T12

Laplace transform
Fourier transform and Laplace transform
Poles and zeros
Region of convergence
Important properties

Characterization of LTI System
Pole – Zero Cancellation

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad \text{where} \qquad s = \sigma + j\omega$$

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

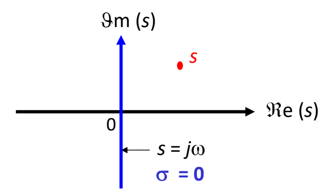
$$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}|\, dt < \infty$$

Relation between Fourier Transform and Laplace Transform

Stable **Stable and Unstable**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(s)|_{s=j\omega}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

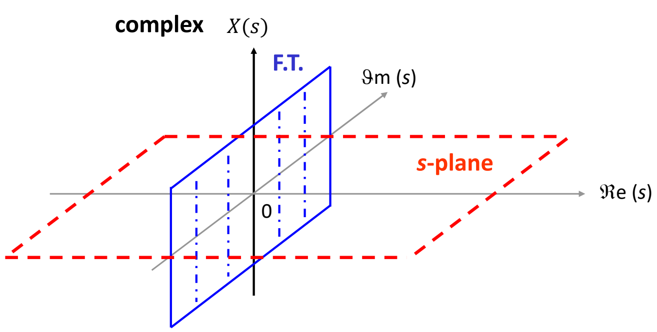


Poles and Zeros

$$X(s) = \frac{(s+a)(s+b)\ldots}{(s+c)(s+d)\ldots}$$

Poles ~ cause $X(s)$ to be infinity (i.e. s = – c , s = – d ... etc)

Zeros ~ cause $X(s)$ to be zero (i.e. s = – a , s = – b ... etc)

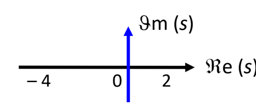


Region of Convergence (ROC)

- The region of s such that the Laplace transform is finite

i.e. $\boxed{x(t)\, e^{-\sigma t}}$ absolute integrable

- Important information to determine $x(t)$

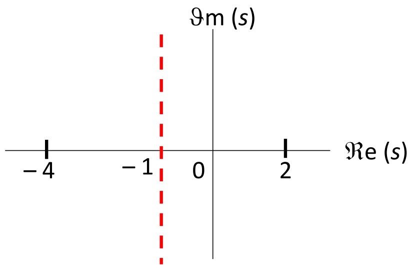


e.g. $x(t) = e^{-t}\, u(t)$ $\qquad X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

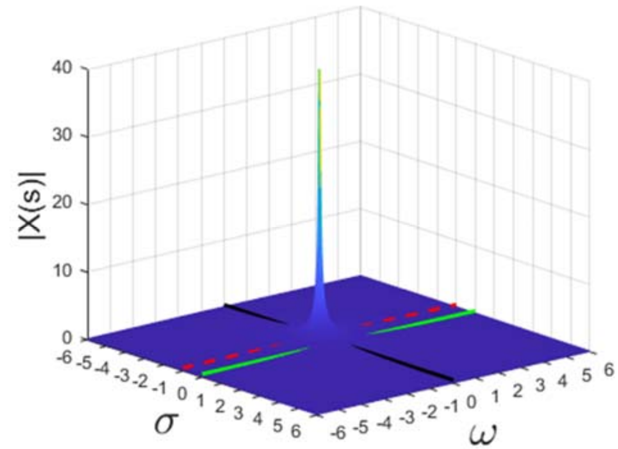
a) Find $X(-4)$ $\qquad X(-4) = \int_0^{\infty} e^{-t} e^{4t} dt = \int_0^{\infty} e^{3t} dt = \infty$

b) Find $X(2)$ $\qquad X(2) = \int_0^{\infty} e^{-t} e^{-2t} dt = \int_0^{\infty} e^{-3t} dt = \frac{1}{3}$

Question : What is right-sided signal ? Left-sided ? Two-sided ? Finite duration ?

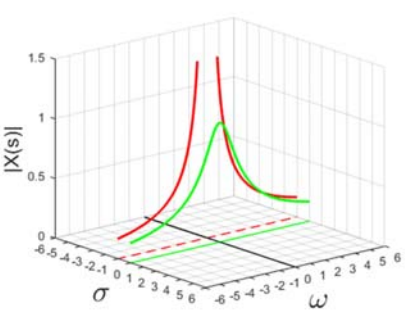


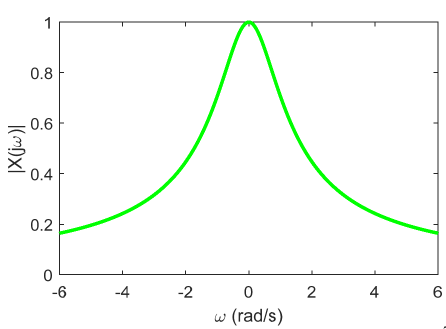
$x(t) = e^{-t}\, u(t)$ $\qquad X(s) = \dfrac{1}{s+1}$ $\qquad Re\{s\} > -1$



$$x(t) = e^{-t}\, u(t)$$

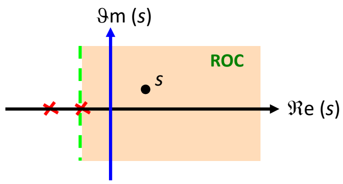
$s = \sigma + j\omega$ $\qquad X(s) = \dfrac{1}{s+1}$ $\qquad s = j\omega$ $\qquad X(j\omega) = \dfrac{1}{j\omega + 1}$



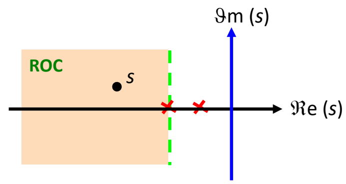


Important Properties

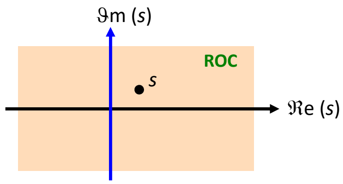
1. The ROC of a rational Laplace transform does not contain any poles.
2. If $x(t)$ is of finite duration and is absolute integrable, then the ROC is the entire complex plane.
3. If $x(t)$ is **right sided**, its ROC is the **right-half plane**.
4. If $x(t)$ is **left sided**, its ROC is the **left-half plane**.
5. If $x(t)$ is **two sided**, its ROC consists of a **strip** in the complex plane.
6. For a rational Laplace transform, the ROC is from the rightmost pole to positive infinity if $x(t)$ is right sided, while the ROC is from the leftmost pole to negative infinity if $x(t)$ is left sided.



Right-sided signal



Left-sided signal

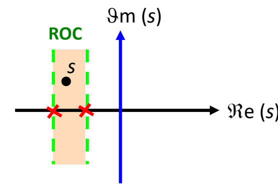


$$X(s) = \int_{-\infty}^{\infty} \boxed{\boldsymbol{x(t)}\ e^{-\sigma t}}\, e^{-j\omega t}\, dt$$

Finite duration and absolute integrable

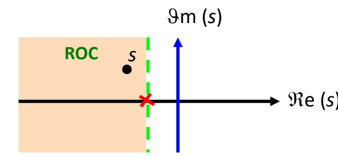
$\boldsymbol{e^{-t}u(t)}$ e^{3t}

$\boldsymbol{e^{-t}[u(t) - u(t-1)]}$ e^{3t}

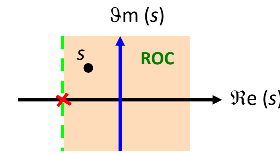


Two-sided signal

Left-sided + **Right-sided**



Left-sided signal



Right-sided signal

Question : No ROC ?

e.g. Notice that an absolute integrable signal $x(t)$ has a pole at $s = 2$.

a) Could $x(t)$ be finite duration ? X

b) Could $x(t)$ be left sided ? ✓

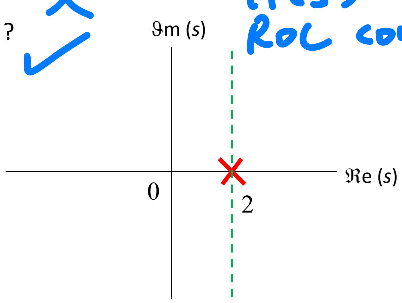
c) Could $x(t)$ be right sided ? X

d) Could $x(t)$ be two sided ? ✓

$H(j\omega)$ ✓

$H(s)$ ✓

ROC contains $j\omega$ axis ✓



e.g. Given : $\dfrac{d^3y(t)}{dt^3} + 2\dfrac{d^2y(t)}{dt^2} + 2\dfrac{d\,y(t)}{dt} + y(t) = \dfrac{d\,x(t)}{dt} + 2\,x(t)$

a) Obtain the system function

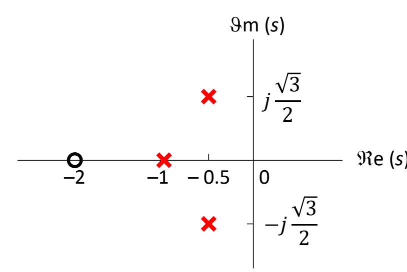
$$s^3Y(s) + 2\,s^2Y(s) + 2\,sY(s) + Y(s) = s\,X(s) + 2\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^3 + 2\,s^2 + 2s + 1}$$

b) Plot the zeros and pole

$$H(s) = \frac{s+2}{(s+1)(s^2+s+1)}$$

$$= \frac{s+2}{(s+1)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

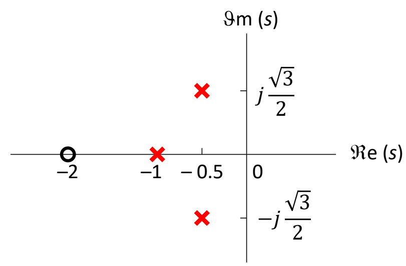


c) How many possible ROC ?

$Re\{s\} < -1$

$Re\{s\} > -0.5$

$-1 < Re\{s\} < -0.5$



d) Obtain $h(t)$ for each ROC

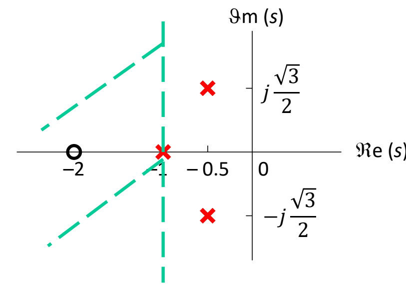
$$H(s) = \frac{s+2}{(s+1)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{A}{S+1} + \frac{B}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}} + \frac{C}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

$$A = 1$$

$$B = -\frac{1}{2}+j\frac{\sqrt{3}}{2}$$

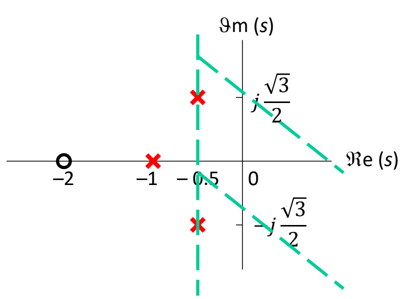
$$C = -\frac{1}{2}-j\frac{\sqrt{3}}{2}$$



ROC : $Re\{s\} < -1$

$$h(t) = -A\,e^{-t}\,u(-t) - Be^{-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)t}\,u(-t) - Ce^{-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)t}\,u(-t)$$

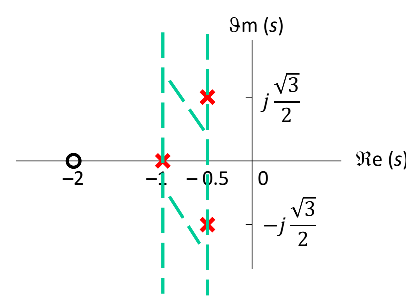
Left-sided



ROC : $Re\{s\} > -0.5$

$$h(t) = A\,e^{-t}\,u(t) + Be^{-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)t}\,u(t) + Ce^{-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)t}\,u(t)$$

Right-sided



ROC : $-1 < Re\{s\} < -0.5$

$$h(t) = A\,e^{-t}\,u(t) - Be^{-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)t}\,u(-t) - Ce^{-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)t}\,u(-t)$$

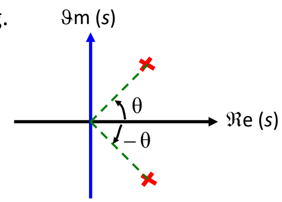
Two-sided

Question : What is the requirement on the LT if this is a real signal ?

$x(t)$ is real ➡ $x(t) = x^*(t)$

➡ $X(s) = X^*(s^*)$

e.g.

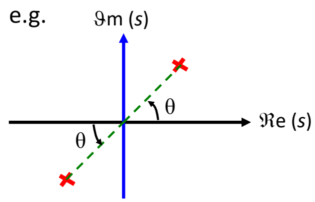


Question : Real and Even function ?

$x(t)$ is even ➡ $x(t) = x(-t)$

➡ $X(s) = X(-s)$

e.g.



e.g. Given the following facts about $x(t)$. Determine $X(s)$ and its ROC.

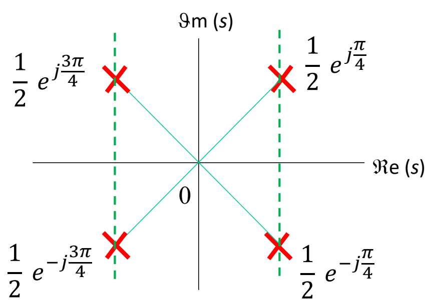
(1) $x(t)$ is real and even. 2-sided

(2) $X(s)$ has four poles and no zeros in the finite s-plane.

(3) $X(s)$ has a pole at $s = \frac{1}{2}\, e^{j\frac{\pi}{4}}$

(4) $\int_{-\infty}^{\infty} x(t)\, dt = 4$ $= H(0)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$



Question : ROC ?

$$X(0) = \int_{-\infty}^{\infty} x(t) dt = 4$$

$$A = 4\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$X(s) = \frac{A}{\left(s - \frac{1}{2}\, e^{j\frac{\pi}{4}}\right)\left(s - \frac{1}{2}\, e^{-j\frac{\pi}{4}}\right)\left(s - \frac{1}{2}\, e^{j\frac{3\pi}{4}}\right)\left(s - \frac{1}{2}\, e^{-j\frac{3\pi}{4}}\right)}$$

Characterization of LTI System
Pole – Zero Cancellation

Characterization of LTI System

1. For a **causal** system, the impulse response is right sided. ROC is on the RHP.

   $h(t) = 0$ for $t < 0$ (the converse is not true)

   For an **anti-causal** system, the impulse response is left sided. ROC is on the LHP.

   $h(t) = 0$ for $t > 0$ (the converse is not true)

2. For a **stable** system, the ROC of the system function $H(s)$ includes the entire $j\omega$-axis.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

3. For a **causal** and **stable** system, all the poles of $H(s)$ lie on the left-half of the s-plane.

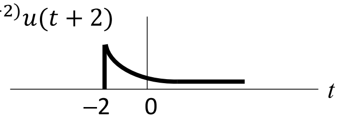
   Combine (1) & (2)

$e^{-t}u(t)$

$$H(s) = \frac{1}{s+1}$$

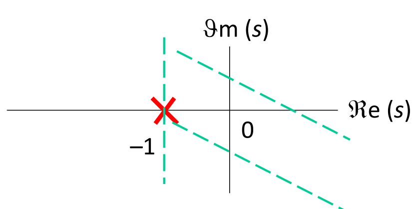
ROC : $Re\{s\} > -1$

$e^{-(t+2)}u(t+2)$

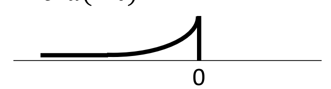


$$H(s) = \frac{e^{2s}}{s+1}$$

ROC : $Re\{s\} > -1$



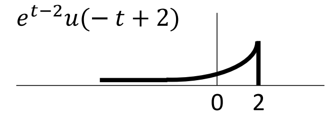
$e^{t}u(-t)$



$$H(s) = -\frac{1}{s-1}$$

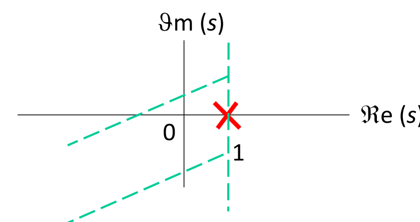
ROC : $Re\{s\} < 1$

$e^{t-2}u(-t+2)$



$$H(s) = -\frac{e^{-2s}}{s-1}$$

ROC : $Re\{s\} < 1$



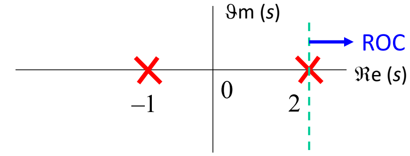
Pole-Zero Cancellation

e.g. Given the system function of a causal system :

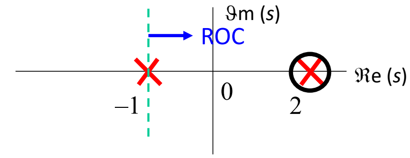
$$H(s) = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$$

a) Is it a stable system ?
b) If not, how to make the system to be stable ?

$$H(s) = \frac{1}{(s+1)(s-2)}$$

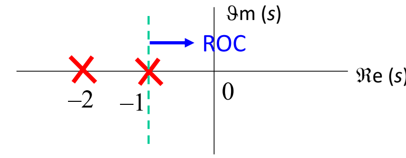


$$H_1(s) = \frac{(s-2)}{(s+1)(s-2)}$$

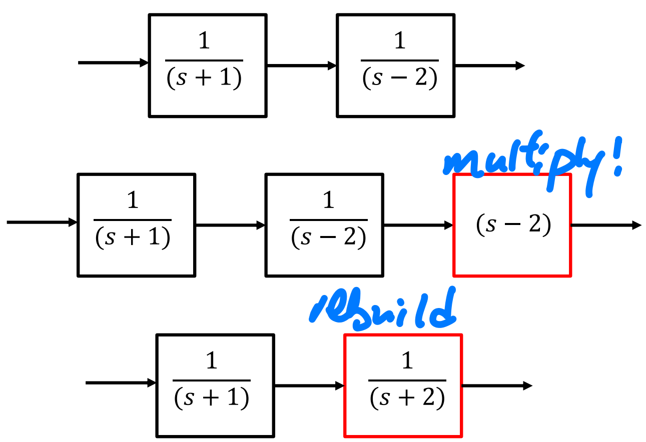


Physical implementation ?

$$H_2(s) = \frac{1}{(s+1)(s+2)}$$



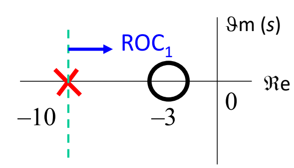
Physical implementation ?

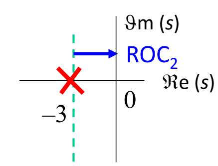


e.g. Convolution

Given two causal systems :

$$H_1(s) = \frac{s+3}{s+10} \qquad H_2(s) = \frac{1}{s+3}$$

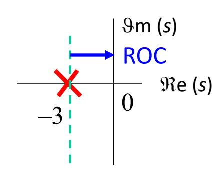




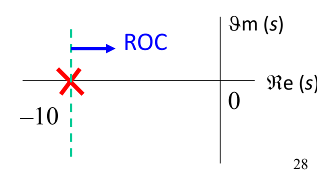
$$h(t) = h_1(t) * h_2(t)$$

Question : ROC ?

$\text{ROC}_1 \cap \text{ROC}_2$



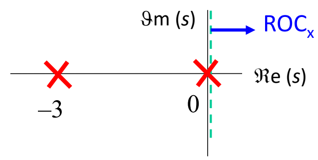
$$H(s) = \frac{1}{s+10}$$



e.g. Differentiation

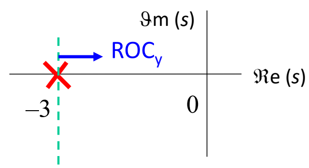
Given a right-sided signal :

$$X(s) = \frac{1}{s(s+3)}$$



$$y(t) = \frac{d}{dt}x(t)$$

$$Y(s) = (s)\frac{1}{s(s+3)} = \frac{1}{(s+3)}$$



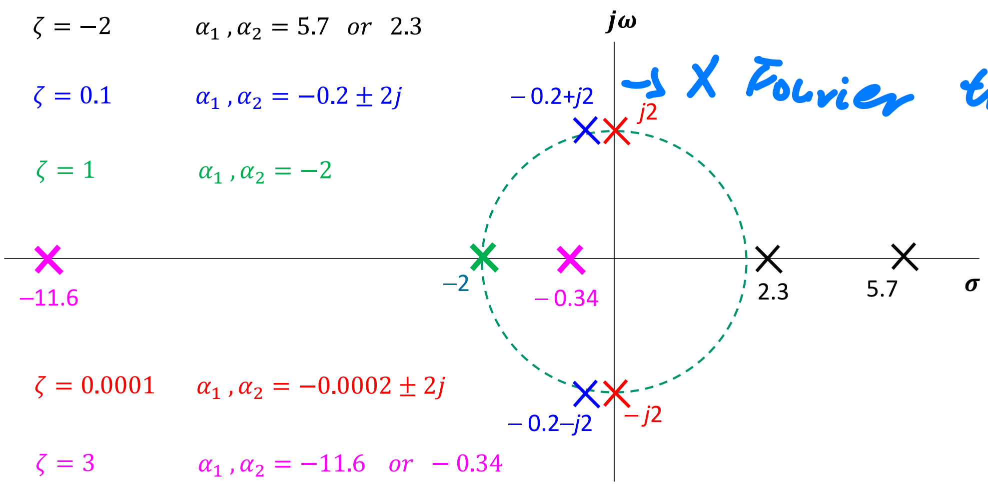
Question : ROC ?

e.g. $\omega_n = 2$ $\quad \alpha_1, \alpha_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$ $\quad$ Causal LTI system

$\zeta = -2$	$\alpha_1, \alpha_2 = 5.7 \ \ or \ \ 2.3$
$\zeta = 0.1$	$\alpha_1, \alpha_2 = -0.2 \pm 2j$
$\zeta = 3$	$\alpha_1, \alpha_2 = -11.6 \ \ or \ \ -0.34$
$\zeta = 1$	$\alpha_1, \alpha_2 = -2$
$\zeta = 0.0001$	$\alpha_1, \alpha_2 = -0.0002 \pm 2j$

Question : Why to check the locations of poles and zeros ?

$\zeta = -2$	$\alpha_1, \alpha_2 = 5.7 \ \ or \ \ 2.3$
$\zeta = 0.1$	$\alpha_1, \alpha_2 = -0.2 \pm 2j$
$\zeta = 1$	$\alpha_1, \alpha_2 = -2$
$\zeta = 0.0001$	$\alpha_1, \alpha_2 = -0.0002 \pm 2j$
$\zeta = 3$	$\alpha_1, \alpha_2 = -11.6 \ \ or \ \ -0.34$



→ X Fourier transform