# Lecture 6

## Properties and Characterization of LTI Systems (Deduction)

(Ref: Chapter 2 O&W)

I. Commutative, Distributive and Associative Properties of Convolution and LTI systems

II. Characterizing LTI system by Impulse Response

III. Convolution Sum/Integral Example

# I. Properties of Convolution/LTI Systems

- First, we will show that convolution which describes the input/output relationship of LTI systems is *commutative*, *distributive*, and *associative*.
- These properties imply that we can arrange LTI systems identically in many different ways.

# 1. Commutative Property

Convolution is ***commutative*** ; i.e., “$x$ convolving with $h$” is the same as “$h$ convolving with $x$”

$$x(t) * h(t) = h(t) * x(t); \qquad x[n] * h[n] = h[n] * x[n]$$

When convolving, order of the two operands does not matter!

***Proof*** **(for CT)**:

1. let $\tau' = t - \tau;\;\; d\tau = -d\tau'$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad = -\int_{\infty}^{-\infty} x(t-\tau')h(\tau')d\tau'$$

2. Reverse limits of integration and cancel minus sign

$$= \int_{-\infty}^{\infty} x(t-\tau')h(\tau')d\tau' \qquad = h(t) * x(t)$$

$\tau'$ is just a dummy variable

# Commutative Property – Couponed Bond Revisited

- The commutative property should be quite obvious from our couponed bond example.

We can compute the payout in 2019 by enumerating over the years of bond purchases:

$$y[n] = \sum_k x[k]h[n-k]$$

$\sum_k (\cdot)$ means summing over all relevant $k$'s

Here $k$ is the year of the input

$$y[2019] = \underbrace{x[2015]h[2019-2015]}_{\text{input in 2015}} + \underbrace{x[2016]h[2019-2016]}_{\text{input in 2016}} + \underbrace{x[2017]h[2019-2017]}_{\text{input in 2017}}$$

$$+ \underbrace{x[2018]h[2019-2018]}_{\text{input in 2018}} + \underbrace{x[2019]h[2019-2010]}_{\text{input in 2019}} \underbrace{+\ldots}_{\text{and so on}}$$

Or we can enumerate over how many years ago an investment was made:

$$y[n] = \sum_k h[k]x[n-k]$$

Here $k$ is the number of years since the bond purchase

$$y[2019] = \underbrace{h[0]x[2019-0]}_{\text{input 0 year ago}} + \underbrace{h[1]x[2019-1]}_{\text{input 1 year ago}} + \underbrace{h[2]x[2019-2]}_{\text{input 2 years ago}}$$

$$+ \underbrace{h[3]x[2019-3]}_{\text{input 3 years ago}} + \underbrace{h[4]x[2019-4]}_{\text{input 4 years ago}} \underbrace{+\ldots}_{\text{and so on}}$$

The commutative property means that the followings are equivalent:

$$x(t) \longrightarrow \boxed{\text{System } h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$h(t) \longrightarrow \boxed{\text{System } x(t)} \longrightarrow y(t) = h(t) * x(t)$$

We can look at signals as systems and systems as signals. Signals and systems are *interchangeable*.

From now on, we will also say "$x$ convolving with $h$" or "$h$ convolving with $x$" *interchangeably* .

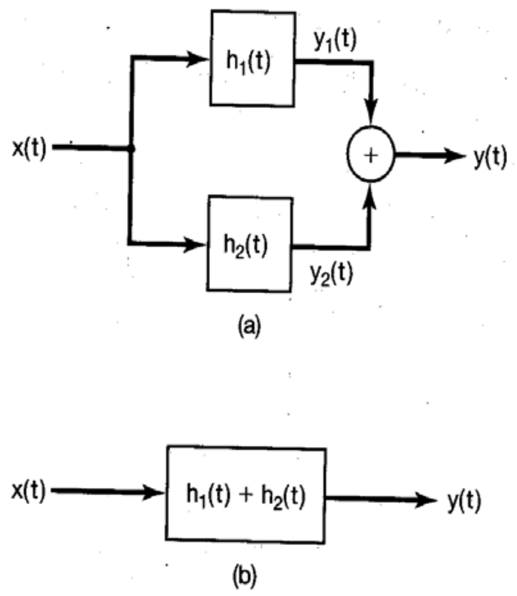
# 2. Distributive Property

Convolution is ***distributive***, which means:

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

*This implies we can add many systems in parallel into one or break one system into many in parallel.*




*add two systems in parallel into one or break one system into two in parallel*

**Figure 2.23** distributive prop for a parallel int systems.

Proof is trivial because *integration/summation is distributive*:

***Proof:***

$x(t) * \{h_1(t) + h_2(t)\}$ Convolution with a sum

$$= \int_{-\infty}^{\infty} x(\tau)\{h_1(t-\tau) + h_2(t-\tau)\}d\tau$$

=Multiply-and-integral of a sum

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau$$

=Sum of the individual multiply-and-integrals
=Sum of the individual convolutions

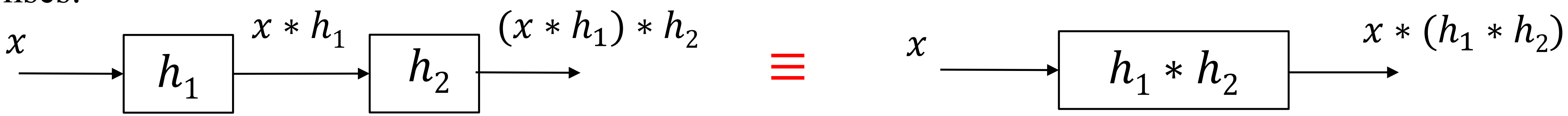
# 3. Associative Property

When we convolve multiple signals in sequence, the order of association does not matter: it does not matter which signal we convolve with which signal first.

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\} = x(t) * h_1(t) * h_2(t)$$

$$\underbrace{\{x[n] * h_1[n]\} * h_2[n]}_{x \text{ first convolving with } h_1\text{, then } h_2.} = \underbrace{x[n] * \{h_1[n] * h_2[n]\}}_{x \text{ convolving with } h_1 \text{ convolved with } h_2 \text{ first}} = \underbrace{x[n] * h_1[n] * h_2[n]}_{\text{No need to specify which convolve with which first}}$$

This means we can combine two systems in cascade into one by convolving the individual impulse responses:




The above is obviously true.
Imagine you have a black box containing two LTI systems in series inside. Ask yourself two questions:
1 - Is the black box LTI?
2 - If yes, what is its impulse response?
For algebraic proof, see Reference 6.1 at the end.

With the commutative and associative properties, it means that when cascading two LTI systems, the order of the cascade does not matter

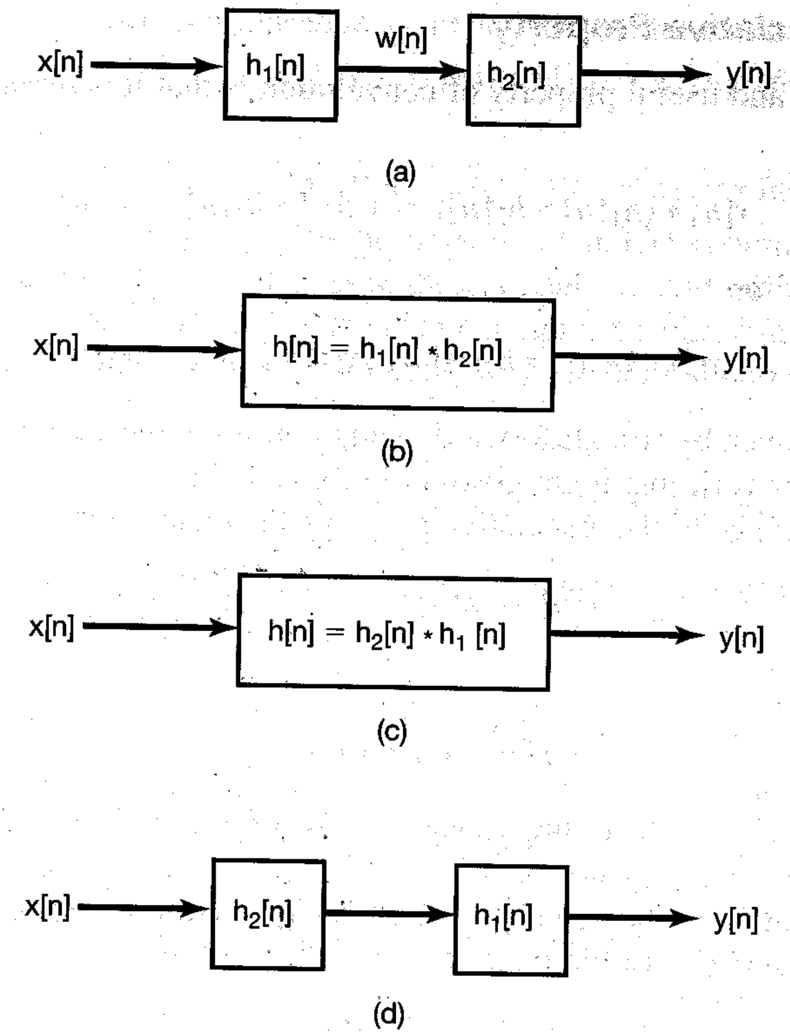



**Figure 2.25** convolution and and the commu series interconn

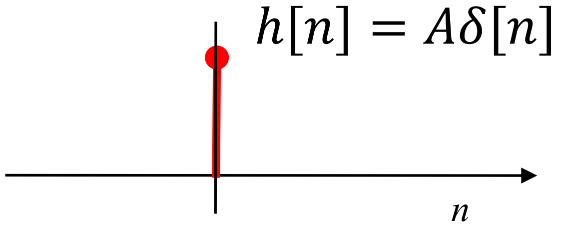
Processing the signal with system $h_1$ first followed by $h_2$ is the same as processing it with $h_2$ first followed by $h_1$

# II. Characterizing LTI system by Impulse Response

## 1. Memoryless/Has Memory

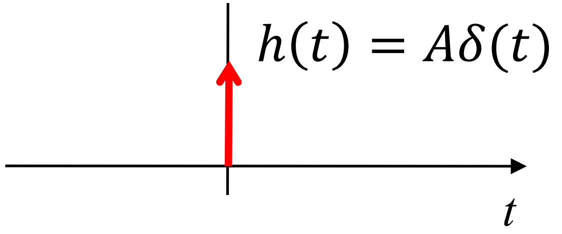
- An LTI system is **memoryless** iff $h[n] = 0$ for $n \neq 0$; $h(t) = 0$ for $t \neq 0$
- This means the only memoryless LTI system is an amplifier, with impulse response being a scaled impulse at time zero: $h[n] = A\delta[n], h(t) = A\delta(t)$.

Memoryless DT & CT Response




$h[n-k]$ nonzero only for $k = n$

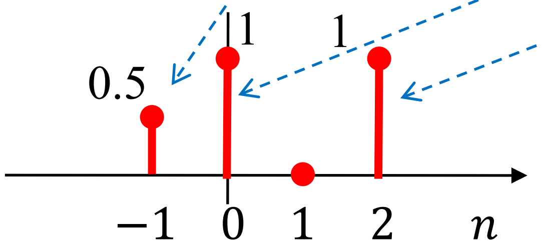
$$y[n] = \sum_k x[k]h[n-k] = x[n]h[0] = Ax[n]$$



# Example Impulse Responses with Memory

One way to specify a DT impulse response is to list its values at different times

I. $h[n] \;\; = \underline{0.5\delta[n+1]} + \underline{\delta[n]} + \underline{\delta[n-2]}$



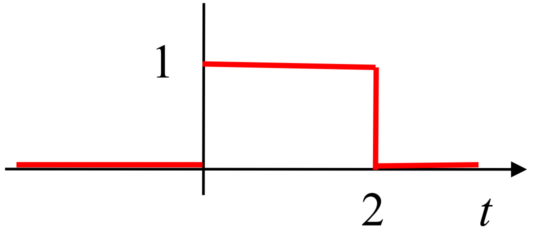

$\Rightarrow y[n] = 0.5x[n+1] + x[n] + x[n-2]$

Input at 1 time unit into future

Input at 2 time units in the past

$h(t)$ is our familiar window integrator

II. $h(t) = u(t) - u(t-2)$ $\quad \Rightarrow \; y(t) = \int_{t-2}^{t} x(\tau)d\tau$




integrates input over the preceding 2 time units

## 2. Causality

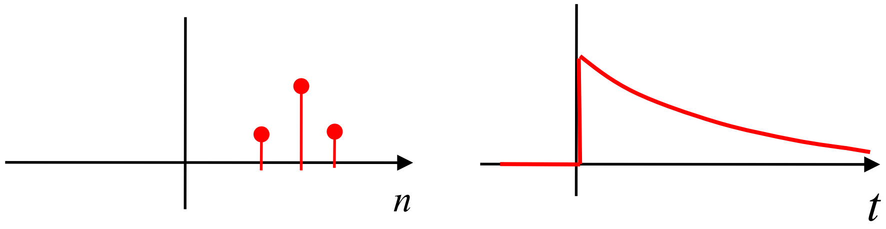
- An LTI system is **causal** iff impulse response is zero for all time less than zero:
  $h[n] = 0$ for $n < 0$ ; $h(t) = 0$ for $t < 0$.

Consider CT case $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

For system to be causal, $y(t)$ cannot depend $x(\tau)$ with $\tau > t$

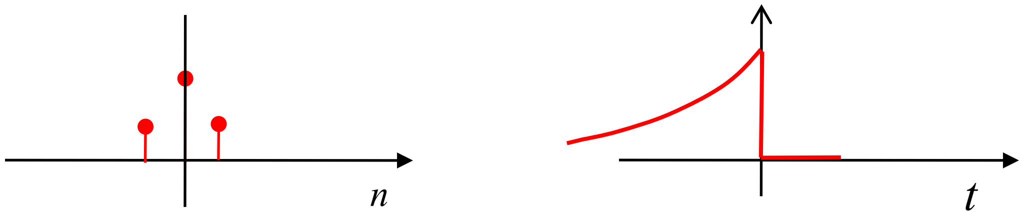
This means $h(t-\tau)$ must be 0 for all $\tau > t$, or $h(\cdot) = 0$ whenever argument is $< 0$

Causal Responses




$h[n] = 0$ for $n < 0$ ; $h(t) = 0$ for $t < 0$

Non-Causal Responses



## 3. Stability

An LTI system is BIBO **stable** iff impulse response is ***absolute integrable*** for CT or ***absolute summable*** for DT, meaning:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \; ; \qquad \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$|\cdot|$ means absolute value or magnitude

Recall that BIBO (Bounded Input Bounded Output stability) means:

if $|x(t)| \leq B < \infty \; \forall t$ then $|y(t)| \leq B' < \infty \; \forall t$

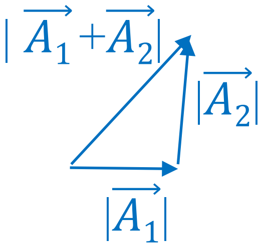
***Proof:***

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|d\tau$$

$$= \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau$$

$$\leq B \int_{-\infty}^{\infty} |h(\tau)|d\tau \; < \infty$$

if $h(t)$ is absolute integrable



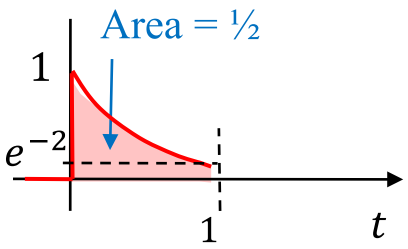

Schwartz Inequality: absolute value/magnitude of sum is no greater than sum of absolute values/magnitudes: $|A_1+A_2| \leq |A_1|+|A_2|$ or $|\int A| \leq \int |A|$

$|A_1 A_2| = |A_1||A_2|$
Magnitude of product = product of magnitudes

$|x(t)| \leq B$

**Examples:**

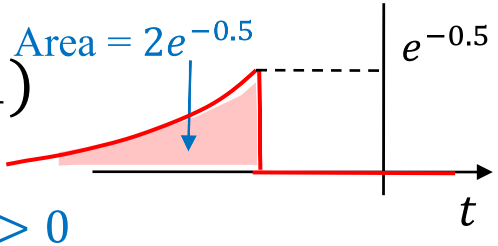
$$h_1(t) = e^{-2t}u(t)$$




casual exponential

$$\int_{-\infty}^{\infty} |h_1(t)|\, dt = \int_{-\infty}^{\infty} e^{-2t}u(t)\, dt = \frac{1}{2}$$

$$h_2(t) = e^{0.5t}u(-t-1)$$

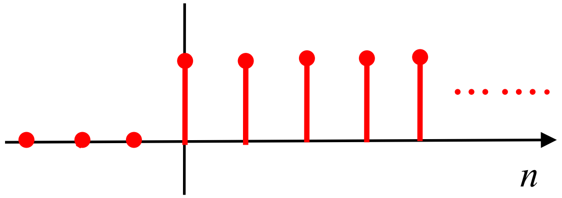



Left-sided exponential

Anti-casual exponential: $= 0\ \forall t > 0$

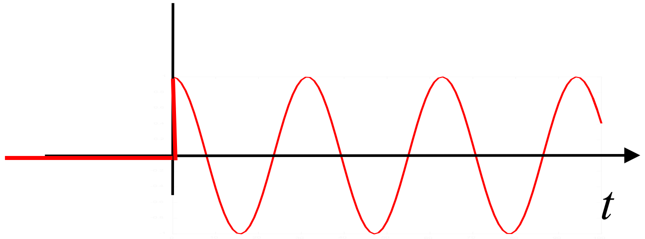
$$\int_{-\infty}^{\infty} e^{0.5t}u(-t-1)\, dt = 2e^{-0.5}$$

$$h_3[n] = u[n]$$




Unit step is not absolute summable

$$h_4(t) = \cos(t)u(t)$$




$|\cos(t)|$ integrates to non-zero over one period.
Therefore $h_4(t)$ is not absolute integrable.

# Unit step response of LTI systems

- If input to an LTI system is the unit step, the output, the unit step response, is the integral/first sum of the impulse response.

**CT** $$u(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$ Integral of impulse response

**DT** $$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n] = \sum_{k=-\infty}^{n} h[k]$$ First sum of impulse response

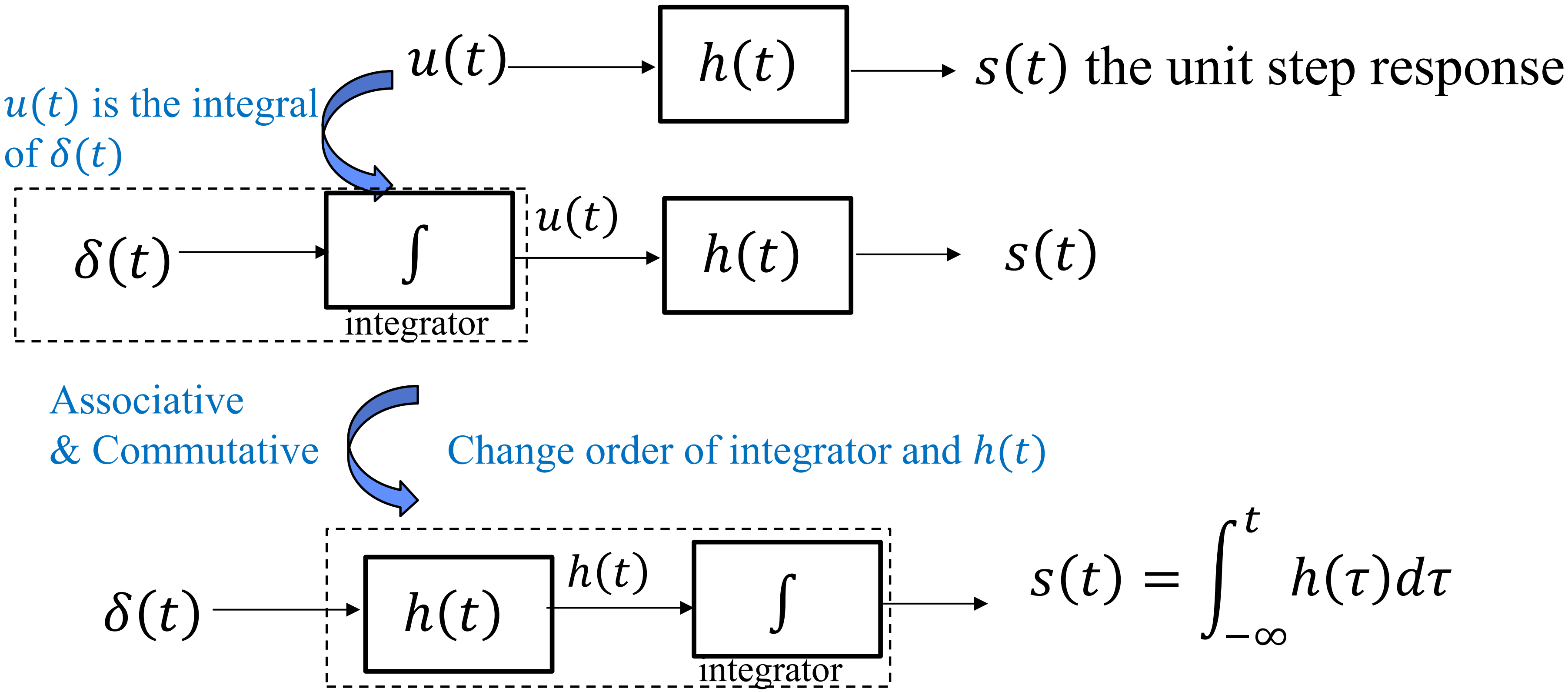
This is because convolving with unit step results in integration/summation:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

# Unit step response from Properties of LTI

We can also deduce the unit step response in the following way:

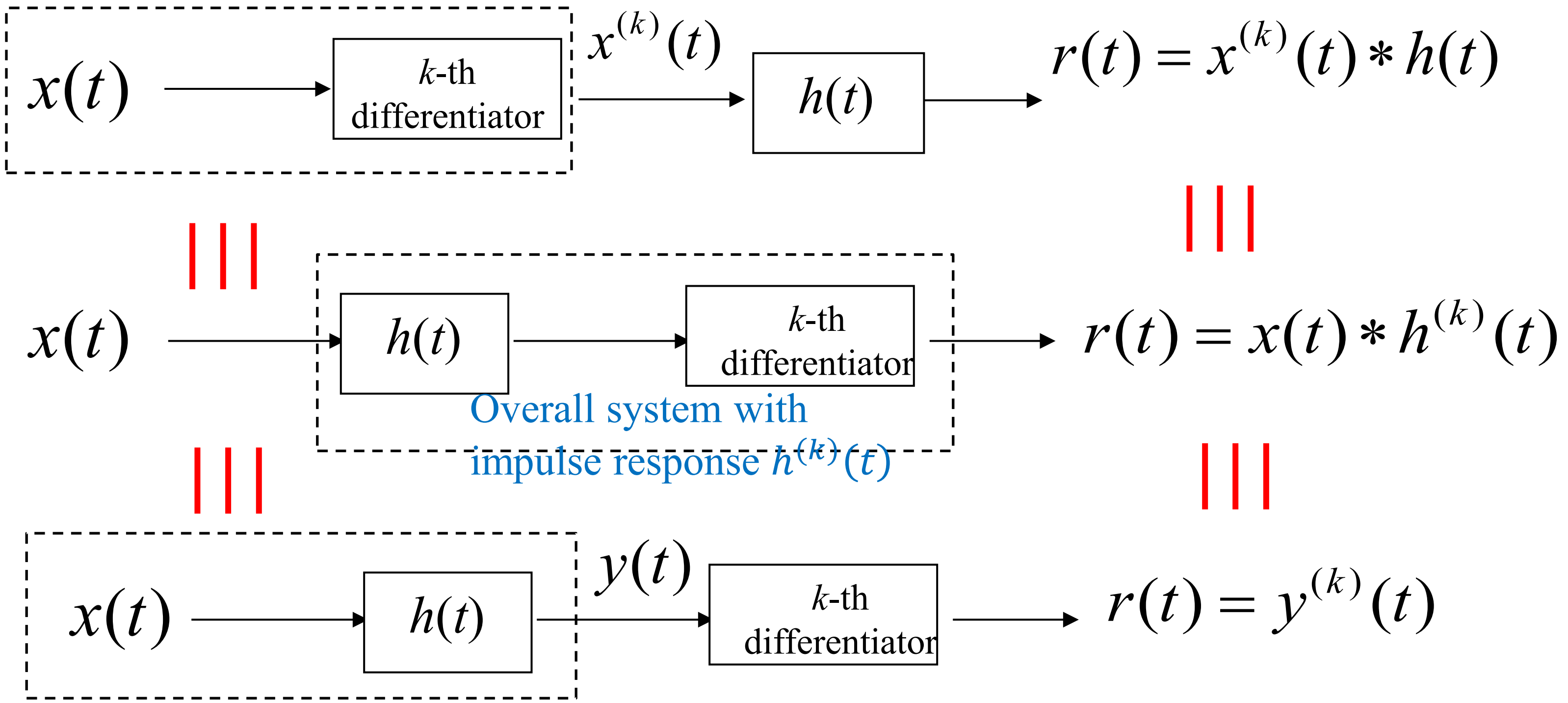



$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Impulse response of this cascaded system is the integral of $h(t)$

# Response to derivative/integral of input

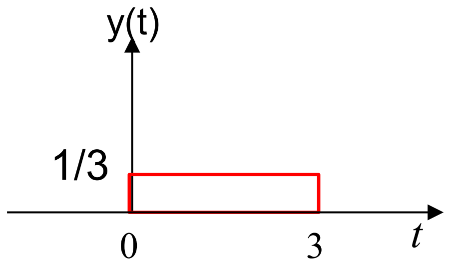
We can replace the input in the previous slide by an arbitrary input $x(t)$, and replace the integrator by any $k$-th order differentiator (negative $k$ means integrator), since differentiator is LTI.




$$y(t) = x(t) * h(t) \Rightarrow$$
$$\frac{dy(t)}{dt} = \left(\frac{dx(t)}{dt}\right) * h(t)$$
$$= x(t) * \left(\frac{dh(t)}{dt}\right)$$

We can apply the differentiator (integrate) anywhere and obtain the same final result – we can differentiate the input, the impulse response, or the output!

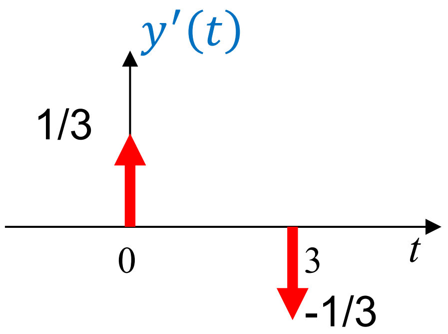
**Example**. $y(t)$ is the output of an LTI system when the input is $x(t)$:




Sketch the output when the input is

$x^{(1)}(t)$ or $x'(t)$, $\frac{dx(t)}{dt}$

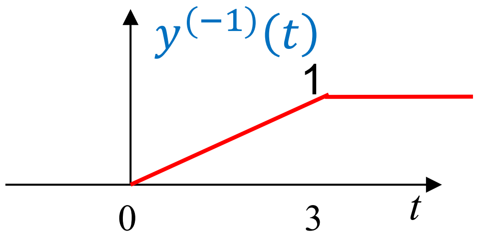
$\Rightarrow$ Output is derivative of $y(t)$

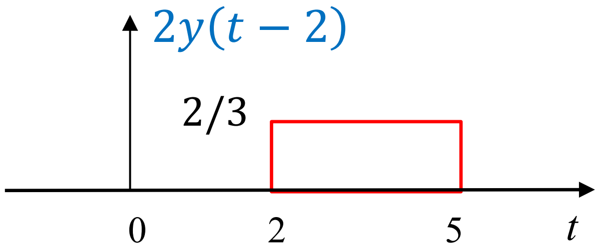


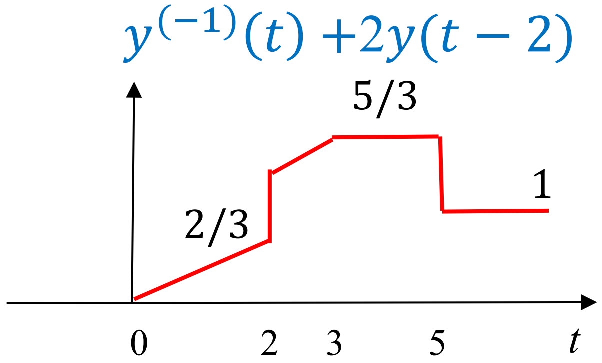

Sketch the output when the input is

$x^{(-1)}(t) + 2x(t-2)$ or $\int_{-\infty}^{t} x(\tau)d\tau + 2x(t-2)$

$\Rightarrow$ Output is $y^{(-1)}(t) + 2y(t-2)$



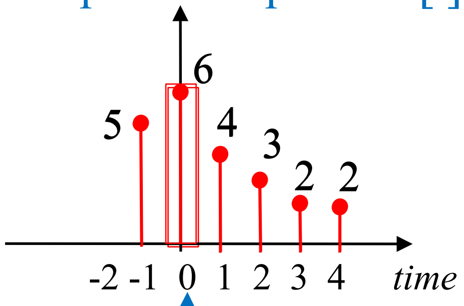



# III. Convolution sum & Integral Examples

- The convolution sum for DT is $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- Computing it conceptually involves the following steps:
  - ✓ Step 1: draw $x[k]$ and $h[k]$ by replacing "$n$" by "$k$"   $k$: variable of summation
  - ✓ Step 2: flip $h[k]$ to get $h[-k]$; $h[-k]$ is called the *convolution kernel*   kernel: essence, core
  - ✓ Step 3: Shift $h[-k]$ by $n$ to obtain $h[n-k]$; i.e., recognize $h[0]$ as anchor of kernel and place it at $n$.
  - ✓ Step 4: Multiple $x[k]$ by $h[n-k]$ and sum over all $k$.

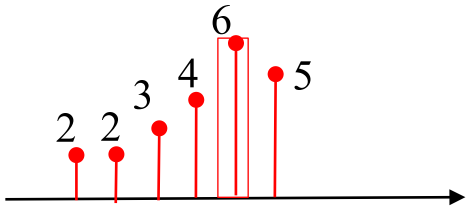
Visualizing the Impulse Response and Convolution kernel

An Impulse Response $h[\cdot]$




$h[0]$: kernel anchor

Corresponding Convolution Kernel




Kernel as a sequence of values

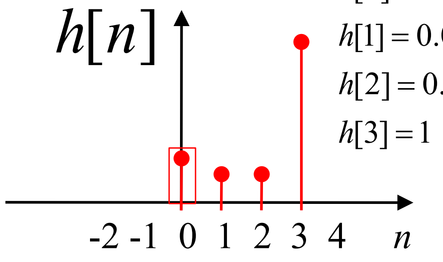
| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 4 | $\boxed{6}$ | 5 |

$h[0]$
kernel anchor

# Couponed Bond Example Revisited

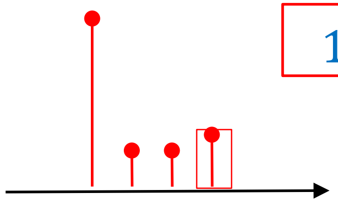
For our couponed bond example, the impulse response and kernel are as shown below:

Impulse Response




coupon bond example:
$h[0]=0.1$
$h[1]=0.08$
$h[2]=0.08$
$h[3]=1$

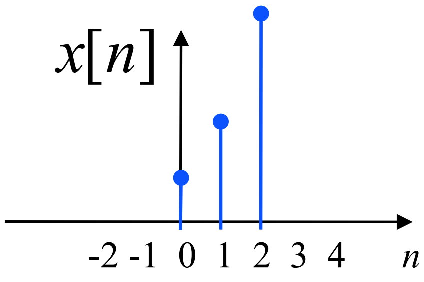
Convolution Kernel



Kernel as a sequence of values

| | | | |
|---|---|---|---|
| 1 | .08 | .08 | $\boxed{.1}$ |

$h[0]$



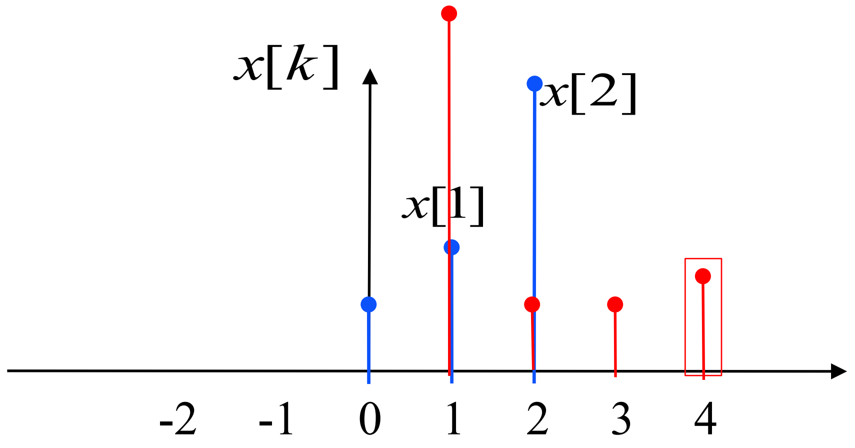

$x[n]$ Input as a sequence of values

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| … … | 0 | 0 | $10 | $20 | $40 | 0 | 0 | … … |

$x[0]$
Year 2015

What is the output at year 2019 ($n = 4$)?

We multiple $x[k]$ with $h[4 - k]$ and add up the sum



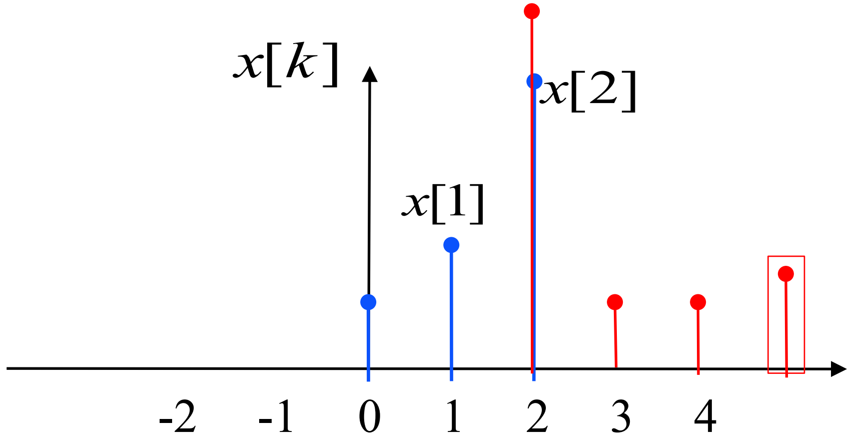

We anchor the kernel at year 2019 and multiply and add:

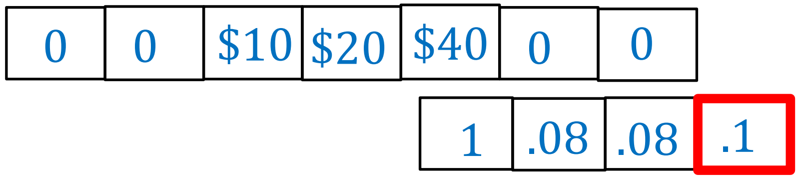
| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | \$10 | \$20 | \$40 | 0 | 0 |
| | | | 1 | .08 | .08 | $\boxed{.1}$ |

$$y[4] = x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0] = \$20 \times 1 + \$40 \times 0.08 = \$23.2$$

What is the output at year 2020 ($n = 5$)?

We shift $h[-k]$ to $h[5-k]$




We anchor the kernel at year 2020:

$$y[5] = x[2]h[3] = \$40 \times 1 = \$40$$

# Convolution Kernel for Image Processing

- In image processing, impulse responses are 2-dimensional. They are also often symmetrical (even), so the convolution kernel is the same as the impulse response. Hence, in image processing, we often describe systems by their kernels in form of 2-D matrices.

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & \boxed{8} & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

Edge Detector Kernel

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & \boxed{4} & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

3x3 Gaussian Blur Kernel

# Doing the Convolution Integral

We follow a similar procedure but have to integrate instead of sum.

Sometimes the math can look messy when we try to come up with closed-form mathematical expression for our results!

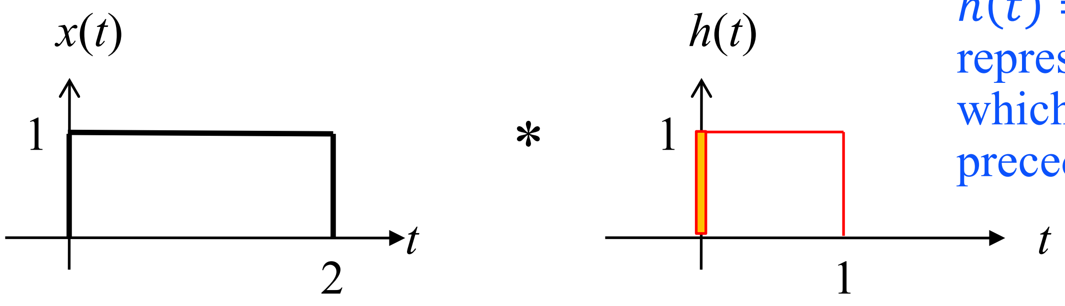
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- ✓ step 1 : draw $x(\tau)$ and $h(\tau)$ by replacing '$t$' by '$\tau$' — $\tau$: variable of integration
- ✓ step 2 : flip $h(\tau)$ to get the kernel $h(-\tau)$ — kernel is time reversal of impulse response
- ✓ step 3 : shift the kernel $h(-\tau)$ by $t$ to get $h(t-\tau)$ — anchor kernel at $t$
- ✓ step 4 : multiply input with shifted kernel and compute the integral

Let's go through some simple examples:

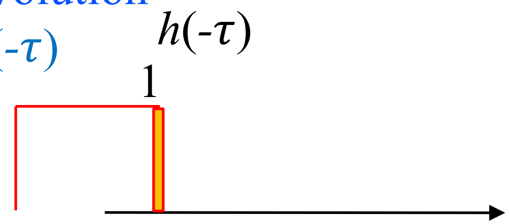
# CT Example 1 – Convolution of Two Windows

Convolve the two window/pulse signals below:



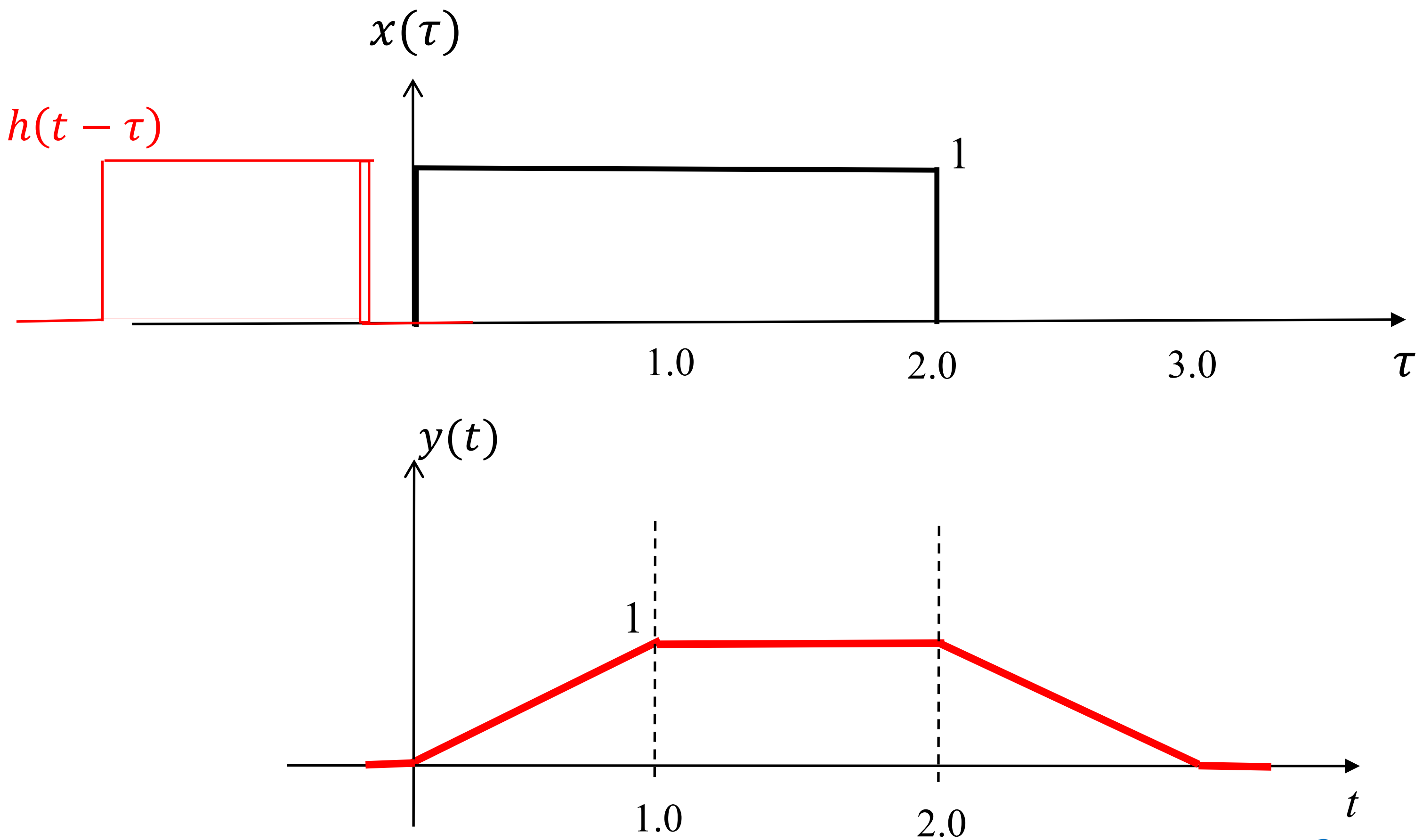

$h(t) = u(t) - u(t-1)$ here represents a window integrator, which integrates $x(t)$ over the preceding 1 time unit interval;

The Convolution Kernel $h(-\tau)$




We anchor the kernel at $t$, multiply with $x(\tau)$, and integrate the product $x(\tau)h(t-\tau)$ over all time.

The animation below demonstrates how the output, given by the area of the product, changes when we vary $t$. The area is indicated by the region shaded red.




$t < 0$: $y(t) = 0$

$0 \le t < 1$: $y(t) = t$

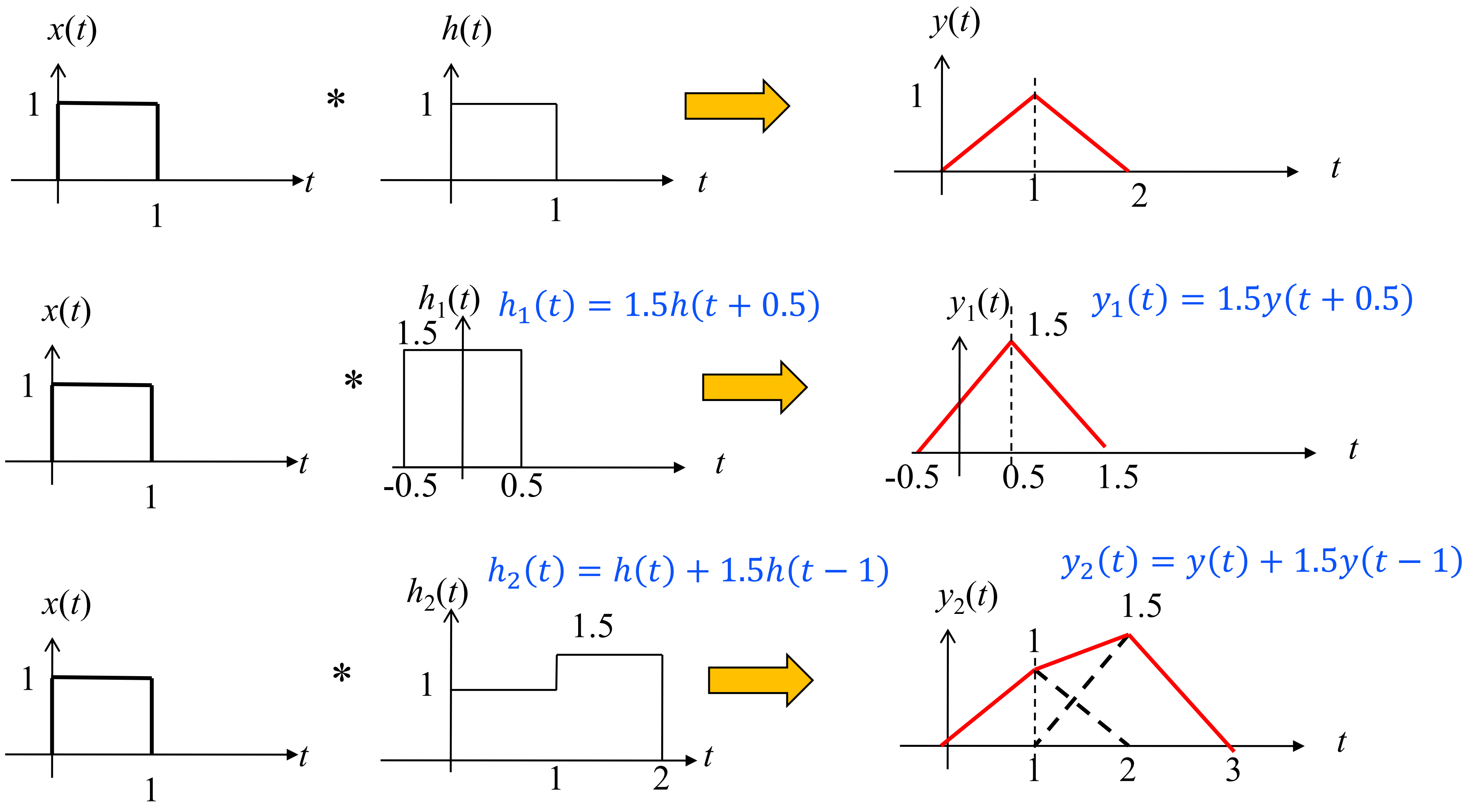
$1 \le t < 2$: $y(t) = 1$

$2 \le t < 3$: $y(t) = 3 - t$

$t \ge 3$: $y(t) = 0$

$$y(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

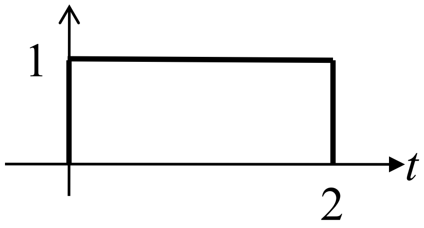
# Variations on example:



# CT Example 2 – Convolution Window with Causal Exponential

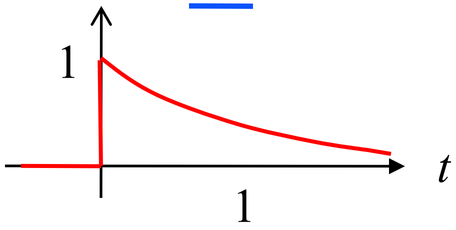
Convolve window with a casual exponential:

$x(t)$



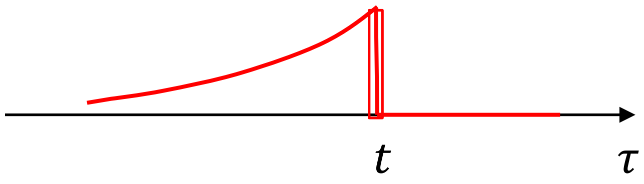

$*$

$h(t) = e^{-t}u(t)$




The convolution kernel is:

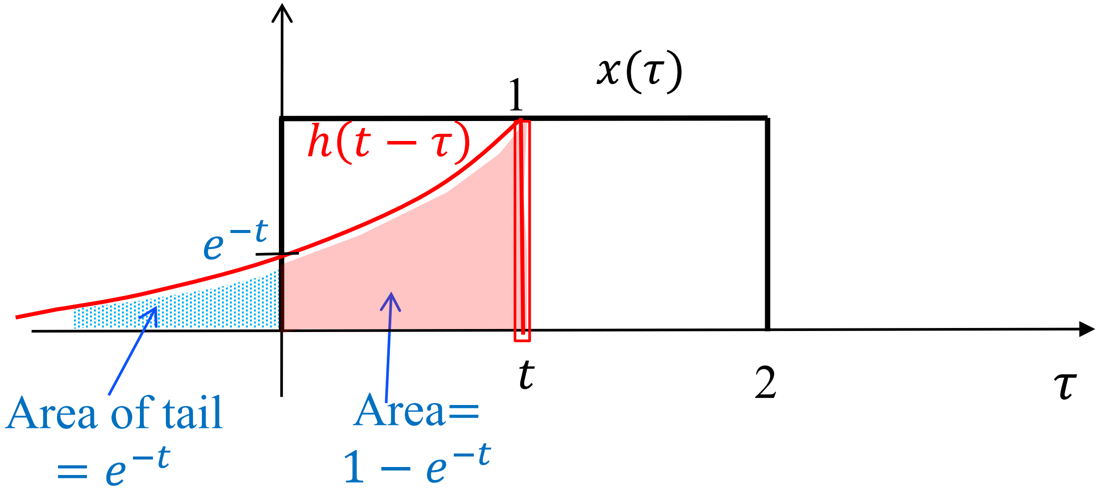
$h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$



Anchor kernel at $t$:

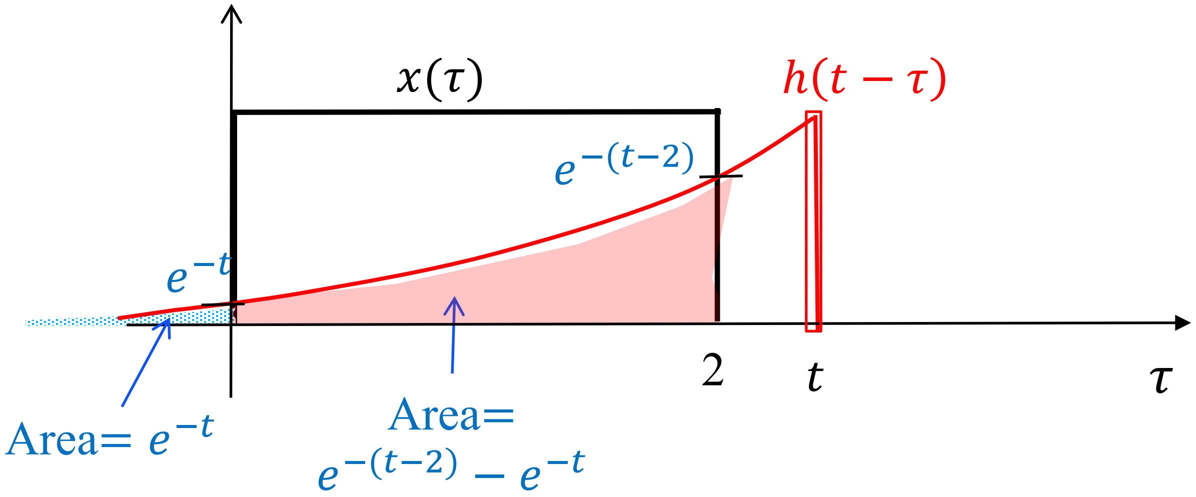
For $0 \leq t < 2$:

$$y(t) = 1 - e^{-t}$$



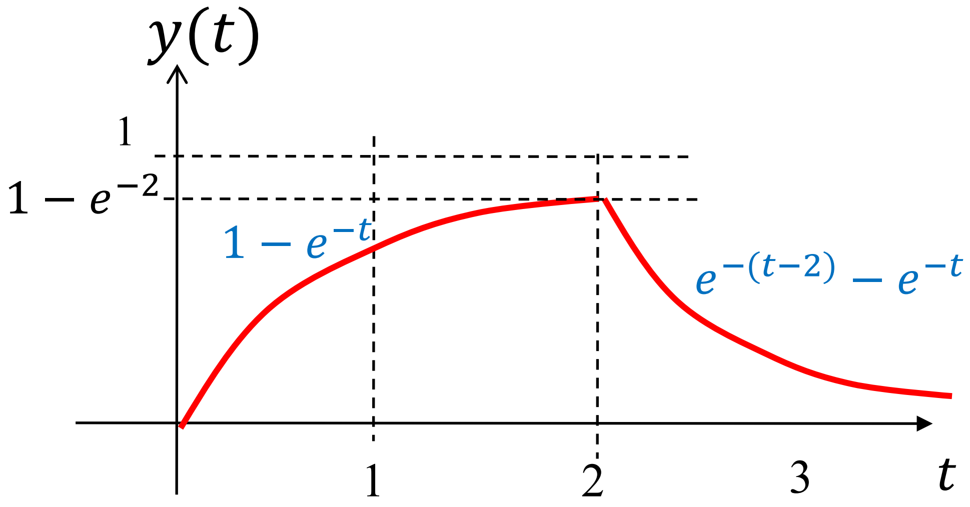

For $t \geq 2$:

$$y(t) = e^{-(t-2)} - e^{-t}$$



Output $y(t)$ in general form:

$$y(t)=\begin{cases} 0 & t<0 \\ 1-e^{-t} & 0\le t<2 \\ e^{-(t-2)}-e^{-t} & t>2 \end{cases}$$



# Self-Test – Lecture 6

1. State the commutative, distributive, and associative property of convolution.

2. Draw an impulse response that is non-causal

3. What can we say about the impulse response of an LTI system that is BIBO stable?

**Reference 6.1** From Problem 2.43, an algebraic proof of the associative property for CT Case:

- $x$ convolve with "$h$ convolve with $g$ first":

$$x(t) * \{h(t) * g(t)\} = x(t) * \Bigg\{ \underbrace{\int_{\tau=-\infty}^{\infty} h(t-\tau)g(\tau)d\tau}_{g(t) * h(t);\ \text{regard as } r(t)} \Bigg\} = \underbrace{\int_{\sigma=-\infty}^{\infty} x(\sigma) \Bigg( \underbrace{\int_{\tau=-\infty}^{\infty} h(t-\sigma-\tau)g(\tau)d\tau}_{r(t-\sigma)} \Bigg) d\sigma}_{x(t) * r(t)}$$

Use $\sigma$ because we already used $\tau$

$$= \int_{\sigma=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\sigma)h(t-\sigma-\tau)g(\tau)d\tau d\sigma$$

- $x$ convolve with $h$ first, and then with $g$:

$$\{x(t) * h(t)\} * g(t) = \Bigg\{ \underbrace{\int_{\sigma-\infty}^{\infty} x(\sigma)h(t-\sigma)d\sigma}_{x(t) * h(t) \text{ using } \sigma \text{ as variable of integration; regard as } s(t)} \Bigg\} * g(t)$$

$$= \underbrace{\int_{\tau=-\infty}^{\infty} \Bigg( \underbrace{\int_{\sigma=-\infty}^{\infty} x(\sigma)h(t-\sigma-\tau)d\sigma}_{s(t-\tau)} \Bigg) g(\tau)d\tau}_{s(t) * g(t)} = \int_{\sigma=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\sigma)h(t-\sigma-\tau)g(\tau)d\tau d\sigma$$

Same as above!