Ch7.2: Optimal Receiver

Information source and input transducer → Source Coding → Channel Coding → Modulator → Channel → Demodulator (Matched Filter) → Channel Decoding → Source Decoding → Information sink and output transducer

If $p(0) \neq p(1)$
waveform may not be square pulses!

- Questions to be answered:
  - ❑ **BER for General Signals & Receivers: Performance evaluation for general signals**
    - ❑ **Optimum Threshold**
  - ❑ **The Matched Filter: The optimal receiver structure**
    - ❑ **Optimal Receiver**
    - ❑ **Optimal Signals**

Ch7.2: Matched Filter

Summary/Outline

Target: minimize Pe

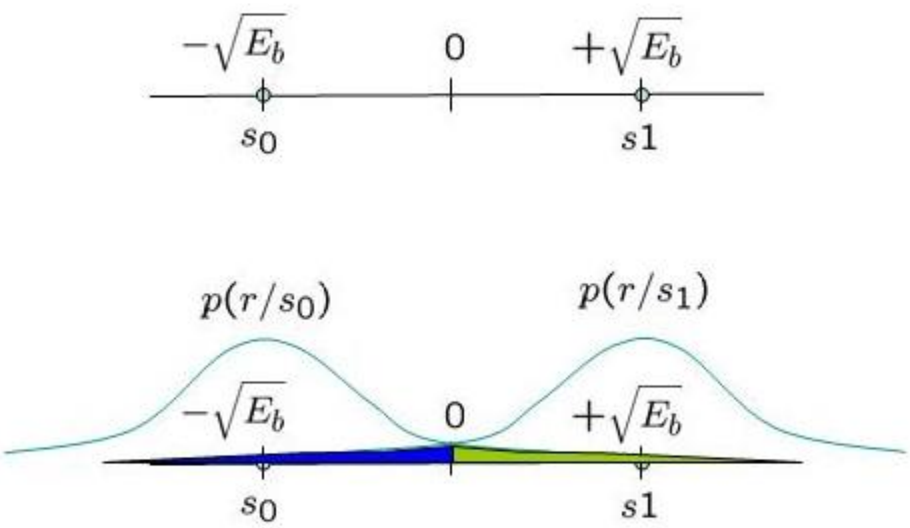
- We have introduced very important concepts
  - Transmission of bits
  - Baseband binary communications
  - Computation of error probability P_e as a function of signal energy per bit to noise density E_b/N_o.

Optimal!

- The above discussion were based on some assumptions (e.g. integrate-dump receiver)
- We will now talk about the ***optimal receiver structure***, i.e. a receiver that can minimize the error probability P_e for a given E_b/N_o

Ch7.2: Optimal Receiver

- **Error Probability for General Signals & Receivers**
  - **Optimum Threshold**
- Input-Output Relation
- The Matched Filter
  - Optimal Receiver
  - Examples
  - Optimal Signals

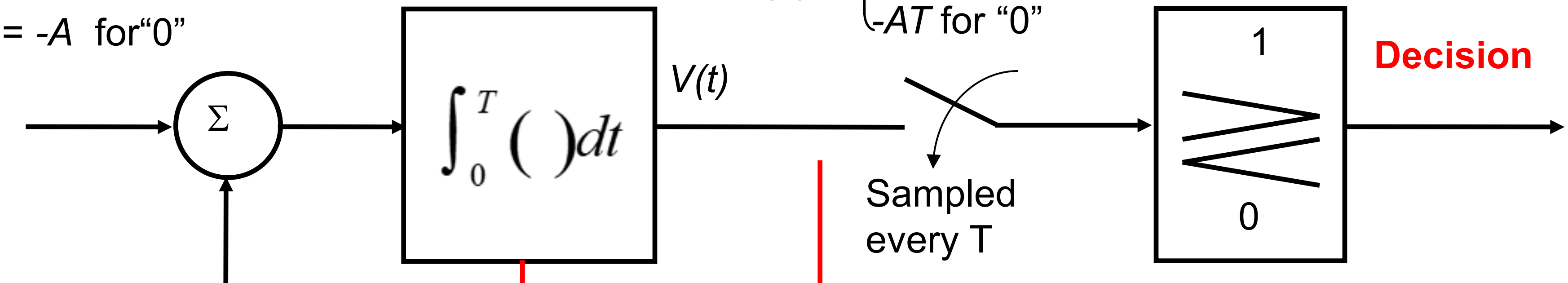


General Receiver Structures

$s_1(t) = A$ for "1" Now!

$s_0(t) = -A$ for"0"

$V(T)= \begin{cases} AT \text{ for "1"} \\ -AT \text{ for "0"} \end{cases}$



Threshold = 0

AWGN: $n(t) \sim S_N(f) = \frac{N_0}{2}$

General waveform

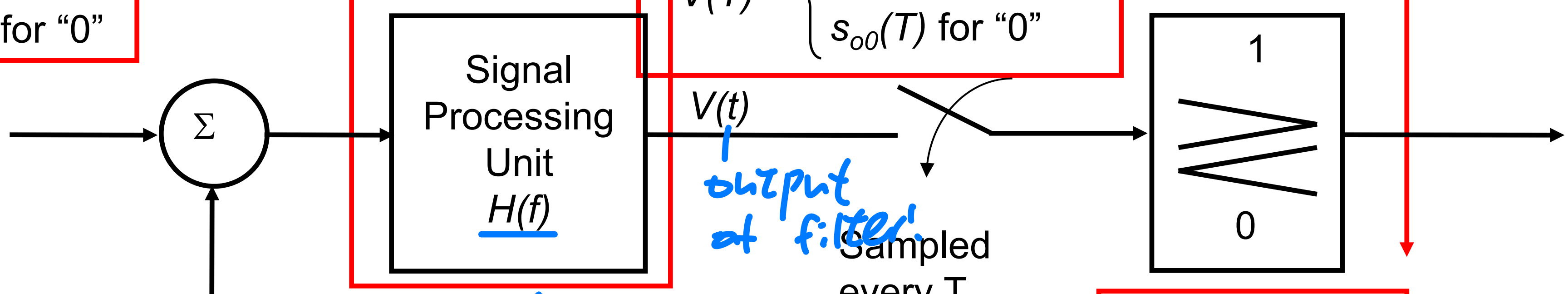
不一定是 square!

$s_1(t)$ for "1"
$s_0(t)$ for "0"

不用 specify! General!

Subscript "o" for output

$V(T)= \begin{cases} s_{o1}(T) \text{ for "1"} \\ s_{o0}(T) \text{ for "0"} \end{cases}$



filter!

AWGN: $n(t) \sim S_N(f) = \frac{N_0}{2}$

Threshold = V_D

general!

Generalization

1. Input Signal	$s_1(t) = A$ for "1" $s_0(t) = -A$ for "0"	$s_1(t)$ for "1" $s_0(t)$ for "0"
2. Signal Processing Unit	Integration and dump	Generalized Linear Filter $H(f)$
3. Output Signal	AT for "1" $-AT$ for "0"	$s_{o1}(t)$ for "1" $s_{o0}(t)$ for "0"
4. Decision threshold	0	V_D

Only they are the same!

5. Input Noise	AWGN: $S_N(f) = \frac{N_0}{2}$	AWGN: $S_N(f) = \frac{N_0}{2}$
6. Sampling Period	T	T

Performance Evaluation

- Let's write the input signal as: (the duration of the signal is *T*)

$$s(t) = \begin{cases} s_0(t) & \text{"0" sent} \quad 0 \le t < T \\ s_1(t) & \text{"1" sent} \quad 0 \le t < T \end{cases}$$

- The sampled signal and noise at the output of the linear filter *H(f)* is (subscript "**o**" stands for **output)**

แล้ว AT / -AT

Gaussian (mean = 0)

Sampled every bit period T

$$V\left(\boxed{T}\right) = \begin{cases} \underline{s_{o1}\left(\boxed{T}\right)} + N & \text{"1" sent} \\ \underline{s_{o0}\left(\boxed{T}\right)} + N & \text{"0" sent} \end{cases}$$

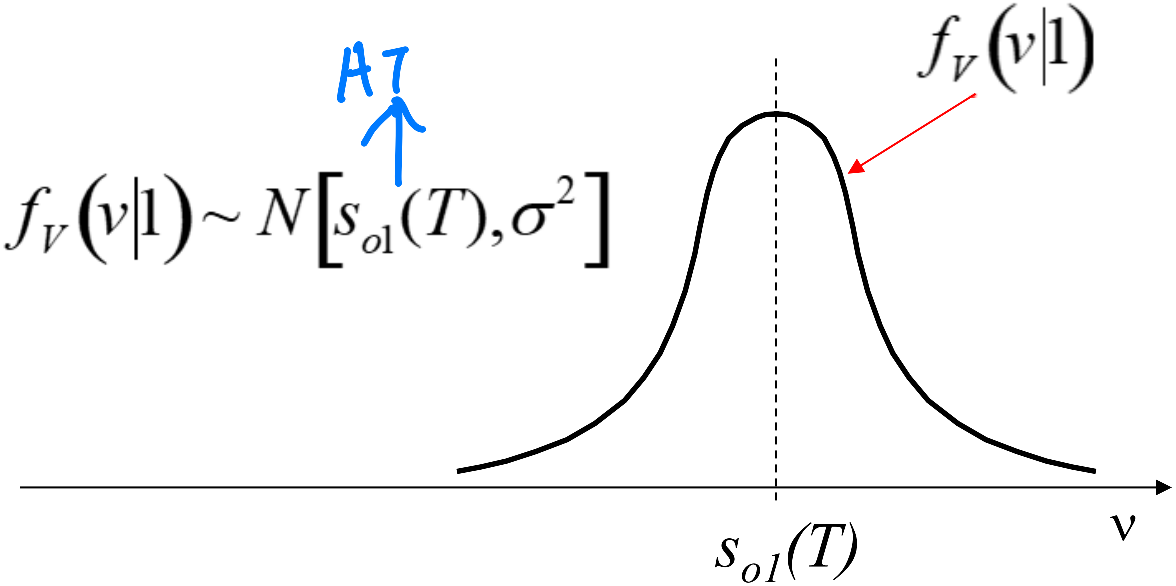
T = bit period!

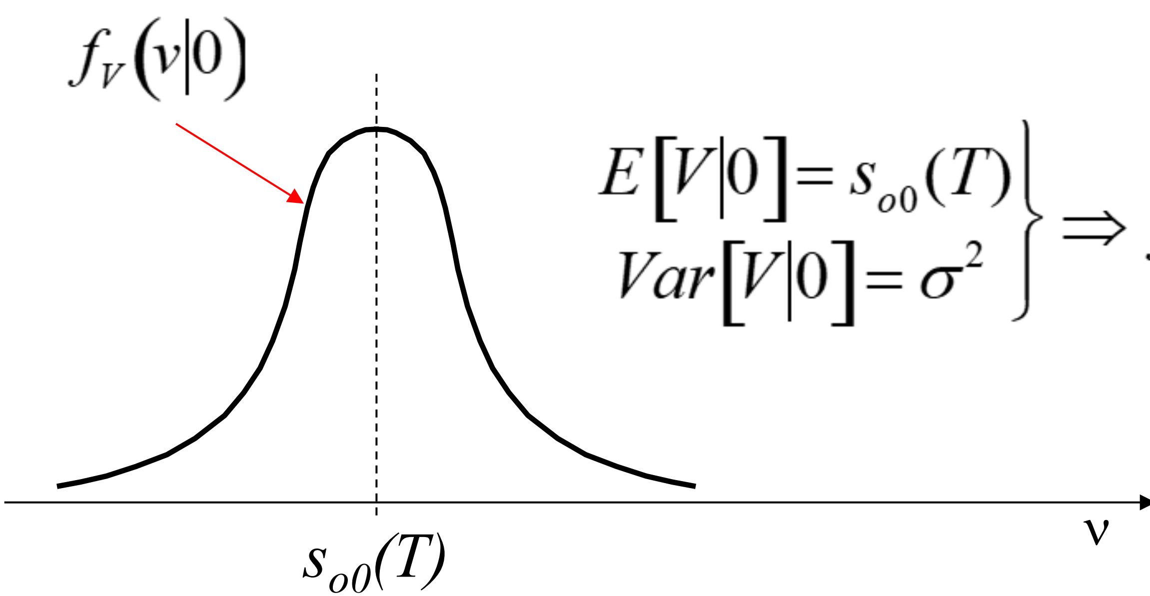
- The noise component of the output *N* should be Gaussian distributed with zero mean and variance of σ^2 (*not known at this stage*).

Performance Evaluation

AT

$$\left.\begin{aligned} E[V|1] &= s_{o1}(T) \\ Var[V|1] &= \sigma^2 \end{aligned}\right\} \Rightarrow f_V(v|1) \sim N[s_{o1}(T), \sigma^2]$$





$$\left.\begin{aligned} E[V|0] &= s_{o0}(T) \\ Var[V|0] &= \sigma^2 \end{aligned}\right\} \Rightarrow f_V(v|0) \sim N[s_{o0}(T), \sigma^2]$$

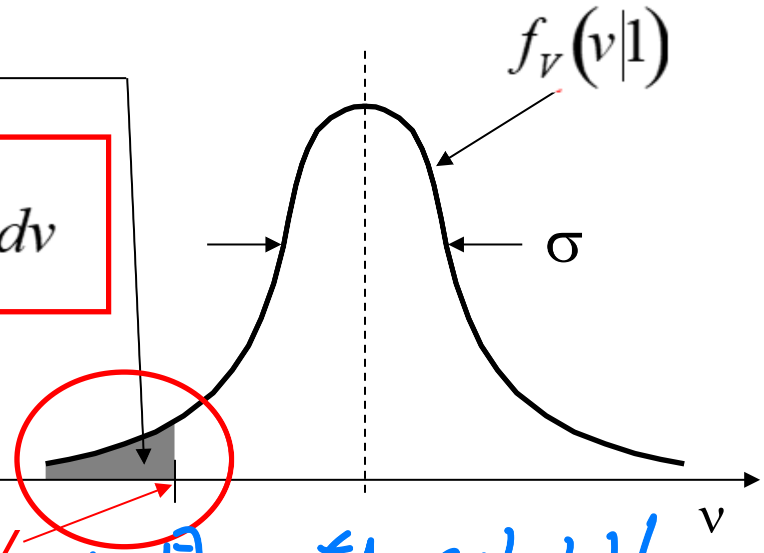
−AT

Performance Evaluation

$$P(E\,|\,1) = P(0 \text{ received}\,|\,1 \text{ sent})$$

$$\boxed{P(E\,|\,1) = P(V < V_D\,|\,1 \text{ sent}) = \int_{-\infty}^{V_D} f_V(v|1)\,dv}$$

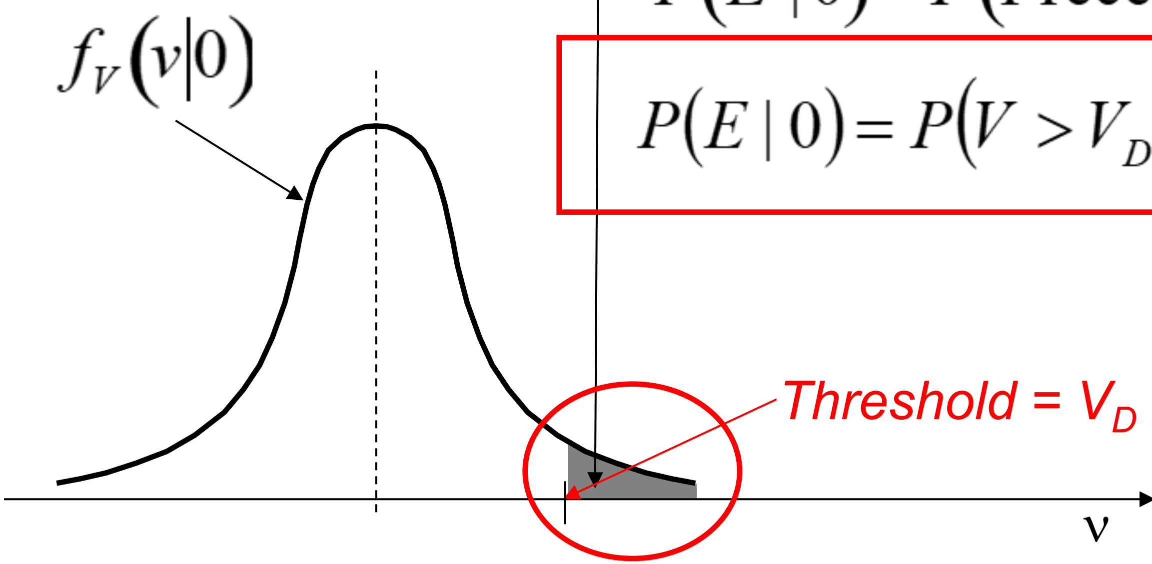
$$\Phi\left(\frac{V_D - s_{01}(T)}{\sigma}\right)$$



不是0 threshold!

$$P(E\,|\,0) = P(1 \text{ received}\,|\,0 \text{ sent})$$

$$\boxed{P(E\,|\,0) = P(V > V_D\,|\,0 \text{ sent}) = \int_{V_D}^{\infty} f_V(v|0)\,dv}$$



Performance Evaluation

$$P(E|0)=\int_{V_D}^{\infty} f_V(v|0)dv=\int_{V_D}^{\infty}\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(v-s_{o0})^2}{2\sigma^2}}dv$$

$$\boxed{\text{Transformation: } x=\frac{v-s_{o0}}{\sigma}\Rightarrow dx=\frac{dv}{\sigma}}$$

$$P(E\,|\,0)=\int_{\frac{V_D-s_{o0}}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \quad = \boxed{Q\left[\frac{V_D-s_{o0}(T)}{\sigma}\right]}$$

$$P(E\,|\,1)=Q\left[\frac{s_{o1}(T)-V_D}{\sigma}\right]$$

Can be obtained through similar steps

One more step after the transformation: $x' = -x$

$$v \mid 1 \sim N(s_{01}, \sigma^2)$$

$$v \mid 0 \sim N(s_{00}, \sigma^2)$$

$$P(E \mid 1) = P(v < v_D \mid 1) = P\left(\overbrace{\frac{v - s_{01}}{\sigma}}^{N(0,1)} < \frac{v_D - s_{01}}{\sigma} \,\middle|\, 1 \right)$$

$$= \Phi\left(\frac{v_D - s_{01}}{\sigma} \right) = Q\left(\frac{s_{01} - v_D}{\sigma} \right)$$

$$P(E \mid 0) = P(v > v_D \mid 0) = P\left(\overbrace{\frac{v - s_{00}}{\sigma}}^{N(0,1)} > \frac{v_D - s_{00}}{\sigma} \,\middle|\, 0 \right)$$

$$= Q\left(\frac{v_D - s_{00}}{\sigma} \right) = \Phi\left(\frac{s_{00} - v_D}{\sigma} \right)$$

Performance Evaluation

Error probability

$$P_e = P(E|1)P(1) + P(E|0)P(0)$$

$$P_e = Q\left[\frac{s_{o1}(T) - V_D}{\sigma}\right]P(1) + Q\left[\frac{V_D - s_{o0}(T)}{\sigma}\right]P(0)$$

$\frac{1}{2}$ $\frac{1}{2}$

VIP (Very ImPortant assumption)

Assume they are same!

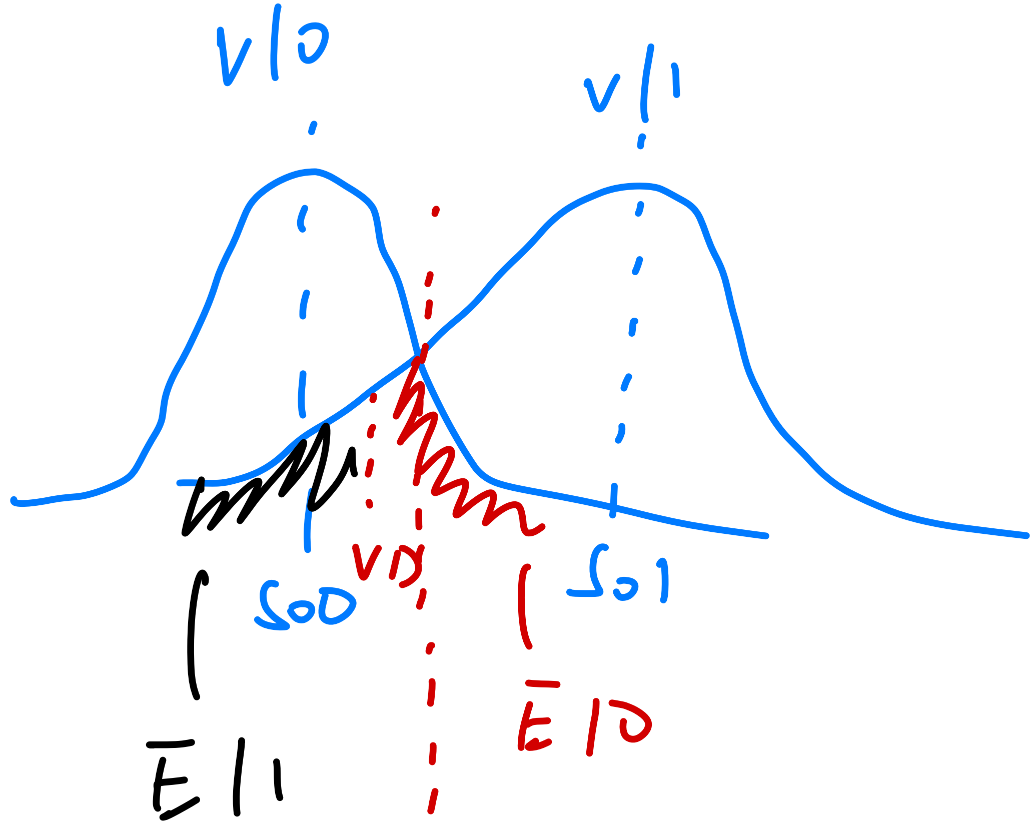
1. $\boxed{P(1) = P(0) = 1/2}$

2. $$\boxed{V_{D,opt} = \frac{s_{o1}(T) + s_{o0}(T)}{2}}$$

$$= \left[Q\left(\frac{s_{o1} - V_D}{\sigma}\right) + Q\left(\frac{V_D - s_{o0}}{\sigma}\right)\right]\frac{1}{2}$$

$\Rightarrow$ select V_D to minimize P_e

Optimum decision threshold → best P_e



= Summation of two parts!

optimal v10:

$$= \frac{s00 + s01}{2} \text{ (mid-point)}$$

not depends on sigma!

Optimum Threshold

- When $P(0) = P(1) \equiv \frac{1}{2}$

$\boxed{\text{First optimum: Optimum Threshold}}$

mid-point!

$$\boxed{V_{D,opt} = \frac{s_{o0}(T) + s_{o1}(T)}{2}}$$

Thus, when $V_D = V_{D,opt}$

$\boxed{\text{Physical meaning?}}$

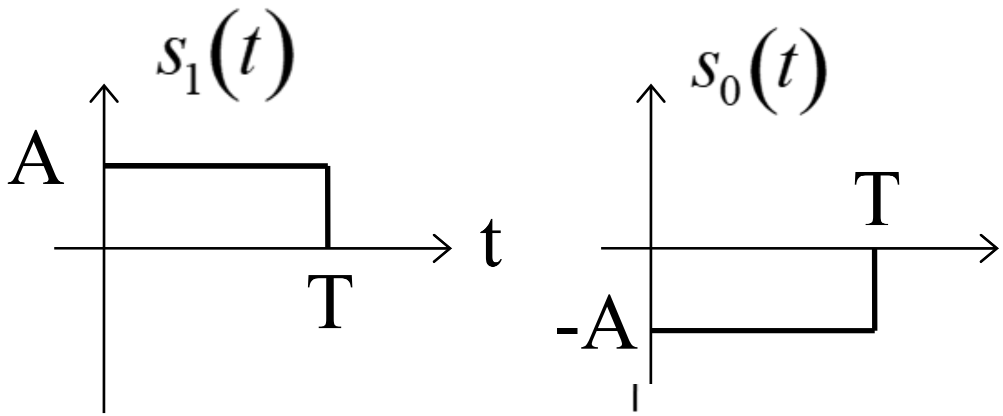
two points larger! ⇒ difference larger ↓ Pe ↓

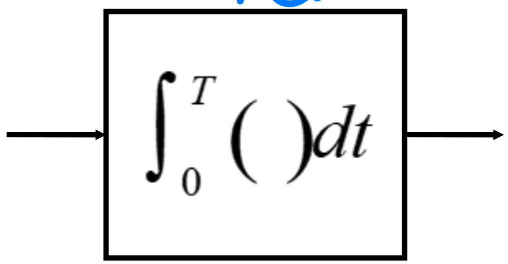
$$\boxed{P_e = Q\left[\sqrt{\frac{(s_{o1}(T) - s_{o0}(T))^2}{4\sigma^2}}\right]}$$

Performance Evaluation: Example

- **Example**: *Assume signal processing ~ Integrate-and-dump and that $s_k(t)$ are as follows, then*

Apply general result!





$$s_{o0} = -AT, \qquad s_{o1} = AT, \qquad \sigma^2 = \frac{N_0 T}{2}$$

- **So**

$$\frac{(s_{o1} - s_{o0})^2}{4\sigma^2} = \frac{4A^2T^2}{4\sigma^2} = \frac{2A^2T}{N_0} \triangleq \frac{2E_b}{N_0}$$

$$\therefore P_e = Q[Z] = Q\left[\sqrt{\frac{2E_b}{N_0}}\right]$$

Optimal Receiver

- We have obtained error probability for arbitrary linear filter $h(t) \leftrightarrow H(f)$,

$$P_e = Q\left[\sqrt{\frac{\left(s_{o1} - s_{o0}\right)^2}{4\sigma^2}}\right] = Q\left[\sqrt{\frac{\zeta^2}{4}}\right]$$

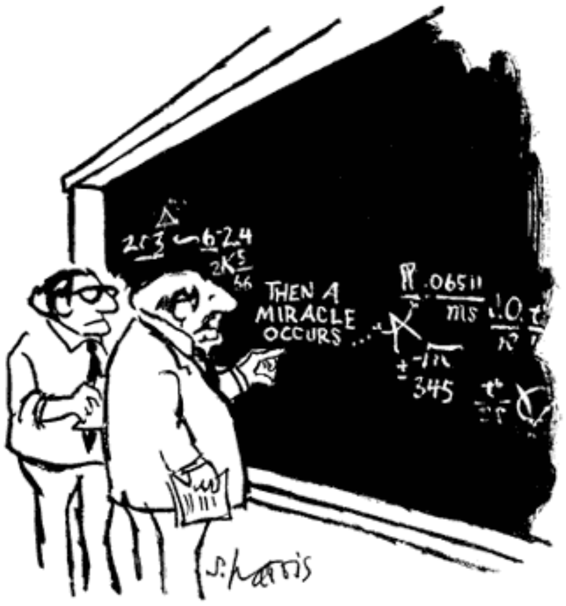
$$\frac{s_{o1} - s_{o0}}{\sigma} = \zeta$$

- Recall that ***Q[z]* gets smaller as *z* increases**.
- We have yet to put any specification on *H(f)*.
- An optimal filter/receiver *H(f)* can be found such that **ζ is maximized and therefore P_e is minimized**
- We call such filter a ***Matched Filter.***

details next lecture!

Ch7.2: Optimal Receiver

- Error Probability for General Signals & Receivers
  - Optimum Threshold
- **Input-Output Relation**
- The Matched Filter
  - Optimal Receiver
  - Examples
  - Optimal Signals

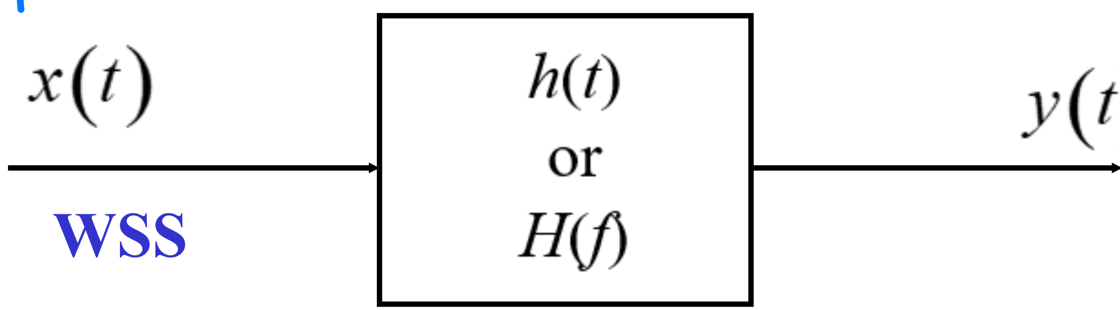


"I think you should be more explicit here in step two."

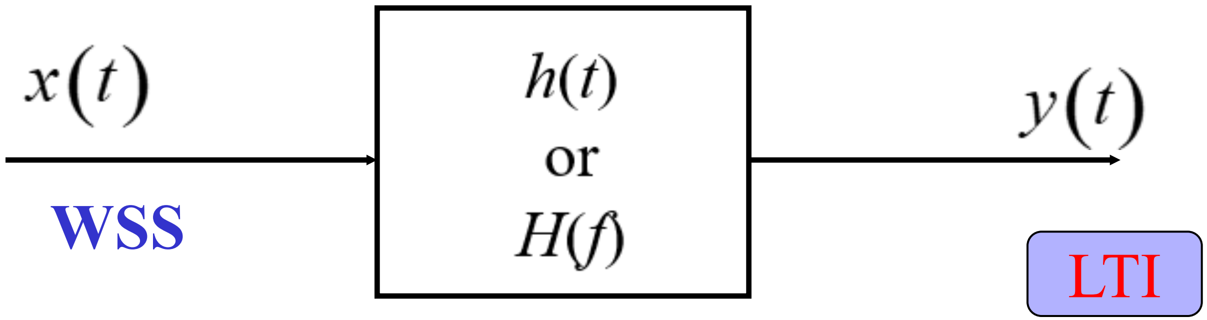
Input and Output Relationships

- In order to design the optimal receiver, we need to understand how a linear filter will affect the signal and noise, respectively.
- The signal has a fixed waveform. Thus, the input and output relationships are easy to obtain.
- However, the noise is a random process. Then, what will be the input and output relationships if a wide sense stationary (WSS) random process (AWGN) goes through a linear filter?

input with noise!



Input and Output Relationships for PSD



$$y(t) = \int_{-\infty}^{\infty} h(s)x(t-s)ds$$

$$R_{xy}(\tau) = E\left[x(t)y(t+\tau)\right]$$ cross correlation

$$= E\left[x(t)\int_{-\infty}^{\infty} h(s)x(t+\tau-s)ds\right]$$

$$= \int_{-\infty}^{\infty} h(s)E\left[x(t)x(t+\tau-s)\right]ds$$

$$= \int_{-\infty}^{\infty} h(s)R_x(\tau-s)ds$$ WSS

Cross Correlation between x(t) and y(t)

- Thus,

FT
$$R_{xy}(\tau) = h(\tau) * R_x(\tau)$$
$$S_{XY}(f) = H(f) S_X(f)$$

Power Spectral Density = FT of Correlation Function

- Likewise, can show that

$$R_{yx}(\tau) = h(-\tau) * R_x(\tau)$$
$$S_{YX}(f) = \underline{H^*(f) S_X(f)}$$

Time reversal properties of F.T. applied for H(f)

Input and Output Relationships

Autocorrelation

$$R_y(\tau) = E\left[y(t)y(t+\tau)\right] = h(\tau) * R_{yx}(\tau)$$

derive by yrself!

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$$

Expression for $R_{YX}(\tau)$ from last page

FT

$$S_Y(f) = H(f)H^*(f)S_X(f)$$

FT on both side

PSD

$$\boxed{S_Y(f) = \left|H(f)\right|^2 S_X(f)}$$

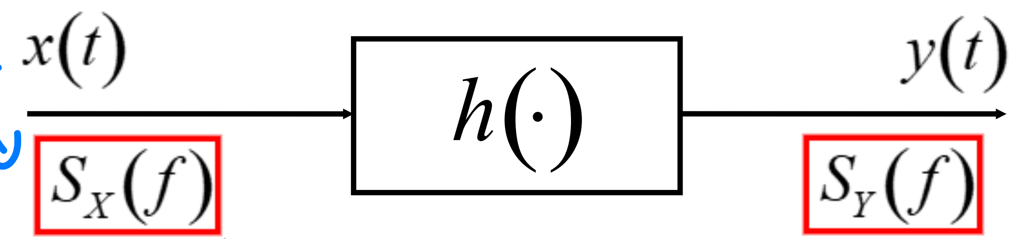
simple relationship!

know noise

the power!

Key Result: PSDs of the input and output random processes

FT

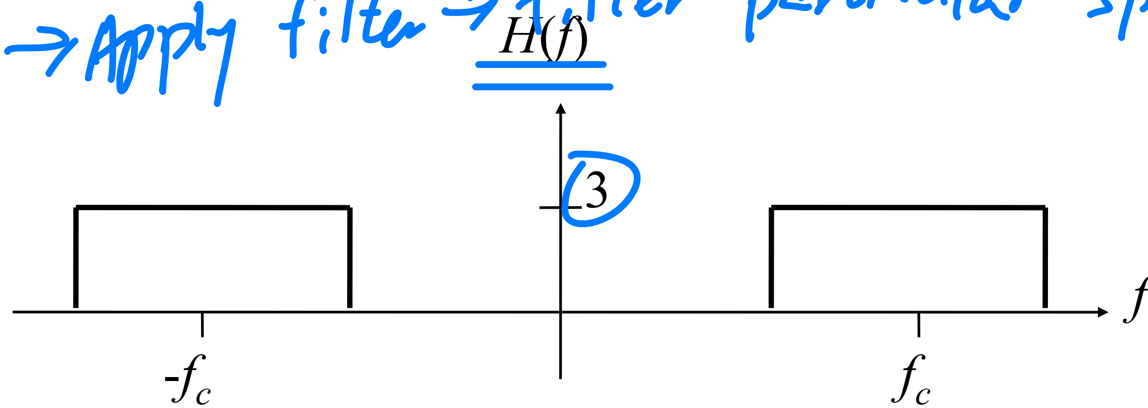


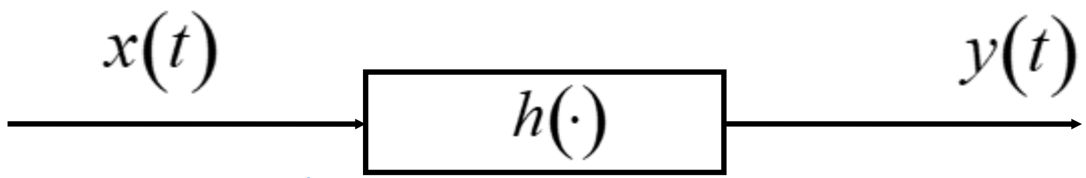
Analysis the noise

Output PSD → $$S_Y(f) = \left|H(f)\right|^2 S_X(f)$$ ← Input PSD

Input and Output Relationships: Example

AWGN → Apply filter → filter particular spectrum

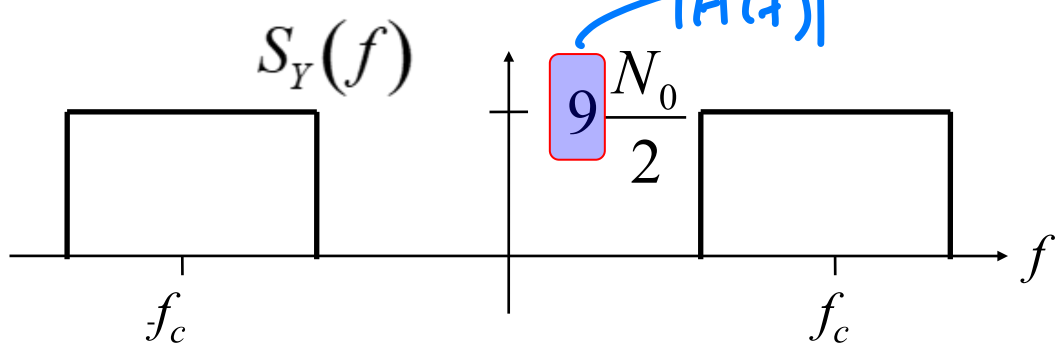




AWGN noise!

- If $S_X(f) = N_0/2$ then, what is $S_Y(f)$?

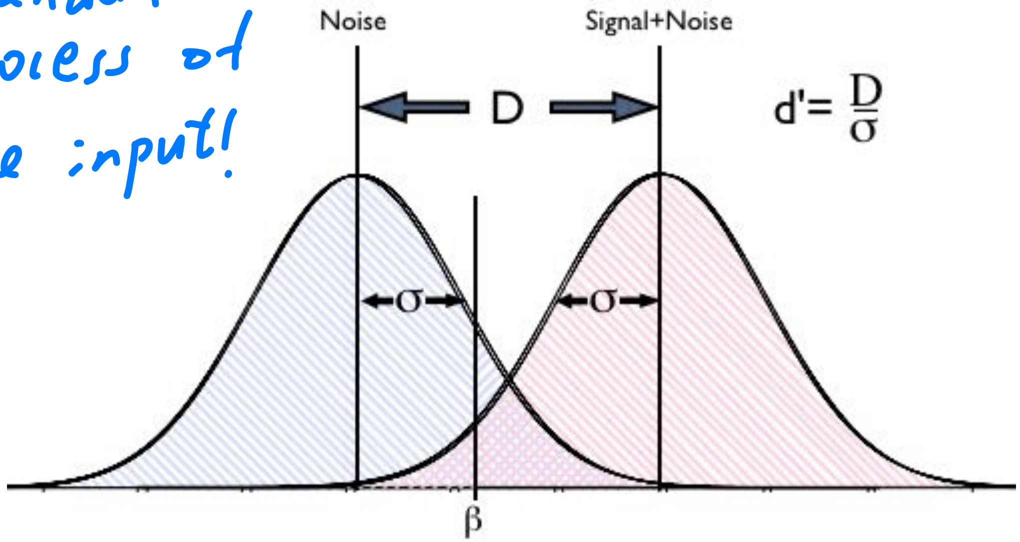
$|H(f)|^2$



Ch7.2: Optimal Receiver

- Error Probability for General Signals & Receivers
  - Optimum Threshold
- Input-Output Relation
- **The Matched Filter**
  - **Optimal Receiver**
  - Examples
  - Optimal Signals

Random process of the input!



Optimal Receiver

- We want to find a receiver structure that can minimize the BER

$$\boxed{P_e = Q\left[\sqrt{\frac{(s_{o1}(T) - s_{o0}(T))^2}{4\sigma^2}}\right]}$$

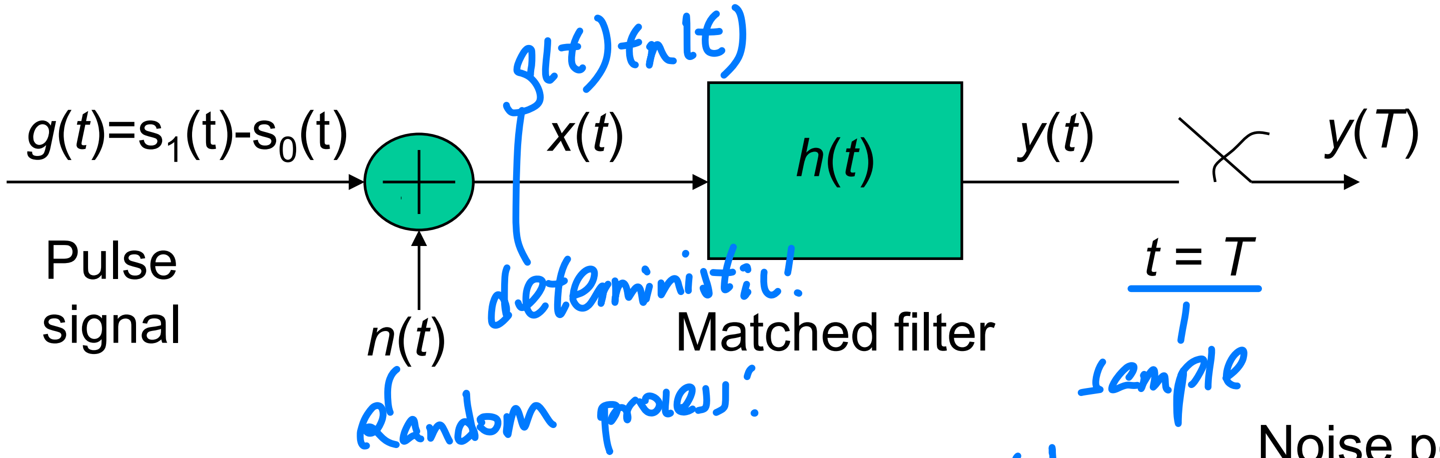
Q function
反轉;

- This is equivalent to maximize the term

$$\zeta^2 = \frac{(s_{o1}(T) - s_{o0}(T))^2}{\sigma^2} = \frac{|g_o(t)|^2}{\sigma^2}$$

where $s_{o1}(T) - s_{o0}(T)$ can be regarded as the output of the filter to input signal $g(t) = s_1(t) - s_0(t)$ at time T.

Matched Filter Derivation



- **Noise** $w(t) = \underline{n(t) * h(t)}$

input!

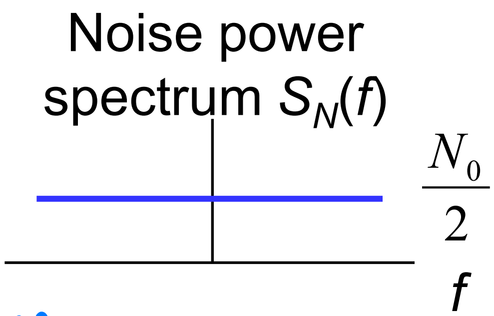
output!

PSD

$$S_W(f) = \underbrace{S_N(f)}_{\text{AWGN}} \underbrace{S_H(f)}_{\text{Filter}} = \frac{N_0}{2} |H(f)|^2$$

= 'noise $S_N(t)$

PSD plug in!

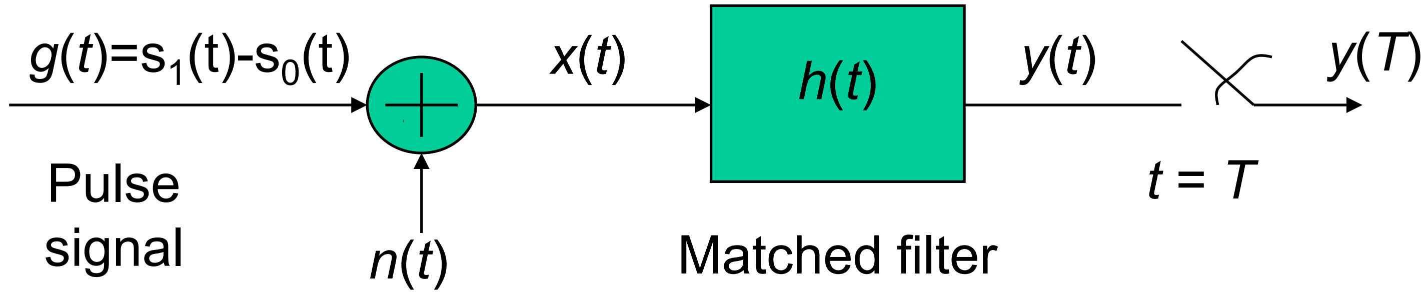


$$\underline{\underline{\sigma^2 = E\{w^2(T)\}}} = \boxed{\int_{-\infty}^{\infty} S_W(f)\, df} = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df$$

分母

noise power!

Matched Filter Derivation



input!

- **Signal** $g_o(t) = g(t) * h(t) \qquad G_o(f) = H(f)G(f)$

inverse FT

$$g_o(t) = \int_{-\infty}^{\infty} \boxed{H(f)\,G(f)}\, e^{j2\pi f t}\, df$$

$$|\, g_o(T)\,|^2 = \left| \int_{-\infty}^{\infty} H(f)\,G(f)\, e^{j2\pi f T}\, df \right|^2$$

Matched Filter Derivation

Max this!

- Find $h(t)$ that maximizes pulse peak SNR

signal power!

$$\zeta^2 = \frac{\left| \int_{-\infty}^{\infty} \underbrace{H(f)}_{x(f)} \underbrace{G(f)\, e^{j2\pi f T}}_{y(f)} \, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df}$$

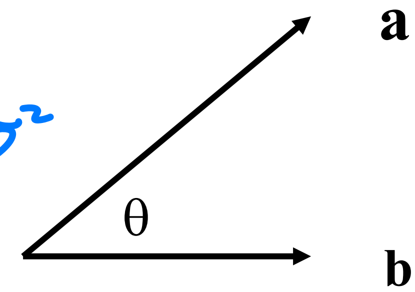
$= |g_o(t)|^2$

$= \sigma^2$

$\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

useful



- Schwartz's inequality

same dimension, inner product

For vectors: $|\boldsymbol{a}^T\boldsymbol{b}| \le ||\boldsymbol{a}||\,||\boldsymbol{b}|| \leftrightarrow \cos(\theta) = \frac{\boldsymbol{a}^T\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} \le 1$

For functions: $\left|\int_{-\infty}^{\infty} x(t)y^*(t)\,dt\right|^2 \le \int_{-\infty}^{\infty} |x(t)|^2 dt \int_{-\infty}^{\infty} |y(t)|^2 dt$,

and the lower bound reached iff $x(t) = ky(t)$, $\forall k \in C$.

complex functions

Apply this to maximize!

For functions

$$\Big|\underbrace{\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt}_{\text{inner product}}\Big|^2 \le \underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{norm}^2} \ \underbrace{\int_{-\infty}^{\infty} |y(t)|^2 dt}_{\text{norm}^2}$$

Matched Filter Derivation

|exp| = 1

- Let $x(f) = H(f)$ and $y(f) = G^*(f)e^{-j2\pi fT}$
- Thus, $|\int_{-\infty}^{\infty} H(f)G\,(f)\,e^{j2\pi fT}df|^2 \leq \int_{-\infty}^{\infty}|H(f)|^2 df \int_{-\infty}^{\infty}|G(f)|^2 df$, which further gives $|\int_{-\infty}^{\infty} x(f)\,y^*(f)\,df| \leq \int \cdots \int \cdots$

$$\zeta^2 = \frac{|\int_{-\infty}^{\infty} H(f)\,G(f)\,e^{j2\pi fT}\,df\,|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df} \leq \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,df$$

independent on H(f)!

- The maximum $\zeta^2_{\max} = \frac{2}{N_o}\int_{-\infty}^{\infty}|G(f)|^2 df$ occurs when

  ? x(f) y(f)

  $H_{opt}(f) = kG^*(f)e^{-j2\pi fT}$ by Schwartz's inequality.

  constant!

- Hence, $h_{opt}(t) = kg(T-t)$. — matched filter

  " typically " same as signal!

Matched to what?

apply for any waveform!

- Let $g(t)=[s_1(t)-s_0(t)]$

Impulse Response of Matched Filter

$$\boxed{h_{opt}(t)=s_1(T-t)-s_0(T-t)}$$

optimal threshold!

Second optimum: Optimal Receiver

- The matched filter matches to the difference between the two pulses utilized to transmit bit "1" and bit "0", respectively.
- What can we see in frequency domain?

matched filter!

Receiver Block Diagram

Filter:

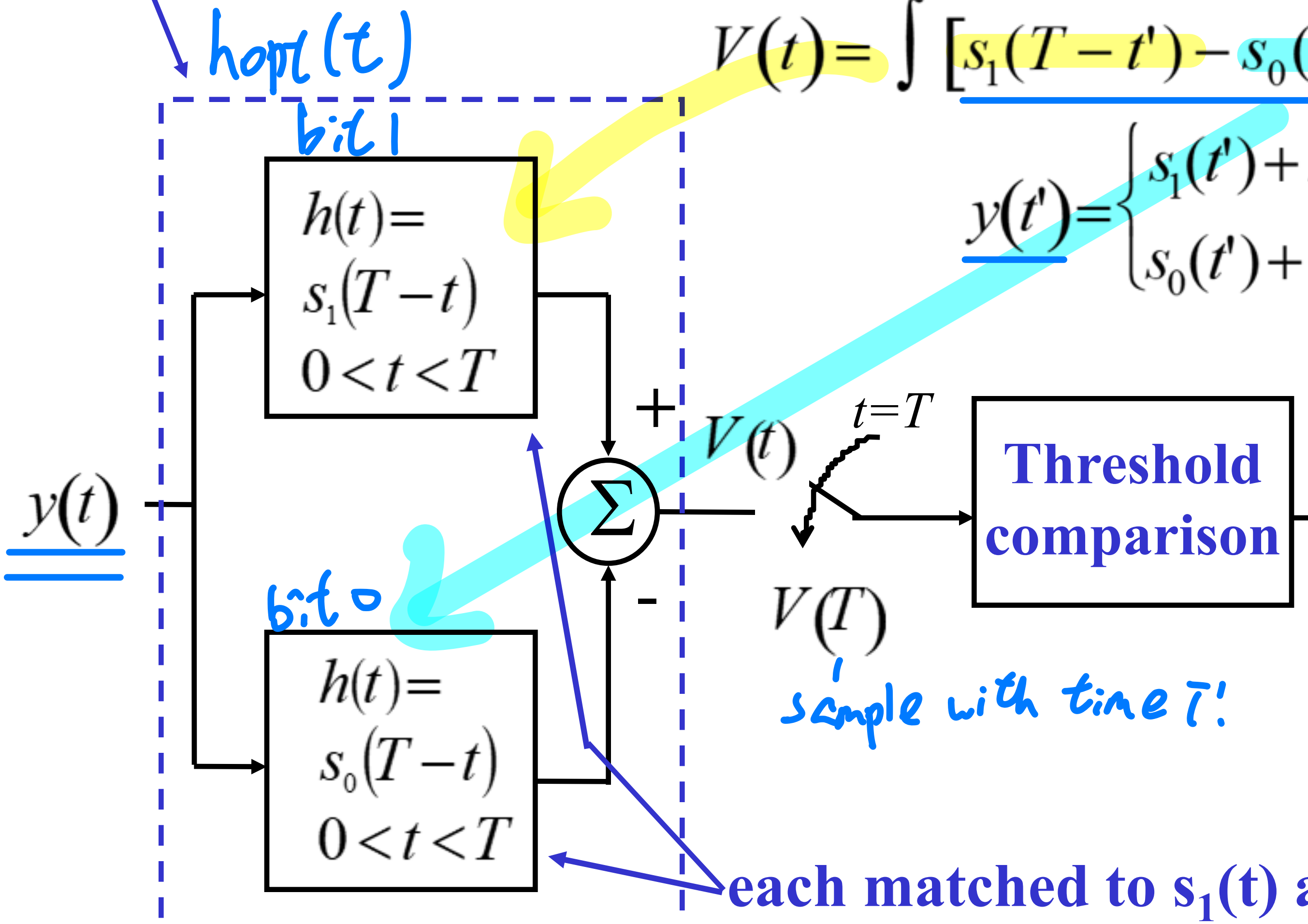
$$h_{opt}(t) = [s_1(T-t) - s_0(T-t)] \quad (k = 1)$$

$$V(t) = \int h_{opt}(t')y(t-t')dt'$$

output = hopt(t) * y(t) input

$$V(t) = \int [s_1(T-t') - s_0(T-t')]y(t-t')dt'$$

$$y(t') = \begin{cases} s_1(t') + n(t') & \text{for “1”} \\ s_0(t') + n(t') & \text{for “0”} \end{cases}$$



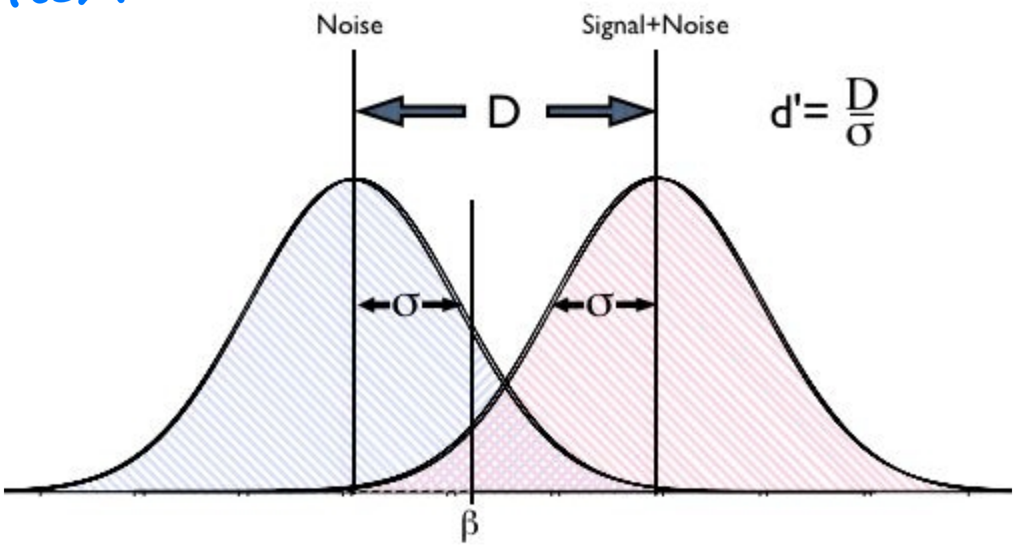
Decision:

$V(T) > V_{D,opt}$: "1"

$V(T) < V_{D,opt}$: "0"

Ch7.2: Optimal Receiver

- Error Probability for General Signals & Receivers
  - Optimum Threshold
- Input-Output Relation
- **The Matched Filter** another filter!
  - Optimal Receiver
  - **Examples**
  - Optimal Signals

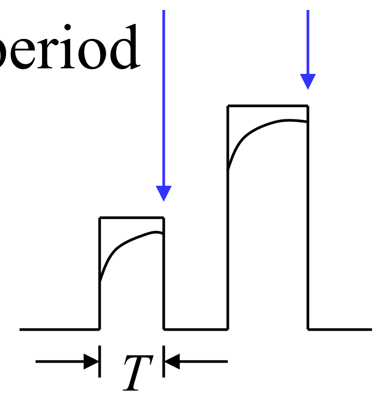


Matched Filter for Rectangular Pulse

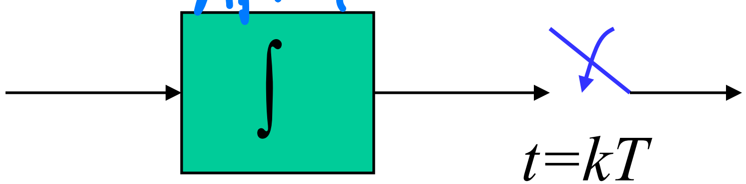
- Matched filter for causal rectangular pulse has an impulse response that is a causal rectangular pulse.
- Convolve input with rectangular pulse of duration T sec and sample result at T sec is same as to:
  - First, integrate for T sec → in 7.1
  - Second, sample at symbol period T sec
  - Third, reset integration for next time period
- Integrate and dump circuit

optimal when rectangular pulse!

Sample and dump

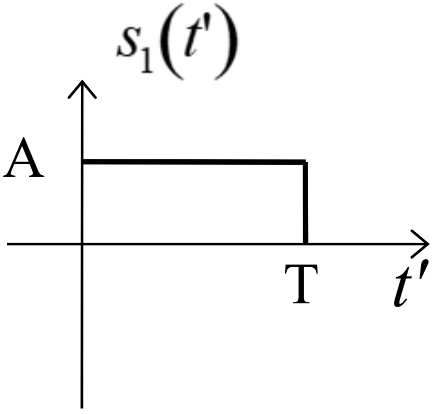


Apply filter = integrate!



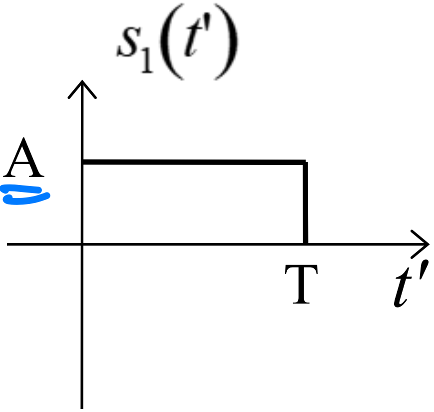
Matched Filter: Example

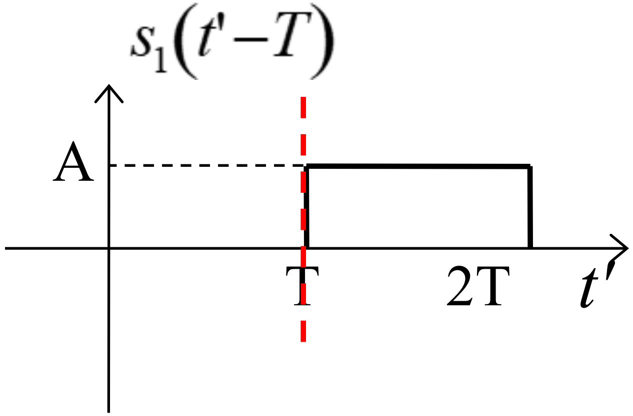
Antipodal Baseband Signal



Matched Filter: Example

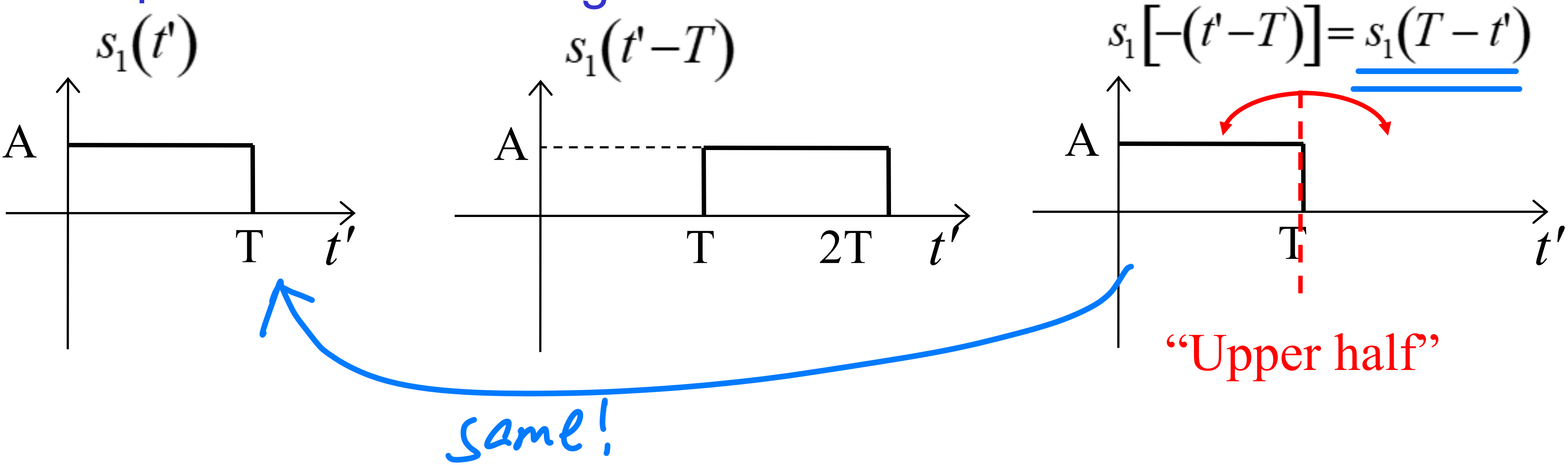
Antipodal Baseband Signal





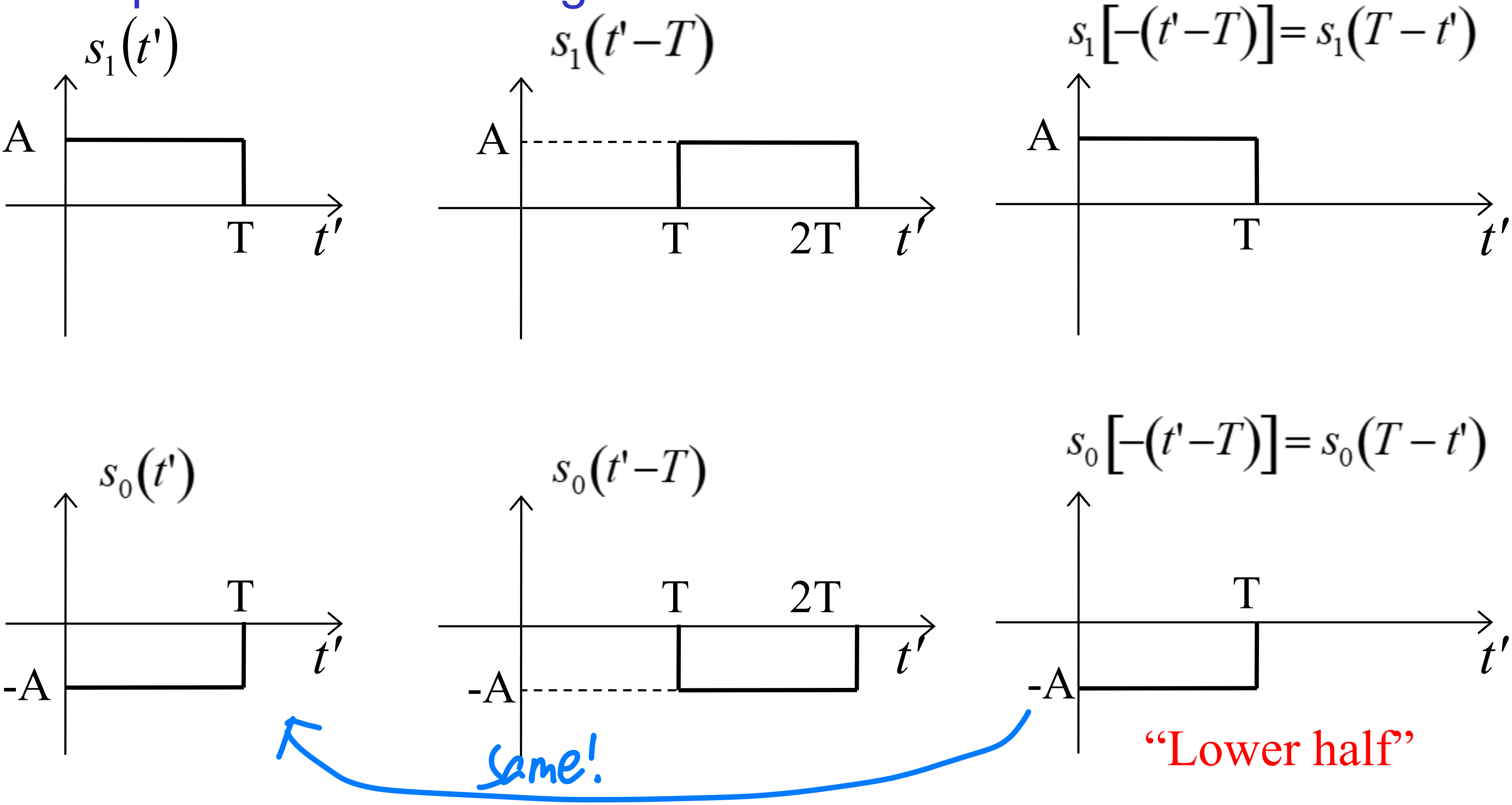
Matched Filter: Example

Antipodal Baseband Signal



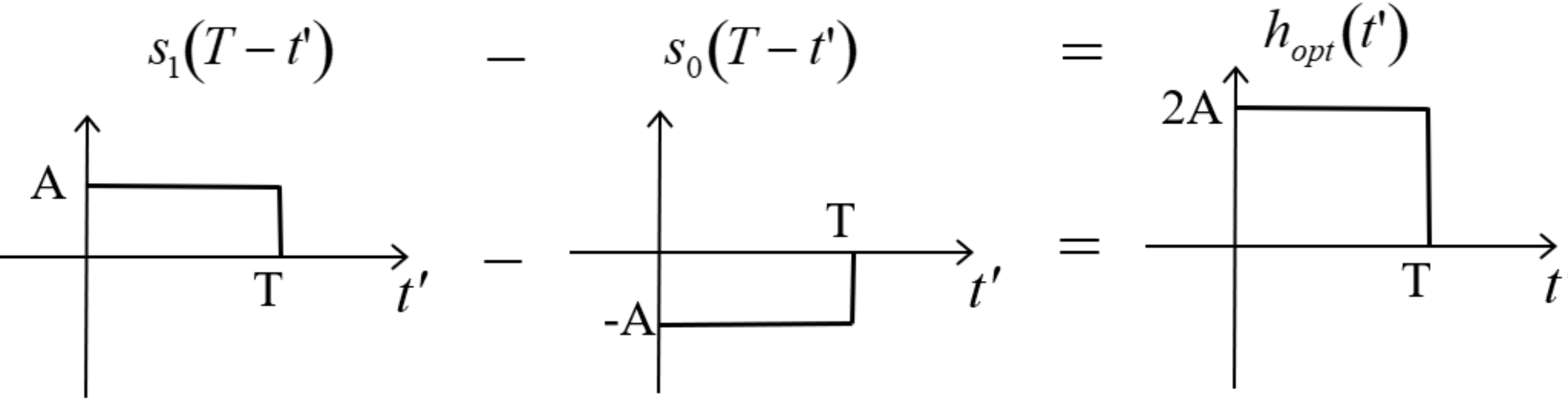
Matched Filter: Example

Antipodal Baseband Signal



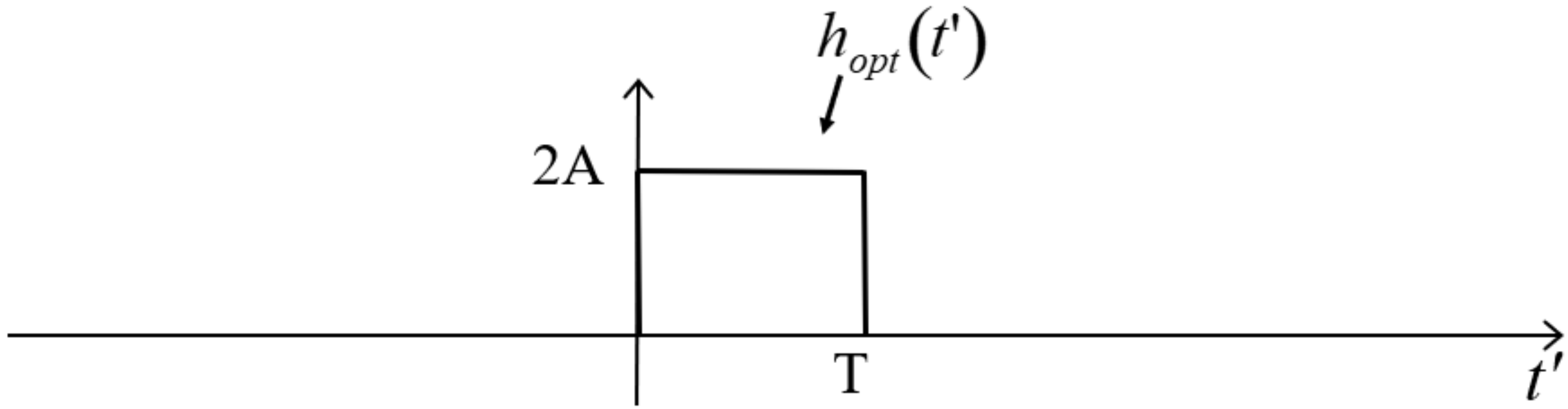
$$h_{opt}(t) = [s_1(T-t') - s_0(T-t')]$$

overall = difference!



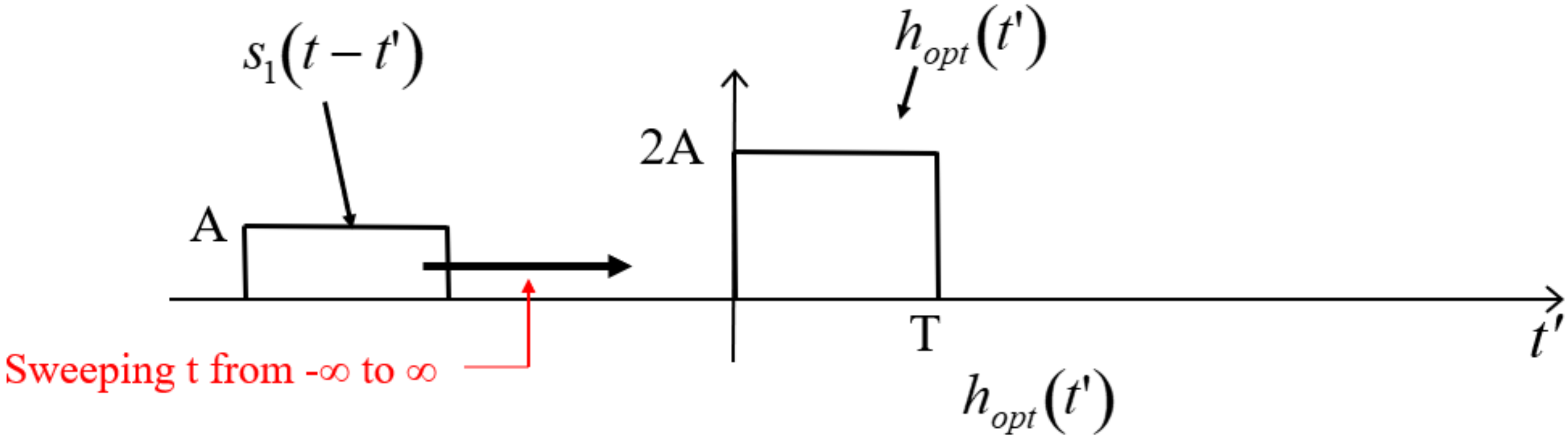
$$V(t) = \int \left[\boxed{s_1(T-t') - s_0(T-t')}\right] y(t-t')\,dt'$$

Case 1: $y(t') = s_1(t')$ "1"



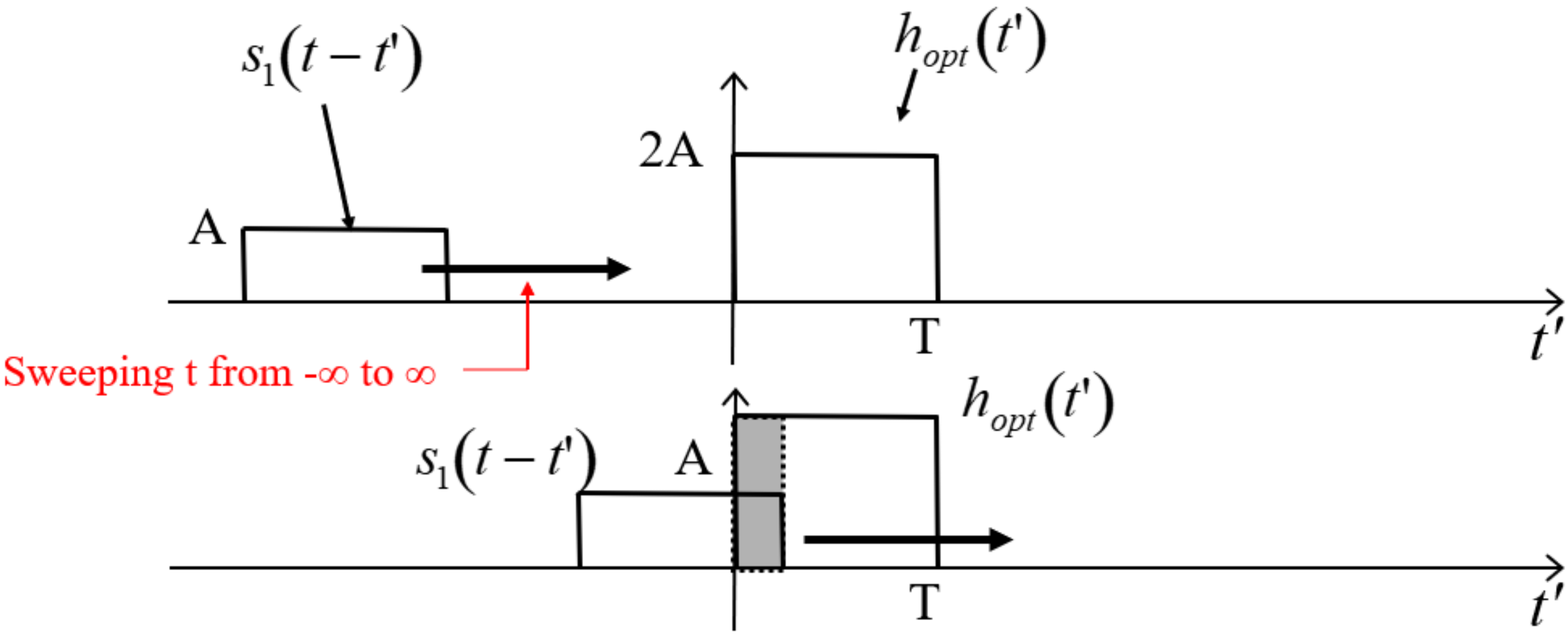
$$V(t) = \int \left[s_1(T-t') - s_0(T-t')\right] \boxed{y(t-t')}\, dt'$$

Case 1: $\boxed{y(t') = s_1(t')}$ "1"



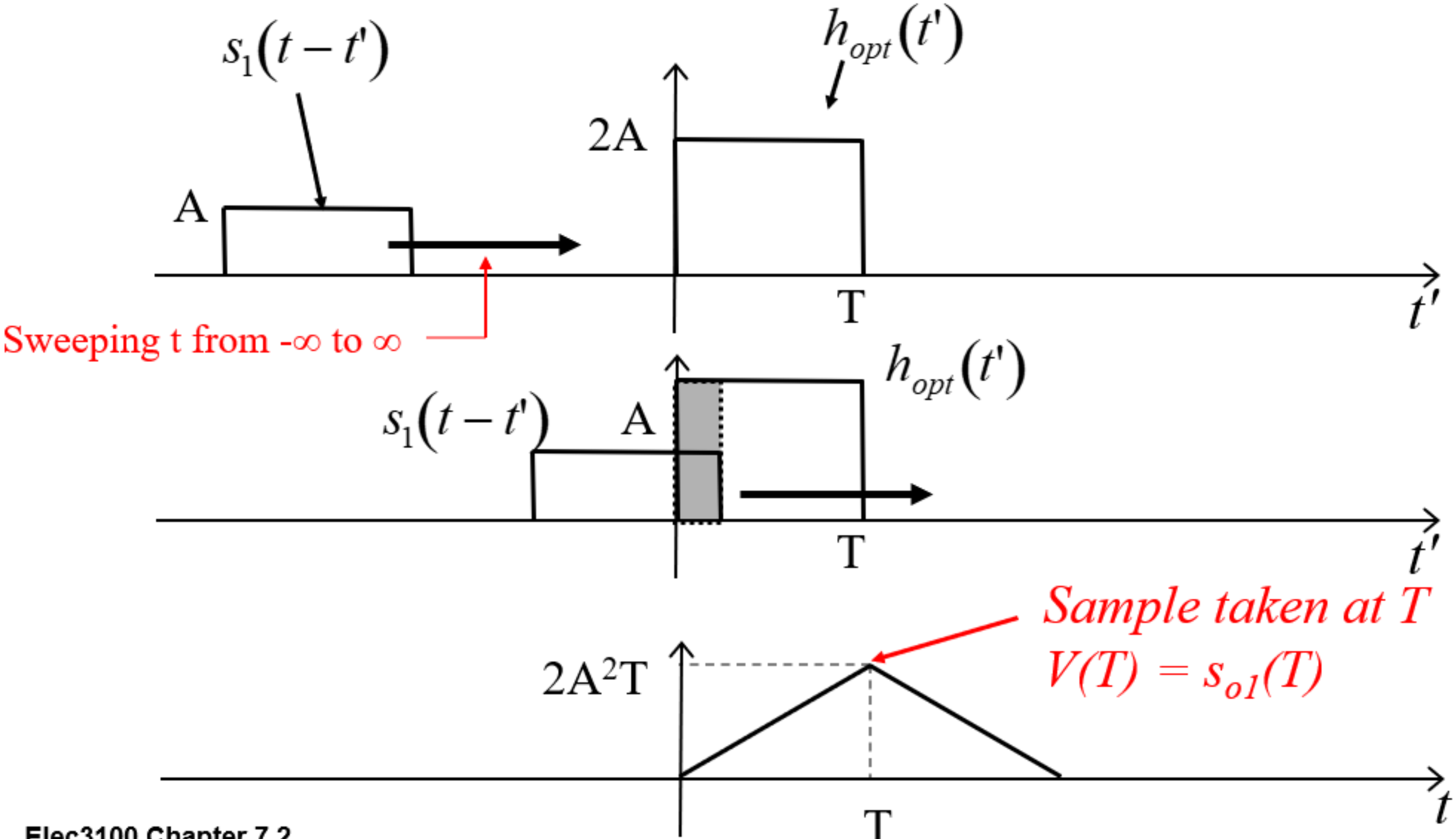
$$V(t) = \int \left[\boxed{s_1(T-t')} - s_0(T-t') \right] y(t-t')\,dt'$$

Case 1: $y(t') = s_1(t')$ "1"



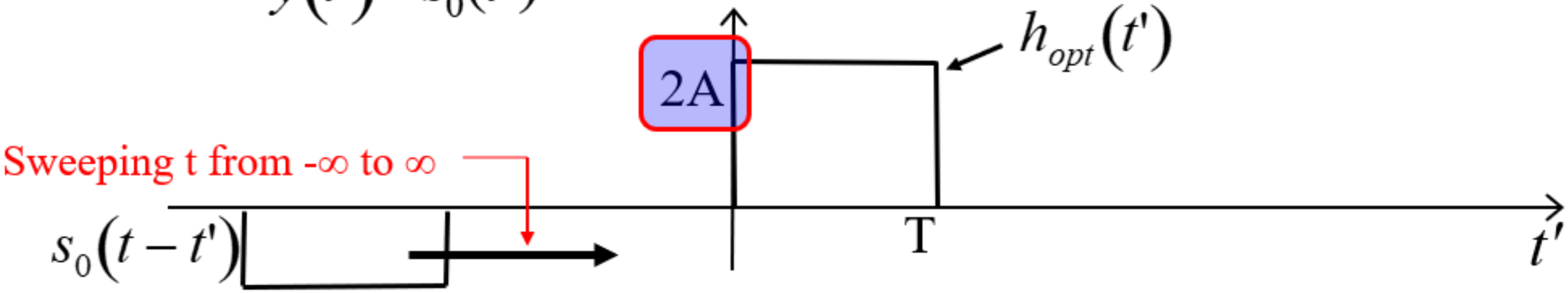
$$V(t) = \int \left[s_1(T-t') - s_0(T-t')\right] y(t-t')\,dt'$$

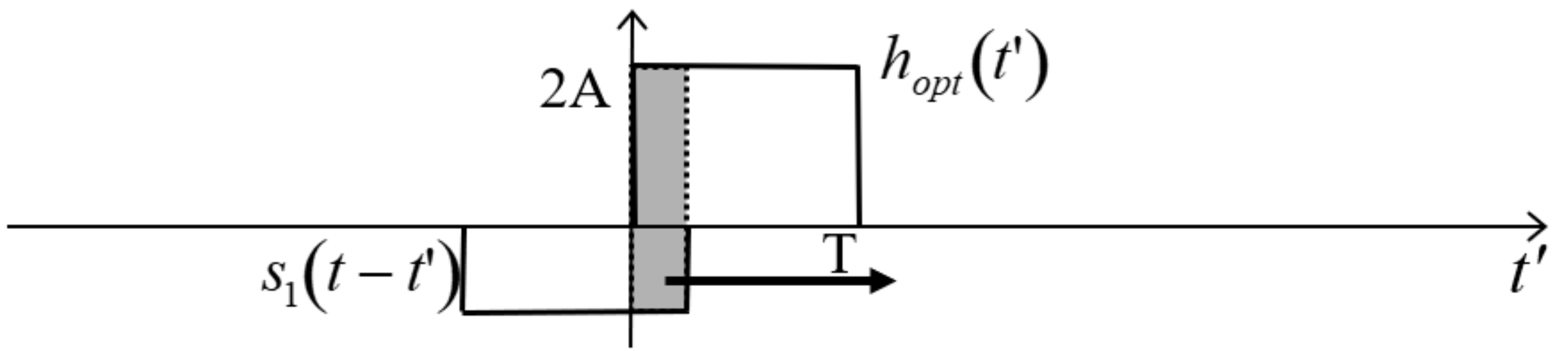
Case 1: $y(t') = s_1(t')$ "1"

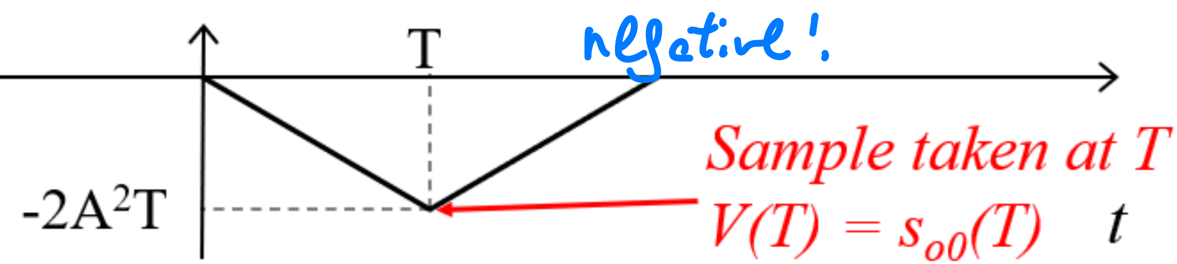


$$V(t) = \int \left[s_1(T - t') - s_0(T - t') \right] y(t - t') dt'$$

Case 2: $y(t') = s_0(t')$ “0”





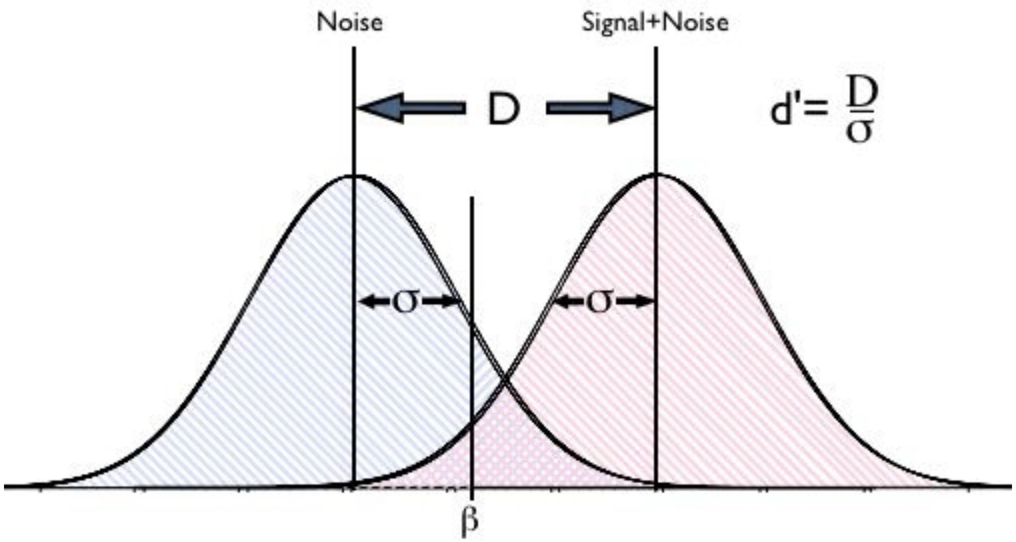


Ch7.2: Optimal Receiver

- Error Probability for General Signals & Receivers
  - Optimum Threshold

- Input-Output Relation

- **The Matched Filter**
  - Optimal Receiver
  - Examples
  - **Optimal Signals**



Optimal Signal

BER = Q(ζ)

From Schwarz inequality,

Parseval's Theorem

$$\zeta_{\max}^2 = \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,df = \frac{2}{N_0}\int_{-\infty}^{\infty}\boxed{g^2(t)}\,dt$$

$$\zeta_{\max}^2 = \frac{2}{N_0}\int_0^T\boxed{\left(s_1(t)-s_0(t)\right)^2}dt = \frac{2\boxed{E_g}}{N_0}$$

Want this be large;

(E_g not same as E_b)

Third optimum: Optimal Signal

$$\boxed{E_g = \int_0^T\left(s_1(t)-s_0(t)\right)^2 dt}$$

$$= \int_0^T s_1^2(t)\,dt + \int_0^T s_0^2(t)\,dt - 2\int_0^T s_1(t)s_0(t)\,dt$$

$$= E_1 + E_0 - 2\rho_{10}\sqrt{E_1E_0}$$

Physical meaning?

Want negative correlated!

where ρ_{10} is the correlation coefficient

$$-1 \le \rho_{10} = \frac{1}{\sqrt{E_1E_0}}\int_0^T s_1(t)s_0(t)\,dt \le 1$$

not only Energy, but also correlation!

Cross correlation! −1 ⇒ optimal!

Optimal Receiver: Matched Filter

- The **Optimal Filter** is the **Matched Filter**

$$h_{opt}(t) = s_1(T-t) - s_0(T-t)$$

$$P_e = Q\left[\sqrt{\frac{(s_{o1}-s_{o0})^2}{4\sigma^2}}\right] = Q\left[\sqrt{\frac{\zeta_{\max}^2}{4}}\right]$$

$$= Q\left[\sqrt{\frac{E_g}{2N_0}}\right]$$

maximize this!

$$E_g = E_1 + E_0 - 2\rho_{10}\sqrt{E_1 E_0}$$

$\boxed{\text{NOT } E_b}$

Summary

Optimize Your Time
Don't just manage it!
Optimal Thinking

We have three things to optimize:

- **O1**: Optimal Signal Es
  Make the two signals as **dissimilar** as possible (Why?)

Rx

- **O2**: Optimal Processing Unit
  Maximize the **difference** between two signals (How?)

- **O3**: Optimum Threshold (We like **symmetry**, right?)
  Make good use of the signal difference

In your opinion, which one is the most important?

Next time → optimal ratio