Lecture 21

Chapter 9: System Characterization from Laplace transform (Analysis)

I. Inverse Laplace Transform by Partial Fraction

II. Causality and Stability

III. Properties of Laplace Transform & Differentiation System Revisited

I. Inverse Laplace Transform by Partial Fraction (9.3)

- Recall that $X(s=\sigma+j\omega) = FT\{x(t)e^{-\sigma t}\}$.
- That means we can apply inverse FT to $X(s)$ along a fixed σ to obtain:

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\sigma+j\omega)e^{j\omega t}d\omega \quad \text{IFT}$$

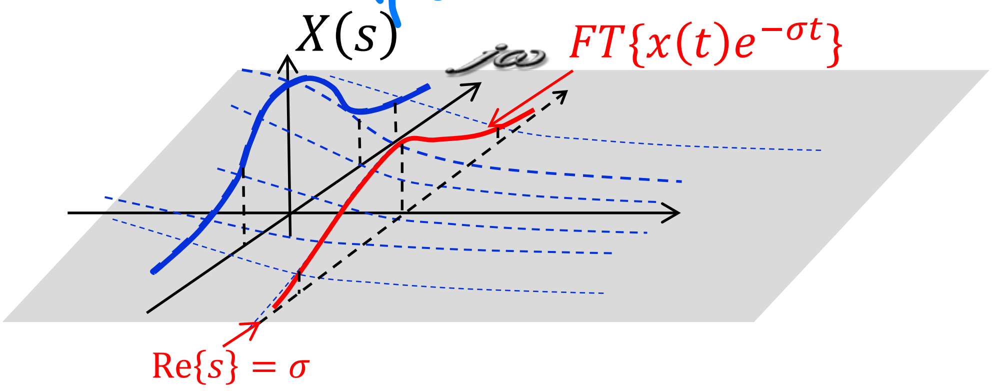
Always do rational form!

Multiplying both sides by $e^{\sigma t}$, we obtain:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega$$

$$= \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds \quad ✓$$

$s=\sigma+j\omega \;\; \Rightarrow d\omega = \frac{1}{j}ds$ A superposition of e^{st}



the inverse LT integral, which is a *contour integral* on the complex s-plane

- For LTs in rational form, instead of contour integration, we typically use ***partial fraction expansion*** to do the inverse transform.

Transforms in Rational Form

Again, a rational Laplace transform is one that is in the form $N(s)/D(s)$. It can be expressed in three alternate forms as we have seen for rational Fourier transform:

1. **<u>The Polynomial Form</u>** which will allow us to write the differential equation

or $H(s)$

$$X(s) = \frac{b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0} \quad \frac{N(s)}{D(s)}$$

– degree smaller than D(s)

want this

We usually assume the order of $N(s)$ < order of $D(s)$ so that $X(s) \to 0$ when $|s| \to \infty$

2. **The Factored Form** which tells us the locations of the poles (α_k) and zeros (β_i):

Form 2a:

zeros

$$X(s) = \frac{b_{N-1}\prod_{i=1}^{N-1}(s-\beta_i)}{\prod_{k=1}^{N}(s-\alpha_k)}$$

poles

a_N assumed normalized to 1

or Form 2b:

$$X(s) = \frac{A\prod_{i=1}^{N-1}\left(1-\frac{s}{\beta_i}\right)}{\prod_{k=1}^{N}\left(1-\frac{s}{\alpha_k}\right)}$$

$1-\frac{s}{\alpha_k}=0$ when $s=\alpha_k \Rightarrow$ Hence a pole

The advantage of Form 2b is function value at $s=0$ is explicitly given:

$$X(0) = A \quad \text{by Form 2b}$$

$$X(0) = \frac{b_{N-1}\prod_{i=1}^{N-1}(-\beta_i)}{\prod_{k=1}^{N}(-\alpha_k)} \quad \text{by Form 2a}$$

3. <u>The Partial Fraction Form</u> which with ROC allows you to find the inverse transform

residues

$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s - \alpha_k}$$

Only the poles are explicit in this form

Inverse LT with Partial Fraction

- With a rational LT in partial fraction form, we can write down the inverse transform as a sum of causal and anti-causal exponentials:

$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s-\alpha_k} \quad \Longrightarrow x(t) = \sum_{k=1}^{N} \chi_k c_k \, e^{\alpha_k t} u(\chi_k t)$$

$u(t)$ or $u(-t)$

where χ_k is a sign variable:
$$\chi_k = \begin{cases} +1 & \text{if ROC is right of } \alpha_k \\ -1 & \text{if ROC is left of } \alpha_k \end{cases}$$

- As a reminder, the following helps us find the residue values c_k:

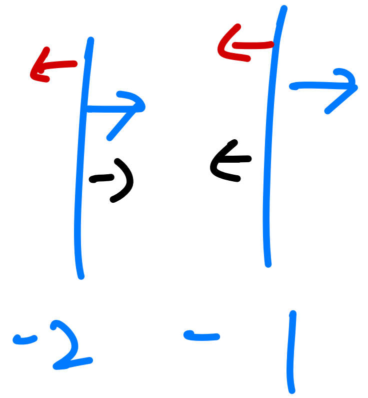
$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s-\alpha_k} \quad \Longrightarrow \quad c_k = (s-\alpha_k)X(s)\Big|_{s=\alpha_k}$$

9.3 The inverse Laplace transform

- **Example 9.9:** Suppose we're given

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} > -1$$

$\alpha_1 = -1 \quad \alpha_2 = -2$



$(e^{-t} - e^{-2t})u(t)$

As discussed in Example 9.8, there are two poles: -1 and –2.

- Partial fraction expansion gives:

$$X(s) = \frac{c_1\,1}{(s+1)} - \frac{c_2\,1}{(s+2)}, \qquad \mathrm{Re}\{s\} > -1$$

$c_1 = \left.\frac{1}{(s+2)}\right|_{s=-1} = 1; \quad c_2 = \left.\frac{1}{(s+1)}\right|_{s=-2} = -1$

- Since the overall ROC of $X(s)$ is the intersection of the two individual ROCs, it implies ROCs of the two terms are both a right-half plane and the inverse transform of both terms are right-sided.

$-(e^{-t} - e^{-2t})\,u(-t)$

ROC to the right of both α_1 and α_2

Therefore, $x(t) = (e^{-t} - e^{-2t})u(t)$

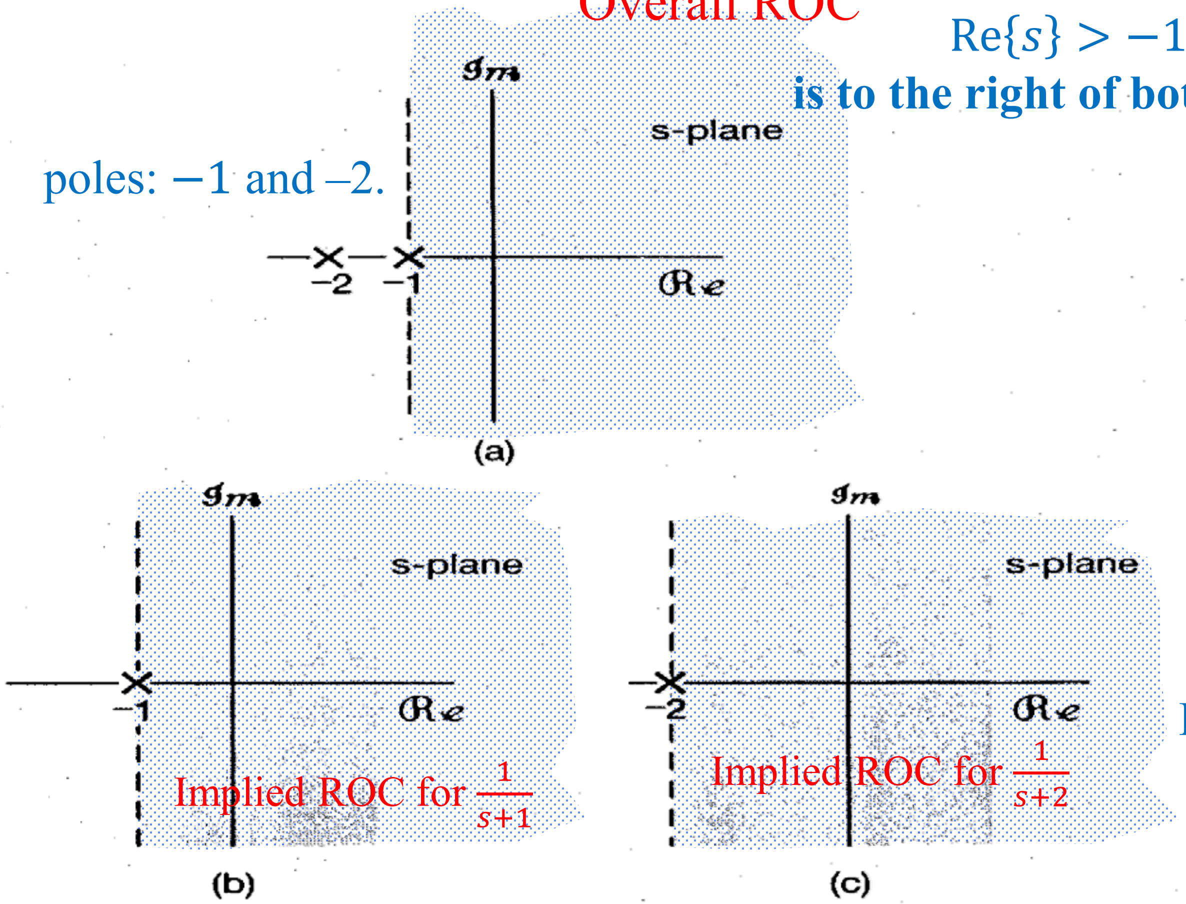
$-e^{-t}u(-t) - e^{-2t}u(t)$

Overall ROC

$\mathrm{Re}\{s\} > -1$

is to the right of both poles

poles: -1 and -2.



Both poles lead to a causal inverse LT

Figure 9.14 Construction of the ROCs for the individual terms in the partial-fraction expansion of $X(s)$ in Example 9.8: (a) pole-zero plot and ROC for $X(s)$; (b) pole at $s = -1$ and its ROC; (c) pole at $s = -2$ and its ROC.

- **Example 9.10:** Suppose we're given

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} < -2$$

Then

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+2)}, \qquad \mathrm{Re}\{s\} < -2$$

which is the same as Example 9.9 but with a different ROC.

- The overall ROC is to the left of the pole at $s = -1$.
  Therefore the inverse LT for $\frac{1}{s+1}$ should be $-e^{-t}u(-t)$ Anti-causal

- The overall ROC is also to the left of the pole at $s = -2$.
  Therefore the inverse LT for $\frac{1}{s+2}$ should be $-e^{-2t}u(-t)$

- Therefore, $x(t) = (-e^{-t} + e^{-2t})u(-t)$

- **Example 9.11:** Suppose we're given

A strip

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}, \qquad -2 < \mathrm{Re}\{s\} < -1$$

This time, the overall ROC is to the left of the pole at $s = -1$ and to the right of the pole at $s = -2$.

- Therefore the inverse Laplace transform is

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

Lecture 21

Chapter 9: System Characterization from Laplace transform

I. Inverse Laplace Transform through Partial Fraction Expansion

II. Causality and Stability

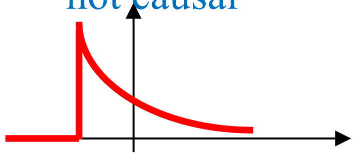
III. Properties of Laplace Transform & Differentiation System Revisited

II. Causality and Stability (9.7)

9.7.1 Causality

- The impulse response of a causal LTI system is right-sided. Hence the ROC of the system function for a causal system must be a right-half plane (RHP).
- However, ROC being RHP means impulse response is right-sided but not necessarily causal.
- But if the system function is rational and ROC is a RHP, then the system is causal, because the inverse of a rational LT must be a combination of causal and anti-causal exponentials.

Right-sided but not causal



- **Examples:**

ROC a right-half plane $\Rightarrow$ Impulse response right-sided

$H(s)$ rational +ROC a RHP $\Rightarrow$ Impulse response is causal

rational

$$H(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} > -1 \quad \rightarrow \text{Causal}$$

RHP

✓

ROC is not a RHP. Impulse response is not causal.

$$H(s) = \frac{1}{(s+1)(s+2)}, \qquad -2 < \mathrm{Re}\{s\} < -1 \quad \rightarrow \text{Non-causal}$$

A strip

✗

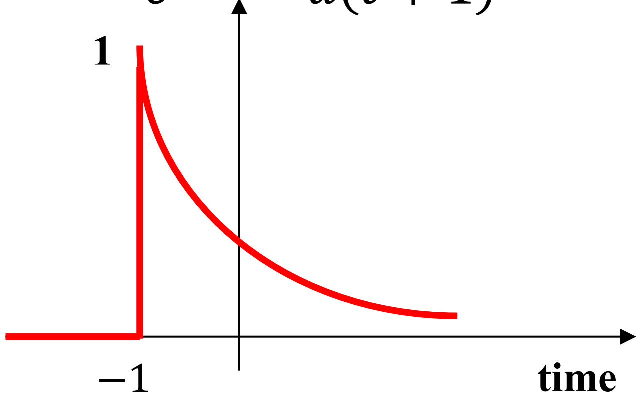
→ causal → non causal!?!

9.7.1 Causality (cont.)

If $H(s)$ is not rational, a right-sided impulse response is not necessarily causal, as illustrated below

$e^{-t}u(t)$ time advanced by 1

- **Example**: $h(t) = e^{-(t+1)}u(t+1)$, a causal exponential shifted to left by 1.



$e^{-(t)}u(t)$ time advanced by 1

We can show its LT is:

Not a polynomial; not rational

$$e^{-(t+1)}u(t+1) \overset{L}{\leftrightarrow} H(s) = \frac{e^{s}}{s+1}, \qquad \mathrm{Re}\{s\} > -1$$

RHP

$h(t)$ is nonzero for $-1 < t < 0$, hence the system is not causal, but its ROC is a RHP.

ROC being a RHP does not imply causality in this case because $H(s)$ is not rational.

9.7.2 Stability

An LTI system is BIBO stable if and only if the ROC of its system function includes the $j\omega$-axis.

It is because all the following conditions are equivalent:

- ROC of system function includes the $j\omega$-axis, central idea!
- The Fourier transform converges, $FT\{h(t)\} = H(j\omega) = H(s)|_{s=j\omega} < \infty \ \forall\omega$
- $h(t)$ is absolute integrable: $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ $h(t)$ absolute integrable $\Rightarrow H(j\omega) < \infty \ \forall\omega$
- Frequency response of the system exists,

Frequency response = $H(j\omega)$. The existence of frequency response means all sinusoidal inputs leads to a bounded output; i.e., $|H(j\omega)| < \infty \ \forall\omega$

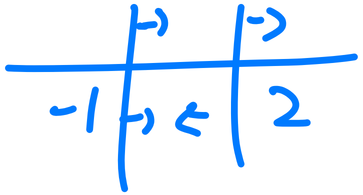
- **<u>Example 9.20</u>:**

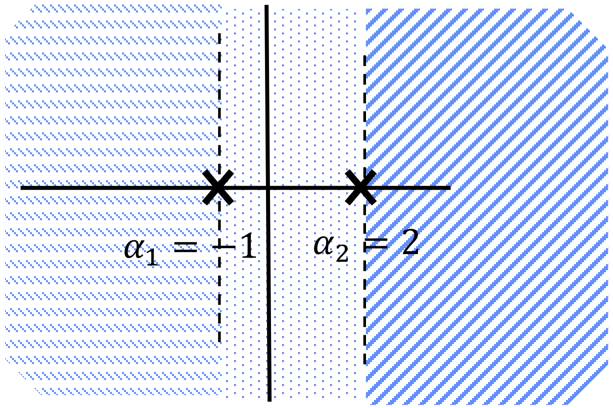
$c_1 = \left.\frac{s-1}{(s-2)}\right|_{s=-1} = \frac{-2}{-3} = \frac{2}{3}$ $\quad c_2 = \left.\frac{s-1}{(s+1)}\right|_{s=2} = \frac{1}{3}$

Let

$$H(s) = \frac{\overset{\beta_1 = 1}{s-1}}{(s+1)(s-2)} = \frac{2/3}{(s+1)} + \frac{1/3}{(s-2)}$$

$\alpha_1 = -1 \quad \alpha_2 = 2$

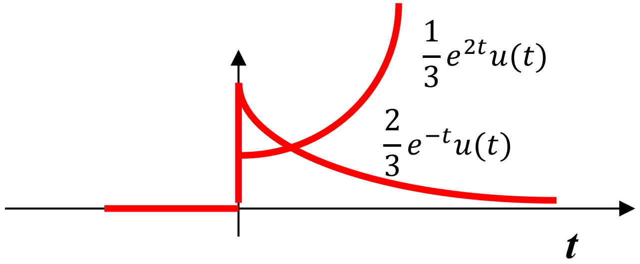




There are poles with two different real parts. Therefore, there can be three different $h(t)$'s corresponding to the three possible ROC's

causal inverse for both terms — RHP

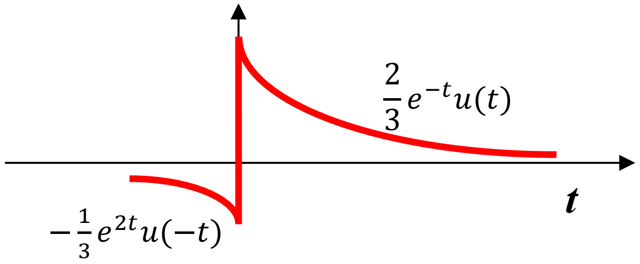
1. $h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$ for $\mathrm{Re}\{s\} > 2$ X



Combination of causal and anti-causal inverses — strip

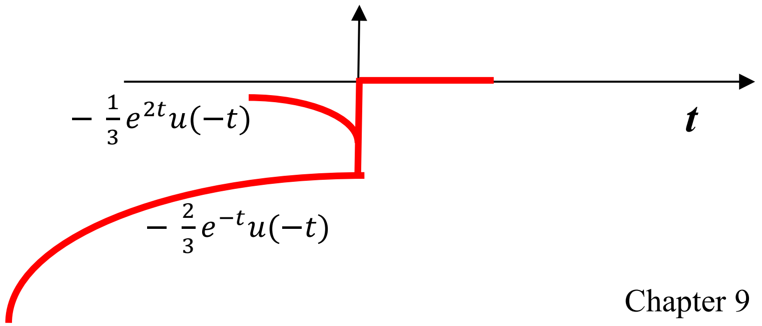
2. $h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$ for $-1 < \mathrm{Re}\{s\} < 2$ ✓

Only this case is stable as the ROC includes the $j\omega$-axis



Anti-causal inverse for both terms — LHP

3. $h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)$ for $\mathrm{Re}\{s\} < -1$ X



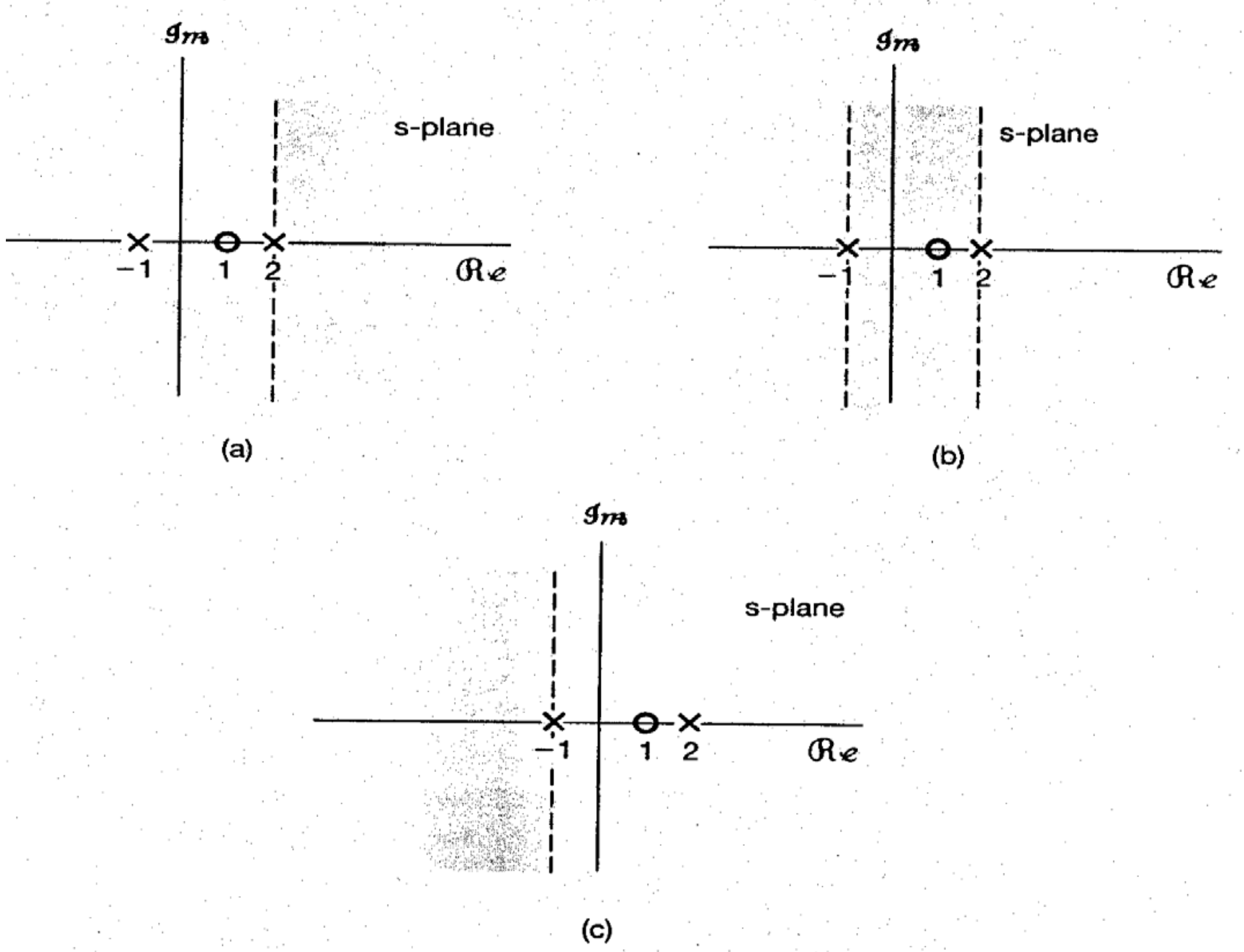


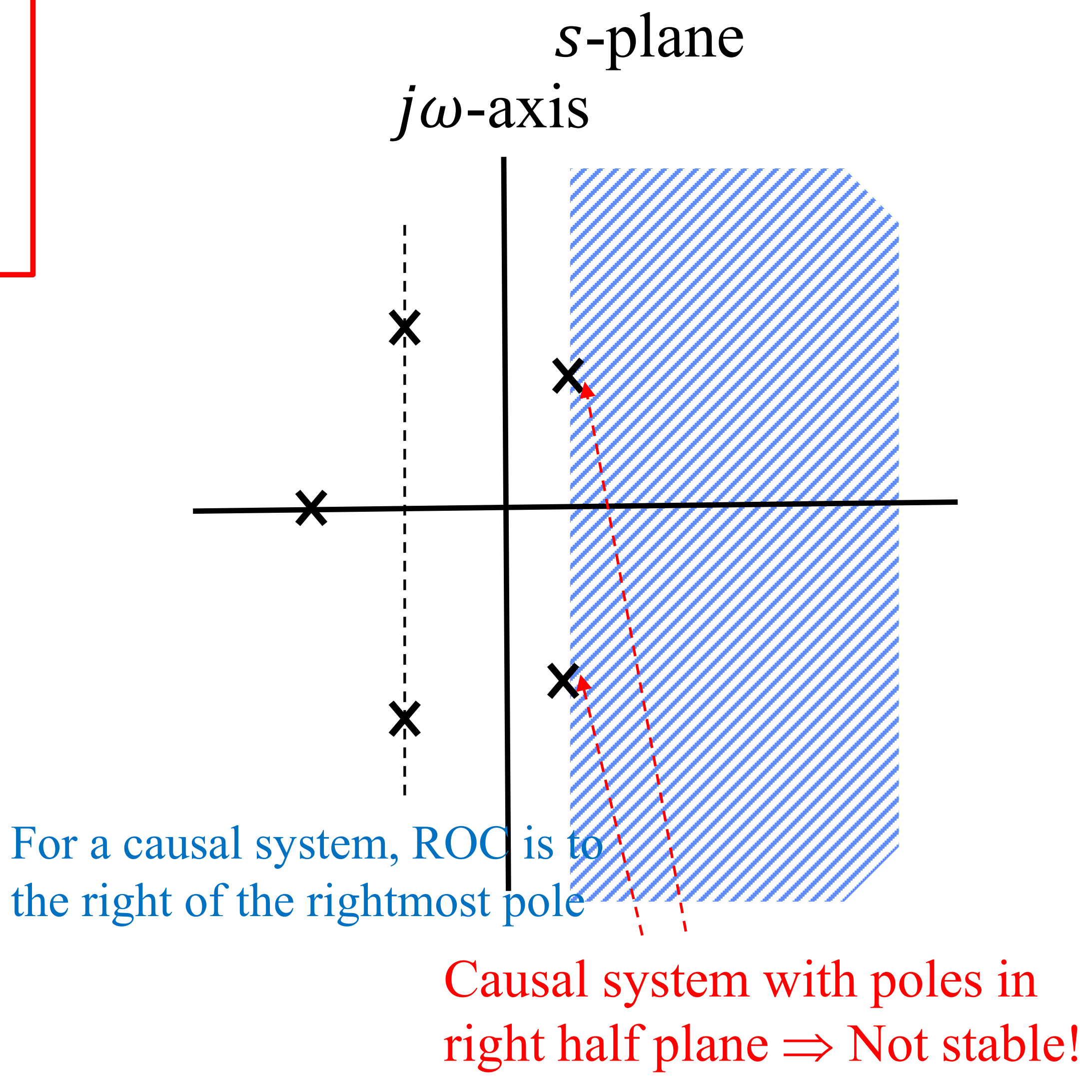
Figure 9.25 Possible ROCs for the system function of Example 9.20 with poles at $s = -1$ and $s = 2$ and a zero at $s = 1$: (a) causal, unstable system; (b) noncausal, stable system; (c) anticausal, unstable system.

Poles in RHP!!

- A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane, i.e., the rightmost pole of $H(s)$ must be to the left of the $j\omega$ axis.

The ROC for a causal system is the intersection of RHP's, and therefore is to the right of the rightmost pole, and the ROC must include the $j\omega$-axis for system to be stable.

- What about an anti-causal system (an anti-causal system is one such that $h(t) = 0 \; \forall t > 0$)? An anti-causal system is stable iff all poles are to the right of the $j\omega$-axis.



Examples: Determine if the following four systems are causal, stable.

$$H_1(s) = \frac{1}{(s+1)(s+2)}; \quad \mathrm{Re}\{s\} > -1$$

causal, stable.

$$H_2(s) = \frac{1}{(s+1)(s+2)}; \quad -2 < \mathrm{Re}\{s\} < -1$$

Not causal, not stable.

$$H_3(s) = \frac{(s-3)^2}{(s-1)(s+2)}; \quad -2 < \mathrm{Re}\{s\} < 1$$

Not causal, stable.

$$H_4(s) = \frac{s-5}{s^2+5s+6}; \quad \mathrm{Re}\{s\} > -2$$

Causal, stable.

Lecture 21

Chapter 9: System Characterization from Laplace transform

I. Inverse Laplace Transform through Partial Fraction Expansion

II. Causality and Stability

III. Properties of Laplace Transform & Differentiation System Revisited

III. Properties of Laplace transform (9.5)

- FT is just a special case of LT:

s, the complex frequency, captures rate of exponential growth and freq of oscillation in one variable

$s = j\omega;\ \ \mathrm{Re}\{s\} = 0$

LT: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt;$

FT: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

LT and FT convert signals from time domain to frequency domain

- Hence, properties of LT are similar to those of FT, and can be proven in similar ways.
- But for LT we must also consider how the ROC is affected in different situations

Because the ROC conveys information about stability, right-sidedness/left-sidedness

Table 9.1 - Properties of LT

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{Re}\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Weighted sum in time ↔ weighted sum in frequency

Time shift by τ ↔ multiplication by $e^{s\tau}$

Convolution in time ↔ multiplication in freq

Differentiation/integration in time ↔ multiplication/division by freq in freq

For LT of weighted sum to converge, LT of individual terms must converge

ROC may be larger than the intersection of individual ROCs because of pole cancellations

Error in some editions of text

AKW

The Key LT Property – The Differentiation Property

- To derive the differentiation property, we differentiate both sides of the inverse LT integral:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(\overbrace{\frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds}^{\text{Inverse LT integral}}\right) = \overbrace{\frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} sX(s)\, e^{st}\, ds}^{\text{Inverse LT integral for } sX(s)}$$

limits of integration are independent of t;
need to differentiate integrand only

$$\frac{d}{dt}e^{st} = s\, e^{st}$$

Therefore:

$$\frac{d}{dt}x(t) \overset{L}{\leftrightarrow} sX(s)$$

If $X(s)$ has a pole at $s = 0$, multiplying $X(s)$ by s cancels this pole and may expand the ROC (if this pole bounds the ROC). Therefore, the new ROC is at least the original ROC.

Tables of LT pairs (Reference Only)

Laplace transforms of various elementary functions are similar in form to their Fourier transforms, with s replacing $j\omega$.

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

LCCDE (Linear Constant Coefficient Differential Equation) Revisited

- Applying Laplace transforms to both sides of an LCCDE, and using the linearity and differentiation property of LT, we obtain:

$$LT\left\{\frac{d}{dt}x(t)\right\} = sX(s)$$

$$\Rightarrow LT\left\{\frac{d^k}{dt^k}x(t)\right\} = s^k X(s)$$

$$\underbrace{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}}_{\text{LCCDE}} \Rightarrow \sum_{k=0}^{N} a_k\, s^k Y(s) = \sum_{k=0}^{M} b_k\, s^k X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k\, s^k}{\sum_{k=0}^{N} a_k\, s^k} = \frac{N(s)}{D(s)}$$

We have a system function $H(s)$ in rational form. We have seen this earlier in frequency response. We are simply replacing $j\omega$ by s.

- We can reformulate our 2nd order ODE in terms of s:

$$y^{(2)}(t) + a_1 y^{(1)}(t) + a_0 y(t) = b_1 x^{(1)}(t) + b_0 x(t)$$

$$\Rightarrow \ H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{b_1 s + b_0}{(s-\alpha_1)(s-\alpha_2)} = \frac{c_1}{(s-\alpha_1)} + \frac{c_2}{(s-\alpha_2)}$$

Poles: quadratic roots

Partial Fraction Expansion

- Now we can think of the roots of $D(s)$ as poles of the system function. Visualizing the location of the poles helps us understand the characteristics of the system.

- Recall the definition of natural frequency ω_n and ζ parameter :

ζ characterizes the poles: ${a_1}^2 > 4a_0$ if $|\zeta| > 1$

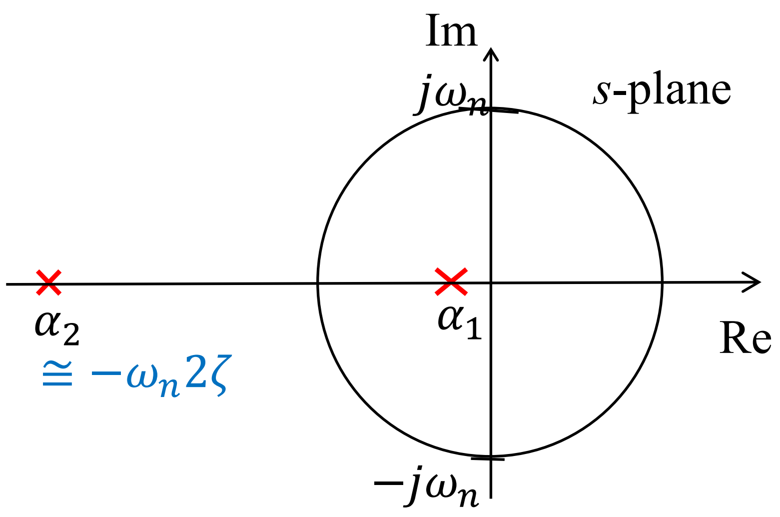
$$\boxed{\omega_n = \sqrt{a_0};\ \zeta = \frac{a_1}{2\sqrt{a_0}}} \qquad \boxed{\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_0}}{2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)}$$

scaling in frequency

ω_n provides a scaling in frequency, and ζ characterizes the system

$\zeta > 1$: Both roots are real and < 0 ; system has no oscillation and is stable for causal systems.

1. $\zeta \gg 1$ Over-damped



near 0

If $\zeta \gg 1$, then one root is at $\alpha_2 = -\omega_n(2-\varepsilon)\,\zeta$ where $\epsilon = 1 - \sqrt{1 - \frac{1}{\zeta^2}}$

but the other root is at $\alpha_1 = -\omega_n \varepsilon\zeta = 0^-$. Means "a little smaller than"

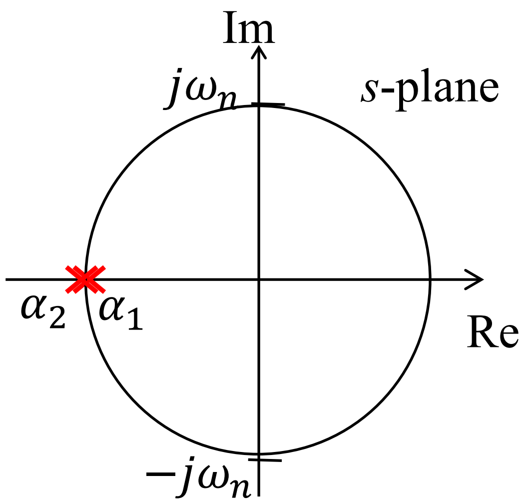
The pole at 0^- means the response decays very slowly in time. The system is overdamped.

$$\boxed{\omega_n = \sqrt{a_0};\ \zeta = \frac{a_1}{2\sqrt{a_0}}}$$

$$\boxed{\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_0}}{2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)}$$

2. $\zeta = 1$ ($\alpha_1 = \alpha_2 = -\omega_n$)

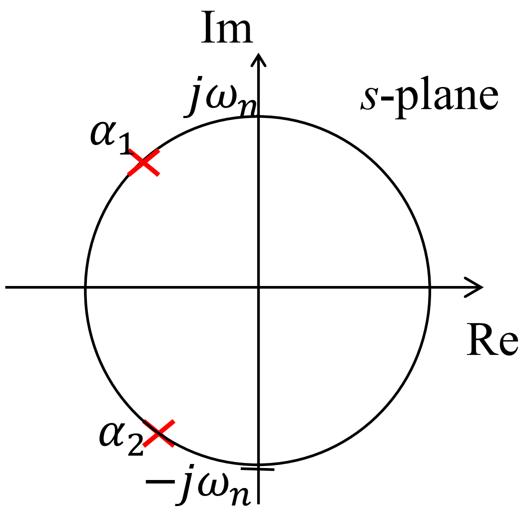
Critically damped



$\zeta = 1$: the roots are equal and the impulse response decays at the fastest rate

3. $1 > \zeta > 0$

Oscillatory



$\alpha_1 = \alpha_2{}^*$;

$\alpha_1\ \alpha_2 = a_0 = \omega_n{}^2$

$|\alpha_1| = |\alpha_2| = \omega_n$

so α_1, α_2 are on circle of radius ω_n

$1 > \zeta > 0$: the roots become complex and system is oscillatory

4. $\zeta \to 0^+$ Under-damped

Means "a little greater than"

Impulse response is a damped oscillation that decays very slowly. System may have large response when stimulated at the natural frequency:

$$\cos(\omega_1 t) \to |H(j\omega_1)| \cos(\omega_1 t + \measuredangle H(j\omega_1))$$

Magnitude response   Phase response

For a pole-only 2nd-order system,

$$H(j\omega_1) = \frac{c}{(j\omega_1 - \alpha_1)(j\omega_1 - \alpha_2)}$$

The magnitude response is:

$$|H(j\omega_1)| = \frac{c}{|j\omega_1 - \alpha_1||j\omega_1 - \alpha_2|}$$

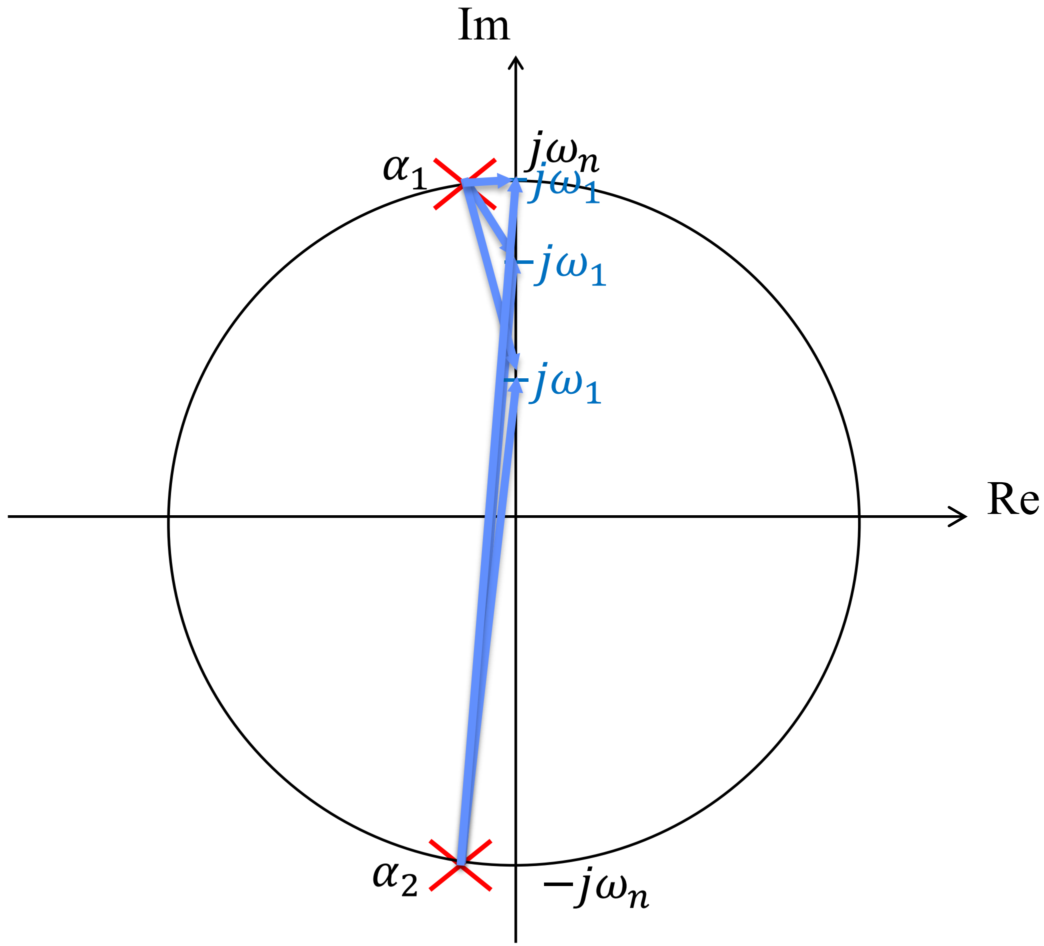
Distant to α_1 Distant to α_2

Real part of the root is very small: root is very close to the $j\omega$ axis, near $\pm j\omega_n$

$$\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

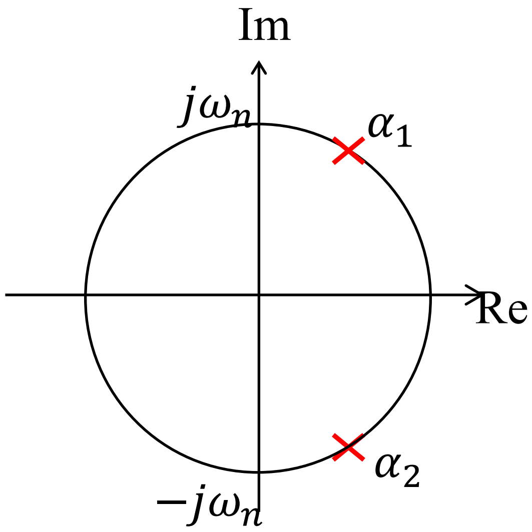
Radical is near $\sqrt{-1} = j$

$\mathrm{Re}\{\alpha_1\} = \mathrm{Re}\{\alpha_2\} = -\zeta\omega_n$ if roots are complex



5. $\zeta \leq 0$

Unstable



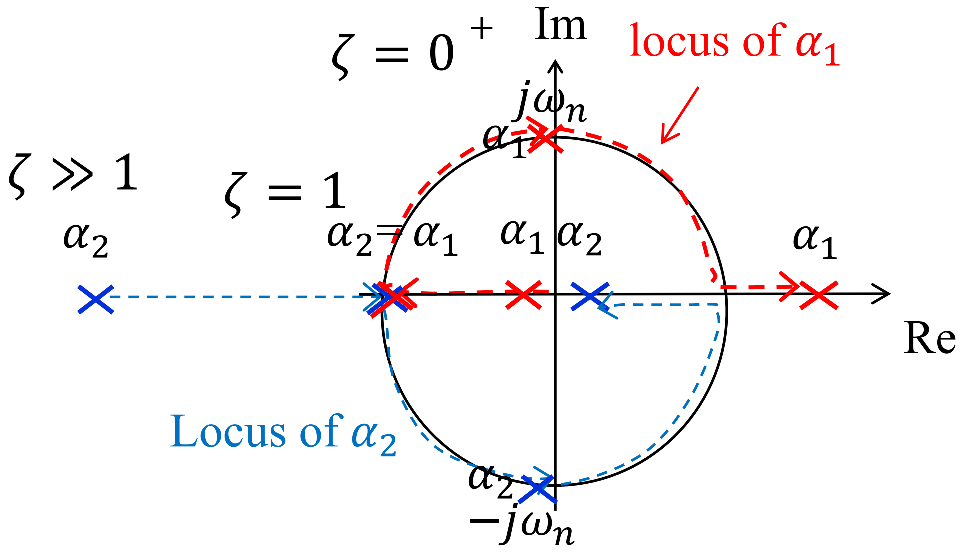
$\mathrm{Re}\{\alpha_1\}, \mathrm{Re}\{\alpha_2\} \geq 0$

Impulse response grows or stays flat toward the right in time

Unstable for causal system

Root loci plot in feedback systems

- For the 2nd-order system we can trace the trajectories (loci) of the two roots as shown below.



- How do we keep a robot walking? How do we keep the Segway in balance?
- We use *feedback control* and we need to understand the root loci in these systems!