Ch2.1: Discrete-Time Signals

Information source and input transducer

Source Coding

Channel Coding

Modulator

Channel

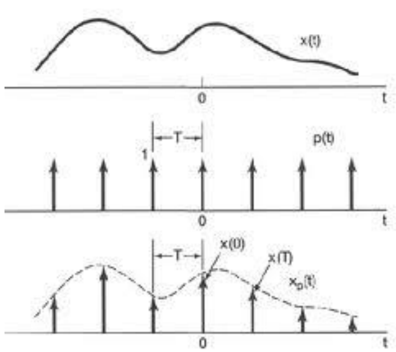
Demodulator (Matched Filter)

Channel Decoding

Source Decoding

Information sink and output transducer

Ch 2: DT Signals and Systems



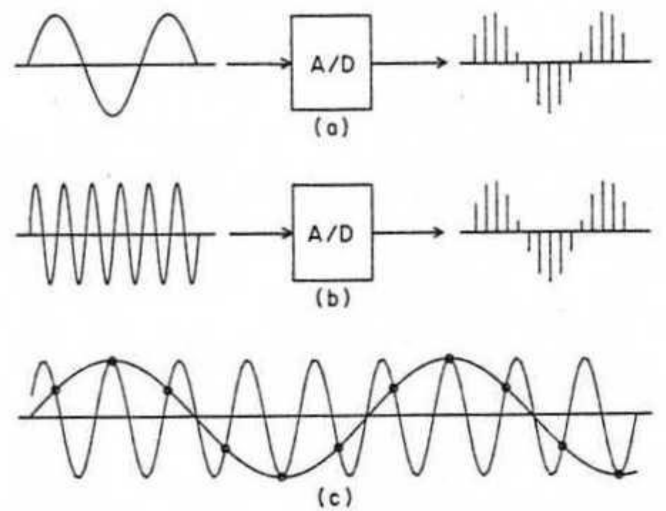
- **Questions to be answered:**
  - **The Sampling Process**: How to represent a continuous-time signal by a discrete-time signal without losing any information. (Why do we want to do this?)
  - **Discrete-Time Signals**: Representation (in time-domain), operations, and classifications of discrete-time signals.
  - **Typical Sequences**: Widely utilized sequences that can be utilized to represent other sequences.
  - **Correlation of Signals**: How similar two signals are? How can we utilize this?

Reading: Mitra (Ch2.1 – 2.3 & 2.9)

Analog signal $\xrightarrow{\text{sampling}}$ DT signal

Ch2.1: Discrete-Time Signals

- **The Sampling Process**
- Discrete-Time Signals
  - Time-Domain Representation
  - Operations on Sequences
  - Classification of Sequences
- Typical Sequences
- Correlation of Signals



→ $\boxed{\text{ADC}}$ → $\boxed{\text{DSP}}$ → $\boxed{\text{DAC}}$ →

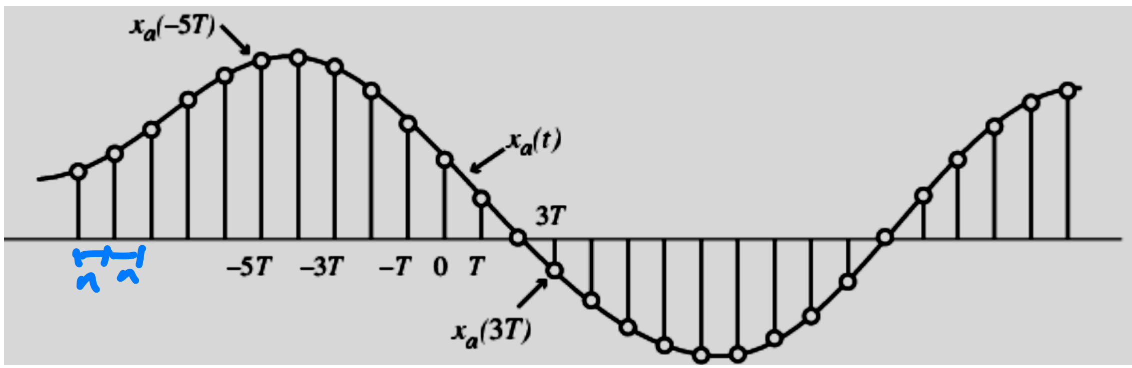
Digital signal processing

Digital Analog conversion

$A\cos(\Omega_0 t + \phi)$

Sampling: From Continuous to Discrete

- Often, a discrete-time sequence $x[n]$ is developed by **uniformly** sampling a continuous-time signal $x_a(t)$.



- The relation: $x[n] = x_a(t)|_{t=nT} = x_a(nT), n = \cdots, -1,0,1, \ldots$
- Time variable *t* is related to time variable *n* as

$F_T \triangleq \frac{1}{T}$

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

F_T

where F_T and Ω_T denoting the **sampling frequency** and **sampling angular frequency**.

$\Omega_T = 2\pi F_T$

most time we this

http://en.wikipedia.org/wiki/Angular_frequency

Sampling Process

frequency   Angular frequency

- Consider the continuous-time signal

$$x(t) = A\cos(2\pi f_o t + \phi) = A\cos(\Omega_o t + \phi).$$

amplitude   nT   phrase   normalized

- The corresponding discrete-time signal is

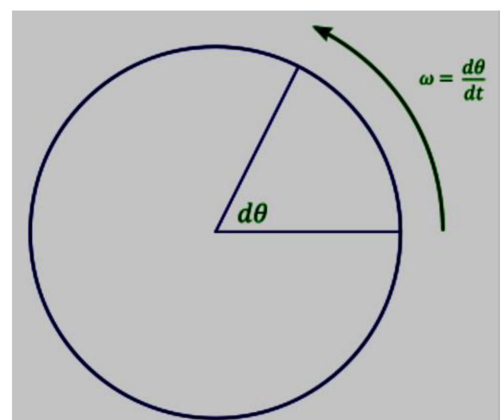
$t \to nT$

$$\underline{x[n]} = A\cos(\Omega_o nT + \phi) = A\cos\left(\frac{2\pi\Omega_o}{\Omega_T}n + \phi\right) = \underline{A\cos(\omega_o n + \phi)}$$

sampling period

where $\omega_o = \frac{2\pi\Omega_o}{\Omega_T} = \Omega_o T$ is the **normalized digital angular frequency**.

- If the unit of T is second, then
  - Unit of ω_o is radians/sample
  - Unit of Ω_o is radians/second
  - Unit of f_o is hertz (Hz).



Why Digital?

- Most of the physical signals are analog in nature
- Why bother to convert to digital?
- Robustness to Noise and Degradation!

continuous signal very hard to reproduce!

- Analog Signals:
  - Information is embedded in the waveform
  - Any distortion on the waveform —> distortion of the information

Digital → can use error detection ... methods

- Digital Signals:
  - Information is embedded in the "bits" behind the waveform.
  - Distortion on the waveform may not result in distortion of the bits

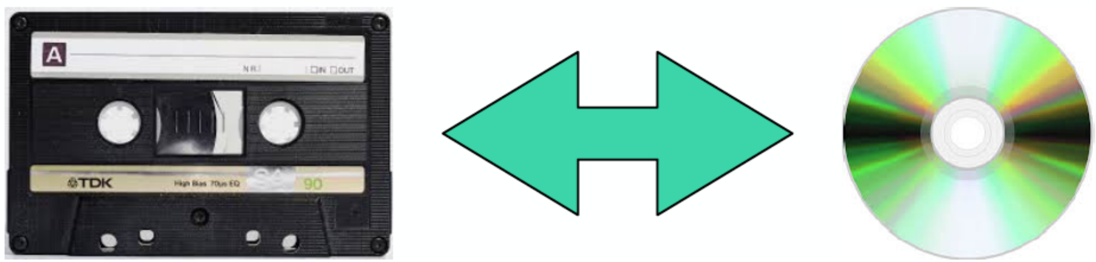
Detected as 1!

Robustness of Digital

compress coding

Digital → Analog

- Storage Applications
  - Audio Tape vs CD



- Telecommunication Applications
  - Long distance telephone has been very challenging
  - Resolved through digital transmission of signals

Sampling Process: Aliasing (Time Domain View)

$W_0 = \Omega_0 T = \frac{\Omega_0}{F_s}$

- Consider three continuous signals

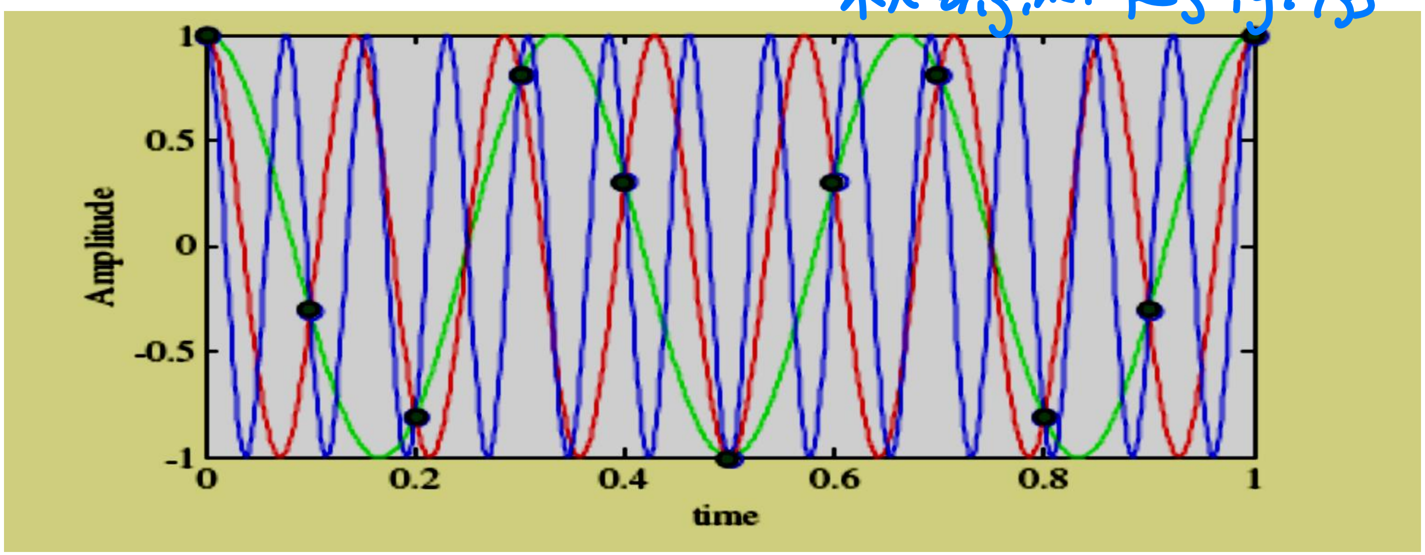
$$g_1(t) = cos(6\pi t),\ g_2(t) = cos(14\pi t), g_3(t) = cos(26\pi t).$$

$T = 0.1s$

- Sampling them at a rate of 10Hz generates

$$g_1(n) = cos(0.6\pi n),\ g_2(n) = cos(1.4\pi n), g_3(n) = cos(2.6\pi n)$$

不知 original 是 g_1, g_2, g_3



- Each sequence has exactly the same value for any given *n*.

Sampling Process: Aliasing (Time Domain View)

$\cos(c+2\pi n) = \cos c$

- This can be verified by observing that

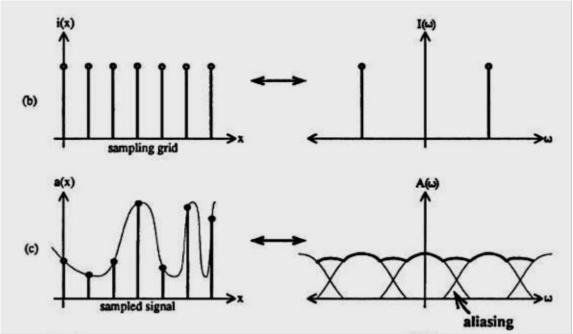
$$g_2(n) = cos(1.4\pi n) = \cos((\underline{2\pi - 0.6\pi})n) = \cos(0.6\pi n)$$
$$g_3(n) = cos(2.6\pi n) = \cos((2\pi + 0.6\pi)n) = \cos(0.6\pi n)$$

- As a result, the three sequences are **identical** and it is difficult to associate a unique continuous-time function with each of them.

- This phenomenon of a continuous-time signal of a higher frequency acquiring the identity of a sinusoidal sequence of a lower frequency after sampling is called **aliasing**.

- **How can we solve this problem?**

$f_s > 2f_{max}$



Sampling Theorem – Time Domain View

- Recall $\omega_o = \frac{2\pi\Omega_o}{\Omega_T}$.

- Thus, if $|\Omega_T| > 2|\Omega_o|$, then the normalized digital angular frequency ω_o obtained by sampling the parent continuous-time signal will be in the range $-\pi < \omega_o < \pi$. ⇨ No aliasing

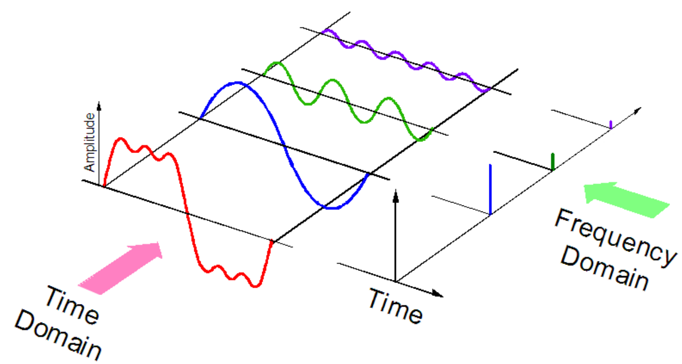
  no ambiguity

- Otherwise, if $|\Omega_T| < 2|\Omega_o|$, ω_o will foldover into a lower digital frequency $\omega_o = \langle 2\pi\Omega_o/\Omega_T \rangle_{2\pi}$ in the range $-\pi < \omega_o < \pi$.

- Thus, **to prevent aliasing the sampling frequency $\boldsymbol{\Omega_T}$ should be greater than 2 times of the frequency $\boldsymbol{\Omega_o}$**.

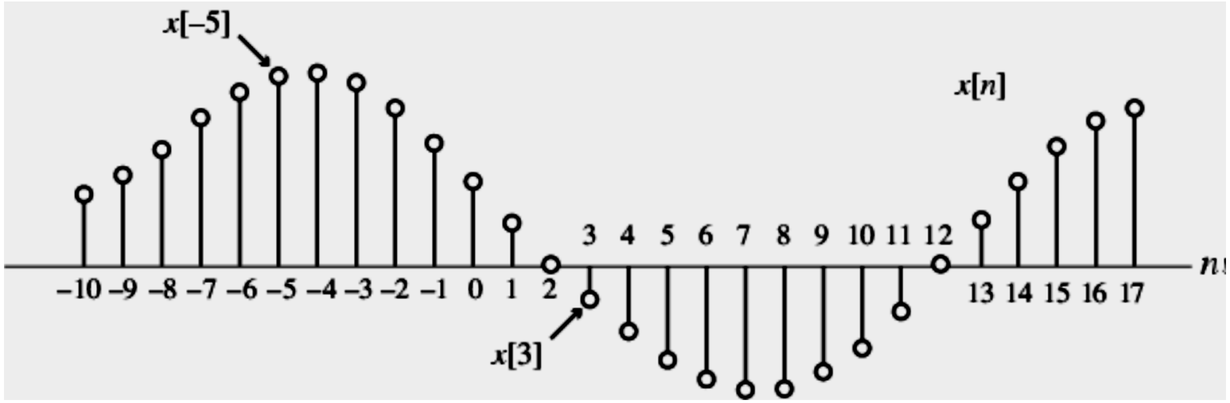
Ch2.1: Discrete-Time Signals

- The Sampling Process
- Discrete-Time Signals
  - **Time-Domain Representation**
  - Operations on Sequences
  - Classification of Sequences
- Typical Sequences
- Correlation of signals



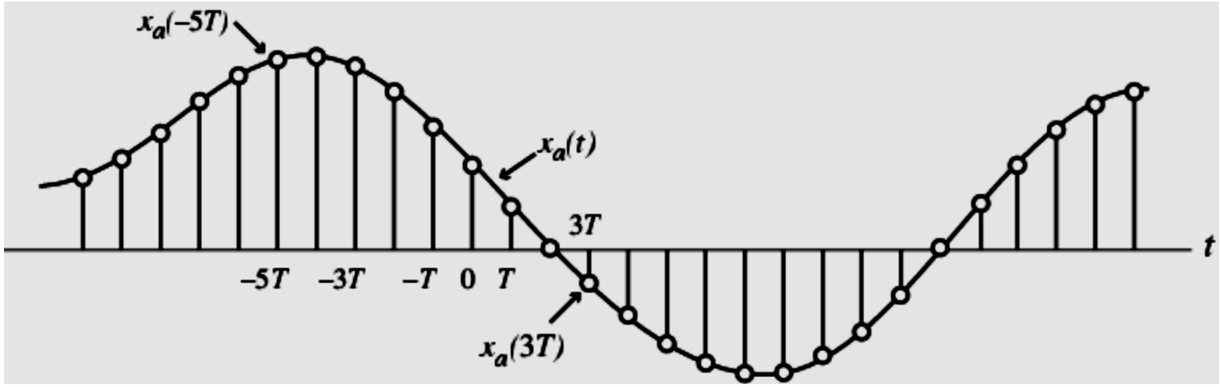
Time-Domain Representation

- Discrete-time signals are represented as sequences of numbers, called **samples.**
- Sample value of a typical sequence is denoted as $x[n]$ with n being an **integer**.
- Discrete-time signal is represented by $\{x[n]\}$.



Generating Discrete-Time Signals

- In some applications, a discrete-time signal $\{x[n]\}$ may be generated by periodically sampling a continuous-time signal $x_a(t)$ at uniform intervals of time.
- The n-th sample is given by $x[n] = x_a(t)|_{t=nT} = x_a(nT)$
- The spacing T is called the sampling interval/period.
- Reciprocal of T is called sampling frequency: $F_T = \frac{1}{T}$.



Real and Complex Sequence

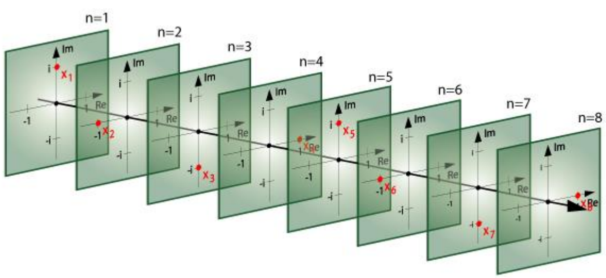
- $x[n]$ is called the n-th **sample** of the sequence, no matter whether $\{x[n]\}$ has been obtained by sampling.

if all elements are all real

- $\{x[n]\}$ is a **real sequence** if $x[n]$ is real for **all** values of n. Otherwise, $\{x[n]\}$ is complex.

- A complex sequence $\{x[n]\}$ can be written as $\{x[n]\} = \{x_{re}[n]\} + j\{x_{re}[n]\}$ where $\{x_{re}[n]\}$ and $\{x_{im}[n]\}$ denote the real and imaginary parts.

- Normally, braces are ignored to denote a sequence if there is no ambiguity.



Example

- Let $x[n] = \cos(0.25n)$ and $y[n] = exp(j0.3n)$ denote two sequences.

  → for any n, will get a real sequence!

  $x[n]$: real $y[n]$ = complex

  – Are they real or complex sequences?
  – Find the real and imagine parts of both signals.
  – Find the complex conjugate sequence.

b) $\mathrm{Re}\{x[n]\} = \cos(0.25n)$

$\mathrm{Im}\{x[n]\} = 0$

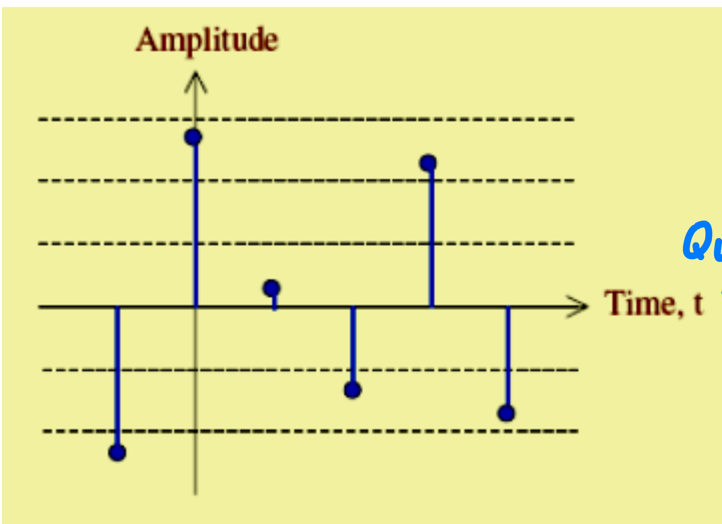
c) $x[n]^* = \cos(0.25n)$

$y[n]^* = \cos(0.3n) - j\sin(0.3n)$

Types of Discrete-Time signals

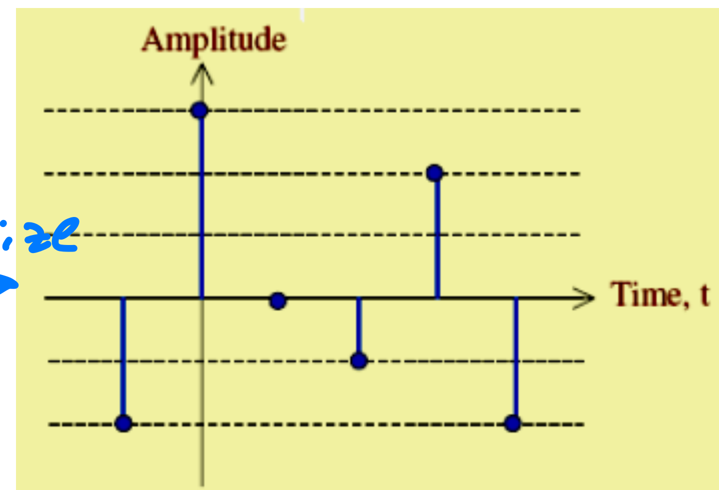
- **Sampled-data signals**: Samples are continuous-valued.
- **Digital signals**: Samples are discrete-valued.
- Digital signals are normally obtained by quantizing the sample values by rounding or truncation.

analog



A sampled - data signal

Quantize



A digital signal

Length of Discrete-Time signals

- A discrete-time signal may be a **finite-length or infinite-length sequence**.

from N_1 to N_2

- Finite-length sequence is only defined for finite time interval: $-\infty < N_1 \leq n \leq N_2 < \infty$ where the length is $N = N_2 - N_1 + 1$. A length-N sequence is referred to as a N-point sequence.

- **Example:** $x[n] = n^2, -3 \leq n \leq 4$, what is the length? 8

- The length of a finite-length sequence can be increased by **zero-padding**, i.e., by appending it with zeros.

- **Example:** $x[n] = \begin{cases} n^2, -3 \leq n \leq 4 \\ 0, 5 \leq n \leq 8 \end{cases}$, what is the length?

Add 4 '0's → length = 12

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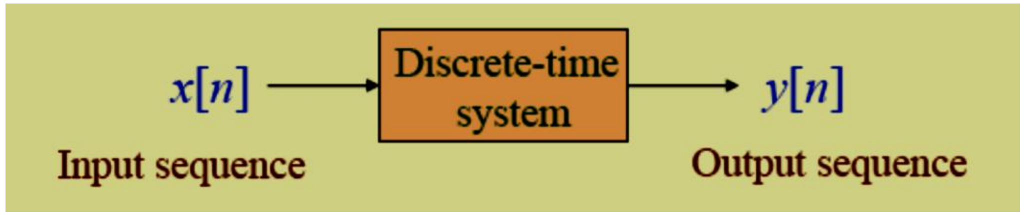
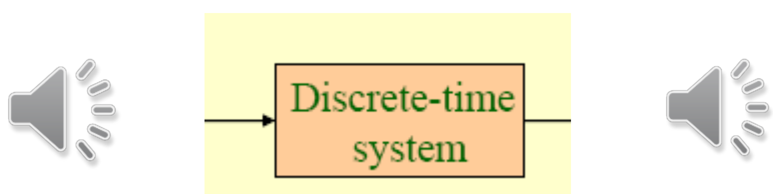
Operations on Sequences



- **Purpose:** Develop another sequence with more desirable properties.

- **Example:**
  - Input signal: Signal corrupted by additive noise
  - Design a discrete-time system to generate an output by removing the noise component.

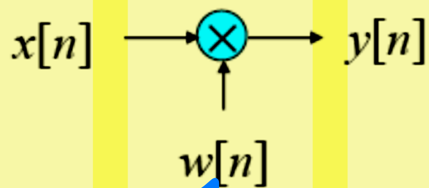


Basic Operations

不是multiplication!

- Product (Modulation) operation:

  – Modulator



$$y[n] = x[n] \cdot w[n]$$

this is designed by me

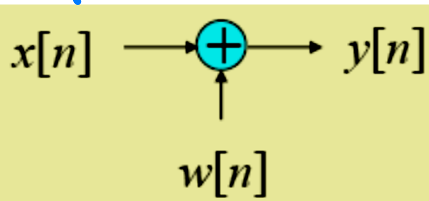
- Application: Develop a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length called the Window Sequence. The process is called Windowing.

  x[n] -3 3

  only interest in subsequence

- Addition operation

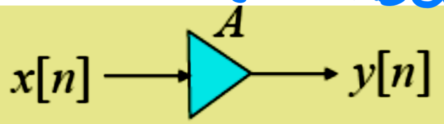
  – Adder



$$y[n] = x[n] + w[n]$$

Basic Operations

- Multiplication operation: number
  - Multiplier

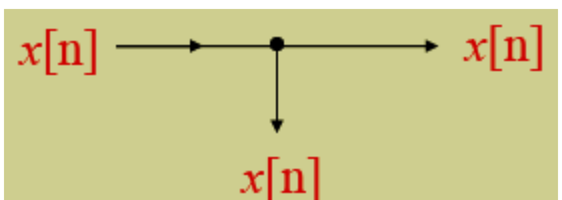


$$y[n] = A \cdot x[n]$$

- Time-shifting operation:

  堵阿

  $y[n] = x[n - N]$, where N is an integer.
  - N>0, delaying operation
  - N<0, advance operation.
- Time reversal (folding) operation: $y[n] = x[-n]$.
- Branching operation: Provide multiple copies of a sequence.



Basic Operations: Examples

- **Example 1**: Consider two sequences

length: 5!

$$a[n] = \{3, 4, 6, -9, 0\}, 0 \le n \le 4$$
$$b[n] = \{2, -1, 4, 5, -3\}, 0 \le n \le 4$$

Determine the new sequence generated from Product, Addition, and Multiplication (A=2).

Product: {6, -4, 24, -45, 0} Addition: {5, 3, 10, -4, -3} ...

- **Example 2**: Consider two sequences

$$a[n] = \{3, 4, 6, -9, 0\}, 0 \le n \le 4$$
$$f[n] = \{-2, 1, -3\}, 0 \le n \le 2 \rightarrow \text{Do zero-padding}$$

Determine the new sequence generated from Product, Addition, and Multiplication (A=0.5).

Basic Operations: Ensemble Averaging

- Signal model: Let d_i denote the noise vector corrupting the i-th measurement

zero mean, independent

$$\mathbf{x}_i = \mathbf{s} + \mathbf{d}_i$$

measure few times, take average

- Operation: Averaging over K measurements → Find mean

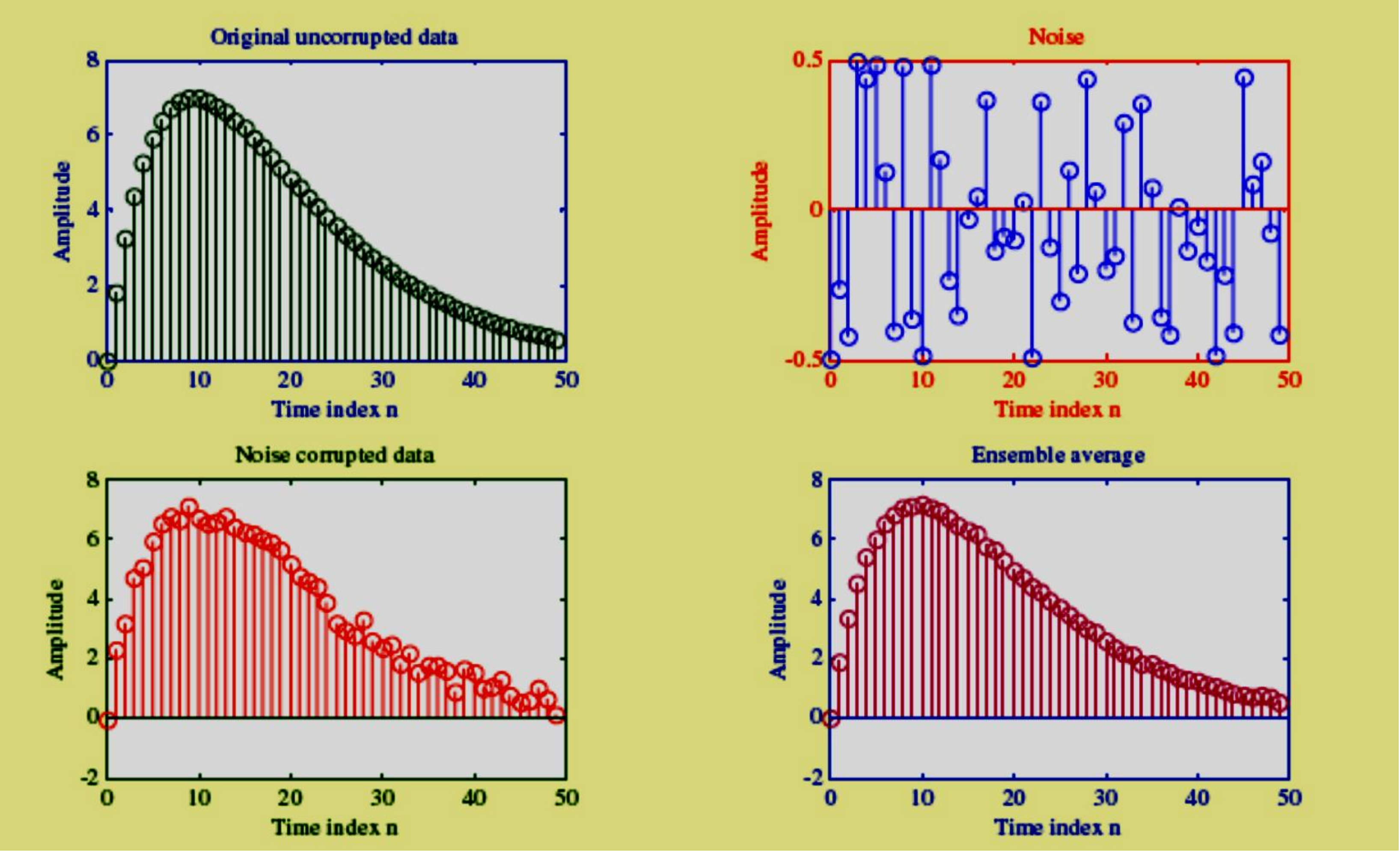
$$\mathbf{x}_{ave} = \frac{1}{K}\sum_{i=1}^{K}\mathbf{x}_i = \mathbf{s} + \frac{1}{K}\sum_{i=1}^{K}\mathbf{d}_i \rightarrow E[d] \approx 0$$

output is this

K↑ → $\frac{1}{K}\sum_{i=1}^{K} d_i$ close to mean 否则 $-E[d]$

- Purpose: Improving the quality of measured data corrupted by additive random noise.
- Assumption: The measured data remains essentially the same from one measurement to next, while additive noise is random.

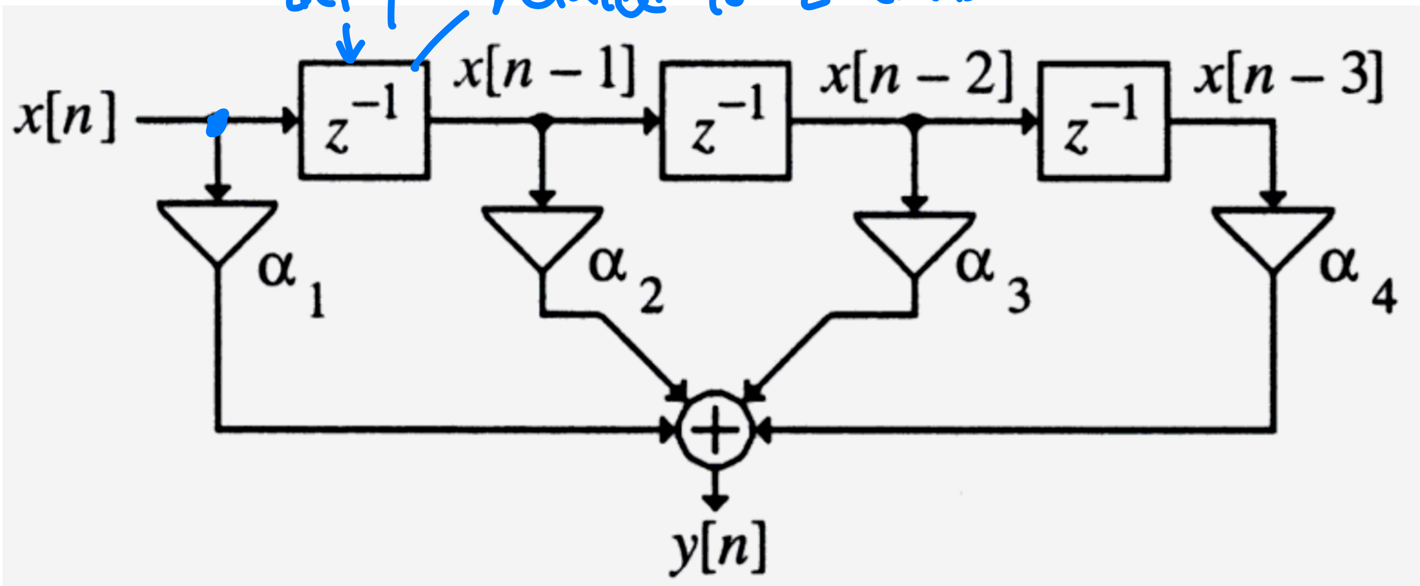
Basic Operations: Ensemble Averaging



3rd topic

Combinations of Basic Operations

delay related to z-transform



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

Add different components

What are the basic operations utilized?

Sampling Rate Alteration

for some reason → change the sequence

- **Definition:** Generating a new sequence $y[n]$ with a sampling rate F'_T higher or lower than that of the sampling rate F_T of a given sequence $x[n]$. F_T
- Sampling rate alteration ratio $R = \frac{F'_T}{F_T}$
- If $R > 1$, the process is called **interpolation**. → higher sampling frequency
  - Interpolator = up-sampler + DT system ← more sample
- If $R < 1$, the process is called **decimation**. → lower sampling freq.
  - Decimator = DT system + down-sampler → less sample

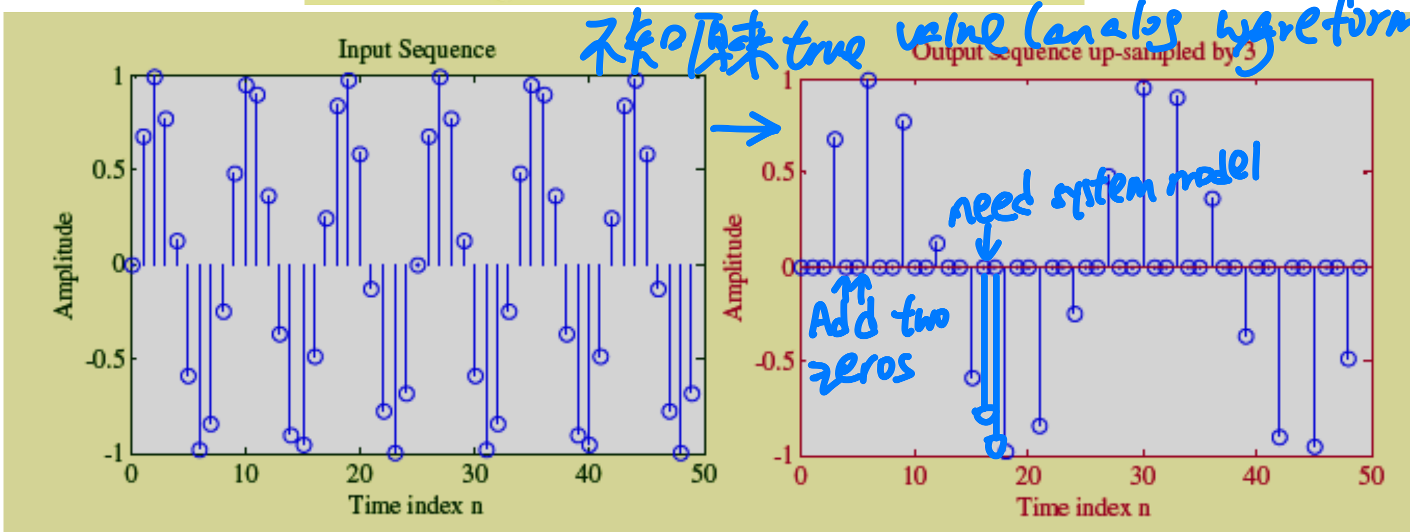
higher Sampling

Operations: Up-Sampling

- In up-sampling by an integer factor of L>1, L-1 equidistant zero-valued samples are inserted by the up-sampler between each two consecutive samples.

↙ Expand oringal value

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$



What kinds of DT systems do we need? Why?

X Further discussion

The DT system following the up-sampler replaces the inserted zero-valued samples with more appropriate values given by a linear combination of samples of x[n].

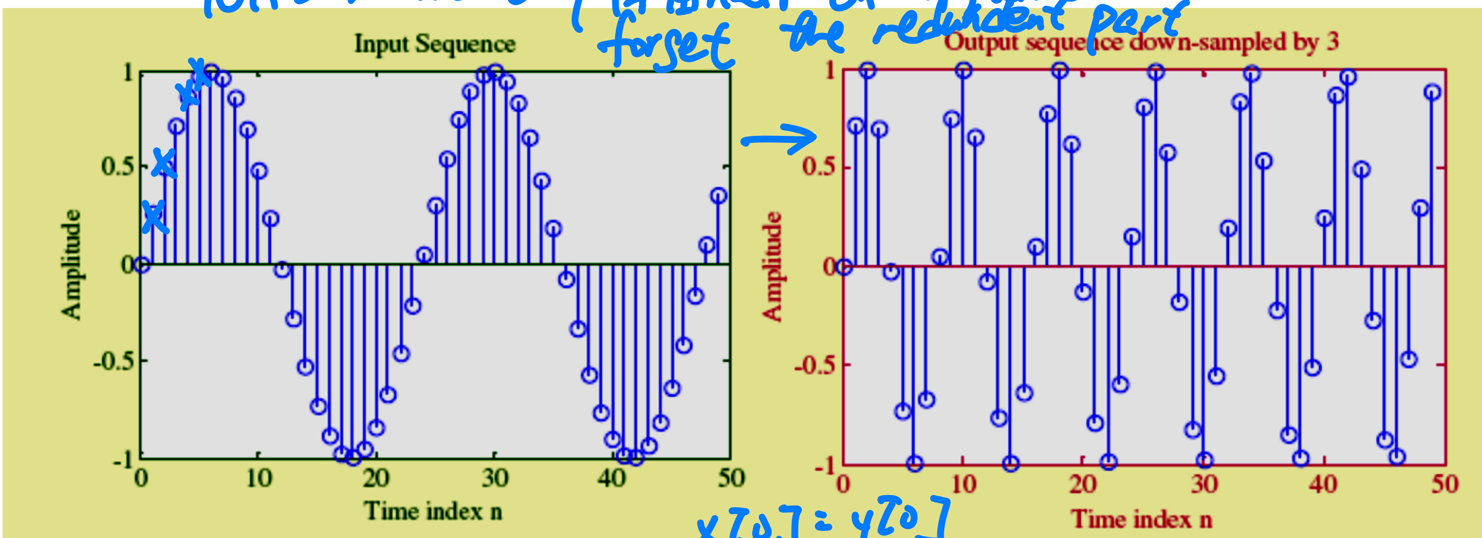
before up-sample, apply system over this sequence → get more reasonable missing value

Operations: Down-Sampling

- In down-sampling by an integer factor of M>1, every M-th samples of the input sequence are kept and the other samples are removed

$$y[n] = x[nM]$$

10Hz → 5Hz Only 保留 half-of-sequence
forget the redundant part



What kinds of DT systems do we need? Why?

The DT system preceding the down-sampler ensures that the input signal is appropriately band-limited to prevent aliasing that is caused by the down-sampling operation.

May have some effect of the original signal → need to compensate

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  - **Classification of Sequences**
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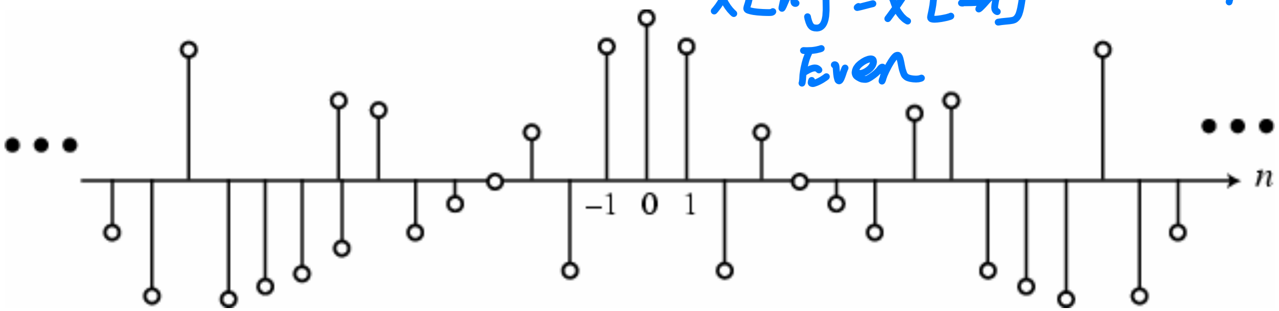
for infinite-length sequence

Sequence Classification: Symmetry

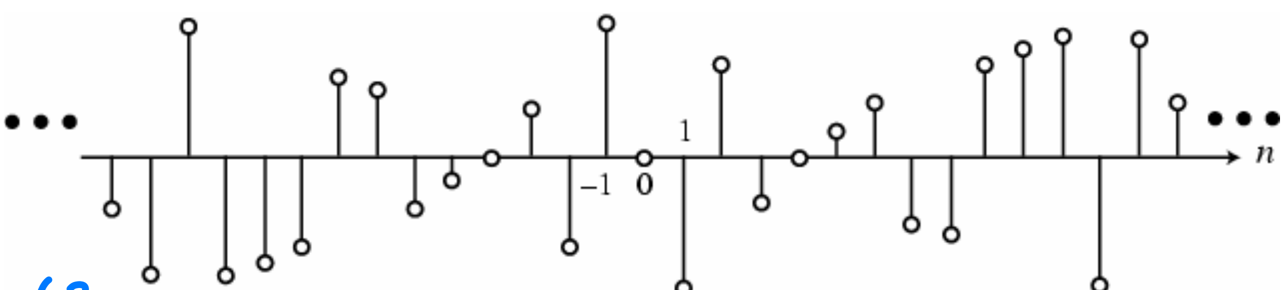
- **Conjugate-symmetric** sequence: $x[n] = x^*[-n]$. What if it is real?

$x[n] = x[-n]$ Even

$x[n] = x[-n]$



- **Conjugate-antisymmetric** sequence: $x[n] = -x^*[-n]$. If real?



$2+3j$

- What is $x[0]$ for different cases?

$= \cos(t) + j\sin(t)$

CS $x[n] = x^*[-n]$ $x[0] = x^*[0]$ real

CAS $x[n] = -x^*[-n]$ $x[0] = -x^*[0]$ pure imaginary

→ have special properties

Representing a Complex Sequence

- Any complex sequence can be expressed as a **sum** of its conjugate-symmetric part and its conjugate-antisymmetric part: $x[n] = x_{cs}[n] + x_{ca}[n]$ where

Easy check!

$$x_{cs}[n] = \frac{1}{2}(x[n] + x^*[-n]) \qquad x_{cs}[n] = x_{cs}^*[n]$$

$$x_{ca}[n] = \frac{1}{2}(x[n] - x^*[-n]) \qquad x_{ca}[n] = -x_{ca}^*[n]$$

- What about a real sequence?

$g[-n] = \{3, -j2, -5-j6, 4-j2, -2+j3, 1+j4, 0\}$

$g^*[-n] = \{3, 2j, -5+j6, 4+j2, -2-j3, 1-j4, 0\}$ -ve side length == +ve side length

- **Example:** Consider a length-7 sequence defined for $-3 \leq n \leq 3$, $g[n] = \{0, 1+j4, -2+j3, 4-j2, -5-j6, -j2, 3\}$.
  Determine its conjugate sequence, and its conjugate-symmetric and conjugate-antisymmetric parts.

Finite-Length Sequence

- A length-N sequence $x[n], 0 \leq n \leq N-1$, can be expressed as $x[n] = x_{pcs}[n] + x_{pca}[n]$ where

$$x_{pcs}[n] = \frac{1}{2}(x[n] + x^*[< -n >_N]), 0 \leq n \leq N-1,$$

$$x_{pca}[n] = \frac{1}{2}(x[n] - x^*[< -n >_N]), 0 \leq n \leq N-1,$$

  are the **periodic conjugate-symmetric** and **periodic conjugate-antisymmetric** parts, respectively.

- For a real sequence, they are called the **periodic even** $x_{pe}[n]$ and **periodic odd** parts $x_{po}[n]$, respectively.

$r = \langle -n \rangle_N \in [0, N-1]$, result must between

$= -n + k \cdot N$

e.g. $N = 4$

$\langle -1 \rangle_4 = \langle -1 + 4 \rangle_4 = 3$ 1nk Rotation

$p_{cs} = \{-2+5j, 1+2.5j, 1-2.5j, -2-5j\}$

p_{ca}

Example

$u^*[\langle -n \rangle_N] = \{-5+j6, 4+j2, -2-j3, 1-j4)$

- Consider a length-4 sequence defined for $0 \leq n \leq 3$

$$u[n] = \{1 + j4, -2 + j3, 4 - j2, -5 - j6\}.$$

Determine its conjugate-symmetric part.

- **Solution:** Its conjugate sequence is given by

$$u^*[n] = \{1 - j4, -2 - j3, 4 + j2, -5 + j6\}.$$

Next, determine the modulo-4 time-reversed version

$$u^*[< -n >_4] = u^*[-n + 4]$$

For example, $u^*[< -0 >_4] = u^*[0] = 1 - j4$

Thus, $u^*[< -n >_4] = \{1 - j4, -5 + j6, 4 + j2, -2 - j3\}$

Finally, $$x_{pcs}[n] = \frac{1}{2}(x[n] + x^*[< -n >_N])$$

Periodic Conjugate-Symmetric

- A length-N sequence $x[n], 0 \leq n \leq N-1$, is called a **periodic conjugate-symmetric** sequence if

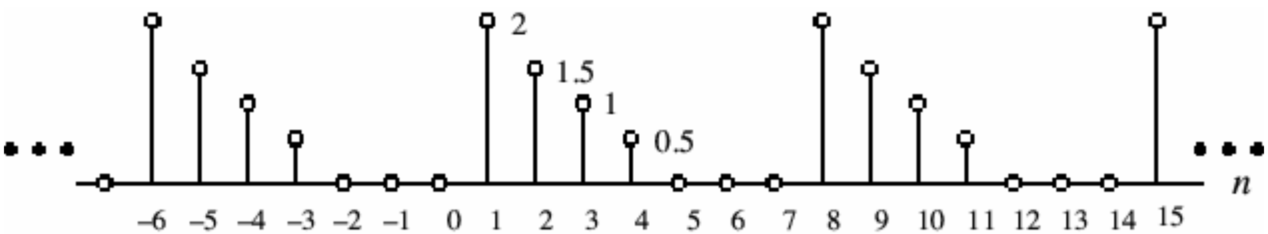
$$x[n] = x^*[< -n >_N] = x^*[< N-n >_N]$$

and is called a **periodic conjugate-antisymmetric** sequence if

$$x[n] = -x^*[< -n >_N] = -x^*[< N-n >_N]$$

Periodic Sequence

- A sequence $\tilde{x}[n]$ satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called a **periodic** with period N where N is a positive integer and k is any integer.
- The smallest value of N is called the **fundamental period**.



- A sequence not satisfying the periodicity condition is called an **aperiodic sequence**.

Energy and Power Signals

- Total **energy** of a sequence $x[n]$ is defined by

$$\mathcal{E}_x[n] = \sum_{n=-\infty}^{\infty} |x[n]|^2$$ power/energy

- An infinite-length sequence with finite sample values may or may not have finite energy. A finite-length sequence with finite sample values has finite energy.

- The **average power of an aperiodic sequence** is defined as $P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$. Energy

total 2k+1 項

- The **average power of a periodic sequence** $\tilde{x}[n]$ with a period N is given by $P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$.

only consider in one period

Example

- **Example:** Consider the causal sequence $x[n]$ defined by

$$x[n] = \begin{cases} 3(-1)^n, n \geq 0 \\ 0, n < 0 \end{cases}$$

Determine the energy and power.

- **Solution:**

Energy = ? infinite! $\sum_{n=0}^{\infty} 9 = \infty$

Its average power is given by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1}\left(9\sum_{n=0}^{K} 1\right) = \lim_{K\to\infty} \frac{9(K+1)}{2K+1} = 4.5$$

forget about convergence!

Energy and Power Signals

- An infinite energy signal with finite average power is called a **power signal**.

∞ energy
finite power

- **Example-** A periodic sequence which has a finite average power but infinite energy.

- A finite energy signal with zero average power is called an **energy signal**.

finite energy
0 power

- **Example-** A finite-length sequence which has finite energy but zero average power.

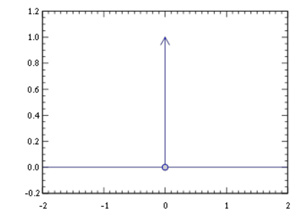
Ch2.1: Discrete-Time Signals

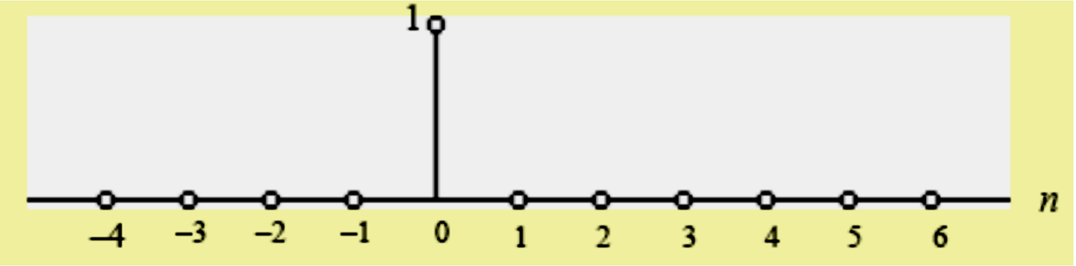
- The Sampling Process
- Discrete-Time Signals
  - Time-Domain Representation
  - Operations on Sequences
  - Classification of Sequences
- **Typical Sequences**
- Correlation of signals

Basic Sequences

$\delta[n]$

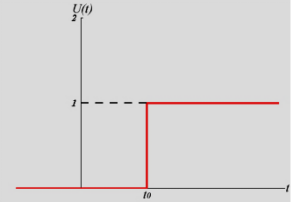
- **Unit sample** sequence $x[n] = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$

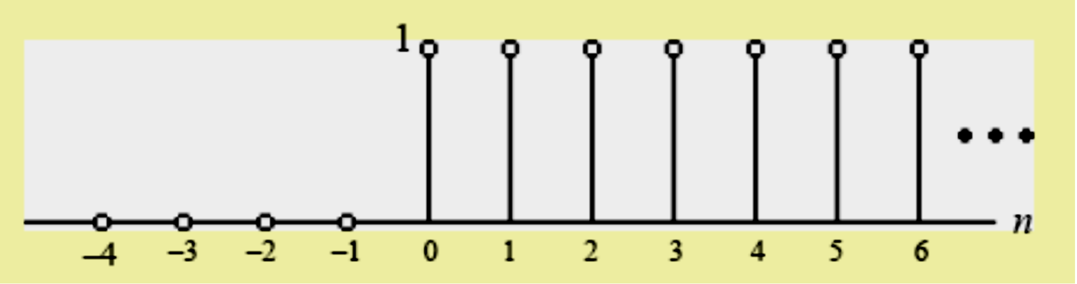




$\delta[n] = u[n] - u[n-1]$

- **Unit step** sequence: $u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$





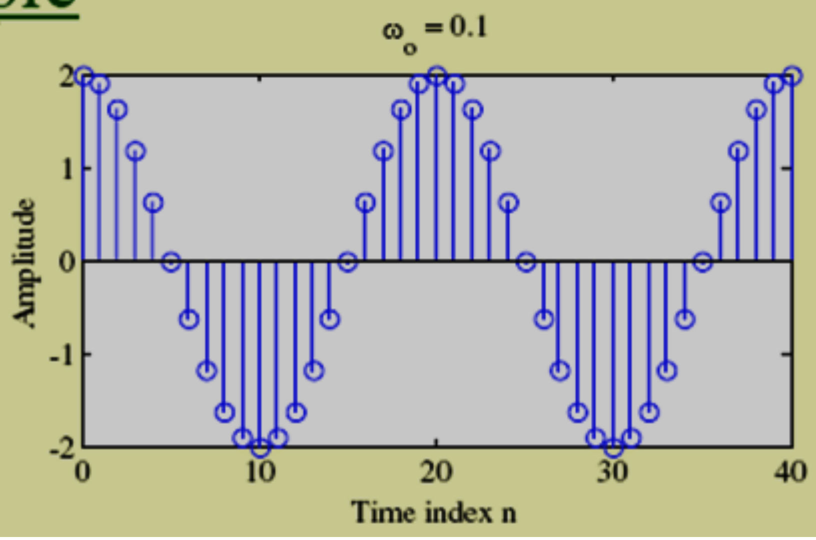
Basic Sequences

- **Real sinusoidal sequence** -

$$x[n] = A\cos(\omega_o n + \phi)$$

where A is the **amplitude**, ω_o is the **angular frequency**, and ϕ is the **phase** of $x[n]$

Example -



Basic Sequences

- **Exponential sequence -**

$|\alpha| = 1$

$$x[n] = \underline{A\alpha^n}, \quad -\infty < n < \infty$$

where A and α are real or complex numbers

- If we write $\alpha = e^{(\sigma_o + j\omega_o)}, A = |A|e^{j\phi}$,

then we can express

$$x[n] = |A|e^{j\phi}e^{(\sigma_o + j\omega_o)n} = \underline{x_{re}[n] + j\,x_{im}[n]},$$

where

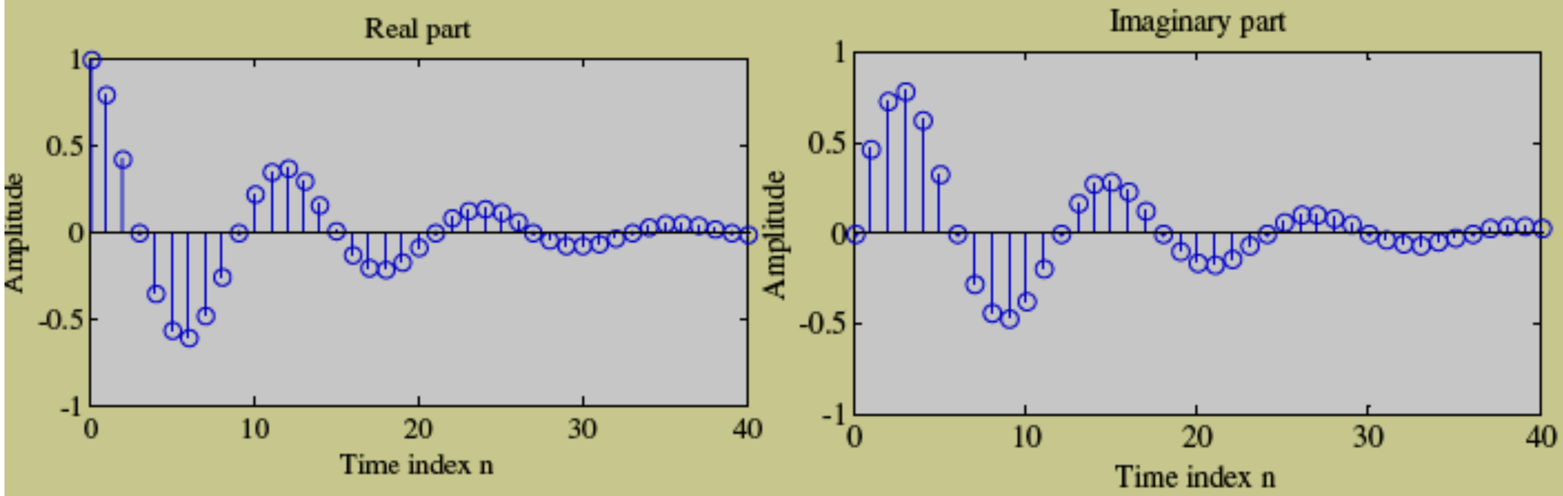
$$\underline{x_{re}[n] = |A|e^{\sigma_o n}\cos(\omega_o n + \phi)},$$

$$\underline{x_{im}[n] = |A|e^{\sigma_o n}\sin(\omega_o n + \phi)}$$

- Oscillate
- magnitude unchanged!

Basic Sequences

- $x_{re}[n]$ and $x_{im}[n]$ of a complex exponential sequence are real sinusoidal sequences with constant $(\sigma_o = 0)$, growing $(\sigma_o > 0)$, and decaying $(\sigma_o < 0)$ amplitudes for $n > 0$



$$x[n] = \exp(-\tfrac{1}{12} + j\tfrac{\pi}{6})n$$

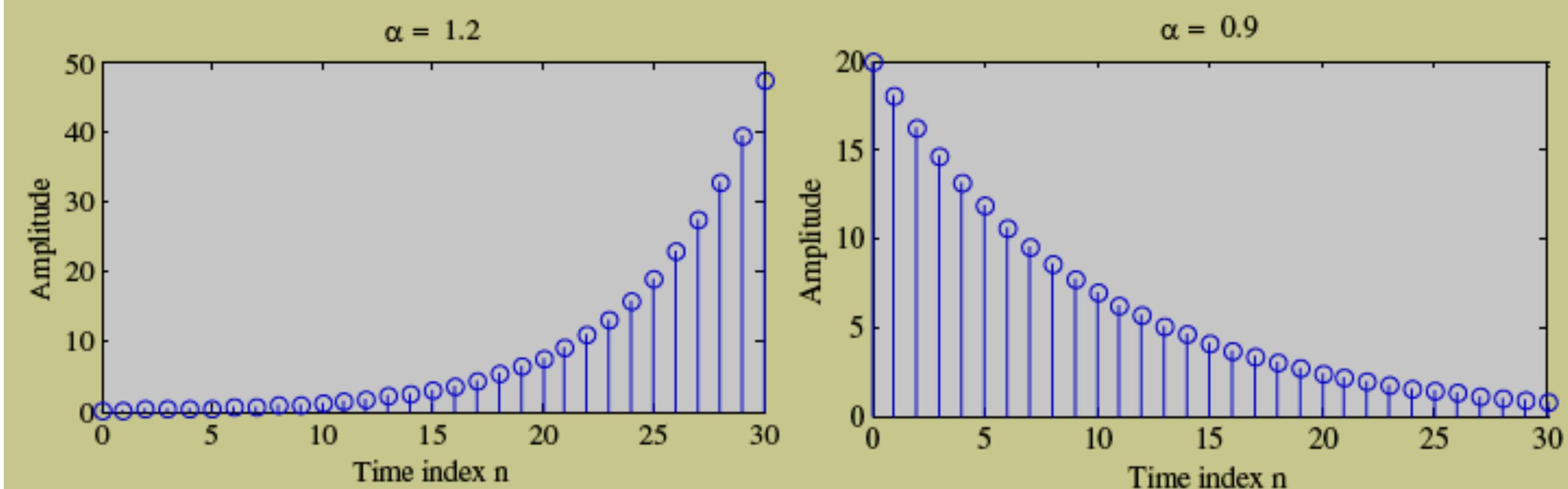
$\cos(-\frac{1}{12}n) + j\sin(\frac{\pi}{6}n)$

Basic Sequences

- **Real exponential sequence -**

$$x[n] = A\alpha^n, \; -\infty < n < \infty$$

where A and α are real numbers



Basic Sequences

may not be periodic!!!

- Sinusoidal sequence $A\cos(\omega_o n + \phi)$ and complex exponential sequence $B\exp(j\omega_o n)$ are periodic sequences of period N if $\omega_o N = 2\pi r$ where N and r are positive integers
- Smallest value of N satisfying $\omega_o N = 2\pi r$ is the **fundamental period** of the sequence
- To verify the above fact, consider

$$x_1[n] = \cos(\omega_o n + \phi)$$

$$x_2[n] = \cos(\omega_o (n + N) + \phi)$$

$x_1[n+N]$ 一定要一樣!

Basic Sequences

$\omega_o N = 2\pi r$

- Now $x_2[n] = \cos(\omega_o(n+N)+\phi)$

  $= \cos(\omega_o n+\phi)\cos\omega_o N - \sin(\omega_o n+\phi)\sin\omega_o N$

  $x_1[n]$ 1 0

  which will be equal to $\cos(\omega_o n+\phi) = x_1[n]$ only if

  $$\sin\omega_o N = 0 \text{ and } \cos\omega_o N = 1$$

- These two conditions are met if and only if

  $$\omega_o N = 2\pi r \quad \text{or} \quad \frac{2\pi}{\omega_o} = \frac{N}{r}$$

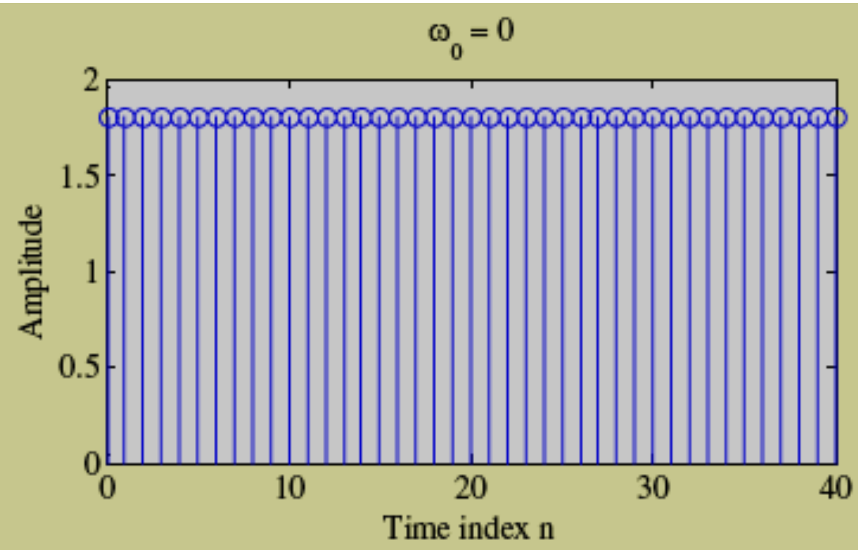
  rational number

Basic Sequences

- If $2\pi/\omega_o$ is a noninteger rational number, then the period will be a multiple of $2\pi/\omega_o$
- Otherwise, the sequence is **aperiodic**
- Example - $x[n] = \sin(\sqrt{3}n + \phi)$ is an aperiodic sequence

$\frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{3}} \notin \mathbb{Q}$

Basic Sequences



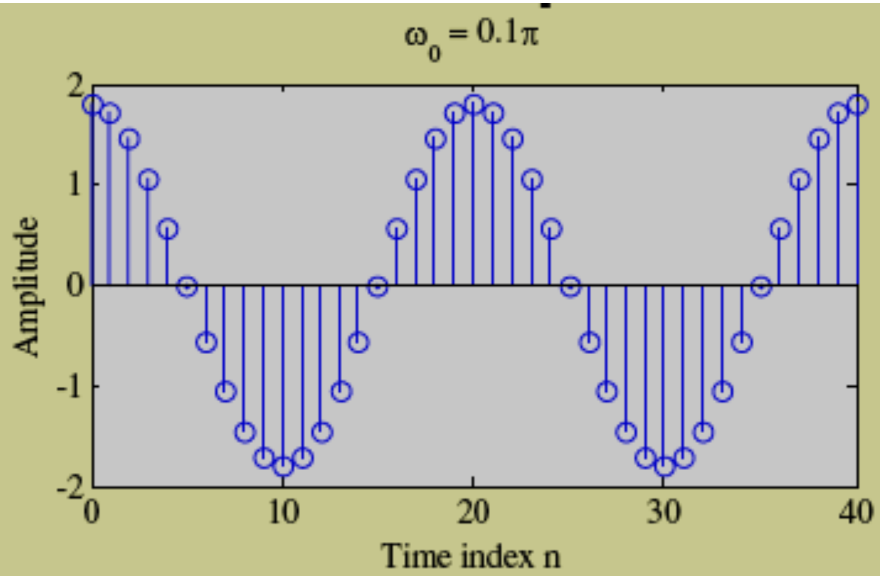
Periodic as 1

- Here $\omega_o = 0$

- Hence period $N = \dfrac{2\pi r}{0} = 1$ for $r = 0$

take limit!

Basic Sequences



- Here $\omega_o = 0.1\pi$
- Hence $N = \dfrac{2\pi r}{0.1\pi} = 20$ for $r = 1$

$\frac{2\pi r}{\omega_0} \neq$ int ?

Basic Sequences

- **Property 1** – Consider $x[n] = \exp(j\omega_1 n)$ and $y[n] = \exp(j\omega_2 n)$ with $0 < \omega_1 < \pi$ and $2\pi k < \omega_2 < 2\pi(k+1)$ where k is any positive integer. If $\omega_2 = \omega_1 + 2\pi k$, then $x[n] = y[n]$.

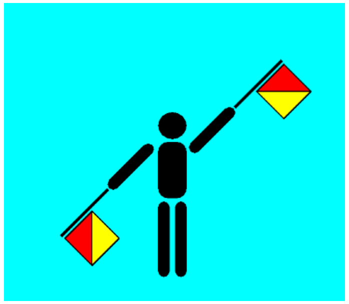
???

low freq

low freq.

- Because of Property 1, a frequency ω_o in the neighborhood of $\omega = 2\pi k$ is indistinguishable from a frequency $\omega_0 - 2\pi k$ in the neighborhood of $\boldsymbol{\omega = 0}$ and a frequency in the neighborhood of $\omega = \pi(2k+1)$ is indistinguishable from a frequency $\omega_0 - 2\pi k$ in the neighborhood of $\boldsymbol{\omega = \pi}$.

High or **Low** frequency?



Basic Sequences

- **Property 2**- The frequency of oscillation of $A\cos(\omega_o n)$ increases as ω_o increases from 0 to π, and then decreases as ω_o increases from π to 2π. Thus, frequencies in the neighborhood of $\omega = 0$ are called low frequencies, and frequencies in the neighborhood of $\omega = \pi$ are called high frequencies.

- Frequencies in the neighborhood of $\omega = 2\pi k$ are called low frequencies, and frequencies in the neighborhood of $\omega = \pi(2k+1)$ are called high frequencies.

- **Example:** Which is a low-frequency signal?

low frequency

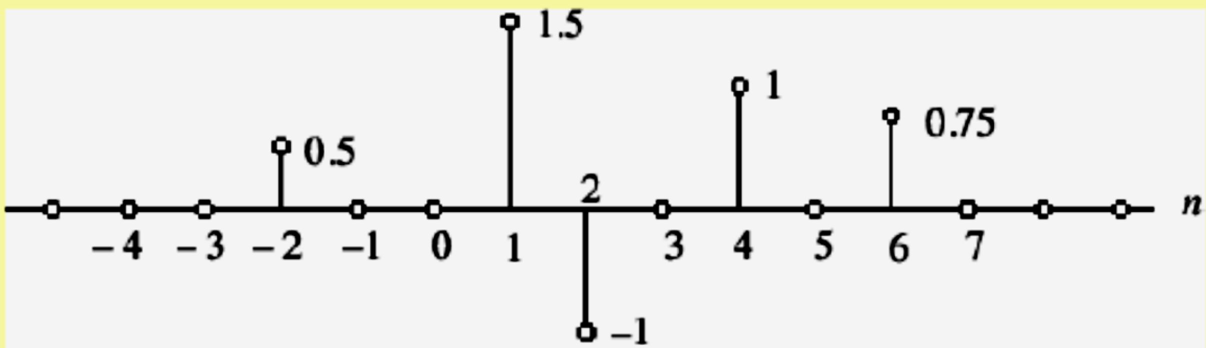
$$v_1[n] = \cos(0.1\pi n) = \cos(1.9\pi n)$$

high frequency

$$v_2[n] = \cos(0.8\pi n) = \cos(1.2\pi n)$$

Basic Sequences

- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions



$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\delta[n-6]$$

 Linear property

Ch2.1: Discrete-Time Signals

- The Sampling Process
- Discrete-Time Signals
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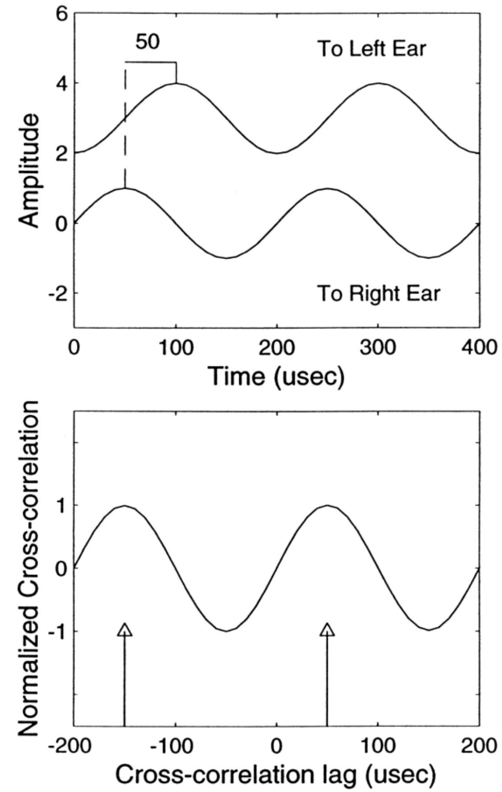
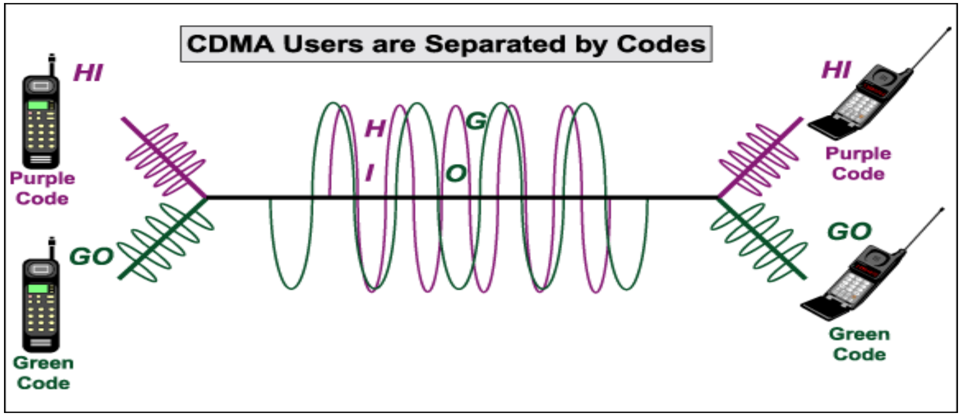
  - Operations on Sequences
  - Classification of Sequences
- Typical Sequences
- **Correlation of signals**



Correlation of Signals

- There are applications where it is necessary to compare one reference signal with one or more signals to determine the **similarity** between the pair and to determine additional information based on the similarity.
- For example, in digital communications, a set of data symbols are represented by a set of unique discrete-time sequences.
- If one of these sequences has been transmitted, the receiver has to determine which particular sequence has been received by comparing the received signal with every member of possible sequences from the set.

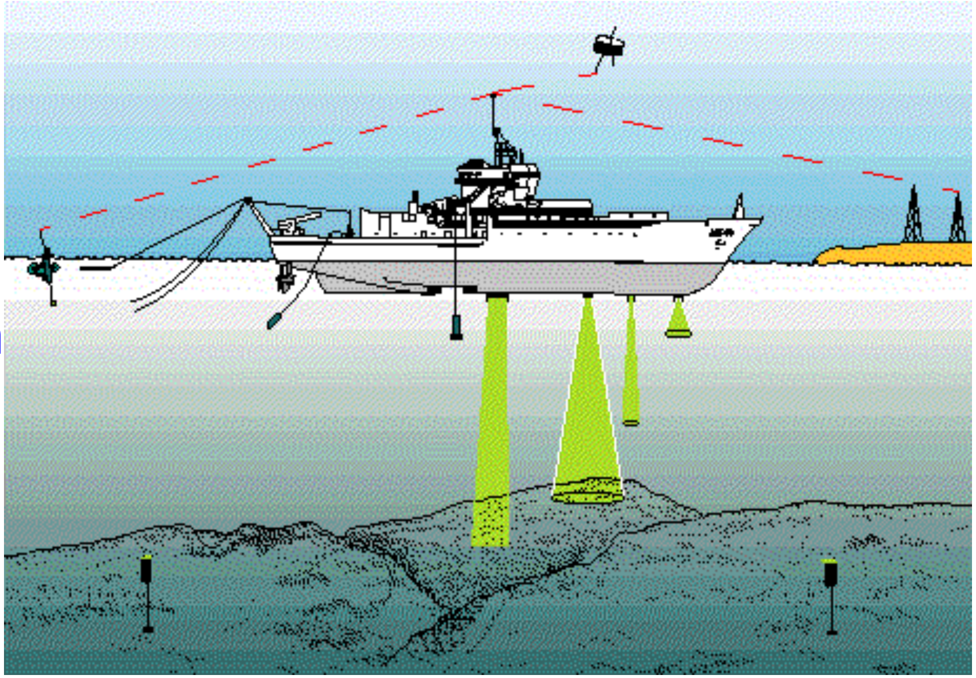
transmit
→ 2 bits
'0', '1'



Correlation of Signals

- Similarly, in **radar and sonar** applications, the received signal reflected from the target is a delayed version of the transmitted signal and by measuring the delay, one can determine the location of the target.

- The detection problem gets more complicated in practice, as often the received signal is corrupted by **additive random noise**.



Correlation of Signals

finite energy
0 power

- A measure of similarity between a pair of energy signals, $x[n]$ and $y[n]$, is given by the **cross-correlation sequence** defined by

align them
⇒ give us highest similarity

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l], l = 0, \pm 1, \pm 2, \ldots$$

- The parameter l is called **lag**, indicating the time-shift between the pair of signals. The **ordering** in the subscripts xy specifies that $x[n]$ is the reference while $y[n]$ being shifted with respect to $x[n]$.
- We can obtain

shift x,y

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l] = \sum_{m=-\infty}^{\infty} y[m+l]x[m] = r_{xy}[-l]$$

Correlation of Signals

- The **autocorrelation sequence** of $x[n]$ is given by

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x^*[n-l], l = 0, \pm 1, \pm 2, \ldots$$

← Energy 趋于 0 → correlation sequence condense

- **Note:** $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n]$ denotes the energy of $x[n]$.

- From $r_{yx}[l] = r_{xy}[-l]$, we know $r_{xx}[l] = r_{xx}[-l]$, implying that $r_{xx}[l]$ is an even function for real $x[n]$.

- **Question: Can you find the similarity between correlation and convolution?**

Normalized Forms of Correlation

Unit will make a large change!

- Normalized forms of autocorrelation and cross-correlation are given by

0 → very different

$$\rho_{xx}[\ell] = \frac{r_{xx}[\ell]}{r_{xx}[0]}, \qquad \rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}}$$

→ ±1 → very similar

- They are often used for convenience in comparing and displaying
- Note: $|\rho_{xx}[\ell]| \le 1$ and $|\rho_{xy}[\ell]| \le 1$ independent of the range of values of $x[n]$ and $y[n]$

Correlation for Power Signals

- The cross-correlation sequence for a pair of power signals, $x[n]$ and $y[n]$, is defined as

$$r_{xy}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]y[n-\ell]$$

very similar to energy

但又能计 power

- The autocorrelation sequence of a power signal $x[n]$ is given by

$$r_{xx}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]x[n-\ell]$$

$\ell = 0$, $r_{xx}[0]$ = power

Correlation for Periodic Signals

Only consider in one period

- The cross-correlation sequence for a pair of periodic signals of period N, $\tilde{x}[n]$ and $\tilde{y}[n]$, is defined as

$$r_{\tilde{x}\tilde{y}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{y}[n-\ell]$$

- The autocorrelation sequence of a periodic signal $\tilde{x}[n]$ of period N is given by

$$r_{\tilde{x}\tilde{x}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{x}[n-\ell]$$

- Note: Both $r_{\tilde{x}\tilde{y}}[\ell]$ and $r_{\tilde{x}\tilde{x}}[\ell]$ are also periodic signals with a period N