T06

Difference between 3.1 and 3.2
Difference between convolution and periodic convolution
CT filters and DT filters

CT Fourier series to CT Fourier transform
CT Fourier transform for aperiodic signal
CT Fourier transform for LTI system

Difference between 3.1 and 3.2

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

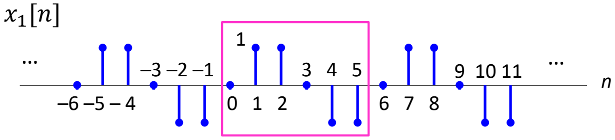
Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$, $y[n]$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	a_k, b_k Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

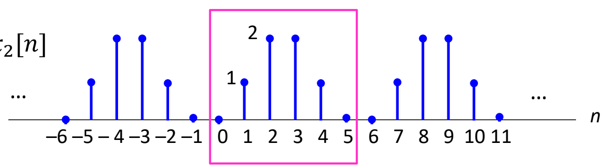
e.g. What is the value of a_0 for each of the following signals ?

$x_1[n]$



$a_0 =$?

$x_2[n]$



$a_0 =$?

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

e.g. Does each of the following signals have F.S. ?

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[n] \qquad y_2[n] = \sum_{k=-\infty}^{n} x_2[n]$$

Difference between Convolution and Periodic Convolution

CT

$$y(t) = h(t) * x(t) \qquad z(t) = x(t) \circledast y(t)$$

$$y(t) = \int_{-\infty}^{\infty} \boldsymbol{h(\tau)}\, \boldsymbol{x(t-\tau)}\, d\tau \qquad z(t) = \int_T \boldsymbol{x(\tau)}\, \boldsymbol{y(t-\tau)}\, d\tau$$

DT

$$y[n] = h[n] * x[n] \qquad z[n] = x[n] \circledast y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \boldsymbol{h[k]}\, \boldsymbol{x[n-k]} \qquad z[n] = \sum_{r=<N>} \boldsymbol{x[r]}\, \boldsymbol{y[n-r]}$$

$$z(t) = x(t)\,y(t) \quad \Longrightarrow \quad c_k = \boxed{\sum_{l=-\infty}^{\infty}} a_l\, b_{k-l}$$

CT

$c_{-\infty}$

$\vdots$

$$c_{-1} = a_{-\infty}b_{-1+\infty} + \cdots + a_0 b_{-1} + a_1 b_{-2} + a_2 b_{-3} + \cdots + a_{\infty} b_{-1-\infty}$$

$$c_0 = a_{-\infty}b_{\infty} + \cdots + a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} + a_3 b_{-3} + \cdots + a_{\infty} b_{-\infty}$$

$$c_1 = a_{-\infty}b_{1+\infty} + \cdots + a_0 b_1 + a_1 b_0 + a_2 b_{-1} + \cdots + a_{\infty} b_{1-\infty}$$

$\vdots$

c_{∞}

$$z[n] = x[n]y[n] \quad \Longrightarrow \quad c_k = \boxed{\sum_{l=<N>}} a_l b_{k-l}$$

DT

e.g. N = 4

$$c_0 = a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} + a_3 b_{-3}$$

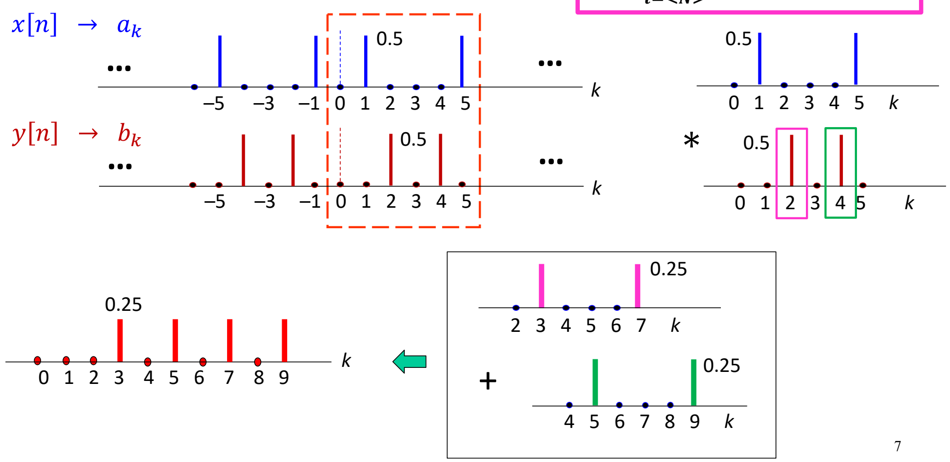
$$c_1 = a_0 b_1 + a_1 b_0 + a_2 b_{-1} + a_3 b_{-2}$$

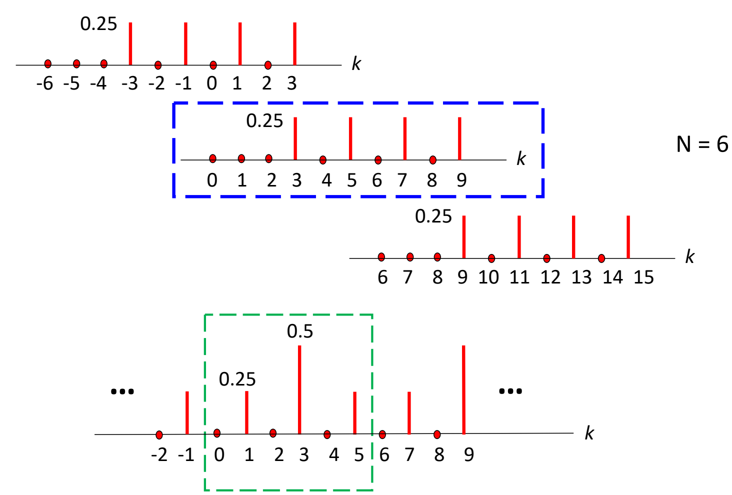
$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_{-1}$$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

e.g. $z[n] = x[n]\,y[n] = \cos\left(\frac{2\pi}{6}n\right)\cos\left(\frac{4\pi n}{6}\right)$

$$\boxed{c_k = \sum_{l=<N>} a_l\, b_{k-l} = a_k \circledast b_k}$$



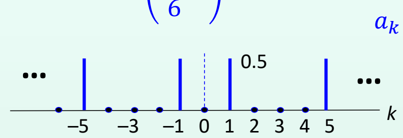


Question : Any other method to obtain / check the answer ?

$$z[n] = \cos\left(\frac{2\pi}{6}n\right)\cos\left(\frac{4\pi n}{6}\right) \qquad \omega_o = \frac{2\pi}{6}$$

$$= \frac{1}{2}e^{j(2)\frac{2\pi}{6}n}\cos\left(\frac{2\pi}{6}n\right) + \frac{1}{2}e^{j(-2)\frac{2\pi}{6}n}\cos\left(\frac{2\pi}{6}n\right)$$

$$c_k = \frac{1}{2}a_{k-2} + \frac{1}{2}a_{k+2}$$



$$c_0 = \frac{1}{2}a_{-2} + \frac{1}{2}a_2 = 0 \qquad c_3 = \frac{1}{2}a_1 + \frac{1}{2}a_5 = 0.5$$

$$a_k = a_{k+N}$$

$$c_1 = \frac{1}{2}a_{-1} + \frac{1}{2}a_3 = 0.25 \qquad c_4 = c_{-2} = c_2^* = 0$$

$$a_k = a_{-k}^*$$

$$c_2 = \frac{1}{2}a_0 + \frac{1}{2}a_4 = 0 \qquad c_5 = c_{-1} = c_1^* = 0.25$$

e.g. You are given the following information about a signal $x[n]$. Determine $x[n]$.

1. $x[n]$ is real and has period $N = 10$.
2. $a_1 = 1 + j$ and $a_k = 0$ for $k = 2$ to 8
3. $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 5$

$$e^{j(9)\frac{2\pi}{10}n} = e^{-j\frac{2\pi}{10}n}$$

Parseval's relation

$$a_9 = a_{-1} = a_1^* = 1 - j = \sqrt{2}\,e^{-j\frac{\pi}{4}}$$

$a_0 \; a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; a_6 \; a_7 \; a_8 \; a_9$

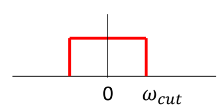
$$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = |a_0|^2 + |a_1|^2 + |a_9|^2 = |a_0|^2 + (\sqrt{2})^2 + (\sqrt{2})^2 = 5 \qquad a_0 = \pm 1$$

$$x[n] = 1 + \sqrt{2}\,e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{10}n} + \sqrt{2}\,e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{10}n} = 1 + 2\sqrt{2}\cos\left(\frac{2\pi}{10}n + \frac{\pi}{4}\right)$$

$$x[n] = -1 + 2\sqrt{2}\cos\left(\frac{2\pi}{10}n + \frac{\pi}{4}\right)$$

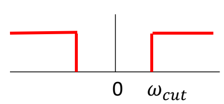
CT Filter

Low-Pass $\quad H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{cut} \\ 0 & otherwise \end{cases}$

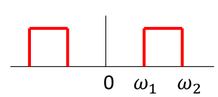


ω_{cut}: cutoff frequency

High-Pass $\quad H(j\omega) = \begin{cases} 0 & otherwise \\ 1 & |\omega| > \omega_{cut} \end{cases}$



Band-Pass $\quad H(j\omega) = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & otherwise \end{cases}$



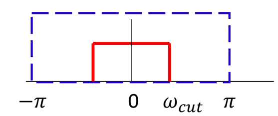
Question : Specification of each filter ?

DT Filter

不可0 otherwise

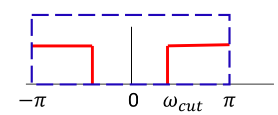
from $-\pi$ to π

Low-Pass $\quad H(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| < \omega_{cut} \\ 0 & \omega_{cut} \le |\omega| \le \pi \end{cases}$

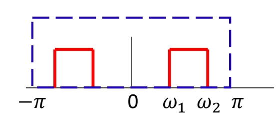


ω_{cut}: cutoff frequency

High-Pass $\quad H(e^{j\omega}) = \begin{cases} 0 & 0 \le |\omega| < \omega_{cut} \\ 1 & \omega_{cut} \le |\omega| \le \pi \end{cases}$

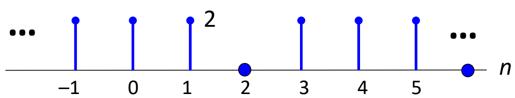


Band-Pass $\quad H(e^{j\omega}) = \begin{cases} 0 & 0 \le |\omega| < \omega_1 \\ 1 & \omega_1 \le |\omega| < \omega_2 \\ 0 & \omega_2 \le |\omega| \le \pi \end{cases}$



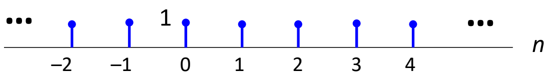
Question : Specification of each filter ?

e.g. Given the following periodic signal $x[n]$:



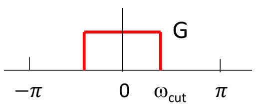
$$x[n] = \frac{3}{2} + \cos\left(\frac{\pi}{2}n\right) - \frac{1}{2}\cos(\pi n)$$

If $x[n]$ is applied to an LTI system to give the following output $y[n]$,



$y[n]$ **?**

a) Sketch a possible frequency response.

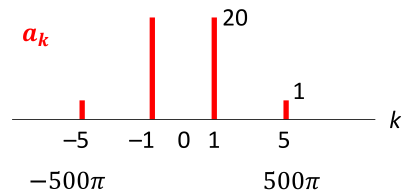


G = **?**

ω_{cut} **?**

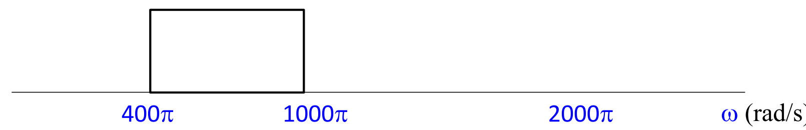
b) What is the type of this filter ? Specification of this filter?

e.g. A real signal $x(t)$ has FS coefficients $\boldsymbol{a_k}$ shown below.



Assume
$\omega_0 = 100\pi \ rad/s$

Available frequency band for transmission



Question : How to make $x(t)$ fit in the available frequency band ?

CT Fourier series to CT Fourier transform
CT Fourier transform for aperiodic signal
CT Fourier transform for LTI system

CT Fourier Series

$\omega_o = \frac{2\pi}{T}$ T = finite

$\omega = k\omega_o$

$$x(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{a_k}\, e^{jk\frac{2\pi}{T}t}$$

$$\boldsymbol{a_k} = \frac{1}{T}\int_0^T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

Periodic Signals

CT Fourier Transform

$\omega_o = \frac{2\pi}{T}$ $T \rightarrow -\infty$ to ∞

$k\omega_o \rightarrow \omega$ (whole w axis)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{X(j\omega)}\, e^{j\omega t}\, d\omega$$

continuous function!

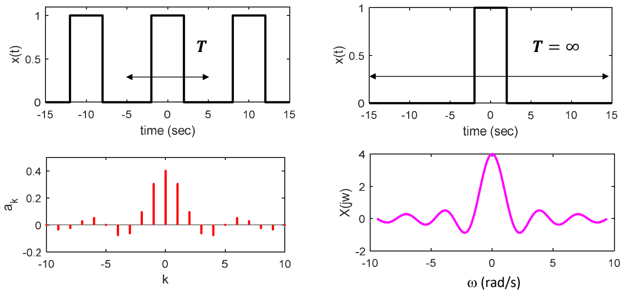
$$\boldsymbol{X(j\omega)} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

new name

Aperiodic Signals

Question :
- What is the difference between $\boldsymbol{a_k}$ and $\boldsymbol{X(j\omega)}$?
- What is the requirement on $\boldsymbol{x(t)}$ to give a valid **FT** ?

e.g.



$T_1 = 2 \quad T = 10$ $\qquad$ $T_1 = 2 \quad T = \infty$

X(0) = Area!

ak
DC

F.S. $\boxed{a_k = \dfrac{\sin(k\omega_o T_1)}{k\pi}}$

F.T. $\boxed{X(j\omega) = \dfrac{2\sin(\omega T_1)}{\omega}}$

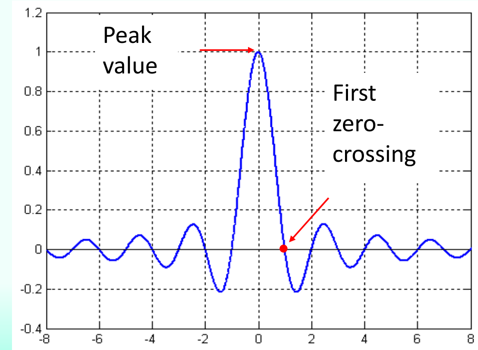
Question : What is the difference between a_0 and $X(j0)$?

not DC term

$$\mathrm{sinc}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}$$

$$\boxed{\lim_{\omega\to 0}\frac{\sin(\pi\omega)}{\pi\omega} = \frac{\pi\omega}{\pi\omega} = 1}$$

$\sin\theta \to \theta$



$$\mathrm{sinc}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega} = 0$$

$$\sin(\pi\omega) = 0$$

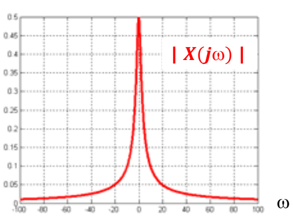
ω = 1

CT Fourier Transform for Aperiodic Signal

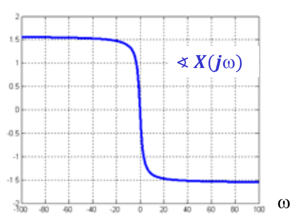
$x(t) \quad \xleftrightarrow{\text{F.T.}} \quad X(j\omega) = |X(j\omega)|\, e^{j\sphericalangle X(j\omega)}$

magnitude $\qquad$ phase

e.g. $x(t) = e^{-2t}u(t) \quad \xleftrightarrow{\text{F.T.}} \quad X(j\omega) = \dfrac{1}{2 + j\omega}$



Magnitude spectrum



Phase spectrum

$$|X(j\omega)| = \frac{1}{\sqrt{4+\omega^2}} \qquad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

e.g. $x(t) \quad \xleftrightarrow{\text{F.T.}} \quad X(j\omega) = \begin{cases} \omega & |\omega| \le 10 \\ 0 & otherwise \end{cases}$

Plot magnitude spectrum and phase spectrum.
Is $x(t)$ a real signal ?

No

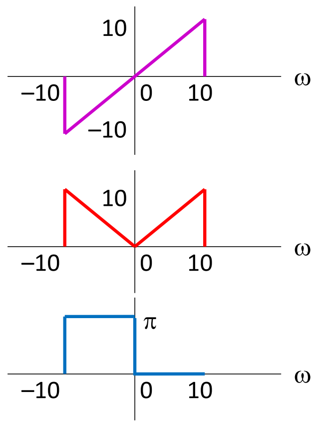
e.g. $x(t) \quad \xleftrightarrow{\text{F.T.}} \quad X(j\omega) = \begin{cases} 4e^{-j2\omega} & |\omega| \le 10 \\ 0 & otherwise \end{cases}$

Plot magnitude spectrum and phase spectrum.
Is $x(t)$ a real signal ?

even $\qquad$ odd

Yes!

$$X(j\omega) = \begin{cases} \omega & |\omega| \le 10 \\ 0 & otherwise \end{cases}$$



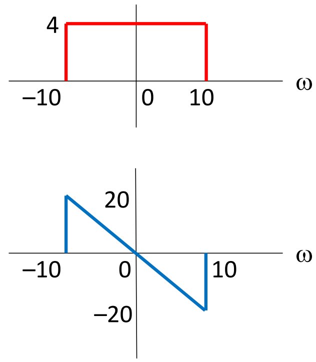
Is $x(t)$ a real signal ?

$$X(j\omega) = \begin{cases} 4e^{-j2\omega} & |\omega| \le 10 \\ 0 & otherwise \end{cases}$$

Magnitude spectrum

$|X(j\omega)|\, e^{j \measuredangle X(j\omega)}$

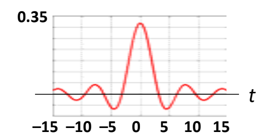
Phase spectrum



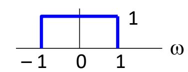
Is $x(t)$ a real signal ?

CT Fourier Transform for LTI System

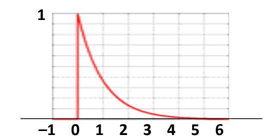
e.g. $h(t) = \dfrac{\sin(t)}{\pi t}$



$$H(j\omega) = \begin{cases} 1 & |\omega| \le 1 \\ 0 & otherwise \end{cases}$$



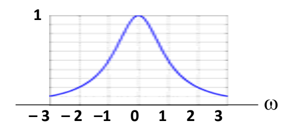
e.g. $h(t) = e^{-t}u(t)$



$$H(j\omega) = \frac{1}{1 + j\omega}$$

$|H(j\omega)|$

Magnitude response



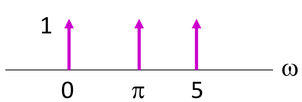
Question : Phase response ? Is it a causal system ?

e.g. Given : $X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$

and $h(t) = u(t) - u(t-2)$

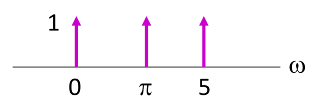
a) Is $x(t)$ periodic ?
b) Is $x(t) * h(t)$ periodic ?

$X(j\omega)$



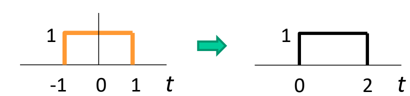
$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi} e^{j\pi t} + \frac{1}{2\pi} e^{j5t}$$

$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$



HCF (π , 5) = **?**

$h(t) = u(t) - u(t-2)$



$$H(j\omega) = \frac{2\sin\omega}{\omega} e^{-j\omega}$$

$H(j0) = 2$

$H(j\pi) = 0$

$H(j5) = \dfrac{2\sin 5}{5} e^{-j5} \ne 0$

$\mathbf{Y(j\omega)}$