MATH2411 questions

Midterm Multiple Choice

Question: If A, B are two events, $P(A) = P(B) = 0.5$, $P(A \cup B) = 1$, which of the following is true?
A. $A \cup B$ is sample space
B. $A \cap B$ is null
C. $P(A^c \cup B^c) = 1$
D. $P(A - B) = 1$

Solution:

Step 1: Understanding the given probabilities

From inclusion-exclusion:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$1 = 0.5 + 0.5 - P(A \cap B)$$

$$1 = 1 - P(A \cap B) \implies P(A \cap B) = 0$$

Step 2: Important distinction

We know $P(A \cap B) = 0$, but this means $A \cap B$ has **probability zero**, not necessarily that it's the empty set. This is crucial for evaluating the options.

Step 3: Constructing a counterexample

Consider the uniform distribution on sample space $S = [0, 1]$.

Define:

$$A = (0, 0.5], \quad B = [0.5, 1]$$

Then:

- $P(A) = 0.5$, $P(B) = 0.5$
- $A \cup B = (0, 1]$, so $P(A \cup B) = 1$
- $A \cap B = \{0.5\}$, $P(A \cap B) = 0$

This satisfies all given conditions.

Step 4: Evaluate each option

A. $A \cup B$ **is sample space**
In our counterexample: $A \cup B = (0, 1] \neq [0, 1] = S$
The point 0 is in S but not in $A \cup B$.
So this is **FALSE**.

B. $A \cap B$ **is null**
"Null" in set theory means empty set.
In our counterexample: $A \cap B = \{0.5\} \neq \emptyset$
So this is **FALSE**.

C. $P(A^c \cup B^c) = 1$
By De Morgan's Law: $A^c \cup B^c = (A \cap B)^c$

$$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 0 = 1$$

This holds **regardless** of the specific example, as long as $P(A \cap B) = 0$.
So this is **TRUE**.

D. $P(A - B) = 1$

$$A - B = A \setminus (A \cap B)$$

In our counterexample: $A - B = (0, 0.5)$, $P(A - B) = 0.5$
In general: $P(A - B) = P(A) - P(A \cap B) = 0.5 - 0 = 0.5$
So this is **FALSE**.

Conclusion:

Only option C is always true.

$$\boxed{C}$$

Midterm Long Question

Question: A company has m staff. At the later 3 consecutive days, each staff can choose one day as holiday. Assume those m staff choose independently, what is the probability for at least one day nobody chooses it as holiday?

Solution:

Step 1: Define events

Let A_i ="Day i has nobody choose it" for $i = 1, 2, 3$

We want $P(A_1 \cup A_2 \cup A_3)$

Step 2: Probabilities of individual events

Each staff picks one of 3 days with probability 1/3

$$P(A_1) = \left(\frac{2}{3}\right)^m \quad \text{(all } m \text{ staff avoid day 1)}$$

Similarly: $P(A_i) = \left(\frac{2}{3}\right)^m$ for $i = 1, 2, 3$

Step 3: Probabilities of intersections

$$P(A_1 \cap A_2) = \left(\frac{1}{3}\right)^m \quad \text{(all must pick day 3)}$$

$$P(A_1 \cap A_3) = \left(\frac{1}{3}\right)^m \quad \text{(all must pick day 2)}$$

$$P(A_2 \cap A_3) = \left(\frac{1}{3}\right)^m \quad \text{(all must pick day 1)}$$

$$P(A_1 \cap A_2 \cap A_3) = 0 \quad \text{(impossible, no day chosen)}$$

Step 4: Inclusion-Exclusion

$$P\left(\bigcup_{i=1}^{3} A_i\right) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$$

$$= 3\left(\frac{2}{3}\right)^m - 3\left(\frac{1}{3}\right)^m + 0$$

Final Answer:

$$\boxed{3\left(\frac{2}{3}\right)^m - 3\left(\frac{1}{3}\right)^m}$$

Comments

This is not a good idea to use complement to execute this question. The counting will then becomes complicated. Which may result in getting 0 out of 5 marks.

Final Short Question

Question:

$$\frac{(X_1 - 2X_2)^2}{a} + \frac{(3X_3 - 4X_4)^2}{b} \sim \chi^2(m)$$

X_1, X_2, X_3, X_4 are samples from $N(0, 4)$.
Find the values of a, b, m.

Solution:

Step 1: Distribution of $X_1 - 2X_2$

Let $U = X_1 - 2X_2$:

- Mean: $0 - 2 \times 0 = 0$
- Variance: $\text{Var}(X_1) + 4\text{Var}(X_2) = 4 + 4 \times 4 = 4 + 16 = 20$

So $U \sim N(0, 20)$

Step 2: Distribution of $3X_3 - 4X_4$

Let $V = 3X_3 - 4X_4$:

- Mean: 0
- Variance: $9\text{Var}(X_3) + 16\text{Var}(X_4) = 9 \times 4 + 16 \times 4 = 36 + 64 = 100$

So $V \sim N(0, 100)$

Step 3: Standardize to $N(0, 1)$

$$\frac{U}{\sqrt{20}} \sim N(0,1) \implies \frac{U^2}{20} \sim \chi^2(1)$$
$$\frac{V}{10} \sim N(0,1) \implies \frac{V^2}{100} \sim \chi^2(1)$$

Step 4: Independent chi-squares

Since X_1, X_2 are independent of X_3, X_4, $U^2/20$ and $V^2/100$ are independent $\chi^2(1)$ variables.
Their sum:

$$\frac{U^2}{20} + \frac{V^2}{100} \sim \chi^2(2)$$

Step 5: Match with given form

Given form is:

$$\frac{U^2}{a} + \frac{V^2}{b} \sim \chi^2(m)$$

Comparing: $a = 20$, $b = 100$, $m = 2$

Final Answer:

$$\boxed{a = 20,\ b = 100,\ m = 2}$$

Final Long Question

Question: If $X \sim N(0, \sigma^2)$

Part (a): Find the MME of σ^2 and check its unbiasedness

Solution:

Method of moments uses $E[X^2] = \sigma^2$.

The first sample moment for X^2:

$$\hat{\sigma}^2_{\text{MM}} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Unbiasedness check:

$$E[\hat{\sigma}^2_{\text{MM}}] = E[X_1^2] = \sigma^2$$

So it is unbiased.

Part (b): Find CDF of $|X|$ in terms of CDF of X, $F(x)$

Let $Y = |X|$. For $y \geq 0$:

$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y)$$

For $X \sim N(0, \sigma^2)$, symmetry gives $F_X(-y) = 1 - F_X(y)$:

Proof: $F_X(-y) = \int_{-\infty}^{-y} f_X(t)dt = \int_{y}^{\infty} f_X(t)dt = 1 - F_X(y)$

Thus:

$$F_Y(y) = F_X(y) - [1 - F_X(y)] = 2F_X(y) - 1, \quad y \geq 0$$

For $y < 0$, $F_Y(y) = 0$.

Part (c): Find PDF of $|X|$ in terms of PDF of X, $f(x)$

Differentiate the CDF:

$$f_Y(y) = \frac{d}{dy}[2F_X(y) - 1] = 2f_X(y), \quad y \geq 0$$

and $f_Y(y) = 0$ for $y < 0$.

Explicitly:

$$f_Y(y) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-y^2/(2\sigma^2)}, \quad y \geq 0$$

Part (d): Find $E[|X|]$

Let $Z \sim N(0, 1)$, so $X = \sigma Z$.

Then $|X| = \sigma |Z|$.

Known result for half-normal: $E[|Z|] = \sqrt{\frac{2}{\pi}}$

Thus:

$$E[|X|] = \sigma E[|Z|] = \sigma\sqrt{\frac{2}{\pi}}$$

Part (e): Construct MME of σ based on $|X|$

From part (d): $E[|X|] = \sigma\sqrt{\frac{2}{\pi}}$

So $\sigma = E[|X|]\sqrt{\frac{\pi}{2}}$

Method of moments estimator:

$$\hat{\sigma}_{\text{MM}} = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{n}\sum_{i=1}^{n} |X_i|$$

Part (f): Check unbiasedness of $\hat{\sigma}_{\text{MM}}$

$$E[\hat{\sigma}_{\text{MM}}] = \sqrt{\frac{\pi}{2}} \cdot E[|X_1|] = \sqrt{\frac{\pi}{2}} \cdot \sigma\sqrt{\frac{2}{\pi}} = \sigma$$

Thus $\hat{\sigma}_{\text{MM}}$ is unbiased for σ.

Summary of Long Question Answers:

1. $\hat{\sigma}^2_{\text{MM}} = \frac{1}{n}\sum X_i^2$, unbiased
2. $F_Y(y) = 2F_X(y) - 1$ for $y \geq 0$, 0 otherwise
3. $f_Y(y) = 2f_X(y)$ for $y \geq 0$, 0 otherwise
4. $E[|X|] = \sigma\sqrt{2/\pi}$
5. $\hat{\sigma}_{\text{MM}} = \sqrt{\pi/2} \cdot \frac{1}{n}\sum |X_i|$
6. $\hat{\sigma}_{\text{MM}}$ is unbiased

Note: The midterm MC question correction shows the importance of distinguishing between "probability zero" and "empty set" in probability theory.