Teaching Schedule

geometric angle!

» **Consider M-ary Digital Transmission**
- Introduction
- Signal Space Concepts
- Basis Vectors/functions
- Determination of an orthogonal basis set (Gram-Schmidt Orthogonalization)

- need to say that the linear receiver is the optimal among ALL receivers!
(universal optimal !!!) from geometric domain aspect!

lego example

time domain: describe the features one by one

freq domain: sinusoid lego. describe one by one
uncountable
(:) universal for all signals
:(not efficient, need all time)

-[illegible] take a chunk:

Motivation

- **Did we really obtain an optimal binary demodulator?**
  - We obtained an optimal threshold VT and optimal filter (Matched Filter).
  - Does that really mean "optimal"?
  - What we have done is only optimal w.r.t. the considered structure of the receiver.
  - However, why the optimal binary demodulator has to have such structure?
- **What about M-ary modulation?**
  - We could in fact increase the bit rate by transmitting more information bits per modulation symbol.
  - How to design the modulator and demodulator?

several approach → different performance

- **How to compare different modulation schemes?**
  - Messy equations, difficult to obtain useful insights.

Signal Space Concepts and Signal Representation

It turns out that the key to analyzing and understanding the performance of digital transmission is the realization that signals used in communications can be expressed and visualized graphically.

Thus

We need to understand signal space concepts as applied to digital communications

$s(t) \rightsquigarrow \vec{s}$ $\quad S$ (signal space)

(Geometric Domain)

$S = \text{span}\{i, j\}$

~orthogonal basis

$\langle i, j \rangle = 0 \quad -(1)$

$\|i\|^2 = \|j\|^2 = 1 \quad -(2)$

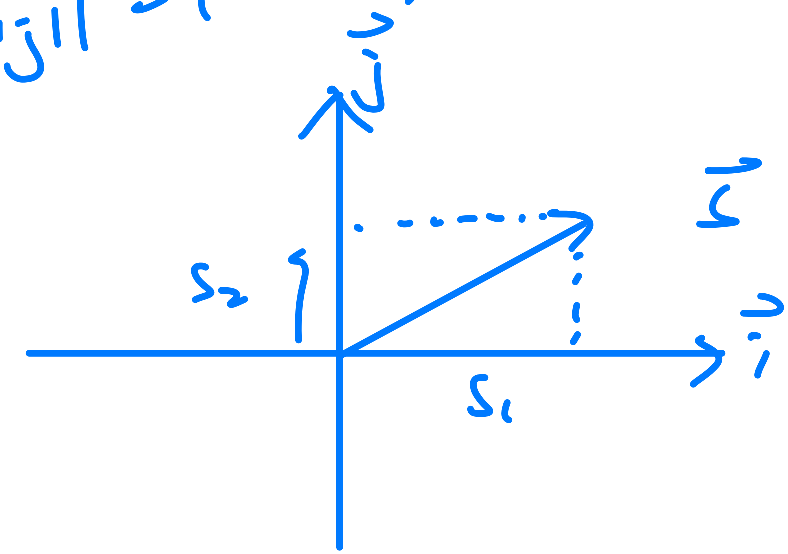
step 1

$S = \text{span}\{\phi_1(t), \phi_2(t)\}$

step 2 :

$s_1 = ?$

$s_2 = ?$



$\vec{S} = \vec{s_1} + \vec{s_2}$

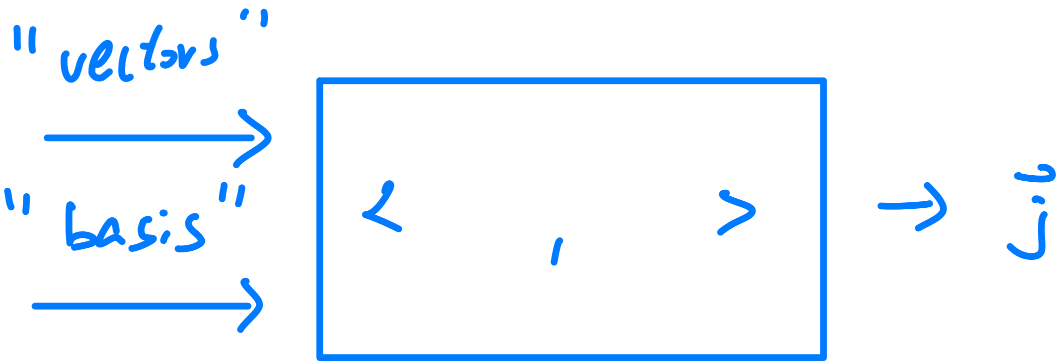
$\langle \vec{s}, \vec{i} \rangle$

$= \langle s_1\vec{i} + s_2\vec{j}, \vec{i} \rangle$

$= \langle s_1\vec{i}, \vec{i} \rangle + \langle s_2\vec{j}, \vec{i} \rangle$

$= s_1 \cdot 1 + s_2 \cdot 0$

$= s_1$



step 1:

$$\{s_1(t), s_2(t) \cdots s_N(t)\} \in S \triangleq \text{Span}\{\phi_1(t), \phi_2(t)\}$$

$\phi_1(t), \phi_2(t)$ are the orthogonal basis

$$\langle \phi_i(t), \phi_j(t) \rangle = 0 \quad \forall i \neq j$$

$$\|\phi(t)\| = 1$$

Step 2:

"$s_1(t)$" $\longrightarrow$ $\vec{s}_1 = (s_{11}, s_{12})$

coordinate

$$s_{11} = \langle s_1(t), \phi_1(t) \rangle$$

$$s_{12} = \langle s_1(t), \phi_2(t) \rangle$$

$$S = \text{span}\{\phi_1(t), \phi_2(t)\}$$

orthonormal basis

Time Domain $\xrightarrow{\text{1 to 1 correspondance}}$ Geometric Domain

$\longleftarrow$

$\vec{s} = (s_1, s_2)$

flexible, representation, can represent all signal inside the space, not universal

Overall Objectives/Goals

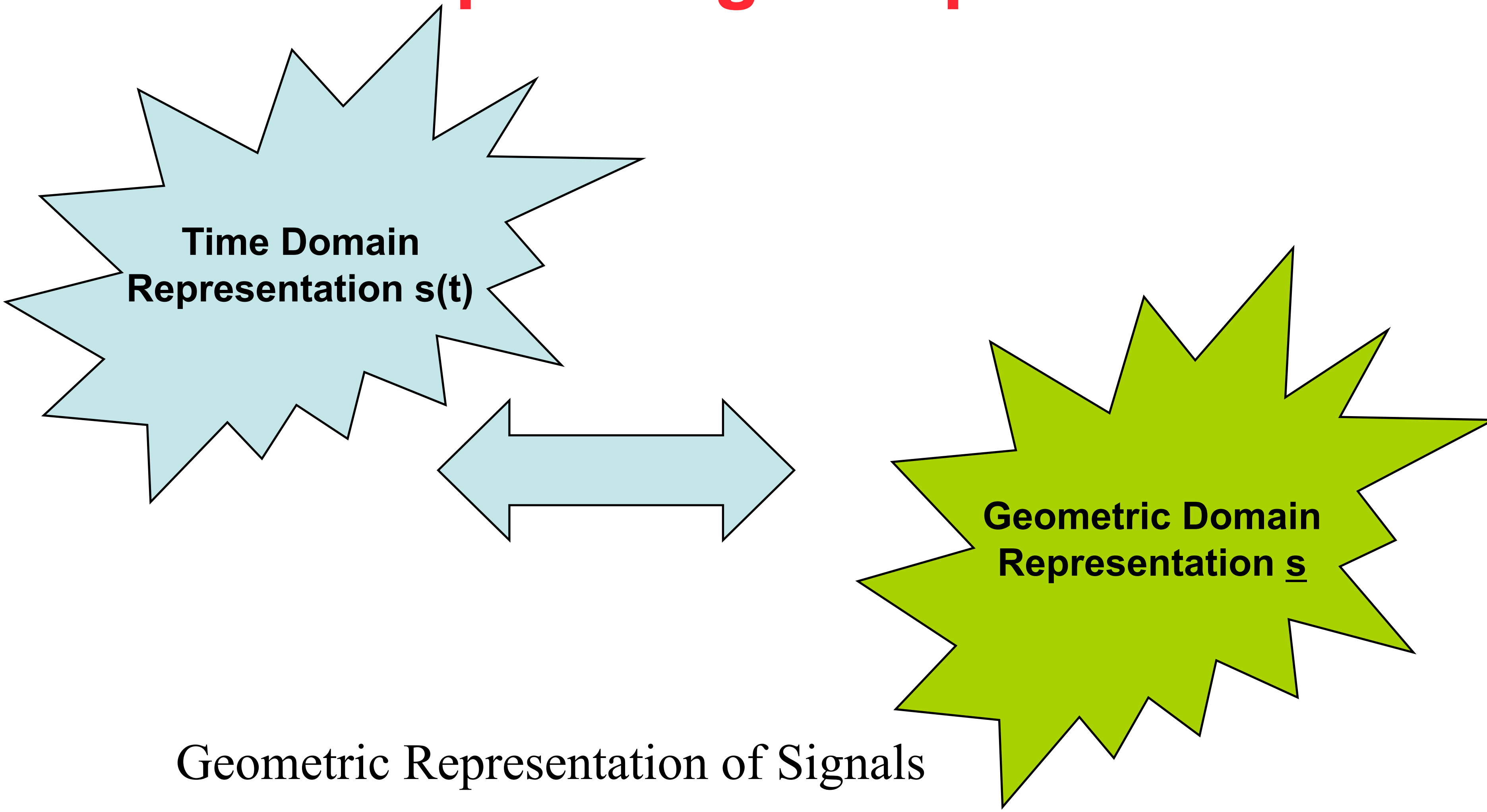
- To analyze the problem of digital signal detection ***from a fundamental point of view***.

- To understand the digital modulation and demodulation from a geometric perspective
  - Easy to understand
  - Useful design insights can be obtained without too much math
  - Concept of Signal Space

Signal Space Concepts

- **Signal space concepts will allow a more general way of looking at modulation schemes.**
- **By choosing an appropriate set of axis for our signal constellation, one can:**
  - **Design modulation types which have desirable properties**
  - **Construct optimal receivers for a given modulation technique**
  - **Analyze the performance of modulation schemes using very general techniques.**

Concept of Signal Space



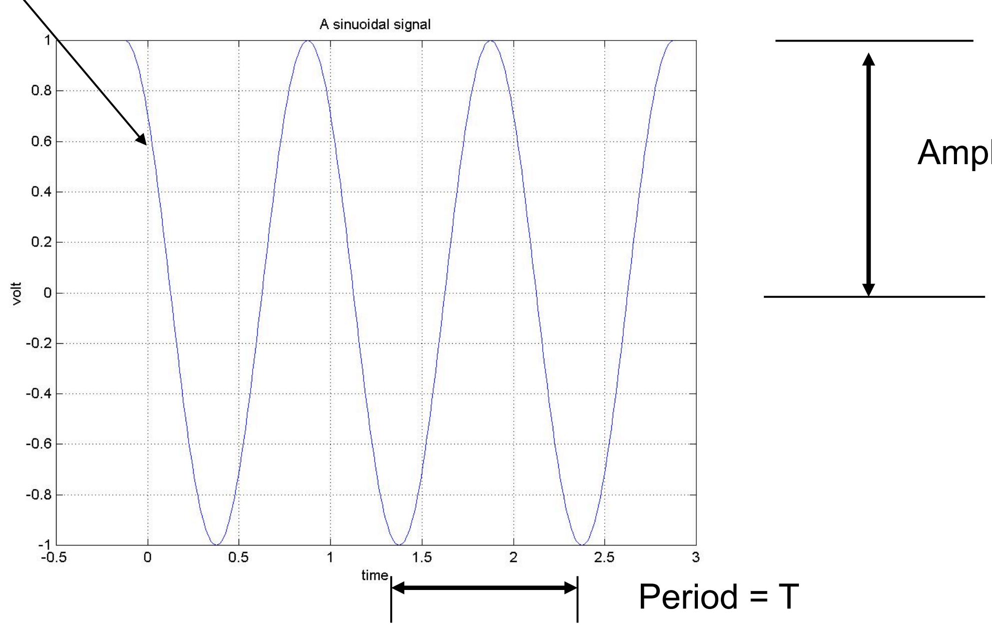
Geometric Representation of Signals

Representation of Signals

- (1) Time Domain:
  Signal is represented by a function in time, s(t).
  Waveform (the shape of the function) could be observed.
  Periodic Signal

Starting
Phase = 30

S(t + T) = s(t) for all t. T = period
Frequency (cycle per second) = 1/T (Hz).



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Representation of Signals

- Frequency Domain:

  Signal could be represented by a function of frequency S(f) as well.

  For some aperiodic signals,

  could be decomposed into components of "sin" and "cos".

  Each component has different (amplitude, frequency, phase).

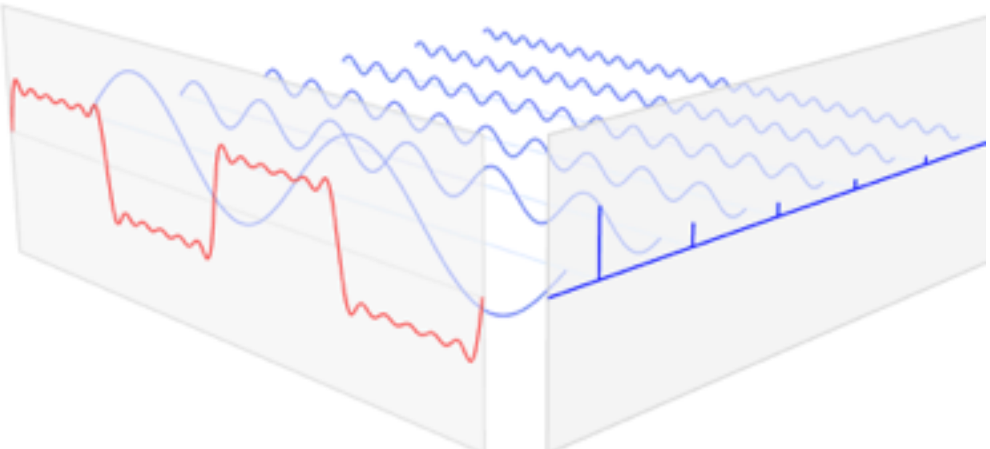
Fourier Transform - Transform Equation:

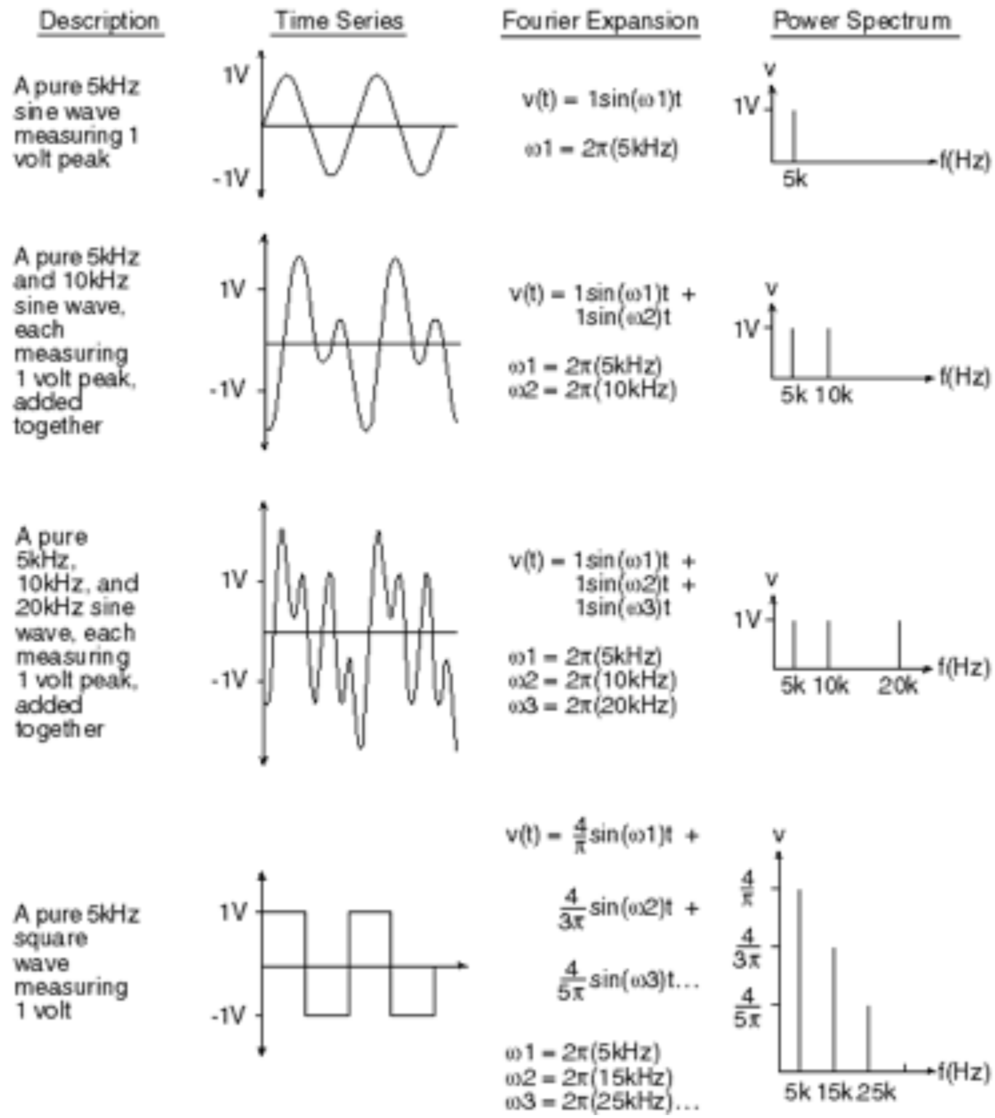
$$F(f) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi ft) dt$$

Fourier Transform - Analysis Equation:

$$f(t) = \int_{-\infty}^{\infty} F(f) \exp(j2\pi ft) df$$

Frequency Decomposition





Representation of Signals

- **Geometric Domain (Signal space)**
  - **Signal s(t) is represented as a "vector" s (with coordinates)**
  - **For a vector to be meaningful, we need to define the space first**
    - » **What is the "frame-of-reference"?**
    - » **The "frame-of-reference" is defined by "x-axis", "y-axis",.....**

Define the "**frame-of-reference**"

- Geometric Domain (Signal Space)
  - Signal could be represented by a point in a space.
  - Step 1: Given a set of M signals, $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ define a D-dim signal space with basis $\{\phi_1(t), \phi_2(t), \ldots, \phi_D(t)\}$.
  - Step 2: Find out the coordinates of each signals by: $s_i(t) \rightarrow \vec{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,D})$

$$s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

Geometric Representation of Signals

- **Time Domain (x(t)), Frequency Domain (X(f)), Geometric Domain (x) are just different views looking at the same coin.**
  - **The physical characterization of the coin will be the same no matter you are computing from which domains**
  - $$E = \int_0^T |x(t)|^2 dt \text{ (Time Domain Energy)}$$
    $$E = \int_{-\infty}^{\infty} |H(f)|^2 df \text{ (Frequency Domain Energy)}$$
    $$E = <\vec{x}, \vec{x}> = \|\vec{x}\|^2 \text{ (Geometric Domain Energy)}$$

$\downarrow$

length square of vector

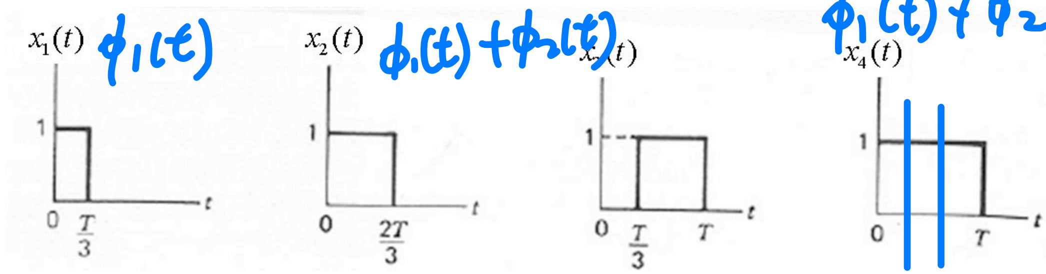
$<\vec{s}_1, \vec{s}_1> = \|s_1\|^2 = \int_0^{T_s} |s_1(t)|^2 dt = E_s$

Example 1

not universal, but efficient

- **Consider 4 signals**

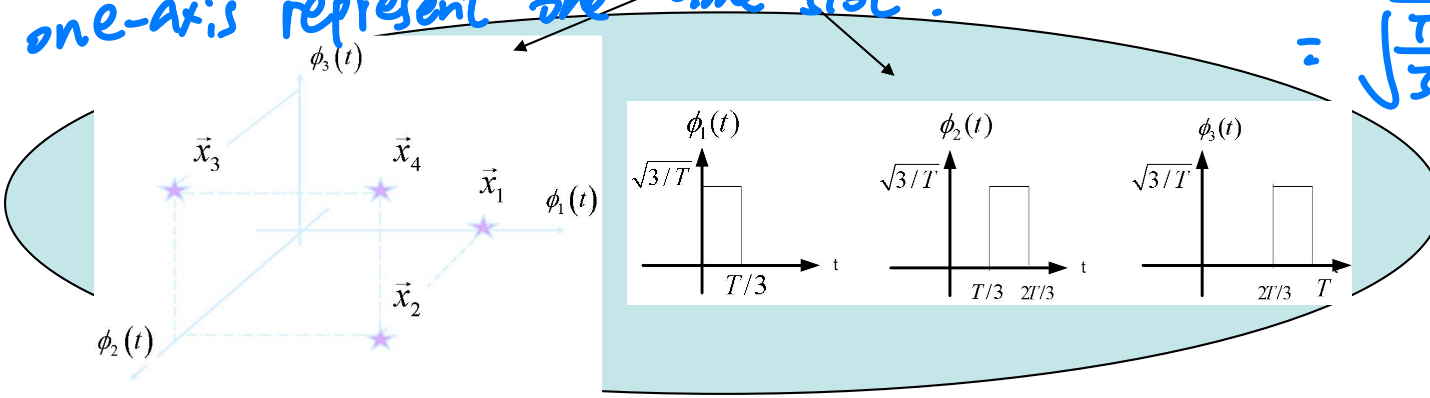
$\phi_1(t)$ $\phi_1(t)+\phi_2(t)$ $\phi_1(t)+\phi_2(t)+\phi_3(t)$



Find the orthonormal basis functions (orthonormal axis) of the Signal Space that contains the 4 signals.

$\sqrt{\frac{3}{T}} \times \sqrt{\frac{T^2}{9}} = \sqrt{\frac{T}{3}}$

one-axis represent one time slot!



Example 1

$\vec{x_1} = (\sqrt{\frac{T}{3}}, 0, 0)$

$\vec{x_2} = (\sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}}, 0)$

Example 1 (Geometric Representation of 4 time domain signals) *Let $x_1(t), x_2(t), x_3(t), x_4(t)$ be 4 time domain signals with duration $T_s = T$ as illustrated in Figure 2.3*

$\vec{x_3} = (0, \sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}})$

$\vec{x_4} = (\sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}})$

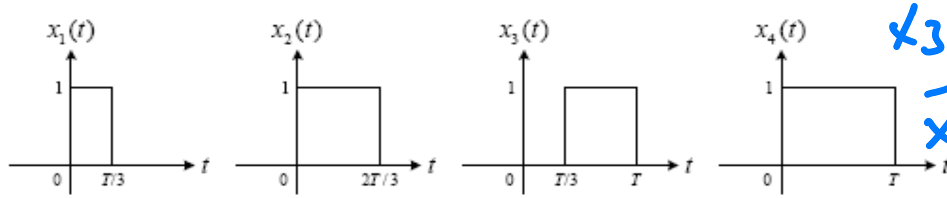


Figure 2.3 Illustration of the time domain waveforms of the 4 signals in example 1.

$E_4 = 3T$, !

必須 normalization

↓

不然必死!!!

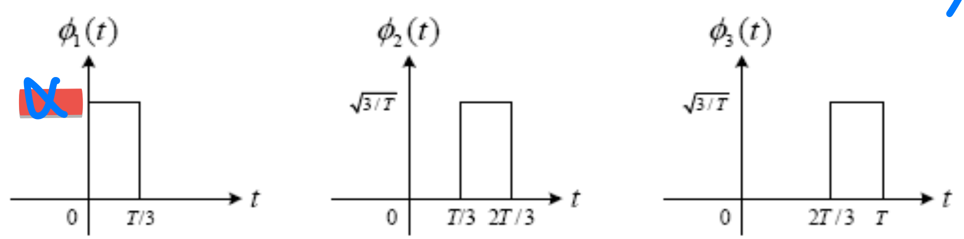


Figure 2.4 Basis functions of the signal space in example 1.

Energy:

$\alpha^2 \cdot T/3 = 1$

$\alpha = \sqrt{\frac{3}{T}}$

Is this space unique?

no!!!,

- can define other spaces!

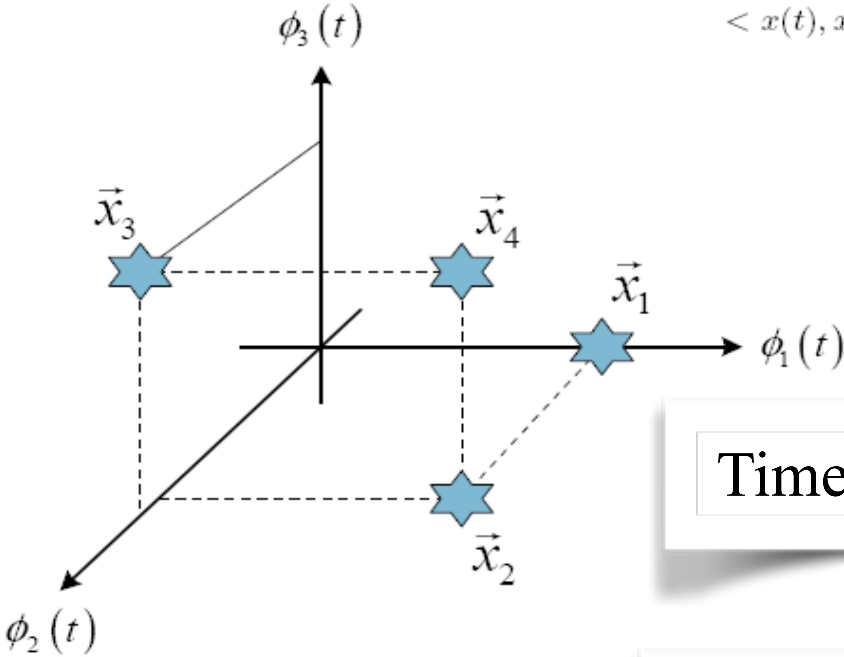
Example 1

$$\boxed{x_1(t) = \sqrt{T/3}\phi_1(t) + 0\phi_2(t) + 0\phi_3(t)}$$

Following the step 2, the coordinates of the 4 time domain signals are given by $\vec{x}_1 = (\sqrt{T/3}, 0, 0)$, $\vec{x}_2 = (\sqrt{T/3}, \sqrt{T/3}, 0)$, $\vec{x}_3 = (0, \sqrt{T/3}, \sqrt{T/3})$ *and* $\vec{x}_4 = (\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3})$ *as illustrated in Figure 2.5*

$$< x(t), x(t) >= \int_0^{T_s} |x(t)|^2 dt = E = \|\vec{x}\|^2.$$

$$P = \frac{1}{T_s}\|\vec{x}\|^2.$$



Time Domain $\boxed{\{x_1(t), x_2(t), x_3(t), x_4(t)\}}$

Basis Function
$\{\phi_1(t), \phi_2(t), \phi_3(t)\}$

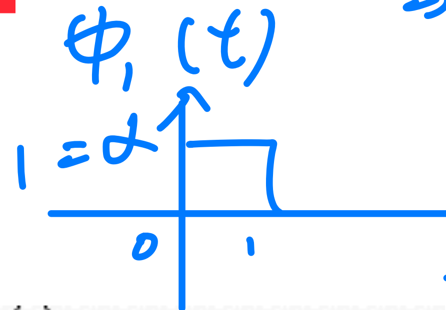
Geometric Domain $\boxed{\{\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}\}}$

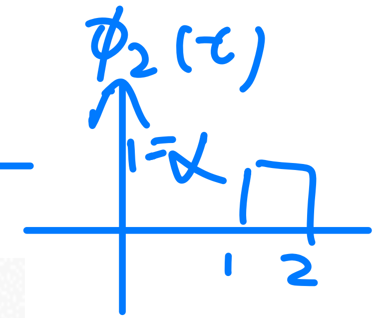
Example 2

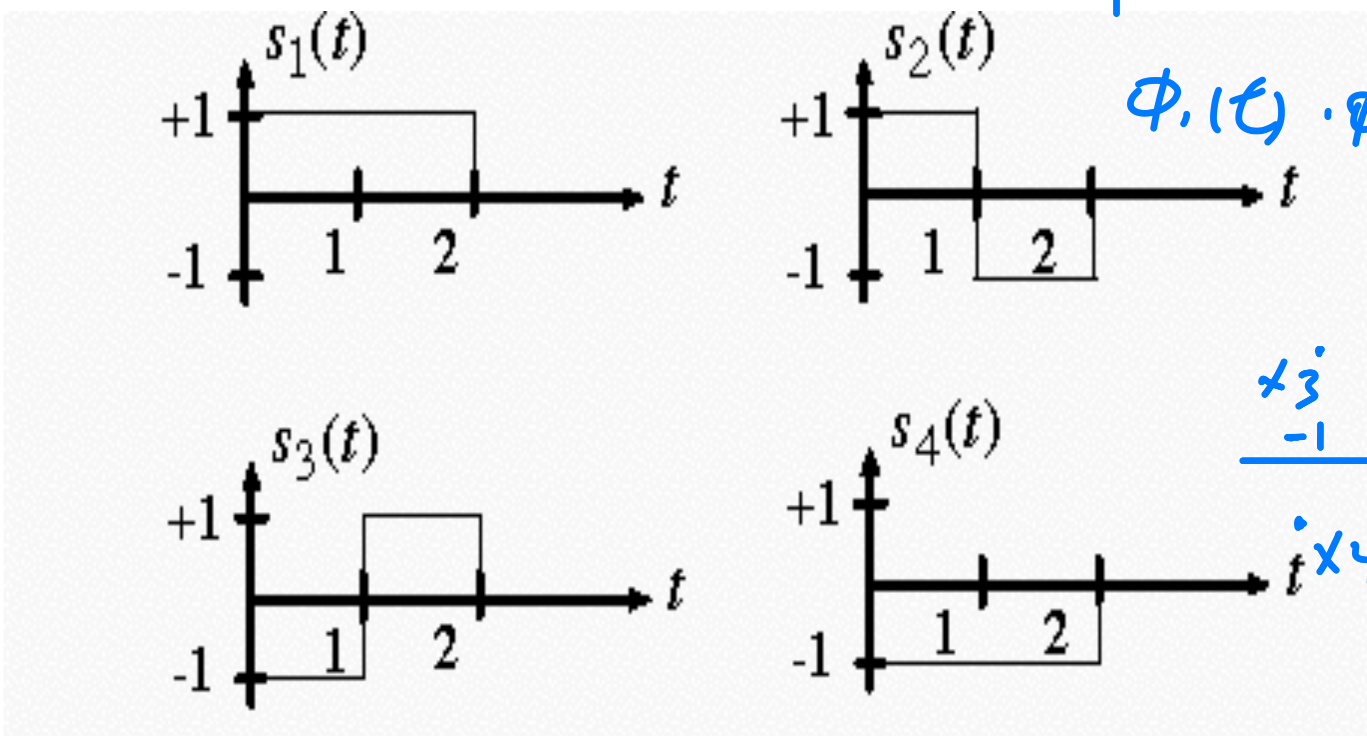
$\int_0^2 |\phi_i(t)|^2 dt = 1$

$\Rightarrow \alpha = 1$

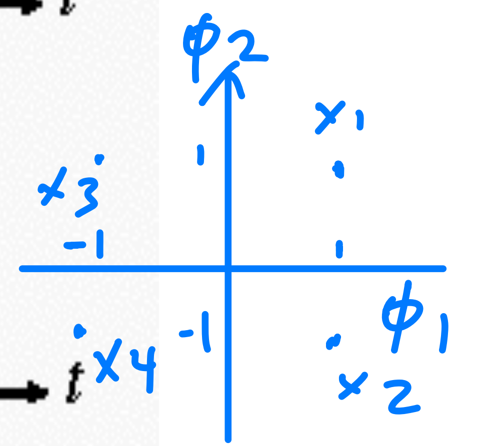
Consider the following signal set:







$\phi_1(t) \cdot \phi_2(t) = 0!$



Basis Functions

- **By inspection, the signals can be expressed in terms of the following functions:**

$$f_1(t) = rect(t - 0.5)$$

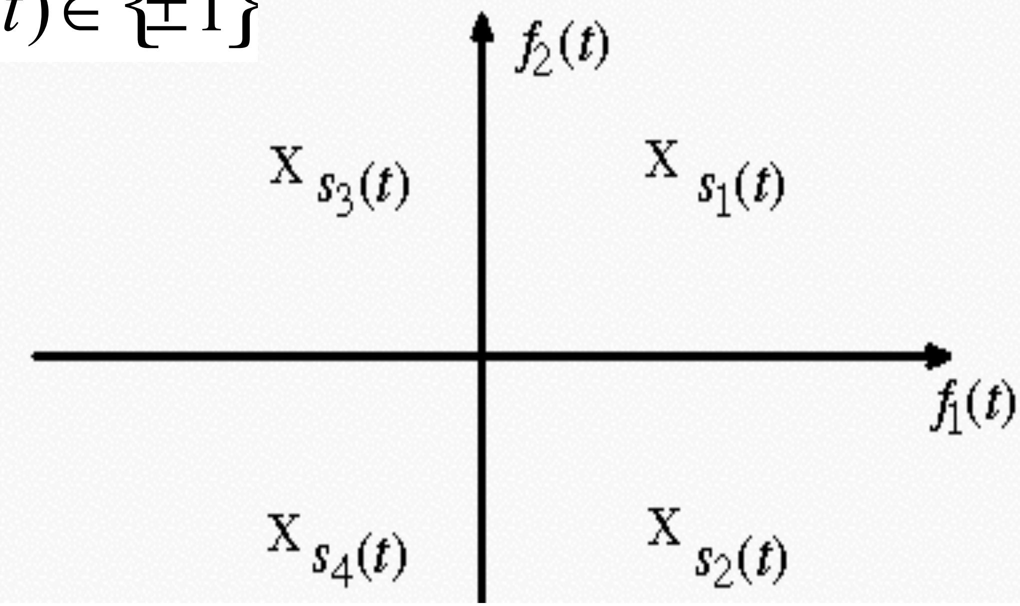
$$f_2(t) = rect(t - 3/2)$$

- **These are known as basis functions.**

Constellation Diagram

一样的!

$$s_1(t),\ldots,s_4(t)\in\{\pm 1\}$$



$$X = \{\vec{x}_1 \cdots \vec{x}_M\} \to \boxed{GSO} \to \{\phi_1, \cdots \phi_D\}$$

$$X \subset \operatorname{span}\{\phi_1 \cdots \phi_D\}$$

$$X = \{x_1(t) \cdots x_M(t)\} \to \boxed{GSO} \to \{\phi_1 \cdots \phi_D\}$$

very simple!

Signal Space and Basis Functions

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In the previous examples, we were able to guess the correct basis functions.
- In general, is there any method which allows us to find a complete orthonormal basis for an arbitrary signal set?
  - Gram-Schmidt Orthogonalization (GSO) Procedure

Vector Space

- **A vector space V over a field F is a set of "abstract objects" called "vectors".**
  - **The elements of V are called "Vectors".**
  - **The elements of F are called "Scalars".**
  - **Two basic "binary operations" (1) Vector additions; (2) Scalar Multiplications that satisfy the following AXIOMS**
    - » **Associativity of Addition: u + (v + w) = (u + v) + w** order x matter
    - » **Commutativity of Addition: u + v = v + u** $\vec{0} \in V$
    - » **Identity Elements of Addition: There exists 0 \in V s.t. 0 + u = u for all u \in V.**
    - » **Inverse Elements of Addition: For every v\in V, there exists -v \in V s.t. v + (-v) = 0**
    - » **Distributivity of Scalar Multiplication (w.r.t. Vector Addition):** a(**u+v**) **=** au **+** a**v**
    - » **Distributivity of Scalar Multiplication (w.r.t. Field Addition): (**a + b)**u =** au **+** b**u.**
    - » **Compatibility of scalar multiplication:** a(b**v) =** (ab)**v**
    - » **Identity element of scalar multiplication: there exists** 1 \in F s.t. 1**v = v for all v \in V.**

Vector Space Examples

- **Coordinate Space over Real elements:-**
  - **V = {**$(a_1, a_2, ..., a_n)$: $a_i \in R$} a vector space can be composed of n-tuples of real numbers. (Field = R)
- **Coordinate Space over Complex elements:-**
  - **V = {**$(a_1, a_2, ..., a_n)$: $a_i \in C$} a vector space can be composed of n-tuples of complex numbers. (Field = C)
- **Function Space (Signal Space):-**
  - V = Functions from any fixed domain to F also forms a vector space.
  - e.g. Functions of time --> R (signal space) is a vector space.

what if vectors are signals?

Inner Product Space

- A vector space (V,F) does not have notion of geometry (or topology) need to satisfy some rules!!!
  - Notion of distance? (Two vectors are close or far away from each other)
  - Notion of topology? (open set, closed set, limits)
  - Notion of geometry? (Circle??)
  - All these requires “norm” 사이즈? norm!!! $\|\vec{x} - x_a\| = r$
  - Notion of angle? (angle between two vectors)
  - All these requires “inner product”
- A vector space (V,F) with an “inner product” is called “inner product space”
  - Inner Product is a mapping <u,v>: V x V --> F that satisfy the following axioms
    - » <u,v> = <v,u>*
    - » <u+v, w> = <u,w> + <v,w>
    - » <au,v>=a<u,v>
    - » <u,u> >= 0 and <u,u> = 0 iff u = 0.

distance → angle → geometric!!!

↓ limit

$\lim_{n\to\infty} a_n = a$ $\lim_{n\to\infty} \vec{x}_n = \vec{x}$

$$\lim_{n \to \infty} a_n = a$$

$\forall \varepsilon > 0, \exists N$ s.t. $|a_n - a| < \varepsilon \quad \forall n > N$

For any $\varepsilon > 0$, $\exists N$ s.t. $\|\vec{x}_n - \vec{x}\| < \varepsilon \quad \forall n > N$

$$\vec{x} \longrightarrow \boxed{\|\ \ \|} \longrightarrow \mathbb{R}$$

Geometric Concepts in Inner Product Space

- **Length of a vector:**
  - $||v||^2 = <v,v>$
- **Distance between two vectors:**
  - $||v-w||^2=<(v-w),(v-w)>$
- **Angle between two vectors:**

$$\cos\theta = \frac{<\mathbf{v},\mathbf{w}>}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

- **Orthogonal vectors: $<v,w> = 0$**
- **Circle (x_c, r): $||x - x_c|| = r$**
- **Limit of a sequence:**

$\lim_{n\to 0} \mathbf{v}_n = \mathbf{v}$

For any $\epsilon > 0$, there exists n_0 such that for all $n > n_0$, $\|\mathbf{v}_n - \mathbf{v}\| < \epsilon$

Vectors and Space Concepts

max. number of linearly independent vectors!

- **An** n-dimensional space S **is defined by a set of n basis vectors (e_1, $e_{2,}$ … e_n);**
  - **S = span (e_1, $e_{2,}$ … e_n);**

$\Rightarrow$ **Any vector $\underline{a}$ can be written as**

$$\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$$

n = dimension = maximum number of *linearly independent vectors* in the vector space

- Notation:

Coordinate Representation of vector a.

$$\underline{\mathbf{a}} = \sum_{i=1}^{n} a_i \underline{e}_i \Leftrightarrow \underline{\mathbf{a}} = (a_1, a_2, \ldots, a_n)$$

- Definitions:

1) **Inner Product** : $\langle \underline{a}, \underline{b} \rangle = \underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i$

2) $\underline{a}$ and $\underline{b}$ are **Orthogonal** ($\perp$) if

$$\underline{a} \cdot \underline{b} = 0$$

3) $\|\underline{a}\| = \sqrt{\langle \underline{a}, \underline{a} \rangle} = \sqrt{\sum_{i=1}^{n} a_i^2}$

$=$ **Norm of $\underline{\mathbf{a}}$**

4) A set of vectors are orthonormal **if they are mutually $\perp$ and all have unity norm.**

$$\text{So if } (\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n) \sim \text{Orthonormal basis}$$

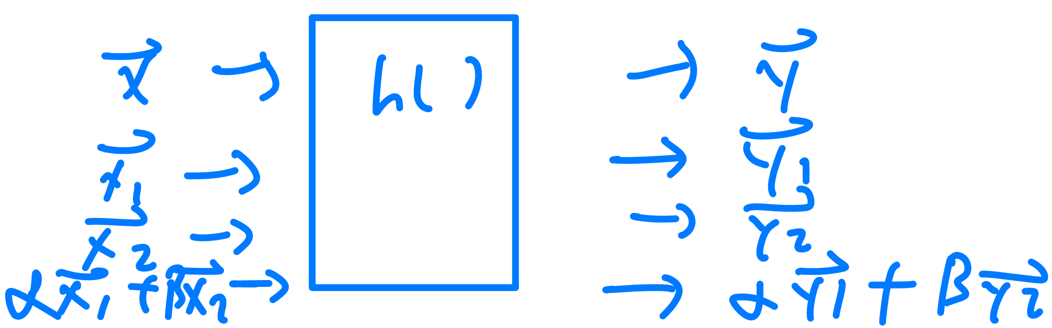
$$\Rightarrow a_i = \underline{a} \cdot \underline{e}_i \quad \text{or} \quad \underline{a} = \sum_{i=1}^{n} (\underline{a} \cdot \underline{e}_i) \underline{e}_i$$

- **5) A transformation h(·) is said to be** Linear **if**

$$h(\alpha \underline{a} + \beta \underline{b}) = \alpha h(\underline{a}) + \beta h(\underline{b})$$

$\forall \alpha, \beta \in$ **IR and** $\forall$**a and b.**

$h(): V \to V$



$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_n \vec{a}_n = \vec{0}$

$\alpha_1 \cdots \alpha_2 \cdots = 0$ (全0: 只有0!!)

6) $\underline{a}_1$, $\underline{a}_2$, …, $\underline{a}_n$ are independent **if none of these vectors can be written as a linear combination of the others.**

7) Triangular Inequality**:**

For any vectors $\underline{a}$ and $\underline{b}$,

$$\|\underline{a}+\underline{b}\| \le \|\underline{a}\| + \|\underline{b}\|$$

With equality iff

$$\underline{a} = k\underline{b} \qquad \text{for some } k \in \mathbb{R}$$

8) Cauchy – Schwartz Inequality:

$$|\underline{a} \cdot \underline{b}| \le \|\underline{a}\| \cdot \|\underline{b}\|$$

with equality if $\underline{a} = k\underline{b}$

$\cos\theta = \dfrac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \|\vec{b}\|}$

$\cos\theta \le 1 \Rightarrow |\langle a, b\rangle| \le \|\vec{a}\| \|\vec{b}\|$

iff $\vec{a} \propto \vec{b}$

$\downarrow$

also can prove in complex field!

9) Pythagorean Relation

if $\underline{a}$ and $\underline{b}$ are $\perp$

$\Rightarrow$

$$\|\underline{a} + \underline{b}\|^2 = \|\underline{a}\|^2 + \|\underline{b}\|^2$$

$$\langle a(t), b(t) \rangle \triangleq \int_0^{T_s} a(t)\, b^*(t)\, dt$$

$$\left| \int_0^{T_s} a(t)\, b^*(t)\, dt \right|^2 \le \int_0^{T_s} |a(t)|^2 dt \cdot \int_0^{T_s} |b(t)|^2 dt$$

Another form of Cauchy-Schwartz Inequality

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2$$

Basis Vectors

- **Let ($\underline{a}_1$, $\underline{a}_2$, …, $\underline{a}_n$) be a set of n vectors. These vectors are** independent **if it is impossible to find constants α_1, α_2, …α_n (<u>not</u>** all zero**) such that**

$$\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \cdots + \alpha_n \underline{a}_n = 0$$

- **In an** n-dim **space, we can have at most n independent vectors**

Signal Space Concepts

- Basic Idea: **Any entity that can be represented by n-tuple is an n-dim Vector** $\Rightarrow$ If a finite-duration signal (Ts) can be represented by n-tuple, then it is a vector.

- **Let $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ be n finite duration signals (Ts)**

- **Consider a finite-duration signal x(t) and suppose that**

$$x(t)=\sum_{i=1}^{n} x_i \varphi_i(t)$$

liner independent 就好
不必 orthogonal!

- **If every signal can be written as above** $\Rightarrow$ **$\{\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)\}$ ~** basis and have n-dim space

$$x(t) \Leftrightarrow \mathbf{x}=(x_1,\ldots,x_n) \text{ with respect to basis } \{\varphi_1(t),\ldots\varphi_n(t)\}$$

- **Define "dot-product" as** $$< x(t), y(t) >= \int_0^{T_s} x(t) y^*(t) dt$$

- **Basis set {φ$_k$(*t*)}^{n} is an** orthogonal **set if**

$$\int_{-\infty}^{\infty} \varphi_j(t)\varphi_k(t)dt = \begin{cases} 0 & j \neq k \\ k_j & j = k \end{cases}$$

- **If k$_j$≡1 ∀*j* ⇒ {φ$_k$(*t*)} is an** orthonormal **set. In this case,**

$$x_k = \int_{-\infty}^{\infty} x(t)\varphi_k(t)dt$$

$$x(t) = \sum_{i=1}^{n} x_i \varphi_i(t)$$

$$\underline{x} = (x_1, x_2, \ldots, x_n)$$

Key Property

Given a signal space $\mathcal{S} = span\{\varphi_1(t), ...\varphi_n(t)\}$ and a finite duration signal $x(t) \in \mathcal{S}$

(1) Computing Dot-Product

Let $x(t), y(t) \in \mathcal{S}$, $x(t) \Leftrightarrow \mathbf{x} = (x_1, .., x_n)$, $y(t) \Leftrightarrow \mathbf{y} = (y_1, .., y_n)$. For orthonormal basis, $< x(t), y(t) >= \sum_{i=1}^{n} x_i y_i$

(2) Energy of x(t)

$E_s = \int_0^{T_s} |x(t)|^2 dt$ (Time Domain Method)

$E_s = \int_{-\infty}^{\infty} |X(f)|^2 df$ (Frequency Domain Method)

$E_s = \|\mathbf{x}\|^2 =< x(t), x(t) >$ (Geometric Domain Method)

Geometric Domain Representation

- Geometric Domain (Signal Space)
  - Signal could be represented by a point in a space.
  - Step 1: Given a set of M signals, $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ define a D-dim signal space with basis $\{\phi_1(t), \phi_2(t), \ldots, \phi_D(t)\}$.
  - Step 2: Find out the coordinates of each signals by: $s_i(t) \rightarrow \vec{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,D})$

Easy, just find projection

$$s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

- **Question 1) How to find the signal space (basis signals) that contains** $\{s_1(t), \ldots, s_M(t)\}$
- **Question 2) How to find the coordinate of each signal?**

Step 1) Gram-Schmidt Orthogonalization for Vectors

- Given a set of M vectors $\{\vec{x}_1,\ldots,\vec{x}_M\}$ the G-S procedure allows one to find out the "orthonormal basis" $\{\vec{\phi}_1,\ldots,\vec{\phi}_M\}$ of the signal space (with the minimum dimension) to contain all the M vectors.
  - **Step 1:** $\vec{\phi}_1 = \dfrac{\vec{x}_1}{\left\|\vec{x}_1\right\|}$
  - **Step 2:** $\vec{v}_2 = \vec{x}_2 - \left\langle \vec{x}_1, \vec{x}_2 \right\rangle \vec{x}_1, \vec{\phi}_2 = \dfrac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$
  - **Step m:** $\vec{v}_m = \vec{x}_m - \sum_{i=1}^{m-1} \left\langle \vec{\phi}_i, \vec{x}_m \right\rangle \vec{\phi}_i, \vec{\phi}_m = \dfrac{\vec{v}_m}{\left\|\vec{v}_m\right\|}$

Projection of $\vec{x}_m$ on the current vector space spanned by $\{\vec{\phi}_1, \ldots, \vec{\phi}_{m-1}\}$

小心，有typo!!!

The process continues until m=M or $\vec{\phi}_m = \vec{0}$ for some $m \in [1, M]$

- Similarly, for signal space, vector = signal.
  - Given a set of M "signals" (vectors), we can use the same GS procedure to find out the "orthogonal basis" (basis signals) of the signal space (with min dimension) to contain all the M signals.
  - Use the same procedure except with the understanding that $\left\langle x(t), y(t) \right\rangle = \int_0^{T_s} x(t) y^*(t) dt$

$X = \{\vec{x}_1, \cdots \vec{x}_M\} \rightarrow \boxed{GSO} \rightarrow \{\vec{\phi}_1 \cdots \vec{\phi}_D\}$

iterative process

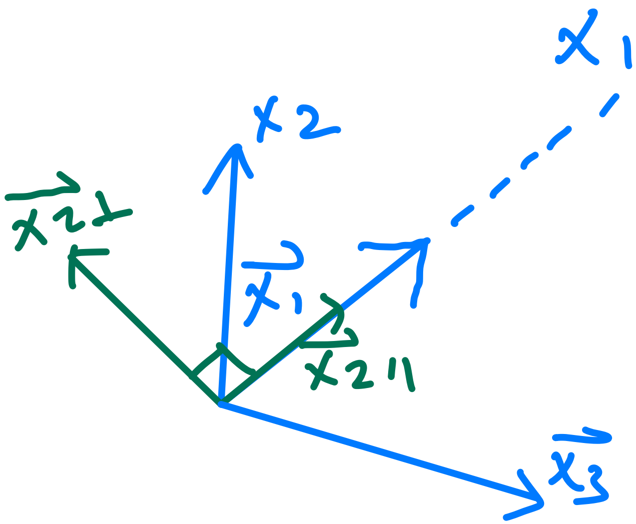
$D \leq M$ (worst case M)

Step 2:

Step 1:

$\vec{\phi}_1 = \dfrac{\vec{x}_1}{||\vec{x}_1||}$

$\vec{\phi}_2 = \dfrac{x_2^{\perp}}{||\vec{x}_2^{\perp}||}$



$\vec{x}_2 = \vec{x_2}^{\parallel} + \vec{x_2}^{\perp}$

$x_2^{\parallel} = \langle x_2, \vec{\phi}_1 \rangle \vec{\phi}_1$

$\Rightarrow x_2^{\perp} = \vec{x}_2 - \langle \vec{x}_2, \vec{\phi}_1 \rangle \vec{\phi}_1$

Step 3:

- Basis set is not unique! As $X = \{\vec{x}_1, \vec{x}_2 \ldots \vec{x}_M\}$ have different permutation, the steps will probably do different things!!!

Summary of GSO

- 1st basis function is a normalized version of 1st signal.
- Remaining basis functions are found by removing portions of signals which are correlated to previous basis functions, and normalizing the result.
- This procedure is repeated until all basis functions are found.

Step 2) Computing the Coordinates

Given the orthonormal basis $\{\phi_1(t), ..., \phi_D(t)\}$ that contains the M finite duration signals $\{s_1(t), .., s_M(t)\}$,

$$s_i(t) \Leftrightarrow \mathbf{s_i} = (s_{i1}, ..., s_{iD})$$

$$s_{ij} =< s_i(t), \phi_j(t) >= \int_0^{T_s} s_i(t)\phi_j^*(t)dt$$

Example

- Consider the following two signals that are defined on $[0,T)$

  $$s_0(t) = A\cos(2\pi f_c t) \qquad s_1(t) = A\sin(2\pi f_c t)$$

  where fc=n/T with n being an integer.

- Find an orthonomal basis set for these two signals. 2

- Repeat the above problem if we now have M-ary signals where

  $$s_i(t) = A\cos\left(\omega_c t + \frac{2\pi(m-1)}{M}\right), \quad m = 1,2,\ldots,M$$ $\mathbb{R}^2$, 或 complex $\mathbb{C}^1$

  1-1 transform

- What is the dimension of the resulting signal space ? 2

- Express $s_i(t)$ in terms of these basis functions and the signal energy E_s

1.

$$\alpha_0 s_0(t) + \alpha_1 s_1(t) = 0 \quad \forall t$$

$$\langle s_1(t), s_0(t) \rangle = \int_0^T \cos w_c t \sin w_c t \, dt =$$

Consider $\|\phi_0\|^2 = \|\phi_1\|^2 = 1$

$$\phi_0(t) = \alpha \cos(w_c t)$$

$$\phi_1(t) = \alpha \sin(w_c t)$$

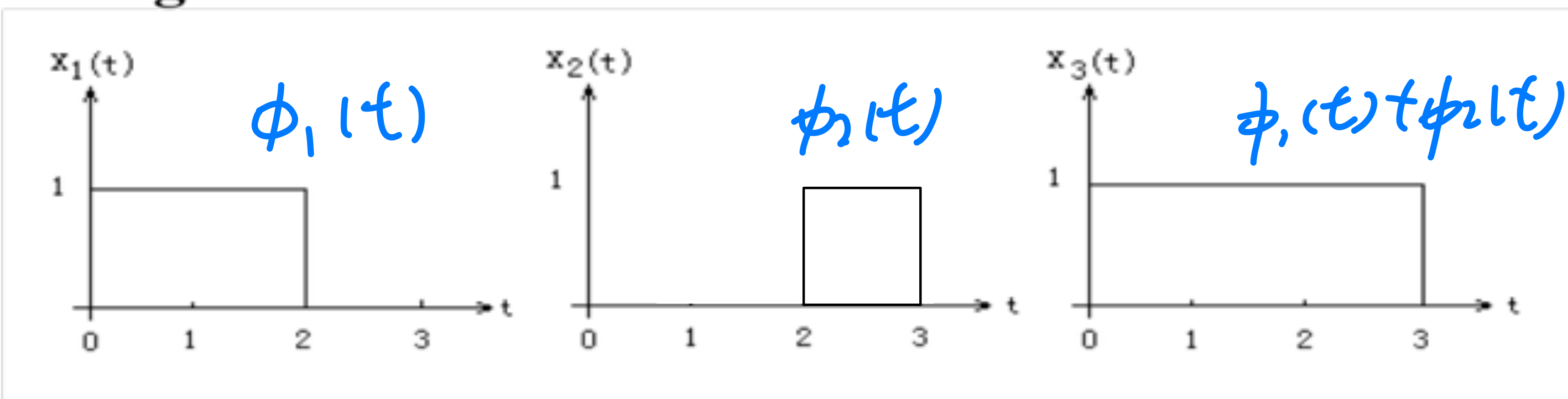
$$\frac{\alpha^2}{2} = 1$$

$$\alpha = \sqrt{2}$$

Example

2D-space is sufficient!

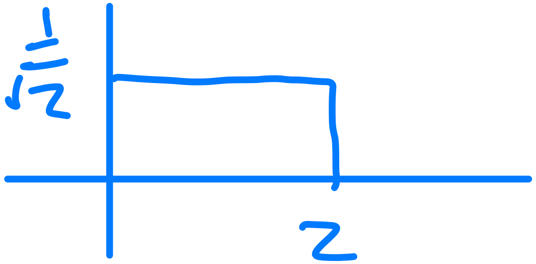
a. Use the Gram-Schmidt procedure to find a set of orthonormal basis functions corresponding to the signals shown below.



b. Express x1, x2, and x3 in terms of the orthonormal basis functions found in Part a.

c. Draw the constellation diagram for the signals

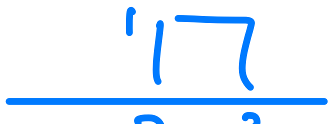
$\phi_1(t) =$

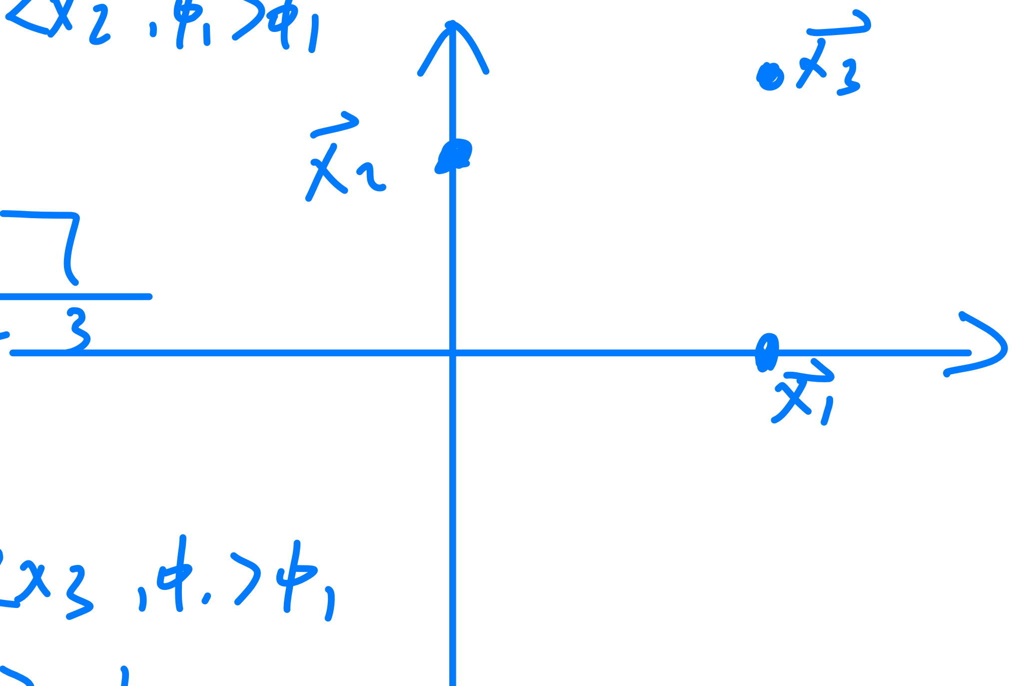


$v_2(t) = x_2 - \langle \vec{x}_2, \phi_1 \rangle \phi_1$

$\vec{v}_2 = \vec{x}_2$

$\phi_2 =$





$v_3(t) = x_3 - \langle x_3, \phi_1 \rangle \phi_1$

$- \langle x_3, \phi_2 \rangle, \phi_2$

$= \cdots$

Every points in $\mathbb{R}^2$ can be one-to-one mapped to $\mathbb{C}^1$ (isomorphism), but need to beware of the operations! like dot products!

In general every points in $\mathbb{R}^n$, each coordinate is in complex number, then can be mapped to $\mathbb{C}^n$, then all operations are applicable!