# Update/Outline

- **In this lecture we will consider**
  - Review Binary Digital Communications
  - Derive Optimum Receiver Structure for Generic Binary Modulations
  - Popular Binary Modulation Schemes

$\boxed{\text{AWGN}}$

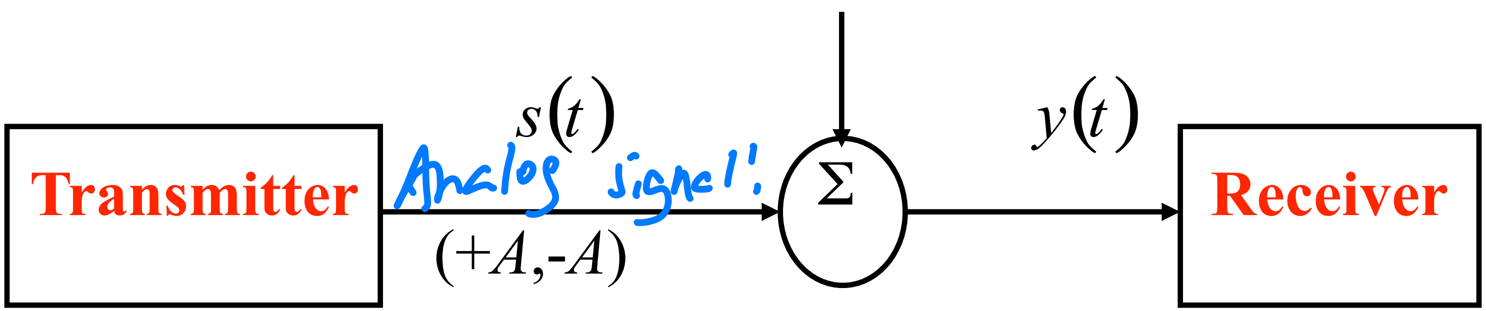
$y(t) = s(t) + n(t)$

# Binary Digital Data Transmission

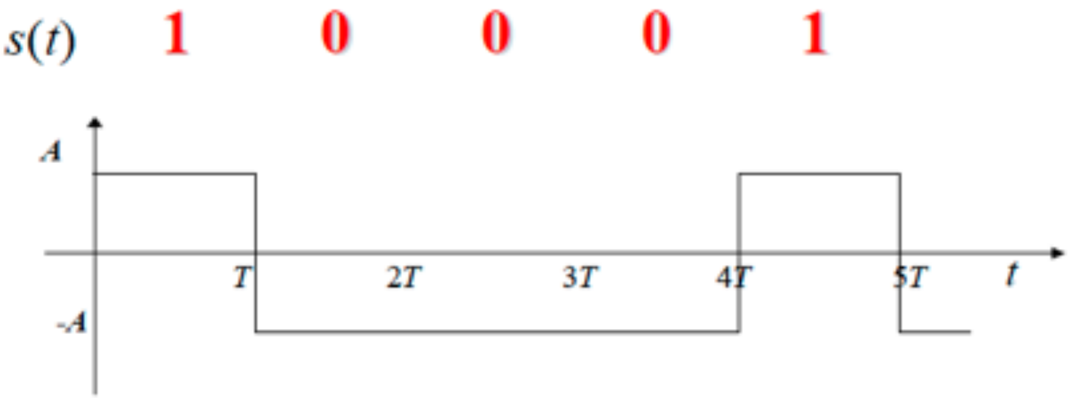
- In **digital systems**, the **signals take discrete values** to represent binary signals: -A, A is used to represent 0 and 1, for example.

- The system model for **baseband binary digital data transmission** is

$$n(t)\text{: PSD} = \frac{1}{2}N_0$$

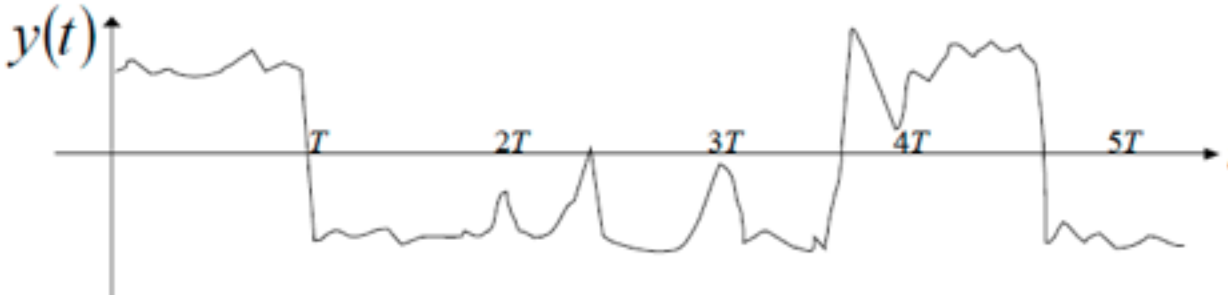


- **Example of a digital signal and transmitted waveform** is




- **Example of a received signal** is

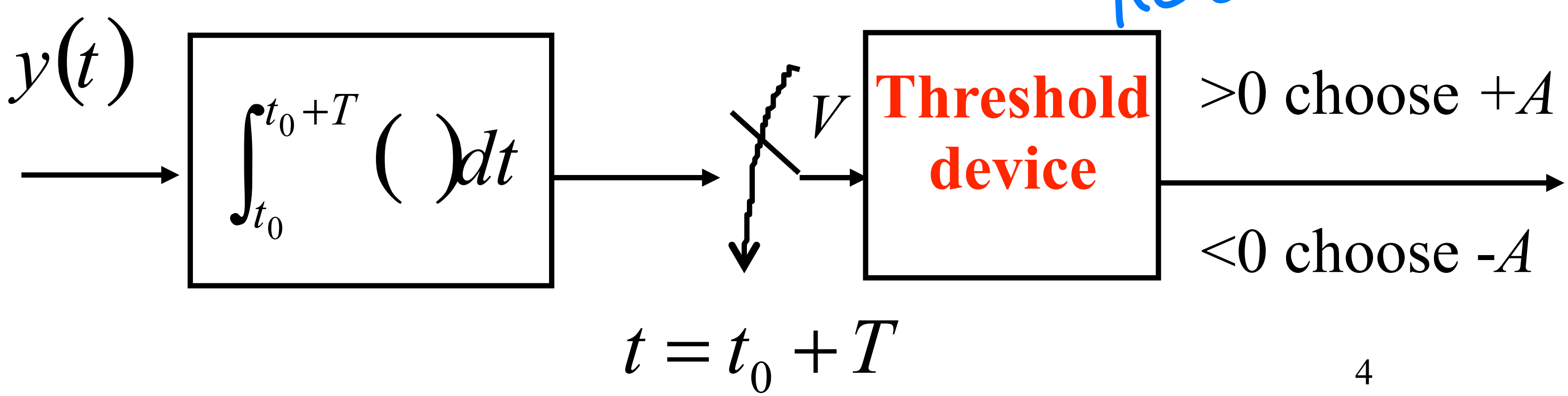
if sample, will easily get wrong



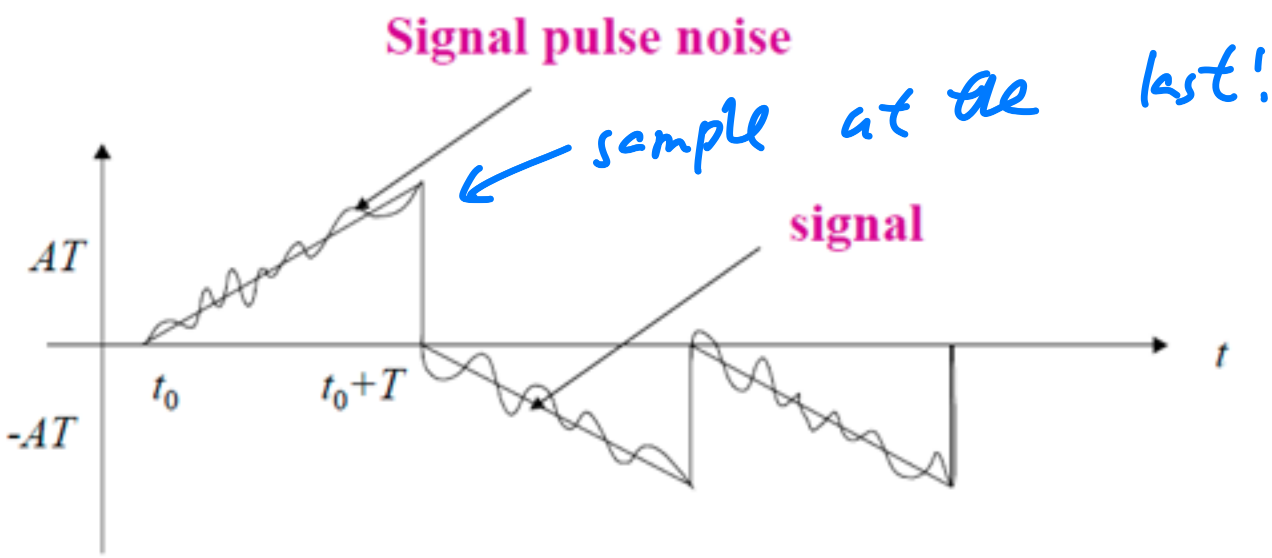
- For digital systems, the **probability of error** is used as a **measure of performance**

- A possible receiver structure for detecting the digital transmitted signals is shown below

Reasonable receiver!

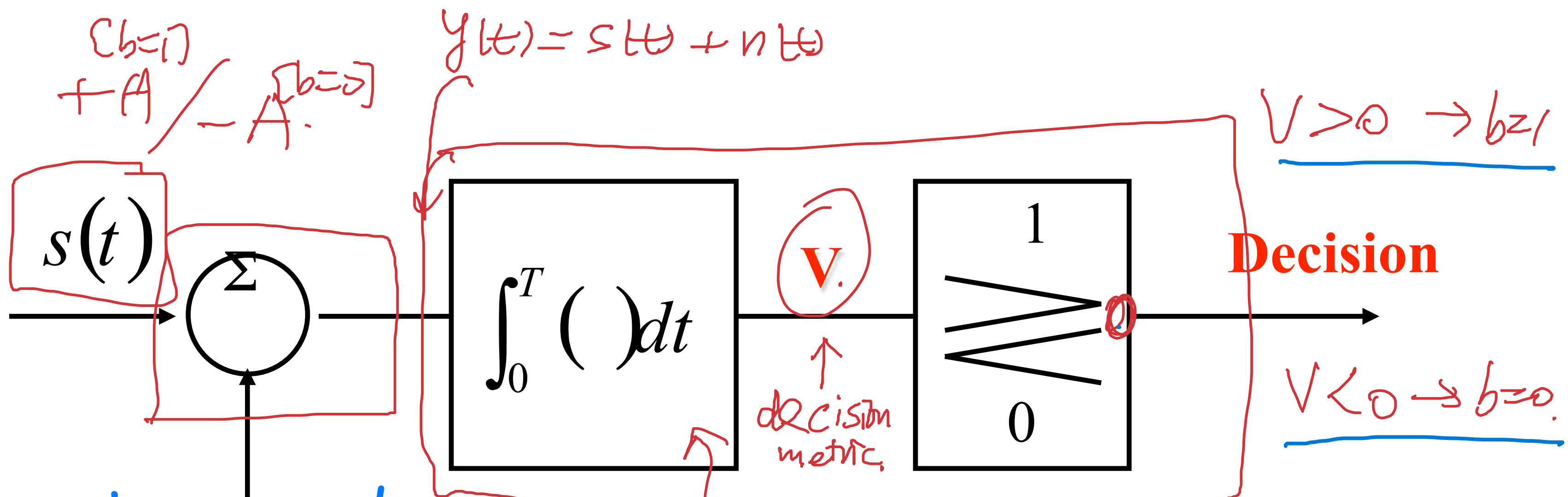


- The **integrator** averages out the noise received so that the output waveform will look like




- Assume noise at the input to the receiver is **AWGN** (**Additive White Gaussian Noise**)

- Based on linear system analysis, the noise output from the integrator will be Gaussian.
- Hence, we have




by receiver!

$$n(t) \sim S_N(f) = \frac{N_0}{2}$$

integrator.

AWGN

RV: $\sim X$ , give the distribution

↙ (strong)

precise description $\Longrightarrow$

① distribution

$F_X(x) = Pr(X \le x)$

$\Longleftarrow$ if gaussian

② pdf

$$f_X(x) = \frac{dF_X}{dx}$$

③ $p(x_1 \le X \le x_2) = F_X(x_2) - F_X(x_1)$

↘ (weaker)

Partial description

① $1^{st}$ order moment

$\mu_X = E(X)$

② $2^{nd}$ order moment

$E[X^2$

Precise description

① Distribution

$F_{X_1,X_2}(x_1,x_2) \stackrel{\Delta}{=} \Pr(X_1 \le x_1, X_2 \le x_2)$

② pdf

$f_{X_1,X_2}(x_1,x_2)$

③ $\Pr(x_1 \le X_1 \le x_2, x_3 \le X_2 \le x_4)$

$= \int_{x_1}^{x_2}\int_{x_3}^{x_4} f_{X_1,X_2}(x_1,x_2)\, d$

Partial description

① $1^{st}$ order moment

$E(X)$ $E(Y)$

② $2^{nd}$ order moment

$E(X)$ $E(Y)$ $E(XY)$

$X_1$ is gaussian

marginal pdf is gaussian, jointly gaussian mean that the jointly pdf is gaussian

Seperable → has two

independence (stronger)

$\Downarrow$ $\quad \Uparrow$ if jointly

uncorrelated (weaker, only in moment domain) $E(XY) = E(X)E(Y)$

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

Random Process $X(t) \sim \{X(t_1) \dots X(t_n)\}$

Precise (strong) $\xrightarrow{\quad}$ Partial (weak)

$\xleftarrow{\cancel{\quad}}$ okay if Gaussian RP

① $F_{t_1 \dots t_n}(x_1, x_2 \dots x_n)$

$\triangleq Pr(X(t_1) \leq x_1, X(t_2) \leq x_2)$

② $f_{t_1 \dots t_n}(x_1 \dots x_n)$

$Pr(X(t_1) \in (x_A^1, x_B^1),$
$X(t_2) \in (x_A^2, x_B^2))$

① $\mu(t) = E(X(t))$

② $E(X(t_1)X(t_2))$
$= R_X(t_1, t_2)$
function of 2 time!

$\boxed{\text{stationary}} \doteq$ (time invariant)

strong   weak

strict-sense stationary

$$F_{t_1+\tau, t_2+\tau \cdots t_n+\tau}(x_1, \cdots x_n) = F_{t_1, t_2 \cdots t_n}(x_1, \cdots x_n) \quad \forall \tau$$

weak-sense stationary

$$m(t) = E(x(t)) = \mu = \text{constant}$$

$$R_x(t_1+\tau, t_2+\tau) \stackrel{?}{=} R_x(t_1, t_2)$$

$$R_x(t_1 - t_2)$$

$$R_x(\Delta t) \triangleq E(x(t)\, x(t+\Delta t))$$

if Gaussian $\Rightarrow$ reversible!

$R_X(\Delta t) \xrightarrow{F} S_X(f)$ → specify the 2nd order moment!

$$\boxed{\begin{array}{l} x(t) \sim X(f) \\ x(0) = \int_{-\infty}^{\infty} X(f)\,df \end{array}}$$

$$R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\,df$$

↓ power

↓ summing up all the frequencies

⇓

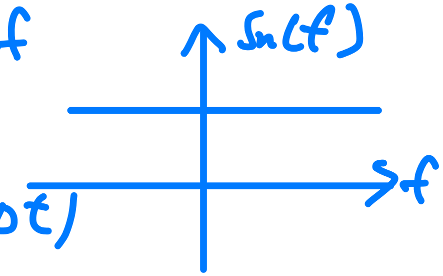
$S_X(f) \sim$ PSD

most random process! → white (specification of 2nd order moment)

AWGN

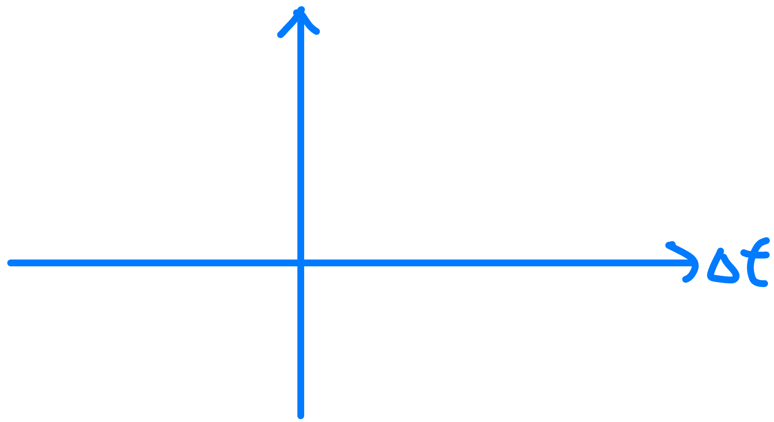
↑ stationary zero mean

$S_n(f) = \frac{N_0}{2} \quad \forall f$




$R(\Delta t) = \frac{N_0}{2}\delta(\Delta t)$

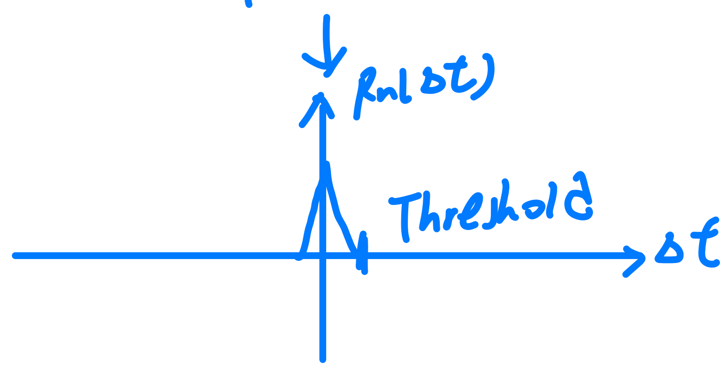
$$R_n(\Delta t) = \frac{N_0}{2}\delta(\Delta t) \triangleq E(n(t), n(t+\Delta t))$$




How likely I can predict the future!

- 但 cannot predict future,
  R 自然 correlation will be very large!

In reality cannot generate that




Raining day:

- Discrete time
- have correlation
- not realizable for continuous time

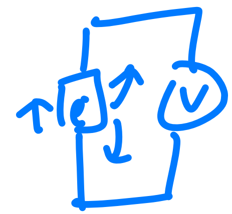
$\boxed{CLT}$

↓

many many iid

random walk!

Assume 1D current ↑, induced a voltage!




thermal noise

$n(t) = \sum_i x_i(t)$

why can measure it

- Random walk due to $e^-$ in conduction band!

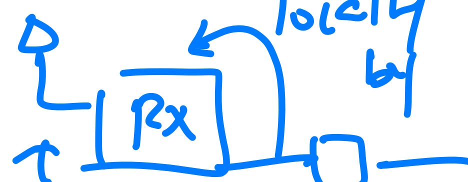
$\boxed{Tx}$

$s(t) =$

$s(t) + n_{tx}(t)$

↑

100 dB attenuation!




locally generated by this! in the resistor!

$y(t) = s(t) + n(t)$

$n_{rx}(t)$

↓

∴没有 attenuation!!!

$$s(t) = \begin{cases} A & 0 \le t < T \\ -A & 0 \le t < T \end{cases}$$

if "1" transmitted

if "0" transmitted

not Gaussian

random variable

distribution?

channel noise

$$V = \int_0^T [s(t) + n(t)]\,dt$$

data

y(t)

Gaussian R.V.

$$= \begin{cases} AT + N & \text{if "1" is sent} \\ -AT + N & \text{if "0" is sent} \end{cases}$$

still have Randomness!

V is conditionally Gaussian

With

1st order moment
2nd order moment.

$$\boxed{N = \int_0^T n(t)\,dt}$$

$Pr(E) = Pr(E|A)P(A) + Pr(E|B)\cdot P(B)$.

# Problem - Error Probability

$P_e \triangleq Pr[\text{"Error"}]$

Event:

conditioning gaussian, completely!

$$P_e = P(E|1\text{ sent})P(1\text{ sent}) + P(E|0)P(0)$$

total probability:

fully Gaussian

Prior Probability

Conditional Error Probability

$E(V|1) = AT + E(\int_0^T n(t)dt) = \int_0^T \cancel{v(t)}\, dt$ AT

$= AT + \int_0^T E(n(t))dt + \int_0^T n(t)\, dt$

$= AT + 0$

$$\sigma_v^2 = E[(v-AT)^2 \mid 1] = E(N^2)$$

$$= E\left(\int_0^T\int_0^T n(t)\,n(t')\,dt\,dt'\right)$$

white noise:

$$= \int_0^T\int_0^T E(n(t)\,n(t'))\,dt\,dt'$$

$$= \int_0^T \boxed{\int_0^T \frac{N_0}{2}\,\delta(t-t')\,dt}\,dt'$$

$$= \int_0^T \frac{N_0}{2}\,dt'$$

$$= \frac{N_0}{2}T \qquad -(2)$$

$$\left(\sum_{i=1}^{N} x_i\right)^2 = \sum_i \sum_j x_i x_j$$

Sifting property

$$\int_0^T g(t)\,\delta(t-t_0)\,dt = g(t_0)$$

# Error Probability Computation

- **We have**

$$V = \int_0^T \left[s(t) + n(t)\right]dt$$

$$= \begin{cases} AT + N & \text{if "1" is sent} \\ -AT + N & \text{if "0" is sent} \end{cases}$$

- **V is Gaussian with variance** $\sigma^2$ (which is the **noise N variance**).

Pr[ RVs > const. ]

$\rightarrow P(E|1) = Pr(V \le 0 \,|\, 1)$

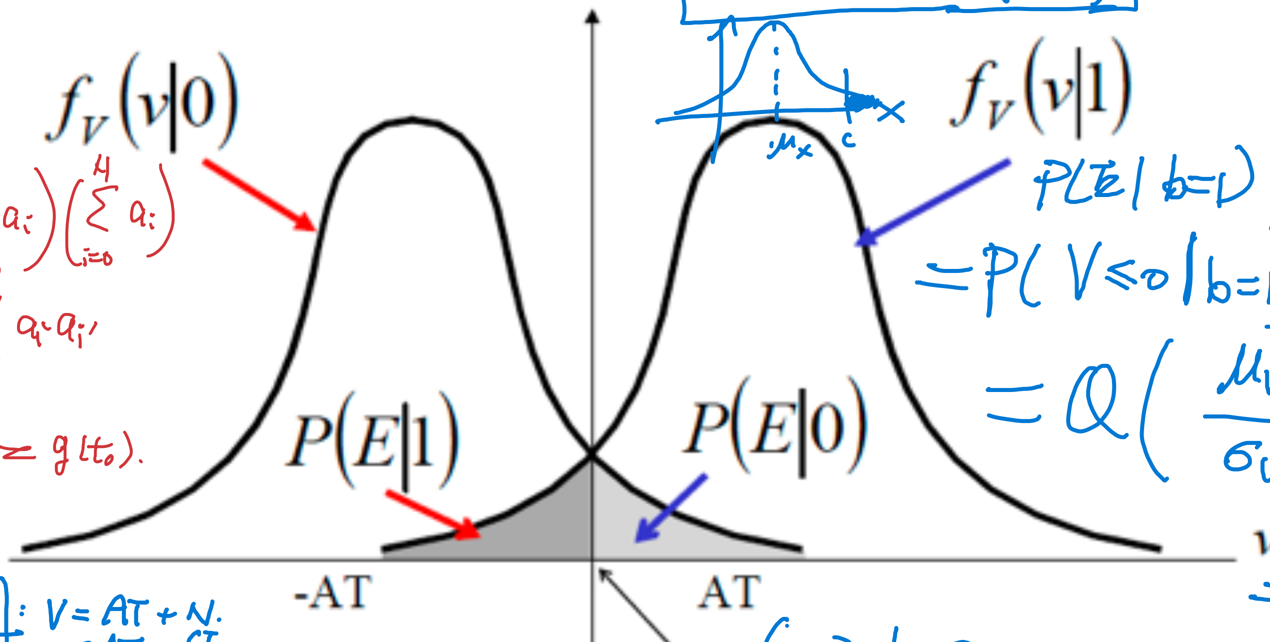
- **To compute** $P_e$, **we need to compute**

$P(E|0) = Pr(V \ge 0 \,|\, 0)$

P(E|0) and P(E|1)

$X \sim N(\mu_x, \sigma_x^2)$

$$\boxed{P_r(X \ge c) = Q\left(\frac{c-\mu_x}{\sigma_x}\right)} *$$




$$\left(\sum_{i=0}^{N} a_i\right)^2 = \left(\sum_{i=0}^{N} a_i\right)\left(\sum_{i'=0}^{N} a_{i'}\right) = \sum_{i}\sum_{i'} a_i a_{i'}$$

Sifting Theorem

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)\,dt = g(t_0).$$

$$P(E|b=1) = P(V \le 0|b=1) = Q\left(\frac{\mu_V}{\sigma_V}\right) = Q\sqrt{\frac{(AT)^2}{(N_0T/2)}} = Q\sqrt{\frac{2A^2T}{N_0}}.$$

$\boxed{\text{Consider 1 is tx}}$: $V = AT + N = AT + \int_0^T n(t)\,dt.$

- $E[V] = AT + E[N] = AT \triangleq \mu_V$.
- $\sigma_V^2 \triangleq E[(V-\mu_V)^2] = E(N^2) = E\left[\left(\int_0^T n(t)dt\right)\left(\int_0^T n(t)dt\right)\right]$

$$= E\left[\int_0^T\int_0^T n(t)n(t')\,dt\,dt'\right] = \int_0^T\int_0^T E[n(t)n(t')]\,dt\,dt'$$

$$= \int_0^T\int_0^T \frac{N_0}{2}\delta(t-t')\,dt\,dt' = \int_0^T \frac{N_0}{2}\,dt' = \frac{N_0T}{2} *$$

Case 2: $b=0$

$V = -AT + N.$

**Threshold=0**

- $\mu_V = -AT$, $\sigma_V^2 = N_0T/2$.

$$P(E|b=0) = P_r(V \ge 0|b=0) = Q\sqrt{\frac{2A^2T}{N_0}}.$$

$$P(E) = P(E|1)P(1 \quad sent) + P(E|0)P(0 \quad sent).$$

With
$$P(E|1) = P(V < 0|\,1\,\text{sent}) = \int_{-\infty}^{0} f_V(v|1)\,dv$$

- The **key** to estimating the error probabilities is to **find out more about the distribution of the noise**

- The **noise mean** can be calculated as

$$E[N] = E\left[\int_0^T n(t)\,dt\right] = \int_0^T \underbrace{E[n(t)]}_{0}\,dt = 0$$

- The **noise variance** is

$$Var[N] = E[N^2] = E\left[\left\{\int_0^T n(t)dt\right\}^2\right]$$

$$= \int_0^T \int_0^T E[n(t)n(v)]dtdv$$

$$= \int_0^T \int_0^T R_n(t-v)dtdv$$

$$= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-v)dtdv \qquad \therefore \text{White noise}$$

$$= \int_0^T \frac{N_0}{2}dv$$

$$= \boxed{\frac{N_0 T}{2} \equiv \sigma^2}$$

- Thus, we know that the **output noise from the integrator** **will have** the following **Gaussian distribution**

$$\therefore N \sim \mathbf{N}\left(0, \sigma^2\right)$$

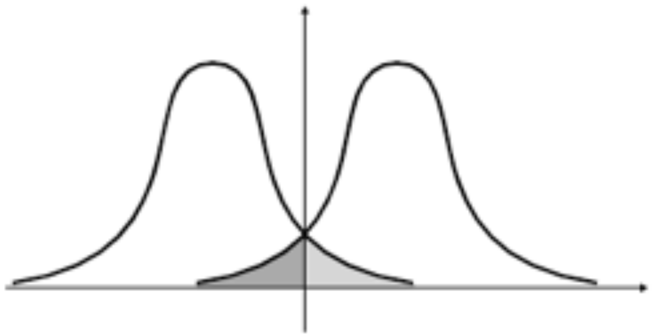
$$\Rightarrow f_n(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right)$$

- Now,

$$P(E|0) = P(V > 0|0 \text{ sent}) = \int_0^\infty f_V(v|0)\,dv$$

- **Due to symmetry**, we have

$$P(E|0) \equiv P(E|1)$$



- If "0" and "1" are **equally likely**, **then** $P(0) \equiv P(1) = \frac{1}{2}$

$$\Rightarrow \qquad P_e = P(E|0) = P(E|1)$$

- **Hence**

$$P_e = P(E|0) = \int_0^{\infty} f_V(v|0)\,dv = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(v+AT)^2}{2\sigma^2}}\,dv$$

one bit

$$E_b = \bar{E}_s = \frac{1}{2}(A^2T + A^2T)$$

$$= A^2T$$

M-ary modulation $\Rightarrow$ these two not the same

$$\Rightarrow P_e = Q\sqrt{\frac{2A^2T}{N_0}} = Q\sqrt{\frac{2E_s}{N_0}}$$

$$= \sqrt{\frac{2\bar{E}_b}{N_0}}$$

- know, understand to derive from first principal!

**VIP Transformation**

- Let $x = \dfrac{v + AT}{\sigma} \Rightarrow dx = \dfrac{dv}{\sigma}$

$$P_e = \int_{\frac{AT}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \triangleq Q\left[\frac{AT}{\sigma}\right]$$

- where

need know how to derive this equation!

$$\boxed{\mathrm{Q}[x] \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt}$$

**Q[.] function**

$Pe = Q\sqrt{\frac{A^2T^2}{(n_0 T/2)}} = Q\sqrt{\frac{2A^2T}{n_0}}$

eta

waveform will not be any of +A, -A

need to express this more generic for any waveform)

# Q-function

The Q-function is a standard form for expressing error probabilities without closed form solution.

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

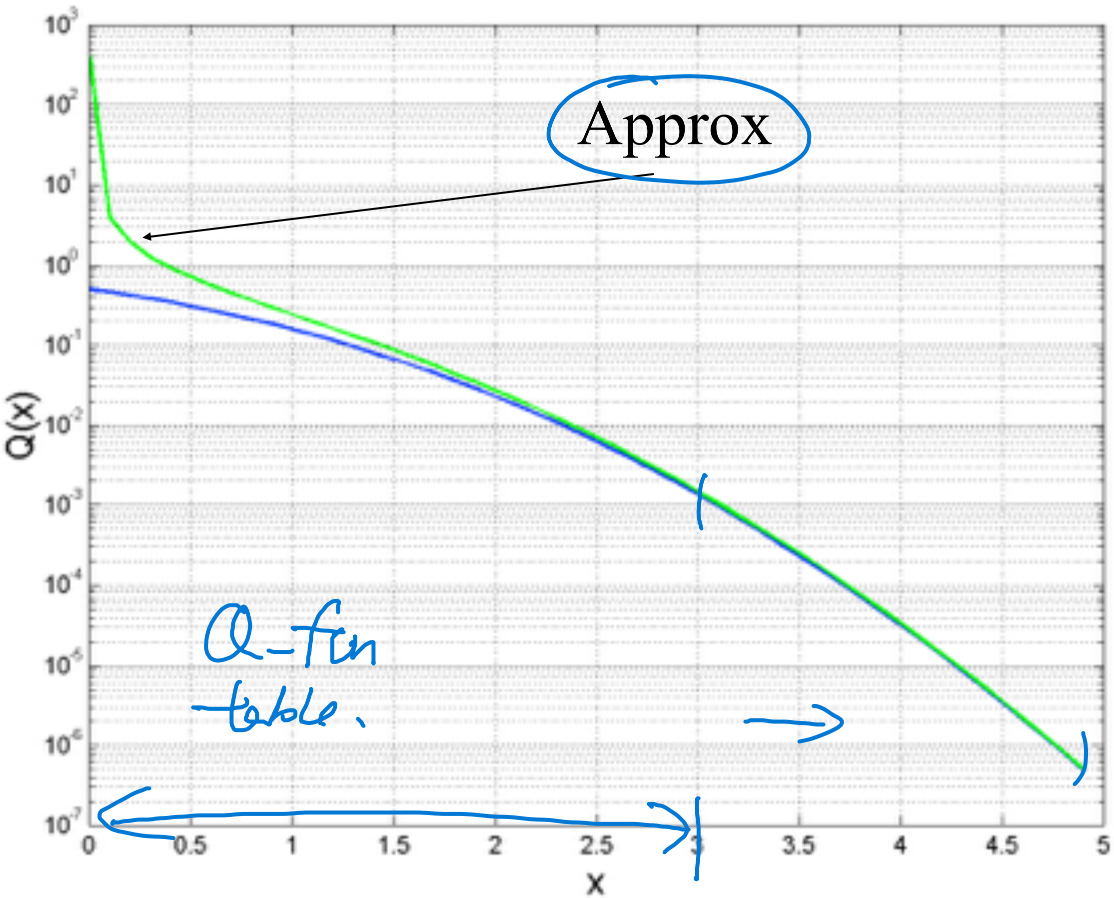
where erfc(.) is complementary error function.

$$Q(x) = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}\left[1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} + \cdots + \frac{(-1)^n \cdot 1\cdot 3\cdots(2n-1)}{x^{2n}}\right]$$

$$\approx \frac{1}{x\sqrt{2\pi}} \boxed{e^{-x^2/2}}, \text{ for } x \geq 3$$

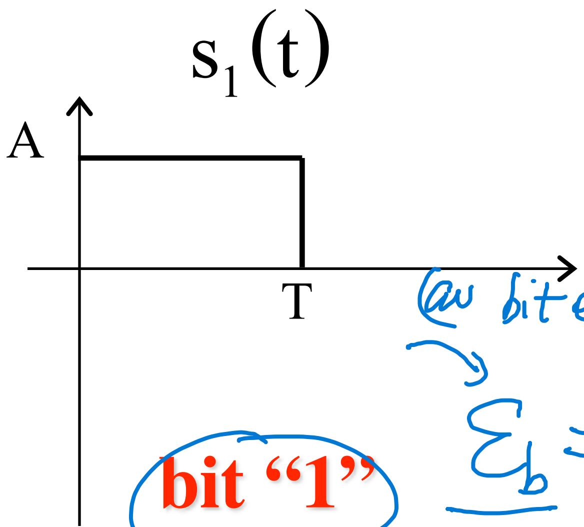
decrease with x, exponentially fast!!!

# Plot



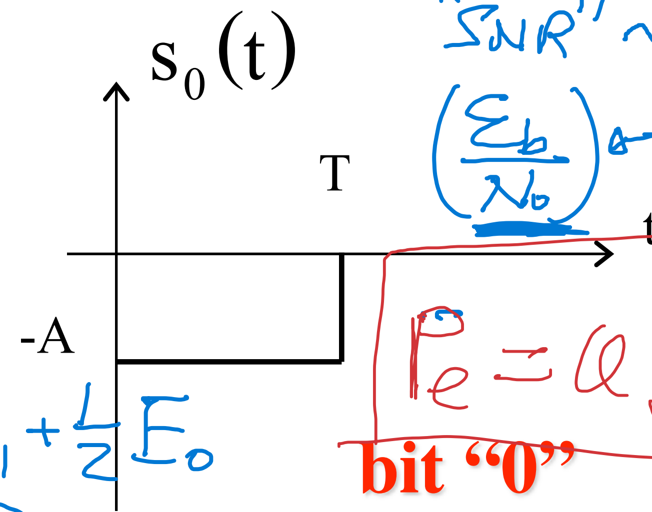
- Next note that one can represent $s(t)$ as $s_0(t)$ ("0" sent) or $s_1(t)$ ("1" sent).

$$\boxed{P_e = Q\sqrt{\frac{2A^2T}{N_0}}}$$




(av bit energy)

$$\underline{\mathcal{E}_b} = \frac{1}{2}E_1 + \frac{1}{2}E_0 = A^2T$$




"SNR" ~ normalized on per bit basis.

$$\left(\frac{\mathcal{E}_b}{N_0}\right)$$

$$\boxed{P_e = Q\sqrt{\frac{2\mathcal{E}_b}{N_0}}}$$

$$E_1 \triangleq \int_0^T s_1^2(t)\,dt = A^2T \qquad E_0 \triangleq \int_0^T s_0^2(t)\,dt = A^2T$$

**Energy**

$$\therefore E_b \underline{\underline{\Delta}} \text{ Energy/bit} = E_0 P(\text{"0" sent}) + E_1 P(\text{"1" sent})$$

$$= \frac{1}{2}(E_0 + E_1)$$

$$But \qquad E_1 = E_0 = A^2 T.$$

- We **can use** these **energy calculations** to **find a more general result** for the error probabilities. i.e.,

$$P_e = Q\left[\frac{AT}{\sigma}\right] = Q\left[\sqrt{\frac{A^2T^2}{\sigma^2}}\right]$$

where

$$\sigma^2 = \frac{N_0 T}{2}$$

- Therefore, $$P_e = Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

$$\equiv Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

~ **SNR** (normalized per bit)

$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]$$

- where $$\operatorname{erfc}(u) = 1 - \operatorname{erf}(u)$$

**Complementary Error Function**

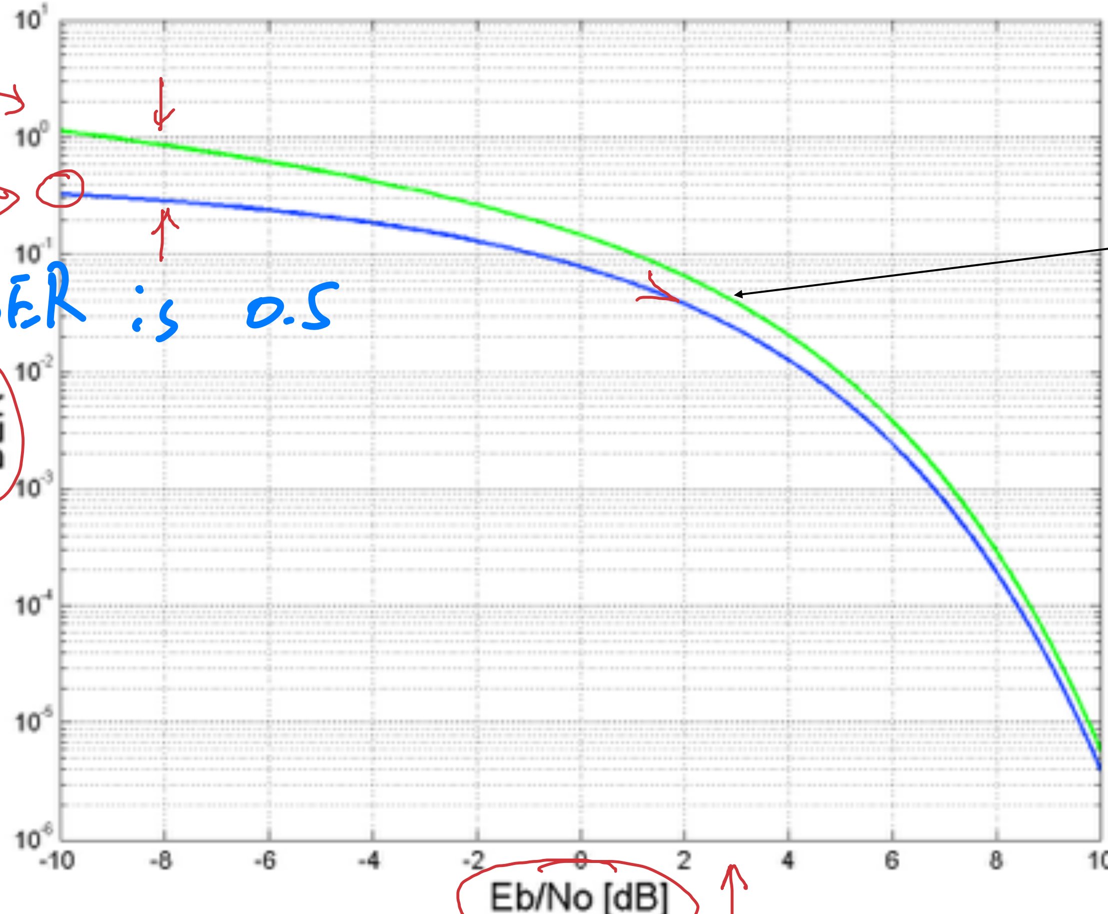
$$\operatorname{erf}(u) \triangleq \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2}\, dt$$

- A graph of $P_e$ for **baseband signaling** is

1

0.5

max BER is 0.5




Approx

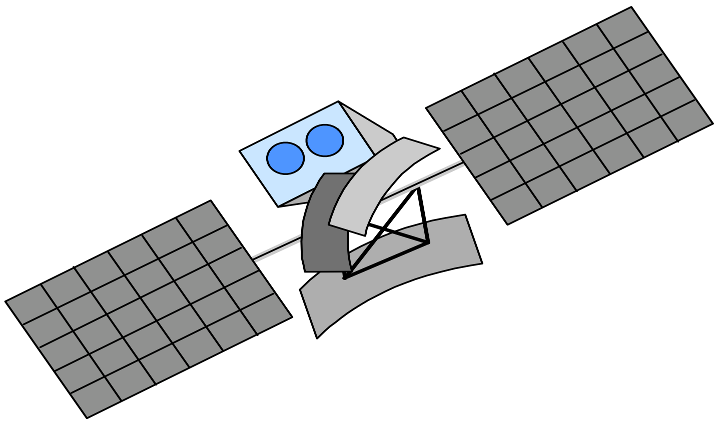
asymptotic slope

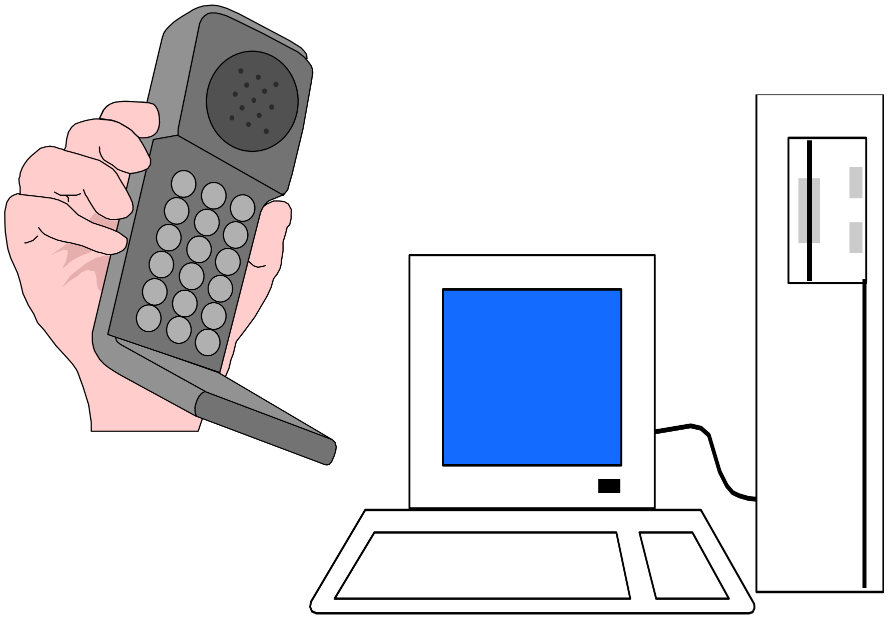
very efficient

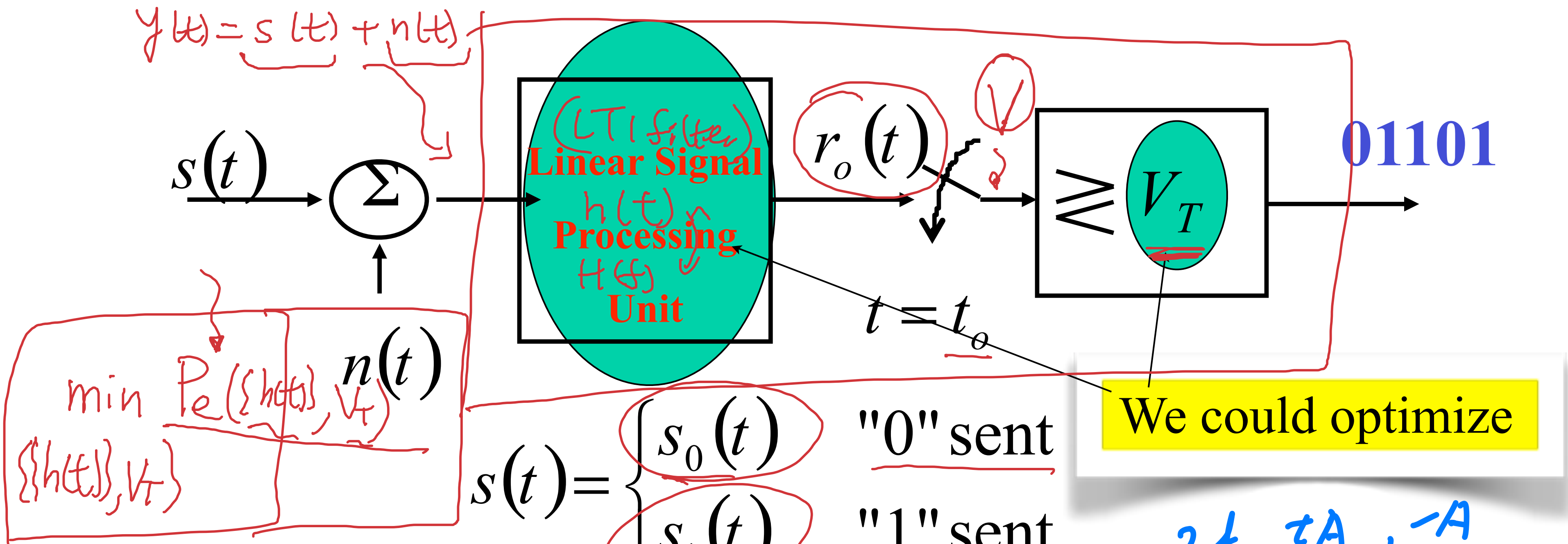
$$Z = \frac{E_b}{N_0} \qquad Pe = Q\left[\sqrt{2Z}\right] = \frac{1}{2}\operatorname{erfc}\left[\sqrt{Z}\right]$$

# Optimum Receiver Structure

- So far we considered a simple ad-hoc receiver structure
- What is the optimum receiver structure we can have that will minimize $P_e$
- The ***optimum receiver structure*** is the Matched Filter







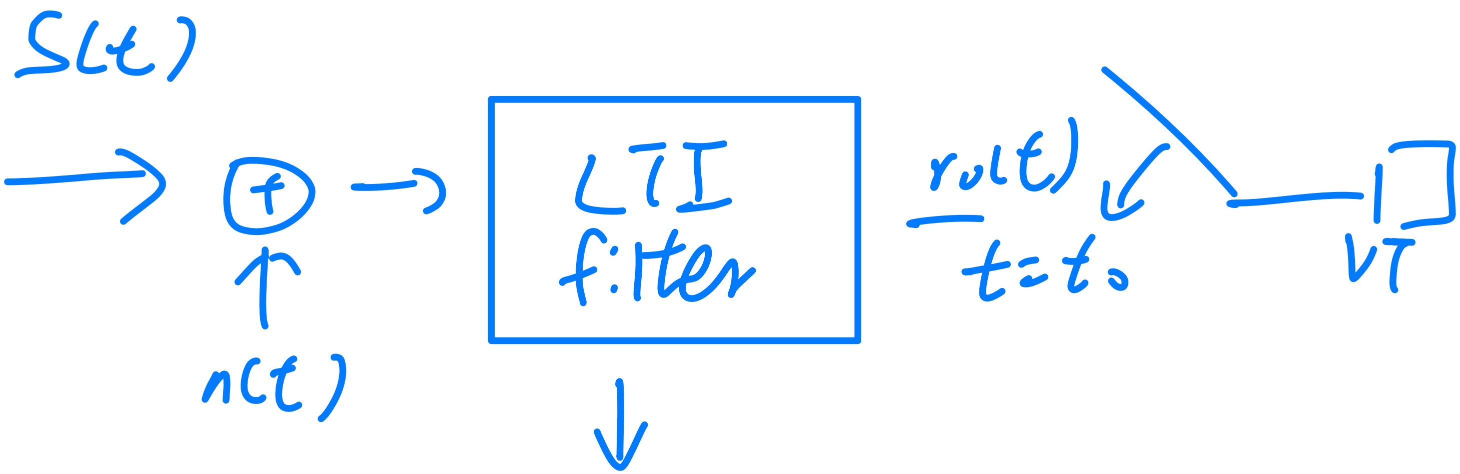

$$s(t)=\begin{cases} s_0(t) & \text{"0" sent} \\ s_1(t) & \text{"1" sent} \end{cases}$$

2t +A, -A
:) integrator
to max. the contrast!

- Assuming that the noise is Gaussian **after** signal processing (zero mean) so that

$$\underbrace{r_o(t)}=\underbrace{s_o(t)}_{\text{Good.}}+\underbrace{n_o(t)}_{\text{Bad.}} \quad \longleftarrow \quad \text{r.p.}$$

where the "**o**" subscript stands for **output**




min. error prob.

target: $\underset{\{h(t)\},\, V_T}{\arg\min}\ P_e(\{h(t)\}, V_T)$

high school $f(x) = x^2 + 2$

$f^* = 2 = \min_x f(x)$

$x^* = 0 = \arg\min_x f(x)$

$$\min_{x} \boxed{f(x)} \rightarrow \text{objective function}$$

$\uparrow$
variables

optimality condition $\quad \dfrac{\partial f}{\partial x_1} = 0$

$$\frac{\partial f}{\partial x_2} = 0$$

$$\vdots$$

$$(h(t)^*, VT^*) = \underset{\{h(t)\}, VT}{\operatorname{argmin}} \; pe(\{h(t)\}, VT)$$

$\downarrow$
function of time. functional space

$s_{o0} = s_o * h(t)$

At $t = t_o$ , $r_o = s_o + n_o$

$$= \begin{cases} s_{o0} + n_o & \text{if "0" sent} \\ s_{o1} + n_o & \text{if "1" sent} \end{cases}$$

with

$$\boxed{n_o \sim N(0, \sigma^2)}$$

$\sigma^2 = ?$

$\sigma^2$'s PSD $= |H(f)|^2 S_x(f)$

$= E_s \frac{N_0}{2}$

**Assume "0" sent, then**

$E[r_o|0] = s_{o0}$ **and** $Var[r_o|0] = \sigma^2$ **Hence,**

$$f(r_o|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_o - s_{o0})^2}{2\sigma^2}}$$

$N(s_{o0}, \sigma^2)$

$$P(E|0) = P(r_o > V_T) = \int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r_o - s_{o0})^2}{2\sigma^2}} dr = Q\left(\sqrt{\frac{(V_T - s_0)^2}{\sigma^2}}\right)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

xo

$$(h(t)^*, V_T^*) = \arg\min P_e(h(t), V_T)$$

$$P_e = P(E|1)P(1) + P(E|0)P(0)$$

$$P(E|0) = P(r_x = 1 | 0)$$
$$= P(r_0 > V_T | 0) = Q\left(\frac{V_T - s_{00}}{\sigma}\right)$$

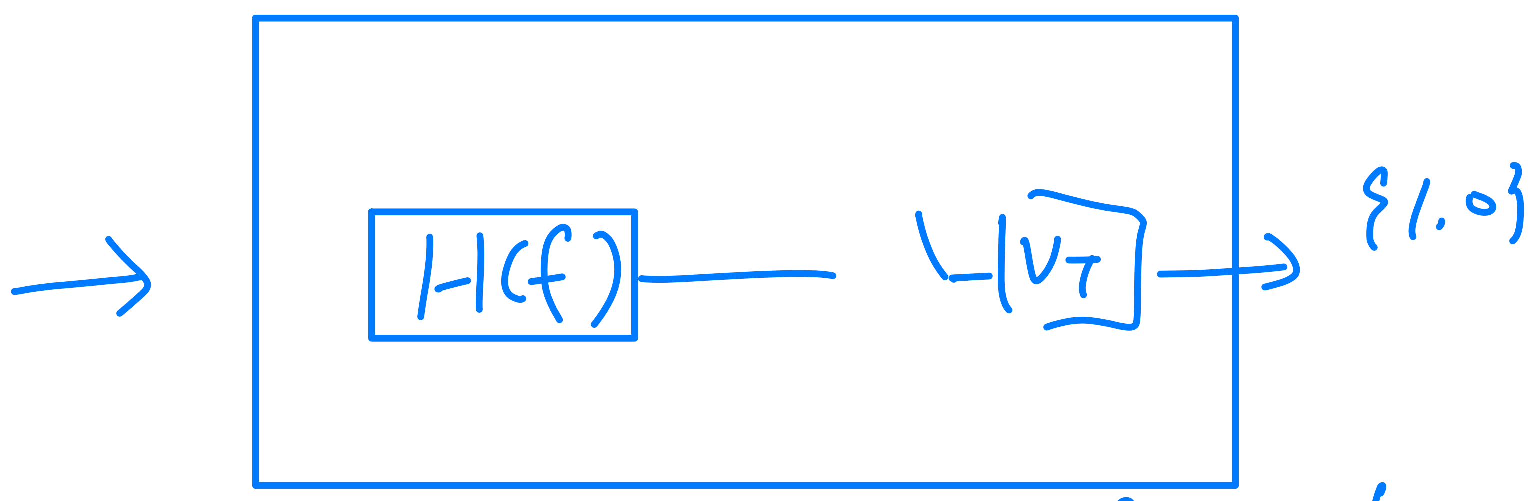
$r_0$ conditionally Gaussian

Convolution:
weighted sum of gaussian also gaussian
although infinite sum

$$n(t) \rightarrow \boxed{h(t)} \xrightarrow{n_0(t)}$$

$$E(n_0) = 0$$
$$E(n_0^2) = \sigma^2$$

$$P(X \geq x_0) = Q\left(\frac{x_0 - \mu_X}{\sigma}\right)$$




family of rx

$$\underset{(H(f),\, V_T)}{\operatorname{argmin}} \; P_e\left(H(f), V_T^*\right) = \underset{H(f)}{\operatorname{argmin}} \; Q\sqrt{\frac{(s_{01}-s_{00})^2}{4\sigma^2}}$$

$$V_T^{opt} = \frac{s_{01}+s_{00}}{2}$$

inner problem

$$\Rightarrow \underset{H(f)}{\operatorname{argmax}} \frac{(s_{o1} - s_{o\phi})^2}{\sigma^2}$$

$$= \underset{H(f)}{\operatorname{argmax}} \frac{\int |G(f) H(f) e^{j\omega t_0} df|^2}{|\int S_n(f) H(f) df|^2}$$

use Cauchy Swartz inequality

$$\frac{|\int G(f) H(f) e^{j 2\pi f t} df|^2}{\int S_n(f) |H(f)|^2 df} \leq P_{max}$$

tight

independent on $H(f)$

$$\left|\int X(f)\,Y(f)\,df\right|^2 \le \int |X(f)|^2\,df \int |Y(f)|^2\,df$$

iff $X(f) \propto Y^*(f)$

choose $Y(f) = \sqrt{S_n(f)}\,H(f)$

$$X(f)\,Y(f) = G(f)\,H(f)\,e^{j2\pi f t_0}$$

$$X(f) = \frac{G(f)\,e^{j2\pi f t_0}}{\sqrt{S_n(f)}}$$

$$P \le \frac{\int \frac{|G(f)|^2}{S_n(f)}\,df \;\cancel{\int S_n(f)\,|H(f)|^2\,df}}{\cancel{\int S_n(f)\,|H(f)|^2\,df}}$$

$$\Rightarrow \quad P_{max} = \int \frac{|G(f)|^2}{S_n(f)} df$$

$$\frac{G(f)e^{j2\pi f t_0}}{\sqrt{S_n(f)}} \propto \sqrt{S_n(f)}\, H^*(f)$$

$$H(f) = \frac{K G(f) e^{-j2\pi f t_0}}{S_n(f)}$$

- **Let** $t = \dfrac{r_o - s_{o0}}{\sigma}$ **, then**

$$P(E|0) = \int_{\frac{V_T - s_{o0}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= Q\left[\frac{V_T - s_{o0}}{\sigma}\right]$$

- **Assume "1" sent, then**

$N(s_{o1}, \sigma^2)$

$$P(E|1) = P(r_o < V_T) = \int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(r_o - s_{o1})^2}{2\sigma^2}} dr$$

$$= Q\left[\frac{s_{o1} - V_T}{\sigma}\right]$$

$\min_{\{\{h(t)\}, V_T\}} P_e(V_T, \{h(t)\}) = \min_{\{h(t)\}} \left[ \boxed{\min_{V_T} P_e(V_T, \{h(t)\})} \right]$

$= \min_{\{h(t)\}} \underbrace{P_e^*(\{h(t)\})}$

# Bit Error Probability

$\min_{x,y} f(x,y)$

{(Primal Decomposition)

$\min_y \left[ \underbrace{\min_x f(x,y)}_{\text{inner problem}} \right]$ (outer problem)

$$P_e = P(0) Q\left[\frac{V_T - s_{o0}}{\sigma}\right] + P(1) Q\left[\frac{s_{o1} - V_T}{\sigma}\right]$$

$P(E|0)$ $P(E|1)$

- **Question**: **What is the optimal threshold,** $V_{T_{opt}}$ **,which minimizes** $P_e$ **?**

$\min_x f(x)$

$\boxed{\frac{df}{dx} = 0}$ (Optimality Condition) $\to x^*$

- **Solution**: $V_{T_{opt}} = \arg\min_{V_T} P_e(V_T)$ inner problem

$$P_e = P(0)\int_{V_T}^{\infty} f_{r_o}(r_o|0)\,dr_o + P(1)\int_{-\infty}^{V_T} f_{r_o}(r_o|1)\,dr_o$$

- **Set** $\boxed{\frac{dP_e}{dV_T} = 0} \Rightarrow V_{T_{opt}}$

# Primal Decomposition

$$\min_{x,y} f(x,y)$$

$$= \min_{y} \left[ \min_{x} f(x,y) \right]$$

inner problem
treat $y$ as constant

$f^*(y)$

$\{h(t)\}$ many dimensions :(

$VT$ 1D only!
easy to tackle

$$\min_{\{h(t)\}} \left[ \min_{VT} Pe\,\{h(t)\}, VT \right]$$

freedom!

integrator → change to a LTI filter!

$h(t)$, parameter behind the filter!

# Optimal Threshold

natural log!

$$V_{T_{opt}} = \boxed{\frac{s_{o0}+s_{o1}}{2}} + \boxed{\frac{\sigma^2}{s_{o1}-s_{o0}}\cancel{\log\frac{P(0)}{P(1)}}}$$

- For the **important case** where $P(0)=P(1)\equiv\frac{1}{2}$ we have

now find h(t)
$p_e^*(\{h(t)\})$
$= p_e(\{h(t), v_T)$
still needs to solve outer product!

$$V_{T_{opt}} = \left(\frac{s_{o0}+s_{o1}}{2}\right)$$

$E[r_0 | b=1]$
$E[r_0 | b=0]$

# Minimum Bit Error Probability

Thus, $\boxed{P_e = Q\left[\frac{s_{o1} - s_{o0}}{2\sigma}\right]}$ when $\underline{V_T = V_{T_{opt}}}$, $P(0) = P(1) \equiv \frac{1}{2}$

$$P_e = Q\left[\sqrt{\underbrace{\frac{(s_{o1} - s_{o0})^2}{4\sigma^2}}_{Z}}\right]$$

$\boxed{\min_{\{h(t)\}} P_e^*(\{h(t)\})}$.

$\min_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n)$

Optimality Conditions:

$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$

$\Rightarrow (x_1^*, x_2^*)$

- **As $Z\uparrow$, $P_e\downarrow$. Hence, to minimize $P_e$, one must maximize $Z$.**

**Thus, must find optimum signal processing unit represented by a Filter,** $h_{opt}(t)$

$$\min_{\{h(t)\}} Q\sqrt{\frac{(s_{01}-s_{0\phi})^2}{4\sigma^2}}$$

$$\max_{\{h(t)\}} \frac{(s_{01}-s_{0\phi})^2}{\sigma^2}$$

affects those two!

$s_\phi(t)$, $s_1(t)$ $\rightarrow$ $\boxed{h\{t\}}$ $\rightarrow$ $s_{0\phi}(t)$, $s_{01}(t)$

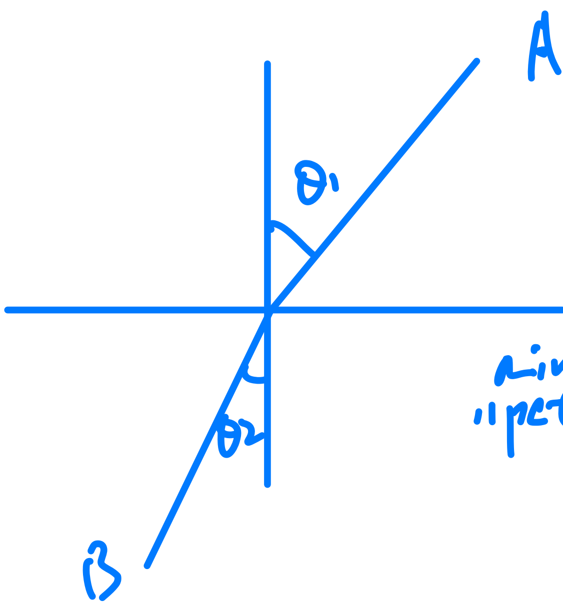
$n(t)$ $\qquad$ $n_0(t)$

functional optimization

function of function

$\Downarrow$

$\boxed{\text{Calculus of variations, Euler conditions!}}$




min "path", optimal ("path")

$\uparrow$ (1)

$\max f(x) = f_{x=x^*}$

$\downarrow$ (2)

inequality.

$f(x) \le f \quad \forall x$

equality holds if $x^* = x^*$

Find tight upper bound under some x!

$\int h(f)H(f)e^{j2\pi f t_0}\,df$

$$\max_{\{h(t)\}} \frac{(s_{o1}-s_{o0})^2}{\sigma^2} \le p^*$$

under some choice of $h(t)$!

$p^*$ independent on $h(t)$

$$\frac{s_1(t)}{s_0(t)} \longrightarrow \boxed{\{h(t)\}} \xrightarrow[s_{o0}(t)]{s_{o1}(t)} \; \frac{s_{o1}}{s_{o0}}$$

$\uparrow \mathcal{F}^{-1}$

$$\frac{S_1(f)}{S_0(f)} \longrightarrow \boxed{H(f)} \longrightarrow \begin{array}{l} S_{o1}(f) = S_0(f)H(f) \\ S_{o0}(f) = S_1(f)H(f) \end{array}$$

$$s_{o1}(t_0) = \mathcal{F}^{-1} S_{o1}(f) = \mathcal{F}^{-1}[S_1(f)H(f)]$$

$$= \int S_1(f)H(f)e^{j2\pi f t_0}\,df$$

$$s_{o0}(t_0) = \int S_0(f)H(f)e^{j2\pi f t_0}\,df$$

$$\boxed{\text{Let } G(f) \triangleq S_1(f) - S_0(f)}$$

$$n(t) \longrightarrow \boxed{h(t)} \longrightarrow n_o(t) \text{ — } n_o$$

$$\mathcal{F} \downarrow$$

$$S_n(f) \longrightarrow \boxed{H(f)} \longrightarrow$$

$$S_{n_o}(f) = S_n(f)\,|H(f)|^2$$

$$\sigma^2 = R_{n_o}(0) = \int_{-\infty}^{\infty} S_{n_o}(f)\,df$$

$$= \int_{-\infty}^{\infty} S_n(f)\,|H(f)|^2\,df$$

Let

$$\rho = \frac{\left(s_{o1} - s_{o0}\right)^2}{\sigma^2}$$

- **To minimize $P_e$, one needs to maximize $\rho$.**

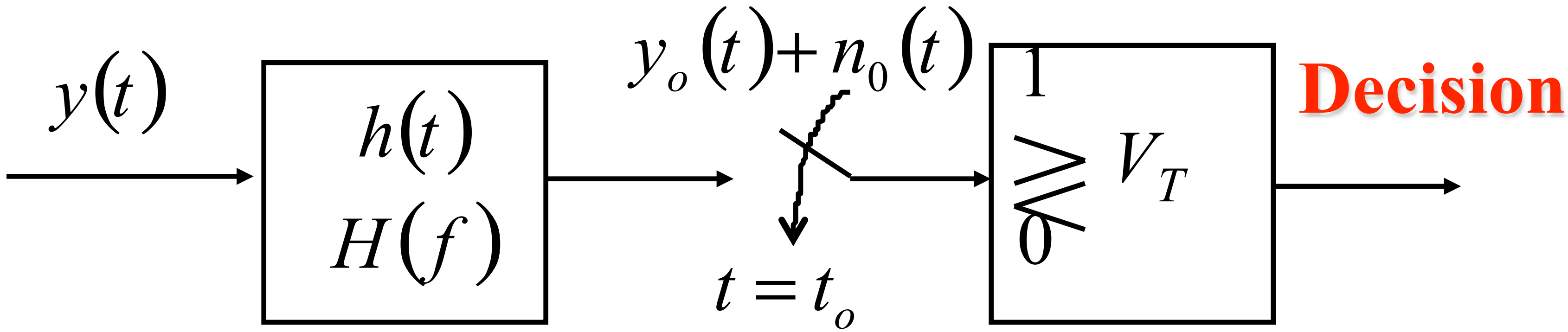
  $\Rightarrow$ **By finding the "Optimal" or best filter that maximizes $\rho$, we obtain the best performance.**

$$h_{opt}(t) = \arg\max_{h(t)} \rho(h(t))$$

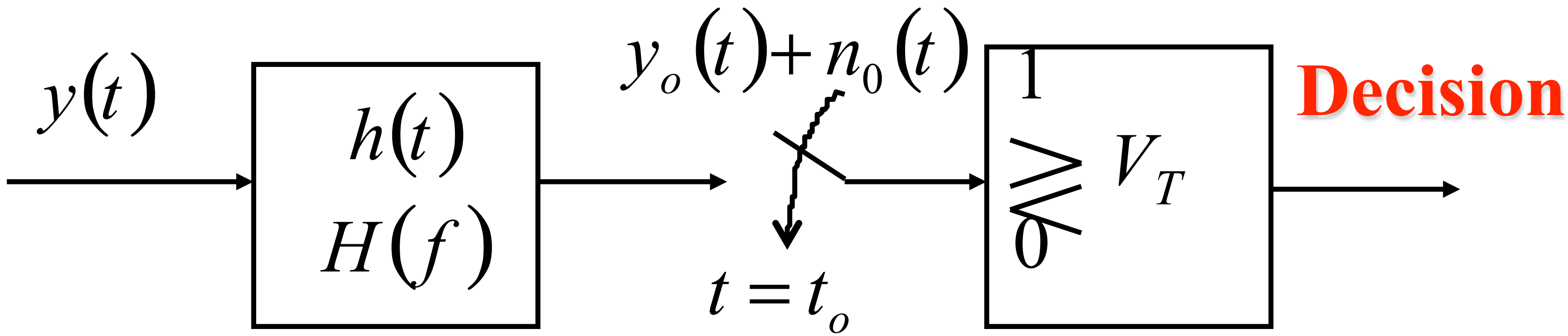
- Given **"0" transmitted** $\rightarrow$ $s_0(t); t \in [0, T]$

- Given **"1" transmitted** $\rightarrow$ $s_1(t); t \in [0, T]$

# Optimal Receiver

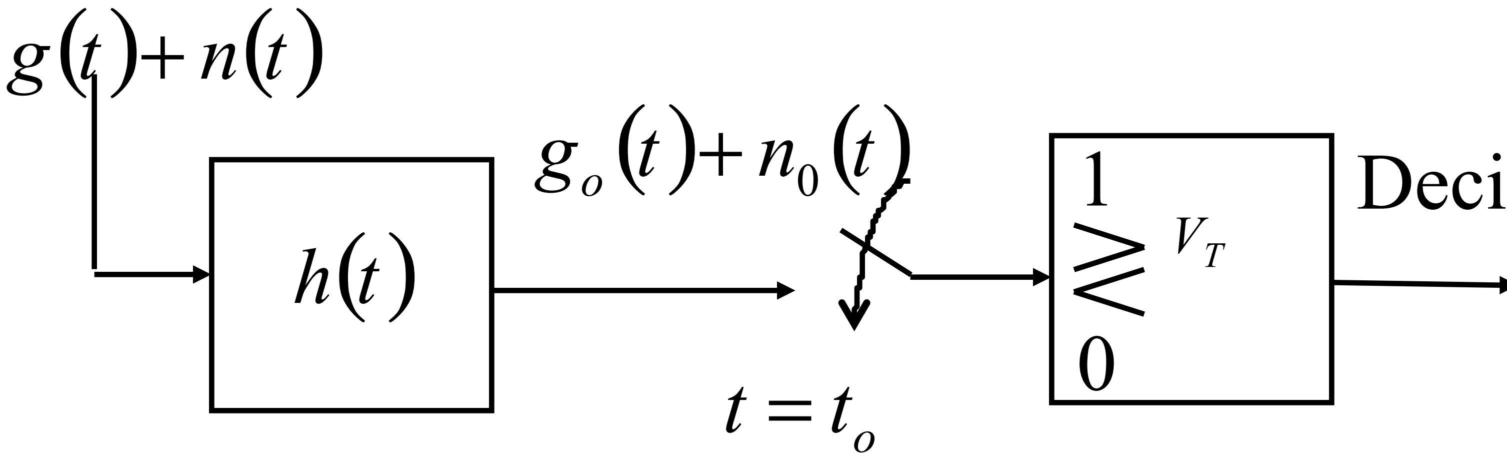



- $$y(t)=\begin{cases} s_0(t)+n(t) \\ or \\ s_1(t)+n(t) \end{cases}$$ **& Define** $g(t)=s_1(t)-s_0(t)$




- $$y(t)=\begin{cases} s_0(t)+n(t) \\ \quad or \\ s_1(t)+n(t) \end{cases}$$ **Define** $$g(t)=s_1(t)-s_0(t)$$

- Let $SNR \triangleq \overline{g_0^2(t)} \big/ \overline{n_0^2(t)}$. Then,




- Let

$$SNR\Big|_{output} \underset{=}{\Delta} \frac{E\left[g_o^2(t)\right]}{E\left[n_o^2(t)\right]}$$

- Instantaneous

$$SNR \underset{=}{\Delta} SNR_I = \frac{g_o^2(t)}{n_o^2(t)}$$

- **Problem**: Find *h(t)* such that *SNR$_I$* is maximized at *t = t$_o$*.

$$h_{opt}(t) = \arg\max_{h(t)} \rho(h(t))$$

$$H_{opt}(f) = \arg\max_{H(f)} \rho(H(f))$$

- **Solution**:

$$g_o(t) = \int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi ft}\, df$$

$$\sigma^2 = \int_{-\infty}^{\infty} |H(f)|^2 S_n(f)\, df$$

$$SNR_I\Big|_{t=t_o} \triangleq \rho = \frac{\left|\int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi f t_o}\, df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_n(f)\, df}$$

# Schwarz's Inequality

- Given $x(t)$ and $y(t)$

tight upper bound!

$$\left|\int_{-\infty}^{\infty} X(f)Y(f)df\right|^2 \le \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df$$

with <u>equality</u> iff

$$X(f) = KY^*(f)$$

where K is some constant.

Next let $X(f) = \dfrac{G(f)e^{j2\pi f t_o}}{\sqrt{S_n(f)}}$ $\quad Y(f) = H(f)\sqrt{S_n(f)}$

$$\Rightarrow X(f)Y(f) = G(f)H(f)e^{j2\pi f t_o}$$

Using Schwarz inequality we get

$$\rho \leq \frac{\boxed{\displaystyle\int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_n(f)}df} \displaystyle\int_{-\infty}^{\infty} |H(f)|^2 S_n(f) df}{\displaystyle\int_{-\infty}^{\infty} |H(f)|^2 S_n(f) df}$$

$$\Rightarrow \rho \le \int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_n(f)}\,df$$

----**Maximum** SNR.

with Equality iff

$$\boxed{\begin{gathered} H_{opt}(f) = K\frac{G^*(f)e^{-j2\pi f t_o}}{S_n(f)} \\ \Rightarrow \rho_{opt} = \int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_n(f)}\,df \end{gathered}}$$

$S_n(f) = \frac{N_0}{2}$

# Optimal Receiver

$$\rho \le \int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_n(f)}\, df$$

with Equality iff

$$H_{opt}(f) = K \frac{G^*(f) e^{-j2\pi f t_o}}{S_n(f)}$$

$$\Rightarrow \rho_{opt} = \int_{-\infty}^{\infty} \frac{|G(f)|^2}{S_n(f)}\, df$$

**Note**:

- Gaussian assumption made
- **No White noise assumption made**

# Optimum Receiver: Special Case

Assume a **White noise input**, then $S_n(f) = \frac{N_0}{2}$

$$\therefore H_{opt}(f) = \frac{2K}{N_0} G^*(f) e^{-j2\pi f t_o}$$

Hence, $h_{opt}(t) = Cg(t_o - t)$ where $C = \frac{2K}{N_0}$

and

$$\rho_{opt} = \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2 df = \frac{2}{N_0}\int_{-\infty}^{\infty} g^2(t)dt$$

$$\triangleq \frac{2E_g}{N_0} = \frac{2\int_0^T (s_1(t) - s_0(t))^2 df}{N_0}$$

- Since $C$ = constant, $\Rightarrow$ WLOG, the **Optimal Filter** is the **Matched Filter**.

$$\rho_{opt} = \frac{(s_{01} - s_{00})^2}{4}$$

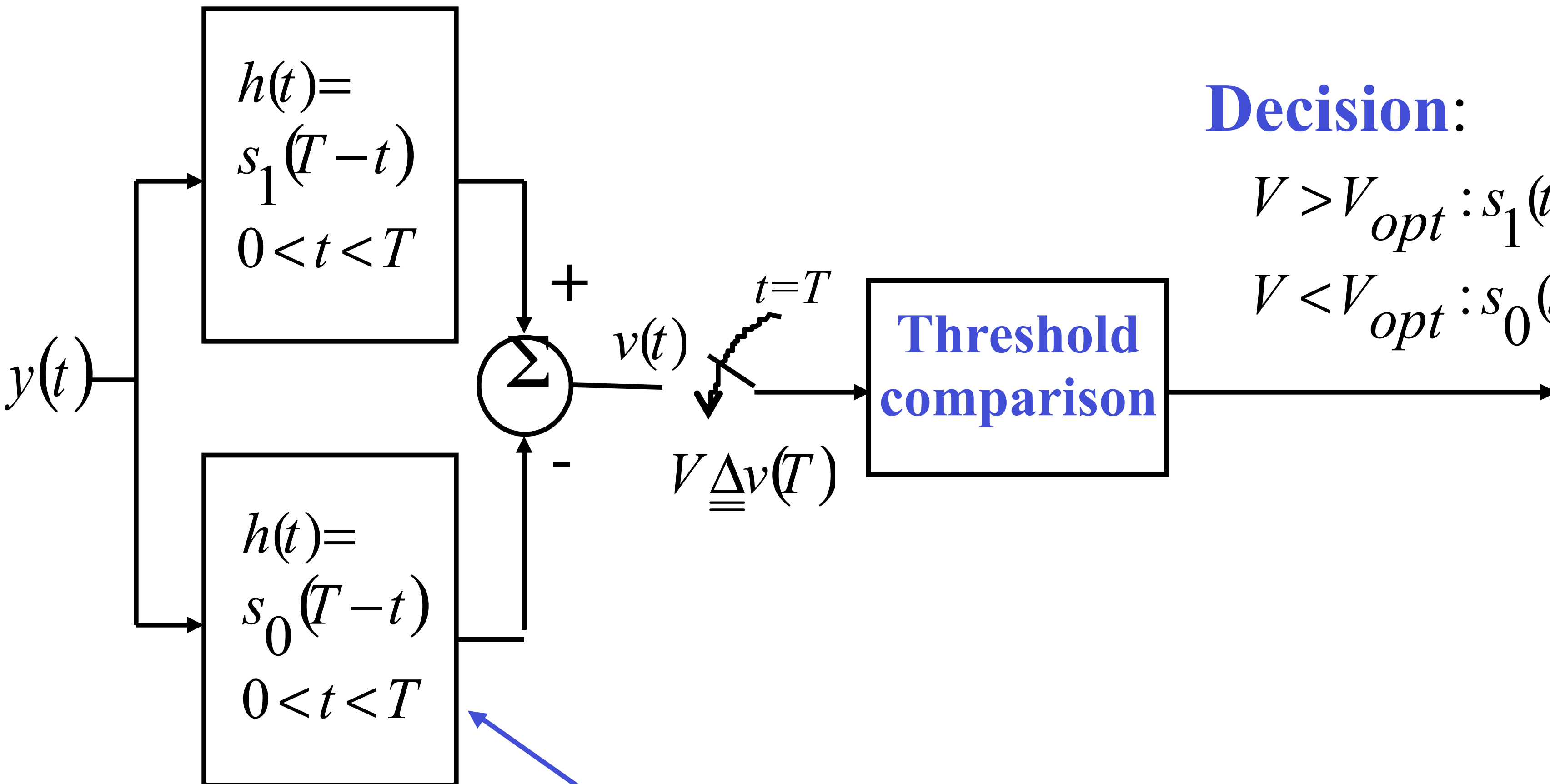
$$h_{opt}(t) = s_1(t_o - t) - s_0(t_o - t)$$

$$Pe = Q\left[\sqrt{\frac{(s_{o1} - s_{o0})^2}{4\sigma^2}}\right] = Q\left[\sqrt{\frac{\rho_{opt}}{4}}\right]$$

$$= Q\left[\sqrt{\frac{E_g}{2N_0}}\right]$$

Energy of $g(t)$ matched to my signal

# Optimum (Matched filter) receiver for binary signaling in white Gaussian noise




**Decision**:

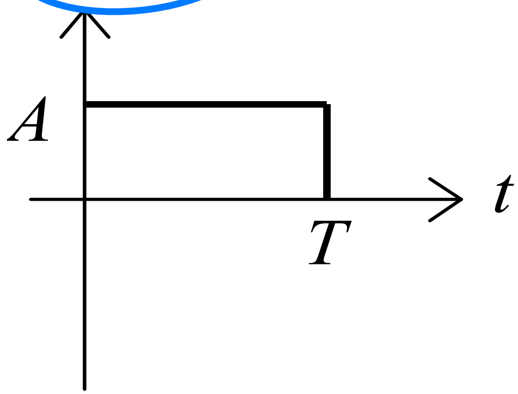
$V > V_{opt} : s_1(t)$

$V < V_{opt} : s_0(t)$

**2 Matched Filters (each matched to $s_1(t)$ and $s_2(t)$)**

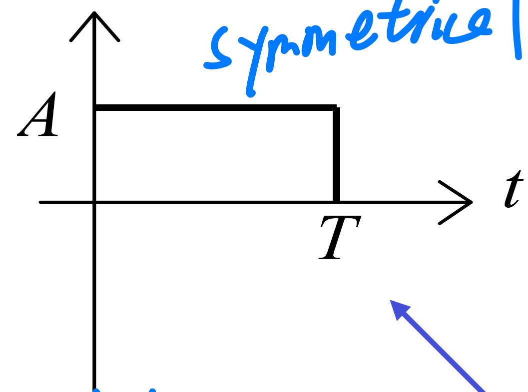
# Matched Filter

$s_1(t)$



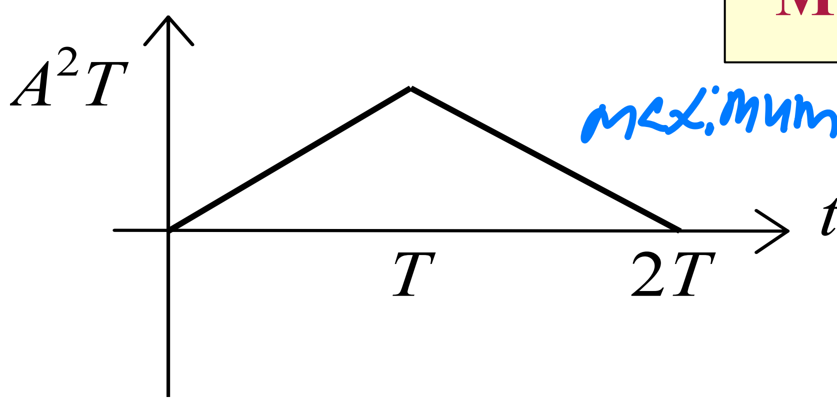

$s_1(T-t) = h_1(t)$

symmetrical!




convolution

$x(t) = s_1(t) * h_1(t)$

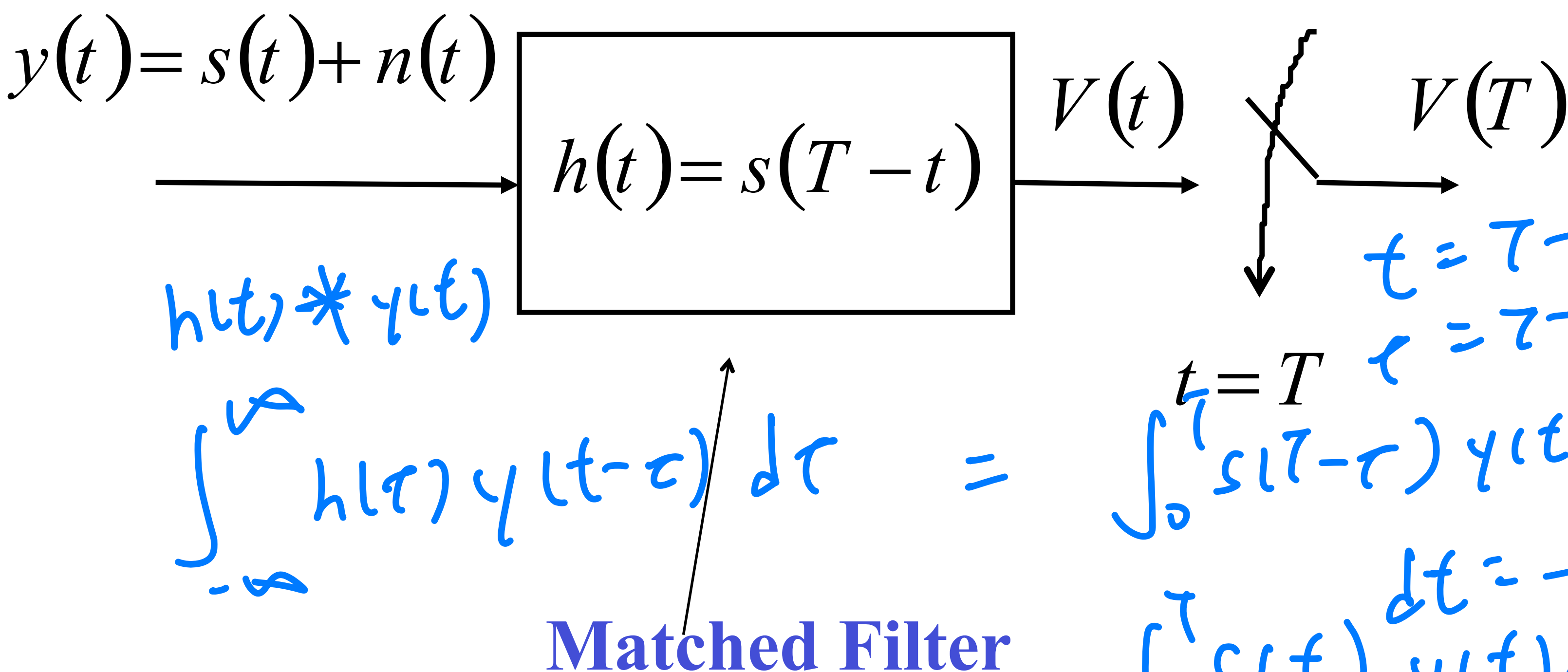



**Matched Filter to $s_1(t)$**

maximum contrast!

# Correlator Receiver

- We know that the optimum receiver structure consists of a matched filter so that




$h(t) * y(t)$

$$\int_{-\infty}^{\infty} h(\tau)\, y(t-\tau)\, d\tau = \int_0^T s(T-\tau)\, y(t-\tau)\, d\tau$$

$t = T - \tau$

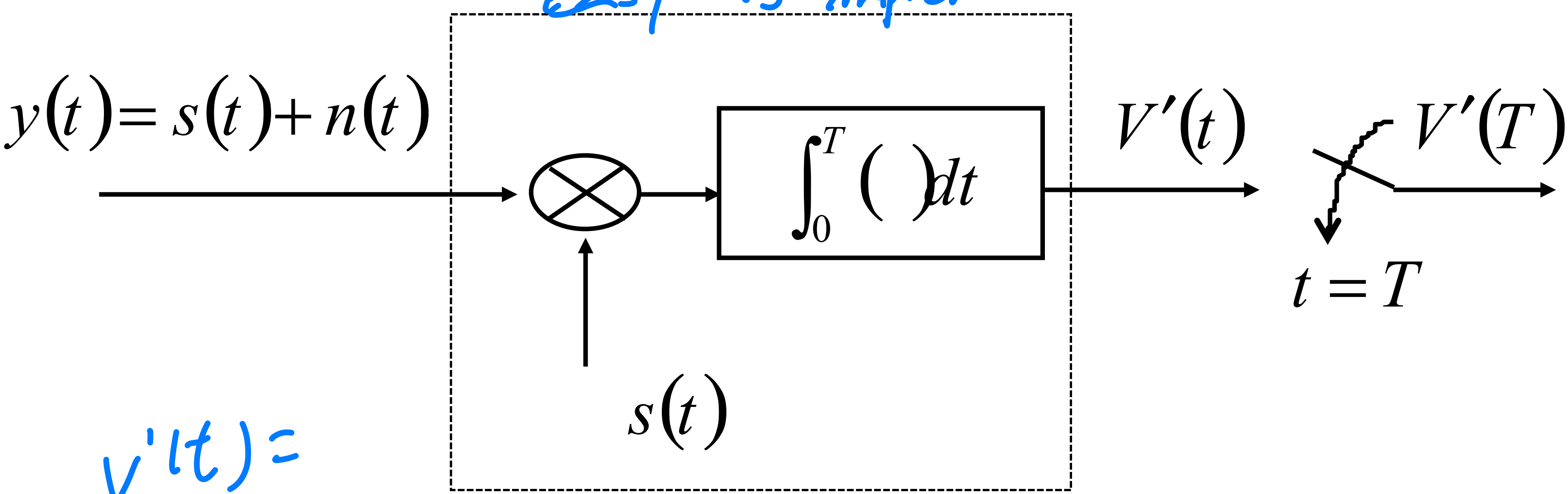
$\tau = T - t$

$dt = -d\tau$

$$= \int_0^T s(t)\, y(t)\, dt$$

- We can also **implement the matched filter as a correlator** as follows:

Easy to implement!




$v'(t) =$

$\int_0^T s(t)^2 + s(t)n(t) dt$

**Correlator Receiver**

**Can easily prove that both receivers are equivalent** (See Lecture notes)

$$V(t) = h(t) * y(t) = \int_0^T s(T-\tau)\, y(t-\tau)\, d\tau$$

$$\begin{cases} h(t) = s(T-\tau) & 0 \le t \le T \\ h(t) = 0 & \text{else} \end{cases}$$

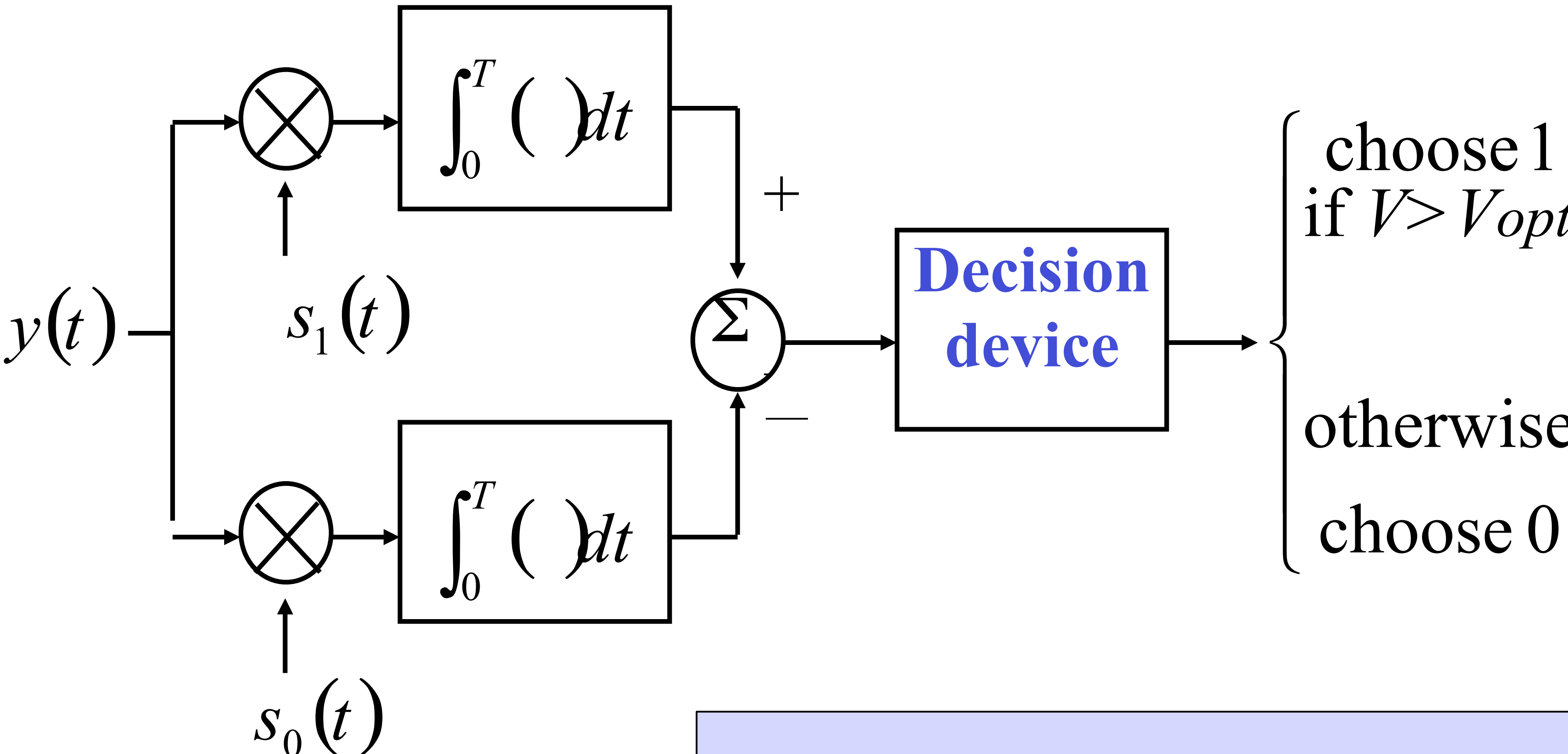
$$\therefore V(T) = \int_0^T s(T-\tau)\, y(T-\tau)\, d\tau$$

$$\equiv \int_0^T s(\alpha)\, y(\alpha)\, d\alpha \qquad (\alpha = T - t)$$

$$\equiv V'(T)$$

$\therefore$ **Both filters are *equivalent*.**

# Optimum (Correlator) receiver for data communications




**Both receivers are equivalent**

# Minimum Bit Error Probability

$$y(t) = \begin{cases} s_1(t) + n(t) \\ \text{or} \\ s_0(t) + n(t) \end{cases} \qquad V_{opt} = \frac{s_{o0} + s_{o1}}{2} \text{ if } P(0) = P(1) = \frac{1}{2}$$

- Let

$$E_1 = \int_0^T s_0^2(t)\,dt \qquad E_2 = \int_0^T s_1^2(t)\,dt$$

- Define

$$\rho_{12} \triangleq \frac{1}{\sqrt{E_1 E_2}} \int_0^T s_0(t)\,s_1(t)\,dt$$

**Correlation Coefficient**

- But $$E_g = \int_0^T s_0^2(t)dt + \int_0^T s_1^2(t)dt - 2\int_0^T s_0(t)s_1(t)dt$$
$$\equiv E_1 + E_2 - 2\sqrt{E_1 E_2}\rho_{12}$$
- Hence,

$$\boxed{P_e = Q\left[\sqrt{\frac{E_g}{2N_0}}\right] = Q\left[\sqrt{\frac{E_1 + E_2 - 2\sqrt{E_1 E_2}\rho_{12}}{2N_0}}\right]}$$

- How shall we select the **correlation coefficient** **to minimize the error probability using the optimum receiver structure**?

# Popular Binary Modulation Schemes

one bit $\longrightarrow$ $\boxed{\text{Bin mod}}$ $\longrightarrow$ $s(t) \in \{s_1(t), s_0(t)\}$

- Antipodal Signaling
- Non-Return to Zero (NRZ)
- Amplitude Shift Keying (ASK)
- Phase Shift Keying (PSK)
- Frequency Shift Keying (FSK)
- Differential PSK (DPSK)

$s_1(t) = +A$

$s_0(t) = -A$

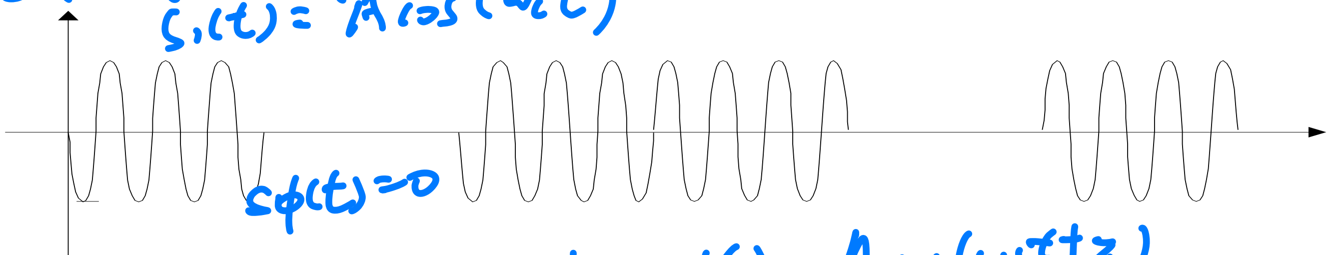
Tx Bits: 1 0 1 1 0 1

**Antipodal baseband signal**

Digital different of AM

$s_1(t) = A\cos(\omega_c t)$

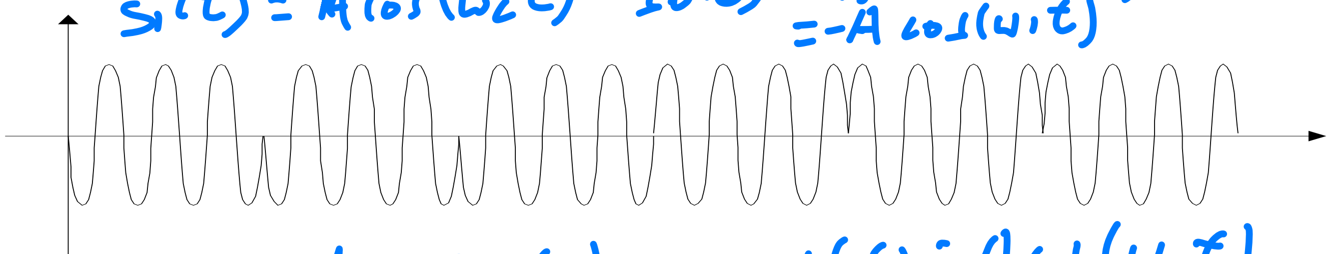
**BASK**



$s_0(t) = 0$

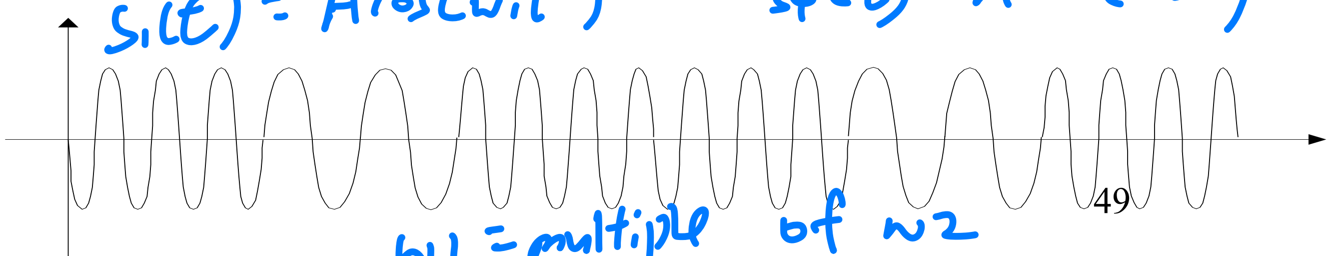
$s_1(t) = A\cos(\omega_c t)$ $\quad s_0(t) = A\cos(\omega_1 t + \pi)$
$= -A\cos(\omega_1 t)$

**BPSK**



$s_1(t) = A\cos(\omega_1 t)$ $\quad s_0(t) = A\cos(\omega_2 t)$

**BFSK**



$\omega_1$ = multiple of $\omega_2$

# Examples of Binary Digital Modulation Schemes

- ***Antipodal Signaling***

$$\boxed{\begin{array}{ll} s_1(t) = A & t \in [0, T] \\ s_0(t) = -A & t \in [0, T] \end{array}}$$

- ***Non-Return to Zero (NRZ)***

$$\boxed{\begin{array}{ll} s_0(t) = A & t \in [0, T] \\ s_1(t) = 0 & t \in [0, T] \end{array}}$$

$E_S = \int_0^T A^2 \cos^2(\omega_c t + \theta_c)\, dT$

$= \frac{A^2 T}{2}$

- **Amplitude Shift Keying (ASK)**

$$s_1(t) = A\cos(\omega_c t + \theta_c) \qquad t \in [0, T]$$
$$s_0(t) = 0 \qquad t \in [0, T]$$

Error probality

$E_b = \frac{A^2 T}{4}$

$$\boxed{P_e = Q\left[\sqrt{\frac{E_g}{2N_0}}\right]} = Q\left[\sqrt{\frac{A^2T/4}{N_0}}\right] = Q\left[\sqrt{\frac{E_b}{N_0}}\right]$$

Not good, express specific in this waveform!

$P_e = Q\left(\sqrt{\frac{A^2 T}{4 N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

- **Phase Shift Keying (PSK) or BPSK**

**VIP**

$$s_0(t) = A\cos(\omega_c t + \theta_c) \qquad t \in [0, T]$$
$$s_1(t) = -A\cos(\omega_c t + \theta_c) \qquad t \in [0, T]$$

$$E_1 = E_2 = \frac{A^2 T}{2} \qquad \Rightarrow \qquad E_b = \frac{A^2 T}{2}$$

$$\rho_{12} = -1$$

**Best Possible performance**

$$P_e = Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

- ***Frequency Shift Keying (FSK)***

$$s_0(t) = A\cos(\omega_1 t + \theta_c) \quad t \in [0, T]$$
$$s_1(t) = A\cos(\omega_2 t + \theta_c) \quad t \in [0, T]$$

Let $f_2 > f_1 \quad \Delta f \triangleq f_2 - f_1$ and $\Delta f = \dfrac{n}{T}$

assume that where *n* is an integer

Frequency separation

with $R = \dfrac{1}{T} \triangleq$ **Source data rate** (bits/sec)

$$\therefore \Delta f = nR$$

Multiple of data rate

- Now,

$$E_g = \int_0^T \left[A\cos(\omega_2 t + \theta_c) - A\cos(\omega_1 t + \theta_c)\right]^2 dt$$

$$= \int_0^T A^2 \cos^2(\omega_2 t + \theta_c)\,dt + \int_0^T A^2 \cos^2(\omega_1 t + \theta_c)\,dt$$

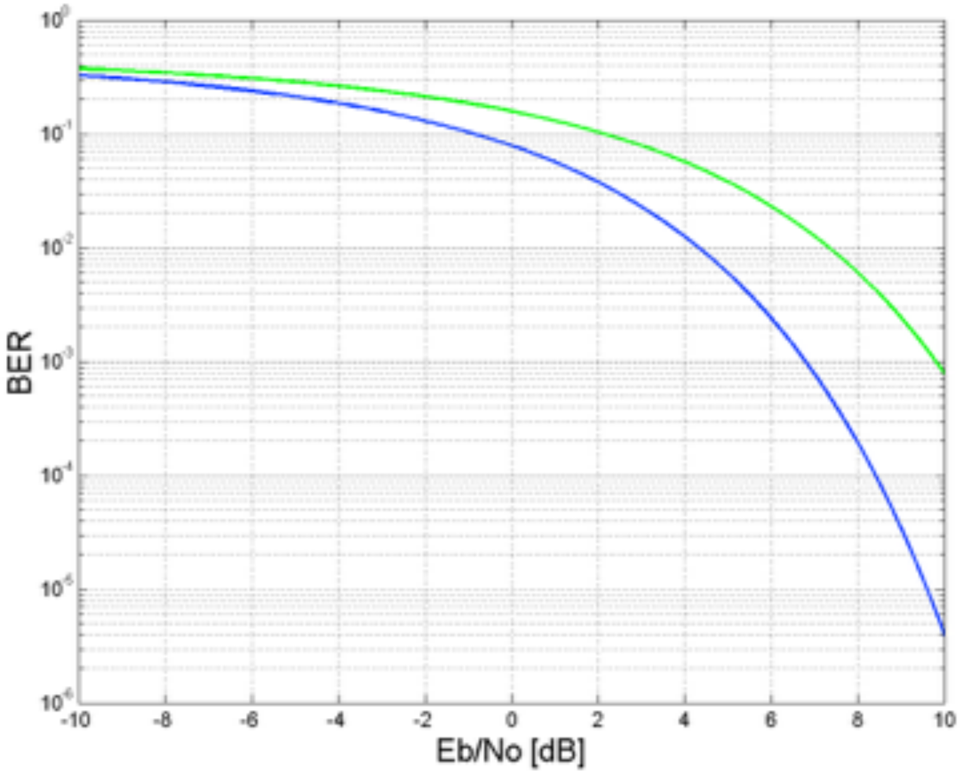
$$-2A^2 \int_0^T \cos(\omega_2 t + \theta_c)\cos(\omega_1 t + \theta_c)\,dt$$

$$= A^2 T - A^2 \int_0^T \cos\left[(\omega_2 - \omega_1)t\right] dt$$

0 Since $\omega_2 - \omega_1 = \dfrac{2\pi n}{T}$

$$P_e = Q\left[\sqrt{\frac{E_g}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2 T}{2N_0}}\right] = Q\left[\sqrt{\frac{E_b}{N_0}}\right]$$

– **Note**:Can easily show that $\rho_{12} \equiv 0$ (actually, it follows from the above). Hence, in this case, $s_0(t)$ and $s_1(t)$ are **ORTHOGONAL**



Did we really get optimal filter?

- why linear, can we use other than threshold metric?

- will come back later!

optimal receiver must be universally optimal!