1. Find the **first five nonzero terms** of the power series solution about $x_0 = 2$ of the initial value problem:

$$\begin{cases} y'' + 2xy' - y = 0, \\ y(2) = 1, \quad y'(2) = -1. \end{cases}$$

[15]

2. Consider the following equation:

$$(x+3)^2 y'' - 5(x+3)y' + 9y = 0.$$

   (a) Identify all the ordinary and singular points; and for singular point(s), identify the type of regular singular or irregular singular point(s). Explain your reasons. [5]

   (b) Find the general solution of the equation. [10]

3. Use Laplace transform to solve the initial value problem:

$$y''(t) + 4y(t) = f(t) + \delta(t - 1.5); \quad y(0) = 0, \quad y'(0) = 0;$$

where

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ (t-4)/6, & 4 \le t < 10, \\ 1, & t \ge 10. \end{cases}$$

[20]

4. Apply the **Laplace transform** to solve the initial value problem:

$$2y'' + y' + 2y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$

[15]

5. Solve the initial value problem:

$$\boldsymbol{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \boldsymbol{x}; \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

[15]

6. Consider the initial value problem:

$$y'' + y' - 6y = 0; \quad y(0) = 3, \quad y'(0) = 0.$$

   (a) Transform the problem into an initial value problem of first-order system of equations.

   (b) Solve the initial value problem obtained in (a) using the method for systems of first-order equations.

   (c) Find the solution y of the original initial value problem through the results obtained in (a) and (b).

[20]

— *End* —

1. Find the **first five nonzero terms** of the power series solution about $x_0 = 2$ of the initial value problem:

$$\begin{cases} y'' + 2xy' - y = 0, \\ y(2) = 1, \quad y'(2) = -1. \end{cases}$$

[15]

Since $P(x) = 1$, $x = 2$ is a regular point

Assume $y = \sum_{n=0}^{\infty} a_n (x-2)^n$

Then $y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n$ $2'$

$y'' = \sum_{n=1}^{\infty} n(n+1) a_{n+1} (x-2)^{n-1}$

$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-2)^n$

$2xy' = 2(x-2)y' + 4y'$

$= 2\sum_{n=0}^{\infty} n a_n (x-2)^n + 4\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n$ $5'$

$\Rightarrow y'' + 2xy' - y = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + 4(n+1) a_{n+1} + 2n a_n - a_n] (x-2)^n = 0$

$\Rightarrow (n+2)(n+1) a_{n+2} + 4(n+1) a_{n+1} + (2n-1) a_n = 0$

$n = 0$: $2a_2 + 4a_1 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0 - a_1$

$n = 1$: $6a_3 + 8a_2 + a_1 = 0 \Rightarrow a_3 = -\frac{2}{3} a_0 + \frac{5}{2} a_1$ $3'$

$n = 2$: $12a_4 + 12a_3 + 3a_2 = 0 \Rightarrow a_4 = \frac{13}{24} a_0 - 2a_1$

Since $a_0 = y(2) = 1$, $a_1 = y'(2) = -1$ $3'$

then $a_2 = \frac{3}{2}$, $a_3 = -\frac{19}{6}$, $a_4 = \frac{61}{24}$

Therefore,

$y = 1 - (x-2) + \frac{3}{2}(x-2)^2 - \frac{19}{6}(x-2)^3 + \frac{61}{24}(x-2)^4 + \cdots$ $2'$ □

2. Consider the following equation:

$$(x+3)^2y'' - 5(x+3)y' + 9y = 0.$$

(a) Identify all the ordinary and singular points; and for singular point(s), identify the type of regular singular or irregular singular point(s). Explain your reasons. [5]

(b) Find the general solution of the equation. [10]

Solution. (a) $P(x) = (x+3)^2$, $Q(x) = -5(x+3)$, $R(x) = 9$.

P, Q, R are polynomials with no common factors.

Let $P(x) = 0$, we have $x_0 = -3$, which is a singular point

Let $P(x) \neq 0$, we have $x_0 \neq -3$, so ordinary points are $\{x \mid x \neq -3\}$. 2

For $x_0 = -3$, $\lim\limits_{x\to x_0} (x-x_0)\dfrac{Q(x)}{P(x)} = -5$

$\lim\limits_{x\to x_0} (x-x_0)^2 \dfrac{R(x)}{P(x)} = 9$. are finite

so $x_0 = -3$ is a regular singular point. 3

(b). This is an Euler equation.

The corresponding characteristic equation is

$r^2 + (-5-1)r + 9 = 0$

$r_1 = r_2 = 3$ 3

So the general solution is 3

$y = c_1|x+3|^3 + c_2|x+3|^3 \ln|x+3|$

4 (2+2)

Laplace transform to solve the initial value problem:

$$y''(t) + 4y(t) = f(t) + \delta(t - 1.5); \quad y(0) = 0, \quad y'(0) = 0;$$

where

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ (t-4)/6, & 4 \le t < 10, \\ 1, & t \ge 10. \end{cases}$$

[20]

Let $Y = L(y)$.

$$f(t) = ((t-4)/6)(u_4(t) - u_{10}(t)) + 1 \times u_{10}(t)$$

$$= (\tfrac{1}{6}(t-4))\, u_4(t) + (-\tfrac{1}{6}(t-10))\, u_{10}(t)$$

4'

$$s^2Y - sy(0) - y'(0) + 4Y = (s^2+4)Y$$
$$= L\{f(t)\} + L\{\delta(t-1.5)\}$$
$$= e^{-4s}\cdot\frac{1}{6}\frac{1}{s^2} + e^{-10s}\cdot\left(-\frac{1}{6}\frac{1}{s^2}\right) + e^{-1.5s}$$

4'

$$\Rightarrow Y = \frac{e^{-4s}}{6s^2(s^2+4)} - \frac{1}{6s^2(s^2+4)}e^{-10s} + \frac{1}{s^2+4}e^{-1.5s}$$

4'

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{(s^2+4)}\right)$$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{4}t - \frac{1}{8}\sin 2t \qquad L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}e^{-4s}\right) = u_4(t)\left(\frac{1}{4}(t-4) - \frac{1}{8}\sin 2(t-4)\right)$$

6'

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}e^{-10s}\right) = u_{10}(t)\left(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)\right)$$

$$L^{-1}\left(\frac{1}{s^2+4}e^{-1.5s}\right) = u_{1.5}(t)\,\frac{1}{2}\sin(2t-3)$$

Therefore

$$y = u_4(t)\left(\frac{1}{24}(t-4) - \frac{1}{48}\sin(2t-8)\right) - u_{10}(t)\left(\frac{1}{24}(t-10) - \frac{1}{48}\sin(2t-20)\right)$$

$$+ \frac{1}{2}u_{1.5}(t)\sin(2t-3)$$

2' □

pply the **Laplace transform** to solve the initial value problem:

$$2y'' + y' + 2y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$

[15]

Solution. Let $Y = \mathcal{L}\{y\}$.

apply Laplace transform to both hands sides of the equation,

we have : $2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2Y = \frac{1}{s-2}$

$\Rightarrow 2(s^2 Y - s y(0) - y'(0)) + (sY - y(0)) + 2Y = \frac{1}{s-2}$

Since $y(0) = 0$, $y'(0) = 1$,

we have $2s^2 Y - 2 + sY + 2Y = \frac{1}{s-2}$

$(2s^2 + s + 2)Y = 2 + \frac{1}{s-2} = \frac{2s-3}{s-2}$

$Y = \frac{2s-3}{(s-2)(2s^2+s+2)}$ 5

$= \frac{A}{s-2} + \frac{Bs+C}{2s^2+s+2}$ 4

$2(s^2 + \frac{1}{2}s + 1)$

$2(s^2 + \frac{1}{2}s + \frac{1}{16} + \frac{15}{16})$

$2(s + \frac{1}{4})^2 + \frac{15}{8}$

$= \frac{2As^2 + As + 2A + Bs^2 + Cs - 2Bs - 2C}{(s-2)(2s^2+s+2)}$

$= \frac{(2A+B)s^2 + (A-2B+C)s + 2A - 2C}{(s-2)(2s^2+s+2)}$

$\Rightarrow \begin{cases} 2A + B = 0 \\ A - 2B + C = 2 \\ 2A - 2C = -3. \end{cases} \Rightarrow \begin{cases} A = \frac{1}{12} \\ B = -\frac{1}{6} \\ C = \frac{19}{12} \end{cases}$ 3

$\Rightarrow Y = \frac{1}{12} \cdot \frac{1}{s-2} + \frac{-\frac{1}{6}(s + \frac{1}{4} - \frac{39}{4})}{2[(s+\frac{1}{4})^2 + \frac{15}{16}]} = \frac{1}{12} \cdot \frac{1}{s-2} - \frac{1}{12}\left[\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{39}{4} \cdot \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right.$

$\Rightarrow y = \mathcal{L}^{-1}\{Y\} = \frac{1}{12}e^{2t} - \frac{1}{12}e^{-\frac{1}{4}t}\cos(\frac{\sqrt{15}}{4}t) + \frac{39}{48} \cdot \frac{4}{\sqrt{15}} e^{-\frac{1}{4}t}\sin(\frac{\sqrt{15}}{4}t)$.

5. Solve the initial value problem:

$$x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x; \qquad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

[15]

Solution. Let $A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$.

$\det(A - rI) = \det\begin{pmatrix} -\frac{1}{2}-r & 1 \\ -1 & -\frac{1}{2}-r \end{pmatrix} = (-\frac{1}{2}-r)^2 + 1 = 0.$ 1

$\Rightarrow r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i$ 2+2

For $r_1 = -\frac{1}{2} + i$, let $\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$,

let $(A - r_1 I)\vec{\xi} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, 3

then $-i\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = i\xi_1 \Rightarrow \vec{\xi} = \begin{pmatrix} \xi_1 \\ i\xi_1 \end{pmatrix}$.

Choose $\vec{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

The $\vec{\xi} \cdot e^{r_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(\frac{1}{2}+i)t} = e^{-\frac{1}{2}t}\begin{pmatrix} 1 \\ i \end{pmatrix}(\cos t + i\sin t)$

$= e^{-\frac{1}{2}t}\begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}$

$Re\{\vec{\xi} e^{r_1 t}\} = e^{-\frac{1}{2}t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$. $Im\{\vec{\xi} e^{r_1 t}\} = e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

So the general solution is

$\vec{x} = e^{-\frac{1}{2}t}\left[c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$ 4

Since $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

we have $c_1 = 1$, $c_2 = -1$ 3

So $\vec{x} = e^{-\frac{1}{2}t}\left[\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right] = e^{-\frac{1}{2}t}\begin{pmatrix} \cos t - \sin t \\ -\sin t - \cos t \end{pmatrix}$

6. Consider the initial value problem:

$$y'' + y' - 6y = 0; \quad y(0) = 3, \quad y'(0) = 0.$$

(a) Transform the problem into an initial value problem of first-order system of equations.

(b) Solve the initial value problem obtained in (a) using the method for systems of first-order equations.

(c) Find the solution y of the original initial value problem through the results obtained in (a) and (b). [20]

(a) Let $y = x_1$; $y' = x_2$

Then $x_1' = y' = x_2$

$x_2' + x_2 - 6x_1 = 0$

Let $\vec{x} = (x_1, x_2)^T$

Therefore. $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ 6x_1 - x_2 \end{pmatrix} = A\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

6'

Where $A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$

b) $\det(A - rI) = \begin{vmatrix} -r & 1 \\ 6 & -1-r \end{vmatrix} = r^2 + r - 6 = (r+3)(r-2)$

$\Rightarrow r_1 = -3 , r_2 = 2$

2'

$\Rightarrow (A + 3I)\vec{z}_1 = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3z_1 + z_2 = 0 \Rightarrow \vec{z}_1 = (1, -3)^T$

$(A - 2I)\vec{z}_2 = \begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

4'

$-2z_1 + z_2 = 0 \Rightarrow \vec{z}_2 = (1, 2)^T$

General solution: $\vec{x} = c_1 e^{-3t}\begin{pmatrix}1\\-3\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$

$$\vec{x}(0) = \begin{pmatrix}c_1+c_2\\-3c_1+2c_2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \Rightarrow \begin{cases} c_1 = \frac{6}{5} \\ c_2 = \frac{9}{5}\end{cases}$$

$$\Rightarrow \vec{x} = \frac{6}{5}e^{-3t}\begin{pmatrix}1\\-3\end{pmatrix} + \frac{9}{5}e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$$

4'

(c) $y = x_1 = \frac{6}{5}e^{-3t} + \frac{9}{5}e^{2t}$ 4' $\square$