**Question 1:** Find the real-valued equilibrium solutions of the following differential equation: [*4 points*]

$$y' = (2 + y^2)(y^2 - 2\pi y)$$

**Solution:**
The equilibrium solutions are the solutions of the equation $y' = 0$. Therefore, we have:

$$y' = (2 + y^2)(y^2 - 2\pi y) = 0$$
$$= y \cdot (2 + y^2)(y - 2\pi) = 0$$

By solving the above equation, we get the equilibrium solutions as $\underline{y = 0}$ and $\underline{y = 2\pi}$.

**Question 2:** Given the following differential equation:

$$y' = 7y - 6$$

**(a)** Find the equilibrium solution of the differential equation. [*4 points*]
**Solution:**
The equilibrium solution is the solution of the equation $y' = 0$. Hence we have:

$$y' = 7y - 6 = 0$$
$$7y = 6$$
$$y = \frac{6}{7}$$

Therefore, the equilibrium solution is $\underline{y = \frac{6}{7}}$.

**(b)** Find the non-equilibrium solution of the differential equation using the method of Calculus. [*4 points*]
**Solution:**
The differential equation is a first-order linear differential equation. Hence we can solve it using the method of Calculus. The general solution of the differential equation is given by:

$$y' = 7y - 6$$
$$\frac{dy}{dx} = 7y - 6$$
$$\frac{dy}{7y - 6} = dx$$
$$\frac{dy}{y - \frac{6}{7}} = 7dx$$
$$\int \frac{dy}{y - \frac{6}{7}} = \int 7dx$$
$$\ln \left| y - \frac{6}{7} \right| = 7x + C$$
$$y - \frac{6}{7} = e^{7x + C}$$
$$y = e^{7x + C} + \frac{6}{7}$$
$$y = Ce^{7x} + \frac{6}{7}$$

Therefore, the non-equilibrium solution of the differential equation is $\underline{y = Ce^{7x} + \frac{6}{7}}$ for $y \neq \frac{6}{7}$.

**(c)** Find the general solution expression for ALL solutions and specify the range of the parameter in your general solution expression corresponding to the equilibrium solution found in part (a) and the non-equilibrium solution found in part (b). [*2 points*]
**Solution:**
The general solution expression for all solutions is given by:

$$y = Ce^{7x} + \frac{6}{7}$$

The range of the parameter $C$ corresponding to the equilibrium solution $y = \frac{6}{7}$ is $C = 0$. The range of the parameter $C$ corresponding to the non-equilibrium solution $y = Ce^{7x} + \frac{6}{7}$ is $C \neq 0$. Therefore, the general solution expression for all solutions is $\underline{y = Ce^{7x} + \frac{6}{7}}$ for $C \neq 0$.

**Question 3:** Find a fundamental set of solutions for the following differential equation: [*10 points*]

$$y'' + 8y' + 16y = 0$$

**Solution:**
The characteristic equation of the differential equation is given by:

$$r^2 + 8r + 16 = 0$$
$$(r+4)^2 = 0$$
$$r = -4$$

Hence the general solution of the differential equation is given by:

$$y = e^{-4t}(c_1 + c_2 t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

Therefore, the fundamental set of solutions for the differential equation is $\underline{y_1 = e^{-4t}}$ and $\underline{y_2 = te^{-4t}}$.


**Question 4:** Solve the following initial value problem: [*12 points*]

$$y'' - 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

**Solution:**
The characteristic equation of the differential equation is given by:

$$r^2 - 4r + 5 = 0$$
$$r = \frac{4 \pm \sqrt{16 - 20}}{2}$$
$$= \frac{4 \pm \sqrt{-4}}{2}$$
$$= \frac{4 \pm 2i}{2}$$
$$= 2 \pm i$$

Hence the general solution of the differential equation is given by:

$$y = e^{2t}(c_1 \cos(t) + c_2 \sin(t)) = c_1 e^{2t} \cos(t) + c_2 e^{2t} \sin(t)$$

Finding the first derivative of $y$:

$$y' = 2c_1 e^{2t} \cos(t) - c_1 e^{2t} \sin(t) + 2c_2 e^{2t} \sin(t) + c_2 e^{2t} \cos(t)$$

Given that $y(0) = 0$ and $y'(0) = 1$, we have:

$$y(0) = c_1 \cos(0) + c_2 \sin(0) = 0$$
$$= c_1$$
$$y'(0) = 2c_1 \cos(0) - c_1 \sin(0) + 2c_2 \sin(0) + c_2 \cos(0) = 1$$
$$= 2c_1 + c_2 = 1$$

We have $c_1 = 0$ and $2c_1 + c_2 = 1$. Hence $c_2 = 1$.
Therefore, the solution of the initial value problem is $\underline{y = e^{2t} \sin(t)}$.

**Question 5:** Find the general solution of the following differential equation:

$$y'' - 2y' - 3y = 2e^{3t} - \sin(2t) + t$$

**Solution:**

The characteristic equation of the homogeneous part of the differential equation is given by:

$$r^2 - 2r - 3 = 0$$
$$(r - 3)(r + 1) = 0$$
$$r = 3, -1$$

Hence the general solution of the homogeneous part of the differential equation is given by:

$$y_h = c_1 e^{3t} + c_2 e^{-t}$$

Next we find the particular solution of the non-homogeneous part of the differential equation.

First we find the particular solution of the differential equation $y'' - 2y' - 3y = 2e^{3t}$. The particular solution of this differential equation is given by:

$$y_p = Ae^{3t}$$
$$y_p' = 3Ae^{3t}$$
$$y_p'' = 9Ae^{3t}$$

Substituting the above expressions into the differential equation, we get:

$$9Ae^{3t} - 2(3Ae^{3t}) - 3(Ae^{3t}) = 2e^{3t}$$
$$9Ae^{3t} - 6Ae^{3t} - 3Ae^{3t} = 2e^{3t}$$
$$0 = 2e^{3t}$$

The above equation is not possible. Hence we multiply the particular solution by $t$ to get the new particular solution:

$$y_p = tAe^{3t}$$
$$y_p' = Ae^{3t} + 3Ate^{3t}$$
$$y_p'' = 6Ae^{3t} + 9Ate^{3t}$$

Substituting the above expressions into the differential equation, we get:

$$6Ae^{3t} + 9Ate^{3t} - 2(Ae^{3t} + 3Ate^{3t}) - 3(tAe^{3t}) = 2e^{3t}$$
$$6Ae^{3t} + 9Ate^{3t} - 2Ae^{3t} - 6Ate^{3t} - 3tAe^{3t} = 2e^{3t}$$
$$4Ae^{3t} = 2e^{3t}$$
$$4A = 2$$
$$A = \frac{1}{2}$$

Therefore, the particular solution of the differential equation $y'' - 2y' - 3y = 2e^{3t}$ is $y_p = \frac{1}{2}te^{3t}$.

Second we find the particular solution of the differential equation $y'' - 2y' - 3y = -\sin(2t)$. The particular solution of this differential equation is given by:

$$y_p = A\sin(2t) + B\cos(2t)$$
$$y_p' = 2A\cos(2t) - 2B\sin(2t)$$
$$y_p'' = -4A\sin(2t) - 4B\cos(2t)$$

Substituting the above expressions into the differential equation, we get:

$$-4A\sin(2t) - 4B\cos(2t) - 2(2A\cos(2t) - 2B\sin(2t)) - 3(A\sin(2t) + B\cos(2t)) = -\sin(2t)$$
$$-4A\sin(2t) - 4B\cos(2t) - 4A\cos(2t) + 4B\sin(2t) - 3A\sin(2t) - 3B\cos(2t) = -\sin(2t)$$
$$(-4A + 4B - 3A)\sin(2t) + (-4B - 4A - 3B)\cos(2t) = -\sin(2t)$$
$$(-7A + 4B)\sin(2t) + (-7B - 4A)\cos(2t) = -\sin(2t)$$

By comparing the coefficients of the terms, we get:

$$-7A + 4B = -1$$
$$-7B - 4A = 0$$

Solving the above system of equations, we get $A = \frac{7}{65}$ and $B = -\frac{4}{65}$. Therefore, the particular solution of the differential equation $y'' - 2y' - 3y = -\sin(2t)$ is $y_p = \frac{7}{65}\sin(2t) - \frac{4}{65}\cos(2t)$.

Third we find the particular solution of the differential equation $y'' - 2y' - 3y = t$. The particular solution of this differential equation is given by:

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Substituting the above expressions into the differential equation, we get:

$$2A - 2(2At + B) - 3(At^2 + Bt + C) = t$$
$$2A - 4At - 2B - 3At^2 - 3Bt - 3C = t$$
$$-3At^2 - 4At - 3Bt + 2A - 2B - 3C = t$$

By comparing the coefficients of the terms, we get:

$$-3A = 0$$
$$-4A - 3B = 1$$
$$2A - 2B - 3C = 0$$

Solving the above system of equations, we get $A = 0, B = -\frac{1}{3}, C = \frac{2}{9}$. Therefore, the particular solution of the differential equation $y'' - 2y' - 3y = t$ is $y_p = -\frac{1}{3}t + \frac{2}{9}$.
Therefore, the general solution of the differential equation is given by:

$$y = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{2}te^{3t} + \frac{7}{65}\sin(2t) - \frac{4}{65}\cos(2t) - \frac{1}{3}t + \frac{2}{9}$$

**Question 6:** Solve the following initial value problem: [*12 points*]

$$ty' - 2y - 7t^2 = 0, \quad y(-1) = 2$$

**Solution:**
Given the differential equation $ty' - 2y - 7t^2 = 0$, we can rewrite the equation as $y' - \frac{2}{t}y = 7t$.
The integrating factor of the above differential equation is given by:

$$\mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln(t)} = \frac{1}{t^2}$$

Let $g(t) = 7t$. The solution of the differential equation is given by:

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)} \left( \int \mu(t)g(t)dt \right) \\
&= t^2 \left( \int 7t \cdot \frac{1}{t^2} dt \right) \\
&= t^2 \left( \int \frac{7}{t} dt \right) \\
&= t^2(7\ln(t) + C) \\
&= 7t^2\ln(|t|) + Ct^2
\end{aligned}$$

Given that $y(-1) = 2$, we have:

$$\begin{aligned}
7(-1)^2\ln(|-1|) + C(-1)^2 &= 2 \\
7\ln(1) + C &= 2 \\
7(0) + C &= 2 \\
C &= 2
\end{aligned}$$

Therefore, the solution of the initial value problem is $\underline{\underline{y = 7t^2\ln(|t|) + 2t^2}}$.

**Question 7:** Given the following differential equation:
$$\frac{dy}{dx} = \frac{y\cos(x)}{3y^4 + 5}$$

**(a)** Find all the solutions of the differential equation. [*10 points*]

**Solution:**

There are two types of solutions for the differential equation. The first type of solution is $y = 0$ which is the equilibrium solution. The second type of solution is the non-equilibrium solution where $y \neq 0$. The differential equation can be rewritten as:

$$\frac{dy}{dx} = \frac{y\cos(x)}{3y^4 + 5}$$

$$(3y^3 + \frac{5}{y})dy = \cos(x)dx$$

$$\int (3y^3 + \frac{5}{y})dy = \int \cos(x)dx$$

$$\frac{3}{4}y^4 + 5\ln|y| = \sin(x) + C$$

$$y^4 + \frac{20}{3}\ln|y| = \frac{4}{3}\sin(x) + C$$

Therefore, the solutions of the differential equation are $\underline{\underline{y = 0}}$ and $\underline{\underline{y^4 + \frac{20}{3}\ln|y| = \frac{4}{3}\sin(x) + C}}$.

**(b)** Find the particular solution of the differential equation that satisfies the initial condition $y(0) = 1$. [*3 points*]

**Solution:**

Given that $y(0) = 1$, we have:

$$(1)^4 + \frac{20}{3}\ln|(1)| = \frac{4}{3}\sin(0) + C$$

$$1 + 0 = 0 + C$$

$$C = 1$$

Therefore, the particular solution of the differential equation that satisfies the initial condition $y(0) = 1$ is:

$$\underline{\underline{y^4 + \frac{20}{3}\ln|y| = \frac{4}{3}\sin(x) + 1}}$$

**(c)** Find the particular solution of the differential equation that satisfies the initial condition $y(0) = 0$. [*2 points*]

**Solution:**

Given that $y(0) = 0$, since $y = 0$ is the equilibrium solution.

The particular solution of the differential equation that satisfies the initial condition $y(0) = 0$ is $\underline{\underline{y = 0}}$.

**Question 8:** Given a solution $y_1(x) = \frac{1}{x}$ of the following differential equation:

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0 \quad \text{for} \quad x > 0$$

**(a)** Find the second linearly independent solution $y_2(x)$ of the differential equation such that $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of this equation for $x > 0$ by applying the method of reduction of order. **[12 points]**

**Solution:**

Given that $y_1(x) = \frac{1}{x}$ is a solution of the differential equation, we can find the second linearly independent solution $y_2(x)$ by applying the method of reduction of order. Let $y_2(x) = v(x)y_1(x) = v(x)\frac{1}{x}$. The first and second derivatives of $y_2(x)$ are given by:

$$y_2(x) = v(x)\frac{1}{x}$$

$$y_2'(x) = v'(x)\frac{1}{x} - v(x)\frac{1}{x^2}$$

$$y_2''(x) = v''(x)\frac{1}{x} - v'(x)\frac{1}{x^2} - v'(x)\frac{1}{x^2} + v(x)\frac{2}{x^3} = v''(x)\frac{1}{x} - 2v'(x)\frac{1}{x^2} + v(x)\frac{2}{x^3}$$

Substituting $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ into the differential equation, we get:

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$$

$$v''(x)\frac{1}{x} - 2v'(x)\frac{1}{x^2} + v(x)\frac{2}{x^3} + \frac{3}{x}\left(v'(x)\frac{1}{x} - v(x)\frac{1}{x^2}\right) + \frac{1}{x^2}\left(v(x)\frac{1}{x}\right) = 0$$

$$v''(x)\frac{1}{x} - 2v'(x)\frac{1}{x^2} + 3v'(x)\frac{1}{x^2} + v(x)\frac{2}{x^3} - 3v(x)\frac{1}{x^3} + v(x)\frac{1}{x^3} = 0$$

$$v''(x)\frac{1}{x} + v'(x)\frac{1}{x^2} = 0$$

$$xv''(x) + v'(x) = 0$$

The above differential equation is a first-order linear differential equation. The integrating factor of the above differential equation is given by:

$$\mu(x) = v'(x) = \frac{dv}{dx}$$

Therefore, the general solution of the differential equation is given by:

$$xv''(x) + v'(x) = 0$$

$$x\mu'(x) + \mu(x) = 0$$

$$x\frac{d\mu}{dx} = -\mu$$

$$\frac{d\mu}{\mu} = -\frac{dx}{x}$$

$$\int \frac{d\mu}{\mu} = -\int \frac{dx}{x}$$

$$\ln|\mu| = -\ln|x| + C$$

$$\mu = \frac{1}{x}e^C = \frac{C}{x}$$

Therefore, the general solution of the differential equation is given by:

$$xv''(x) + v'(x) = 0$$

$$\frac{d}{dx}(xv'(x)) = 0$$

$$xv'(x) = C$$

$$v'(x) = \frac{C}{x}$$

$$v(x) = C_1\ln|x| + C_2$$

Therefore, the second linearly independent solution $y_2(x)$ is given by:

$$\underline{\underline{y_2(x) = (C_1\ln|x| + C_2)\frac{1}{x}}} \quad \text{for} \quad x > 0$$

**(b)** Compute the Wronskian of $y_1(x)$ and $y_2(x)$ and verify that the solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the differential equation for $x > 0$. [3 points]

**Solution:**

Given that $y_1(x) = \frac{1}{x}$, $y_2(x) = (C_1\ln|x| + C_2)\frac{1}{x}$, $y_1'(x) = -\frac{1}{x^2}$, and $y_2'(x) = C_1\frac{1}{x^2} - C_1\ln|x|\frac{1}{x^2} - C_2\frac{1}{x^2}$.

The Wronskian of $y_1(x)$ and $y_2(x)$ is given by:

$$
\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{x} & (C_1\ln|x| + C_2)\frac{1}{x} \\ -\frac{1}{x^2} & C_1\frac{1}{x^2} - C_1\ln|x|\frac{1}{x^2} - C_2\frac{1}{x^2} \end{vmatrix} \\
&= \frac{1}{x}\left(C_1\frac{1}{x^2} - C_1\ln|x|\frac{1}{x^2} - C_2\frac{1}{x^2}\right) - \left(-\frac{1}{x^2}\right)\left((C_1\ln|x| + C_2)\frac{1}{x}\right) \\
&= \frac{C_1}{x^3} - \frac{C_1\ln|x|}{x^3} - \frac{C_2}{x^3} + \frac{C_1\ln|x|}{x^3} + \frac{C_2}{x^3} \\
&= \frac{C_1}{x^3}
\end{aligned}
$$

Therefore, the Wronskian of $y_1(x)$ and $y_2(x)$ is $W(y_1, y_2) = \frac{C_1}{x^3}$.

Since the Wronskian is not equal to zero, the solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the differential equation for $x > 0$.

**(c)** Find the general solution of the differential equation for $x > 0$. [2 points]

**Solution:**

Since $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the differential equation for $x > 0$, the general solution of the differential equation is given by:

$$y(x) = C_1\frac{1}{x} + (C_2\ln|x| + C_3)\frac{1}{x} \quad \text{for} \quad x > 0$$

$-$ **End of the Midterm Exam** $-$