Lecture 2
Basic Characterization and Manipulation of Signals

(Ref: Chapter 1 O&W)

I. Basic Manipulation of Signals (Language – Math, notation)

II. Some Characterization of Signals: 1. Even/Odd Signals, 2. Finite Duration, Infinite Duration, Right-Sided and Left-Sided Signals

III. Periodic Signals and Poisson Sum

IV. Some example signals: exponential, sinusoidal, and unit step

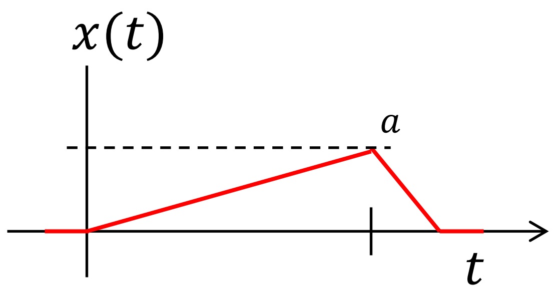
I. Basic Manipulation of Signals

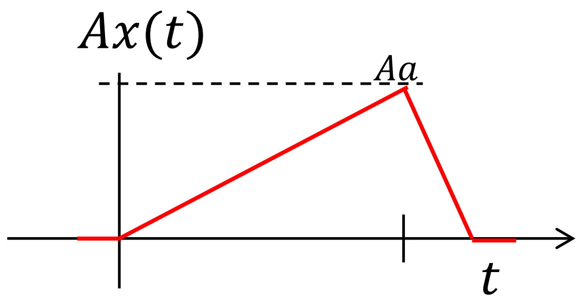
CT and DT signals $x(t), x[n]$, are just functions of time.

The most basic way that we can manipulate a signal is to multiply it by a constant – **Amplification/Scaling**

1. ***Multiplying signal by constant***

Multiplying a signal by a constant A means multiplying each value of the signal by A.

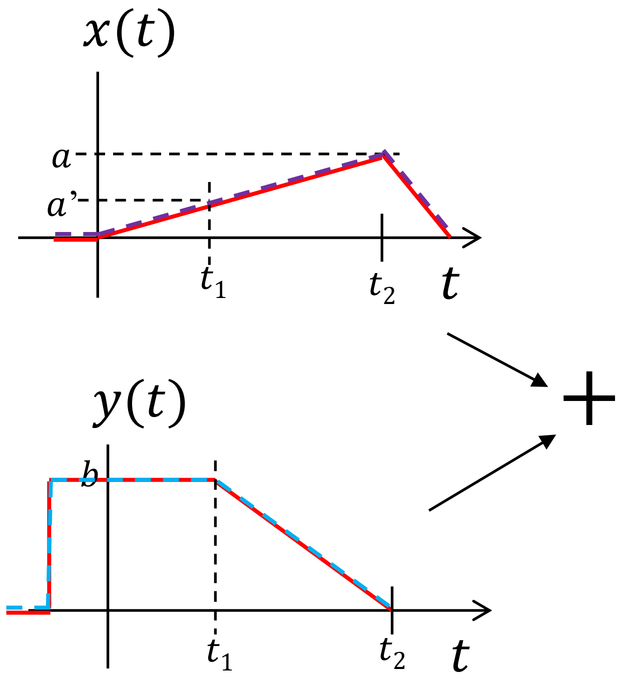




We can also add or multiply two signals together.

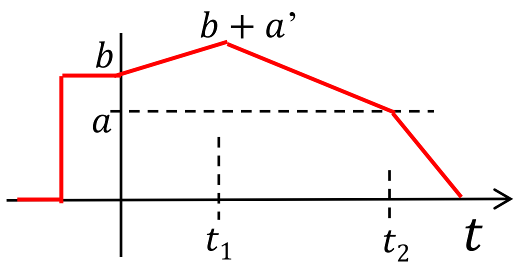
2. *Adding/multiplying signals together*

Adding/multiplying two signals just means adding/multiplying the values of the two signals time instant by time instant



Adding two signals

$x(t) + y(t)$



Note that adding two straight line segments produces a straight line segment

Mathematical representation of a straight line:

$$x(t) = \alpha_1 + \beta_1 t$$

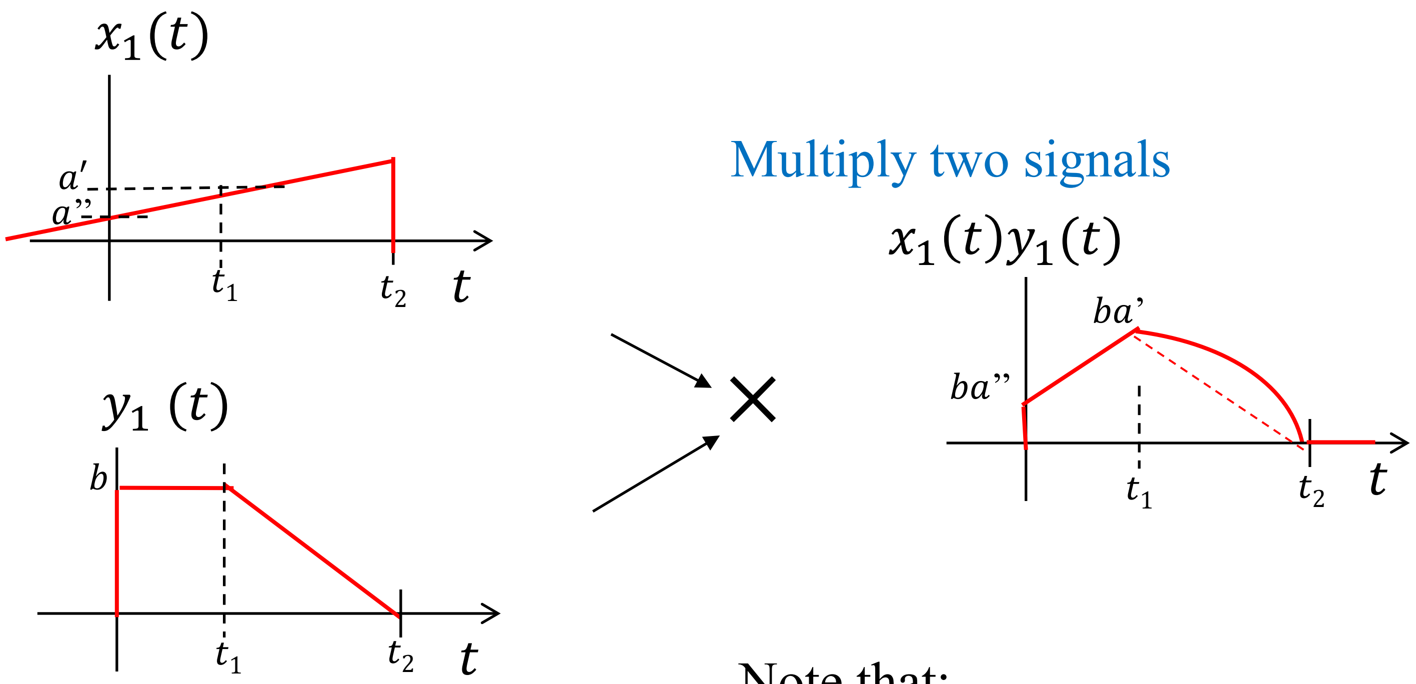
intercept   slope

If $y(t) = \alpha_2 + \beta_2 t$, then

$$x(t) + y(t) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)t$$

which is also a straight line

Multiplying:



Note that:

- Multiplying by zero gives zero (*gating*)
- Multiplying by a constant scales a curve
- Multiplying two straight line segments produces a quadratic curve

Transformation of the Independent Variable

Often, we need to manipulate signals in time :

3a. ***Time Shifting***

3b. ***Time Reflection/Reversal***

3c. ***Time Scaling***

3a. Time shifting: CT: $x(t - t_0)$, or DT: $x[n - n_0]$,

Example: Let $y[n] = x[n - 3]$.

y is x delayed by 3 time units:

$$y[3] = x[0]$$

$$y[5] = x[2], \text{ etc.}$$

delay

Value of x will show up in y 3 time units later

Conversely, if $y[n] = x[n + 4]$, y is x advanced by 4 time units

We usually express time shift in the form of $x[n - n_0]$ because usually we think of time delay.

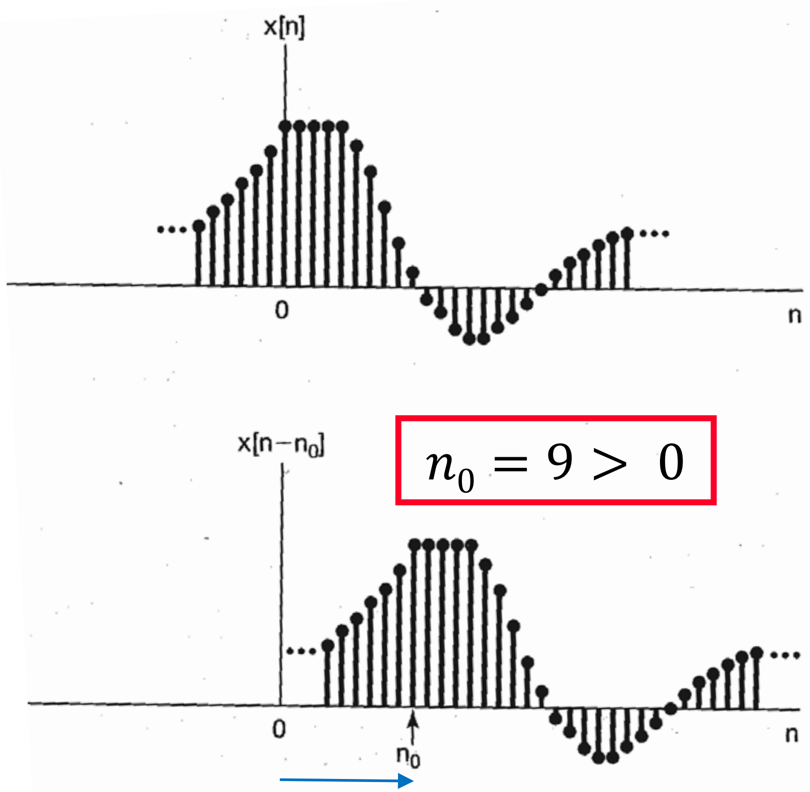
When we delay a signal, we shift it in time to the right. If we time advance a signal, we shift it to the left.

Minus **– shift to the right =** ***time delay***

e.g.: $\boldsymbol{y[n] = x[n-9]}$

Plus **– shift to the left =** ***time advance***

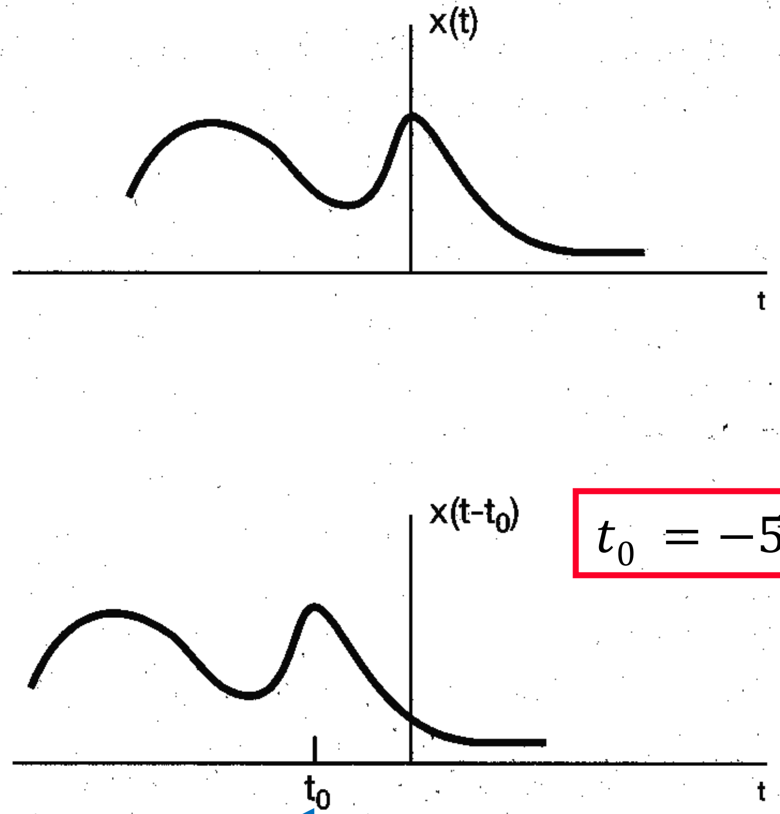
e.g.: $\boldsymbol{y(t) = x(t+5)}$



$n_0 = 9 > 0$

delay by n_0

Figure 1.8 D
related by a time
$n_0 > 0$, so that x
verson of $x[n]$ (i.
occurs later in x[



$t_0 = -5 < 0$

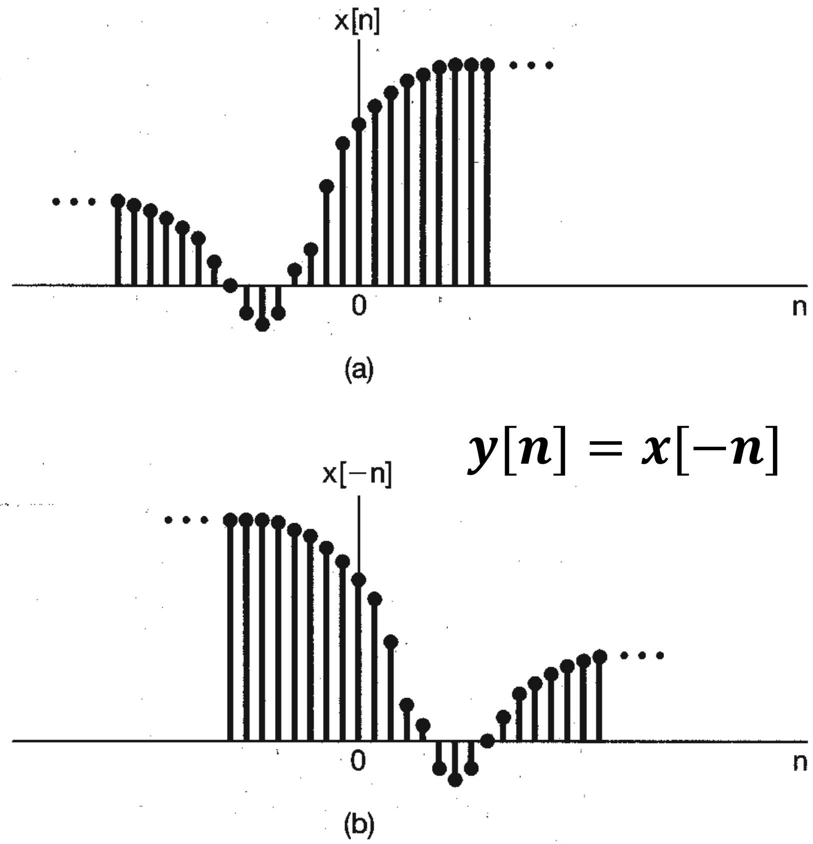
Time advance by t_0

Figure 1.9 Continuous-time signals related

3b. Time Reflection (or Time Reversal):

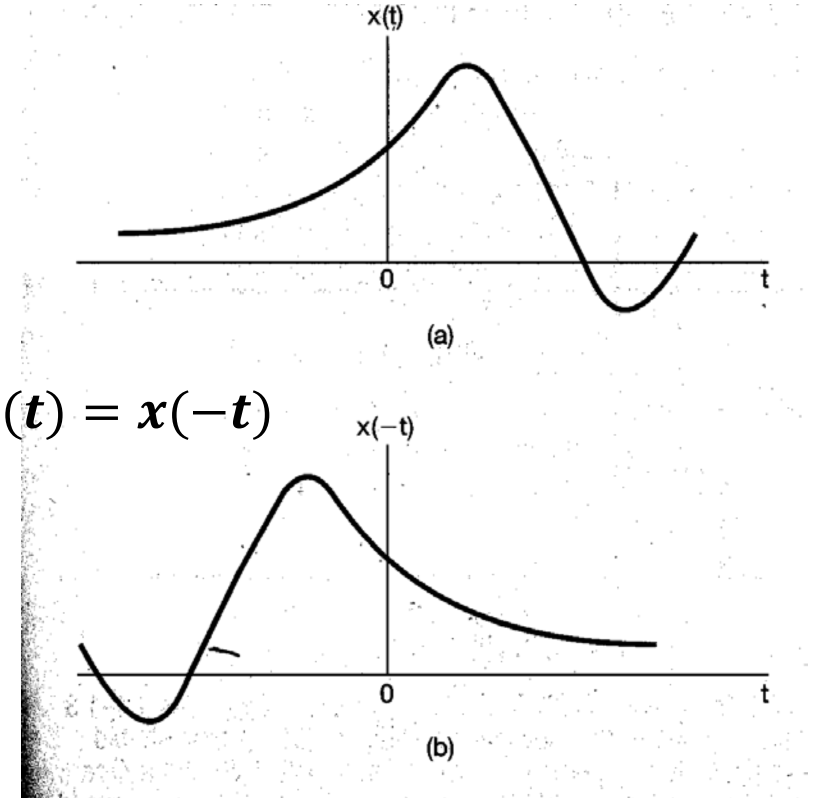
$$\textbf{DT: } y[n] = x[-n] \textbf{ , or CT: } y(t) = x(-t)$$

$y[n]$ is a reflection of $x[n]$ around the time origin.



$$y[n] = x[-n]$$

Figure 1.10 (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.



$$y(t) = x(-t)$$

Figure 1.11 (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.

3c. Time scaling $\quad y(t) = x(\alpha t)$

(For now we will consider time scaling *for CT only*. Time scaling for DT is important but not as straightforward)

$|\alpha| > 1$ means ***time compression***
$\Rightarrow$everything happening faster

$|\alpha| < 1$ means ***time dilation/expansion***
$\Rightarrow$everything happening more slowly

$\alpha < 0$ means in addition a ***time reversal***

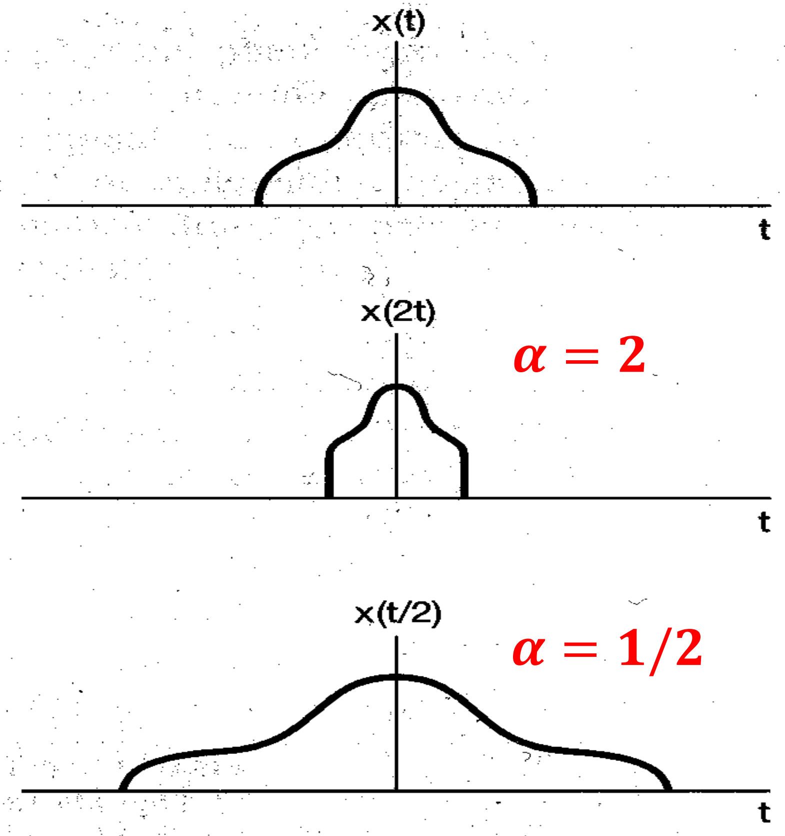


Figure 1.12 Continuous-time signals related by time scaling.

Combination of Shifting and Time Scaling

Example 1.1-1.3 For $x(t)$ below, plot $y_1(t) = x(-t+1)$ and $y_2(t) = x\left(\frac{3}{2}t+1\right)$.

$\cancel{}$ Time scaling Time shifting

For the expressed form of y_1 and y_2 above, we should *shift first* and then time reverse/scale:

x(-t)

x(-t+1)

⚠ Time shift must be the first step

Shift first and scale:

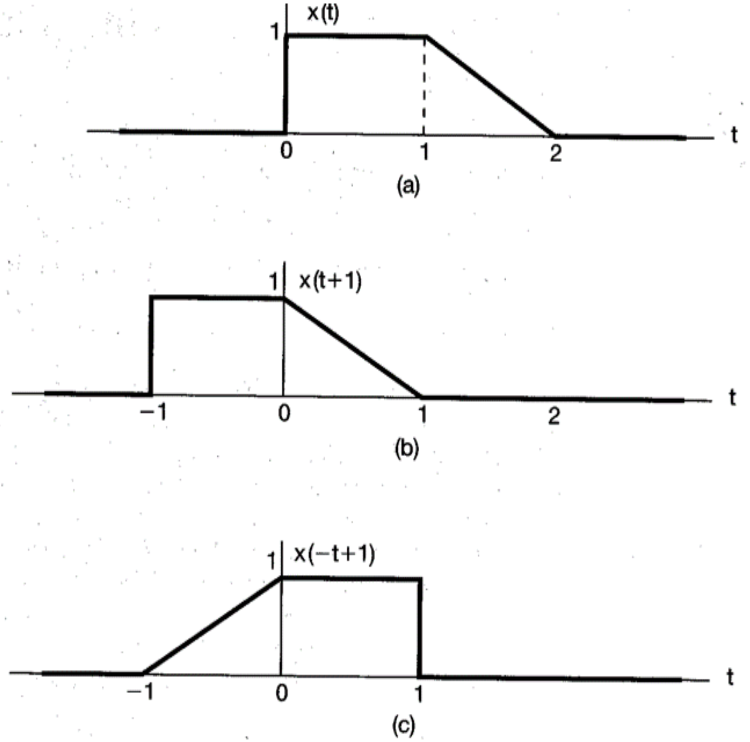


Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted

First, we shift $x(t)$ to left (time advance) by 1 to get an intermediate signal

$$g_1(t) = x(t+1)$$

Then, we time reverse g_1 to get y_1:

$$g_1(-t) = x(-t+1) = y_1(t)$$

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II) Some Characterization of Signals: 1. Even/Odd Signals, 2. Finite Duration, Infinite Duration, 3. Right-Sided and Left-Sided Signals (language – keywords)

III) Periodic Signals and Poisson Sum

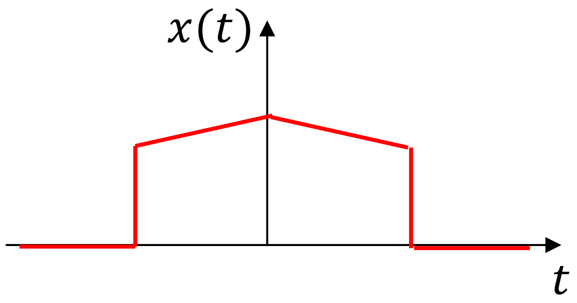
IV) Some example signals: exponential, sinusoidal, and unit step

1. Even and Odd Signals

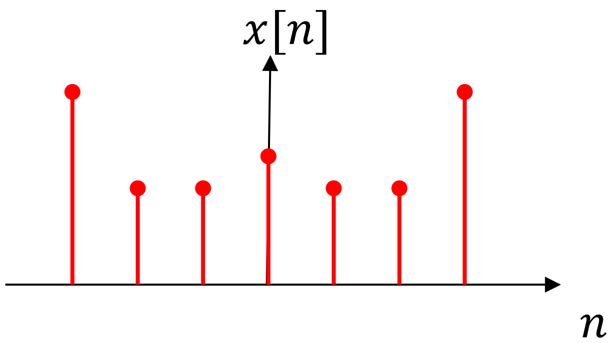
- **Even signal**: One that is unchanged under time reversal. Signal is even if:

$$\text{CT: } x(-t) = x(t) \qquad \text{or DT: } x[\underbrace{-n}_{\text{time reversal}}] = \underbrace{x[n]}_{\text{unchanged}}$$

A CT Even Signal



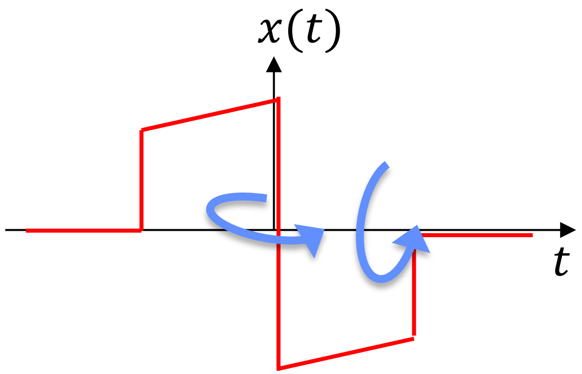
A DT Even Signal

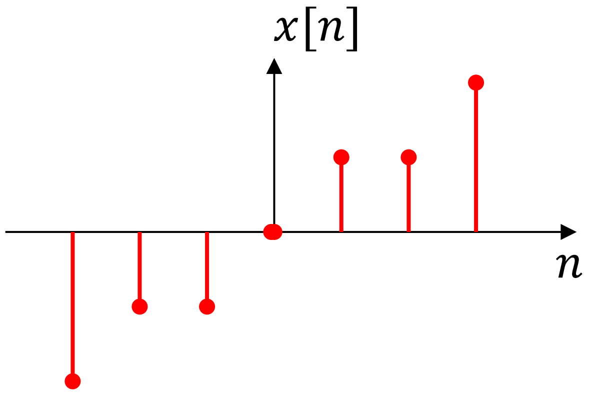


- **Odd signal**: One that is negated under time reversal. Signal is odd if:

$$x(-t) = -x(t) \text{ or } x[-n] = -x[n]$$

time reversal   negation





Any odd signal must be equal to zero at time zero,

since $x(-0) = -x(0)$ which means $x(0) = -x(0) \Rightarrow x(0) = 0$

1. neither

2. $x(t) = \begin{cases} 2(t-1) & \forall -1 < t < 1 \\ 4-(t-1)^2 & \forall 1 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$

$x(t) = \begin{cases} 2(t/2-1) & \forall -1/2 < t < 1/2 \\ 4-(t/2-1)^2 & \forall 1/2 \le t \le 3/2 \\ 0 & \text{otherwise} \end{cases}$

???

Decomposition of Signal into Even and Odd Parts

- Any signal can be viewed as the sum of an even part and an odd part:

$$x(t) = x_{even}(t) + x_{odd}(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\}$$

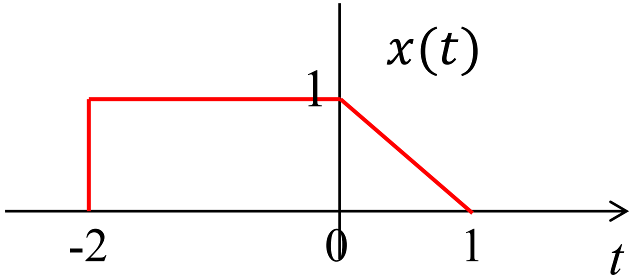
- We can find the even and odd parts by the *half-sum* and *half difference* of $x(t)$ and its time-reversed signal $x(-t)$

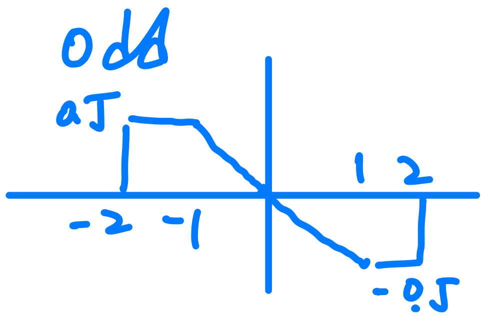
$$x_{even}(t) = \frac{x(t) + x(-t)}{2} \qquad x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

half-sum *half-difference*

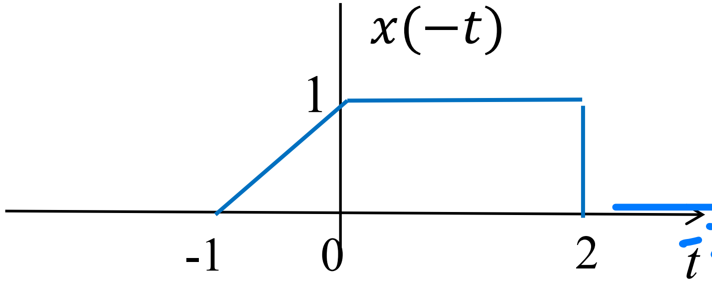
Example

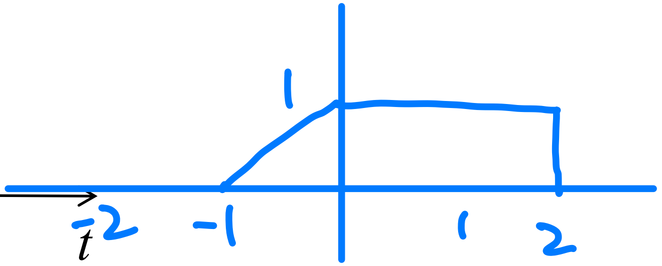
- Sketch the even and odd parts of $x(t)$ shown:

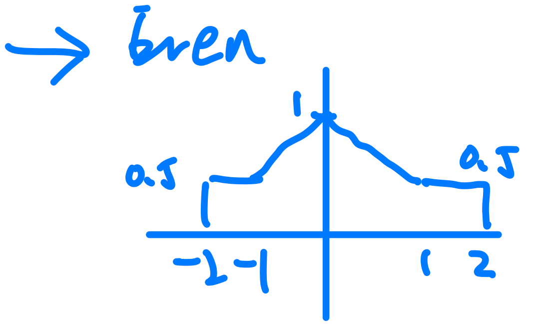




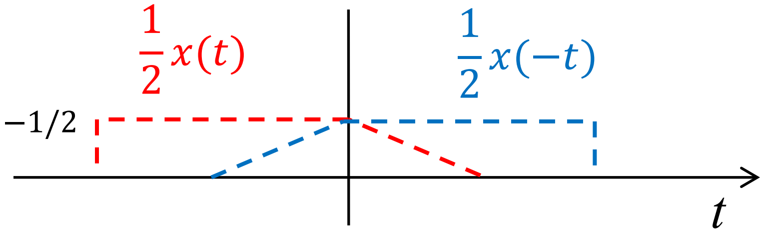
We first sketch $x(-t)$:

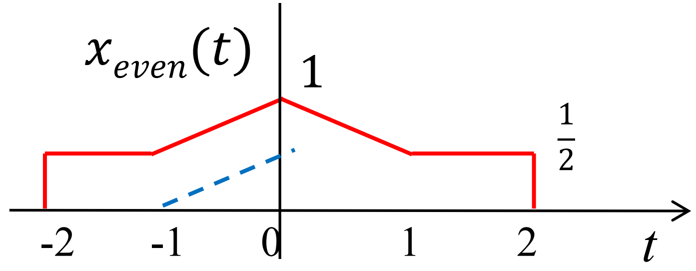




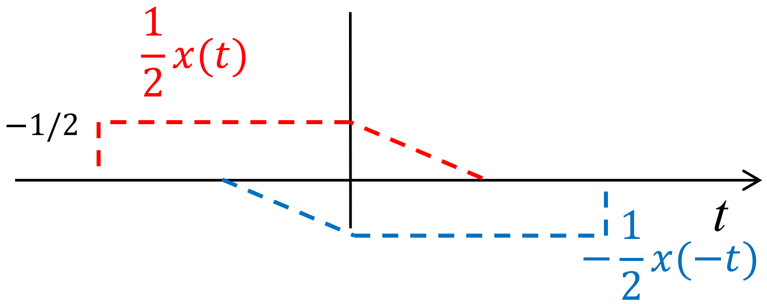


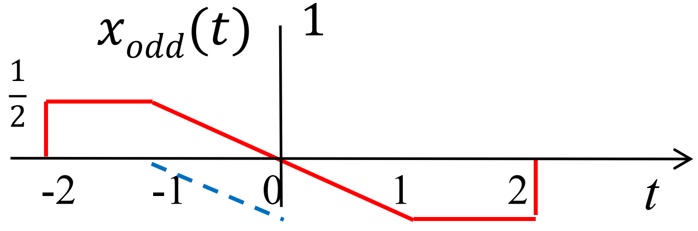
Half sum of $x(t)$ and $x(-t)$:





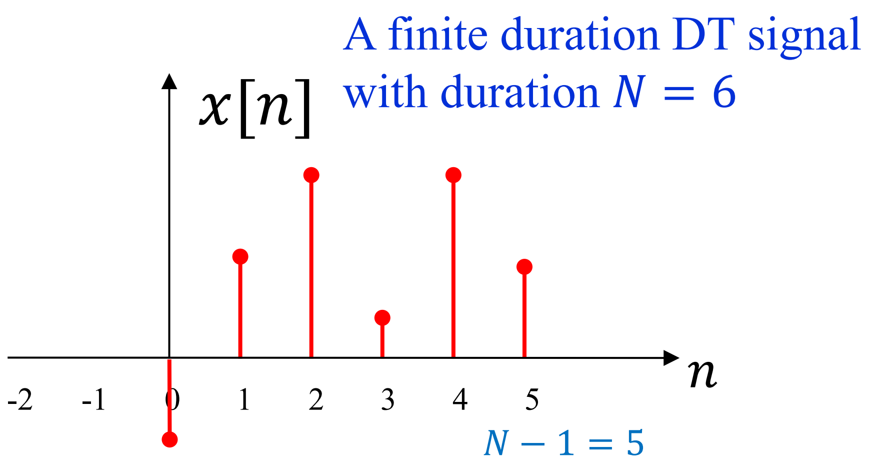
Half difference of $x(t)$ and $x(-t)$:

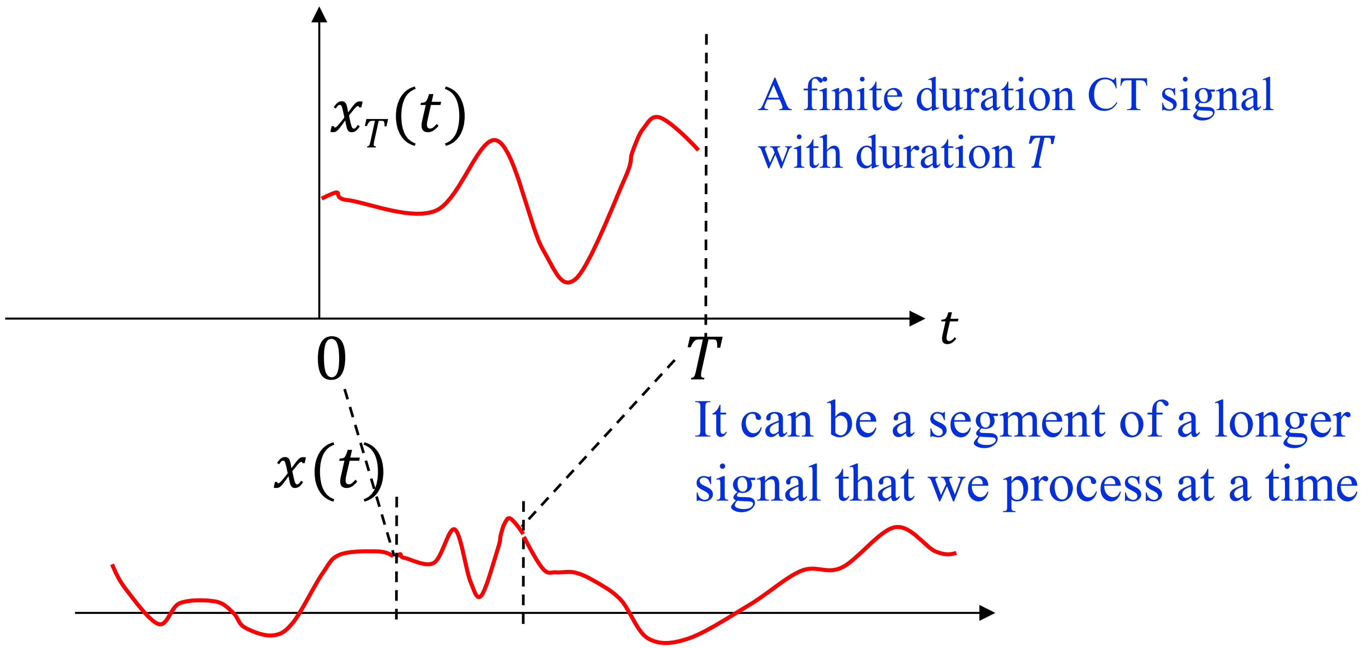




2. Infinite Duration and Finite Duration Signals

- An ***infinite duration*** signal that lasts "forever".
- If a signal lasts only for a finite period of time, we say that it is of ***finite duration***. We also say that it has ***finite support***.
- A finite duration signal could be what we extract from a longer duration signal to process at a time.

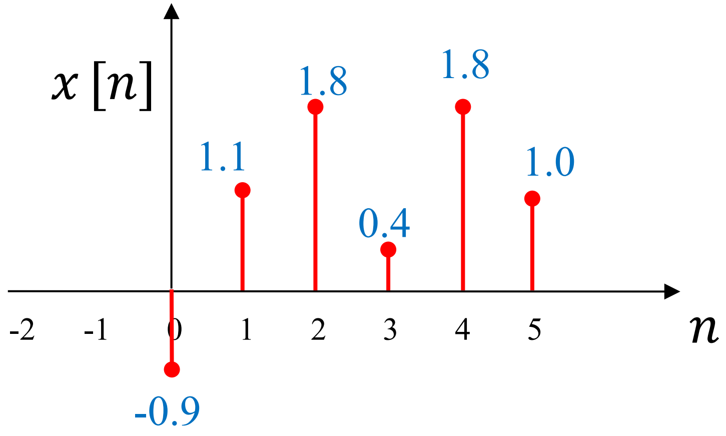




Finite Duration DT Signal as Vector

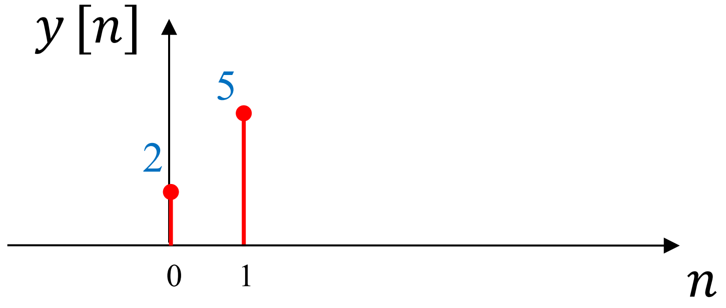
- A finite duration DT signal is a sequence of N complex numbers. We can treat it as a complex N-vector.

A finite duration DT signal with duration $N = 6$



$$\vec{x} = \begin{bmatrix} -0.9 \\ 1.1 \\ 1.8 \\ 0.4 \\ 1.8 \\ 1.0 \end{bmatrix}$$

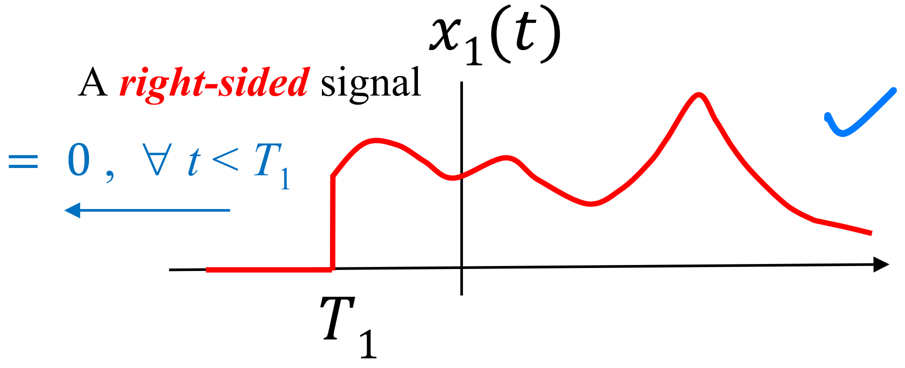
A finite duration DT signal with duration $N = 2$

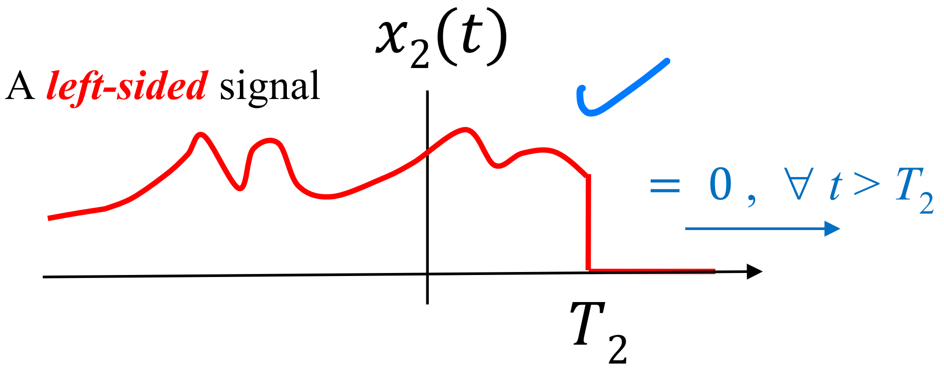


$$\vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

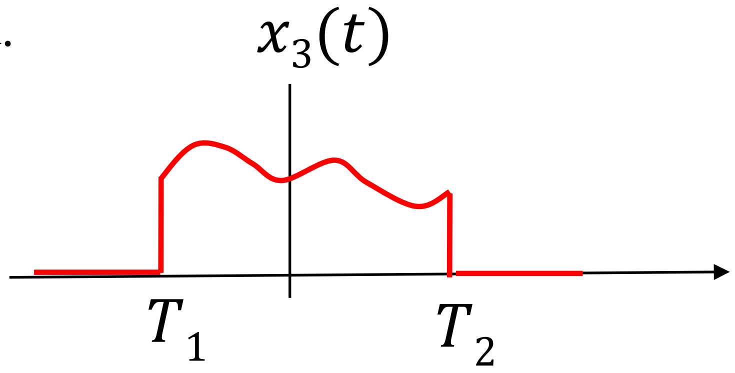
3. Right-Sided and Left-Sided Signals

- A signal is ***right-sided*** if there is a time T_1 for which $x(t) = 0$, $\forall\ t < T_1$:
  signal with *initial rest*. 开始停!
  for all
- A signal is ***left-sided*** if there is a time T_2 for which $x(t) = 0$, $\forall\ t > T_2$:
  signal with *final rest*.



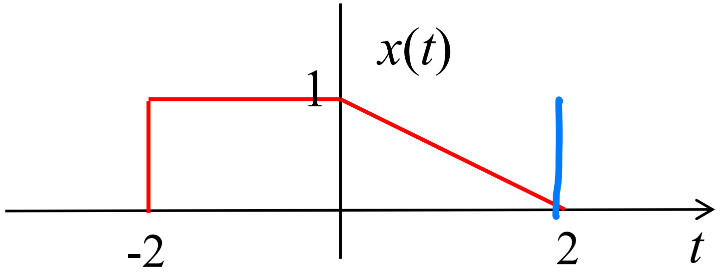


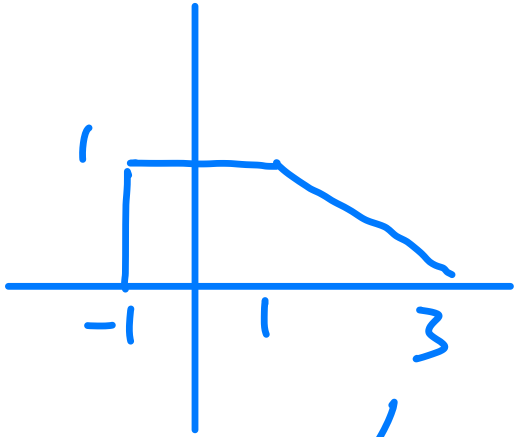
- A finite duration signal is both right-sided and left-sided.



Review Questions

1. For the signal $x(t)$ shown:



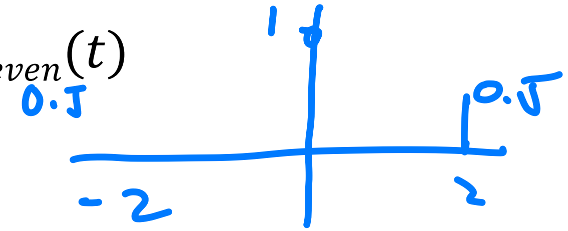


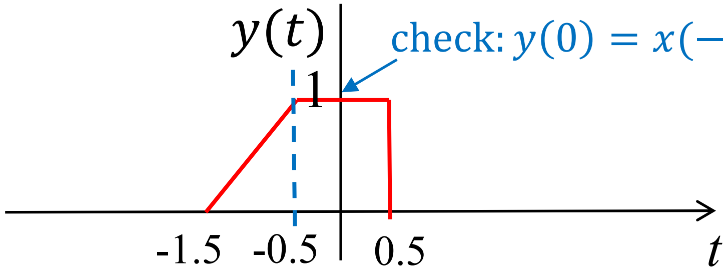
2. Time reversal and compressed by 2

i) Draw $y(t) = x(-2t - 1)$

1. Delay by 1

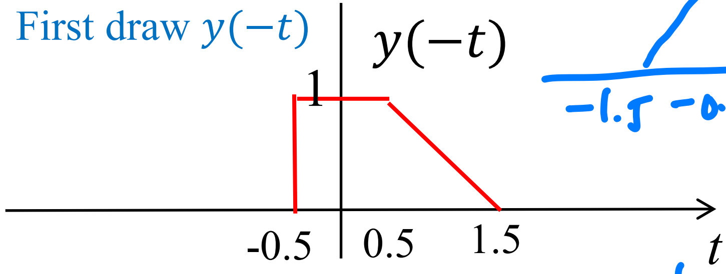
ii) Draw $y_{even}(t)$

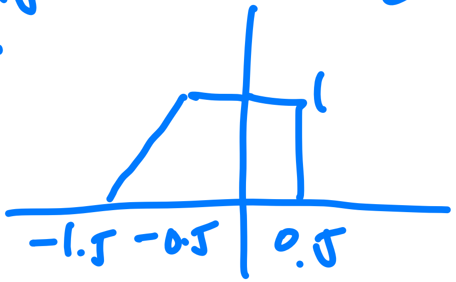


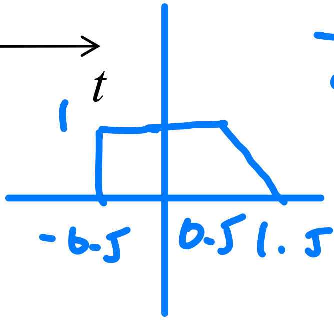


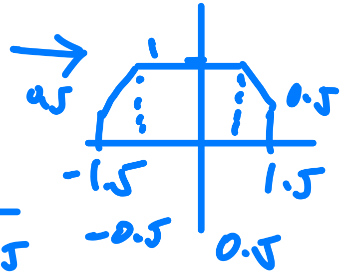
check: $y(0) = x(-1)$

First draw $y(-t)$

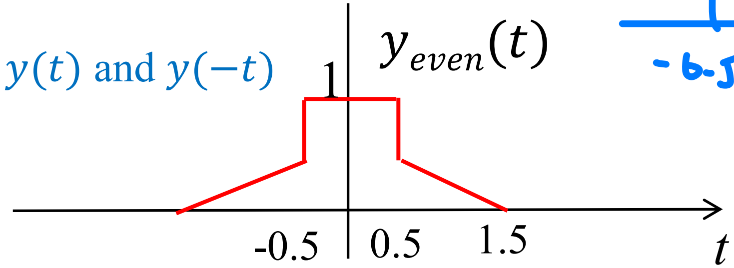








Even part as half sum of $y(t)$ and $y(-t)$



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III) Periodic Signals and Poisson Sum (Language – math notations & keywords)

IV) Some example signals: exponential, sinusoidal, and unit step

III. Periodic Signals and Poisson Sum

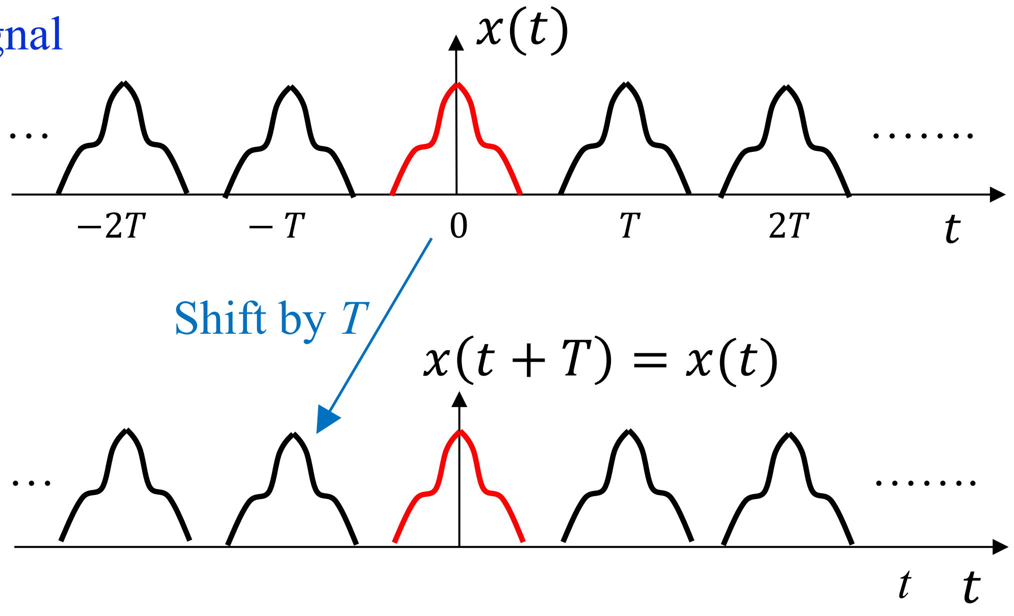
- A ***Periodic signal*** is one that is unchanged (invariant) after a given time shift time-shift:

for all

Definition: A CT signal $x(t)$ is ***periodic*** with period T, or *T-periodic*, if $x(t+T) = x(t)\ \forall t$

A DT signal $x[n]$ is ***periodic*** with period N, or ***N-periodic*** if $x[n+N] = x[n]\ \forall n$

A T-periodic CT signal



Periodic Signals - Continue

- A T-periodic signal must also be kT-periodic for any integer k:

  This is because if $x(t+T) = x(t)\ \forall t$,

  shifting both sides of above gives $x((t+T)+T) = x(t+T) = x(t)\ \forall t$,

  or $x(t+2T) = x(t)\ \forall t$ So $x(t)$ is $2T$-periodic

  and we can repeat the argument above

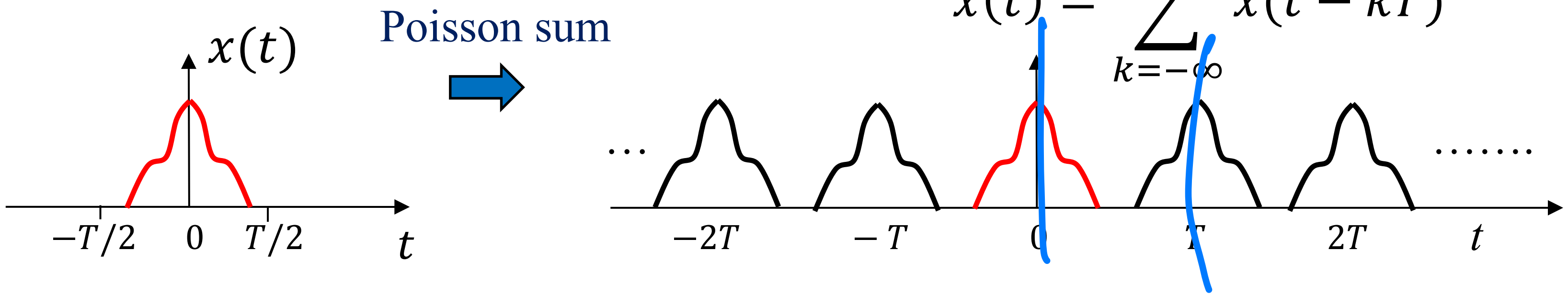
- ***Aperiodic signal*** is a signal that is not periodic.

Periodic Signal as a Poisson Sum

- A ***Poisson sum***, or ***periodic extension***, is the sum of an infinite number of time-shifted copies of another finite-duration or infinite-duration signal.
- Consider a finite-duration signal $x(t)$ shown below.

Its Poisson sum is the signal: $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$$



- Since $x(t)$ is finite-duration, it is equal to one "chunk" of its Poisson sum:

$$x(t) = \begin{cases} \tilde{x}(t) & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

Poisson Sum/Periodic Extension of any Signal

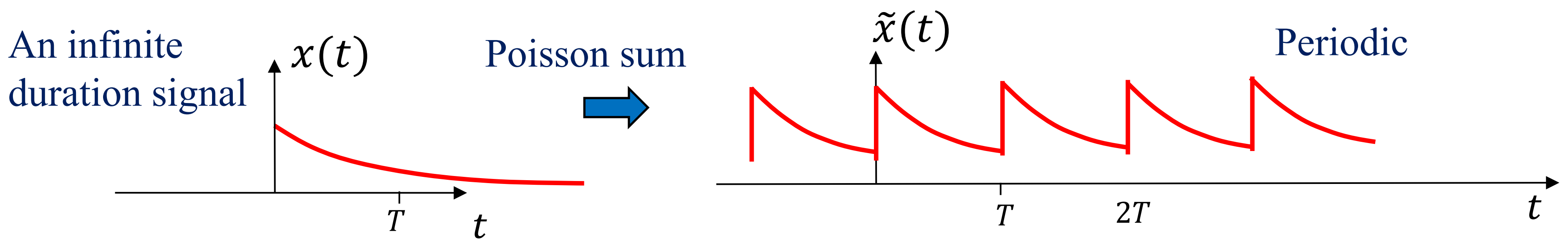
- The Poisson sum of any signal, finite-duration or infinite-duration, *is always periodic*.

Proof: Let $\tilde{x}(t)$ be a Poisson sum/periodic extension of any $x(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

If we shift $\tilde{x}(t)$ by T, it remains unchanged because:

$$\tilde{x}(t+T) = \sum_{k=-\infty}^{\infty} x((t+T) - kT) = \sum_{k=-\infty}^{\infty} x(t-(k-1)T) = \sum_{k=-\infty}^{\infty} x(t-kT) = \tilde{x}(t) \quad \forall t$$



- A caveat is that if $x(t)$ is itself periodic, then its Poisson sum would blow up and is not very meaningful.

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III) Periodic Signals and Poisson Sum

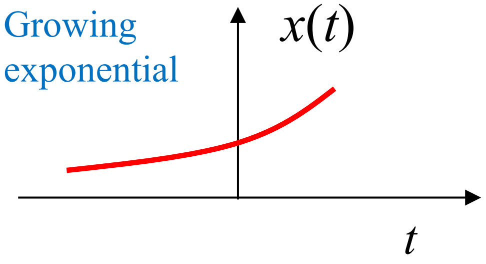
IV) Some Example Signals: Exponential, Sinusoid, and Unit Step
(Language – math notations, simple relations & keywords)

1. The Real Exponential

The signal $x(t) = e^{at}$ where a is called the *growth constant* and for now is real.

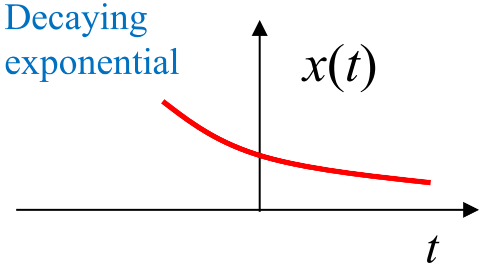
$a > 0$

Growing exponential



$a < 0$

Decaying exponential



- The exponential is common in natural phenomena because it is unchanged under differentiation except for *multiplication by the growth constant*:

$$\frac{de^{at}}{dt} = ae^{at}, \quad \frac{d^2e^{at}}{dt^2} = a^2e^{at}, \ldots\ldots\ldots, \frac{d^ke^{at}}{dt^k} = a^ke^{at}$$

- In fact, time delay also becomes multiplication by a constant:

$$x(t-\tau) = e^{a(t-\tau)} = e^{-a\tau}e^{at} = e^{-a\tau}x(t)$$

Time shift by $-\tau$

multiplication by $e^{-a\tau}$

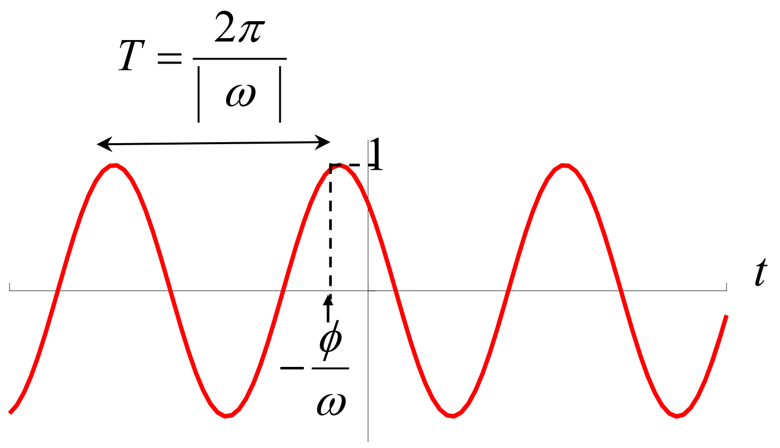
2. The Real Sinusoid

The signal $x(t) = \cos(\omega t + \phi)$

It is a periodic signal representing an ***oscillation***.

Phase change per unit time

- ω is the *angular frequency* (in radian/sec), and
- ϕ is an *offset angle/phase*.



- We can alternatively express a sinusoid in terms of *ordinary frequency* $f = \frac{\omega}{2\pi}$ which is in unit of Hertz, or cycles/sec:

$$x(t) = \cos(2\pi f t + \phi)$$

- The sinusoid is periodic with period equal to the reciprocal of the ordinary frequency, or reciprocal of the angular frequency times 2π:

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

Sine, Cosine, and Derivative

- The sine function is the same as cosine except for a 90° phase lag $\left(\phi = -\frac{\pi}{2}\right)$.

$$\sin(\omega t) = \cos(\omega t - \pi/2)$$

- The *derivative of a sinusoid is also a sinusoid at the same frequency*, but there is a multiplication by the angular frequency and a phase advance of 90° :

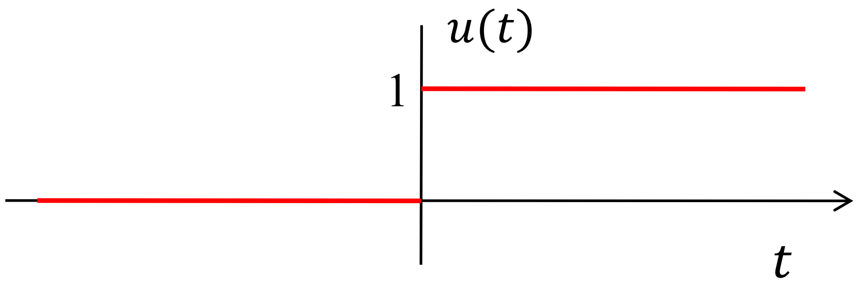
$$\frac{d\cos(\omega t)}{dt} = -\omega\sin(\omega t) = -\omega\cos(\omega t - \pi/2) = \omega\cos(\omega t + \pi/2)$$

$$\frac{d\sin(\omega t)}{dt} = \omega\cos(\omega t) = \omega\sin(\omega t + \pi/2)$$

3. The Unit Step Signal

Denoted $u(t)$, the CT unit step is defined as:

$$\boxed{u(t) = \begin{cases} 0, & t < 0 \\ 1 & t > 0 \end{cases}}$$



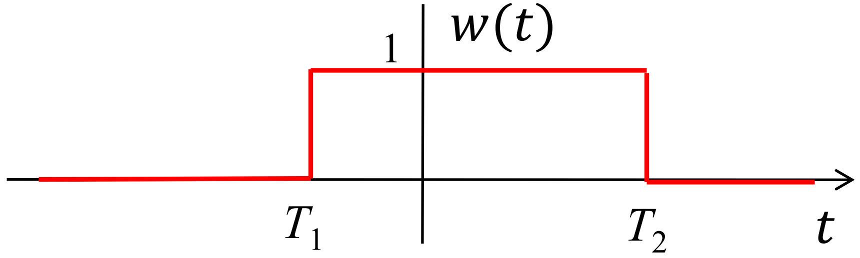
- The unit step represents a signal that is switched on at time 0.

- What is $u(t)$ at $t = 0$?
  Mathematically, $u(0)$ is undefined because $u(t)$ is discontinuous at $t = 0$.
  Physically, the unit step is an *idealization* because nothing can be switched on truly in no time.

Window/Rectangular Signal as Difference of Unit Steps

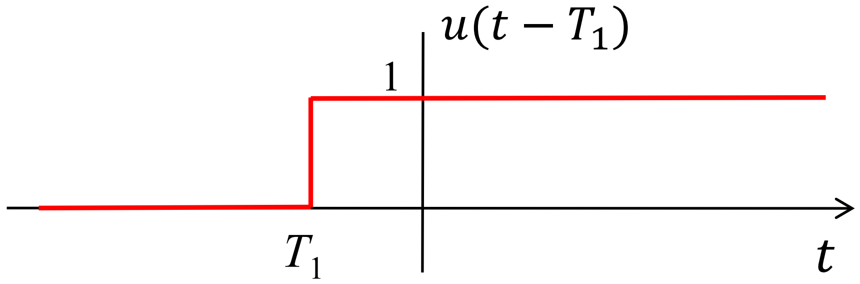
A window/rectangular signal is equal to 1 in the time interval (T_1, T_2) and is equal to 0 otherwise.

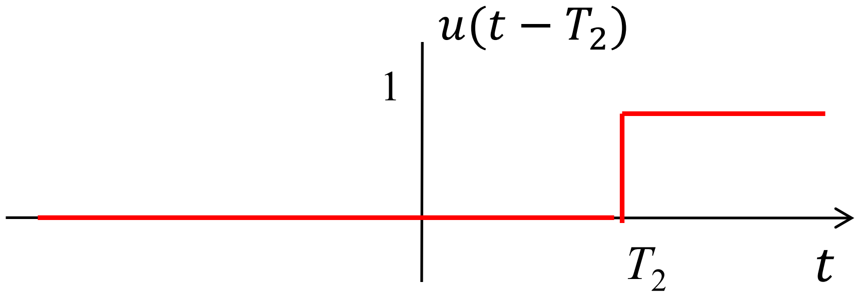
$$\boxed{w(t) = \begin{cases} 1, & T_1 < t < T_2 \\ 0 & otherwise \end{cases}}$$



We can represent the above window as the difference of two shifted unit steps:

$$\boxed{w(t) = u(t - T_1) - u(t - T_2)}$$

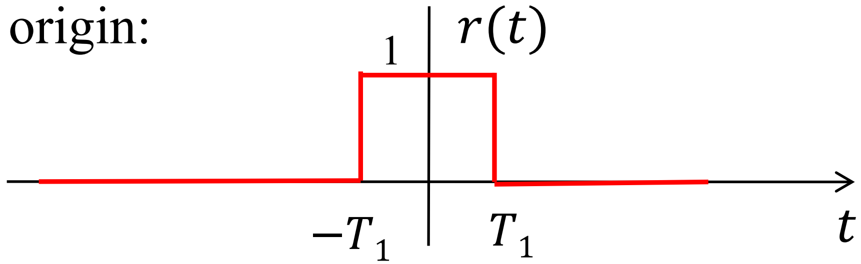




Example – The Rectangular Wave

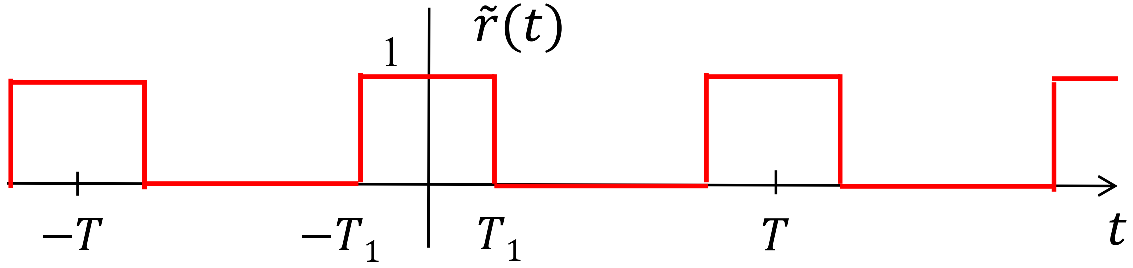
A window/rectangular signal can also be viewed as a pulse.
The following pulse is a window centered at the origin:

$$r(t) = u(t + T_1) - u(t - T_1)$$



We can use a Poisson sum of the pulse signal to represent a periodic rectangular/square wave:

$$\boxed{\tilde{r}(t) = \sum_{k=-\infty}^{\infty} r(t - kT) = \sum_{k=-\infty}^{\infty} [u(t + T_1 - kT) - u(t - T_1 - kT)]}$$



Example – Data Communication Signal

- In data communication, we vary (*modulate*) the individual signal pulses in a string of pulses to convey 0 and 1 bits.
- Each modulated pulse is called a *symbol*.
- One mathematical representation of a data stream signal is:

Window/pulse for kth symbol

$$\boxed{s(t) = \sum_{k=-\infty}^{\infty} \alpha_k r(t - kT)}$$

"Value" of kth symbol

- Possible values for α_k could be $\{0, A\}$, $\{-A, +A\}$, $\{-A, 0, A\}$, $\{-A, -0.33A, 0.33A, A\}$ etc., where A is a voltage level. All these choices are used in different "keying" schemes in different communication systems.