T04

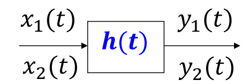
Examples of Convolution Integral
Convolution Kernel

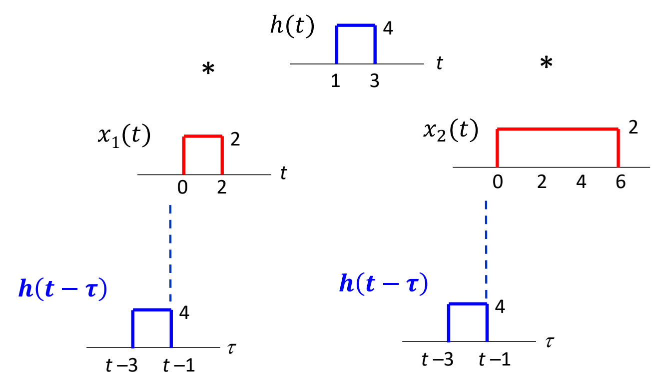
Inner product and projection
Eigenfunction, System function and Frequency response (CT)
Eigenfunction, System function and Frequency response (DT)

Frequency domain

Fourier analysis for CT signals and systems
Fourier series coefficient

e.g.



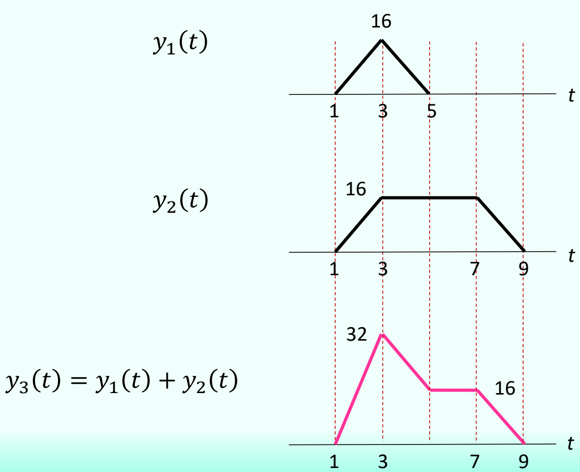


$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

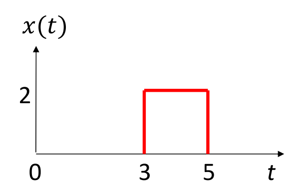
$y_1(t) = x_1(t) * h(t) =$ **?**

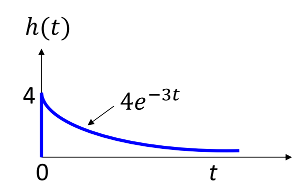
$y_2(t) = x_2(t) * h(t) =$ **?**

Question : What is the output if the input is the sum of $x_1(t)$ and $x_2(t)$?

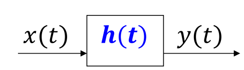


e.g. Given : $x(t) = 2\,u(t-3) - 2\,u(t-5)$ $\quad h(t) = 4\,e^{-3t}\,u(t)$





Sketch $y(t)$

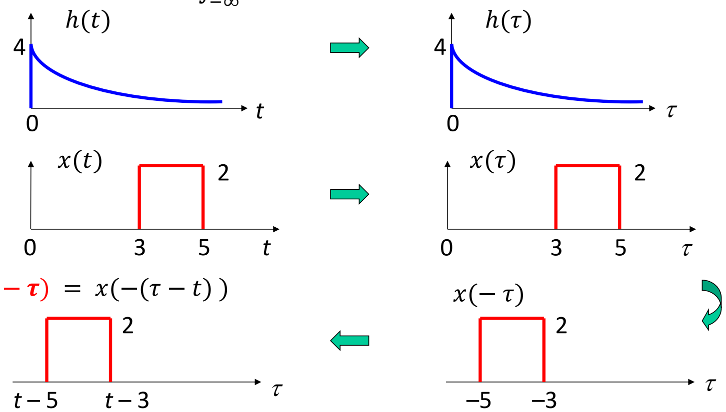


$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Step 1 :

Step 2 :

$x(t-\tau) = x(-(\tau - t))$

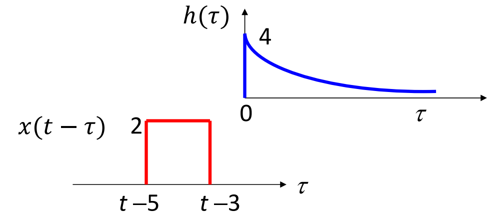


Step 3 : Perform multiplication and integration

(a) For $\boxed{t-3<0}$

$t<3$

$$\Rightarrow \quad y(t) = 0$$

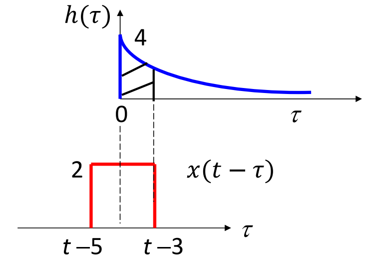


(b) For $\boxed{t-3 \geq 0 \text{ and } t-5 \leq 0}$

$3 \leq t \leq 5$

$$\Rightarrow \quad y(t) = \boxed{\int_{0}^{t-3}} (2)(4e^{-3\tau})d\tau$$

$$= \frac{8}{3}\left(1 - e^{-3(t-3)}\right)$$

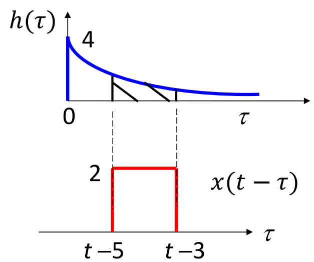


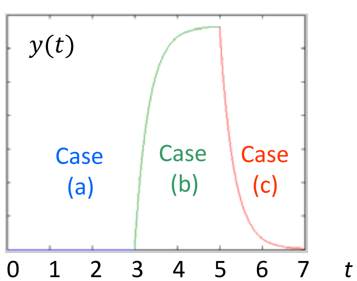
(c) For $\boxed{t-5>0}$

$t>5$

$$\Rightarrow \quad y(t) = \boxed{\int_{t-5}^{t-3}} (2)(4e^{-3\tau})d\tau$$

$$= \frac{8}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right)$$





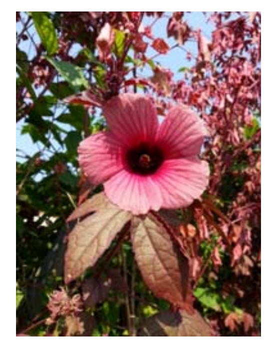
Convolution Kernel

e.g.

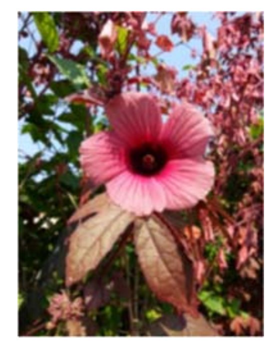
5-by-5 matrix or mask

$$\begin{bmatrix} 0.0400 & 0.0400 & 0.0400 & 0.0400 & 0.0400 \\ 0.0400 & 0.0400 & 0.0400 & 0.0400 & 0.0400 \\ 0.0400 & 0.0400 & 0.0400 & 0.0400 & 0.0400 \\ 0.0400 & 0.0400 & 0.0400 & 0.0400 & 0.0400 \\ 0.0400 & 0.0400 & 0.0400 & 0.0400 & 0.0400 \end{bmatrix}$$

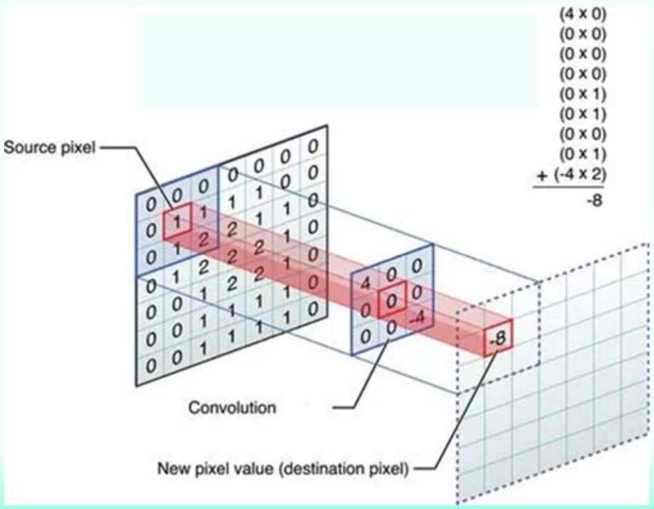
Before



After



Question : What does this convolution kernel do?



Eigenfunction, System function and Frequency response (CT)
Eigenfunction, System function and Frequency response (DT)

Frequency domain

Inner product and projection

Fourier analysis for CT signals and systems
Fourier series coefficient

Eigenfunction, System Function and Frequency Response (CT)

$x(t) = e^{st}$ → LTI System $h(t)$ → $y(t) = x(t) * h(t)$ Time domain

Eigenfunction   Convolution

$\boxed{s = \sigma + j\omega}$

System function

$x(t) = e^{s_o t}$ $\boldsymbol{H(s)} = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$ $y(t) = \boldsymbol{H(s_o)}\, e^{s_o t}$

$\boldsymbol{s} = s_o$ Eigenvalue

$\sigma = 0$

Frequency response (stable system)

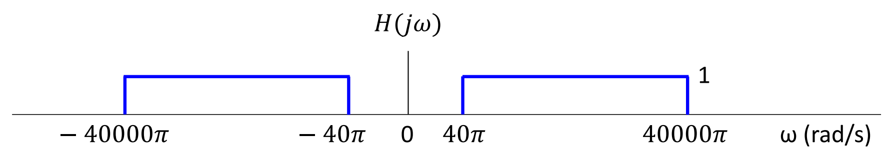
Frequency domain

$x(t) = e^{j\omega_o t}$ $\boldsymbol{H(j\omega)} = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau$ $y(t) = \boldsymbol{H(j\omega_o)}\, e^{j\omega_o t}$

$\boldsymbol{\omega} = \omega_o$

e.g. A CT signal $x(t)$ is applied to a CT LTI stable system.

$$x(t) = \cos(20\pi t) + \cos(300\pi t) + \cos(1000\pi t) + \cos(4000\pi t) + \cos(60000\pi t)$$

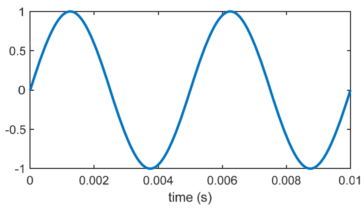


a) What are the corresponding eigenvalues ?
b) What is the output ?
c) What does this system do ?
d) Practical case related to this mathematical model ?

e.g.

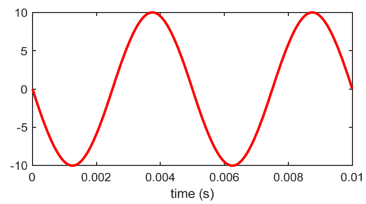
$\boldsymbol{x(t)} = \sin(\omega_o t)$ → **LTI System** $h(t)$ → $\boldsymbol{y(t)} = -10\sin(\omega_o t)$

$\boldsymbol{y(t)} = 10\sin(\omega_o t - \pi)$



eigenvalue ?

$\boldsymbol{H(j\omega_o)}$



Question :
Why is the eigenvalue complex for general case?
What does this system do ?

$-10 = 10\, e^{j\pi}$

$$10\sin(\omega_o t - \pi) = 10\sin(\omega_o t)\cos(\pi) - 10\cos(\omega_o t)\sin(\pi) = -10\sin(\omega_o t)$$

Eigenfunction, System Function and Frequency Response (DT)

$x[n] = z^n$ → **LTI System** $h[n]$ → $y[n] = x[n] * h[n]$ Time domain

Eigenfunction

$\boxed{z = re^{j\omega}}$

System function

$x[n] = \boldsymbol{z_o}^n$ $\quad \boldsymbol{H(z)} = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$ $\quad y[n] = \boldsymbol{H(z_o)}\, \boldsymbol{z_o}^n$

$\boldsymbol{z} = z_o$ Eigenvalue

$r = 1$

Frequency response (stable system)

Frequency domain

$x[n] = e^{j\boldsymbol{\omega_o} n}$ $\quad \boldsymbol{H(e^{j\omega})} = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$ $\quad y[n] = \boldsymbol{H(e^{j\omega_o})}\, e^{j\omega_o n}$

$\boldsymbol{\omega} = \omega_o$

e.g. A DT signal $x[n]$ is applied to a DT LTI and stable system.

$\boxed{x[n] = 3 + \cos(\pi n)}$ → **System function** $H(z) = z^{-1}$ → $y[n]$

a) Represent $x[n]$ as the sum of eigenfunctions.

$x[n] = 3\, e^{j(0)n} + e^{j(\pi)n}$

$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$

Question : $\quad z = e^{j\omega}$
What is the frequency response ?

b) Find the eigenvalue for each eigenfunction.

$H(e^{j\omega}) = e^{-j\omega}$ $\qquad H(e^{j0}) = 1$ $\qquad H(e^{j\pi}) = e^{-j\pi} = -1$

c) Write down the expression of the output $y[n]$.

$y[n] = 3\, e^{j(0)n} - e^{j(\pi)n} = 3 - \cos(\pi n) = \ 3 + \cos(\pi(n-1))$

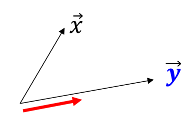
d) What does this DT LTI system do ?

Inner Product and Projection

Inner Product for real / complex vectors : $\sum_i x_i\, y_i^*$

Project $\vec{x}$ onto $\vec{\boldsymbol{y}}$:

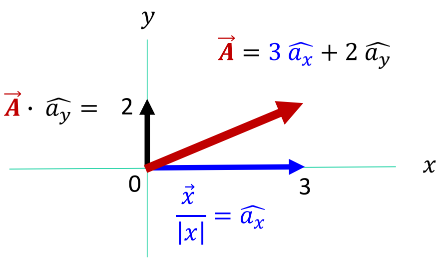
$$\langle \vec{x}, \vec{y} \rangle \frac{\vec{y}}{|\vec{y}|^2} = \frac{\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle}{|\vec{\boldsymbol{y}}|^2}\, \vec{\boldsymbol{y}}$$



Inner product $\quad$ Projection coefficient $\quad$ Direction

Question : What is the meaning if the inner product of two vectors is zero ?

Question : Any difference between dot product and inner product ?

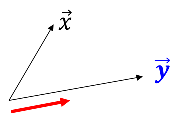


Dot product

$$\vec{A}\cdot\widehat{a_x} = \vec{A}\cdot\frac{\vec{x}}{|x|} = 3$$

$$3\,\widehat{a_x} = \vec{A}\cdot\frac{\vec{x}}{|x|}\,\frac{\vec{x}}{|x|} = \frac{\vec{A}\cdot\vec{x}}{|x|^2}\,\vec{x}$$

Inner product



$$\langle\vec{x},\vec{y}\rangle\frac{\vec{y}}{|\vec{y}|^2} = \frac{\langle\vec{x},\vec{y}\rangle}{|\vec{y}|^2}\,\vec{y}$$

e.g. Find the projection coefficient if vector x is projected on to vector y.

$$\frac{\langle\vec{x},\vec{y}\rangle}{|\vec{y}|^2}$$

$\vec{x} = (2+j\,,1-j)$

$\vec{y} = (1+j\,,2-j)$

$\vec{y}^* =$ **?**

$\langle\vec{x},\vec{y}\rangle = \sum_i x_i\,y_i^* = 6-2j$

$|\vec{y}|^2 = \sum_i y_i\,y_i^* = 7$

$\vec{x} = \left(e^{-j\frac{\pi}{2}}, e^{-j\frac{3\pi}{2}}\right)$

$\vec{y} = \left(e^{j\frac{\pi}{2}}, e^{j\pi}\right)$

$\vec{y}^* =$ **?**

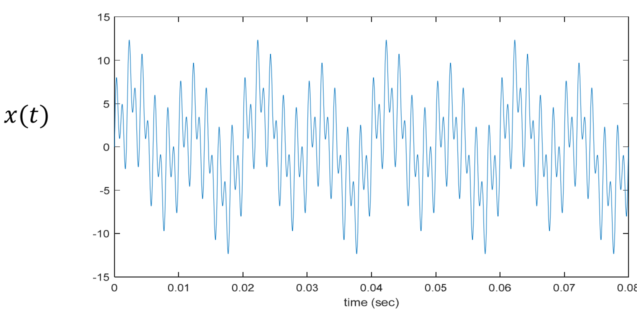
$\langle\vec{x},\vec{y}\rangle = -1-j = \sqrt{2}\,e^{-j\frac{3\pi}{4}}$

$|\vec{y}|^2 = 2$

Question : Can we use inner product to analyze signals ?

$$\sum_i x_i\,y_i^* \Rightarrow \int_{-\infty}^{\infty} x(t)\,y^*(t)$$

e.g. A real periodic signal $x(t)$ is plotted below according to a given data file.

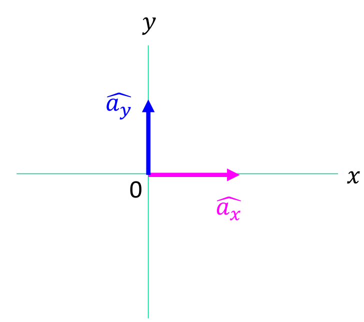


$$\omega_o = \frac{2\pi}{T}$$

Question :
- What is the fundamental ordinary frequency (in Hz) ?
- How many frequency components contained in $x(t)$?
- How to know the angular frequency (in rad/s) of each frequency component ?

i.e. Decomposition

Dot Product



Vector A

$\vec{A}\cdot\widehat{a_x} = 3$ Non-zero

$\vec{A}\cdot\widehat{a_y} = 2$ Non-zero

$\vec{A} =$ **?**

Vector B

$\vec{B}\cdot\widehat{a_x} = 0$ Zero

$\vec{B}\cdot\widehat{a_y} = 32$ Non-zero

$\vec{B} =$ **?**

Question : Expression of vector A and B ?

Inner Product

$$< x(t) \, , e^{j(0)\frac{2\pi}{T}t} > = 0$$

$$< x(t) \, , e^{j(1)\frac{2\pi}{T}t} > \neq \mathbf{0}$$

$$< x(t) \, , e^{j(2)\frac{2\pi}{T}t} > = 0$$

$$< x(t) \, , e^{j(3)\frac{2\pi}{T}t} > = 0$$

$$\vdots$$

$$< x(t) \, , e^{j(10)\frac{2\pi}{T}t} > \neq \mathbf{0}$$

$$< x(t) \, , e^{j(k)\frac{2\pi}{T}t} > = 0 \quad \text{for } k > 10$$

What does it mean ?

$$< x(t) \, , e^{jk\frac{2\pi}{T}t} > \neq \mathbf{0} \quad \text{for } k \geq 0 \text{ and } k = 1, 10$$

$$< x(t) \, , e^{jk\frac{2\pi}{T}t} > = 0 \quad \text{for } k \geq 0 \text{ and otherwise}$$

Fourier Analysis for CT Signals and LTI Systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

Time-domain

Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{j k\omega_o t}$$

F.S. coefficient

Periodic

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{\boxed{a_k \, H(jk\omega_o)}}^{b_k} \, e^{j k\omega_o t}$$

F.S. coefficient

$H(j\omega)$
Frequency response

$H(jk\omega_o)$
Eigenvalue

Frequency-domain

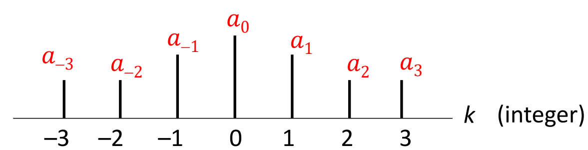
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{j k\omega_o t}$$

Complex / Real periodic Signal

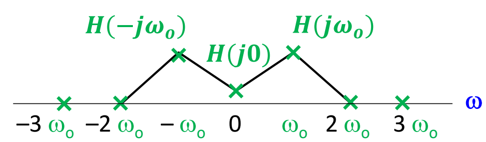
A sum of Complex / Real sinusoids

$\vdots$		$y(t)$ $\vdots$
$a_{-1} \, e^{-j\,\omega_o t}$		$H(-j\omega_o) \, a_{-1} \, e^{-j\,\omega_o t}$
a_0	$\longrightarrow \boxed{\text{LTI}} \longrightarrow$ $H(j\omega)$	$H(j0) \, a_0$
$a_1 \, e^{j\,\omega_o t}$		$H(j\omega_o) \, a_1 \, e^{j\,\omega_o t}$
$a_2 \, e^{j\,2\omega_o t}$		$H(j2\omega_o) \, a_2 \, e^{j\,2\omega_o t}$
$\vdots$		$\vdots$

e.g. The FS of the input and the frequency response is given below.

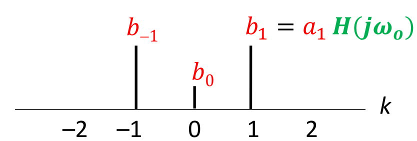


Question :
- What is the meaning of k ?

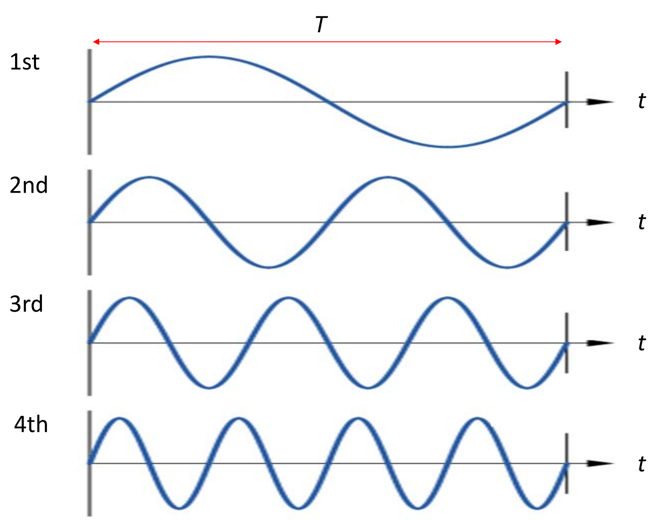


$$\omega = k\omega_o \qquad \omega_o = \frac{2\pi}{T}$$

Sketch the FS of the output



$$b_k = a_k \, H(jk\omega_o)$$



Fourier Series Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \boxed{e^{jk\frac{2\pi}{T}t}}$$

Basis function

Inner Product $= \sum_i x_i\, y_i^*$

$$\Rightarrow \quad a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

For any **real** and periodic signal : $x(t) = x^*(t)$

$$a_k = a_{-k}^*$$

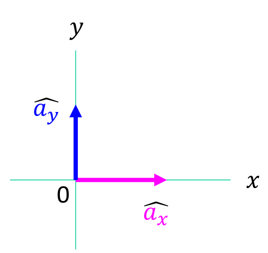
$$x(t) = a_0 + 2\sum_{k=1}^{\infty} |a_k| \cos\left(k\frac{2\pi}{T}t + \measuredangle a_k\right)$$

Question :
- Why is the orthogonality important ?

$e^{jk\frac{2\pi}{T}t}$

Orthogonality

Dot Product



Inner Product

$\widehat{a_x} \cdot \widehat{a_y} = 0 \quad \Leftrightarrow \quad < e^{jk\frac{2\pi}{T}t}, e^{jm\frac{2\pi}{T}t} > = 0$

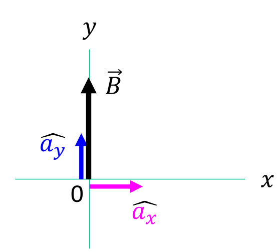
for $k \neq m$

$\widehat{a_x} \cdot \widehat{a_x} \neq 0 \quad \Leftrightarrow \quad < e^{jk\frac{2\pi}{T}t}, e^{jm\frac{2\pi}{T}t} > \neq 0$

for $k = m$

Decomposition

Dot Product



$\vec{B} = 2\,\widehat{a_y}$

Inner Product

$\vec{B} \cdot \widehat{a_x} = 0 \quad \Leftrightarrow \quad \frac{1}{T}\int_0^T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt = 0$

for $k \neq -10, -1, 1, 10$

$\vec{B} \cdot \widehat{a_y} = 2 \quad \Leftrightarrow \quad \frac{1}{T}\int_0^T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt \neq 0$

for $k = -10, -1, 1, 10$

$$\left.\begin{array}{l} a_{-10} \neq 0 \\ a_{-1} \neq 0 \\ a_1 \neq 0 \\ a_{10} \neq 0 \end{array}\right.$$

Question : What does it mean ?

e.g. A periodic signal is given below.

$$x(t) = 2 + 4\sin(100\pi t) + \cos(400\pi t)$$

Question : How to find all a_k of $x(t)$?

$a_0 = $ **?**

Approach 1 :

$$\cos(\omega_o t + \theta) = \frac{e^{j(\omega_o t + \theta)} + e^{-j(\omega_o t + \theta)}}{2}$$

$$\sin(\omega_o t + \theta) = \frac{e^{j(\omega_o t + \theta)} - e^{-j(\omega_o t + \theta)}}{j2}$$

a_k $\boxed{e^{jk\frac{2\pi}{T}t}}$

Basis function

Approach 2 :

$$\sin(\omega_o t + \theta) = \cos\left(\omega_o t + \theta - \frac{\pi}{2}\right)$$

$$\cos\left(\omega_o t + \theta - \frac{\pi}{2}\right) = \frac{e^{j(\omega_o t + \theta - \frac{\pi}{2})} + e^{-j(\omega_o t + \theta - \frac{\pi}{2})}}{2}$$

$$a_k = a_{-k}^*$$

$\boxed{e^{jk\frac{2\pi}{T}t}}$

Basis function

$$\sin(\omega_o t) = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{j2} = \left(\frac{1}{2j}\right)e^{j\omega_o t} + \left(-\frac{1}{2j}\right)e^{-j\omega_o t}$$

$$\omega_o = \frac{2\pi}{T} \qquad \frac{1}{2j} = -\frac{j}{2} \qquad -\frac{1}{2j} = \frac{j}{2} \qquad j^2 = -1$$

$$\cos(\omega_o t) = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} = \left(\frac{1}{2}\right)e^{j\omega_o t} + \left(\frac{1}{2}\right)e^{-j\omega_o t}$$

$$\frac{1}{2} \qquad \frac{1}{2}$$

Question :
- What is the meaning of a_k ?
- Why is it complex ?