Revision notes for Differential equation

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0. Preface

This note is for revision for the course MATH2351 in hkust. All contents are based on different lecture notes. The contents are for revision and reference only. Please don't expect that this puny note can substitute the lectures and the lecture notes directly. Last but the most important point, please treasure this note. I have really dedicated much effort and time into it. Please don't sell it!

Wish you can have a good grade

Enjoy:)

By the kindest author,

CP

1. Linear first order ODE

1.1 Separable first-order differentiable equations

$$g(y)\frac{dy}{dx} = f(x)$$

$$\int_{y_0}^{y} g(y)dy = \int_{x_0}^{x} f(x)dx$$

Example

$$y' = 4y - 61$$

$$\frac{dy}{dx} = 4y - 61$$

$$\frac{dy}{y - \frac{61}{4}} = 4\,dx$$

$$\ln\left(y - \frac{41}{4}\right) = 4x + C$$

$$y - \frac{41}{4} = e^{4x}e^{C}$$

$$y = ke^{4x} + \frac{41}{4}, where\ k > 0$$

1.2 Equilibrium solution

Given a differential equation

$$y' = f(y)$$

The equilibrium solution(s) are all value(s) of y such that $y' = 0$

1.3 Linear first order ode

$$\frac{dy}{dx} + p(x)y = g(x)$$

Multiplying both side with the integration factor $\mu(x)$.

$$\mu(x)(\frac{dy}{dx} + p(x)y) = \mu(x)g(x)$$

Idea: Use the trick of product rule,

$$\mu(x)(\frac{dy}{dx} + p(x)y) = \frac{d}{dx}(\mu(x)y(x))$$

$$\mu\frac{dy}{dx} + \mu py = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$$

Rearranging the terms,

$$pdx = \frac{d\mu}{\mu}$$

$$\int_{x_0}^{x} pdx = \int_{1}^{\mu}\frac{d\mu}{\mu}$$

$$\mu(x) = e^{\int_{x_0}^{x} p(x)dx}$$

After deriving the solution for finding $\mu(x)$, it is possible to solve the $y(x)$.

$$\mu(x)(\frac{dy}{dx} + p(x)y) = \mu(x)g(x)$$

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)g(x)$$

$$\int_{y_0}^{y}\mu(x)y(x)dy = \int_{x_0}^{x}\mu(x)g(x)dx$$

It is notice that the term $\boldsymbol{\mu(x)y(x)}$ depends on x instead of y.

$$\mu(x)y(x) - \mu(x) - y_0 = \int_{x_0}^{x}\mu(x)g(x)dx$$

$$y(x) = \frac{1}{\mu(x)}(y_0 + \int_{x_0}^{x}\mu(x)g(x)dx)$$

2. Linear second order ODE

2.1 The principal of superposition

In a differential equation

$$\ddot{x} + p(t)\dot{x} + q(t) = 0$$

Definition of principal of superposition:

Suppose $x = X_1(t)$ and $x = X_2(t)$ are the solution.

Then $x = C_1X_1(t) + C_2X_2(t)$ is also the solution.

This is an easy proof.

Please notice that $X_1(\mathrm{t}) \neq \mathrm{Constant} \times X_2(\mathrm{t})$

Otherwise, the solution collapse to $x = C_1X_1(t)$.

2.2 The Wronskian

In a differential equation

$$\ddot{x} + p(t)\dot{x} + q(t) = 0$$

$x = C_1X_1(t) + C_2X_2(t),\ x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$

Then x must fulfill the following system of equations:

$$\begin{cases} C_1X_1(t_0) + C_2X_2(t_0) = x_0 \\ C_1\dot{X}_1(t_0) + C_2\dot{X}_2(t_0) = u_0 \end{cases}$$

If C_1, C_2 have unique solution, then the determinant of this equation must not be equal to 0.

$$W = \begin{vmatrix} X_1(t_0) & X_2(t_0) \\ \dot{X}_1(t_0) & \dot{X}_2(t_0) \end{vmatrix} \neq 0$$

This determinant is also called the **Wronskian**.

So, it implies that $X_1(\mathrm{t}) \neq \mathrm{Constant} \times X_2(\mathrm{t})$

It is a very good way to check out the answer.

2.3 Homogeneous second order ode

Homogeneous second order ode is defined as:

$$a\ddot{x} + b\dot{x} + cx = 0$$

$$a, b, c \text{ are constants and } a \neq 0$$

Guessing game, let $x=e^{rt}$

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

This equation collapse to a quadratic equation, this quadratic equation is also called **characteristic equation**.

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then r can be

I. Distinct real roots

II. Conjugate imaginary roots

III. Repeated real roots

Case 1: Distinct real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

By the principle of superposition,

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case 2: Conjugate imaginary roots

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} = \lambda + i\mu$$

$$r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a} = \lambda - i\mu$$

$$x_1 = e^{(\lambda+\mu i)t} = e^{\lambda t}(cos(\mu t) + sin(\mu t)), x_2 = e^{(\lambda-\mu i)t} = e^{\lambda t}(cos(\mu t) - sin(\mu t))$$

However, we don't want complex function. How to change them to real functions?

We can use the trick of the principle of superposition

$$x_1 + x_2 = 2e^{\lambda t}cos(\mu t)$$

It shows that x=$e^{\lambda t}cos(\mu t)$ is also a solution

$$x_1 - x_2 = 2ie^{\lambda t}sin(\mu t)$$

It shows that x=$e^{\lambda t}sin(\mu t)$ is also another solution

By the principle of superposition,

$$C_1e^{\lambda t}cos(\mu t) + C_2e^{\lambda t}sin(\mu t)$$

Case 3: Repeated real roots

$$r = \frac{-b}{2a}$$

$$x_1 = x_2 = e^{rt}$$

However, does it make sense?

Just check the Wronskian!

$$W = \begin{vmatrix} X_1(t_0) & X_2(t_0) \\ \dot{X}_1(t_0) & \dot{X}_2(t_0) \end{vmatrix} = \begin{vmatrix} e^{rt_0} & e^{rt_0} \\ re^{rt_0} & re^{rt_0} \end{vmatrix} = 0\ ???$$

So x_2 must be $v(t)e^{rt}$

$$a\ddot{x_2} + b\dot{x_2} + cx_2 = 0$$

By substituting the derivatives,

$$a\ddot{v} = 0$$

$$v(t) = C_3 + C_4 t$$

By the principle of superposition,

$$x = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$$

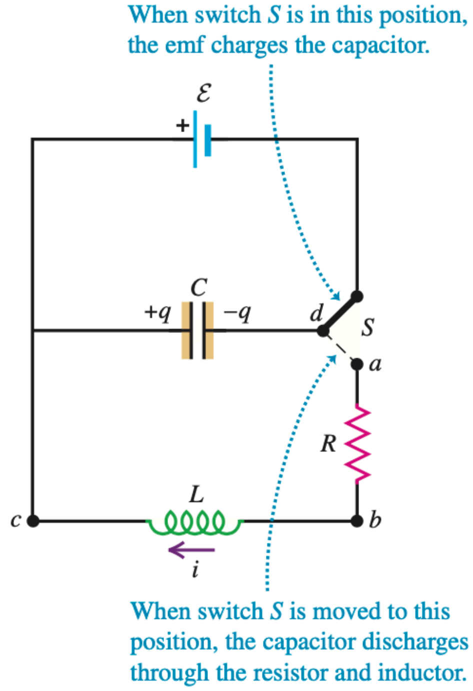
$$W = \begin{vmatrix} e^{rt_0} & t_0 e^{rt_0} \\ re^{rt_0} & e^{rt_0} + rt_0 e^{rt_0} \end{vmatrix} \neq 0$$

This combination of solution is possible.

The trick of multiplying the factor $v(t)$ is called **Reduction of Order**.

This trick can be expanded and solve any second order differential equation

$$\ddot{x} + p(t)\dot{x} + q(t) = 0$$



Example: LRC circuit in general physics 2

This circuit can be modelled as

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

This is a second order differential equation

Solve for characteristic equation, the solutions are complex

$$r = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\left(\frac{1}{LC}\right)}}{2} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$q = C_1 e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right) + C_2 e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right)$$

$$q = Ae^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \phi\right)$$

2.4 Inhomogeneous second order ode

$$\ddot{x} + p(t)\dot{x} + q(t) = g(t)$$

Then the solution is

$$x = C_1X_1(\mathrm{t}) + C_2X_2(\mathrm{t}) + \mathrm{Y}(\mathrm{t})$$

Which $C_1X_1(\mathrm{t}) + C_2X_2(\mathrm{t})$ is the solution of homogeneous solution of

$$\ddot{x} + p(t)\dot{x} + q(t) = 0$$

Also, $\mathrm{Y}(\mathrm{t})$ is the **particular solution** of the equation

$$\ddot{x} + p(t)\dot{x} + q(t) = g(t)$$

We can only find that particular solution by guessing.

Some common particular solution is either one (or union) of the following functions

Exponential functions $x = e^{rt}$

Harmonic functions $x = Asin(\omega t) + Bcos(\omega t)$

3. Series solutions and Euler Equations

3.1 Series solutions

This chapter we consider the differential equations

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

- Ordinary points

$$The\ point\ x0\ is\ called\ ordinary\ point\ if\ P(x_0) \neq 0$$

- Singular points

$$The\ point\ x0\ is\ called\ ordinary\ point\ if\ P(x_0) = 0$$

To solve the differential equations near an ordinary point x_0

We can let

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then we can iterate the recurrence relationship of a_n and find out the final series solution.

3.2 Euler equations

Recall the equation

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

A relatively simple differential equation that has a singular point can be called as **Euler equation**.

$$L[y]=x^2y''+\alpha xy'+\beta y=0$$

α,β are constants, $x_0=0$ is a singular point

To solve this type of differential equation is very simple

Let's let $y=x^r, y'=rx^{r-1}, y''=r(r-1)x^{r-2}$

Substitute back to the original equation $L[y]$.

We can obtain its characteristic equation

$$L[y]=r(r-1)x^r+\alpha rx^r+\beta x^r=0$$

$$r^2+(\alpha-1)r+\beta=0, x>0$$

Solve for r, we can get

$$r=\frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$$

Then r can be

I. Distinct real roots
II. Conjugate complex roots
III. Repeated real roots

Case I, real distinct roots

By the principle of superposition,

$$y(x) = C_1x^{r_1} + C_2x^{r_2}$$

Case II, complex conjugate roots

$$r_1 = \lambda + i\mu, r_2 = \lambda - i\mu, \mu \neq 0$$

Then

$$x^r = e^{lnx^r} = e^{rlnx} = e^{(\lambda \pm i\mu)lnx}$$

$$x^r = e^{\lambda}e^{\pm i\mu lnx} = e^{\lambda}(\cos(\mu lnx) + \mathrm{i}\sin(\mu lnx))$$

By the principle of superposition,

$$y = C_1e^{\lambda}cos(\mu lnx) + C_2e^{\lambda}sin(\mu lnx)$$

Case III, repeated real roots

Use the reduction of order to get the second solution

$$y_2(x) = v(x)x^{r_1}$$

Then we can find that

$$y_2(x) = x^{r_1}\ln(x)$$

By the principle of superposition,

$$y(x) = C_1x^{r_1} + C_2x^{r_1}\ln x$$

The Wronskian of these three cases are not equal to zero.

For the shifted Euler equation

$$(x-x_0)^2y''+\alpha(x-x_0)y'+\beta y=0$$

Let $y=(x-x_0)^r$, solve for the characteristic equation

$$r^2+(\alpha-1)r+\beta=0, x>0$$

The solution of r have the following three cases

I. Distinct real roots,

$$y(x)=C_1|x-x_0|^{r_1}+C_2|x-x_0|^{r_2}$$

II. Conjugate complex roots,

$$y=C_1e^{\lambda}cos(\mu\ln\ |x-x_0|)+C_2e^{\lambda}sin(\mu\ln\ |x-x_0|)$$

III. Repeated real roots,

$$y(x)=C_1|x-x_0|^{r_1}+C_2|x-x_0|^{r_1}\ln|x-x_0|$$

3.3 Classifying singular points

Consider the following differential equation, $P(x),Q(x),R(x)$ are all polynomials

$$P(x)y''+Q(x)y'+R(x)y=0$$

Consider singular point(s) x_0 where $P(x_0)=0$

- Regular singular point

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}\neq\pm\infty\ AND\ \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}\neq\pm\infty$$

- Irregular singular point

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}=\pm\infty\ OR\ \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}=\pm\infty$$

4. Laplace transform

4.1 Definition of Laplace transform

The definition of Laplace transform is as following.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

It is possible to use Laplace Transform to solve differential equation.

See appendix.

4.2 Heaviside step function

The Heaviside step function is defined as following

$$u_c(t) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases}$$

We can model any discontinuous function by using the step function. For example,

$$f(t) = \begin{cases} a(t), t < c \\ b(t), t \geq c \end{cases}$$

$$f(t) = a(t) + u_c(t)(b(t) - a(t))$$

4.3 Impulse function and Dirac delta function

The Impulse function is defined as following

$$g(t) = \begin{cases} very\ big, t_0 - \tau < t < t_0 + \tau \\ 0, otherwise \end{cases}$$

$$\int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt$$

It is notice that

$$g(t) = \begin{cases} c, t_0 - \tau < t < t_0 + \tau \\ 0, otherwise \end{cases}$$

$$\int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} cdt = 2\tau c$$

If $c = \frac{1}{2\tau}$

Then $\int_{-\infty}^{\infty} g(t)dt = 1$

The Dirac delta function is defined as a unit impulse at an arbitrary point t_0, with following properties.

1. $\delta(t - t_0) = 0\ \forall t \neq t_0$
2. $\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1$

4.4 Laplace transform of step function and Dirac delta function

Laplace transform of Heaviside step function is

$$\mathcal{L}\{u_c(t)\} = \int_0^{\infty} e^{-st} u_c(t) dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st} dt$$

$$= \lim_{b\to\infty} [-\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s}]$$

$$= \frac{e^{-cs}}{s}$$

Laplace transform of Dirac delta function is

$$\mathcal{L}\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st} \delta(t - t_0) dt$$

$$= \lim_{\tau\to 0} \int_{t-t_0}^{t+t_0} e^{-st} \delta(t - t_0) dt$$

$$= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{t-t_0}^{t+t_0} e^{-st} dt$$

By some limit and integration rules,

$$= e^{-st_0}$$

5. System of odes

5.1 Introduction of system of odes and method of solving

$$\begin{cases} \dot{x_1} = \mu_{11}x_1 + \mu_{12}x_2 + \ldots + \mu_{1n}x_n \\ \dot{x_2} = \mu_{21}x_1 + \mu_{22}x_2 + \ldots + \mu_{2n}x_n \\ \cdots \end{cases}$$

We can rewrite is as a matrix equation

$$x(\dot{t}) = Ax$$

$$x(\dot{t}) = \begin{pmatrix} \dot{x_1} \\ \vdots \\ \dot{x_n} \end{pmatrix}$$

$$A = \begin{pmatrix} \mu_{11}x_1 & \cdots & \mu_{1n}x_n \\ \vdots & \ddots & \vdots \\ \mu_{n1}x_1 & \cdots & \mu_{nn}x_n \end{pmatrix}$$

How to solve this system of equations?

Let $x = \vec{v}e^{\lambda t}$, which $\vec{v}$ is 1xn matrix(vector)

Then $\dot{x} = r\vec{v}e^{\lambda t}$

$$x(\dot{t}) = Ax$$

$$\lambda\vec{v} = \mathrm{A}\vec{v}$$

It is an eigenvalue problem that we have learnt in Linear Algebra.

Another characteristic equation is obtained.

$$det(A - \lambda I) = 0$$

Solving eigenvalue λ

This equation can be all real roots and/or conjugate complex roots.

Assume that all eigenvalues are distinct real. (We will deal with complex roots in the next session)

We can find all the corresponding eigenvectors $\vec{v}$

Please notice that eigenvectors are not unique, because this equation

$$(A - \lambda I)\vec{v} = 0$$

Has non-trivial solution as $det(A - \lambda I) = 0$.

By the principle of superposition,

$$x = C_1\overrightarrow{v_1} + C_2\overrightarrow{v_2} + \ldots + C_n\overrightarrow{v_n}$$

5.2 Special case-complex eigenvectors

The method is very similar.

Find its corresponding complex eigenvectors.

Rewrite the complex eigenvectors as

$$\vec{v} = \overrightarrow{v_{Real}} + i\overrightarrow{v_{Imaginary}}$$

Use the **Euler Formula**,

$$e^{i\theta} = cos\theta + isin\theta$$

By the principle of superposition,

One possible solution is its real part, while another is its imaginary part.

Example

$$\dot{x} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} x$$

Step 1, find eigenvalues

$$det\begin{pmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{pmatrix} = 0$$

$$\lambda = -\frac{1}{2} \pm i$$

Step 2, find corresponding eigenvectors

For $\lambda_1 = -\frac{1}{2} + i$

$$\begin{pmatrix} -\frac{1}{2} - \lambda_1 & 1 \\ -1 & -\frac{1}{2} - \lambda_1 \end{pmatrix} \vec{v} = \vec{0}$$

$$\vec{v}_1 = \binom{1}{0} + i\binom{0}{1}$$

Step 3, find the solution

$$x_1 = \vec{v}_1 e^{\lambda_1 t} = \left(\binom{1}{0} + i\binom{0}{1}\right) e^{-\frac{t}{2}}(cost + isint)$$

$$x_1 = e^{-\frac{t}{2}}\left(\binom{cost}{-sint} + i\binom{sint}{cost}\right) = e^{-\frac{t}{2}}\binom{cost}{-sint}$$

It is notice that x has imaginary part, but since x has two solutions, the imaginary part will be diminished by the principle of superposition.

Step 4, repeat step 3 by substituting different eigenvectors that were found in step 2

For $\lambda_2 = -\frac{1}{2} - i$

Here has a trick, since λ_1, λ_2 are conjugate, then the final solution should be the superposition of real part and the imaginary part of x_1.

$$x = e^{-\frac{t}{2}}\left(c_1\binom{cost}{-sint} + c_2\binom{sint}{cost}\right)$$

6. Exam questions

6.1 Midterm Exam

1. Find the real-valued equilibrium solutions of

$$y' = (2 + y^2)(y^2 - 2\pi y)$$

(4 points)

2. Given

$$y' = 7y - 6$$

(10 points)

a) Find the equilibrium solution

b) Use the method of calculus to find the non-equilibrium solutions

c) Find the general solution expression for ALL solutions and specify the range of the parameter in your general solution expression corresponding to the equilibrium and non-equilibrium solution found in a) and b).

3. Find a fundamental set of solutions of the equation

$$y'' + 8y' + 16y = 0$$

(10 points)

4. Solve

$$\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 0, \qquad y'(0) = 1 \end{cases}$$

(12 points)

5. Find the general solution of

$$y'' - 2y' - 3y = 2e^{3t} - \sin(2t) + t$$

(20 points)

6. Solve

$$\begin{cases} ty' - 2y - 7t^2 = 0 \\ y(-1) = 2 \end{cases}$$

(12 points)

7. Given

$$\frac{dy}{dx} = \frac{y\,cos\,x}{3y^4 + 5}$$

   a) Find all the solutions of the differential equation. (10 points)
   b) Find the particular solution satisfying the initial condition $y(0) = 1$. (3 points)
   c) Find the particular solution satisfying the initial condition $y(0) = 0$. (2 points)

8. Given a solution $y_1 = \frac{1}{x}$ of the following equation:

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0, \qquad x > 0$$

   a) Apply the method of reduction of order to find another solution y_2 such that y_1 and y_2 form a fundamental set of solutions of this equation for $x > 0$. (12 points)
   b) Compute the Wronskian of y_1 and y_2. (3 points)
   c) Find the general solution of this equation for $x > 0$. (2 points)

6.2 Final Exam

1. Find the first five nonzero terms of the power series solution about $x_0 = 2$ of the initial value problem

$$\begin{cases} y'' + 2xy' - y = 0 \\ y(2) = 1, \qquad y'(2) = -1 \end{cases}$$

(15 points)

2. Consider the following equation

$$(x+3)^2 y'' - 5(x+3)y' + 9y = 0$$

   a) Identify all the ordinary and singular points; and for singular point(s), identify the type of regular singular of irregular singular point(s). Explain your reasons. (5 points)
   b) Find the general solution of the equation. (10 points)

3. Use Laplace transform to solve the initial value problem

$$y''(t) + 4y(t) = f(t) + \delta(t - 1.5)$$
$$y(0) = 0, \qquad y'(0) = 0;$$

where

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ (t-4)/6, & 4 \le t < 10 \\ 1 & t \ge 10 \end{cases}$$

4. Apply the Laplace transform to solve the initial value problem:

$$2y'' + y' + 2y = e^{2t}, y(0) = 0, y'(0) = 1$$

(15 points)

5. Solve the initial value problem:

$$x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(15 points)

6. Consider the initial value problem:

$$y'' + y' - 6y = 0$$

$$y(0) = 3, \qquad y'(0) = 0.$$

(20 points)

a) Transform the problem into an initial value problem of first- order system of equations.

b) Solve the initial value problem obtained in (a) using the method for systems of first-order equations.

c) Find the solution y of the original initial value problem through the results obtained in a) and b).

Appendix: Table of Laplace Transform

Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}, \quad s > 0$
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\dfrac{a}{s^2 + a^2}, \quad s > 0$
6.	$\cos(at)$	$\dfrac{s}{s^2 + a^2}, \quad s > 0$
7.	$\sinh(at)$	$\dfrac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$\cosh(at)$	$\dfrac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16.	$\displaystyle\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$