Signal Processing and Communications Lecture Notes
ELEC 3100 — Digital Communications and Signal Processing

cplam

Figure 1: Hope you can enjoy my notes :)

Contents

1 Introduction to Discrete-Time Signals

1.1 The Sampling Process

A discrete-time sequence $x[n]$ is often obtained by uniformly sampling a continuous-time signal $x_a(t)$:

$$x[n] = x_a(t)\big|_{t=nT_s} = x_a(nT_s)$$

where T_s is the **sampling period**.

- **Sampling Frequency**:
$$F_s = \frac{1}{T_s}$$
- **Normalized Digital Angular Frequency**:
$$\omega_0 = \Omega_0 T = \frac{2\pi\Omega_0}{\Omega_T}$$
where Ω_0 is the angular frequency of the continuous-time sinusoid.
- **Aliasing**: Occurs when $|\Omega_T| < 2|\Omega_0|$, causing higher frequencies to be indistinguishable from lower frequencies.
- **Nyquist Criterion**: To avoid aliasing,
$$F_s > 2f_{\max}$$
where $f_{\max}$ is the highest frequency in the signal.

1.2 Discrete-Time Signal Representation

A discrete-time signal is denoted as $\{x[n]\}$.

- **Finite-Length Signal**: Defined for $N_1 \leq n \leq N_2$ with length
$$N = N_2 - N_1 + 1.$$
- **Infinite-Length Signal**: Defined for all n.
- **Basic Operations**:
  - **Product**: $y[n] = x[n] \cdot w[n]$
  - **Addition**: $y[n] = x[n] + w[n]$
  - **Time Shift**: $y[n] = x[n - n_0]$
  - **Time Reversal**: $y[n] = x[-n]$

1.3 Basic Sequences

- **Unit Sample**:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- **Unit Step**:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- **Exponential**:

$$x[n] = A\alpha^n$$

- **Sinusoidal**:

$$x[n] = A\cos(\omega_0 n + \phi)$$

1.4 Problems

1. **Problem 1.1** — A continuous-time signal is given by $x_a(t) = 3\cos(800\pi t) + 2\sin(1200\pi t) + \cos(2000\pi t)$ (a) Determine the Nyquist rate of $x_a(t)$. (b) If the signal is sampled at $f_s = 1$ kHz, find the discrete-time signal $x[n]$. (c) Which frequency components alias, and to what frequencies?

2. **Problem 1.2** — Classify the following signals as energy, power, or neither: (a) $x[n] = (-0.5)^n u[n]$ (b) $x[n] = e^{j(\pi n/4 + \pi/3)}$ (c) $x[n] = \cos(\pi n/3)u[n]$

3. **Problem 1.3** — Determine whether the following signals are periodic. If periodic, find the fundamental period: (a) $x[n] = \cos(\frac{\pi}{8}n) + \sin(\frac{\pi}{6}n)$ (b) $x[n] = e^{j\frac{3\pi}{5}n} + e^{j\frac{2\pi}{7}n}$

4. **Problem 1.4** — Compute the signal energy and power for: (a) $x[n] = (0.5)^{|n|}$ (b) $x[n] = \cos(\frac{\pi}{4}n)u[n]$

2 Correlation

2.1 Cross-Correlation

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

- **Normalized**:

$$\rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]r_{yy}[0]}}$$

2.2 Auto-Correlation

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

- **Property**: $r_{xx}[0]$ is the signal energy

2.3 Problems

1. **Problem 2.1** — Find the autocorrelation $r_{xx}[l]$ of

$$x[n] = \{1, 2, 3, 2, 1\}$$

   for $-4 \leq l \leq 4$.

2. **Problem 2.2** — Given $x[n] = \{1, 2, 1\}$ and $y[n] = \{1, -1, 1\}$, (a) Compute $r_{xy}[l]$ for $-2 \leq l \leq 2$. (b) Find normalized cross-correlation $\rho_{xy}[l]$ at $l = 0$.

3 Discrete-Time Fourier Transform (DTFT)

3.1 Definition

The DTFT of a sequence $x[n]$ is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Periodicity**: $X(e^{j\omega})$ is periodic with period 2π.
- **Inverse DTFT**:
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$$

3.2 Convergence Condition

- **Sufficient Condition**: If $x[n]$ is absolutely summable,
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$
then $X(e^{j\omega})$ exists for all ω.

3.3 DTFT Properties

- **Linearity**:
$$\alpha x_1[n] + \beta x_2[n]\text{DTFT}\alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$
- **Time Shift**:
$$x[n-n_0]\text{DTFT}e^{-j\omega n_0}X(e^{j\omega})$$
- **Frequency Shift**:
$$e^{j\omega_0 n}x[n]\text{DTFT}X\left(e^{j(\omega-\omega_0)}\right)$$
- **Convolution**:
$$x[n] * h[n]\text{DTFT}X(e^{j\omega})H(e^{j\omega})$$
- **Parseval's Theorem**:
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$$

3.4 Example

Find the DTFT of $x[n] = a^n u[n]$ with $|a| < 1$.

Solution:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

3.5 Problems

1. **Problem 3.1** — Find the DTFT of: (a) $x[n] = \delta[n-2] + \delta[n+2]$ (b) $x[n] = (0.8)^{|n|}$

2. **Problem 3.2** — Given $X(e^{j\omega}) = \text{DTFT}\{x[n]\}$, find the DTFT of: (a) $x[n-3]$ (b) $e^{j\pi n/4}x[n]$ (c) $x[-n]$ (d) $nx[n]$

3. **Problem 3.3** — Find the DTFT of:

$$x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

   Sketch $|X(e^{j\omega})|$ for $M = 3$.

4. **Problem 3.4** — Given $X(e^{j\omega}) = \text{DTFT}\{x[n]\}$, find the DTFT of: (a) $x[n]\cos(\omega_0 n)$ (b) $x[n]\sin(\omega_0 n)$

4 Discrete Fourier Transform (DFT)

4.1 Definition

For a length-N sequence $x[n]$, the N-point DFT is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

where $W_N = e^{-j\frac{2\pi}{N}}$.

- **Inverse DFT (IDFT)**:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad n = 0, 1, \ldots, N-1$$

4.2 Matrix Representation

- **DFT Matrix $\mathbf{D}_N$**:

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

- Then:

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}, \quad \mathbf{x} = \frac{1}{N} \mathbf{D}_N^* \mathbf{X}$$

4.3 Circular Convolution

- **Definition**:

$$y_c[n] = \sum_{m=0}^{N-1} x[m] h[\langle n-m \rangle_N], \quad 0 \leq n \leq N-1$$

- **DFT Property**:

$$y_c[n] = x[n] \circledast h[n] \mathrm{DFT} X[k] H[k]$$

- **Linear Convolution via DFT**:
  1. Zero-pad sequences to length $L \geq N + M - 1$
  2. Compute DFTs, multiply, then IDFT

4.4 Problems

1. **Problem 4.1** — Let $x[n] = \{1, 2, -1, 3\}$ for $n = 0, 1, 2, 3$. (a) Compute the 4-point DFT using the DFT matrix. (b) Verify your result using the IDFT matrix.

2. **Problem 4.2** — Given $x[n] = \{1, 2, 1, 2\}$ and $h[n] = \{1, 0, -1, 0\}$, (a) Find $y_c[n] = x[n] \circledast h[n]$ (4-point circular convolution). (b) Find the linear convolution $y_L[n] = x[n] * h[n]$ and compare.

3. **Problem 4.3** — Let $x[n] = \{1 + j, 2 - j, -1, 3 + j\}$ for $n = 0, 1, 2, 3$. (a) Find the DFT $X[k]$. (b) Find the periodic conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$ and their DFTs.

4. **Problem 4.4** — Given $x[n] = \{1, 2, 3\}$ and $h[n] = \{1, 1\}$, (a) Find linear convolution $y_L[n]$. (b) Find minimum N for DFT-based linear convolution. (c) Compute $y_L[n]$ using DFT/IDFT method.

5 Fast Fourier Transform (FFT)

5.1 Decimation-in-Time (DIT) FFT

Assume $N = 2^\nu$. Split $x[n]$ into even- and odd-indexed subsequences:

$$X[k] = G[k] + W_N^k H[k], \quad k = 0, \dots, N-1$$

where $G[k]$ and $H[k]$ are $N/2$-point DFTs of even and odd samples.

- **Butterfly Computation**:

$$X[k] = G[k] + W_N^k H[k]$$

$$X[k + N/2] = G[k] - W_N^k H[k]$$

5.2 Computational Complexity

- **Direct DFT**: $O(N^2)$
- **FFT**: $O(N \log N)$
- For N-point FFT:
  - Complex multiplications: $\frac{N}{2} \log_2 N$
  - Complex additions: $N \log_2 N$

5.3 Problems

1. **Problem 5.1** — For $N = 256$, (a) How many complex multiplications are required for direct DFT? (b) How many for DIT-FFT? (c) What is the speed-up factor?
2. **Problem 5.2** — Draw the butterfly diagram for an 8-point DIT-FFT. Label all intermediate nodes with appropriate twiddle factors.
3. **Problem 5.3** — Sketch the butterfly diagram for an 8-point Decimation-in-Frequency (DIF) FFT.
4. **Problem 5.4** — Explain how to compute the DFT of two real length-N sequences using a single N-point FFT.

6 Z-Transform

6.1 Definition

The z-transform of $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where $z = re^{j\omega}$.

- **Relation to DTFT**:
$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})$$

6.2 Region of Convergence (ROC)

- **ROC** is the set of z for which $X(z)$ converges.
- **Properties**:
  - ROC is a ring in the z-plane: $r_R < |z| < r_L$
  - ROC cannot contain poles.
  - For finite-length sequences, ROC is entire z-plane except possibly $z = 0$ or $z = \infty$.

6.3 Inverse Z-Transform

1. **Inspection Method**: Use known transform pairs.
2. **Partial Fraction Expansion**: For rational $X(z)$.
3. **Power Series Expansion**: Long division.
4. **Cauchy Integral Theorem** (formal):
$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}\, dz$$

6.4 Z-Transform Properties

- **Linearity**:
$$ax_1[n] + bx_2[n]\mathcal{Z}aX_1(z) + bX_2(z)$$
- **Time Shift**:
$$x[n-n_0]\mathcal{Z}z^{-n_0}X(z)$$
- **Convolution**:
$$x[n] * h[n]\mathcal{Z}X(z)H(z)$$
- **Multiplication by Exponential**:
$$a^n x[n]\mathcal{Z}X\left(\frac{z}{a}\right)$$

6.5 System Function

For an LTI system with impulse response $h[n]$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- **Stability**: All poles inside unit circle.
- **Causality**: ROC is exterior of a circle containing outermost pole.

6.6 Problems

1. **Problem 6.1** — For
$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$
(a) Find poles and zeros. (b) Sketch possible ROCs. (c) For each ROC, determine if the system is causal and/or stable.

2. **Problem 6.2** — Find $h[n]$ for
$$H(z) = \frac{1}{(1 - 0.8z^{-1})(1 + 0.5z^{-1})}, \quad |z| > 0.8$$
using partial fraction expansion.

3. **Problem 6.3** — Given
$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 1.2z^{-1} + 0.4z^{-2}}$$
(a) Find the poles. (b) Determine all possible ROCs and corresponding stability/causality.

4. **Problem 6.4** — Find $X(z)$ and ROC for:
$$x[n] = \begin{cases} 2^n, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

7 FIR Filter Design

7.1 Least Integral-Squared Error Design

Given desired frequency response $H_d(e^{j\omega})$, find finite-length $h[n]$ minimizing:

$$\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 \, d\omega$$

Solution: Truncate ideal impulse response:

$$h[n] = h_d[n], \quad -M \leq n \leq M$$

7.2 Gibbs Phenomenon

Truncation causes oscillations near discontinuities in frequency response.

- **Mitigation**:
  - Use window functions (e.g., Hann, Hamming)
  - Smooth transition bands in specification

7.3 Ideal Filter Impulse Responses

- **Lowpass**:

$$h_{\text{LP}}[n] = \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty$$

- **Highpass**:

$$h_{\text{HP}}[n] = \delta[n] - \frac{\sin(\omega_c n)}{\pi n}$$

- **Bandpass** and **Bandstop** derived from lowpass prototypes.

7.4 Problems

1. **Problem 7.1** — Design a length-7 FIR lowpass filter with cutoff $\omega_c = \pi/2$ using a rectangular window.

2. **Problem 7.2** — Explain why truncating the ideal impulse response causes ripples near the cutoff frequency, and how a Hamming window can reduce them.

3. **Problem 7.3** — Design a length-9 FIR highpass filter with cutoff $\omega_c = 3\pi/4$ using a Hamming window.

4. **Problem 7.4** — Show that an FIR filter has linear phase if its impulse response is symmetric or antisymmetric.

8 Source Coding

8.1 Motivation

Source coding is used to reduce the number of bits required to represent information, enabling efficient storage and transmission. Compression can be:

- **Lossy**: Higher compression but imperfect reconstruction (e.g., JPEG, MP3).
- **Lossless**: Perfect reconstruction (e.g., ZIP, PNG).

8.2 Lossy Source Coding

8.2.1 Pulse Code Modulation (PCM)

- Converts analog signals to digital by **sampling** and **quantization**.
- Quantization error (noise) given by:

$$\overline{e^2} = \frac{\Delta^2}{12}$$

  where Δ is the quantization step size.

- Signal-to-quantization noise ratio (SQNR):

$$\text{SQNR} = \frac{3}{2} \cdot 2^{2N} \quad \text{(for sine wave)}$$

  or in dB:

$$\text{SQNR}_{\text{dB}} \approx 6.02N + 1.76 \quad \text{dB}$$

  where N is the number of bits per sample.

8.2.2 Transform Coding

- Transforms data into another domain (e.g., DCT in JPEG) for energy compaction.
- Enables more efficient quantization by concentrating energy in fewer coefficients.

8.3 Lossless Source Coding

8.3.1 Entropy

- **Entropy** $H(X)$ of a discrete random variable X:

$$H(X) = -\sum_i p(x_i) \log_2 p(x_i) \quad \text{(bits/symbol)}$$

- Represents the theoretical lower bound on average bits per symbol for lossless compression.

8.3.2 Huffman Coding

- Variable-length prefix code that assigns shorter codes to more frequent symbols.
- Average code length L satisfies:

$$H(X) \leq L < H(X) + 1$$

- Constructed via a binary tree merging least probable symbols.

8.3.3 Arithmetic Coding

- Encodes entire message into a single fractional number in $[0, 1)$.
- Achieves compression closer to entropy than Huffman for long sequences.
- More complex but avoids integer bit constraints.

8.3.4 Lempel–Ziv–Welch (LZW)

- Dictionary-based compression that builds a codebook of recurring patterns.
- Does not require prior knowledge of symbol probabilities.
- Used in GIF, TIFF, and early UNIX compression.

8.4 Problems

1. **Problem 8.1** — Given symbols $\{A, B, C, D, E\}$ with probabilities $\{0.25, 0.2, 0.2, 0.2, 0.15\}$: (a) Construct a Huffman tree. (b) Assign codewords. (c) Compute average code length and compare with entropy.

2. **Problem 8.2** — Encode the sequence **ABABABA** using LZW with initial dictionary $\{\phi, A, B\}$. Show dictionary growth and output codewords.

3. **Problem 8.3** — Encode the sequence **CAB** using arithmetic coding with probabilities: $P(A) = 0.4, P(B) = 0.3, P(C) = 0.3$. Show intervals step by step.

4. **Problem 8.4** — Decode the LZW codewords:

$$(0, A), (0, B), (2, A), (1, B)$$

with initial dictionary $\{\phi, A, B\}$.

9 Channel Coding

9.1 Motivation

Channel coding adds redundancy to detect and correct errors introduced during transmission.

9.2 Types of Codes

- **Error-detecting codes**: e.g., parity check, CRC.
- **Error-correcting codes**: e.g., repetition codes, Hamming codes, Reed–Solomon.

9.3 Repetition Codes

- Encode each bit by repeating it n times.
- Decoding via majority voting.
- Code rate: $R = 1/n$.

9.4 Hamming Codes

- Linear block codes with parameters (n, k) where $n = 2^m - 1$, $k = n - m$.
- Example: $(7, 4)$ Hamming code corrects single-bit errors.
- Uses parity-check matrix H for syndrome decoding.

9.5 Reed–Solomon Codes

- Non-binary cyclic codes operating on symbols of size m bits.
- Parameters (n, k) can correct up to $t = \frac{n-k}{2}$ symbol errors.
- Widely used in CDs, DVDs, QR codes, and digital broadcasting.

9.6 Channel Coding in GSM

- Speech frames (260 bits) classified into Class 1A (50 bits, CRC protected), Class 1B (132 bits), and Class 2 (78 bits, uncoded).
- Convolutional code (rate 1/2) applied to Class 1 bits.
- Tail bits added for trellis termination.

9.7 Problems

1. **Problem 9.1** — Given generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

   (a) Encode message $u = [1\ 0\ 1\ 1]$. (b) Find parity-check matrix H. (c) Decode received vector $r = [1\ 0\ 1\ 1\ 0\ 1\ 0]$ assuming at most one error.

2. **Problem 9.2** — For a (7,4) Hamming code with parity-check matrix:

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

   Received: $r = [1\ 0\ 1\ 0\ 1\ 0\ 1]$. (a) Compute syndrome. (b) Locate and correct error.

10 Baseband Communications & Noise

10.1 System Model

- Binary baseband transmission with antipodal signaling:

$$s(t) = \begin{cases} +A, & \text{bit "1"} \\ -A, & \text{bit "0"} \end{cases}$$

- Channel: Additive White Gaussian Noise (AWGN) with PSD $N_0/2$.

10.2 White Gaussian Noise

- Modeled as a zero-mean WSS random process.
- Autocorrelation: $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$.
- PSD: $S_n(f) = \frac{N_0}{2}$ for all f.

10.3 Integrate-and-Dump Receiver

- Suboptimal receiver for antipodal signaling.
- Integrate received signal over bit period T and compare to threshold (zero).
- Output at sampling time:

$$V = \begin{cases} AT + N, & \text{"1" sent} \\ -AT + N, & \text{"0" sent} \end{cases}$$

where $N \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 = \frac{N_0 T}{2}$.

10.4 Error Probability

- For equally likely bits and threshold 0:

$$P_e = P(E|1) = P(E|0) = Q\left(\frac{AT}{\sigma}\right)$$

- Using $E_b = A^2 T$ (energy per bit):

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$.

10.5 Problems

1. **Problem 10.1** — For antipodal signaling with $A = 2$ V, $T = 1$ ms, and $N_0/2 = 10^{-6}$ W/Hz: (a) Find P_e for integrate-and-dump receiver. (b) What A is needed for $P_e \leq 10^{-6}$?

2. **Problem 10.2** — For antipodal signaling with $P(1) = 0.7, P(0) = 0.3$, (a) Find optimal decision threshold V_D. (b) Derive P_e in terms of E_b/N_0.

11 Optimal Receiver

11.1 General Receiver Structure

- Received signal: $y(t) = s_i(t) + n(t)$, $i \in \{0, 1\}$.
- Linear filter $h(t)$ followed by sampler at $t = T$.
- Decision based on threshold V_D.

11.2 Optimum Threshold

For equally likely bits:

$$V_{D,\text{opt}} = \frac{s_{o1}(T) + s_{o0}(T)}{2}$$

where $s_{oi}(T)$ is the filtered signal output at $t = T$.

11.3 Matched Filter

- The filter that maximizes output SNR at sampling instant.
- Impulse response for binary signaling:

$$h_{\text{opt}}(t) = s_1(T - t) - s_0(T - t)$$

- Equivalent to correlating with $s_1(t) - s_0(t)$.
- Maximum output SNR:

$$\zeta_{\max}^2 = \frac{2}{N_0} \int_0^T [s_1(t) - s_0(t)]^2 dt = \frac{2E_g}{N_0}$$

  where E_g is the energy of the difference signal.

11.4 Optimal Signal Design

To minimize P_e, signals should be as dissimilar as possible:

$$P_e = Q\left(\sqrt{\frac{E_g}{2N_0}}\right)$$

For antipodal signals: $E_g = 4E_b$, giving:

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

11.5 Problems

1. **Problem 11.1** — Given

$$s_0(t) = 0, \quad s_1(t) = A, \quad 0 \leq t \leq T$$

   (a) Find matched filter impulse response $h(t)$. (b) Sketch $h(t)$ and output $s_o1(t)$. (c) Derive P_e.

2. **Problem 11.2** — Show that the matched filter output at $t = T$ is equivalent to the correlator output for the signal $s_1(t) - s_0(t)$.

3. **Problem 11.3** — Given

$$p_0 = p, \quad p_1 = 1 - p$$

   Prove that the optimum filter is $h_{\text{opt}}(t) = k(s_1(T-t) - s_0(T-t))$, k is any non-zero constant.

12 Digital Modulation

12.1 Why Modulation?

- Enables transmission over bandpass channels (e.g., wireless).
- Allows frequency-division multiplexing.
- Improves antenna efficiency (smaller antennas at higher frequencies).

12.2 Digital Modulation Schemes

12.2.1 Amplitude Shift Keying (ASK)

$$s(t) = \begin{cases} A\cos(2\pi f_c t), & "1" \\ 0, & "0" \end{cases}$$

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

12.2.2 Binary Phase Shift Keying (BPSK)

$$s(t) = \begin{cases} A\cos(2\pi f_c t), & "1" \\ -A\cos(2\pi f_c t), & "0" \end{cases}$$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

12.2.3 Frequency Shift Keying (FSK)

$$s(t) = \begin{cases} A\cos(2\pi f_1 t), & "1" \\ A\cos(2\pi f_2 t), & "0" \end{cases}$$

For orthogonal FSK with $\Delta f = n/T$:

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

12.3 Correlator Receiver Implementation

- Matched filter can be implemented as a correlator:

$$V_i(T) = \int_0^T y(t)s_i(t)dt$$

- Decision: choose i corresponding to larger $V_i(T)$.
- Equivalent to matched filter but often simpler to implement.

12.4 Problems

1. **Problem 12.1** — Compare BPSK and orthogonal FSK in terms of: (a) Bandwidth efficiency (b) Error probability P_e vs E_b/N_0 (c) Receiver complexity

2. **Problem 12.2** — For binary ASK:

$$s_1(t) = A\cos(2\pi f_c t), \quad s_0(t) = 0, \quad 0 \leq t \leq T$$

   (a) Find matched filter impulse response. (b) Derive P_e in terms of E_b/N_0.

13 Channel Models

13.1 Channel Effects

- **Attenuation**: Signal power reduction with distance.
- **Fading**: Random amplitude fluctuations due to multipath.
- **Intersymbol Interference (ISI)**: Time dispersion causing adjacent symbols to interfere.
- **Noise**: Additive (e.g., AWGN) and multiplicative (e.g., fading).

13.2 Multipath Channels

- Received signal is sum of delayed and scaled copies:

$$y(t) = \sum_k a_k s(t - \tau_k)$$

- Channel impulse response: $h(t) = \sum_k a_k \delta(t - \tau_k)$.
- Frequency-selective if bandwidth $> 1/\tau_{\max}$ (delay spread).

13.3 OFDM (Orthogonal Frequency Division Multiplexing)

- Divides broadband channel into many narrowband orthogonal subcarriers.
- Uses IDFT/DFT for modulation/demodulation.
- Cyclic prefix converts linear convolution to circular convolution, simplifying equalization.
- Robust against frequency-selective fading.

13.4 Rayleigh Fading

- Model for non-line-of-sight multipath with many scattered paths.
- Envelope follows Rayleigh distribution:

$$f(r) = \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)}, \quad r \geq 0$$

- Power is exponentially distributed.

13.5 Problems

1. **Problem 13.1** — A channel has impulse response $h(t) = \delta(t) + 0.5\delta(t - T/2)$. If BPSK with bit duration T is transmitted, sketch the received signal for bit sequence **101**.

2. **Problem 13.2** — Why is a cyclic prefix added in OFDM? What is the minimum length required if channel delay spread is 2 μs and subcarrier spacing is 15 kHz?

3. **Problem 13.3** — Show that subcarriers $e^{j2\pi kt/T}$ and $e^{j2\pi mt/T}$ are orthogonal over interval T if $k \neq m$.

4. **Problem 13.4** — If the envelope r of a received signal follows Rayleigh distribution with $\sigma^2 = 2$, find $P(r < 1)$.

14 Multiplexing and Multiple Access

14.1 Multiplexing

- **Frequency Division Multiplexing (FDM)**: Users assigned non-overlapping frequency bands.
- **Time Division Multiplexing (TDM)**: Users assigned non-overlapping time slots.
- **Wavelength Division Multiplexing (WDM)**: Optical version of FDM.

14.2 Multiple Access

- **FDMA**: Different users use different frequencies (e.g., analog cellular).
- **TDMA**: Different users use different time slots (e.g., GSM).
- **CDMA**: All users share same frequency and time but use orthogonal codes (e.g., IS-95, WCDMA).
- **SDMA**: Spatial separation using antenna arrays (beamforming).

14.3 Code Division Multiple Access (CDMA)

- Each user assigned a unique spreading code (e.g., Walsh–Hadamard).
- Transmitter: $x_i(t) = d_i(t) \cdot c_i(t)$ where d_i is data, c_i is spreading code.
- Receiver correlates with desired user's code to extract data:

$$\hat{d}_i = \int y(t)c_i(t)dt$$

- Processing gain: $G = \frac{T_b}{T_c}$ (ratio of bit duration to chip duration).

14.4 Comparison

- **TDMA**: Requires synchronization, fixed data rate per user.
- **FDMA**: Simple but inflexible, guard bands needed.
- **CDMA**: Soft capacity, resistant to interference and multipath, but requires power control.

14.5 Problems

1. **Problem 14.1** — Given Walsh codes:

$$c_1 = [+1, +1, +1, +1], \quad c_2 = [+1, -1, +1, -1], \quad c_3 = [+1, +1, -1, -1]$$

   (a) Verify orthogonality. (b) If user 1 sends +1, user 2 sends -1, user 3 sends +1, find the composite transmitted signal. (c) Show how to recover user 2's data.

2. **Problem 14.2** — Design a TDMA frame for 4 users, each requiring 10 kbps, with a total channel bandwidth of 50 kHz using QPSK. Specify slot duration, guard time, and frame length.

3. **Problem 14.3** — Three users each require 10 kHz bandwidth. If guard bands are 2 kHz, what total bandwidth is needed for FDMA?

4. **Problem 14.4** — Explain the near-far problem in CDMA and how power control mitigates it.

15 Summary Problems (Comprehensive)

1. **Problem 15.1** — A voice signal bandlimited to 4 kHz is sampled at 8 kHz, quantized with 8-bit PCM, then encoded with a (7,4) Hamming code, modulated using BPSK, and transmitted over an AWGN channel. (a) Find output bit rate. (b) If channel bandwidth is 50 kHz, is transmission possible? (c) Find P_e if $E_b/N_0 = 10$ dB.

2. **Problem 15.2** — A 4 kHz audio signal is sampled at 8 kHz, quantized to 256 levels, Huffman encoded (average code length 1.5× entropy), modulated with QPSK, and transmitted over AWGN with $E_b/N_0 = 8$ dB. (a) Find bit rate. (b) Find overall P_e. (c) Suggest one method to reduce bit rate without increasing P_e.