ELEC2100: Signals and Systems

Lecture 5

Impulse Response (Deduction)

(Ref: Chapter 2 O&W)

I) Impulse Response for DT LTI System

II) The Convolution Sum and Example DT Impulse Responses

III) Impulse Response and the Convolution Integral for CT LTI System

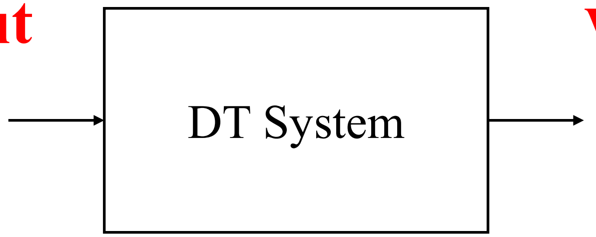
IV) Example CT Impulse Responses

I. Impulse Response for DT LTI System

- In this lecture, we will introduce the concept of ***impulse response*** and show how it fully characterizes any LTI system
- Let's start with the Discrete-Time (DT) case. Given an input $x[n]$, how can we determine the output $y[n]$?

Given an input

$x[n]$

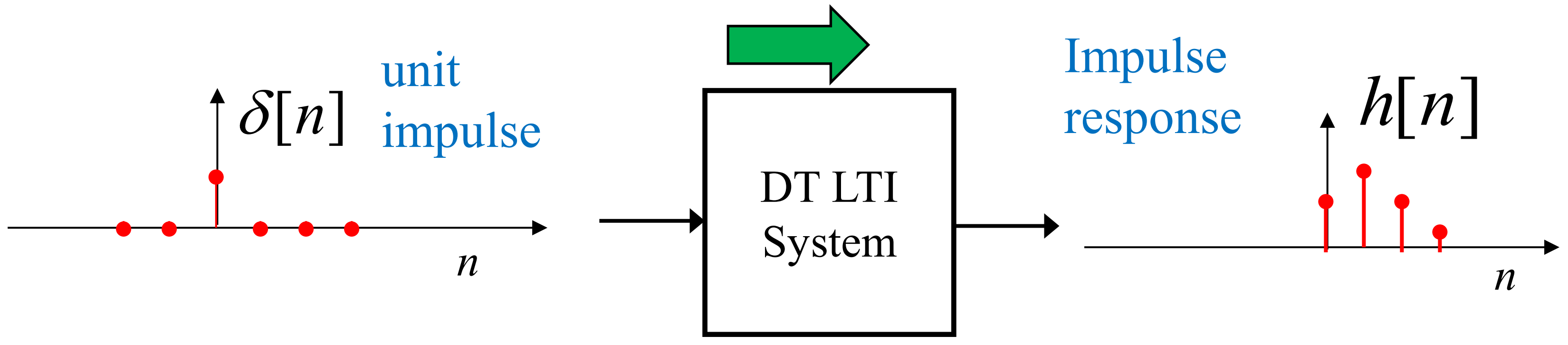


What is the output???

$y[n]???$

DT Impulse Response

- Let $h[n]$ be the output of a DT LTI system when the input is the unit impulse $\delta[n]$. We call $h[n]$ the **impulse response**.

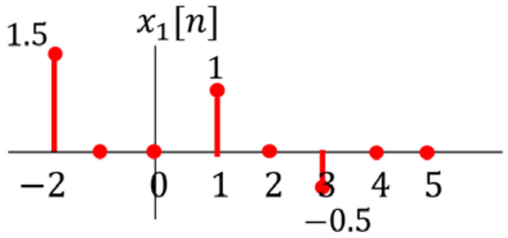


- If we know $h[n]$, we can determine the output of this LTI system for any input.

- Remember, we have shown that *any DT input signal can be viewed as a* ***superposition*** *of* ***shifted*** *unit impulse signals*.

For example, we can describe the signals to the right as:

$x_1[n] = 1.5\delta[n+2] + \delta[n-1] - 0.5\delta[n-3]$



DT Signal as Weighted Sum of Shifted Impulses

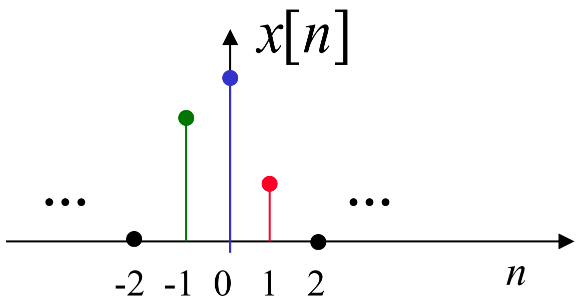
Mathematically, we can express any $x[n]$ as

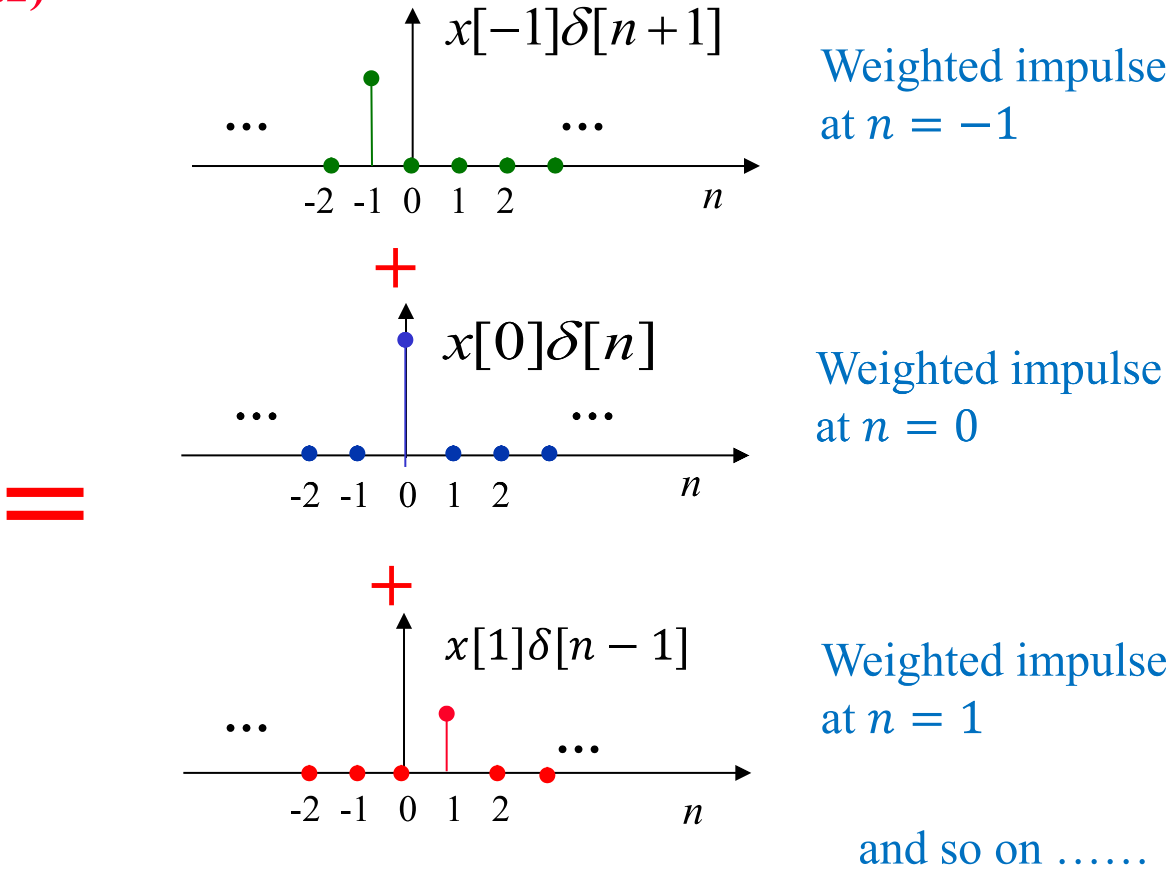
$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} \underline{x[k]}\,\underline{\delta[n-k]}}$$

Eq. (2.2)

weight/magnitude of impulse

impulse at $n = k$





Output of an LTI System

- If system is ***linear***, output must be the same weighted sum of the responses to the individual shifted impulses. Let $h_k[n]$ be the system's response to a $\delta[n-k]$. Then

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n] \qquad \textbf{Eq. (2.3)}$$

Input is a weighted sum of $\delta[n-k]$

Output is a weighted sum of responses to $\delta[n-k]$

- If system is ***time-invariant***, $h_k[n]$ must be $h[n-k]$. Hence:

$h_k[n] = h[n-k]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{\begin{array}{c}\text{Linear}\\ \text{Time-Invariant}\\ \text{(LTI)}\\ \text{System}\end{array}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \qquad \textbf{Eq. (2.6)}$$

- Eq. (2.6) is called the **convolution sum**.
- It ***means*** that $y[n]$, the output of an LTI system at any time n, can be founded by summing up the responses arising from the inputs at all different times k and the response at time n arising from the input at time k is $x[k]h[n-k]$.

Example – Investment in Couponed Bond

Example: When the government or a company wants to borrow money from the public, it issues bonds. A couponed bond is one that offers intermediate payouts before maturation.

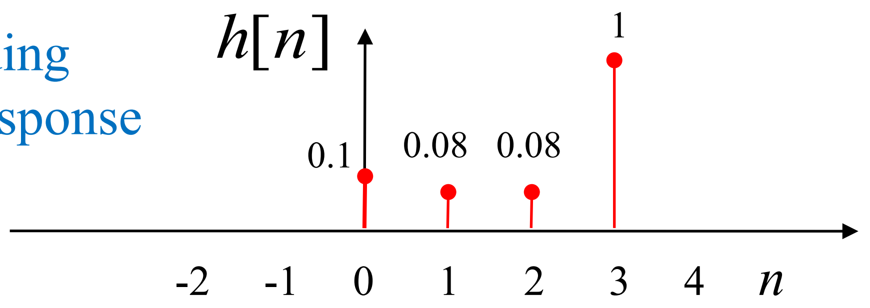
- Assume there is a couponed US government bond that provides the following payout schedule:

10% rebate immediately		⇨ $h[0]=0.1$
8% after one year		⇨ $h[1]=0.08$
8% after two years		⇨ $h[2]=0.08$
100% at the end of the third year		⇨ $h[3]=1$

Thus, you invest \$1, you get \$1.26 back in total over 3 years.
We can think of this bond as an LTI system with the impulse response of:

$$h[n] = 0.1\delta[t] + 0.08\delta[t-1] + 0.08\delta[t-2] + \delta[t-3]$$

Corresponding Impulse Response



- To help pay for your HKUST education, your parents buy for you \$10 of this bond in 2015, \$20 of this bond in 2016, \$40 of this bond in 2017. How much of a payout will you receive in 2019?

$y[2019]$=?

$$y[2019] = x[2015] \times h[4]$$ 2019-2015=4; h[4]=0

$$+x[2016] \times h[3]$$ 2019-2016=3; h[3]=1

$$+x[2017] \times h[2]$$ 2019-2017=2; h[2]=0.08

$$+x[2018] \times h[1]$$ 2019-2018=1; h[1]=0.08

$$+x[2019] \times h[0]$$ 2019-2019=0; h[0]=0.1

$$= \$10 \times 0 + \$20 \times 100\% + \$40 \times 8\% + \$0 \times 8\% + \$0 \times 10\% = \$\mathbf{23.2}$$

The output in year 2019 is simply the sum of the outputs arising from inputs in all different years!
This is what the convolution sum mean!

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

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II. The Convolution Sum and Example Impulse Responses

- We say that the output of an LTI system is the **convolution** of the input with the system's impulse response.

- We use the "$*$" symbol to denote convolution:

$$\boxed{y[n] = \underbrace{x[n] * h[n]}_{\text{convolution}} = \underbrace{\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}_{\text{Convolution sum}}}$$

Convolution = LTI

- ***Conversely***, if the output of a system is given by convolving the input with any function $h[n]$, then the system is LTI! It is because the convolution operation is LTI:

<u>Convolution is Linear:</u>

Convolving a weighted sum with h:

$$\underline{(ax_1[n] + bx_2[n])} * h[n] = \sum_{k=-\infty}^{\infty} \underline{(ax_1[k] + bx_2[k])}\, h[n-k]$$

put weighted sum into convolution sum

$$= a \sum_{k=-\infty}^{\infty} x_1[k]\, h[n-k] + b \sum_{k=-\infty}^{\infty} x_2[k]\, h[n-k] \quad = ax_1[n] * h[n] + bx_2[n] * h[n]$$

break into two weighted sums

= weighted sum of the individual convolutions

<u>Convolution is TI:</u>

Convolving shifted x with h

Replace $x[k]$ by $x[k-m]$

$n-k = n-(k'+m) = (n-m)-k'$

$$x[n-m] * h[n] = \sum_{k=-\infty}^{\infty} x[k-m]\, h[n-k] = \sum_{k'=-\infty}^{\infty} x[k']\, h[(n-m)-k'] = y[n-m]$$

let $g[n] = x[n-m]$

then $g[k] = x[k-m]$

Let $k' = k - m$

This means we are evaluating the output at time $n-m$

Summing over k from $-\infty$ to ∞ means summing over k' from $-\infty$ to ∞ as well, and k' is just a dummy variable for summation.

- So system being LTI means convolution and convolution means LTI!

Example Impulse Response 1: $h[n] = \delta[n]$ the Identity System

- Consider convolving a DT signal $x[n]$ with $h[n] = \delta[n]$:

$$x[n] * h[n] = x[n] * \overset{g[n]}{\delta[n]} = \sum_{k=-\infty}^{\infty} x[k]\,\overset{g[n-k]}{\delta[n-k]} = x[n]$$

Only one non-zero term in the sum because:

$$\delta[n-k] = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

This means that $\delta[n]$ is the ***identity*** function under convolution. Convolving any signal $x[n]$ with $\delta[n]$ results in $x[n]$ unchanged.

- So an LTI system with $h[n] = \delta[n]$ is a *do-nothing* system, or *identity system*.
- $h[n] = A\delta[n]$ is an *amplifier* that multiplies the input by A

Example Impulse Response 2: $h[n] = \delta[n-m]$ is a Delay System

- What about an LTI system with $h[n] = \delta[n-m]$, a shifted impulse?

$h[n-k] = \delta[(n-m)-k]$ since $h[n] = \delta[n-m]$

$$x[n] * h[n] = x[n] * \delta[n-m] = \sum_{k=-\infty}^{\infty} x[k]\,\underline{\delta[(n-m)-k]} = x[n-m]$$

Non-zero and equals 1 only when argument is 0, i.e., $k = n - m$

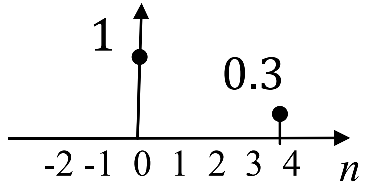
- Convolving any $x[n]$ with $\delta[n-m]$ produces $x[n-m]$.

  $h[n] = \delta[n-m]$ represents a *delay* system that delays the input by m.

Other Example Impulse Responses

For each system below, express output $y[n]$ in terms of input $x[n]$ and explain what the system does.

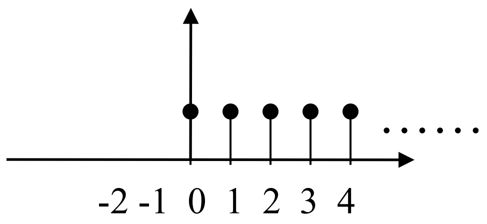
3. $h[n] = \delta[n] + 0.3\delta[n-4]$



$y[n] = x[n] + 0.3x[n-4]$

An echo system

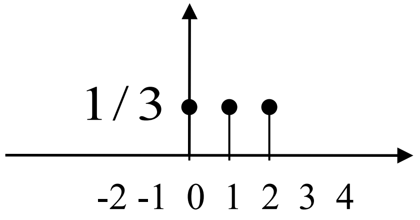
4. $h[n] = u[n]$



$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

An first sum system

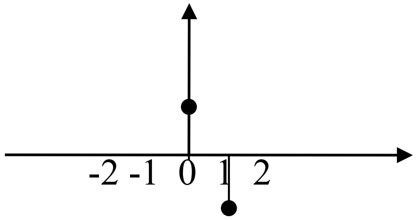
5. $h[n] = \frac{1}{3}(u[n] - u[n-3])$



$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$$

A window averager/smoother

6. $h[n] = \delta[n] - \delta[n-1]$



$$y[n] = x[n] - x[n-1]$$

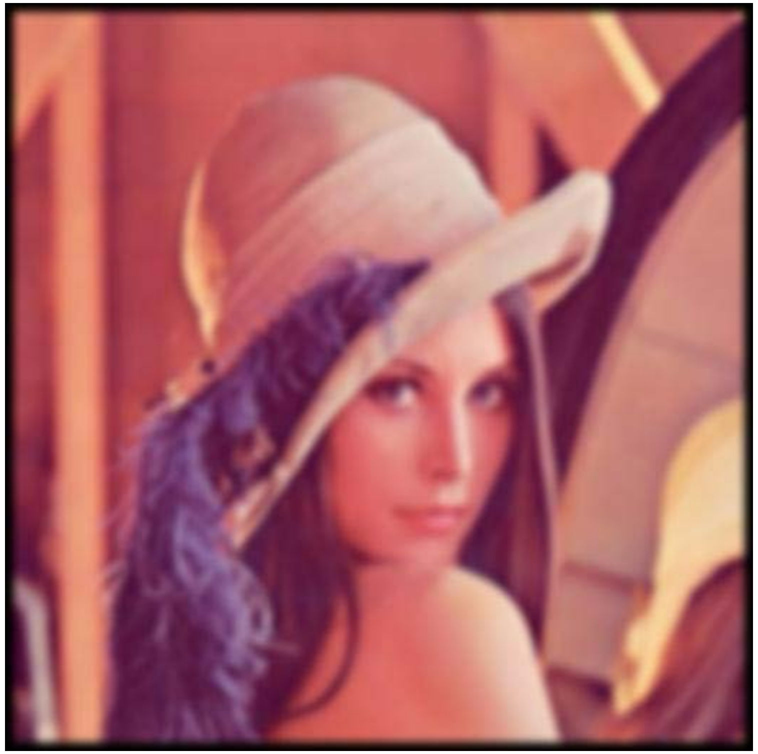
A first difference system

Ripple is convolution

Pretty Lena

A Pretty Picture of Lena

Lena after a smoother



You will learn in the future that we need to smooth a picture first before we down-sample it – i.e., when creating a thumbnail

Lena after a first difference system (differentiator)

Allows us to detect boundaries/outlines of objects in an image

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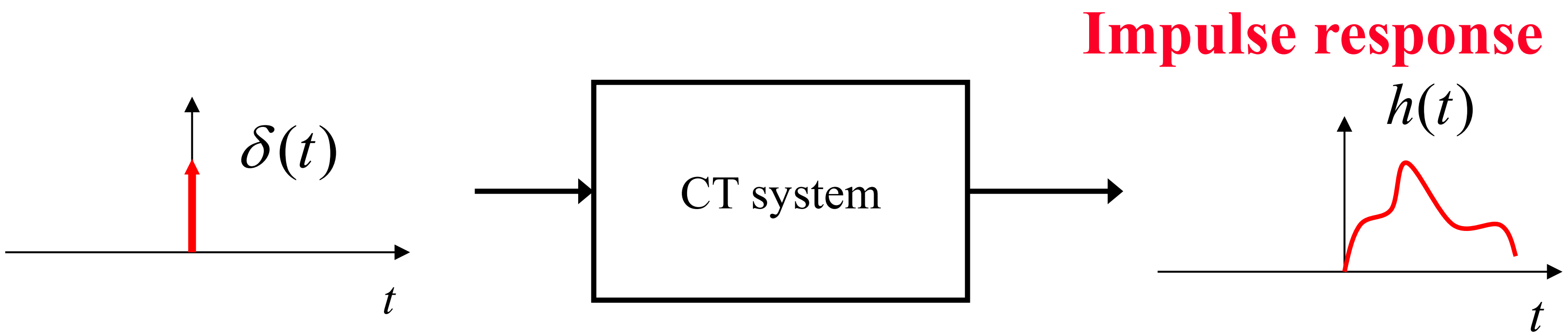
III. Impulse response and Convolution Integral for CT System

- Now, for a CT system, let $h(t)$ be the output when the input is the CT impulse $\delta(t)$
- $h(t)$ is the ***impulse response***



- Next, we will show that any signal $x(t)$ can be viewed as a superposition of shifted impulse signals in the form of an integral.
- We do so by first showing that any CT signal can be *approximated* by a weighted sum of shifted narrow pulses. In the limit these pulses become infinitesimally narrow, the approximation becomes exact and the weighted sum becomes an integral.

Representation of CT Signal as a sum of narrow pulses

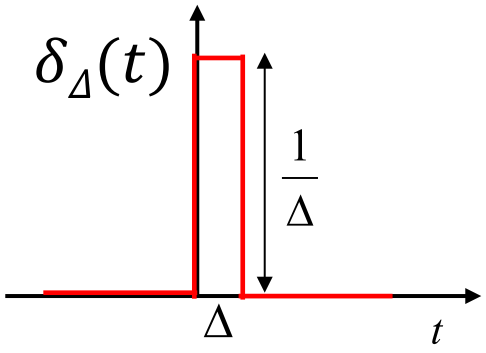
- In Chapter 1, we argued that $\delta(t)$ can be approximated by $\delta_\Delta(t)$, a narrow pulse with width Δ and height $1/\Delta$
- The signal $\delta_\Delta(t)\Delta$ would be a narrow pulse with width Δ and height 1.
- We can approximate a CT signal $x(t)$ using shifted $\delta_\Delta(t)\Delta$ signals by what is called a staircase approximation:

Staircase approximation: Weighted sum of shifted narrow pulses:

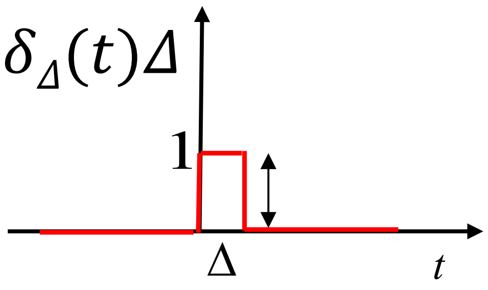
Each term in the sum is a narrow pulse at $k\Delta$ with height given by $x(k\Delta)$

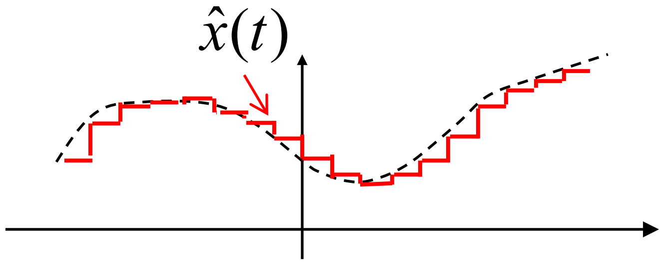
$$\boxed{x(t) \approx \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\,\Delta}$$

Approximate impulse Introduced in Chapter 1



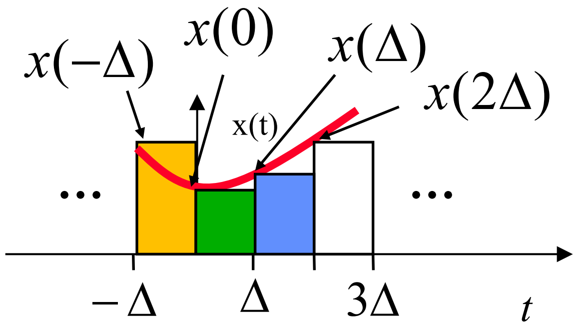
Multiply by Δ gives a narrow pulse with height 1



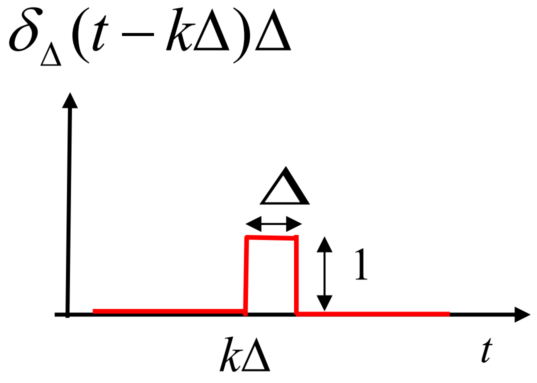


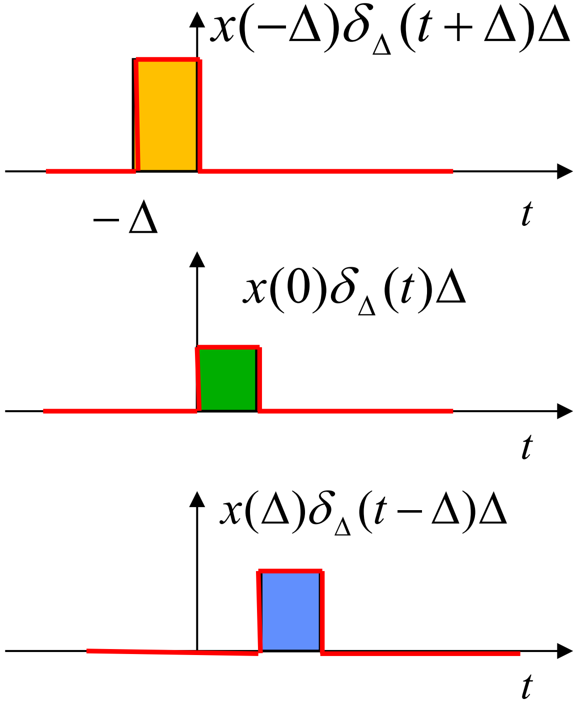
Representation of CT Signal as a sum of narrow pulses

- $\delta_\Delta(t-k\Delta)\Delta$ is a narrow pulse shifted to $k\Delta$.
- We scale the height of each narrow pulse by the value of $x(k\Delta)$ and add all the weighted narrow pulses together to form the staircase approximation.



A shifted narrow pulses with unit height





Next, we take limit of Δ going to zero. Then the approximation becomes exact and the summation becomes an integral:

$$\boxed{x(t)} = \lim_{\Delta \to 0} \hat{x}(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

$$\boxed{= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Eq. (2.27)

$$\lim_{\Delta \to 0} \sum_{k=K_1}^{K_2} g(k\Delta)\Delta = \int_{L_1}^{L_2} g(\tau)d\tau$$

- Observe that Eq. (2.27) looks very similar to Eq. (2.2), $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Here, it means that:

  Any signal $x(t)$ can be viewed as weighted sum (by integrating) of infinitely many shifted impulses $\delta(t - \tau\ )$ with infinitesimally small weights of $x(\tau)d\tau$

- As in the DT case, we can also interpret Eq. (2.27) as a convolution of $x(t)$ with $\delta(t)$, with $\delta(t)$ being the identity function under convolution. Any $x(t)$ convolving with $\delta(t)$ produces $x(t)$ unchanged.

Response of CT LTI System

$$\lim_{\Delta\to 0} \delta_\Delta(t) = \delta(t)$$

- Let $h_\Delta(t)$ be the response of a CT system to the input $\delta_\Delta(t)$. In limit, $\lim_{\Delta\to 0} h_\Delta(t) = h(t)$
- For a CT LTI system, if the input is a superposition of shifted $\delta_\Delta(t)$, the output is the same superposition of shifted $h_\Delta(t)$:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \longrightarrow \boxed{\text{CT LTI system}} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$$

- Now, we take limit $\Delta \to 0$, the input becomes $x(t)$ and the output becomes:

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{CT LTI system}} \longrightarrow$$

$$\boxed{y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Eq. 2.33

Convolution of input with impulse response

The Convolution Integral

- In summary, for a CT LTI system, we can determine the output from the input and the impulse response by:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

- This is called the ***convolution integral*** and we also use the "*" symbol to denote it:

$$\boxed{y(t) = x(t) * h(t)}$$

- Just like the DT case, for an LTI system, the output is the convolution of the input and the system's impulse response!

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IV. Example CT Impulse Responses

Example 1. What is a CT LTI system with $h(t) = \delta(t)$?

Let $x(t)$ be the input, then the output is:

$$y(t) = x(t) * h(t) \quad = x(t) * \delta(t) \quad = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad = x(t)$$

from Eq. (2.27), $x(t)$ as limit of staircase approximation

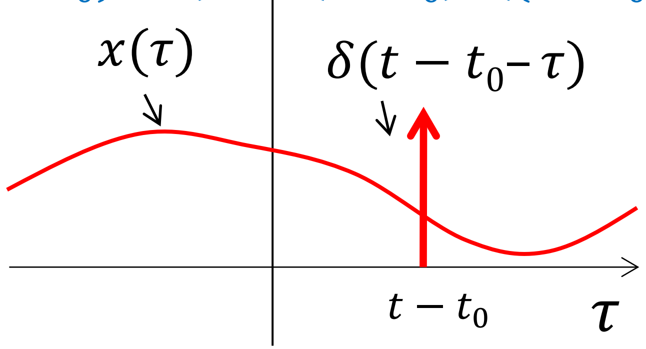
Again, convolving any signal $x(t)$ with $\delta(t)$ yields $x(t)$ unchanged

- LTI system with impulse response $h(t) = \delta(t)$ is a *do-nothing system*, or an *identity system*.

Example 2: $h(t) = \delta(t - t_0)$

As function of τ,
$x(\tau)\delta((t-t_0)-\tau) = x(t-t_0)\delta((t-t_0)-\tau)$

- What about an LTI system with $h(t) = \delta(t - t_0)$?



$h(t-\tau) = \delta((t-t_0)-\tau)$

$$y(t) = x(t) * \delta(t-t_0) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta((t-t_0)-\tau)d\tau$$

Sampling property of impulse!

$$= \int_{\tau=-\infty}^{\infty} x(t-t_0)\delta((t-t_0)-\tau)d\tau = x(t-t_0)$$

Also directly from the sifting property of impulse, Lecture 03-3, Property 6

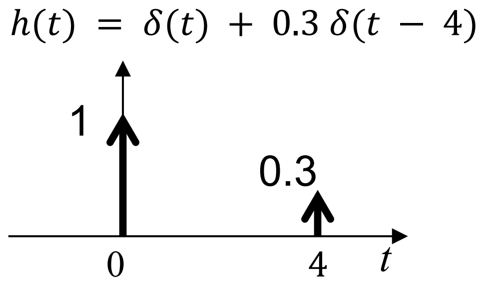
Convolving with a shifted impulse yields a shifted version of the input.

- $h(t) = \delta(t - t_0)$ represents a *time-delay system*. The time shift is $-t_0$. delay is t_0

Other Example CT Impulse Responses

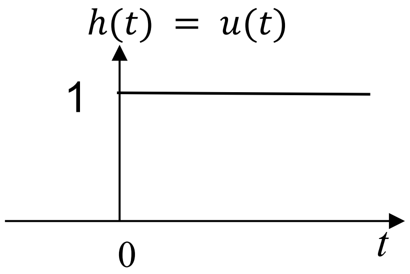
3. Echo/Multipath System: $h(t) = \delta(t) + 0.3\delta(t-4)$

Impulse plus a delayed and attenuated impulse



$$y(t) = x(t) + 0.3x(t-4)$$

4.Integrator: $h(t) = u(t)$

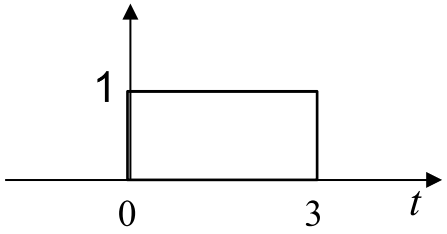


Since $u(t-\tau) = 0$ when $\tau > t$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\underbrace{u(t-\tau)}_{h(t-\tau)=u(t-\tau)}d\tau = \int_{-\infty}^{t} x(\tau)d\tau = \int x(t)$$

5. Windowed Integrator/Smoother

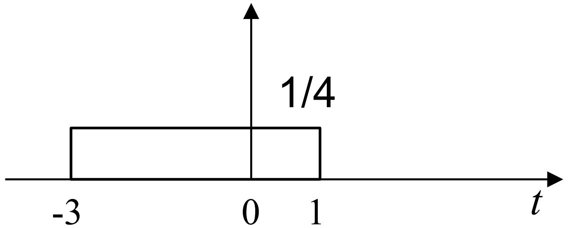
$5a.\ h(t) = (u(t) - u(t-3))$



$$h(t-\tau) = \begin{cases} 1 & 0 < t-\tau < 3 \ \Rightarrow t-3 < \tau < t \\ 0 & otherwise \end{cases}$$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-3}^{t} x(\tau)d\tau$$

$5b.\ h(t) = (u(t+3) - u(t-1))1/4$



$$h(t-\tau) = \begin{cases} 1/4 & -3 < t-\tau < 1 \ \Rightarrow t-1 < \tau < t+3 \\ 0 & otherwise \end{cases}$$

$$y(t) = \frac{1}{4}\int_{t-1}^{t+3} x(\tau)d\tau$$

Non-causal because response is non-zero before $t = 0$

This is a non-causal system since integrating to $t + 3$

CT Differentiator

- Recall that differentiation is LTI. What is the $h(t)$ for a differentiator?

- If the input is $\delta(t)$, the output of a differentiator must be its derivative, $\frac{d\delta(t)}{dt}$.

- The derivative of $\delta(t)$ is called the ***unit doublet***.

- The unit doublet is discussed in Section 2.5.3 of the text. We will not study its mathematical properties. The key is to recognize that differentiation is LTI.

Self-Test – Lecture 5

1. From memory and based on understanding, write down the convolution sum and convolution integral

2. If $y(t) = \int_{t-2}^{t+1} x(\tau)d\tau$, where $y(t)$ is the output to any input $x(t)$. Is the system LTI, and what is the impulse response?