ref: Chapter 4 O&W)

I. From Fourier Series to Fourier Transform

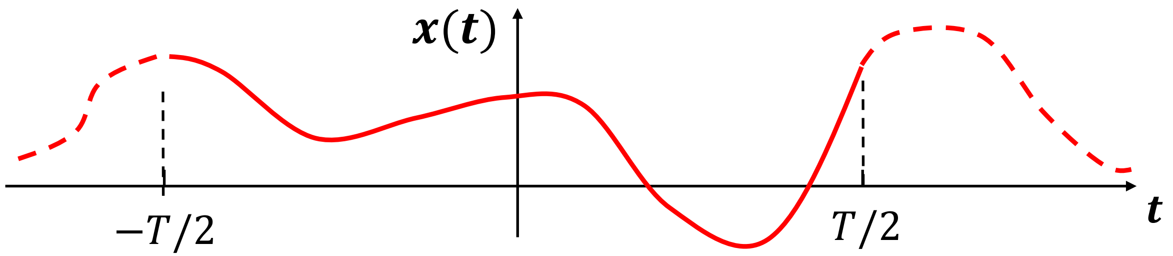
II. Fourier Transform Examples

III. Fourier Transform of Periodic Signals

I. From Fourier Series to Fourier Transform

Often, signals we encounter are aperiodic, and we have to do ***transforms*** instead of series decomposition. Transform and decomposition are primarily the same, except that in transform we do not care about any constant scaling. First, we will show how we can extend Fourier series decomposition to Fourier transform by *regarding aperiodic signals as periodic signals in the limit of period T going to infinity*.

Consider a CT signal $x(t)$:



For $x(t)$ within the interval $(-\frac{T}{2}, \frac{T}{2})$, we can always represent $x(t)$ as a FS sum:

For $-\frac{T}{2} \leq t < \frac{T}{2}$, Periodic!

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

A Fourier series synthesis sum

Analysis integral is an inner product: $< x(t), \phi_k(t) >$

where: $$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

Self-Inner Product of harmonic: $< \phi_k(t), \phi_k(t) > = T$

maybe x periodic!

Define $X_T(j\omega)$ as the inner product integral of $x(t)$ with $e^{j\omega t}$, a complex sinusoid of arbitrary frequency ω, over an interval of duration T:

$$X_T(j\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j\omega t}\, dt$$

This means $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt = \frac{1}{T} X_T\left(j\frac{k2\pi}{T}\right)$ are sampled values of $X_T(j\omega)$ scaled by $\frac{1}{T}$.

Hence, we can rewrite the F.S synthesis expression for $x(t)$ within the interval $\left(\frac{-T}{2}, \frac{T}{2}\right)$ as :

$$x(t) = \underbrace{\sum_{k=-\infty}^{\infty}}_{\text{FS sum}} \overbrace{\frac{1}{T} X_T\left(j\frac{k2\pi}{T}\right)}^{a_k} e^{jk\frac{2\pi}{T}t}$$

Expressing in $\omega_0 = \frac{2\pi}{T}$, we have: $x(t) = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X_T(jk\omega_0) e^{jk\omega_0 t}$ ... ... (1)

Now we take limit of $T \to \infty$, and define the **Fourier transform integral** of $x(t)$ as:

$$X(j\omega) = \lim_{T\to\infty} X_T(j\omega) \quad = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$\pm\frac{T}{2} \to \pm\infty$

Fourier Transform Integral

For Equation (1) in previous slide, when $T \to \infty$, $\omega_0 \to 0$, which means the summation becomes an integral, which is the inverse **Fourier transform integral**:

$$x(t) = \lim_{\substack{T\to\infty \\ \omega_0\to 0}} \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X_T(jk\omega_0) e^{jk\omega_0 t} \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} g(k\Delta)\Delta = \int_{-\infty}^{\infty} g(\alpha)d\alpha$$

Inverse Fourier Transform Integral

Fourier transform for aperiodic signals

By regarding an aperiodic signal as a periodic signal with period $T \to \infty$, we have established the Fourier transform and inverse Fourier transform pair:

- Fourier transform (FT)

Inner product integral

Symbol for FT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad = \Im\{x(t)\}$$

$$< x(t), e^{j\omega t} > = \int x(t)(e^{j\omega t})^* \, dt$$

<u>Analysis Equation</u>

Produces a frequency domain representation, or the spectrum, of the time signal

- Inverse Fourier transform (IFT)

A superposition of complex sinusoids

Symbol for IFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \quad = \Im^{-1}\{X(j\omega)\}$$

A density

<u>Synthesis Equation</u>

Which considers the time signal as a superposition of complex sinusoids, and the spectrum, which is now a ***<u>density</u>***, describes the frequency content in the signal.

Convergence of Fourier transform

- Do all CT signals have a Fourier transform?

  If $X(j\omega) = \mathfrak{J}\{x(t)\}$, and $x_{IFT}(t) = \mathfrak{J}^{-1}\{X(j\omega)\}$ does $x_{IFT}(t) = x(t)$?

- The conditions for the existence and convergence of FT are similar to those for FS and are satisfied by all practical signals (Dirichlet conditions)

  Again convergence is again in the sense of no energy in the error:

  $\int_{-\infty}^{\infty} |e(t)|^2 \, dt = 0$ where $e(t) = x_{IFT}(t) - x(t)$ is the error.

Lecture 12

Continuous-time Fourier Transform

I. From Fourier Series to Fourier Transform

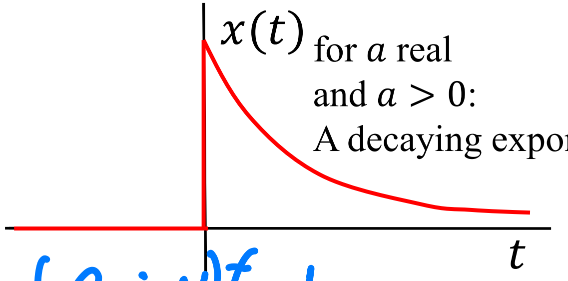
II. Fourier Transform Examples (Analysis)

III. Fourier Transform of Periodic Signals

II. Fourier Transform Examples

1. One-sided causal Exponential

- **Example 4.1:** Consider the ***one-sided exponential*** signal $x(t) = e^{-at}u(t)$
  where a can be complex and $Re\{a\} > 0$



$$\int_0^\infty e^{-at} e^{-j\omega t}\, dt = \int_0^\infty e^{(-a-j\omega)t}\, dt = \left.\frac{e^{(-a-j\omega)t}}{-a-j\omega}\right|_0^\infty = \frac{1}{a+j\omega}$$

Applying the FT integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_0^\infty e^{-at} e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt$$

$u(t)$

Magnitude of terms depends only on real part of exponent: $\left|e^{-(a+j\omega)\infty}\right| = \left|e^{-Re\{a\}\infty}\right| \left|e^{-j(Im\{a\}+\omega)\infty}\right| = e^{-Re\{a\}\infty}$

$\int e^{\alpha t} = \frac{e^{\alpha t}}{\alpha}$

Evaluates to 1 when $t = 0$

Evaluates to 0 when $t = \infty$ if $Re\{a\} > 0$

$$= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{t=0}^{t=\infty} = \frac{e^{-(a+j\omega)0} - e^{-(a+j\omega)\infty}}{(a+j\omega)} = \frac{1}{(a+j\omega)}$$

$\boxed{\text{Integral converges only if } Re\{a\} > 0}$

- If $Re\{a\} < 0$, $x(t)$ is unstable (not absolute integrable), and its FT does not exist.
  We cannot apply FT to deal with unstable signals!

Example 4.1 Continued -

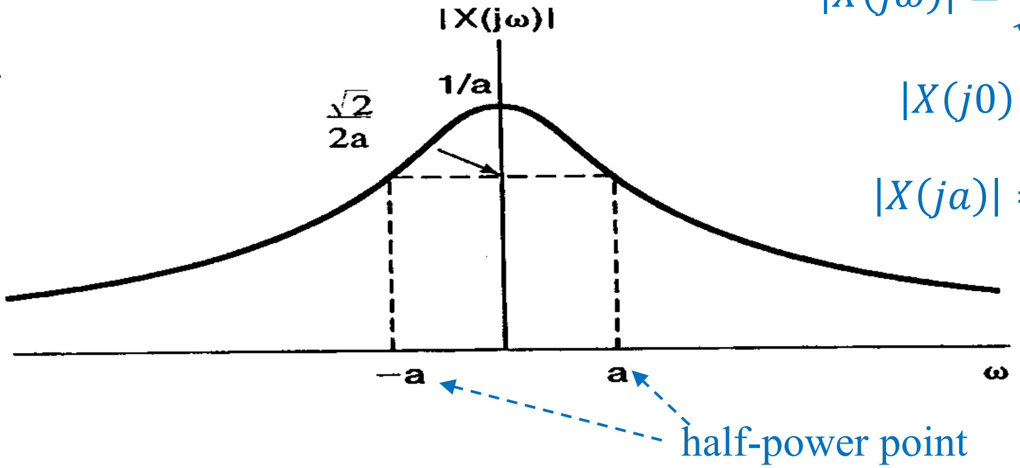
For a real a, we plot the magnitude and phase of the Fourier transform (called the **magnitude** and **phase spectrum** respectively):

$$X(j\omega) = \frac{1}{a + j\omega}, \quad \Re e\{a\} > 0.$$

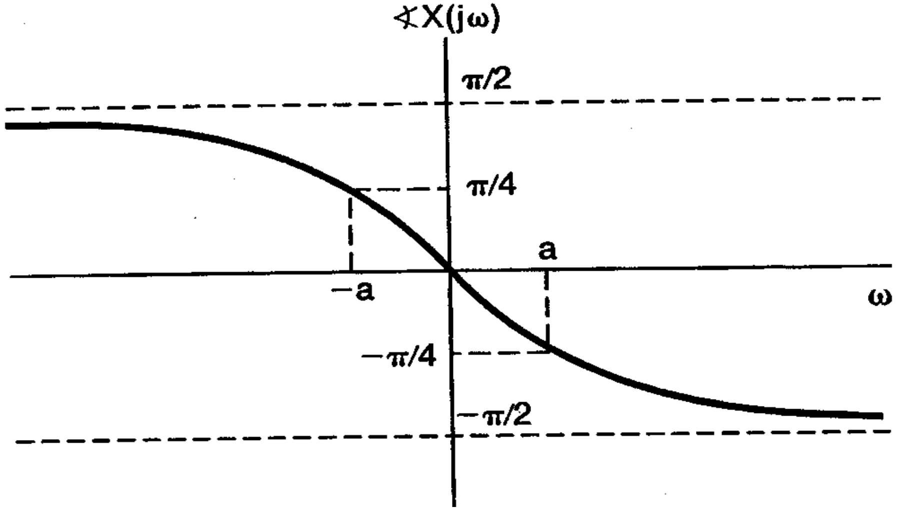
$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$|X(j0)| = \frac{1}{a}$$

$$|X(ja)| = \frac{1}{\sqrt{2}a}$$



$$\sphericalangle X(j\omega) = -\sphericalangle(a + j\omega) = -tan^{-1}\frac{\omega}{a}$$

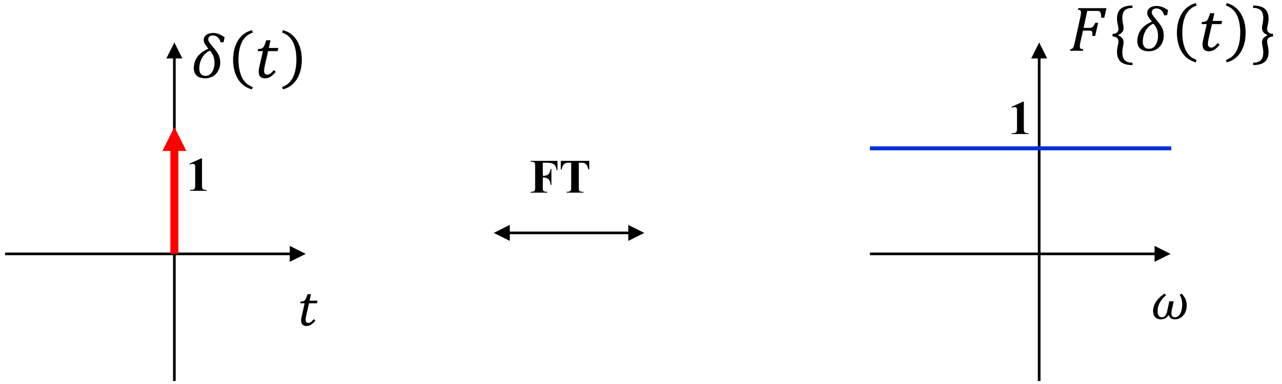


- The exponential constant a specifies the *half-power point*.
- We will see that the ***one-sided exponential*** is the basic signal for understanding the transient behavior of systems. It is also the prototypical response that we can use to construct more complex systems.

Example 2. FT of Impulse Function

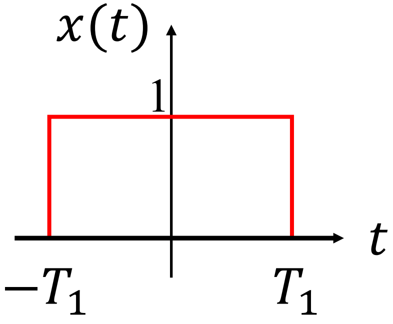
- **Example 4.3** The FT of the unit impulse is the constant 1:

$$x(t) = \delta(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \overset{\delta(t)g(t) = \delta(t)g(0)}{=} \int_{-\infty}^{\infty} \delta(t) \overset{e^{0}=1}{e^{-j0}} dt = 1$$



Example 3. FT of Rectangular Pulse/Window Signal

Example 4.4: The FT of a rectangular pulse/window signal is a sinc function.



$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

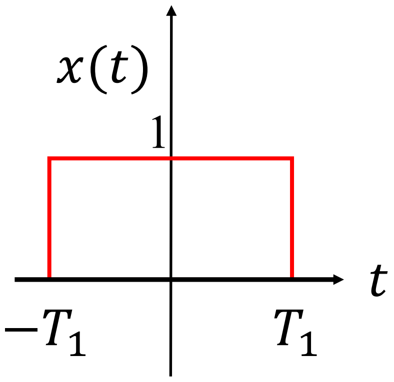
The FT is:

$x(t) = 1 \text{ for } -T_1 < t < T_1$

$$\boxed{X(j\omega)} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

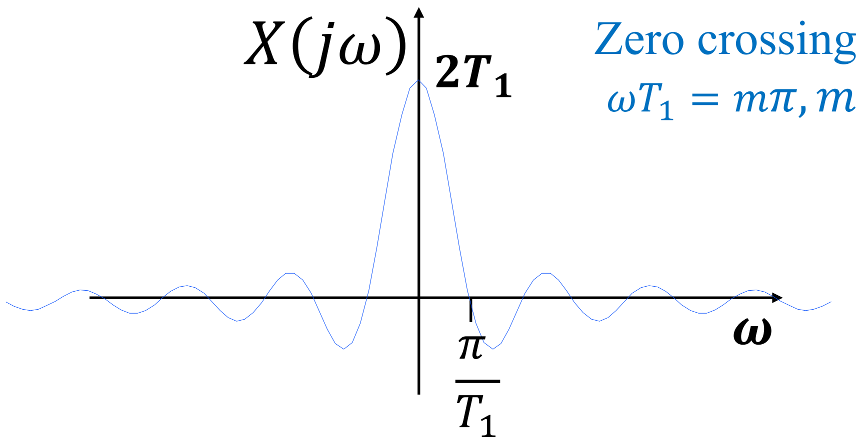
$$\boxed{= 2\frac{\sin\omega T_1}{\omega} \quad = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)}$$

where: $\boxed{\operatorname{sinc}(\theta) = \dfrac{\sin\pi\theta}{\pi\theta}}$



FT

$\longleftrightarrow$



Zero crossing when
$\omega T_1 = m\pi, m = \pm 1, \pm 2 \ldots$

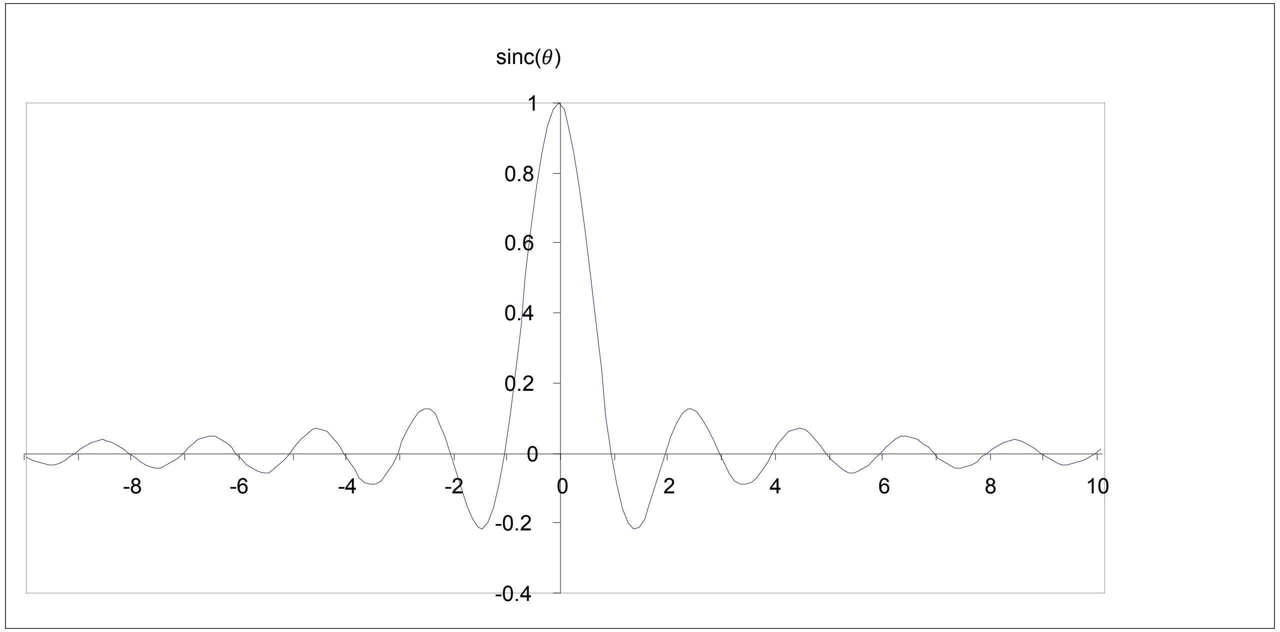
$$u(t + T_1) - u(t - T_1)$$

$$2\frac{\sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

More on the Sinc Function in Normalized Form

- The sinc function in normalized form has height of 1 at zero and zero-crossing at integer values of the argument:

$$\mathrm{sinc}(\theta) = \frac{\sin\pi\theta}{\pi\theta} = \begin{cases} 1 & \theta = 0 \\ 0 & \theta = \pm 1, \pm 2, \ldots \end{cases}$$



Also,

$$\int_{-\infty}^{\infty} \mathrm{sinc}(\theta)d\,\theta = 1$$

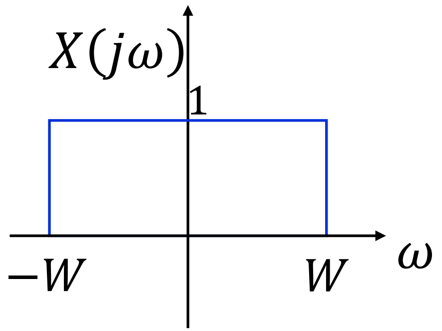
$$\int_{-1}^{\infty} \mathrm{sinc}(\theta)d\,\theta \cong 1.09$$

$\max_{\beta} \int_{\beta}^{\infty} \mathrm{sinc}(\theta)d\,\theta \cong 1.09$ at $\beta = -1$

- You will encounter the sinc function often.

Example 4. IFT of Rectangular Window Spectrum

- **Example 4.5** The Inverse FT of a rectangular window in frequency domain is a sinc function in time domain

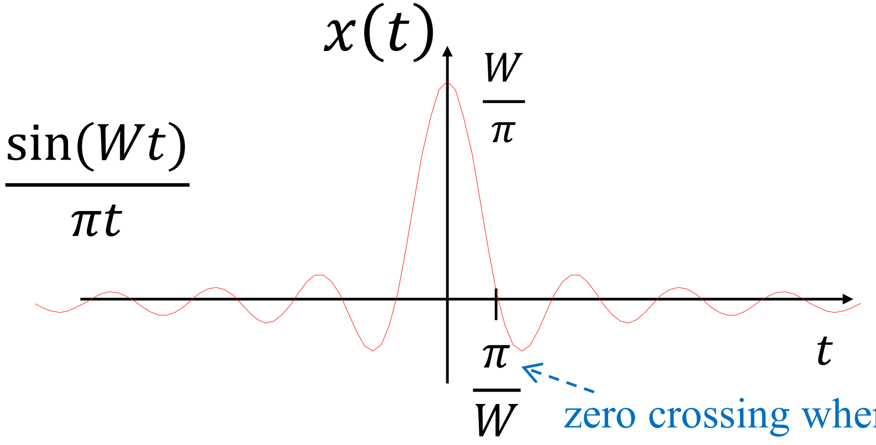


$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

We apply the IFT integral:

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-W}^{W} = \frac{e^{jWt} - e^{-jWt}}{2\pi j t} = \frac{\cancel{2j}}{\cancel{2j}}\frac{\sin(Wt)}{\pi t}$$

$X(j\omega) = 1$
within this interval

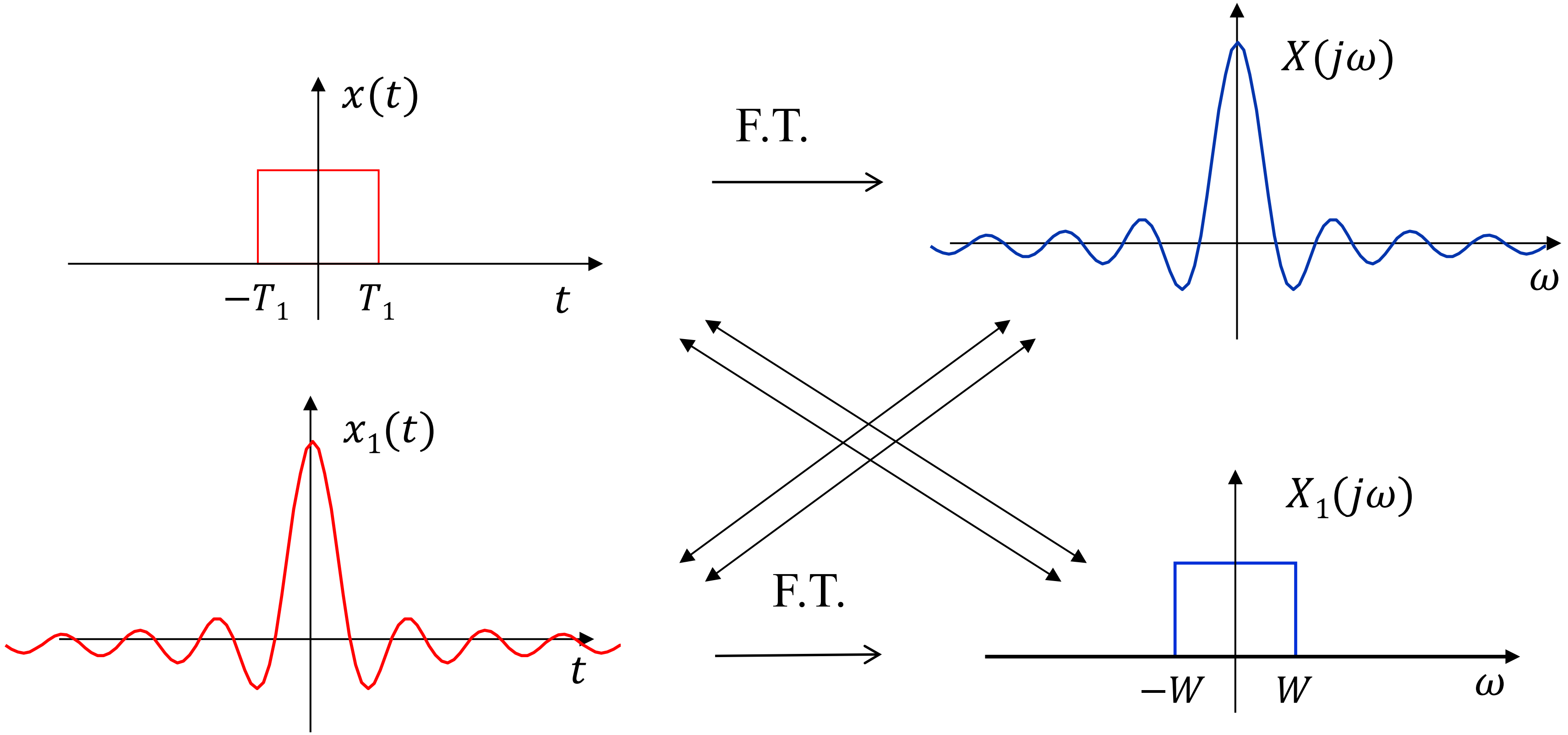


zero crossing when
$Wt = m\pi$; or $t = \frac{m\pi}{W}$

Recall that the frequency response of an LTI system is the FT of its impulse response. If an LTI system has a frequency response $H(j\omega)$ that is a rectangular window, it is an ideal-low pass filter. Hence, the impulse response of an ILPF is a sinc function.
Also, in digital communications, we transmit electrical, radio, or optical pulses to convey bits. Often, we transmit sinc pulses because their spectrum is limited to a frequency range, limiting interference across channels and degradation effects such as dispersion (future subject).

Duality Property

- In the last two examples, we see the rectangular pulse/window and the sinc function are each other's FT and IFT.

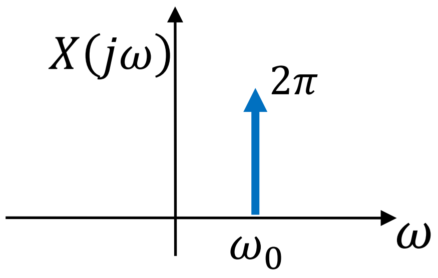


- This is not surprising because of the symmetry in the FT and IFT integrals – they are identical except for a factor 2π and a change in sign. This is known as the **duality property**.

Example 5. FT a of Complex Sinusoid

- In Chapter 3, we see that periodic signals have an FS representation. Do they have an FT representation as well?
- Let us first consider an FT that is a shifted impulse function in frequency:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$



To find its IFT, we apply the IFT (synthesis integral):

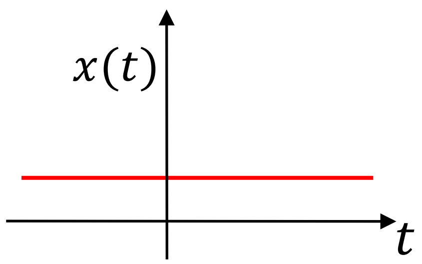
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

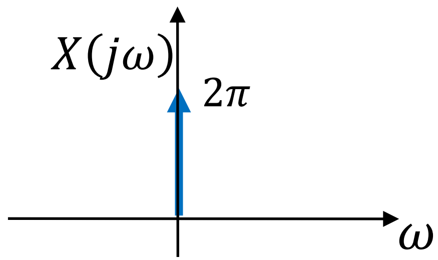
This means that the FT of a complex sinusoid at frequency ω_0 is an impulse in frequency at ω_0 with magnitude 2π:

$$\boxed{e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)}$$

If ω_0 is zero, the complex sinusoid becomes a constant:

$$\boxed{1 \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega)}$$





Lecture 12

Continuous-time Fourier Transform

I. From Fourier Series to Fourier Transform

II. Fourier Transform Examples

III. Fourier Transform of Periodic Signals (Analysis)

III. Fourier transform for periodic signals

- If $x(t)$ is periodic, we learnt in Chapter 3 that it can be written as a Fourier Series sum:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}; \qquad \omega_0 = \frac{2\pi}{T}$$

F.S. synthesis sum

- Linearity of FT is obvious: FT of a sum = sum of the individual FTs. Therefore, FT of the F.S. sum is :

Replace order of summation and integration

$$X(j\omega) = \int_{-\infty}^{\infty} \Bigg(\overbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}^{x(t)} \Bigg) e^{-j\omega t}\, dt = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t}\, dt}_{\text{FT of individual complex sinusoids}}$$

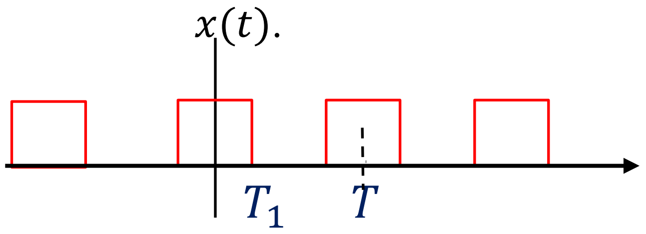
$$\boxed{= \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)}$$

which is a sequence of shifted impulses

- Fourier series can be viewed as the special case of Fourier transform for periodic CT signals. The F.S. coefficients give the magnitude of the shifted impulses. (The factor 2π arises as a matter of convention because of the use of angular frequency.)

Example 6. FT for Periodic Square Wave

- **Example 4.6** Consider once again the periodic square wave $x(t)$.



We can represent its spectrum either by its F.S. coefficients $\{a_k\}$ with knowledge of the fundamental frequency, or by its F.T. :

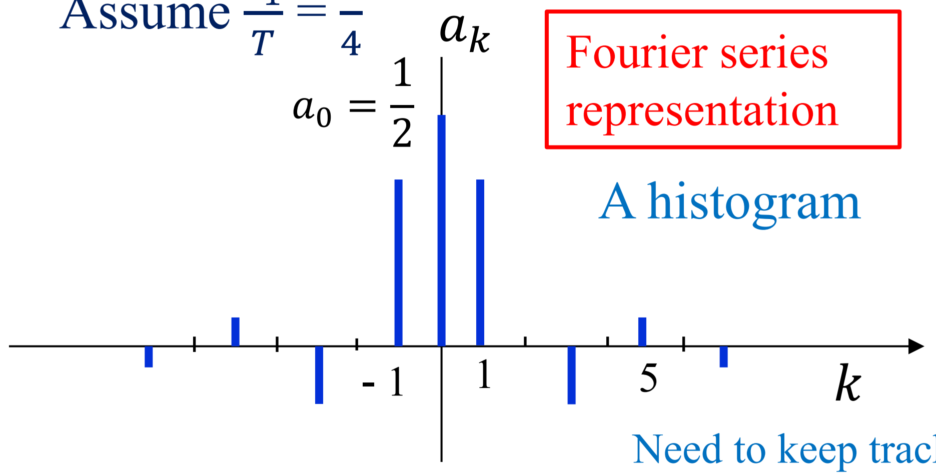
From Example 3.5, Eq (3.44):

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi}$$

FT is a set of impulses at the harmonic frequencies, with magnitudes $2\pi a_k$:

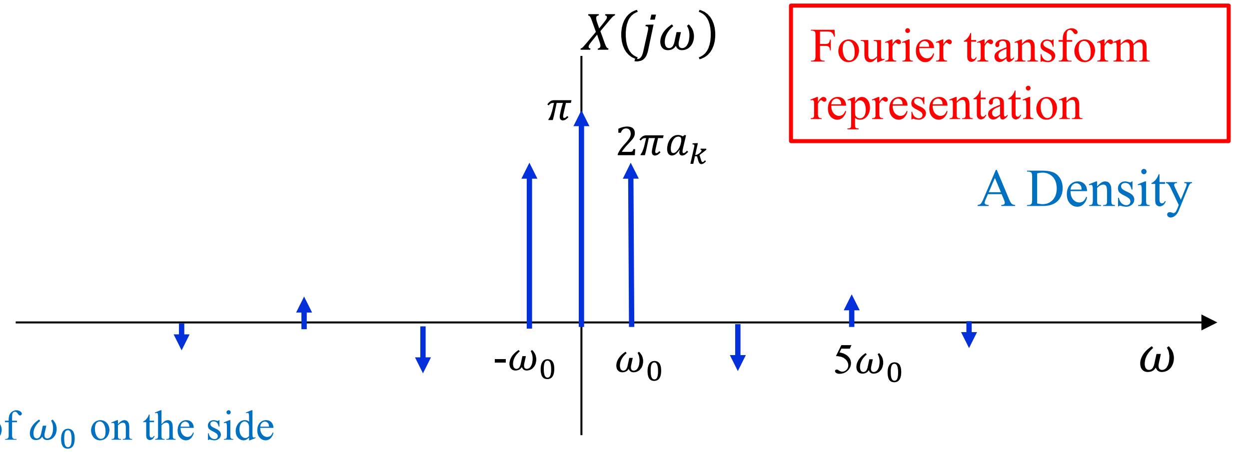
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \frac{\sin(k\omega_0 T_1)}{k\pi} \delta(\omega - k\omega_0)$$

Assume $\frac{T_1}{T} = \frac{1}{4}$



Fourier series representation

A histogram



Fourier transform representation

A Density

Need to keep track of ω_0 on the side

Example 7. FT for real sinusoids

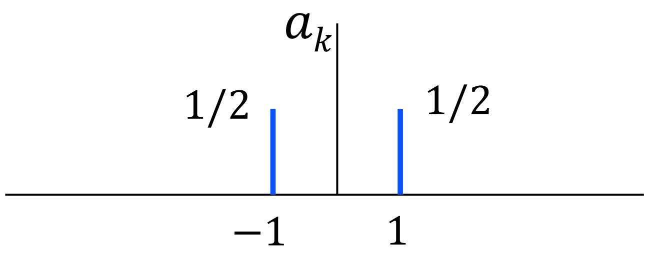
- **Example 4.7:** Consider again both the sine function and the cosine function:

$$x(t) = \cos\omega_0 t$$

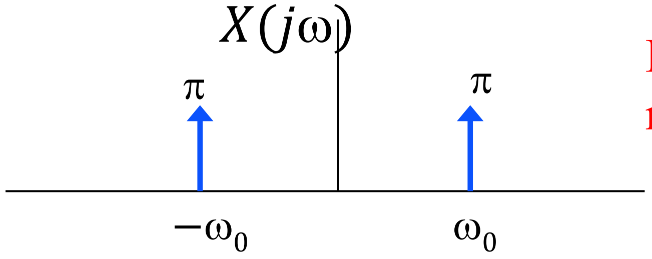
$$\cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$a_1 = \frac{1}{2};\ a_{-1} = \frac{1}{2}$$

$$a_k = 0 \quad \forall\, k \neq \pm 1$$



Fourier series representation



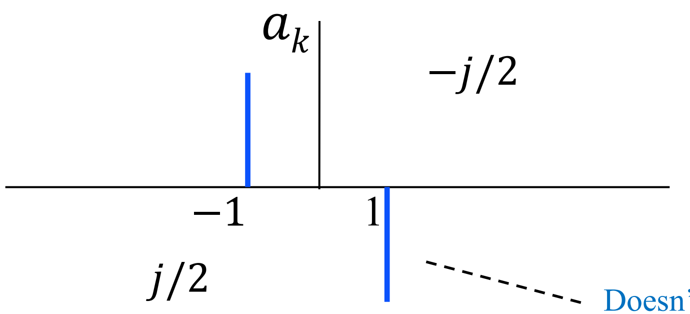
Fourier transform representation

$$x(t) = \sin\omega_0 t$$

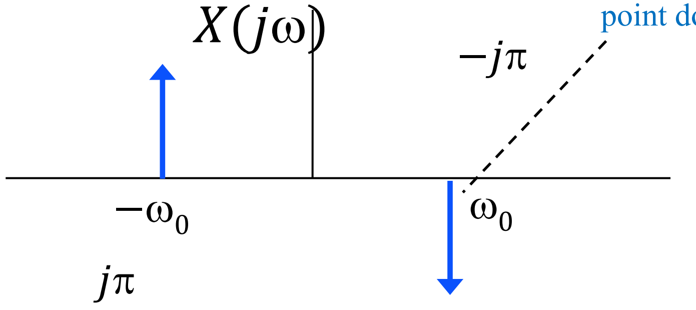
$$\sin\omega_0 t = \cos\left(\omega_0 t - \frac{\pi}{2}\right) = \frac{1}{2}e^{j\omega_0 t}e^{-j\frac{\pi}{2}} + \frac{1}{2}e^{-j\omega_0 t}e^{j\frac{\pi}{2}}$$

$$a_1 = \frac{-j}{2};\ a_{-1} = \frac{j}{2}$$

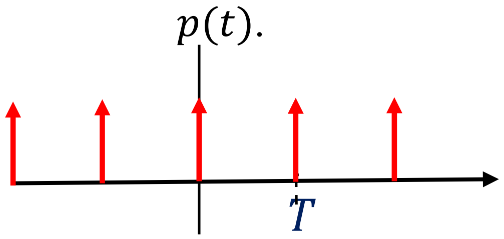
$$a_k = 0 \quad \forall\, k \neq \pm 1$$



Doesn't matter if these point up or point down



Example 8. FT for Periodic Impulse Train



- **Example 4.8** The Periodic Impulse Train $x(t) = p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

Recall from Chapter 3 that its FS coefficients are: $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T}$

This implies its Fourier transform is: $X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{k2\pi}{T}\right)$

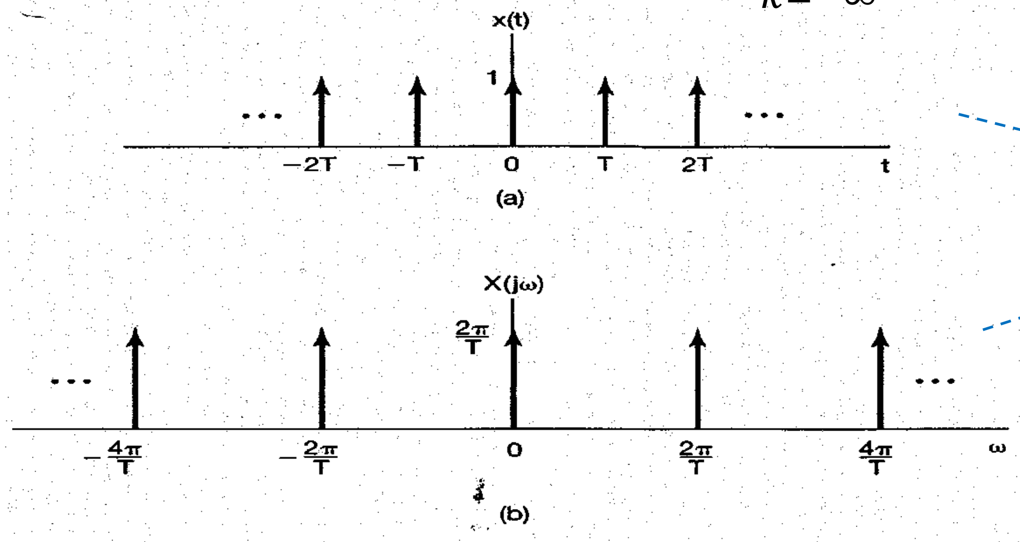


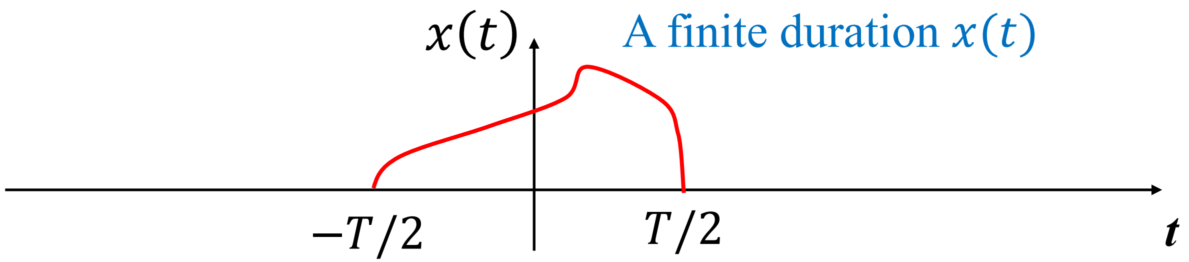
Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

The Fourier transform of a periodic impulse train in time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$.

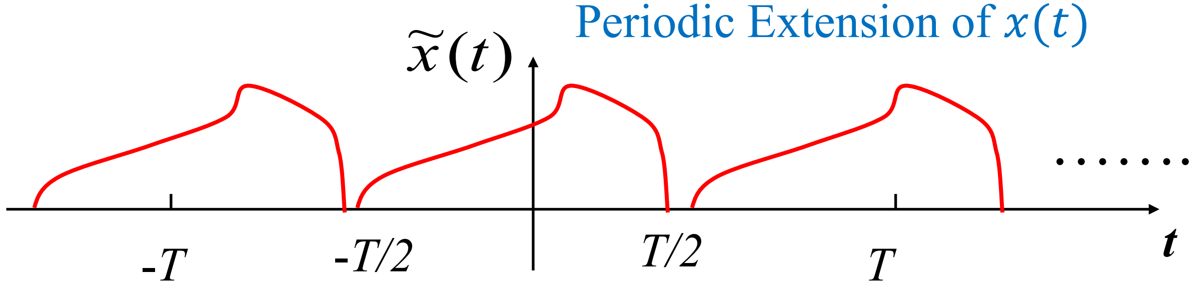
This is an important relation when we talk about sampling

Example 9. FS Coefficients as sample values of FT

- Now, consider a signal $x(t)$ of finite duration T such that $x(t) = 0$ for $|t| > \frac{T}{2}$.



- The Poisson sum $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$ is called a ***periodic extension*** of $x(t)$:



$$\tilde{x}(t) = x(t) \text{ for } -\frac{T}{2} \leq t \leq \frac{T}{2}$$

- The Poisson sum/periodic extension can be regarded as the result of convolution with a periodic impulse train:

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad = x(t) * p(t)$$

- Recall that the Fourier Series **synthesis** and **analysis** equations for the periodic signal $\tilde{x}(t)$ are :

Synthesis $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Analysis $$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)\, e^{-jk\omega_0 t} dt$$

- In the interval (-*T*/2, *T*/2), $\tilde{x}(t) = x(t)$. Outside the interval, $x(t) = 0$. This means:

$x(t) = 0$ outside of the interval anyway

$x(t) = \tilde{x}(t)$ for $-\frac{T}{2} \le t < t$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T} X(jk\omega_0)$$

- The above means the FS coefficients of the *periodic extension* (Poisson sum) of a signal are the sample values of the signal's FT scaled by $1/T$!

FS coefficients of the Poisson sum

$$a_k = \frac{1}{T} X(jk\omega_0)$$

sample values of the FT of the original signal

So for a finite duration signal, sample values of its FT is enough to specify the signal and its FT.

FT Summary

Table 4.2

- $x(t)$ that is periodic has both an FT and an FS.

  The FT is a discrete set of impulses.

- FS coefficients of the periodic extension of $x(t)$ are the sample values of the FT of $x(t)$ (with scaling by $1/T$)

Periodic
Has both
FT and FS

Periodic
Extension

Periodic
Extension

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\ a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—

$$a_k = \frac{2\sin k\omega_0 T_1}{Tk\omega_0} \quad T\omega_0 = 2\pi$$
$$= \frac{\sin k\omega_0 T_1}{k\pi}$$