ELEC2100: Signals and Systems

Lecture 14

Signal: time domain
system:

Discrete-time Fourier transform

(Analysis & Deduction)

(Ref: Chapter 5 O&W)

I. The Discrete-Time Fourier transform (DTFT) (Analysis)

II. DTFT Examples

III. DTFT for Periodic Signals

IV. Properties of DTFT

I. The Discrete-Time Fourier transform (DTFT)

- DTFT is our fourth and last variant of Fourier analysis. It is for aperiodic DT signals.
- Here, we simply start by stating the DTFT synthesis and analysis equations pair:

Discrete-Time Fourier Transform (DTFT)

Synthesis equation

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\,\omega$$

Superposition of complex sinusoids $e^{j\omega n}$ through integration for ω over a range of 2π

Analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

An inner product sum of $x[n]$ with $e^{j\omega n}$

- $X(e^{j\omega})$, the DTFT, is a density function that describe the frequency composition of $x[n]$. It is the spectrum, or frequency domain representation of $x[n]$.

Proof of the DTFT Synthesis/Analysis Pair

- To prove the validity of the synthesis-analysis equation pair, we substitute the analysis equation back into the synthesis equation:

$$\underbrace{\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\,\omega}_{\text{the synthesis equation}} = \frac{1}{2\pi}\int_{2\pi}\Bigg(\underbrace{\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}}_{\text{Analysis equation for } X(e^{j\omega})}\Bigg)e^{j\omega n}d\,\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{2\pi} x[k]e^{j\omega(n-k)}d\,\omega = x[n]$$

use a different variable k

Move $x[k]$ out of the integral

Change order of integration and summation

all terms are zero except when $n-k=0$, or $k=n$, since:

$$\int_{2\pi} e^{j\omega m}d\omega = \begin{cases} 0 & m \neq 0 \\ 2\pi & m = 0 \end{cases}$$

- Hence, superimposing complex sinusoids (through integrating) using $X(e^{j\omega})$ as weight does reproduce $x[n]$.

- The monstrous looking notation $X(e^{j\omega})$ can be ***very scary*** to students. But again $X(e^{j\omega})$ is simply a function of frequency ω!

The Complicated Notation $X(e^{j\omega})$ Explained Again

- Recall that the ***system function*** of a DT LTI system is given by the z-transform of the impulse response:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

The system function provides the eigenvalue when the input is a complex exponential:
$$z_1^n \rightarrow H(z_1)\, z_1^n$$

- The z-transform evaluated at $z = e^{j\omega}$ (i.e., $|z| = 1$) gives the ***frequency response*** which is the DTFT of the impulse response $h[n]$

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega};|z|=1}$$

The frequency response provides the eigenvalue:
$$e^{j\omega n} \rightarrow H(e^{j\omega}) e^{j\omega n}$$

- For any $x[n]$, using the intimidating notation $X(e^{j\omega})$ for its DTFT has the following advantages:
  1. We can use the same function $X(\cdot)$ to refer to z-transform and DTFT
  2. Reminds us that the DTFT is the z-transform $X(z)$ with $z = e^{j\omega}$
  3. Makes clear that the DTFT is 2π-periodic in ω!

Periodicity of DTFT

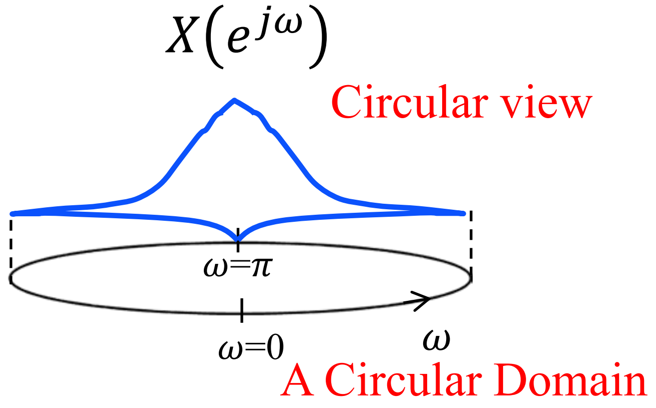
- The notation $X(e^{j\omega})$ reminds that DTFT is 2π-periodic in ω because the value of the argument is unchanged if we add 2π to ω.

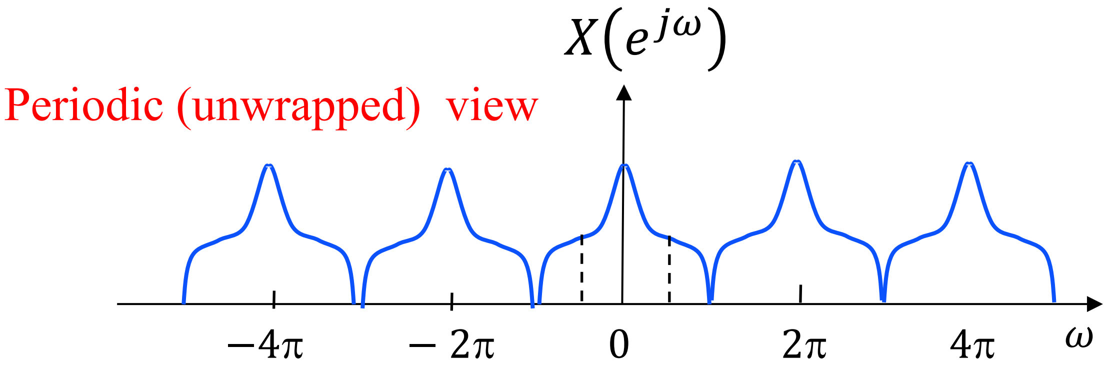
$$e^{j(\omega+2\pi)} = e^{j\omega}\cancelto{1}{e^{j2\pi}} = e^{j\omega}$$

Therefore DTFT must be 2π-periodic $\Rightarrow X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$

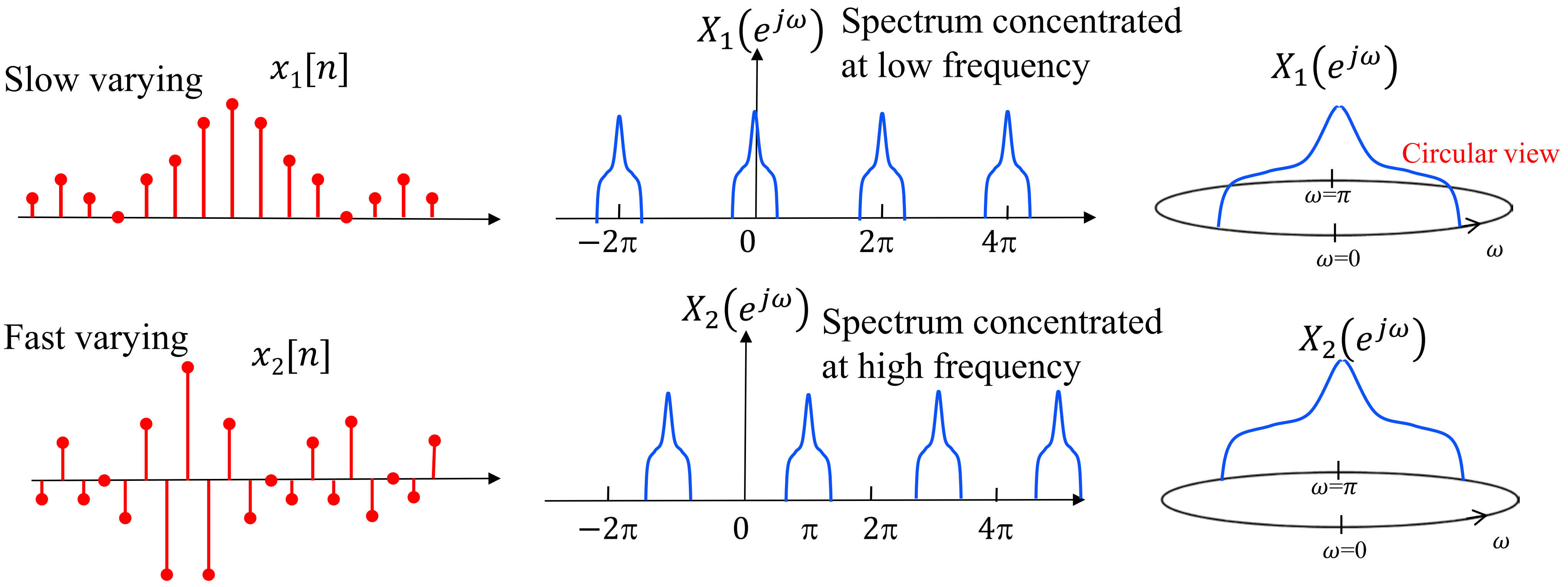
- As discussed before, DT frequency is periodic (actually circular): ω and $\omega + k2\pi$ mean the same frequency.

- Like the FS coefficients for a DT periodic signal, a DTFT is defined over a *circular domain*. For ease of mathematical representation, we allow ω to take arbitrary values and regard $X(e^{j\omega})$ as periodic over ω.





- As $x[n]$ is aperiodic, its spectrum (the DTFT) is a density function rather than a set of discrete frequencies. It is because as the period N becomes infinity, the fundamental frequency approaches zero.
- For DT signals, $\omega = m2\pi$ represents low frequency and $\omega = (2m+1)\pi$ represents high frequency.



Lecture 14

Discrete-time Fourier transform

I. The discrete-time Fourier transform (DTFT)

II. DTFT Examples (Analysis)

III. DTFT for Periodic Signals

IV. Properties of DTFT

II. DTFT Examples

Example 5.1 One-Sided Decaying Exponential

- Find DTFT of $x[n] = a^n u[n]$, $|a| < 1$.

Apply the analysis equation:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \overbrace{a^n u[n]}^{x[n]} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$x[n] = 0$ for $n < 0$

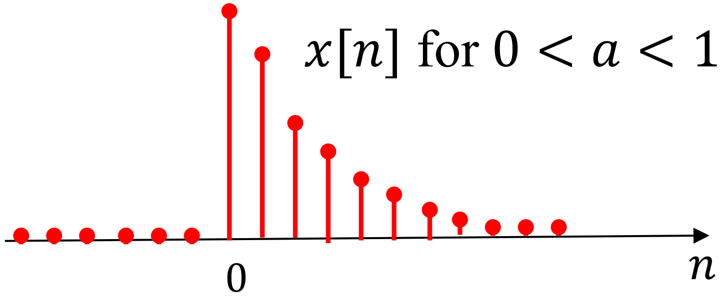
$$= \frac{1}{1 - \underbrace{ae^{-j\omega}}_{2\pi\text{-periodic in } \omega}}$$

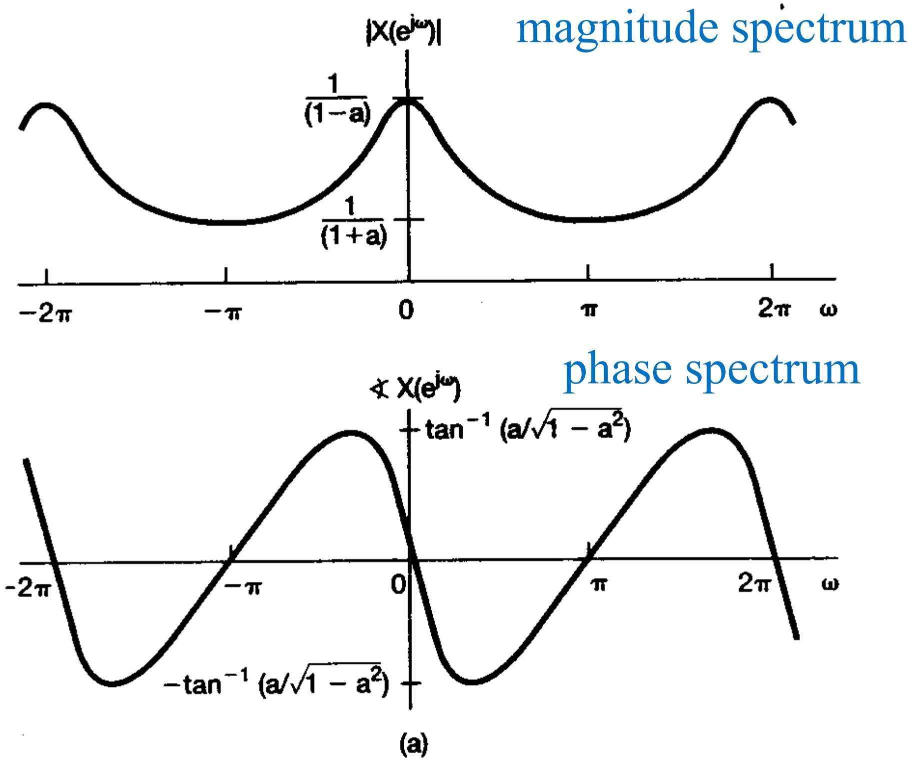
Infinite geometric sum:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

with $\alpha = ae^{-j\omega}$

But if $|a| \geq 1$, sum does not converge; i.e., no DTFT

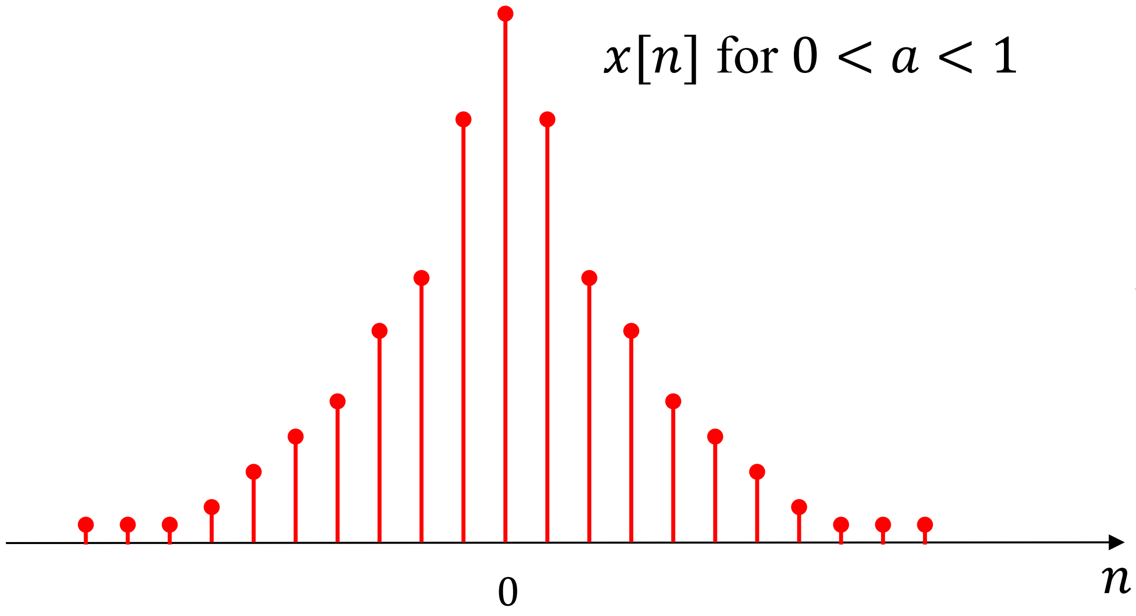




- *See Figure 5.4 for the magnitude and phase of the spectrum.*
- *Note that the expression is 2π-periodic. Mathematical expression for a DTFT should be 2π-periodic in ω.*

Example 5.2 Two-Sided Decaying Exponential

- Example 5.2: Consider $x[n] = a^{|n|}$, $|a| < 1$



$$\sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} (a^{-1} e^{-j\omega})^{n} + \sum_{n=0}^{\infty} (a e^{-j\omega})^{n}$$

- **Example 5.2 (cont.):**

$$\sum_{m=1}^{\infty}\alpha^m = \alpha\sum_{m=0}^{\infty}\alpha^m = \frac{\alpha}{1-\alpha}$$

Break into two sums

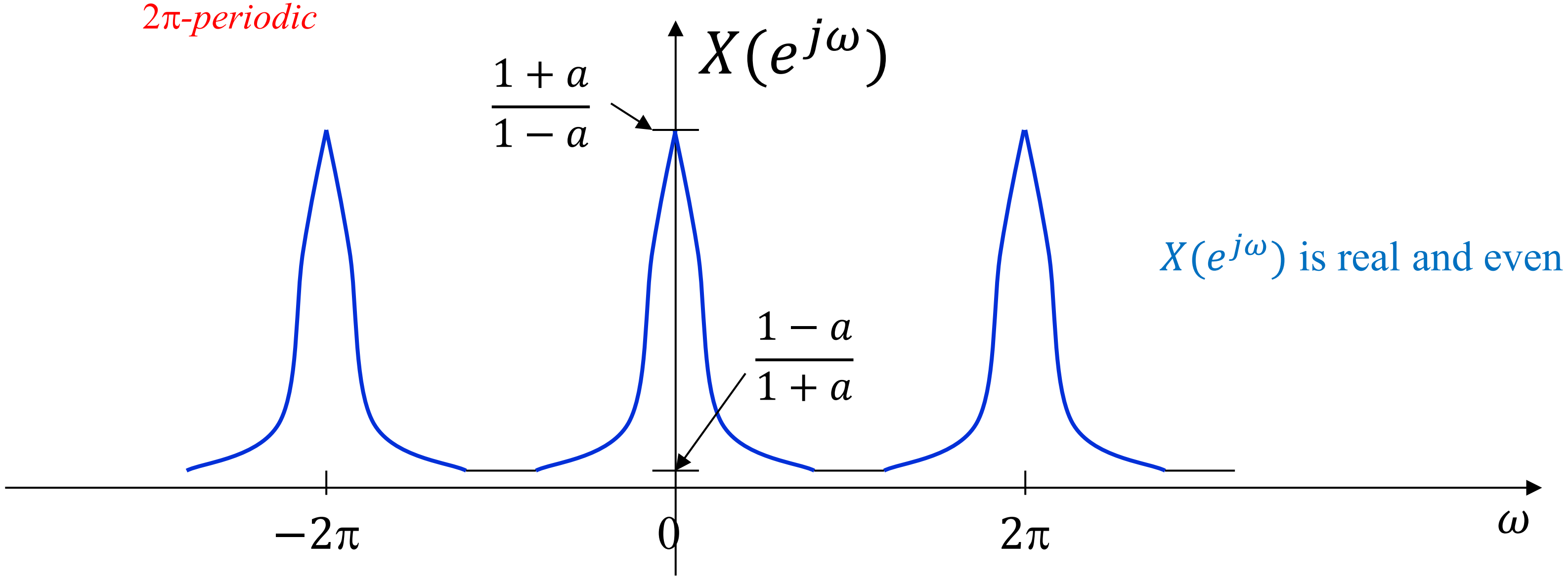
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} = \frac{\left(1-ae^{j\omega}\right) + ae^{j\omega}(1-ae^{-j\omega})}{(1-ae^{-j\omega})(1-ae^{j\omega})} = \frac{1-ae^{j\omega}+ae^{j\omega}-a^2}{1-ae^{-j\omega}-ae^{j\omega}+a^2}$$

$$= \frac{1-a^2}{1-2a\cos(\omega)+a^2}$$

2π-periodic

Note again that the expression is 2π-periodic

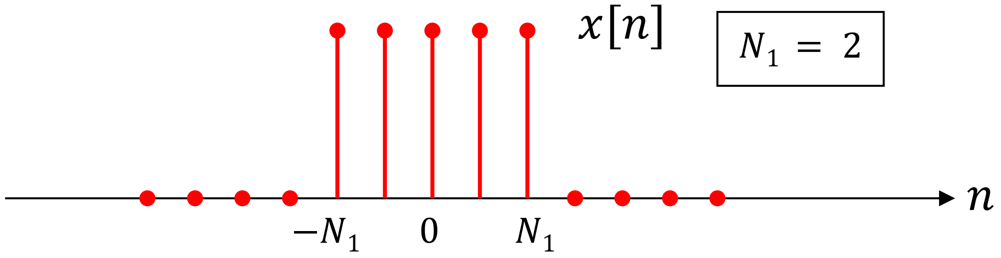


$X(e^{j\omega})$ is real and even because $x[n]$ is real and even

Example 5.3: Rectangular Pulse/Window

$$\sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{[(e^{-j\omega})^{2N_1+1} - 1]e^{j\omega N_1}}{e^{-j\omega} - 1}$$

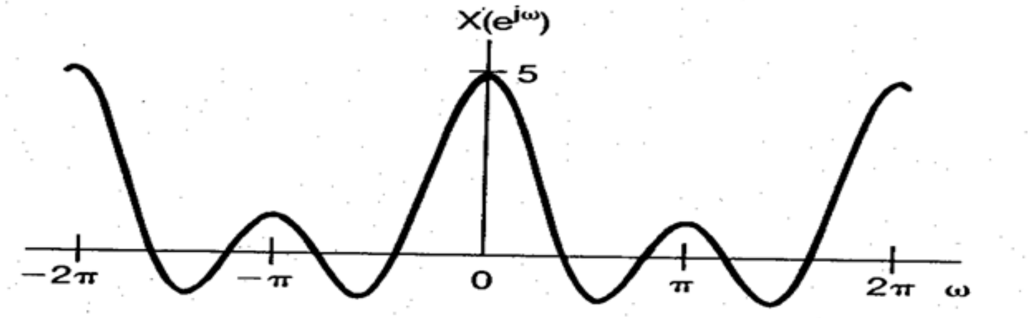
$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$



You may skip the detailed derivation

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = e^{j\omega N_1}\sum_{n=0}^{2N_1} e^{-j\omega n} = \frac{e^{j\omega N_1}\left(1 - e^{-j\omega(2N_1+1)}\right)}{1 - e^{-j\omega}}$$

$$= \frac{e^{j\omega N_1} e^{-j\omega\left(\frac{2N_1+1}{2}\right)}\left(e^{j\omega\left(\frac{2N_1+1}{2}\right)} - e^{-j\omega\left(\frac{2N_1+1}{2}\right)}\right)}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} = \frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$



Convince yourself that the whole expression is 2π-periodic even though the denominator is 4π-periodic. When ω is increased by 2π , both the numerator and denominator are negated:

$$\frac{\sin(\omega+2\pi)\left(N_1+\frac{1}{2}\right)}{\sin\left(\frac{\omega+2\pi}{2}\right)} = \frac{\sin\left(\omega\left(N_1+\frac{1}{2}\right)+N_1 2\pi+\pi\right)}{\sin\left(\frac{\omega}{2}+\pi\right)} = \frac{-\sin\omega\left(N_1+\frac{1}{2}\right)}{-\sin\left(\frac{\omega}{2}\right)}$$

5.1.3 Convergence of DTFT

- We ask ourselves the convergence issue again. Can we always accurately represent a DT signal $x[n]$ by its DTFT?

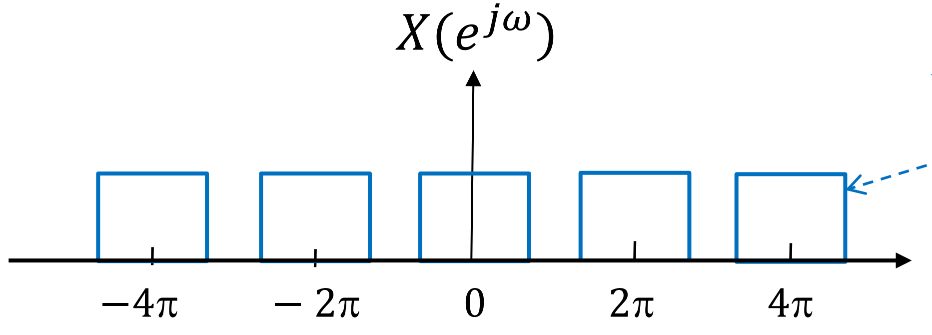
- For DTFT, $x[n]$ is DT and therefore it does not have discontinuities. There is no convergence issue when we synthesize $x[n]$ from its DTFT $X(e^{j\omega})$.

Weight　　Sinusoidal functions in ω

- But if we consider the transform equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

We can view $X(e^{j\omega})$ as a weighted sum of sinusoidal functions in ω. Hence, $X(e^{j\omega})$ should be a smooth function in ω without discontinuities. Hence an $X(e^{j\omega})$ as shown below for an ideal low pass filter is an "idealization":



These edges must be "smooth", but it is simpler to idealize them as discontinuous

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III. 5.2 DTFT for periodic signals

- Recall that for a CT complex sinusoid $e^{j\omega_0 t}$, the CTFT is an impulse in frequency:

$$x(t) = e^{j\omega_0 t} \overset{CTFT}{\longleftrightarrow} X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

- What about the DTFT of a DT complex sinusoid $e^{j\omega_0 n}$?

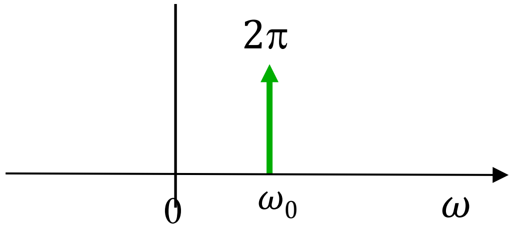
- We expect that it should again be simply an impulse at the given frequency ω_0, but DTFT is over a circular domain and it should be expressed as 2π-periodic, so we have:

$$x[n] = e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 \underbrace{- 2\pi l}_{2\pi\text{-periodic}}) \quad \textbf{(5.18)}$$

A Poisson sum expression!

Spectrum of CT Complex Sinusoid

CTFT

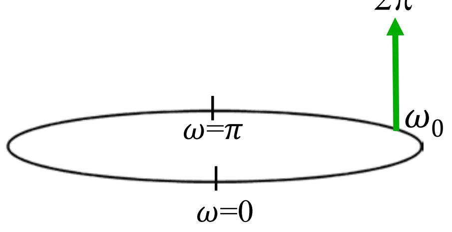


$X(j\omega) = 2\pi\delta(\omega - \omega_0)$

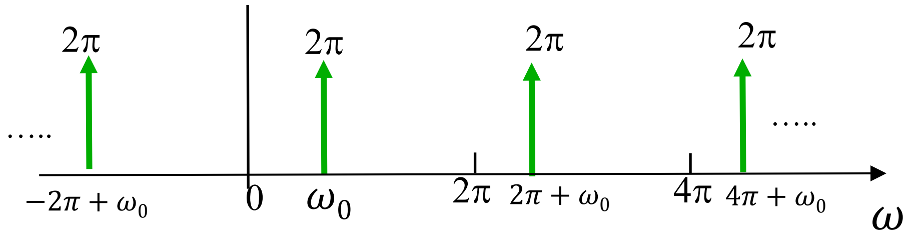
Spectrum of DT Complex Sinusoid

DTFT

Circular view



Unwrapped view



$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- To verify (5.18) is correct, we apply the synthesis equation to the DTFT

$\delta(\omega-\omega_0)g(\omega) = \delta(\omega-\omega_0)g(\omega_0)$

$$\underbrace{\frac{1}{2\pi}\int_{2\pi} \overbrace{\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)}^{X(e^{j\omega})} e^{j\omega n}\,d\omega}_{\text{Synthesis equation}} = \frac{1}{2\pi}\int_{\substack{\text{A } 2\pi \text{ interval}\\ \text{containing } \omega_0}} 2\pi\delta(\omega-\omega_0)e^{j\cancelto{\omega_0}{\omega} n}\,d\omega = e^{j\omega_0 n}$$

No need for summation; there is only one impulse within the interval

There is only one impulse within any interval of width 2π. Choosing the interval that contains ω_0, we evaluate the integral to $e^{j\omega_0 n}$.

- A lazy way to *express* the DTFT of a DT complex sinusoid is to specify it only for $0 \le \omega < 2\pi$:

$$\boxed{X(e^{j\omega}) = 2\pi\delta(\omega-\omega_0) \quad \text{for } 0 \le \omega < 2\pi}$$

The form of the argument of $X(e^{j\omega})$ implies that it is 2π-periodic in ω, so specifying $X(e^{j\omega})$ over one period is sufficient. But the lazy expression is not one that you can conveniently “plug” into another expression.

DTFT for periodic signals

Weighted sum of DT complex sinusoids

- Form Chapter 3, if $x[n]$ is periodic, it has a FS expansion: $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ $\quad \omega_0 = \frac{2\pi}{N}$
- By linearity and (5.18), the DTFT should be:

DTFT of $e^{jk\omega_0 n}$

$x[n]$ made up of N complex sinusoids

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$

three different expressions, all representing the same thing

Since a_k is N-periodic, a simpler expression is:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \qquad \textbf{(5.20)}$$

If lazy, we may also specify $X(e^{j\omega})$ only for $0 \leq \omega < 2\pi$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k\omega_0) \quad 0 \leq \omega < 2\pi$$

Three different ways to express the DTFT

For an N-periodic $x[n]$:

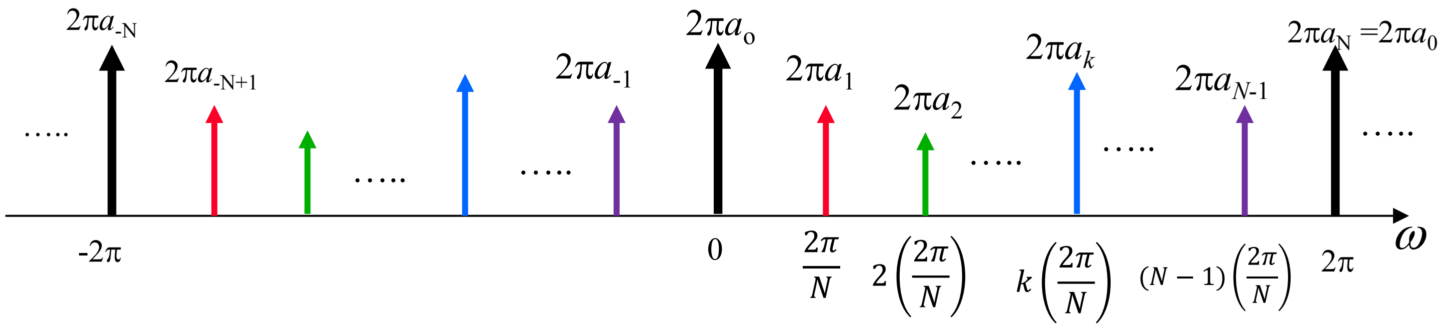
$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \quad \overset{DTFT}{\longleftrightarrow}$$

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$

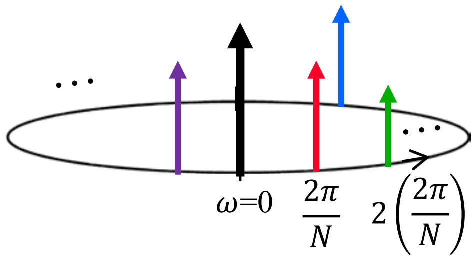
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \qquad \textbf{(5.20)}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k\omega_0) \quad 0 \le \omega < 2\pi$$

Unwrapped view



Circular view

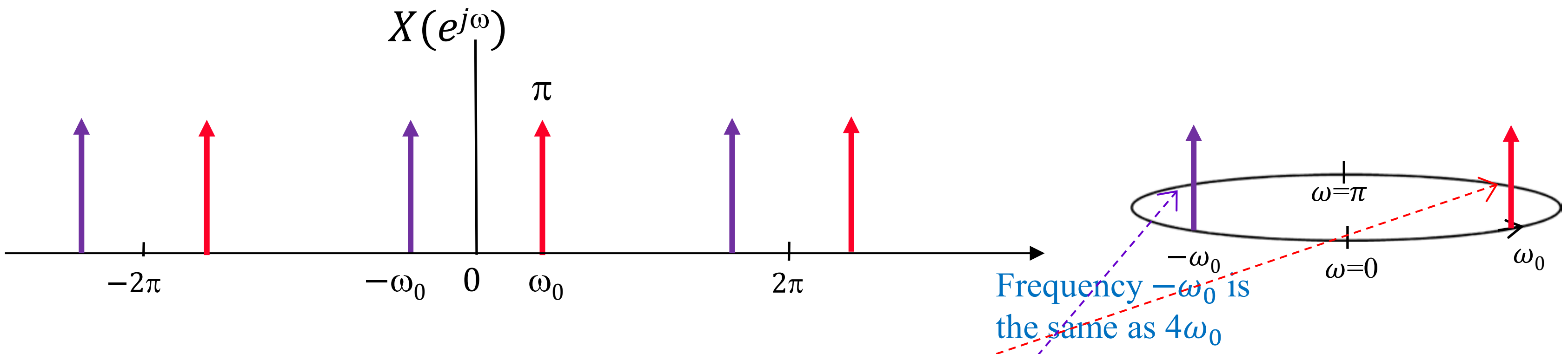


There are actually only N different harmonics

Example 5.5: DTFT of Discrete-Time Sinusoid

Let: $x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \ \omega_0 = \frac{2\pi}{5}$

DTFT is: $$X(e^{j\omega}) = \underbrace{\boxed{\sum_{l=-\infty}^{\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right)}}_{\text{DTFT}} + \underbrace{\boxed{\sum_{l=-\infty}^{\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)}}_{\text{DTFT}}$$



A lazy expression for the DTFT would be:

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right) \quad \text{for } -\pi \le \omega < \pi$$

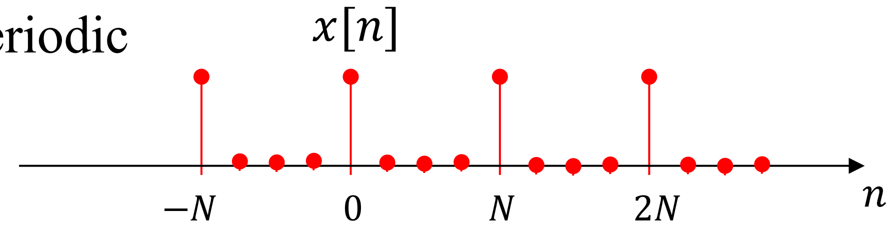
$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega - 4 \times \frac{2\pi}{5}\right) \quad \text{for } 0 \le \omega < 2\pi$$

Example 5.6: DTFT of DT Infinite Impulse Train

Example 5.6 Consider the discrete-time infinite impulse train/periodic impulse train:

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

Poisson sum



$x[n]$ is N-periodic, it has a Fourier series representation and a finite set of F.S. coefficients:

$$x[n] = \sum_{n=<N>} a_k e^{jk\frac{2\pi}{N}n} \qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}; \quad -\infty \le k \le \infty$$

Unwrapped view

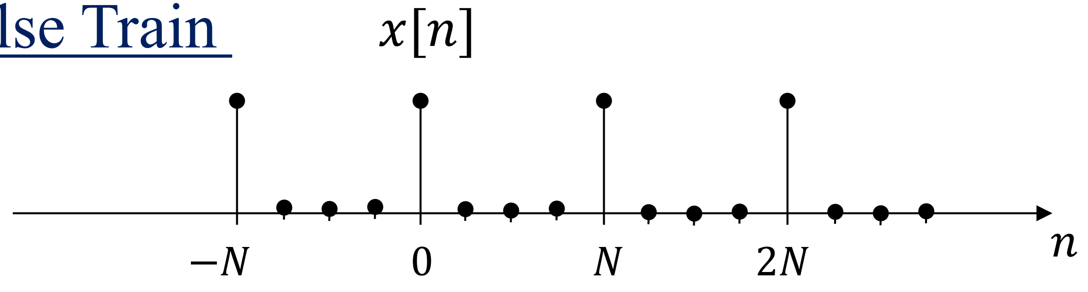
Hence from Eq.(5,20), $x[n]$'s DTFT is:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{k2\pi}{N}\right)$$

So the spectrum of a periodic impulse train in time is again a periodic impulse train in frequency

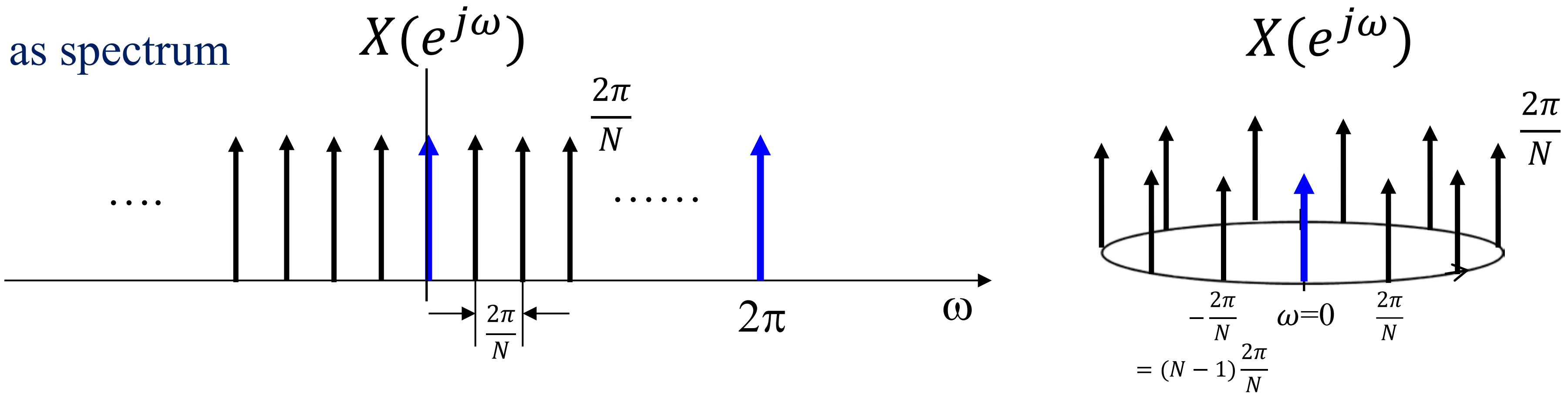
Time Domain

Periodic Impulse Train

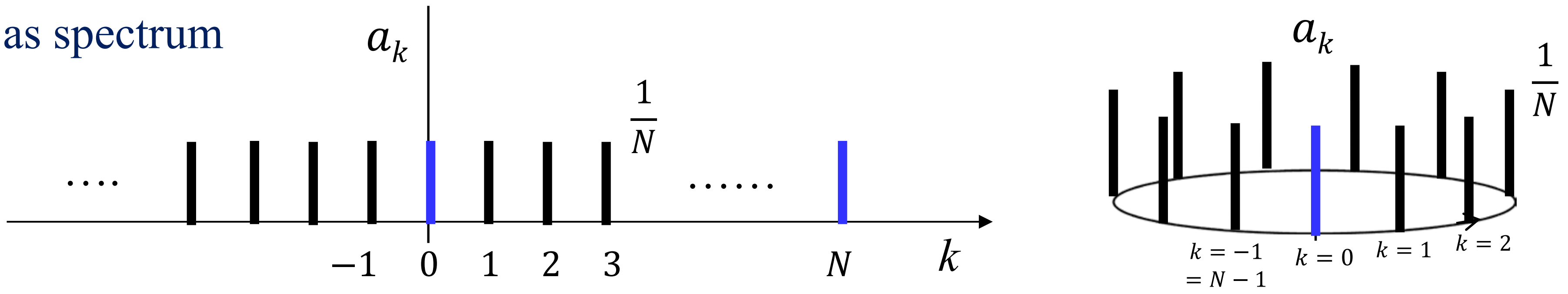


Frequency Domain

DTFT as spectrum



DTFS as spectrum



DTFT often looks complicated, because their mathematical expression must be 2π-periodic

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Poisson sum representation

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$, $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—

<u>$x[n]$ periodic:</u>
Can use either Fourier series expansion or Fourier transform for spectrum

Lecture 14

Discrete-time Fourier transform

I. Decomposition of DT aperiodic signals into complex sinusoids: the discrete-time Fourier transform (DTFT)

II. DTFT Examples

III. DTFT for Periodic Signals

IV. Properties of DTFT (Deduction)

IV. Properties of DTFT (5.3)

- Notation: Often we use the same symbol $\Im$ for CTFT and DTFT, and call DTFT "Fourier Transform" as well:

$$x[n] \overset{\Im}{\leftrightarrow} X(e^{j\omega}) \qquad\qquad X(e^{j\omega}) = \Im\{x[n]\}$$

$$x[n] \overset{DTFT}{\leftrightarrow} X(e^{j\omega}) \qquad\qquad x[n] = \Im^{-1}\{X(e^{j\omega})\}$$

- Various DTFT properties are summarized in Table 5.1. Instead of deriving these properties one by one, we simply compare them against similar properties for FS, DTFS, and FT as reference

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

Linearity

Fourier transforms are linear

All of the Fourier series/transforms are linear. Weighted sum (superposition) of signals in time leads to same weighted sum of spectrums.

CT, Periodic: **CTFS** $Ax(t)+By(t)$ $Aa_k + Bb_k$	CT, Aperiodic: **CTFT** $ax(t)+by(t)$ $aX(j\omega)+bY(j\omega)$
DT, Periodic: **DTFS** $Ax[n]+By[n]$ $Aa_k + Bb_k$	DT, Aperiodic: **DTFT** $ax[n]+by[n]$ $aX(e^{j\omega})+bY(e^{j\omega})$

Time Shifting

Time shift = phase shift

We shift a signal in time, we shift all the sinusoids it contains in phase because time shift of sinusoids can be represented by phase change.

CT, Periodic: **CTFS** $x(t-t_0)$ $e^{-jk\omega_0 t_0} a_k$	CT, Aperiodic: **CTFT** $x(t-t_0)$ $e^{-j\omega t_0} X(j\omega)$
DT, Periodic: **DTFS** $x[n-n_0]$ $e^{-j\frac{2\pi k n_0}{N}} a_k$	DT, Aperiodic: **DTFT** Minus means time delay $x[n-n_0]$ $e^{-j\omega n_0} X(e^{j\omega})$ phase shift = frequency times time shift

Frequency Shifting

Multiply by complex sinusoid = Frequency Shifting

CT, Periodic: **CTFS** $e^{jM\omega_0 t}x(t)$ a_{k-M}	CT, Aperiodic: **CTFT** $e^{j\omega_a t}x(t)$ $X\left(j(\omega-\omega_a)\right)$
DT, Periodic: **DTFS** $e^{j\frac{2\pi Mn}{N}}x[n]$ a_{k-M}	DT, Aperiodic: **DTFT** $e^{j\omega_a n}x[n]$ $X\left(e^{j(\omega-\omega_a)}\right)$

When we multiply $x[n]$ by $e^{j\omega_a n}$, a complex sinusoid at frequency ω_a, we multiply all the complex sinusoids contained in $x[n]$ by $e^{j\omega_a n}$. But multiplying a complex sinusoid by $e^{j\omega_a}$ simply increases its frequency by ω_a, thus shifting the spectrum to the right by ω_a.

Complex Conjugation

Conjugation of signal in time =

Conjugation and frequency reversal of spectrum

CT, Periodic: **CTFS** $x^*(t)$ a^*_{-k}	CT, Aperiodic: **CTFT** $x^*(t)$ $X^*(-j\omega)$
DT, Periodic: **DTFS** $x^*[n]$ a^*_{-k}	DT, Aperiodic: **DTFT** $x^*[n]$ $X^*(e^{-j\omega})$

Time Reversal

Time reversal = frequency reversal

CT, Periodic: **CTFS** $x(-t)$ a_{-k}	CT, Aperiodic: **CTFT** $x(-t)$ $X(-j\omega)$
DT, Periodic: **DTFS** $x[-n]$ a_{-k}	DT, Aperiodic: **DTFT** $x[-n]$ $X(e^{-j\omega})$

Conjugate Symmetry

Transform of a real signal is conjugate symmetric.

CT, Periodic: **CTFS** $x(t)$ real $\Leftrightarrow a_{-k} = a_k^*$ $x(t)$ real & even $\Leftrightarrow a_k$ real & even $x(t)$ real & odd $\Leftrightarrow a_k$ imaginary & odd	CT, Aperiodic: **CTFT** $x(t)$ real $\Leftrightarrow X(j\omega) = X(-j\omega)^*$ $x(t)$ real & even $\Leftrightarrow X(j\omega)$ real & even $x(t)$ real & odd $\Leftrightarrow X(j\omega)$ imaginary & odd
DT, Periodic: **DTFS** $x[n]$ real $\Leftrightarrow a_{-k} = a_k^*$ $x[n]$ real & even $\Leftrightarrow a_k$ real & even $x[n]$ real & odd $\Leftrightarrow a_k$ imaginary & odd	DT, Aperiodic: **DTFT** $x[n]$ real $\Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})^*$ $x[n]$ real & even $\Leftrightarrow X(e^{j\omega})$ real & even $x[n]$ real & odd $\Leftrightarrow X(e^{j\omega})$ imaginary & odd

Time Scaling

Compression/dilation in time = dilation/compression in frequency

But for DT signals time scaling is more complicated to discuss.

CT, Periodic: **CTFS** $x(at)$ $\frac{T}{\lvert a \rvert}$ - Periodic $a_k \longleftarrow$ Values of FS unchanged but frequencies they represent are scaled	CT, Aperiodic: **CTFT** $x(at)$ $\frac{1}{\lvert a \rvert} X\left(j\frac{\omega}{a}\right)$ Compress/Dilate in time = Dilate/Compress of FT and scaling of value
DT, Periodic: **DTFS**	DT, Aperiodic: **DTFT**

Important but more complicated for DT signals; skipped and saved for future courses

Convolution

Convolution/Periodic Convolution in time domain

= Multiplication in frequency domain

For two periodic signals we must apply periodic convolution instead of regular convolution

CT, Periodic: **CTFS** $x(t) \circledast y(t)$ $Ta_k b_k$	CT, Aperiodic: **CTFT** $x(t) * y(t)$ $X(j\omega)Y(j\omega)$
DT, Periodic: **DTFS** $x[n] \circledast y[n]$ $Na_k b_k$	DT, Aperiodic: **DTFT** $x[n] * y[n]$ $X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

Multiplication

Multiplication in time domain

= Convolution/Periodic Convolution in frequency domain

CT, Periodic: **CTFS** $x(t)y(t)$ $a_k * b_k$	CT, Aperiodic: **CTFT** $x(t)y(t)$ $\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
DT, Periodic: **DTFS** $x[n]y[n]$ $a_k \circledast b_k$	DT, Aperiodic: **DTFT** $x[n]y[n]$ $\frac{1}{2\pi}X(e^{j\omega})\circledast Y(e^{j\omega})$

Differentiation/First Difference

Differentiation/First difference = Emphasizing high frequencies

But for DT signal, the highest frequency is π rad/sec.

CT, Periodic: **CTFS** $\frac{d}{dt}x(t)$ $jk\omega_0 a_k$	CT, Aperiodic: **CTFT** $\frac{d}{dt}x(t)$ $j\omega X(j\omega)$
DT, Periodic: **DTFS** $x[n]-x[n-1]$ $\left(1-e^{-jk\frac{2\pi}{N}}\right)a_k$	DT, Aperiodic: **DTFT** $x[n]-x[n-1]$ $\left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$

Integration/Accumulation

Integration/Accumulation= de-emphasizing high frequencies + DC adjustment term

CT, Periodic: **CTFS** $$\int_{-\infty}^{t} x(\tau)d\tau$$ $$\frac{1}{jk\omega_0}a_k$$	CT, Aperiodic: **CTFT** $$\int_{-\infty}^{t} x(\tau)d\tau$$ $$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$
DT, Periodic: **DTFS** $$\sum_{k=-\infty}^{n} x[k]$$ $$\frac{1}{1-e^{-jk\frac{2\pi}{N}}}a_k$$	DT, Aperiodic: **DTFT** $$\sum_{k=-\infty}^{n} x[k]$$ $$\frac{1}{1-e^{-j\omega}}X\left(e^{j\omega}\right)+\pi \underbrace{X(1)}_{X(e^{j0})}\underbrace{\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}_{\text{unwrapped representation}}$$

DC adjustment term for DTFT is periodic.

Zero Frequency/Zero Time

Spectrum at zero frequency is either average or total of time signal.

Signal at zero time is either average or total of spectrum.

CT, Periodic: **CTFS**	CT, Aperiodic: **CTFT**
$a_0 = \frac{1}{T}\int_T x(t)dt$ $x(0) = \sum_{k=-\infty}^{\infty} a_k$	$X(0) = \int_{-\infty}^{\infty} x(t)dt$ $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$
DT, Periodic: **DTFS**	DT, Aperiodic: **DTFT**
$a_0 = \frac{1}{N}\sum_{n=<N>} x[n]$ $x(0) = \sum_{k=<N>} a_k$	$X(1) = X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$ $x[0] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})d\omega$

Parseval's Relation

Power or Energy can be found either in time or frequency domain

Aperiodic Signal: we can work with total energy
Periodic Signal: We can only work with average power since total energy is infinite
Cross-power/energy (cross inner product) of different frequency components is zero.

CT, Periodic: **CTFS** $$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$	CT, Aperiodic: **CTFT** $$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$$
DT, Periodic: **DTFS** $$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{n=<N>} |a_k|^2$$	DT, Aperiodic: **DTFT** $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\,d\omega$$