Lecture 20

Chapter 9: Laplace Transform (Analysis)

I. Laplace Transform and Region of Convergence

II. Examples

III. Properties of Region of Convergence

I. Laplace Transform and Region of Convergence

- Recall that e^{st} is eigenfunction of LTI systems and the Laplace Transform (LT) of the impulse response $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$, called the *system function*, provides the eigenvalue. $e^{st} \rightarrow H(s)e^{st}$

- For a signal $x(t)$, its LT is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt; \qquad x(t) \overset{L}{\leftrightarrow} X(s)$$

$$X(s) = L\{x(t)\}; \quad \text{and} \quad x(t) = L^{-1}\{X(s)\}$$

purely imaginary number

- In *Fourier analysis*, we limit our attention to the case of complex sinusoids where $s = j\omega$, and focus on the integral:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt; \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- We have seen how Fourier analysis has helped us address problems such as modulation, frequency division multiplexing, channel characterization, and sampling. We have also applied *Fourier Transform (FT)* to LCCDEs where we assumed we were dealing with causal systems.

Relationship between LT and FT (9.1)

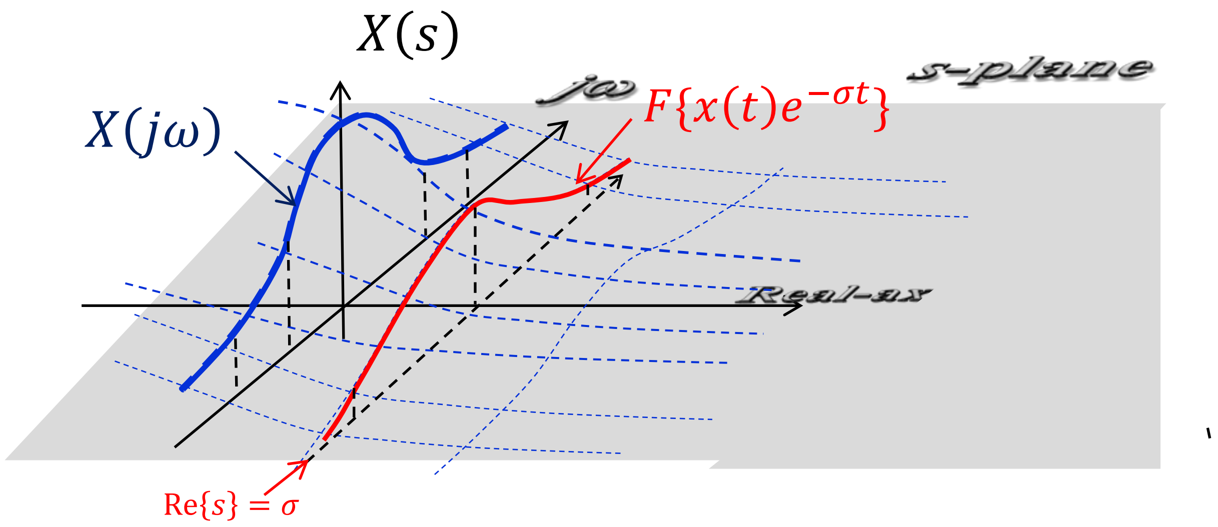
- In Laplace Transform, we extend our attention to the entire s plane. By doing so, we have an additional set of tools for the study of causality, stability, feedback, and transient behavior of systems

- s is a complex number. Let $s = \sigma + j\omega$, then the LT is:

$$e^{-st} = e^{-\sigma t} e^{-j\omega t}$$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt = FT\{x(t)e^{-\sigma t}\}$$

which means that the LT of a signal $x(t)$ evaluated at $s = \sigma + j\omega$ can be viewed as the FT of $x(t)e^{-\sigma t}$.



LT of Casual Exponential

Example 9.1: Determine the LT of $x(t) = e^{-at}u(t)$.

Causal!

We write down the LT integral for $s = \sigma + j\omega$:

$e^{-st} = e^{-\sigma t} e^{-j\omega t}$

$$X(s) = \int_{-\infty}^{\infty} e^{-at} e^{-st} u(t) dt = \int_{-\infty}^{\infty} e^{-(a+\sigma)t} e^{-j\omega t} u(t) dt$$

which is the FT of $e^{-(a+\sigma)t}u(t)$.

From Table 4.2 (p.329) , $FT\{e^{-at}u(t)\} = \dfrac{1}{j\omega + a}$ for $\mathrm{Re}\{a\} > 0$

so that $e^{-at}u(t)$ decays with time and is absolute integrable

Hence,

$$X(s) = FT\{e^{-(a+\sigma)t}u(t)\} = \frac{1}{j\omega + (a+\sigma)} = \frac{1}{s+a} \text{ for } \mathrm{Re}\{a+\sigma\} > 0$$

The condition $\mathrm{Re}\{a+\sigma\} > 0$ can also be written as $\underbrace{\mathrm{Re}\{s\}}_{=\sigma} > -\mathrm{Re}\{a\}$.

ROC!

- **Example 9.1 (cont.):**

In summary:

$$e^{-at}u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

The Laplace transform $X(s)$ converges only for s such that $\mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$

We refer to the values of s for which $X(s)$ converges as the Region of Convergence (ROC) of the Laplace transform.

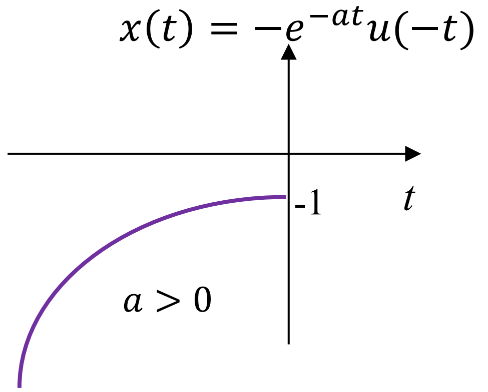
- If $X(s)$ exists for $s = \sigma + j\omega$, it means that the FT of $x(t)e^{-\sigma t}$ converges, which also means $x(t)e^{-\sigma t}$ is *absolute integrable*:

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty$$

LT of Anti-Causal Exponential

A signal $x(t)$ is *anti-causal* if $x(t) = 0 \;\; \forall t > 0$ ✓ positive!

Example 9.2: Find LT of the anti-causal exponential $x(t) = -e^{-at}u(-t)$



Now let's perform the LT integral,

$u(-t)$

for $t = 0$

goes to 0 if $\mathrm{Re}\{s+a\} < 0$

magnitude is always one because exponent is purely imaginary

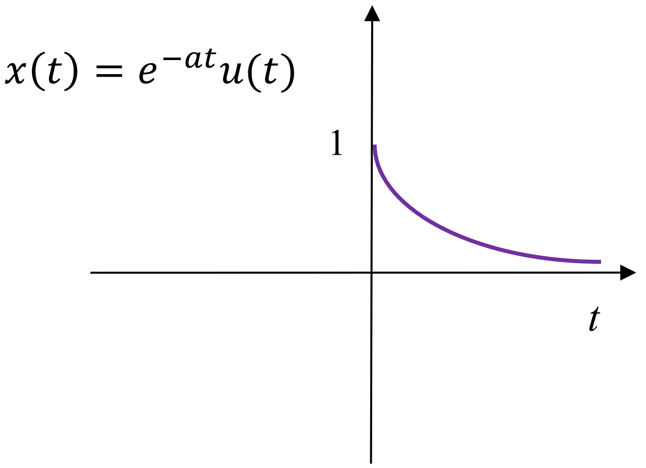
$$X(s) = \int_{-\infty}^{0} -e^{-at}e^{-st}dt = \left.\frac{\cancel{-}e^{-(s+a)t}}{\cancel{-}(s+a)}\right|_{-\infty}^{0} = \frac{1}{s+a} - \frac{e^{-Re\{s+a\}t}e^{-Im\{s+a\}t}}{(s+a)}$$

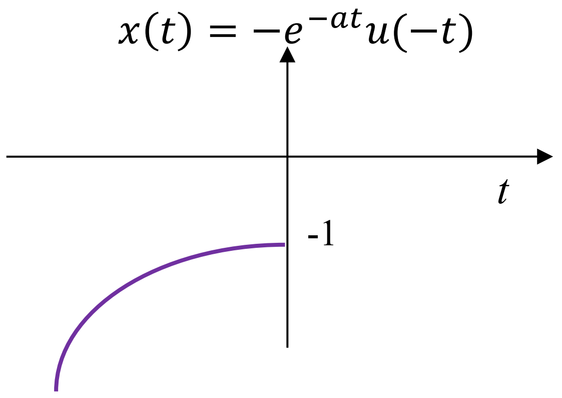
$|e^{j\theta t}|=1$ for θ real

$$= \frac{1}{s+a} \qquad \text{if } \mathrm{Re}\{s+a\} < 0 \quad \text{or } \mathrm{Re}\{s\} < -\mathrm{Re}\{a\}$$

LT of Anti-Causal Exponential

- Note that the Laplace transforms in examples 9.1 and 9.2 have the same algebraic form. The only difference is in the ROC.





$$X(s) = \frac{1}{s+a} \quad \text{for } \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

$$X(s) = \frac{1}{s+a} \quad \text{for } \mathrm{Re}\{s\} < -\mathrm{Re}\{a\}$$

- <u>**Hence, in specifying a Laplace Transform, we must also specify its ROC**</u>.
- The anti-causal signal $x(t) = -e^{-at}u(-t)$ does not appear in Table 4.2. This is because in Fourier Transform we generally focus on causal systems and signals.

Region of Convergence for Example 9.1 and 9.2

$x(t) = e^{-at}u(t)$

Extremely important!!!

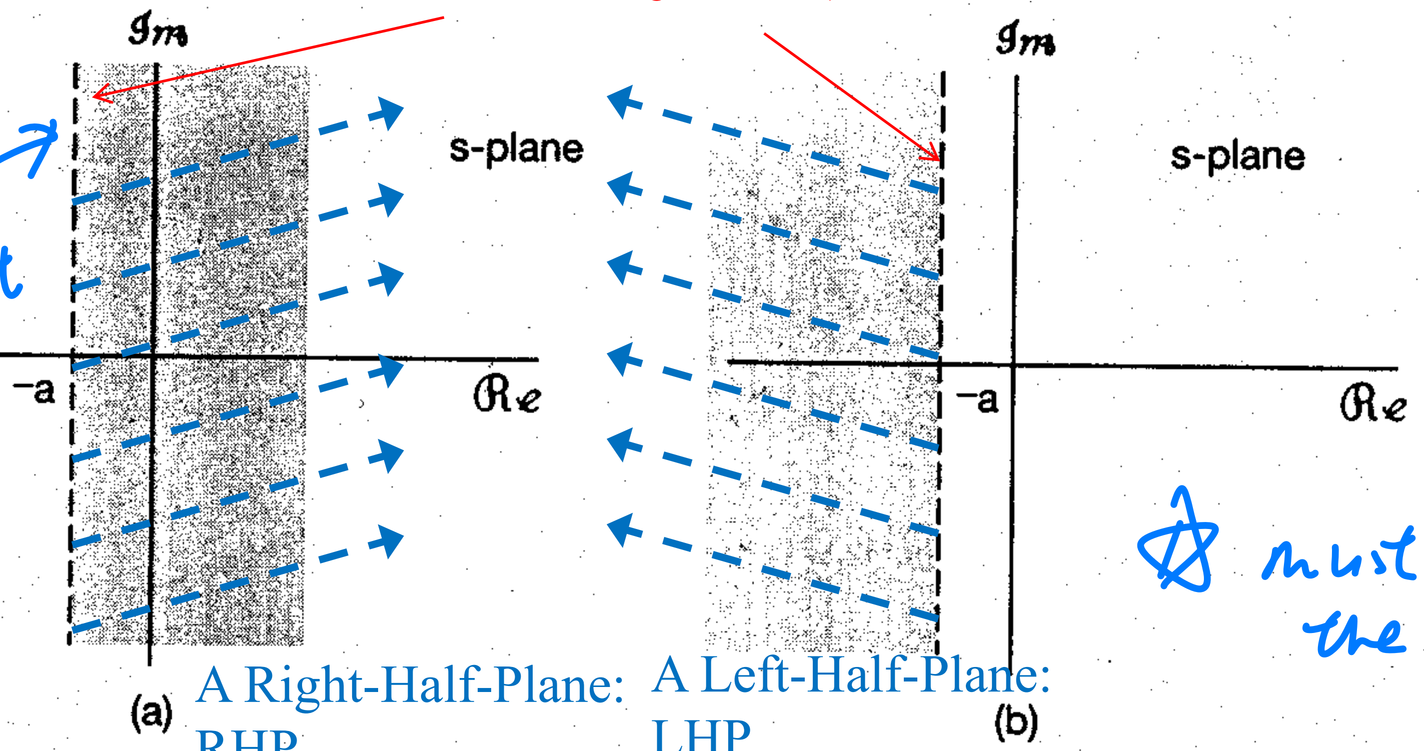
$x(t) = -e^{-at}u(-t)$

$$X(s) = \frac{1}{s+a}\ ; \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

Dashed line means an ***open boundary*** (the boundary is not included in the region of interest - the region shaded)

$$X(s) = \frac{1}{s+a}\ ; \mathrm{Re}\{s\} < -\mathrm{Re}\{a\}$$

boundary still cannot converge!



A Right-Half-Plane: RHP

A Left-Half-Plane: LHP

must specify the ROC!

Figure 9.1 (a) ROC for Example 9.1; (b) ROC for Example 9.2.

Lecture 20

Chapter 9: Laplace Transform

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II. Examples

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II. Examples

Example 9.3 Sum of two real causal exponentials

Consider $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

- Linearity of LT is obvious. From example 9.1

$$\overset{a=2}{3e^{-2t}u(t)} \overset{L}{\leftrightarrow} X(s) = \frac{3}{s+2}_{\,a=2} \quad \text{ROC: Re}\{s\} > -2 \qquad e^{-at}u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s+a} \quad \text{ROC: Re}\{s\} > -\text{Re}\{a\}$$

$$\overset{a=1}{-2e^{-t}u(t)} \overset{L}{\leftrightarrow} X(s) = \frac{-2}{s+1}_{\,a=1} \quad \text{ROC: Re}\{s\} > -1$$

- For the Laplace Transform to converge, both terms must converge. The ROC needs to be the intersection of the individual ROCs. Therefore:

$$3(s+1) - 2(s+2) = s-1$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{L}{\leftrightarrow} X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}, \text{ROC: Re}\{s\} > -1$$

combined!

$$\{Re(s) > -2\} \cap \{Re(s) > -1\}$$
$$= \{Re(s) > -1\}$$

Example 9.4 – Sum of real and complex exponentials

Causal real exponential   Causal damped oscillation

Find LT of $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)\,u(t)$

- We express $x(t)$ as a sum of causal exponentials by decomposing the damped oscillation into the half sum of a conjugate pair of complex exponentials,

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}\right]u(t)$$

- From example 9.1 again:

$$\underbrace{e^{-2t}u(t)}^{a=2} \overset{L}{\leftrightarrow} \frac{1}{s+2},\ a=2 \qquad \text{ROC: } \mathrm{Re}\{s\} > -2 \quad \mathrm{Re}\{a\} = 2$$

$$\underbrace{e^{-(1-3j)t}u(t)}^{a=1-3j} \overset{L}{\leftrightarrow} \frac{1}{s+(1-3j)},\ a=1-3j \qquad \text{ROC: } \mathrm{Re}\{s\} > -1 \quad \mathrm{Re}\{a\} = 1$$

$$\underbrace{e^{-(1+3j)t}u(t)}^{a=1+3j} \overset{L}{\leftrightarrow} \frac{1}{s+(1+3j)},\ a=1+3j \qquad \text{ROC: } \mathrm{Re}\{s\} > -1 \quad \mathrm{Re}\{a\} = 1$$

Intersection is $\mathrm{Re}\{s\} > -1$

Example 9.4 - continue

Therefore:

$$e^{-2t}u(t)+e^{-t}(\cos 3t)u(t) \overset{L}{\leftrightarrow} \frac{1}{s+2}+\frac{1}{2}\left(\frac{1}{s+(1-3j)}\right)+\frac{1}{2}\left(\frac{1}{s+(1+3j)}\right), \; Re(s)>-1$$

Overall ROC is intersection of ROCs of individual terms

Combine the 2nd and 3rd terms

$$=\frac{1}{s+2}+\frac{\cancel{2}(s+1)}{\cancel{2}(s+(1-3j))(s+(1+3j))}$$

$1\times(s+(1+3j))+1\times(s+(1-3j))$
$=2s+2=2(s+1)$

$(s+1)(s+2)$
$=s^2+3s+2$

$$=\frac{(s^2+2s+10)+(s+1)(s+2)}{(s+2)(s^2+2s+10)}$$

$(s+(1-3j))(s+(1+3j))$
$=s^2+2s+10$

$s^2+2s+10+s^2+3s+2$
$=2s^2+5s+12$

$$=\frac{2s^2+5s+12}{(s+2)(s^2+2s+10)}, \quad Re\{s\}>-1$$

LT in Rational Forms and Poles and Zeros

- In Example 9.1 through 9.4, the LTs are all in rational forms:

$$X(s) = \frac{N(s)}{D(s)}$$

- $D(s)$ is the denominator polynomial. Its roots (s: $D(s)$=0) are referred to as the **poles** of $X(s)$. For these values of s, $X(s)$ is infinite. **those s such that**

- $N(s)$ is the numerator polynomial. Its roots (s: $N(s)$=0) are referred to as the **zeros** of $X(s)$. For these values of s, $X(s) = 0$.

- Locations of poles and zeros completely specify the algebraic form of the LT except for a scaling factor.

- We will see that LT in rational form naturally arises in the study of differential equations. Recall that in the study of rational frequency response, the locations of the roots of $D(j\omega)$ tell us a lot about the system.

Pole-Zero Plot of Example 9.3 and 9.4

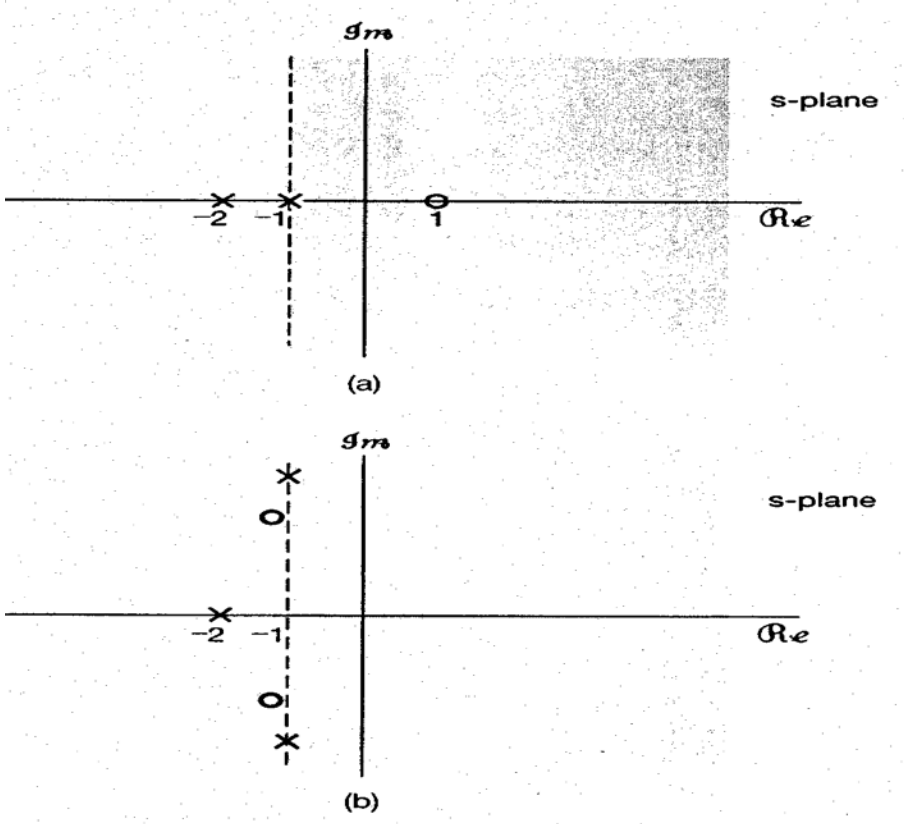

Figure 9.2 s-p
of the Laplace trai
ample 9.3 and (b)
× in these figures
of a pole of the cc
transform—i.e., a
nator. Similarly, e:
i.e., a root of the
shaded regions in

Example 9.3

$$X(s) = \frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+2)(s+1)}, \qquad Re(s) > -1$$

Example 9.4

$$X(s) = \frac{2s^2+5s+12}{(s+2)(s^2+2s+10)}$$

$$= \frac{2\left(s-\left(\frac{-5}{4}+j\frac{\sqrt{71}}{4}\right)\right)\left(s-\left(\frac{-5}{4}-j\frac{\sqrt{71}}{4}\right)\right)}{(s+2)(s+(1-3j))(s+(1+3j))}, Re\{s\} > -1$$

Lecture 20

Chapter 9: Laplace Transform

I. Laplace Transform and Region of Convergence

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III. Properties of Region Of Convergence

- The ROC enables us to convey information concerning causality (right-sidedness/left-sidedness) of signals and stability of systems.
- We will first discuss several properties of the ROC.

Properties of ROC (9.2)

Property 1 **The ROC of an LT consists of strips (vertical regions) parallel to the $j\omega$-axis on the s-plane.**

- The ROC is the set of values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges, or $x(t)e^{-\sigma t}$ is absolute integrable:

$$\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|dt < \infty$$

left or right!

- The condition depends on σ only and hence the ROC is determined by σ only. That is, the ROC must be made up of vertical strips on the s-plane.

not by omega!

不可 contains any pole!

Property 2 **The ROC of a rational LT does not contain any pole.**

- The poles of a rational function is the zeros of its denominator.
- The value of $X(s)$ is infinite at a pole, which means that the LT does not converge at a pole.

Can never include pole!

Property 3 **If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire complex plane.**

nice!

- If a signal has *finite duration* from T_1 and T_2 (it has *finite support*), we can write

$$X(s) = \int_{T_1}^{T_2} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$e^{-st} = e^{-\sigma t}e^{-j\omega t}$

- But, for $t \in [T_1, T_2]$, $|x(t)e^{-\sigma t}| \le |x(t)|\max(e^{-\sigma T_1}, e^{-\sigma T_2})$

不会 infinite!

where

$$\max(e^{-\sigma T_1}, e^{-\sigma T_2}) = \begin{cases} e^{-\sigma T_1} & \sigma > 0 \\ e^{-\sigma T_2} & \sigma < 0 \end{cases}$$

$e^{-\sigma t}$ decaying, hence larger at T_1

$e^{-\sigma t}$ growing, hence larger at T_2

- Hence, if $\int_{-\infty}^{\infty} |x(t)|dt < B$ ($x(t)$ absolute integrable), then

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < B \times \max(e^{-\sigma T_1}, e^{-\sigma T_2})\ ;$$

i.e., $x(t)e^{-\sigma t}$ is absolute integrable for any σ, and the ROC is the entire s-plane.

Property 4 **Suppose $x(t)$ is right-sided. If $Re(s) = \sigma_0$ is in the ROC, then all values of s for which $Re(s) = \sigma_1 > \sigma_0$ will also be in the ROC; i.e., the ROC must be a *right-half plane*.** must extend to infinity to the right

Definition: A signal is right-sided if there exist a T_1 for which $x(t) = 0\ ,\ \forall\, t < T_1$.
A right-sided signal is a signal with initial rest.

- All causal signals are right-sided but right-sided signals are not necessarily causal

Initial rest
$x(t) = 0\ \forall t < T_1$

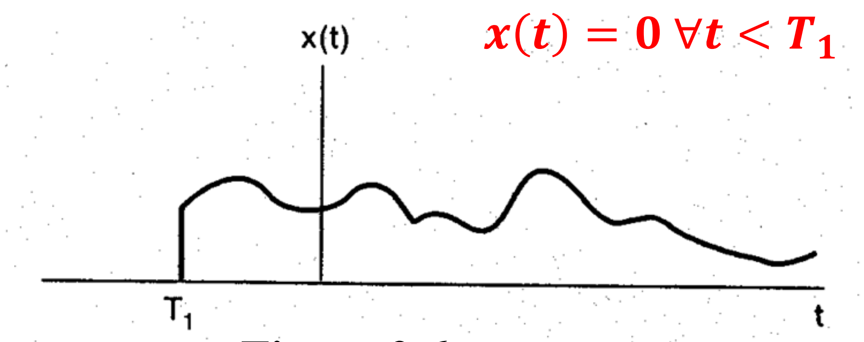
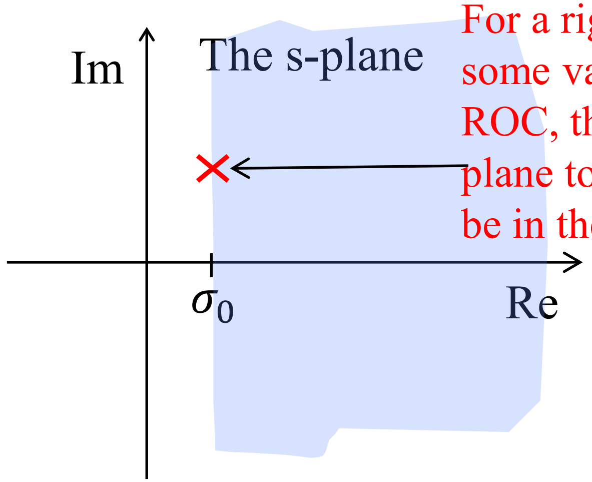


Figure 9.6
A Right-Sided Signal
*But not causal



For a right-sided signal, if some value of s is in the ROC, the entire right-half plane to the right of s must be in the ROC

Property 4 - Proof

- Given $\sigma_1 > \sigma_0$, we can write $x(t)e^{-\sigma_1 t} = x(t)e^{-\sigma_0 t}e^{-\gamma t}$ where $\gamma = \sigma_1 - \sigma_0 > 0$, meaning for all $t > T_1$, $e^{-\gamma t} < e^{-\gamma T_1}$

- Hence,

$e^{-\gamma t} < e^{-\gamma T_1}$

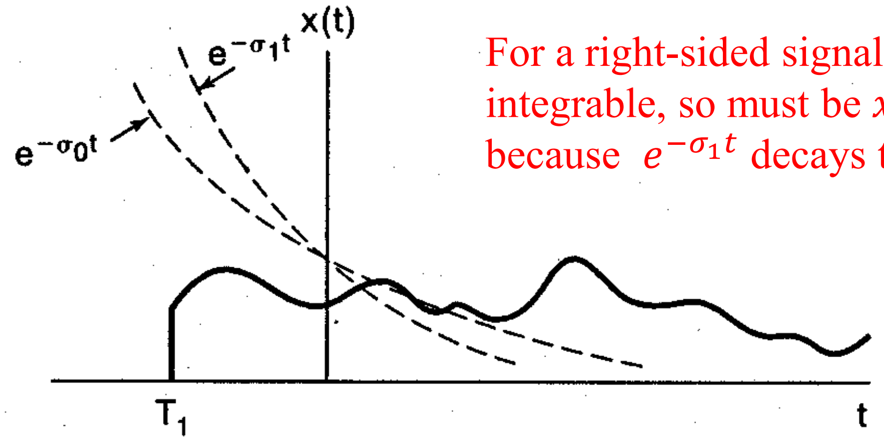
$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma_1 t}|dt = \int_{T_1}^{\infty} |x(t)e^{-\sigma_0 t}e^{-\gamma t}|dt < \int_{T_1}^{\infty} |x(t)e^{-\sigma_0 t}|e^{-\gamma T_1}dt$$

$x(t) = 0 \ \forall \ t < T_1$

$$= e^{-\gamma T_1}\int_{T_1}^{\infty} |x(t)e^{-\sigma_0 t}|dt < \infty$$

if $\int_{-T_1}^{\infty} |x(t)\, e^{-\sigma_0 t}|dt < \infty$

which implies that $s = \sigma_1 + j\,\omega$ lies in the ROC.



For a right-sided signal $x(t)$, if $x(t)e^{-\sigma_0 t}$ is absolute integrable, so must be $x(t)e^{-\sigma_1 t}$ for $\sigma_1 > \sigma_0$. This is because $e^{-\sigma_1 t}$ decays towards the right faster than $e^{-\sigma_0 t}$

Figure 9.7 If $x($ and $x(t)e^{-\sigma_0 t}$ is abs then $x(t)e^{-\sigma_1 t}, \sigma_1 >$ absolutely integrabl

AKW

Property 5 **Suppose $x(t)$ is left-sided. If $Re(s) = \sigma_0$ is in the ROC, then all values of s for which $Re(s) = \sigma_1 < \sigma_0$ will also be in the ROC;**

i.e., the ROC must be a *left-half plane*.

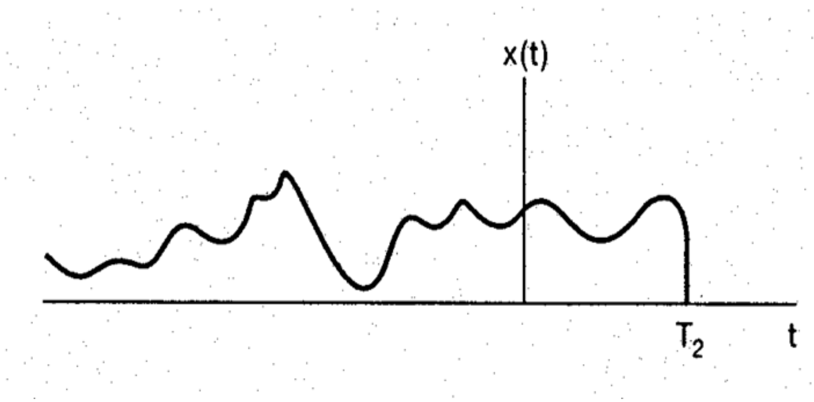
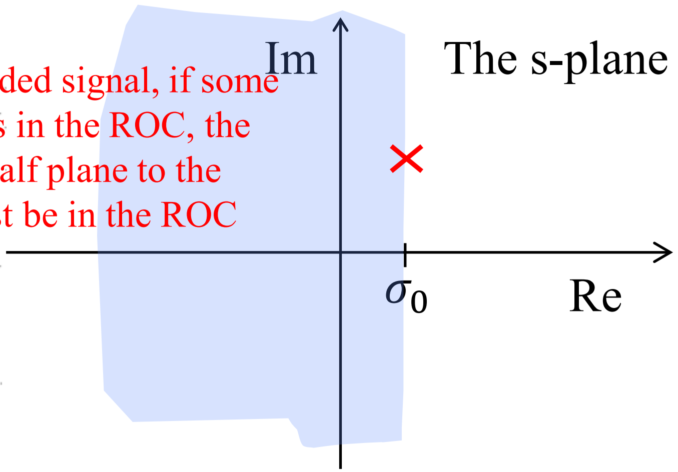


Figure 9.8

For a left-sided signal, if some value of s is in the ROC, the entire left-half plane to the leftof s must be in the ROC



- A left-sided signal is a signal with final rest (i.e., there is some T_2 for which $x(t) = 0,\ \forall\ t > T_2$).
- All anti-causal signals are left-sided but left-sided signals are not necessarily anti-causal.
- We follow same proof as previous slide but with $\sigma_1 < \sigma_0$ and $t < T_2$

Property 6 **Suppose $x(t)$ is two-sided. If $Re(s) = \sigma_0$ is in the ROC, then the ROC consists of a strip in the complex plane that includes the line $Re(s) = \sigma_0$.**

- "Two-sided" means neither right- nor left-sided.
- But we can divide a two-sided signal into a right-sided part and a left-sided part; i.e., let $x(t) = x_L(t) + x_R(t)$ where $x_L(t)$ is left-sided and $x_R(t)$ is right-sided.

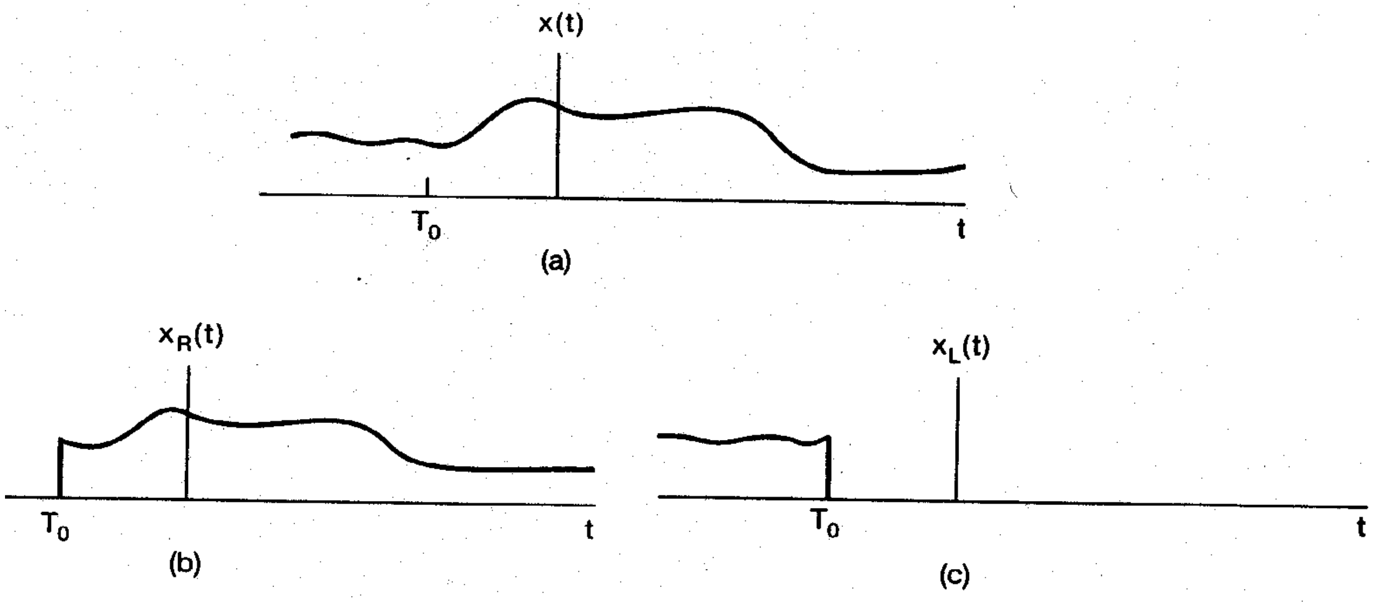


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided sig-

Property 6 - Proof

- By Properties 4 and 5, the ROCs of $x_L(t)$ and $x_R(t)$ are left-half and right-half planes respectively, each encompassing the line $Re(s) = \sigma_0$
- The ROC of $x(t)$ is the intersection of the ROCs of $x_L(t)$ and $x_R(t)$, and is clearly a strip encompassing the line $Re(s) = \sigma_0$

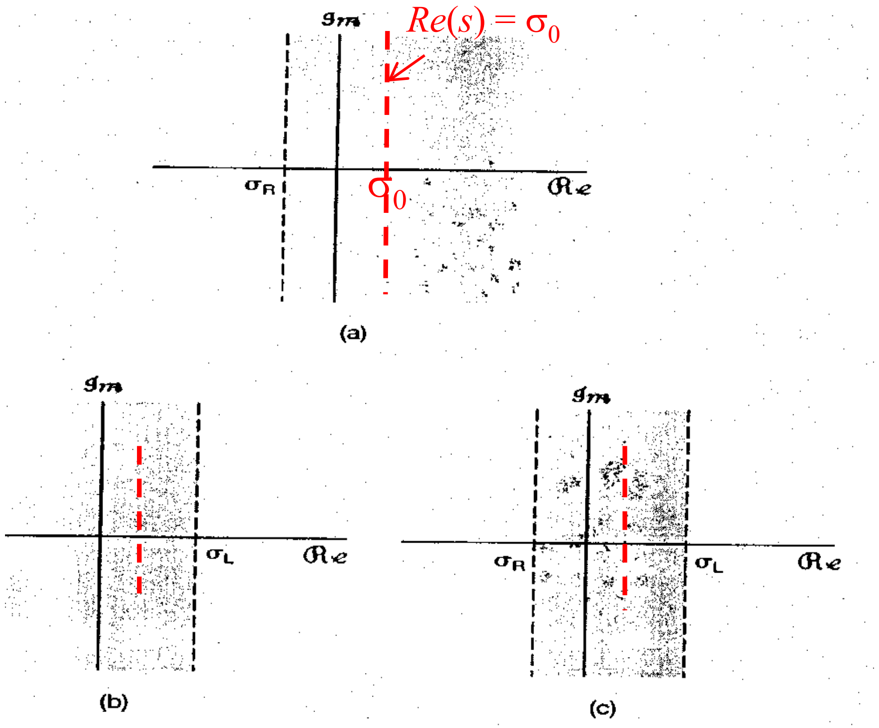


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Fi

Properties of ROC - Remarks

- ***Remarks***

  - A signal either does not have a Laplace transform, or its ROC must fall into one of the four categories covered by Properties 3-6.

  - **The ROC of a Laplace transform must be either:**

i) **the whole *s* plane, or** (finite length)

ii) **a right-half plane, or** left-sided

iii) **a left-half plane, or** right-sided

iv) **a single strip** some two-sided

ROC for Rational LT

Property 7 **For a rational Laplace transform, the ROC is bounded by poles or extends to infinity. No poles are contained in the ROC.**

A signal with rational LT is a linear combination of causal and anti-causal exponentials, the ROC of each being a right-half or left-half plane. the overall ROC is the intersection of the individual ROCs.

Property 8 **Suppose the Laplace transform of $x(t)$ is rational. If $x(t)$ is right sided, the ROC is the right-half plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the left-half plane to the left of the leftmost pole.**

Example 9.7

Example 9.7

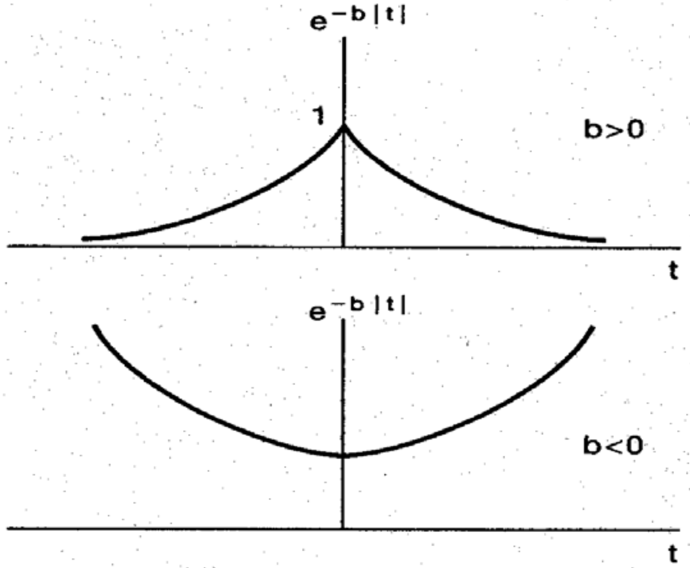
Let

$$x(t) = e^{-b|t|}, \tag{9.47}$$

as illustrated in Figure 9.11 for both $b > 0$ and $b < 0$. Since this is a two-sided signal, let us divide it into the sum of a right-sided and left-sided signal; that is,

$$x(t) = e^{-bt}u(t) + e^{+bt}u(-t). \tag{9.48}$$

two sided



Laplace X exists?

Figure 9.11 Signal $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

From Example 9.1,

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \quad \Re e\{s\} > -b, \tag{9.49}$$

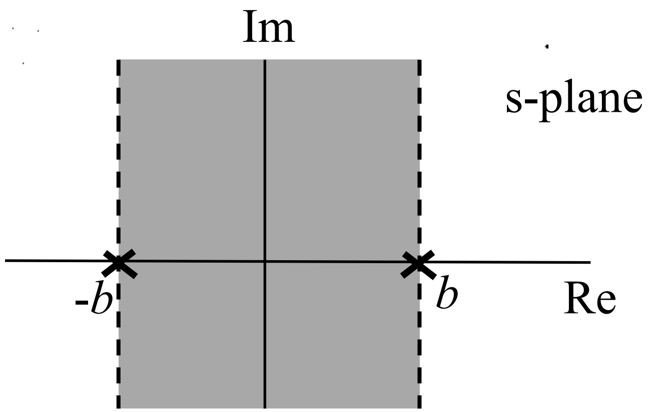
and from Example 9.2,

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \quad \Re e\{s\} < +b. \tag{9.50}$$

Although the Laplace transforms of each of the individual terms in eq. (9.48) have a region of convergence, there is no *common* region of convergence if $b \leq 0$, and thus, for those values of b, $x(t)$ has no Laplace transform. If $b > 0$, the Laplace transform of $x(t)$ is

$$e^{-b|t|} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2}, \quad -b < \Re e\{s\} < +b. \tag{9.51}$$

The corresponding pole-zero plot is shown in Figure 9.12, with the shading indicating the ROC.

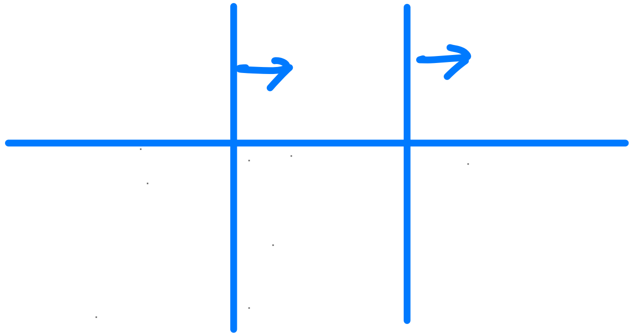


Example 9.8

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \tag{9.52}$$

with the associated pole-zero pattern in Figure 9.13(a). As indicated in Figures 9.13(b)–(d), there are three possible ROCs that can be associated with this algebraic expression, corresponding to three distinct signals. The signal associated with the pole-zero pattern in Figure 9.13(b) is right sided. Since the ROC includes the $j\omega$-axis, the Fourier



ROC : $\mathrm{Re}\{s\} > -1 \;\cup\; \mathrm{Re}\{s\} > -2$

$\Rightarrow \mathrm{Re}\{s\} > -1$

Example 9.8 continued

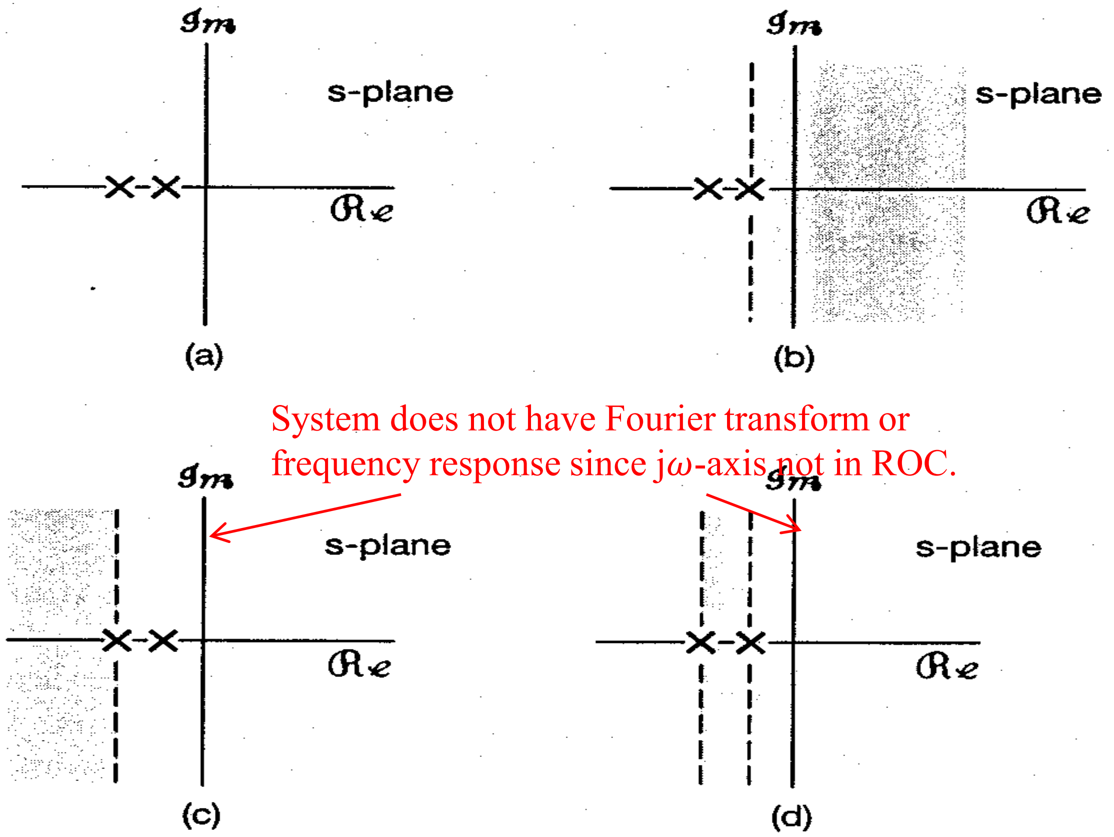


Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a right-sided sequence; (c) ROC corresponding to a left-sided sequence; (d) ROC corresponding to a two-sided sequence.

transform of this signal converges. Figure 9.13(c) corresponds to a left-sided signal and Figure 9.13(d) to a two-sided signal. Neither of these two signals have Fourier transforms, since their ROCs do not include the $j\omega$-axis.