T11

Spectrum analyzer
DTFS, DTFT, DFT and FFT
OFDM

LCC Differential equation
Standard form of frequency response
The second-order system

Time interval
↑ time duration, better freq. resolution

more time duration
⇒ more information (sample points)

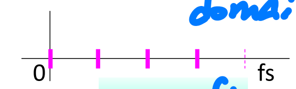
want consider frequency domain

Spectrum Analyzer

Matlab - fft

time domain ↓ frequency domain!

x = [1 2 3 4]
X[k] = [X[0] X[1] X[2] X[3]]



Frequency resolution ? $\frac{fs}{4}$

Frequency resolution = $1/W$ Hz

Highest frequency = $f_s/2$ Hz — sampling thm.

Time duration of a CT signal = *W* in second

Sampling rate = f_s in Hz or T_s in second

0 ~ fs

No. of samples = W / T_s

Frequency interval = f_s / No. of samples = $1/W$ in Hz

what is the frequency interval?

x = [1 2 3 4 5 6 … 16]

Frequency resolution ? $\frac{fs}{16}$

Question : How to improve the frequency resolution if f_s is fixed ? longer time duration

Case 2 have higher freq. resolution ⇒ know more details!

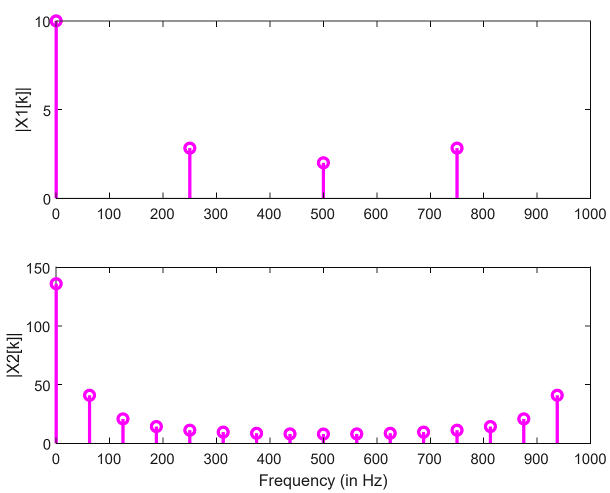
16 sample points!

x1 = [1 2 3 4]

fft(x1)

x2 = [1 2 3 4 5 6 … 16]

fft(x2)



obtain this! fft/length

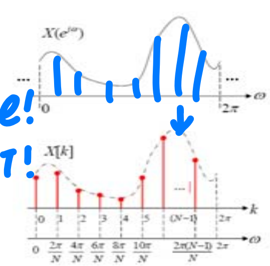
☆ DTFS, DTFT, DFT and FFT

DTFS : $a_k = \frac{1}{N}\sum_{k=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$ ← 只知这个就！

DTFT : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

continuous! computer 不可 handle!

DFT : $X[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{N}} = \boxed{\sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}}$ DFT!



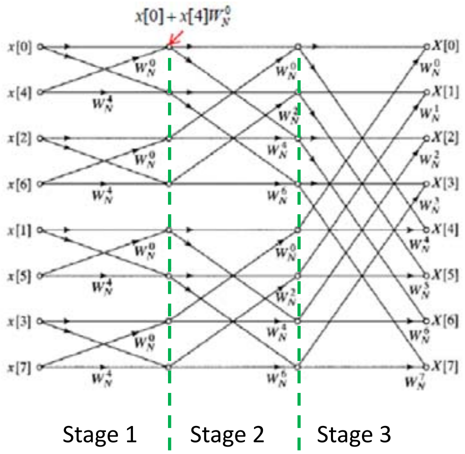
```
For length N input vector x, the DFT is a length N vector X,
with elements
                 N
   X(k) =       sum  x(n)*exp(-j*2*pi*(k-1)*(n-1)/N), 1 <= k <= N.
                n=1
```

from 1 to N!

Matlab

$O(N\log N)$ fft = DFT!

e.g. Number of DFT values N = 8



Direct calculation

$$X[k] = \sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

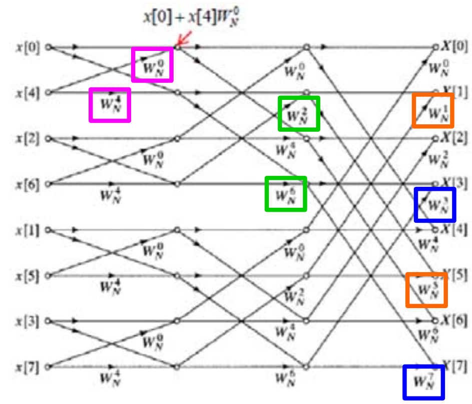
8次!

FFT

$$W_N^k = e^{-j2\pi k/N}$$

Number of stages $= log_2(8) = 3$

N = 8



$$W_N^k = e^{-j2\pi k/N}$$

$\boxed{W_8^0} = 1 \qquad \boxed{W_8^4} = e^{-j2\pi(4)/8} = -1$

$$W_8^2 = e^{-j2\pi(2)/8} = -j$$

$$W_8^6 = e^{-j2\pi(6)/8} = j$$

$$W_8^1 = e^{-j2\pi(1)/8} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^5 = e^{-j2\pi(5)/8} = -\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$W_8^3 = e^{-j2\pi(3)/8} = -\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$W_8^7 = e^{-j2\pi(7)/8} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

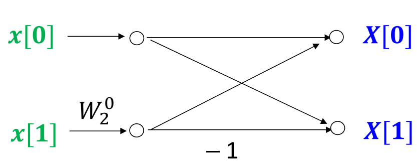
3 stages!

e.g. Number of DFT values N = 2

$$X[k] = \sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$X[0] = x[0]\, e^{-j(0)\frac{2\pi}{2}(0)} + x[1]\, e^{-j(0)\frac{2\pi}{2}(1)}$$

$$X[1] = x[0]\, e^{-j(1)\frac{2\pi}{2}(0)} + x[1]\, e^{-j(1)\frac{2\pi}{2}(1)}$$



1st stage

Direct calculation N^2

Complex multiplication = 2 (2)

Complex addition = 2 $N(N-1)$

FFT 1 Butterfly

Complex multiplication = 1

Complex addition/subtraction = 2

$$X[0] = x[0] + x[1]W_2^0$$

$$X[1] = x[0] - x[1]W_2^0$$

e.g. Number of DFT values N = 8

$$X[k] = \sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

Direct calculation

Complex multiplication = 8 (8) = 64

Complex addition = 8 (7) = 56

FFT 12 Butterflies $\frac{N\log N}{2}$

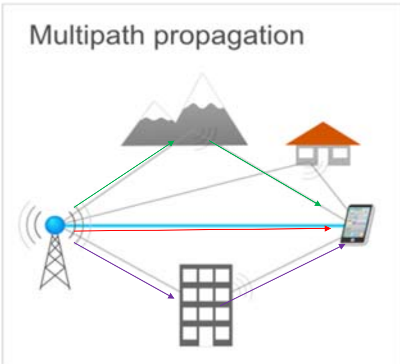
Complex multiplication = 1 (12) = 12

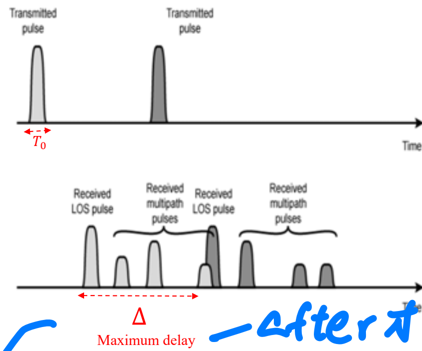
Complex addition/subtraction = 2 (12) = 24

$N\log N$

Reflection

Multipath Propagation and Inter-Symbol-Interference (ISI)





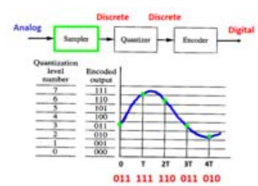
after 才傳送

time gap!

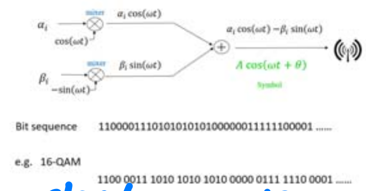
不想 interference!

阿

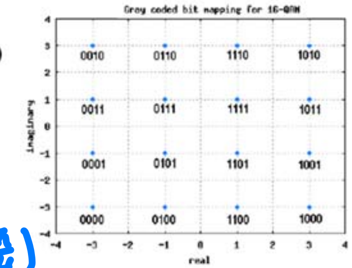
Sampling



I/Q Channel



Constellation diagram

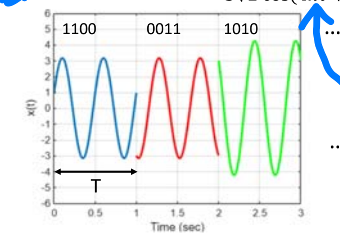


EM wave is analog!

$3\sqrt{2}\cos(4\pi t+\pi/4)$

$\leftarrow 3+3j$

$=\sqrt{18}\cos(\omega t+\frac{\pi}{4})$



transmitted RF signal $= \sum_i \sigma_i r(t-iT)e^{j\omega t}$

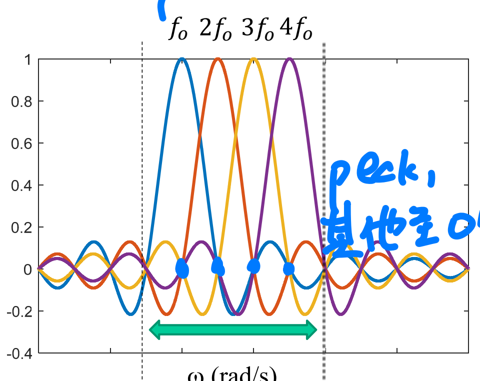
$x(t)$ The information signal that convey bits

The complex sinusoid representing the I- and Q- carriers

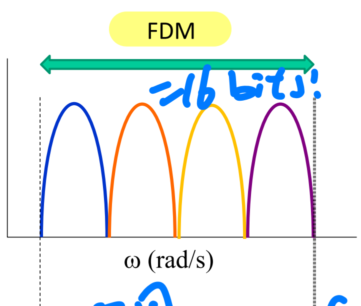
Question : How to use low symbol rate to achieve high bit throughput ?

???. set sub carrier: Each one fourier coefficients, transmit simultaneous

OFDM (Orthogonal Frequency Division Multiplexing)



peak, 其他是0!



=16 bits!

不用 carrier frequency!

Tradeoff: pay more to buy license of bandwidth!

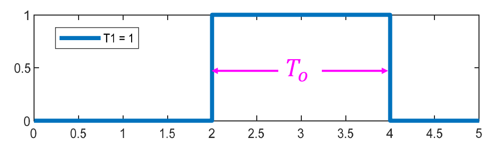
$\dfrac{2\sin\left(\omega\frac{T_o}{2}\right)}{\omega}$ $\qquad \dfrac{(2\pi f_o)T_o}{2}=\pi \qquad f_o=\dfrac{1}{T_o}$

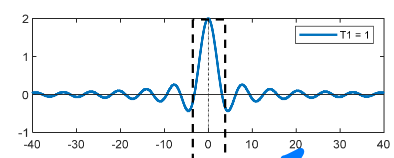
Question : Subcarrier frequency f_o ?

做法:

maximum

zero!





$$x(t)=\begin{cases}1 & |t|<\frac{T_o}{2}\\ 0 & |t|>\frac{T_o}{2}\end{cases} \quad \Leftrightarrow \quad \frac{2\sin\left(\frac{\omega T_o}{2}\right)}{\omega}$$

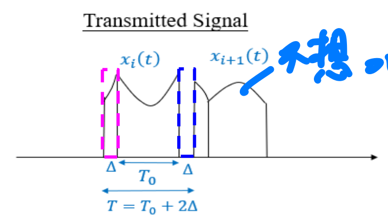
=A

Question : How to estimate T ?

Cyclic Prefix

Guard Time

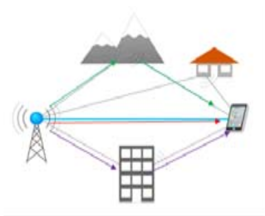
Transmitted Signal



不想 overlap!

Δ = max. delay time

Channel



the i-th symbol

$x_i(t)$

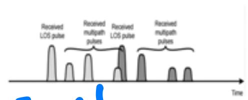
FFT

the k-th subcarrier

$\sigma_{i,k}$

FS coeff.

$b_k = a_k H(jk\omega_0)$



Sum! Delay version!

$$y_i(t) = \rho_0 x_i(t) + \rho_1 x_i(t-\tau_1) + \dots$$

FFT

$$\gamma_{i,k} = \underbrace{\left(\rho_0 + \rho_1 e^{-j\frac{k2\pi}{T_0}\tau_1} + \cdots\right)}_{c_k} \sigma_{i,k}$$

Channel coefficient

can build equalizer.

LCC Differential equation

Standard form of frequency response

The second-order system

LCC Differential Equation

Output $$\boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}}$$ **Input**

LCC Transform ↓ can find h(t) easily!

$$FT\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = FT\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

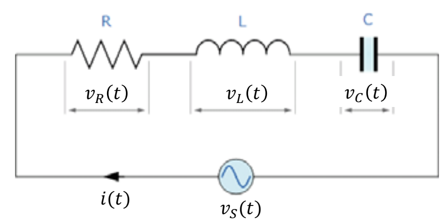
Question :
- What is the frequency response ?
- What is the impulse response ?

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \xrightarrow{\text{Inverse F.T.}} h(t)$$

(Factorization → Partial fraction)

e.g.



$$v_R(t) = Ri(t)$$

$$v_L(t) = L\frac{d}{dt}i(t)$$

$$v_C(t) = \frac{1}{C}\int i(t)dt$$

$$L\frac{d}{dt} i(t) + R\, i(t) + \frac{1}{C}\int i(t)dt = v_S(t)$$

second order!

Output $$\boxed{L\frac{d^2}{dt^2} i(t) + R\frac{d}{dt} i(t) + \frac{1}{C} i(t) = \frac{d}{dt} v_S(t)}$$ **Input**

$$\frac{I}{V_s} = \frac{j\omega}{L(j\omega)^2 + R(j\omega) + \frac{1}{C}}$$

total impedance!

$$Z = \frac{V_s}{I} = j\omega L + R + \frac{1}{j\omega C}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k \, (j\omega)^k}{\sum_{k=0}^{N} a_k \, (j\omega)^k}$$

Matlab

[B, A] = butter(N, Wn)
[H, fh] = freqz(B, A, 1e3, fs)

```
freqz Frequency response of digital filter
  [H,W] = freqz(B,A,N) returns the N-point complex frequency response
  vector H and the N-point frequency vector W in radians/sample of
  the filter:

               jw               -jw              -jmw
        jw  B(e)    b(1) + b(2)e + .... + b(m+1)e
     H(e) = ---- = ------------------------------------
               jw               -jw              -jnw
            A(e)    a(1) + a(2)e + .... + a(n+1)e

  given numerator and denominator coefficients in vectors B and A.
```

B

A

Matlab filter!

[B, A] = butter(N, Wn)
[H, fh] = freqz(B, A, 1e3, fs)

```
butter Butterworth digital and analog filter design.
  [B,A] = butter(N,Wn) designs an Nth order lowpass digital
  Butterworth filter and returns the filter coefficients in length
  N+1 vectors B (numerator) and A (denominator). The coefficients
  are listed in descending powers of z. The cutoff frequency
  Wn must be 0.0 < Wn < 1.0, with 1.0 corresponding to
  half the sample rate.
```


want! at least

Standard Form of Frequency Response (Causal and Stable System)

Polynomial Form

$$H(j\omega) = \frac{b_{N-1}(j\omega)^{N-1} + \cdots + b_1(j\omega) + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0} \qquad \text{(Rational)}$$

Factored form

$$H(j\omega) = \frac{b_{N-1} \prod_{i=1}^{N-1}(j\omega - \beta_i)}{\prod_{k=1}^{N}(j\omega - \alpha_k)}$$

$$\frac{(\ )(\ )(\ )(\ )}{(\ )(\ )(\ )(\ )(\ )}$$

Partial fraction form

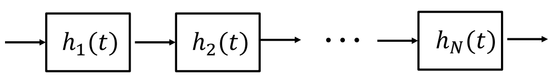
$$H(j\omega) = \sum_{k=1}^{N} \frac{c_k}{j\omega - \alpha_k}$$

look table!

Question : Physical implementation ?

Factored form

$$H(j\omega) = \frac{b_{N-1} \prod_{i=1}^{N-1}(j\omega - \beta_i)}{\prod_{k=1}^{N}(j\omega - \alpha_k)}$$

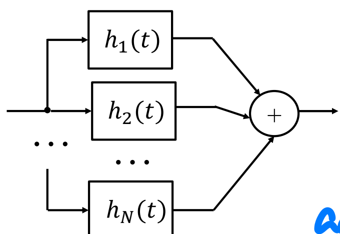


連乘!

$$h(t) = h_1(t) * h_2(t) * \cdots * h_N(t)$$

Partial fraction form

$$H(j\omega) = \sum_{k=1}^{N} \frac{c_k}{j\omega - \alpha_k}$$



add!

$$h(t) = h_1(t) + h_2(t) + \cdots + h_N(t)$$

e.g. The frequency response of an LTI system is given below :

$$H(j\omega) = \frac{(j2\omega + 3)}{(j\omega + 1)(j\omega + 10)}$$

a) Use a differential equation to show the input-output relationship.
b) Find the impulse response.

$$H(j\omega) \; = \frac{j2\omega + 3}{(j\omega + 1)(j\omega + 10)} \; = \; \frac{j2\omega + 3}{(j\omega)^2 + 11j\omega + 10} \; = \; \frac{Y(j\omega)}{X(j\omega)}$$

$$(j\omega)^2 Y(j\omega) + 11j\omega Y(j\omega) + 10Y(j\omega) = 2j\omega X(j\omega) + 3X(j\omega)$$

$$\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) + 10y(t) = 2\frac{d}{dt}x(t) + 3x(t)$$

$$H(j\omega) = \frac{A}{(j\omega + 1)} + \frac{B}{(j\omega + 10)}$$

$$A = \frac{2(-1) + 3}{(-1 + 10)} = \frac{1}{9}$$

$$h(t) = \frac{1}{9}\, e^{-t}u(t) + \frac{17}{9}\, e^{-10t}u(t)$$

$$B = \frac{2(-10) + 3}{(-10 + 1)} = \frac{17}{9}$$

Question : Requirement of using this method ?

先 Division !

$$\frac{j\omega^2}{(j\omega + 1)(j\omega + 2)} \neq \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}$$

$$\frac{A}{j\omega + 1} + \frac{B}{j\omega + 2} = \frac{A(j\omega + 2) + B(j\omega + 1)}{(j\omega + 1)(j\omega + 2)}$$

$$\frac{j\omega^2}{(j\omega + 1)(j\omega + 2)} = 1 \; - \; \boxed{\frac{3j\omega + 2}{j\omega^2 + 3j\omega + 2}}$$

This part is rational.

$$\frac{1}{(j\omega + 1)}\frac{1}{(j\omega + 2)^2} \neq \frac{A}{j\omega + 1} + \frac{B}{(j\omega + 2)^2}$$

$$\frac{A}{j\omega + 1} + \frac{B}{(j\omega + 2)^2} = \frac{A(j\omega + 2)^2 + B(j\omega + 1)}{(j\omega + 1)(j\omega + 2)^2}$$

$$\frac{1}{(j\omega + 1)}\frac{1}{(j\omega + 2)^2} = \left(\frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}\right)\frac{1}{j\omega + 2}$$

$$= \left(\frac{C}{j\omega + 1} + \frac{D}{j\omega + 2}\right) + \frac{B}{(j\omega + 2)^2}$$

focus on second order!

The Second-order System

$$\omega_n = \sqrt{a_0} \quad \text{damping ratio}$$
$$\zeta = \frac{a_1}{2\sqrt{a_0}}$$

e.g. $\omega_n = 2$ $\qquad \alpha_1, \alpha_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$

$\zeta = -2$ $\qquad \alpha_1, \alpha_2 = 2\left(2 \pm \sqrt{2^2 - 1}\right) = 4 \pm \sqrt{3} = 5.7 \;\; or \;\; 2.3$

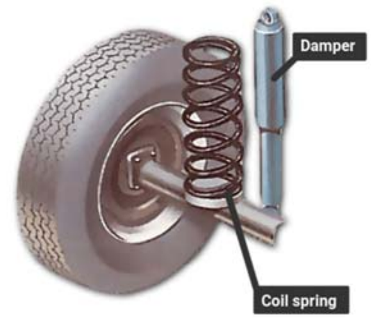
$\zeta = 0.1$ $\qquad \alpha_1, \alpha_2 = 2\left(-0.1 \pm \sqrt{0.1^2 - 1}\right) = -0.2 \pm 2j$

$\zeta = 3$ $\qquad \alpha_1, \alpha_2 = 2\left(-3 \pm \sqrt{3^2 - 1}\right) = -6 \pm 4\sqrt{2} = -11.6 \;\; or \;\; -0.34$ large

$\zeta = 1$ $\qquad \alpha_1, \alpha_2 = 2\left(-1 \pm \sqrt{1^2 - 1}\right) = -2$

$\zeta = 0.0001$ $\qquad \alpha_1, \alpha_2 = 2\left(-0.0001 \pm \sqrt{0.0001^2 - 1}\right) = -0.0002 \pm 2j$

absorb vibration!



$\zeta \approx 1$

e.g. The following LTI system is initially at rest and then an impulse is suddenly applied to this system.

1 $x(t)$ → LTI system $h(t)$ → $y(t) =$ **?** $h(t)$

0 t

a) What is the output if $h(t) = (e^{5.7t} + e^{2.3t})\, u(t)$?

b) What is the output if $h(t) = e^{-0.2t} \cos(2t)\, u(t)$?

c) What is the output if $h(t) = (e^{-11.6t} + e^{-0.34t})\, u(t)$?

d) What is the output if $h(t) = t\, e^{-2t}\, u(t)$? stable!

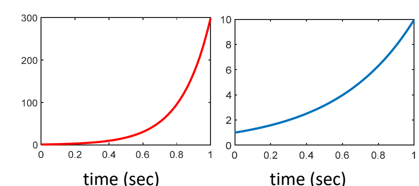
e) What is the output if $h(t) = e^{-0.0002t} \cos(2t)\, u(t)$?

double root

$$\cos(2t) = \frac{e^{j2t} + e^{-j2t}}{2} \qquad h(t) = \frac{1}{2} e^{(-0.2+j2)\,t}\, u(t) + \frac{1}{2} e^{(-0.2-j2)\,t}\, u(t)$$

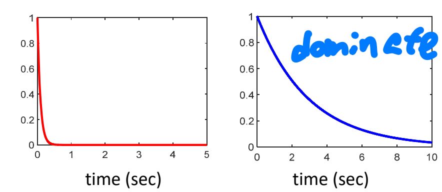
unstable

$y(t) = (e^{5.7t} + e^{2.3t})\, u(t)$



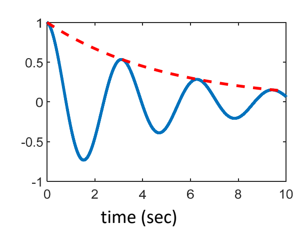
decay very slow!

$y(t) = (e^{-11.6t} + e^{-0.34t})\, u(t)$



oscillation

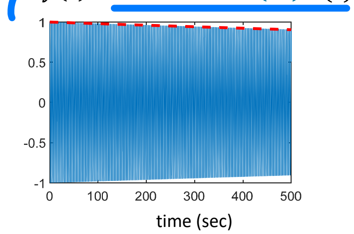
$y(t) = e^{-0.2t} \cos(2t)\, u(t)$



$y(t) = t\, e^{-2t}\, u(t)$

damage the device!

$y(t) = e^{-0.0002t} \cos(2t)\, u(t)$



stable
long to decay!

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{b_1(j\omega)+b_0}{\boxed{(j\omega)^2+a_1(j\omega)+a_0}}$$

While ω_n provides a scaling in frequency, ζ makes explicit whether $a_1{}^2 - 4a_0 \geq 0$ and it also fully characterizes the system. (Recall $\omega_n = \sqrt{a_0}$, $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$ $\boxed{\alpha_1, \alpha_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2-1}))}$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

3a. If $\zeta \leq 0$, system is unstable (since it means $a_1 \leq 0$)

3b. If $|\zeta| < 1$, roots have imaginary part and system is *oscillatory* (if $|\zeta| < 1$ means $a_1{}^2 - 4a_0 < 0$)

3c. If $\zeta \gg 1$, one root is nearly $-2\zeta\,\omega_n$ but the other root is only slightly less than 0. Therefore the impulse response decays very slowly. The system is *over-damped*.

$\zeta \gg 1$ implies $\sqrt{\zeta^2-1} = \zeta - \varepsilon$ where $\varepsilon = 0^+$ <---- *Means just slightly larger than 0*

Hence, $\alpha_1, \alpha_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2-1}) = \omega_n(-2\zeta+\varepsilon), \omega_n(-\varepsilon)$ Close to 0; decay very slowly

3d. If $\zeta \cong 1$, both roots are near $-\omega_n$, and system impulse response decays at the fastest rate possible. System is *critically damped* – desirable in a suspension system.

3e. If $\zeta \to 0^+$, system is *under-damped*. $|H(j\omega)|$ may become very large around ω_n as we will show in a few slides.

$$H(j\omega) = \frac{A}{(j\omega-\alpha_1)} + \frac{B}{(j\omega-\alpha_2)}$$

$$h(t) = A\,e^{\alpha_1 t}u(t) + B\,e^{\alpha_2 t}u(t)$$

$$h(t) = A\,e^{\alpha_1 t}u(t) + B\,e^{\alpha_2 t}u(t) \qquad H(j\omega) = \frac{A}{(j\omega-\alpha_1)} + \frac{B}{(j\omega-\alpha_2)}$$

$$h(t) = (e^{5.7t} + e^{2.3t})\,u(t)$$
$$H(j\omega) = \frac{1}{j\omega-5.7} + \frac{1}{j\omega-2.3}$$

$$h(t) = (e^{-11.6t} + e^{-0.34t})\,u(t)$$
$$H(j\omega) = \frac{1}{j\omega+11.6} + \frac{1}{j\omega+0.34}$$

$$h(t) = e^{-0.2t}\cos(2t)\,u(t)$$
$$H(j\omega) = \frac{1/2}{j\omega+0.2-j2} + \frac{1/2}{j\omega+0.2+j2}$$

$$h(t) = t\,e^{-2t}\,u(t)$$
$$H(j\omega) = \frac{1}{(j\omega+2)^2}$$

$$h(t) = e^{-0.0002t}\cos(2t)\,u(t)$$
$$H(j\omega) = \frac{1/2}{j\omega+0.0002-2j} + \frac{1/2}{j\omega+0.0002+2j}$$

e.g. Given a causal system : $H(j\omega) = \dfrac{1}{(j\omega)^2 + j4\omega + 10}$

$$H(j\omega) = \frac{A}{j\omega+(-\alpha_1)} + \frac{B}{j\omega+(-\alpha_2)}$$

$$\zeta = \frac{4}{2\sqrt{10}} = \frac{2}{\sqrt{10}}$$

$$\omega_n = \sqrt{10}$$

$$\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \omega_n(-\zeta \pm \sqrt{\zeta^2-1})$$
$$= -2 \pm j\sqrt{6} = \sqrt{10}\left(-\frac{2}{\sqrt{10}} \pm \sqrt{\frac{4}{10}-1}\right)$$

$$h(t) = A\,e^{\alpha_1 t}u(t) + B\,e^{\alpha_2 t}u(t) = A\,e^{-2t}\,e^{j\sqrt{6}\,t}\,u(t) + B\,e^{-2t}\,e^{-j\sqrt{6}\,t}\,u(t)$$

Question :

- Stable ?
- Oscillatory ?
- Over-damped ? Under-damped ? Critically damped ?