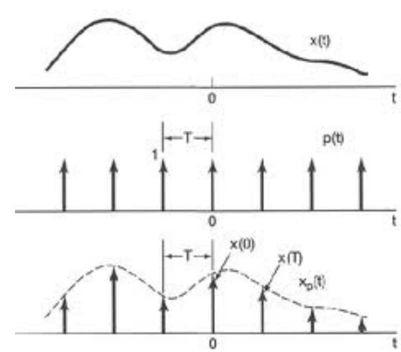
Ch2.2 : Discrete-Time Systems

Information source and input transducer → Source Coding → Channel Coding → Modulator → Channel → Demodulator (Matched Filter) → Channel Decoding → Source Decoding → Information sink and output transducer

Ch2.2: DT Systems

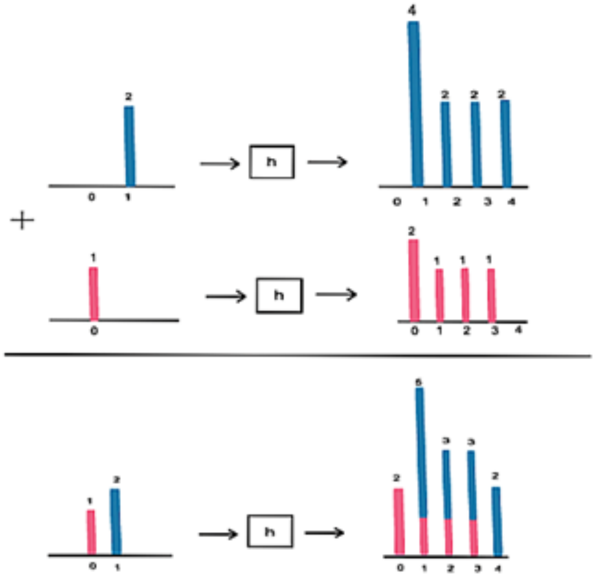
- **Questions to be answered:**
  - **Classification:** What are the types of DT systems?
  - **Time-Domain Characterization of LTI Systems:** Impulse response
  - **Interconnections Schemes:** What happens if we connect several LTI systems together?



Reading: Mitra (Ch4.1 – 4.5)

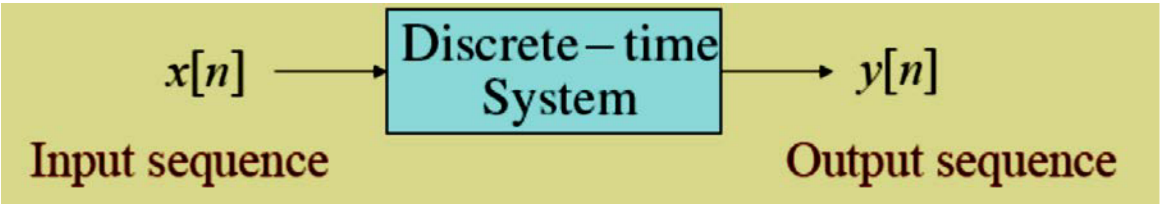
Ch2.2: Discrete-Time Systems

- **Discrete-Time Systems: Classification**
- Time-Domain Characterization of LTI Systems
- Interconnections Schemes



Discrete-Time Systems

- A black-box approach to model "system"
- A discrete-time system processes a given input sequence $x[n]$ to generate an output sequence $y[n]$ with more desirable properties.



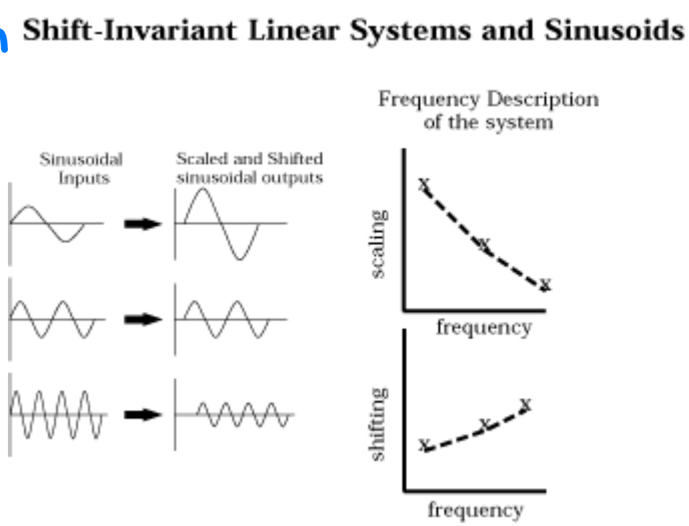
SISO

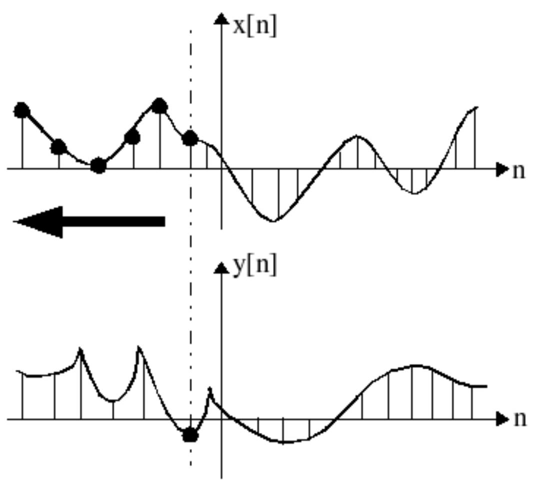
- In most applications, there is a **single-input and single-output**, such as Multiplier, Unit delay, and Unit advance.
- How about Modulator and Adder?

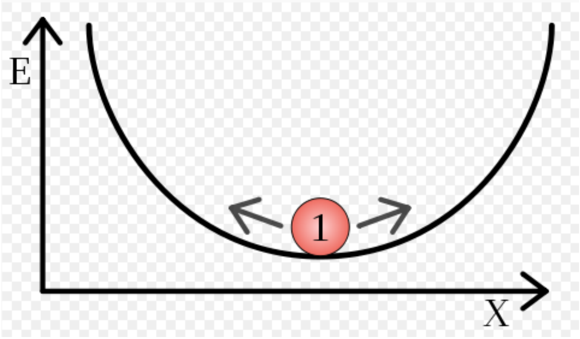
Classifications

LTI system

- Linear System
- Shift-Invariant (Time Invariant) System
- Causal System
- Stable System







Linear Discrete-Time System

- **Linear System:** If $y_1[n]$ and $y_2[n]$ are the outputs due to inputs $x_1[n]$ and $x_2[n]$, respectively, then for an input
$$x[n] = ax_1[n] + bx_2[n],$$
the output is given by
$$y[n] = ay_1[n] + by_2[n]$$
and this holds for any constants a and b.

- **Example:** For an Accumulator, we have
$$y_1[n] = \sum_{l=-\infty}^{n} x_1[l],\ y_2[n] = \sum_{l=-\infty}^{n} x_2[l].$$
Then, the output for $x[n] = ax_1[n] + bx_2[n]$, is
$$y[n] = \sum_{l=-\infty}^{n} x[l] = \sum_{l=-\infty}^{n} ax_1[l] + bx_2[l] = ay_1[n] + by_2[n]$$

Shift-Invariant System

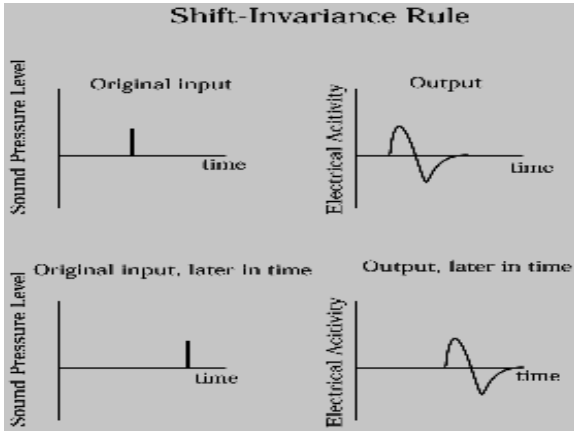
- **Shift-Invariant:** If $y_1[n]$ is the output due to input $x_1[n]$, then for an input $x[n] = x_1[n - n_o]$, the output is given by $y[n] = y_1[n - n_o]$ and this holds for any integer n_o.

shape × change

- If n is related to discrete instants of time, this property is called **time-invariance** property.

- Time-invariance property ensures that for a specified input, the output is **independent** of the time the input is being applied.

time → independent!



Shift-Invariant System

- **Example-** Consider the up-sampler. For an input $x[n]$, the output is given by

$$y_u[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \dots \dots \\ \qquad 0, \quad \text{otherwise} \end{cases}.$$

$\rightarrow y_u[n-n_o]$?

- For an input $x_1[n] = x[n - n_o]$, the output is given by

$n-n_o$ Time invariant

$$y_{1,u}[n] = \begin{cases} x_1\left[\frac{n}{L}\right] = x\left[\frac{n-Ln_o}{L}\right], n = 0, \pm L, \pm 2L, \dots \dots \\ \qquad 0, \quad \text{otherwise} \end{cases}.$$

shift by n_o

$\neq y_u[n-n_o] \rightarrow$ not shift invariant

- However, from the definition of the up-sampler,

不同 Shape

$$y_u[n - n_o] = \begin{cases} x\left[\frac{n-n_o}{L}\right], n = 0, \pm L, \pm 2L, \dots \dots \\ \qquad 0, \qquad \text{otherwise} \end{cases} \neq y_{1,u}[n]$$

Linear Time-Invariant System

- **Linear Time-Invariant (LTI) System:** A system satisfying both the linearity and the time-invariance properties.

- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design.

- Highly useful **signal processing algorithms** have been developed utilizing this class of systems over the last several decades.

Causal System

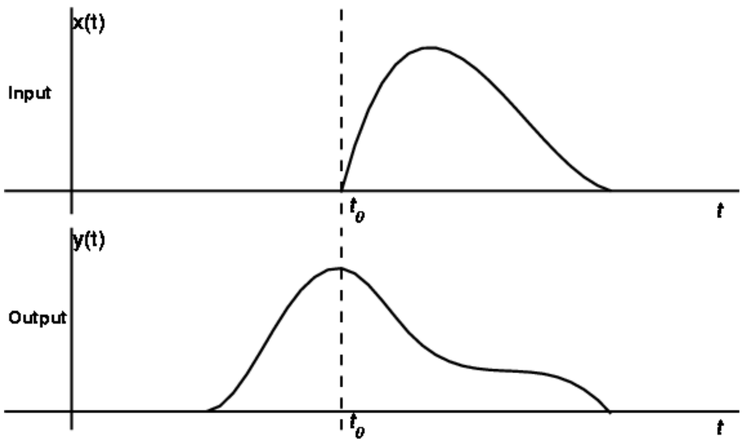
只可 depends on past!

- In a **Causal System**, the n_o-th output $y[n_o]$ depends only on the input samples $x[n]$ for $n \leq n_o$.

不可 future!

- Let $y_1[n]$ and $y_2[n]$ be the responses of a causal discrete-time system to inputs $x_1[n]$ and $x_2[n]$, respectively. Then,
  $x_1[n] = x_2[n]$ for *n* < *N* implies $y_1[n] = y_2[n]$ for *n* < *N*.

- For causal system, changes in output samples do not **precede** changes in the input samples.



Causal System: Examples

- Examples of causal systems:

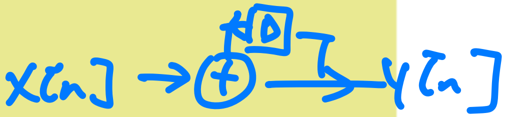
$$y[n]=\alpha_1 x[n]+\alpha_2 x[n-1]+\alpha_3 x[n-2]+\alpha_4 x[n-3]$$

all causal!

$$y[n]=b_0 x[n]+b_1 x[n-1]+b_2 x[n-2]$$
$$+a_1 \underline{y[n-1]}+a_2 \underline{y[n-2]}$$

$$y[n]=y[n-1]+x[n]$$

Already known



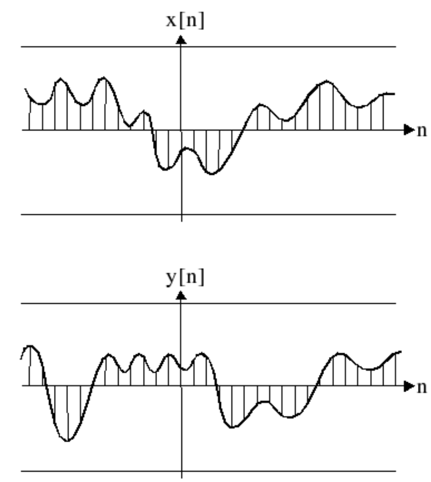
- Examples of noncausal systems:

$$y[n]=x_u[n]+\frac{1}{2}(x_u[n-1]+\underline{x_u[n+1]})$$

$$y[n]=x_u[n]+\frac{1}{3}(x_u[n-1]+\underline{x_u[n+2]})$$
$$+\frac{2}{3}(x_u[n-2]+\underline{x_u[n+1]})$$

Stable System

- There are various definition of stability. We consider here the **bounded-input, bounded-output (BIBO)** stability.
- **Definition:** Let $y[n]$ be the response to an input $x[n]$. If, $|x[n]| \le B_x$ for all values of *n*, then $|y[n]| \le B_y$ for all values of *n*.
- **Example:** The M-point moving average filter is BIBO stable:



$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

For a bounded input $|x[n]| \le B_x$, we have

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right| \le \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]| \le B_x$$

$\le B_x$

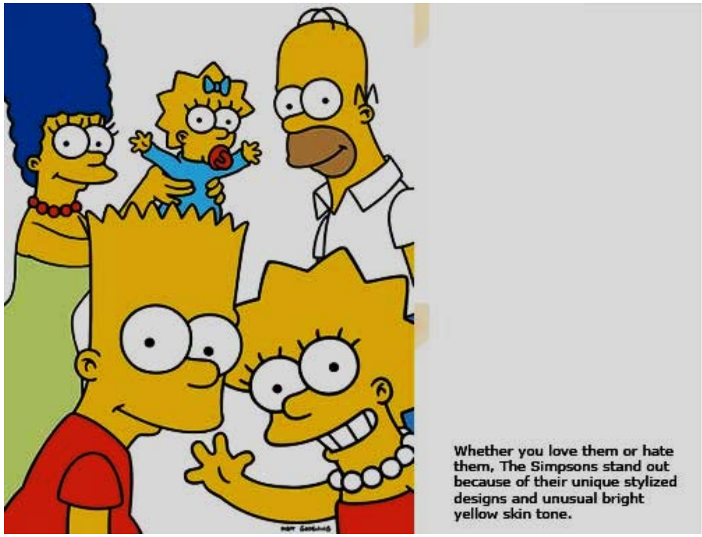
$|a+b| \le |a| + |b|$

Ch2.2 : Discrete-Time Systems

- Discrete-Time Systems: Classification
- **Time-Domain Characterization of LTI Systems**

Mainly focus

- Interconnections Schemes



Impulse and Step Responses

- The response of a discrete-time system to a **unit sample sequence** $\{\delta[n]\}$ is called the **unit sample response** or **impulse response** $\{h[n]\}$.
- The response of a discrete-time system to a **unit step sequence** $\{\mu[n]\}$ is called the **unit step response** or **step response** $\{s[n]\}$.

- **Example:** The impulse response of the discrete-time accumulator $y_1[n] = \sum_{l=-\infty}^{n} x_1[l]$ is obtained by setting $x[n] = \delta[n]$, resulting in

$$h[n] = \sum_{l=-\infty}^{n} \delta[l] = \mu[n]$$

Impulse Responses: Examples

- **Example:** The impulse response of the system
  $$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4\, x[n-3]$$
  is obtained by setting $x[n] = \delta[n]$ resulting in
  $$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4\, \delta[n-3]$$

- **Example:** The impulse response $\{h[n]\}$ of the factor-of-2 interpolator $y[n] = x[n] + \frac{1}{2}(x[n-1] + x[n+1])$ is obtained by setting $x[n] = \delta[n]$ resulting in Use LTI property!
  $$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

Time-Domain Characterization of LTI System

- **Input-Output Relationship:** An LTI discrete-time system is completely characterized by its impulse response. $h[n]$
- We can compute the output for an arbitrary input, if we know the impulse response.
- **How?** Any arbitrary input $x[n]$ can be expressed as a linear combination of delayed and advanced unit sample sequences

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$\delta[n] \rightarrow S \rightarrow h[n]$

LTI

The response to $x[k]\delta[n-k]$ is $x[k]h[n-k]$, thus the total response will be

LTI $\rightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

$x[k]\delta[n-k] \rightarrow S \rightarrow x[k]\cdot h[n-k]$

LTI

Convolution Sum

- The summation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

① ②

is called the **convolution sum** and represented as

$$y[n] = x[n] \circledast h[n].$$

this notation

- **Properties**
  - **Commutative property:**

$$x[n] \circledast h[n] = h[n] \circledast x[n]$$

  - **Associative property:**

$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

  - **Distributive Property:**

$$x[n] \circledast (h[n] + y[n]) = x[n] \circledast h[n] + x[n] \circledast y[n]$$

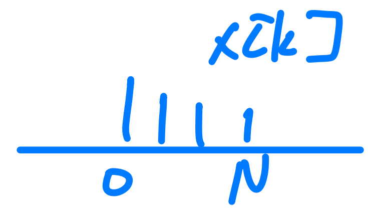
Convolution Sum: Example

- Calculate the convolution of the following two sequences

$$x[n] = \begin{cases} 1, 0 \le n \le 5 \\ 0, \text{otherwise} \end{cases} \text{ and } h[n] = \begin{cases} 1.8 - 0.3n, 0 \le n \le 5 \\ 0, \quad \text{otherwise} \end{cases}.$$

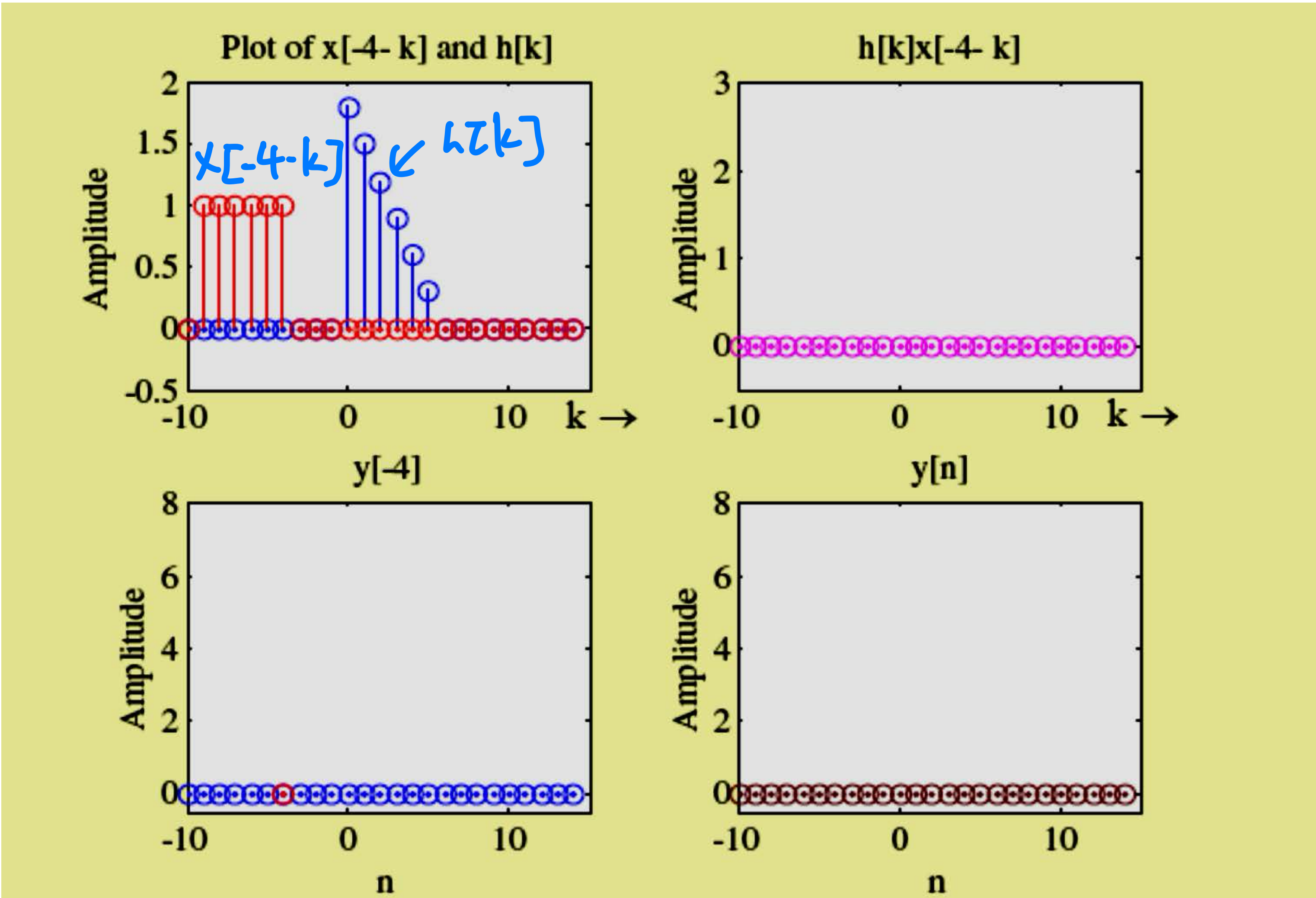
$\sum_{k=0}^{\infty} x[k] \cdot h[n-k]$

sum all to get $y[n]$

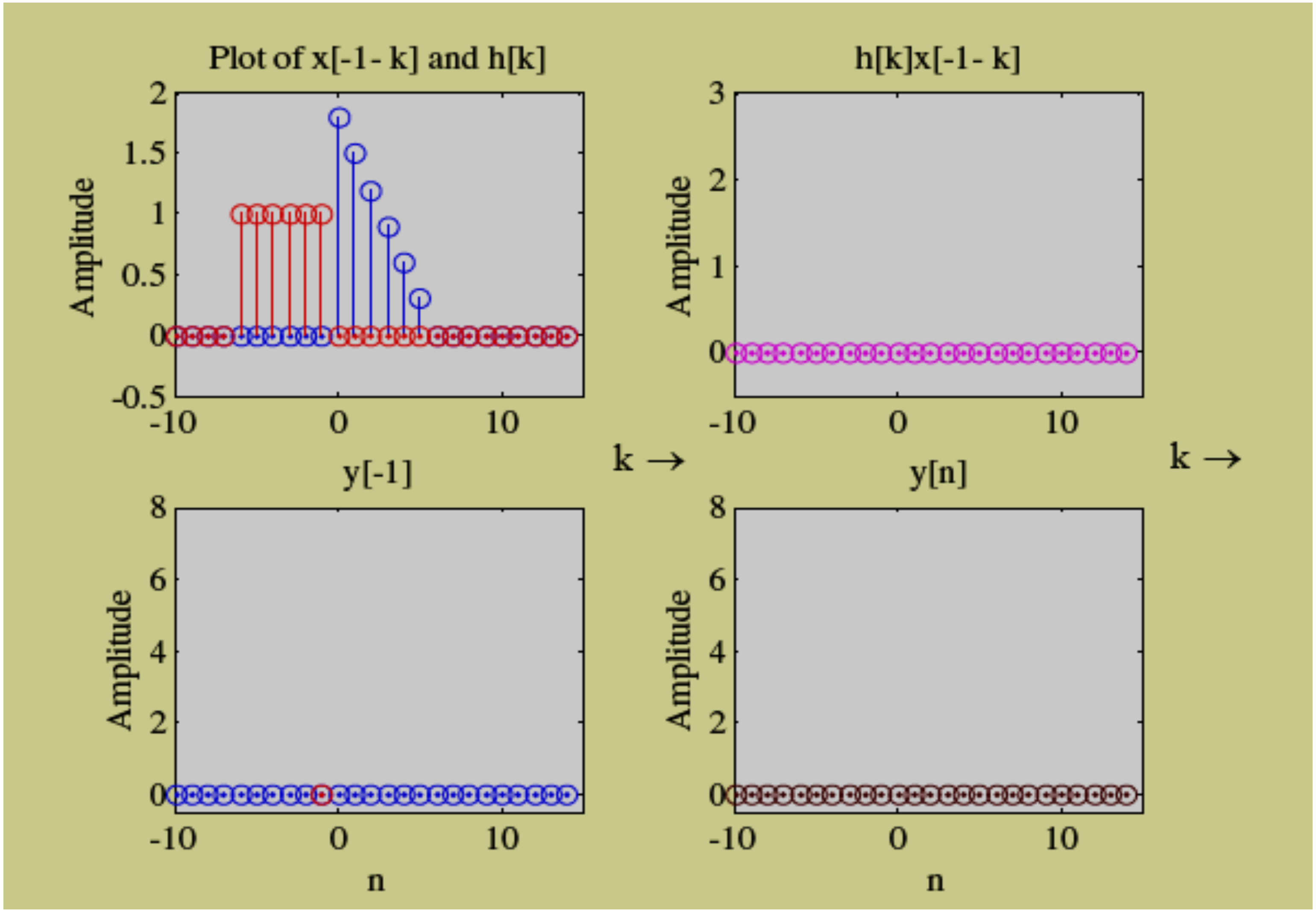


- **Solution:**
  - Obtain one sample $y[n']$
    - Time reverse $h[k]$ to form $h[-k]$
    - Shift $h[-k]$ to the right n' samples to form $h[n' - k]$
    - Form the product $v[k] = x[k]h[n' - k]$
    - Sum all samples $v[k]$ to develop $y[n']$.
  - Repeat the above process to obtain all samples for $\{y[n]\}$

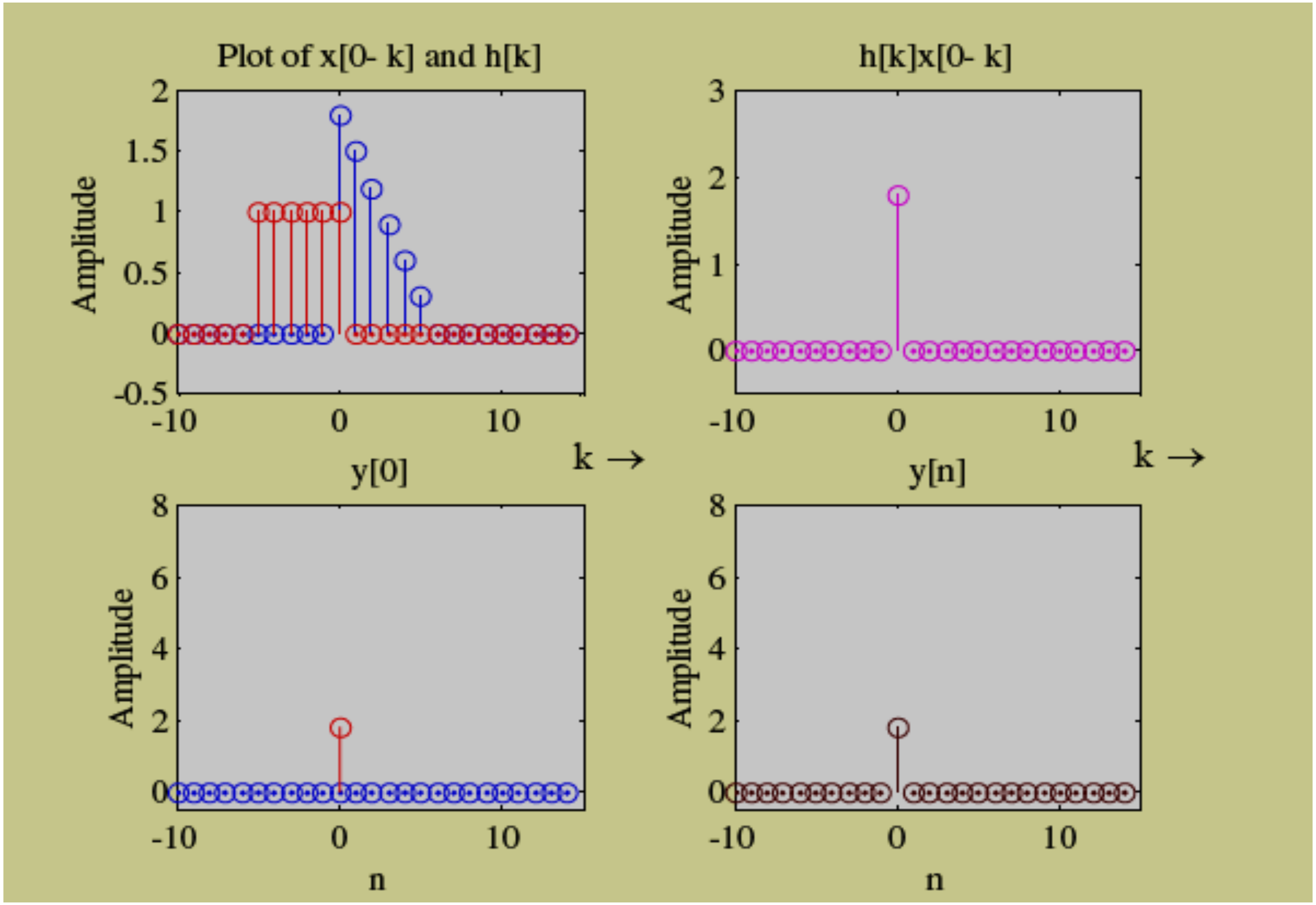
Interpretation: n=-4



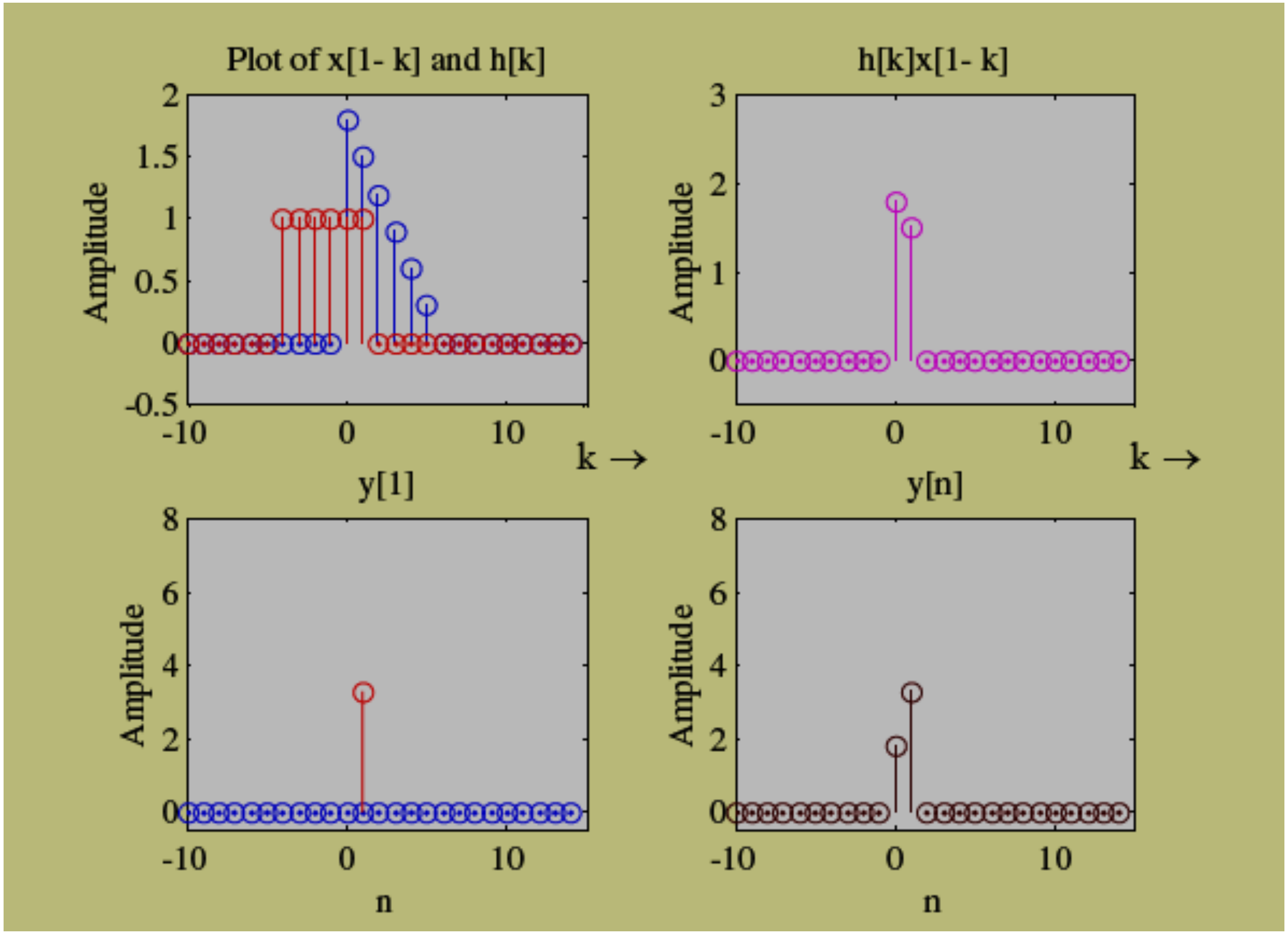
Interpretation: n=-1



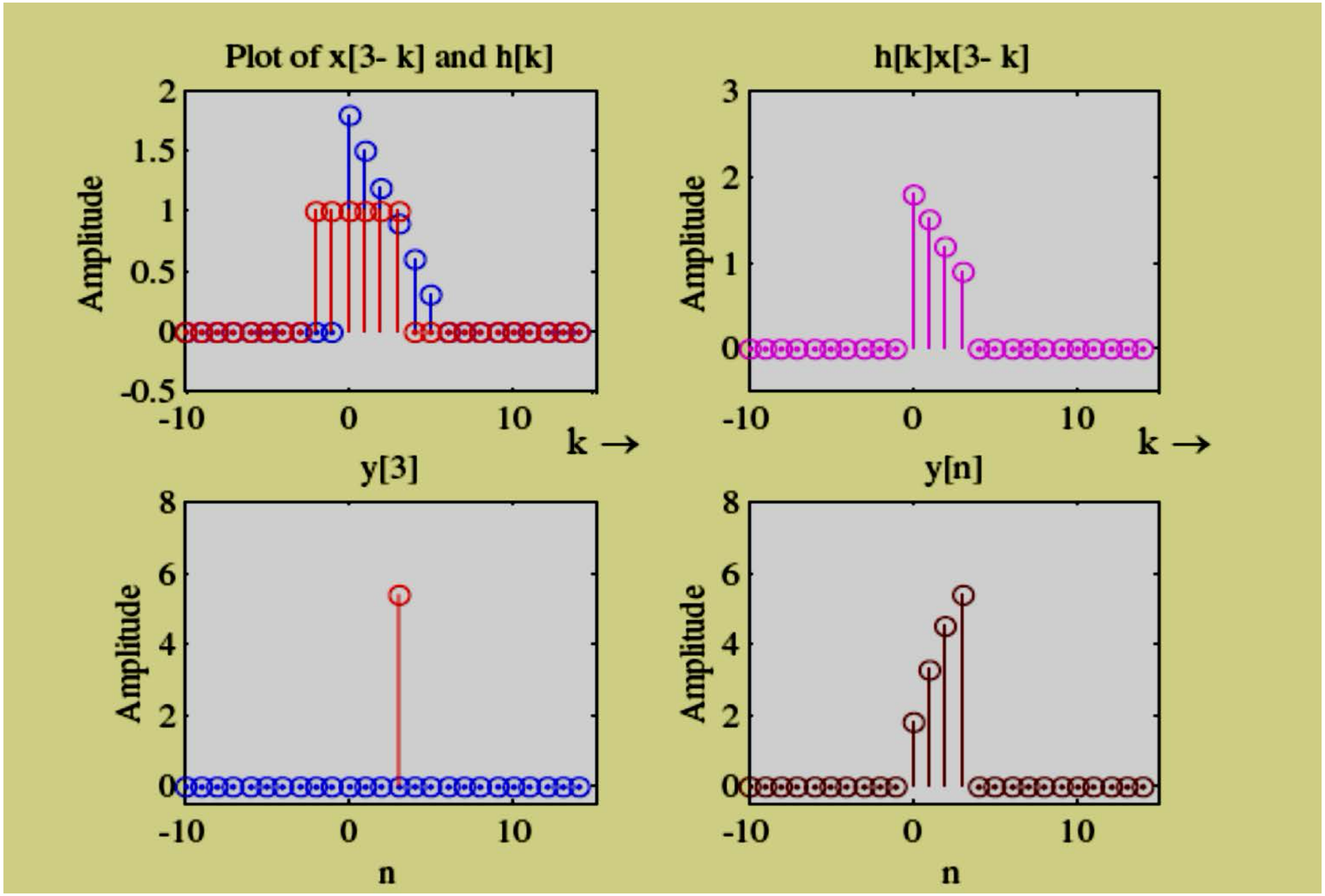
Interpretation: n=0



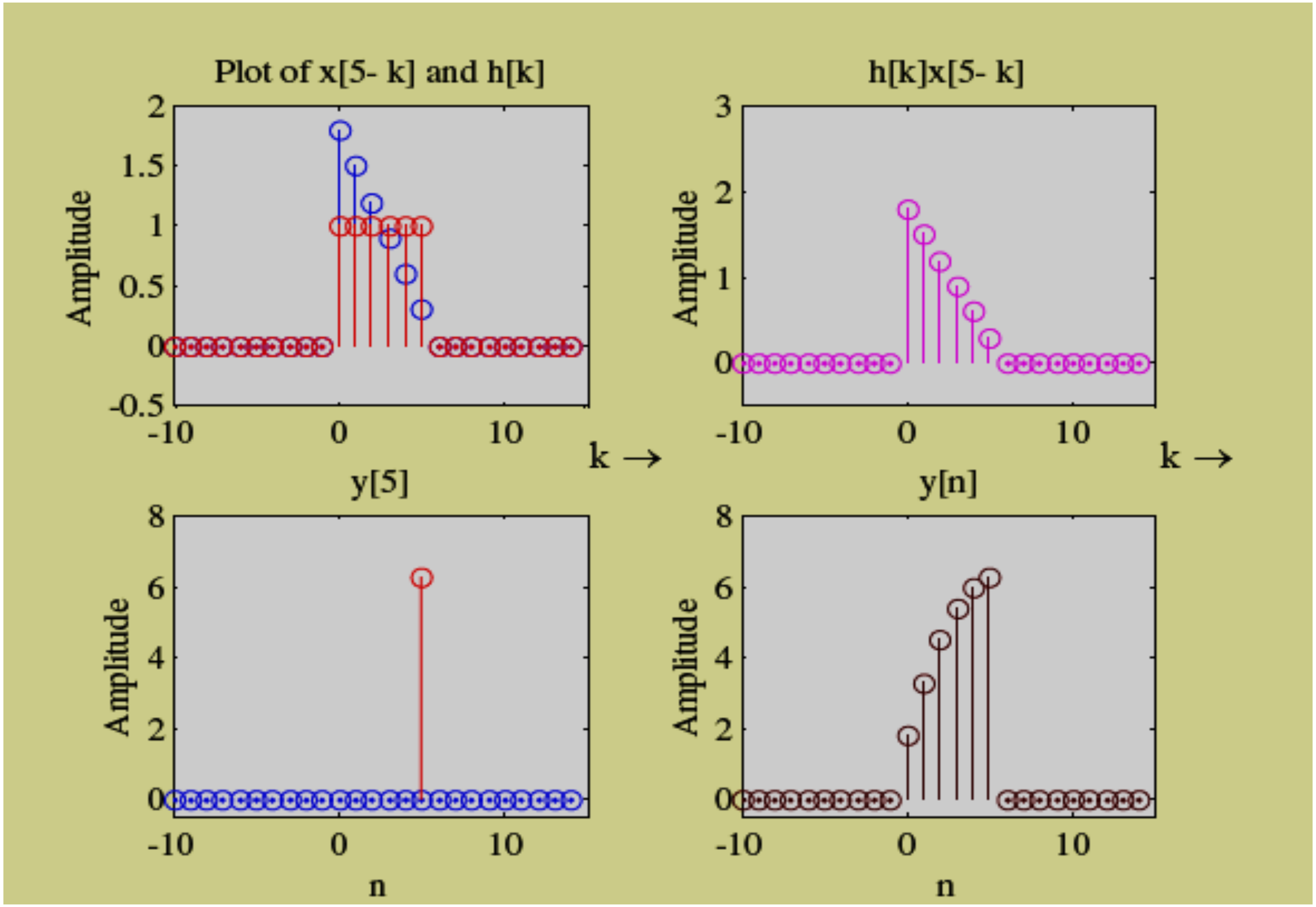
Interpretation: n=1



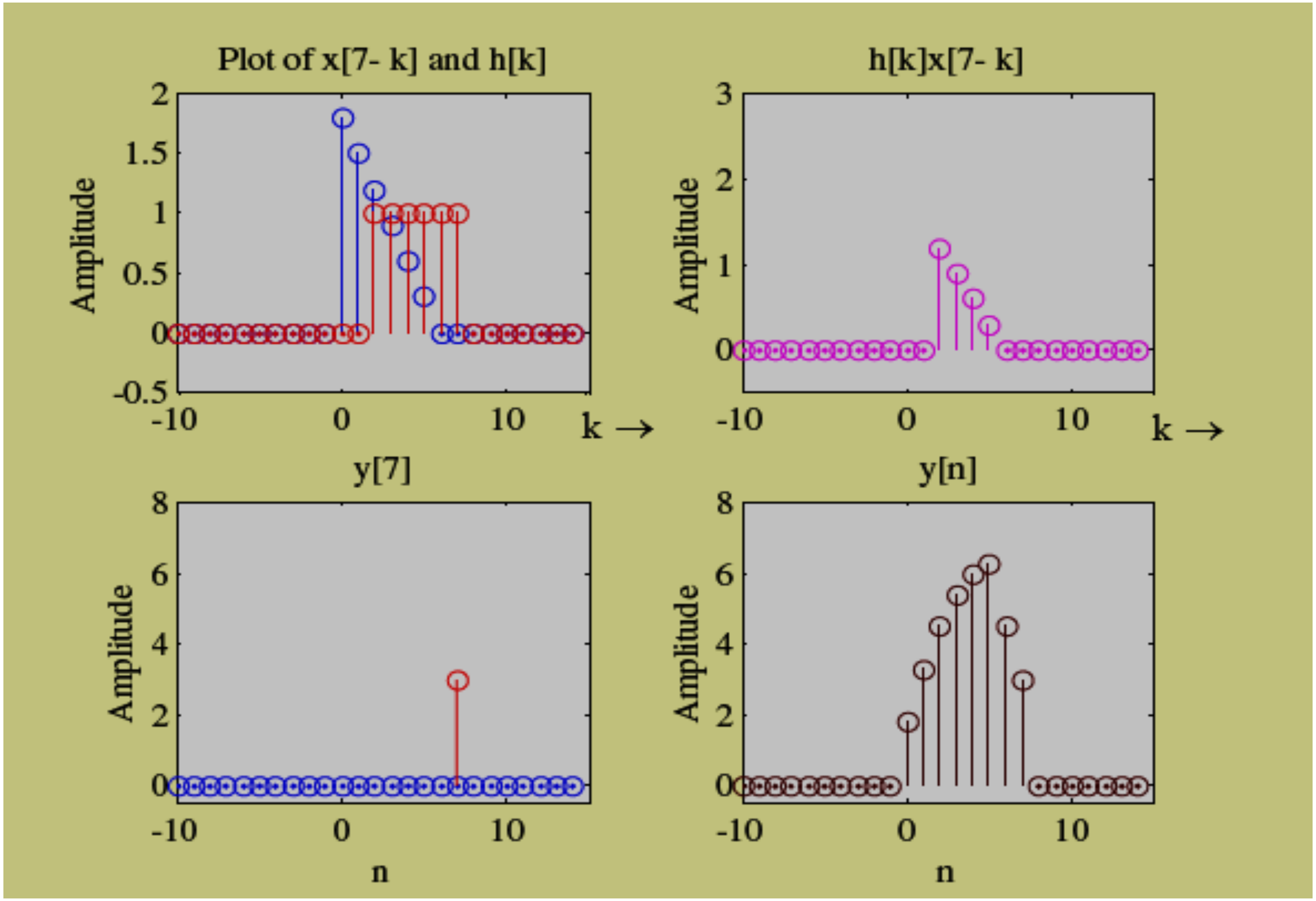
Interpretation: n=3



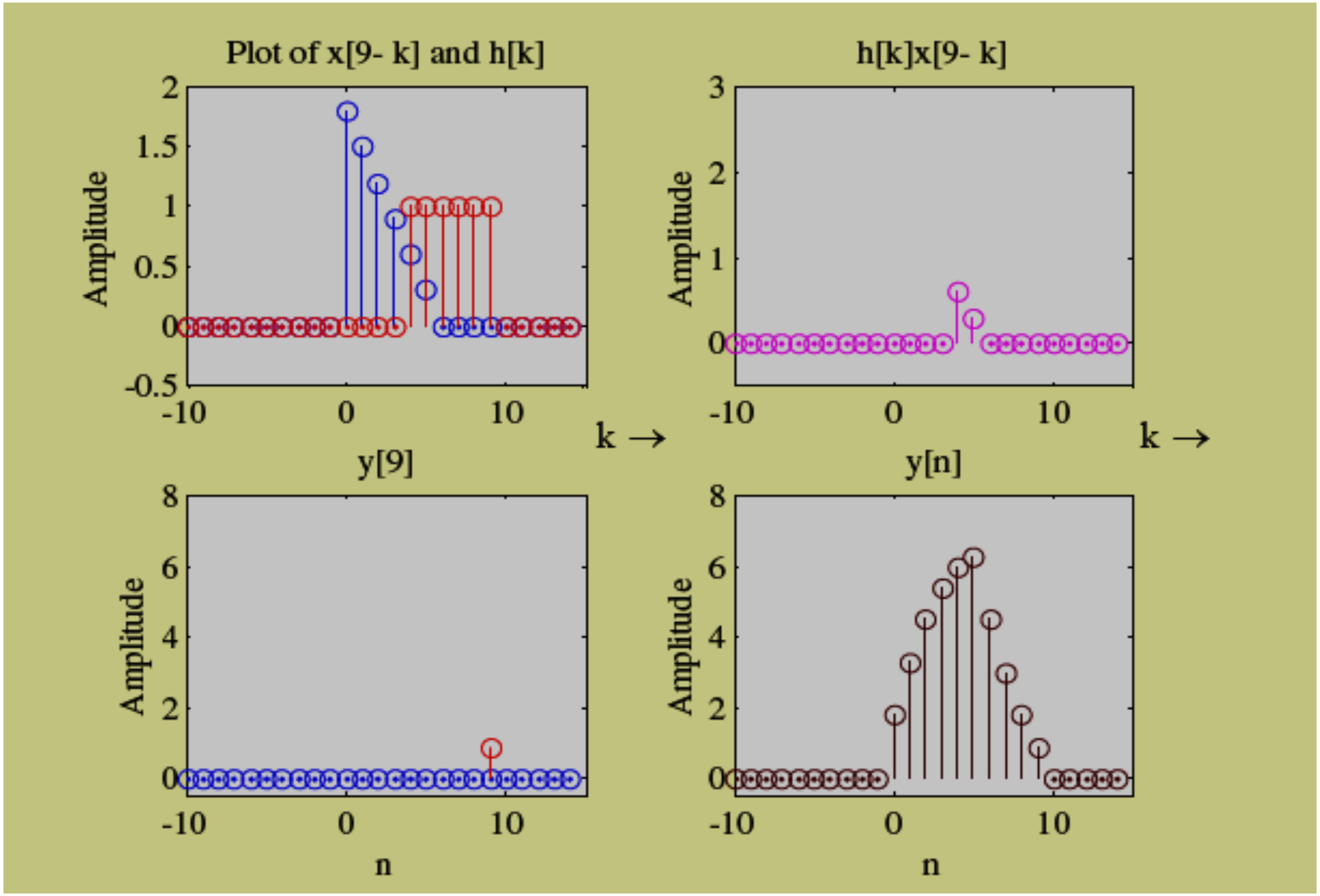
Interpretation: n=5



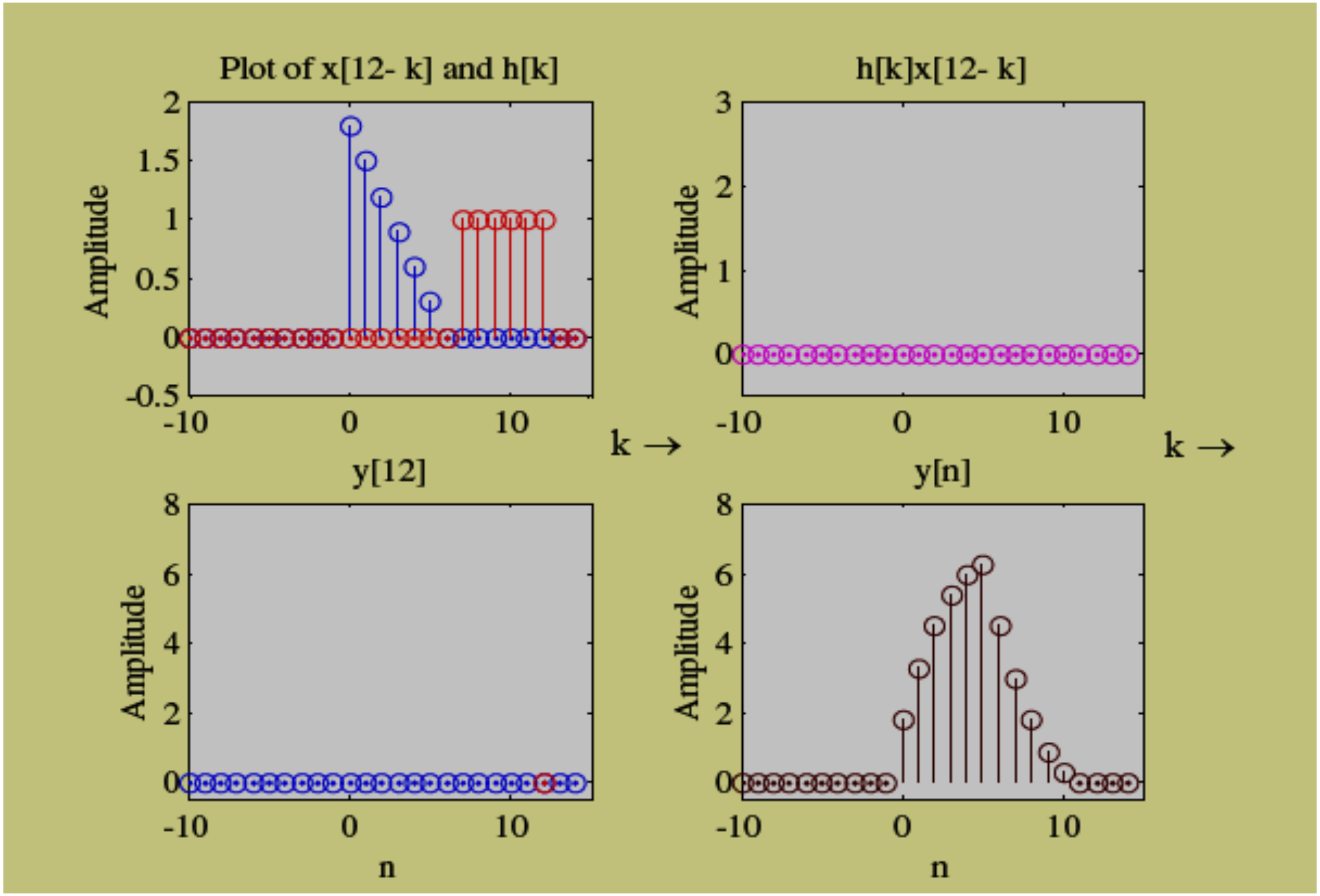
Interpretation: n=7



Interpretation: n=9



Interpretation: n=12



Tabular Method

Use when g[n], h[n] is short!

n:	0	1	2	3	4	5
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$		
$h[n]$:	$h[0]$	$h[1]$	$h[2]$			
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$		
		$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$g[3]h[1]$	
			$g[0]h[2]$	$g[1]h[2]$	$g[2]h[2]$	$g[3]h[2]$
$y[n]$:	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$	$y[5]$

- The samples $y[n]$ generated by the convolution sum are obtained by adding the entries in the column above each sample

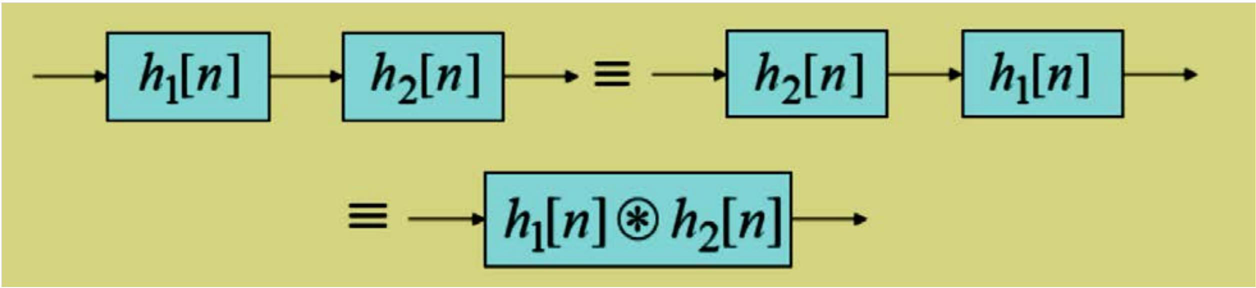
Some fast method

Ch2.2 : Discrete-Time Systems

- Discrete-Time Systems: Classification
- Time-Domain Characterization of LTI Systems
- **Interconnections Schemes**

Interconnection Schemes: Cascade

- **Cascade Connection**



- Impulse response $h[n]$ of the cascade of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by $h[n] = \underline{h_1[n] \circledast h_2[n]}$
- **Note:** The ordering has no effect on the impulse response because of the commutative property of convolution.
- A cascade connection of two stable systems is stable.

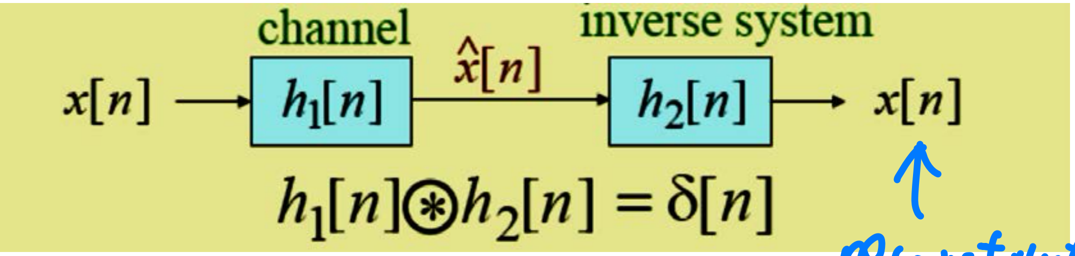
Cascade Connection

- An application of Cascade Connection is to develop an inverse system.
- If a cascade connection satisfies

$$h_1[n] \circledast h_2\,[n] = \delta[n]$$

then the LTI system $h_1[n]$ is said to be the inverse of $h_2[n]$ and vice-versa.

- An application of inverse system concept is in the recovery of a signal $x[n]$ from its distorted version $\hat{x}[n]$.



reconstruct $x[n]$

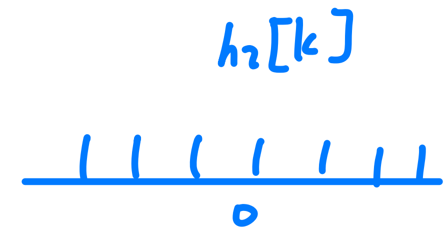
Cascade Connection

Add all samples up to now

- **Example:** Consider the discrete-time accumulator with impulse response $\mu[n]$. Find its inverse system.

- **Solution:** Its inverse system satisfy

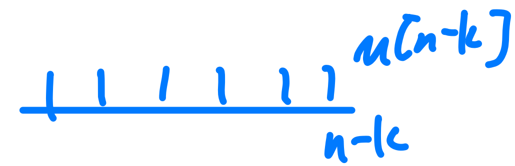
$$\mu[n] \circledast h_2\,[n] \overset{?}{=} \delta[n]$$



It follows that $h_2[n] = 0$, for n<0,

$$h_2[0] = 1,$$

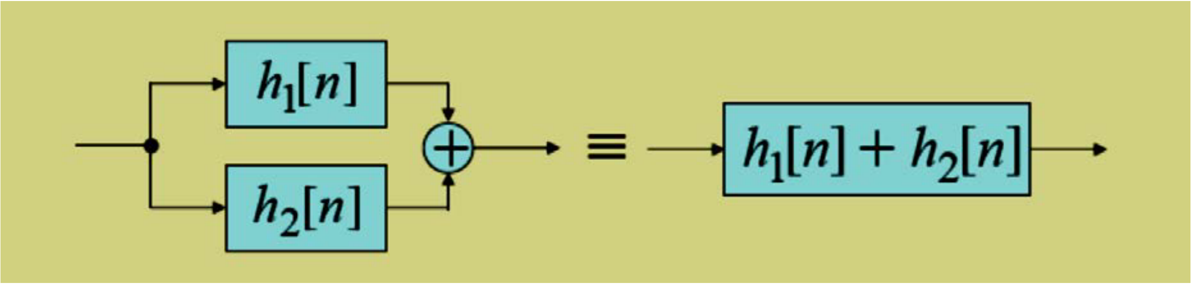
$$\sum_l^n h_2[l] = 0, \text{ for } n \geq 1.$$



Thus, we have $h_2[n] = \delta[n] - \delta[n-1]$, which is called a **backward difference system**.

Interconnection Schemes: Parallel

- **Parallel Connection**



- Impulse response $h[n]$ of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by $h[n] = h_1[n] + h_2[n]$