Lecture 9

Properties of Fourier Series

(Deduction)

(Ref: Chapter 3 O&W)

I. Properties of Fourier Series I: Linearity, Time-Shifting, Time Reversal, Time Scaling

II. Properties of Fourier Series II: Multiplication, Conjugation, Parseval's Relation

III. Fourier Series Examples

I. Properties of Fourier Series I

- We will go through a set of important properties relating operations on signals in time and frequency domains.
- These properties are essential for us to describe, understand and solve many engineering problems.

Notation: A CT periodic signal and its FS representation

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

Properties of FS – 1. Linearity

1. Linearity: superposition in time ⇔ superposition in frequency

Assume $x(t)$ and $y(t)$ are both T-periodic.

Weighted sum of $x(t)$ and $y(t)$ is obviously also T-periodic, and will have FS coefficients that are the weighted sum of the corresponding FS coefficients of $x(t)$ and $y(t)$.

$$x(t) \overset{FS}{\leftrightarrow} a_k \qquad y(t) \overset{FS}{\leftrightarrow} b_k$$

$$\Downarrow$$

$$g(t) = Ax(t) + By(t) \overset{FS}{\leftrightarrow} c_k = Aa_k + Bb_k$$

Table 3.1
(3.58)

<u>Proof:</u> Obvious because all the analysis equations are inner products equations (integrals) and are linear (obey superposition)

$$c_k = \frac{1}{T}\int_T (Ax(t) + B\,y(t))\phi_k^*(t)dt = \frac{A}{T}\int_T x(t)\,\phi_k^*(t)dt + \frac{B}{T}\int_T y(t)\,\phi_k^*(t)dt = Aa_k + Bb_k$$

where $\phi_k^*(t) = e^{-jk2\pi/T}$

is the conjugate of the k-harmonic

All the synthesis equations are also linear.

<u>Interpretation</u>: All Fourier decompositions/synthesis are linear

If we add two signals in time domain, we add their spectrums in frequency domain.

Properties of FS – 2. Time Shifting

<u>2. Time shifting</u>: time shift $\Leftrightarrow$ phase shift in frequency

This property says that *time shifting of signals results in phase shifting of FS coefficients in frequency domain:*

For $x(t)$ is T-periodic, its time-shifted version $y(t) = x(t - t_0)$ is obviously also T-periodic.

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

$$y(t) = x(t - t_0) \overset{FS}{\leftrightarrow} b_k = e^{-jk\omega_0 t_0} a_k$$

$|\cdot| = 1;\ \sphericalangle = -k\omega_0 t_0 = k\omega_0 \times (-t_0)$

Table 3.1
3.5.2

$$|b_k| = |a_k|, \qquad \sphericalangle b_k = \sphericalangle a_k - k\omega_0 t_0$$

No magnitude change

A phase shift proportional to the frequency of the harmonic and the time shift

Proof If we view $x(t)$ as a sum of complex sinusoids: $x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$

then:

$$y(t) = x(t - t_0)$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 (t-t_0)}}_{\text{FS synthesis sum for } x(t-t_0)} = \sum_{k=-\infty}^{\infty} \underbrace{e^{jk\omega_0(-t_0)} a_k}_{b_k} \, \underbrace{e^{jk\omega_0 t}}_{k\text{-th harmonic } \phi_k(t)}$$

Meaning : If we shift a signal in time, we shift all the sinusoids contained in the signal. For a sinusoid, time shift corresponds to a phase shift which is proportional to the time shift times the frequency of the sinusoid

Phase shift = time shift × frequency

Properties of FS – 3. Time Reversal

<u>3. Time reversal</u>: Reversal in time = reversal in frequency

$$x(t) \overset{FS}{\longleftrightarrow} a_k \quad \Rightarrow \quad y(t) = x(-t) \overset{FS}{\longleftrightarrow} b_k = a_{-k}$$

Table 3.1
(3.63)

<u>Proof:</u>

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

Associate minus sign
with k instead

$\phi_{-k}(t)$

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} a_k\, e^{j(-k)\,\omega_0 t}$$

a_k is now the coefficient
for the $-k$-th harmonic: $a_k = b_{-k}$

Properties of FS – 4. Time Scaling

<u>4. Time scaling</u>: Does not change the FS coefficients but scale the underlying frequency

$$\boxed{x(t) \overset{FS}{\longleftrightarrow} a_k \quad \Rightarrow \quad y(t) = x(\alpha t) \overset{FS}{\longleftrightarrow} b_k = a_k}$$

Table 3.1
3.5.4

<u>Proof:</u> $$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(\alpha t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

So time-scaling a signal does not affect the FS coefficients at all, but the frequency that each FS coefficient represents is now scaled by α, and if:

- $\alpha>1$: compression in time $\Leftrightarrow$ expansion (scale up) in frequency
- $0<\alpha<1$: expansion in time $\Leftrightarrow$ compression (scale down) in frequency

Lecture 9

Chapter 3: Properties of Fourier Series

I. Properties of Fourier Series I: Linearity, Time-Shifting, Time Reversal, Time Scaling

II. Properties of Fourier Series II: Multiplication, Conjugation, Parseval's Relation

III. Fourier Series Properties Examples

Properties of FS II – 5. Multiplication

5. Multiplication:

Multiplying two T-periodic signals in time $\Leftrightarrow$ convolving their FS coefficients in frequency.

For $x(t)$ and $y(t)$ both T-periodic

and $x(t) \overset{FS}{\longleftrightarrow} a_k\,,\;\; y(t) \overset{FS}{\longleftrightarrow} b_k$

$$\boxed{g(t) = x(t)y(t) \overset{FS}{\leftrightarrow} c_k = \underbrace{\sum_{n=-\infty}^{\infty} a_n b_{k-n}}_{\text{This is a convolution sum!}}}$$

Table 3.1
(3.64)

Recall: $y[n] = x[n] * h[n]$ means

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Proof:

The multiplication property results from the fact that ***when we multiply two complex sinusoids, we simply add their frequencies***:

$$e^{j\omega_1 t}e^{j\omega_2 t}=e^{j(\omega_1+\omega_2)t} \qquad e^{jm\omega_0 t}e^{jn\omega_0 t}=e^{j(m+n)\omega_0 t}$$

Now $x(t)$ and $y(t)$ is each a sum of harmonics: $x(t)=\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$; $y(t)=\sum_{k=-\infty}^{\infty} b_k\, e^{jk\omega_0 t}$,

multiplying $x(t)$ and $y(t)$ results in a sum of all pairwise products. The k-harmonic in the product is the sum of all products of m- and n- harmonics such that $m+n=k$.

use different variables of summation to avoid confusion

Product of two sums is a double sum of individual products

$$g(t)=x(t)y(t)=\left(\sum_{n=-\infty}^{\infty} a_n\, e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} b_m\, e^{jm\omega_0 t}\right)=\underbrace{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}} a_n b_m e^{jn\omega_0 t}\, e^{jm\omega_0 t}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_n b_m e^{j(m+n)\omega_0 t}=\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_n b_{k-n} e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Where $c_k=\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Multiplying the m- and n- harmonic results in the $m+n$ harmonic

let $k=m+n$
$\rightarrow m=k-n$

The k-harmonic in the product comes from all the different ways of adding n and m to get k

Example $x(t)$ and $y(t)$ both real and T-periodic

$x(t)$ has non-zero FS coefficients $a_1 = 2$, $a_{-1} = 2$, $a_2 = -j$, $a_{-2} = j$

$y(t)$ has non-zero FS coefficients $b_0 = 0.5$, $b_1 = 1$; $b_{-1} = 1$

1. What are the FS coefficients of $g(t) = x(t)y(t)$?

$g(t)$ is real, T-periodic, with $c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$

$$c_0 = b_{-1} \times a_1 + b_0 \times a_0 + b_1 \times a_{-1} = 4$$

$$c_1 = b_{-1} \times a_2 + b_0 \times a_1 + b_1 \times a_0 = 1 - j \qquad c_{-1} = c_1^* = 1 + j$$

$$c_2 = b_{-1} \times a_3 + b_0 \times a_2 + b_1 \times a_1 = 2 - 0.5j \qquad c_{-2} = c_2^* = 2 + 0.5j$$

$$c_3 = b_{-1} \times a_4 + b_0 \times a_3 + b_1 \times a_2 = -j \qquad c_{-3} = j$$

$c_4 = 0$; all other $c_k's$=0

2. From the previous page, the non-zero FS coefficients for $g(t)$ are computed to be:

$$c_0 = 4; \quad c_1 = 1 - j; \quad c_2 = 2 - 0.5j\ ; \quad c_3 = -j$$

$$c_{-1} = 1 + j; \quad c_{-2} = 2 + 0.5j\ ; \quad c_{-3} = j$$

Express $g(t) = x(t)y(t)$ as a sum of real sinusoids.

You should be able to simply write down the answer!

$$g(t) = 4 + 2\sqrt{2}\cos\left(\omega_0 t - \frac{\pi}{4}\right) + 2\sqrt{4.25}\ \cos\left(2\omega_0 t + \arctan\left(\frac{-0.5}{2}\right)\right) + 2\cos\left(3\omega_0 t - \frac{\pi}{2}\right)$$

$\measuredangle c_3 = -\frac{\pi}{2}$

$|c_3| = 1$

$c_0 = 4$

2 times real part

$|c_1| = |1 - j| = \sqrt{2}$

first harmonic

$\measuredangle c_1 = -\frac{\pi}{4}$

$|c_2| = |2 - 0.5j|$
$= \sqrt{4.25}$

$\measuredangle c_2$

$|||$

$2\sin(3\omega_0)$

Properties of FS – 6. Conjugation

6. Conjugation and conjugate symmetry:

i. Conjugating a signal ⇔ conjugating its spectrum and reversing its frequency
ii. Spectrum of real signal is conjugate symmetric

$$x(t) \overset{FS}{\longleftrightarrow} a_k \quad \Rightarrow \quad x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}$$

Table 3.1
(3.65)

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}; \qquad x^*(t) = \sum_{k=-\infty}^{\infty} a^*_k \underbrace{e^{-jk\omega_0 t}}_{\phi_{-k}(t)} = \sum_{k=-\infty}^{\infty} a^*_{-k} e^{jk\omega_0 t}$$

Conjugating $x(t)$ conjugates the FS coefficients as well as negating the frequencies

Hence, if $x(t)$ is real, then $x(t) = x^*(t)$, implying that $a^*_{-k} = a_k$

Properties of FS – 7. Differentiation

7. Differentiation: Differentiation ⇔ multiply by frequency with 90^0 phase shift

$$\boxed{x(t) \overset{FS}{\longleftrightarrow} a_k \quad \Rightarrow \quad y(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\omega_0 a_k}$$

Table 3.1

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}; \qquad y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{jk\omega_0 a_k}_{\text{Coefficient of the } k\text{-harmonic}} \underbrace{e^{jk\omega_0 t}}_{\text{the } k\text{-harmonic}}$$

Differentiation amplifies high frequency.

Properties of FS – 8. Parseval's Relation

8. Parseval's relation: The average power in a periodic signal equals the sum of the powers in all of its harmonics.

$$\boxed{\frac{1}{T}\int_0^T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Table 3.1 (3.67)

Proof: It is a result of the fact that there is no cross-power in two different harmonics; i.e., orthogonality of harmonics.

Energy is self inner product of $x(t)$

$x(t)$ as sum of harmonics

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)^* dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k a_n^* \int_T e^{jk\omega_0 t}\left(e^{jn\omega_0 t}\right)^* dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k a_n^* \int_T e^{j(k-n)\omega_0 t}dt = \frac{1}{T}\sum_{k=-\infty}^{\infty} a_k a_k^* T = \sum_{k=-\infty}^{\infty}|a_k|^2$$

self inner product of $x(t)$ becomes double sum of inner products of individual harmonics

But inner product of two harmonics are zero unless their frequency is the same (Orthogonality of harmonics)

Summary of properties of FS

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

** Will be provided in exams

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III. Fourier Series Examples

Example 3.6

What are d_k, FS coefficients for $g(t)$ as shown?

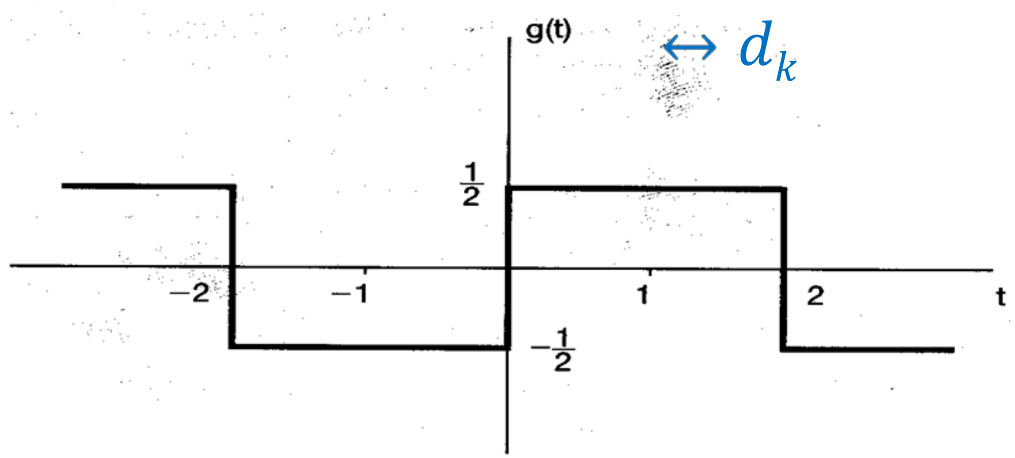
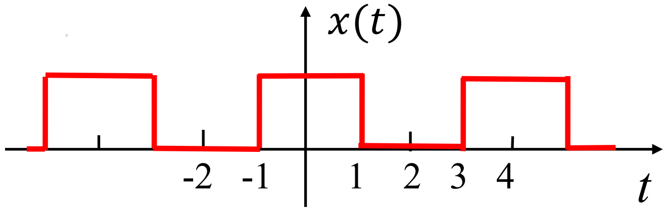


Figure 3.10 Periodic signal for Example 3.6.

We recognize that:

$$g(t) = x(t-1) - 1/2$$



where $x(t)$ is the periodic square wave in Example 3.5 with $T = 4,\ T_1 = 1.$

Recall from Example 3.5 that for $x(t)$: (Or from Table 4.2; more later)

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{1}{k\pi}\sin\left(k\frac{2\pi}{T}T_1\right) = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right); \qquad a_0 = \frac{2T_1}{T} = \frac{1}{2}.$$

So, using Time Shifting and DC Offset, we have:

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2} \quad \text{Time shift} = -1$$

$$d_0 = 0; \quad \text{since } g(t) \text{ has zero DC}$$

$$d_k = \underbrace{a_k e^{jk\frac{\pi}{2}(-1)}}_{\text{Phase shift term}} = a_k(-j)^k$$

$$e^{-j\frac{\pi}{2}} = -j$$

Example 3.7 Integration

What are e_k, FS coefficients for $x_1(t)$ as shown?

$x_1(t) \leftrightarrow e_k$

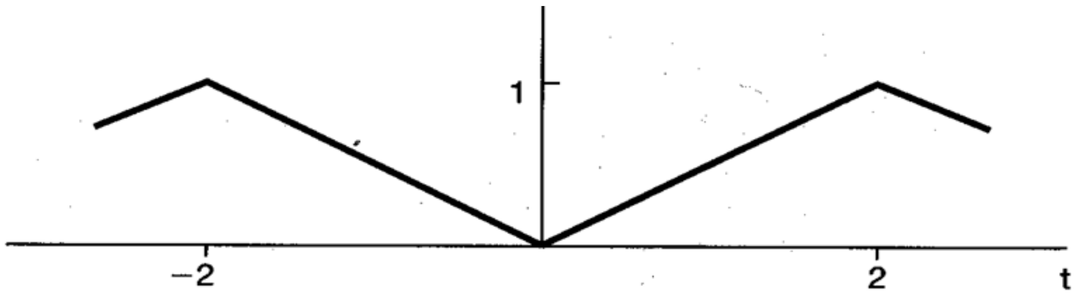


Figure 3.11 Triangular wave signal in Example 3.7.

$g(t)$ in the last slide is the derivative of the triangular wave shown here. Hence,

$$d_k = jk\omega_0 e_k \Rightarrow e_k = \frac{d_k}{jk\omega_0} \text{ for } k \neq 0$$

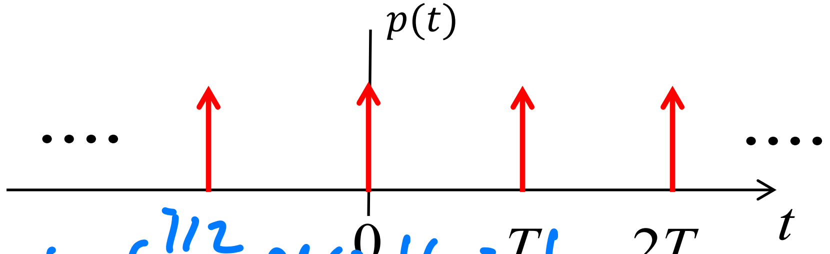
For $k = 0$, we determine the DC term from $x_1(t)$ above:

$$e_0 = \qquad ; \quad e_k =$$

Example 3.8a

The ***periodic impulse train*** $p(t)$ is defined as the ***Poisson sum*** of the CT impulse:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



What are its FS coefficients a_k?

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}$$

Applying the analysis equation, we obtain:

$$\boxed{a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T} \quad \forall k}$$

$\delta(t)g(t) = \delta(t)g(0)$

The periodic impulse train contains *all* harmonics frequencies at equal weight!