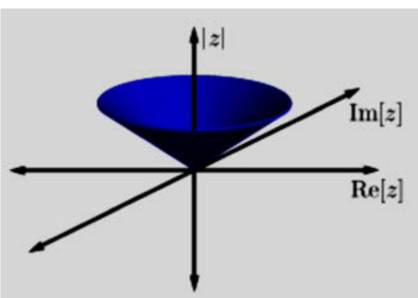
Ch3.4: Z-Transform

design a filter!

Information source and input transducer → Source Coding → Channel Coding → Modulator → Channel → Demodulator (Matched Filter) → Channel Decoding → Source Decoding → Information sink and output transducer

Ch3.4: z-Transform



- Questions to be answered:
  - z-Transform: A general version of DTFT
  - Region of Convergence: Where the transform exists.
  - Inverse z-Transform: Back to time-domain
  - LTI System in Transform Domain: A general version of the frequency response function

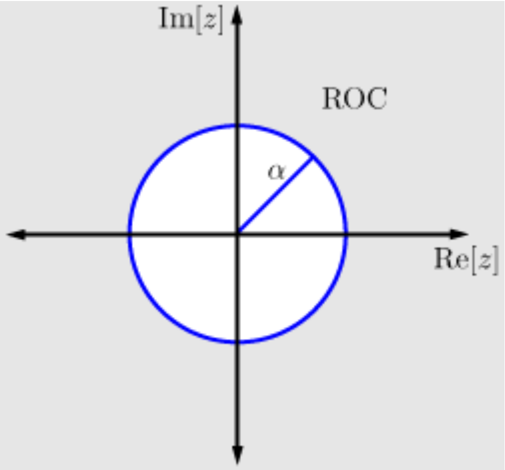
Reading: Mitra (Ch6)

Ch3.4: z-Transform

- **Definition of z-Transform**
- Region of Convergence

much more complicated than IDFT

- Inverse z-Transform
- Z-Transform Properties
- LTI System in Transform Domain



Generalizing DTFT

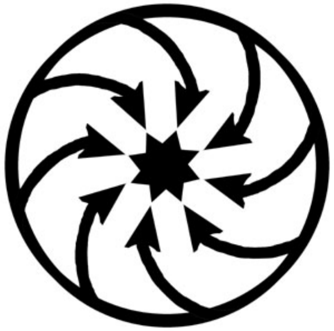
- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems.

absoluable summable sequence

- However, because of the **convergence condition**, in many cases, the DTFT of a sequence may not exist.

- As a result, it is not possible to make use of such frequency-domain characterization in these cases.

- **Question: Can we generalize DTFT?**



z-Transform

DTFT

$X(e^{j\omega}) = \sum x[n] \boxed{e^{-j\omega n}}$

z is continuous, complex value!

- The **z-transform** $X(z)$ of a sequence $x[n]$ is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z is a continuous complex variable. Generally, we can express the complex variable z in polar form as $z = re^{j\omega}$ where $r = |z| > 0$ is the magnitude and ω is the angle of z.

complex value z!

- z-transform gives

z^{-n} pick some value of r

DTFT → r = 1

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\boxed{r^{-n}\ e^{-j\omega n}} = DTFT(\underline{x[n]r^{-n}})$$

z is more general

- In particular, when $r = 1$, then $z = e^{j\omega}$ and the above expression becomes

$$X(z)|_{e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]\ e^{-j\omega n} = DTFT(x[n]) = X(e^{j\omega}) \leftarrow$$

DTFT, special case of z-transform

If RoC contains unit circle → DTFT exists

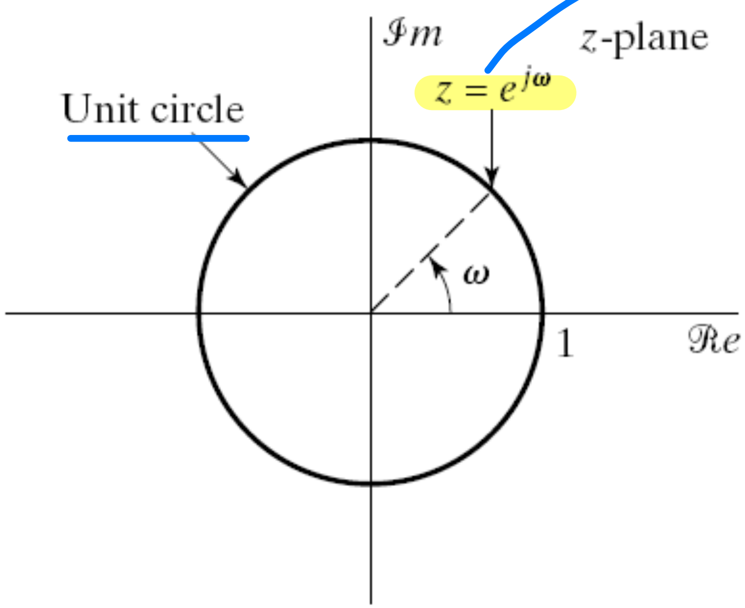
Discrete z-transform

z-Transform

z-transform DTFT

- The relationship between $X(z)$ and $X(e^{j\omega})$ can also be illustrated in the **z-plane** (a real-imaginary plane for complex number).

DTFT



The z-transform evaluated on the unit circle (as ω varies from 0 to 2π) corresponds to the DTFT of $x[n]$.

z-Transform

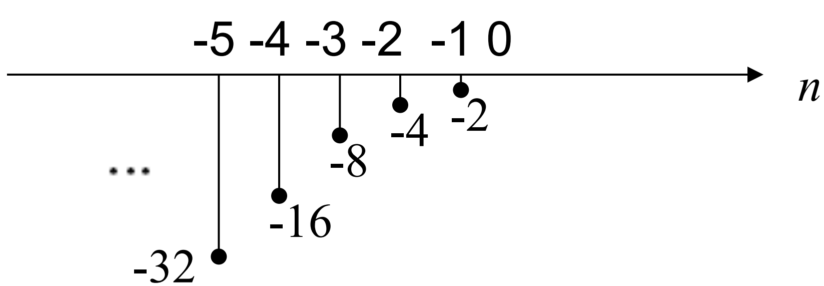
- Main **advantages** of using *z*-transform over DTFT:
  - *z*-transform can encompass a broader class of signals than DTFT
  - Recall that a sufficient condition for convergence of the DTFT is:

$$|X_K(e^{j\omega})| = \left| \sum_{n=-K}^{K} x[n]e^{-j\omega n} \right| \leq \underline{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

absolute summable

不satisfy this condition

  - **Example:** The DTFT of $x[n] = -0.5^n u[-n-1]$ does not exist since it is unbounded in the negative direction:



z-Transform

- On the other hand, the *z*-transform exists if

can pick r so that $X(z)$ is finite!

$$|X(z)| = \left| \sum_{n=-K}^{K} x[n] r^{-n} e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

We can choose region of convergence (ROC) for *z* such that the *z*-transform converges.

non - negative

- **Example:** z-transform of $x[n] = \underline{-0.5^n u[-n-1]}$ exists if $|0.5^{-1}z| < 1$

- **Proof:** $X(z) = \sum_{n=-\infty}^{\infty} -0.5^n u[-n-1] z^{-n}$

non-negative!

$$= -\sum_{n=-\infty}^{-1} (0.5z^{-1})^n$$

$$= -\sum_{m=1}^{\infty} (0.5^{-1}z)^m$$

$$= -\frac{0.5^{-1}z}{1-0.5^{-1}z} = \frac{1}{1-0.5z^{-1}}$$

overall is finite!

$$\sum_{m=1}^{\infty} \alpha^m = \frac{1}{1-\alpha}$$

$|\alpha| < 1$

DTFT X exists → z-transform exists

z-Transform

- Other **advantages** of using *z*-transform:
  - More **convenient notation** than DTFT when dealing with analytical problems

$$z \longleftrightarrow e^{j\omega}$$

  - **Convenient block diagram** representation for implementation of practical systems, Eg., a unit delay system is expressed as:

next chapter!

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n] = x[n-1]$$

delay by 1

  - Useful for determining and solving **difference equations** for discrete-time systems

Application: Filter Design

- The convolution sum description of an LTI discrete-time system with an impulse response $h(n)$ is given by $y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$.

$x[n] = z^n$

- The output to an input z^n is given by

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\
&= \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} z^n \\
&= H(z) z^n
\end{aligned}
$$

'$H(z)$'

how system affect this?

related to $e^{j\omega}$

- For any given input sequence z_0^n, the output will be $H(z_0) z_0^n$, indicating that z^n is an eigen-function for any LTI system.

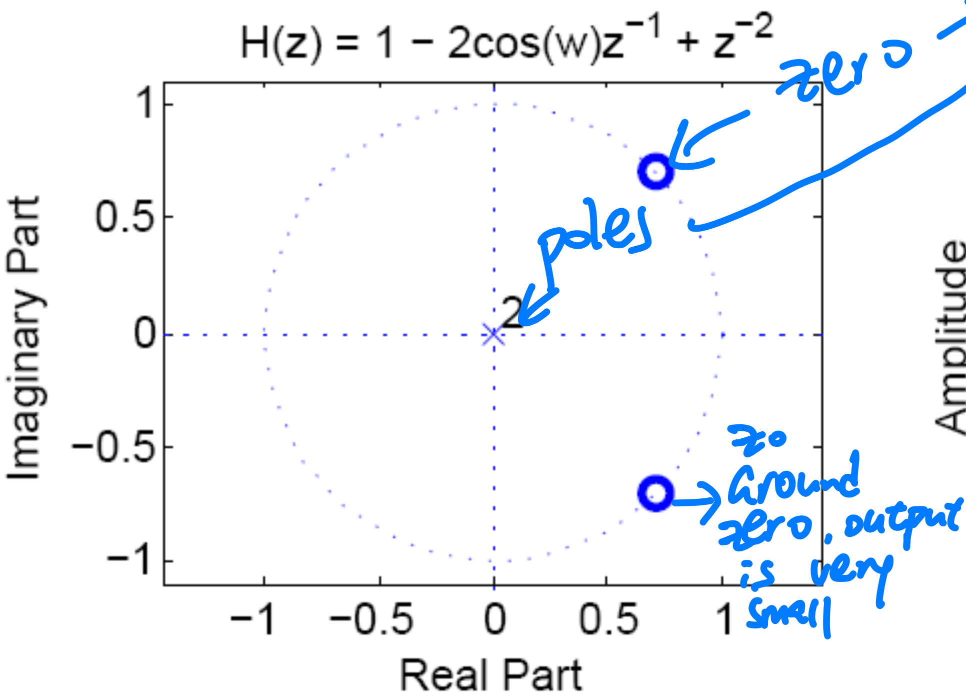
- How can we utilize this property? input $= z^n \rightarrow$ output $=$ scaled z^n

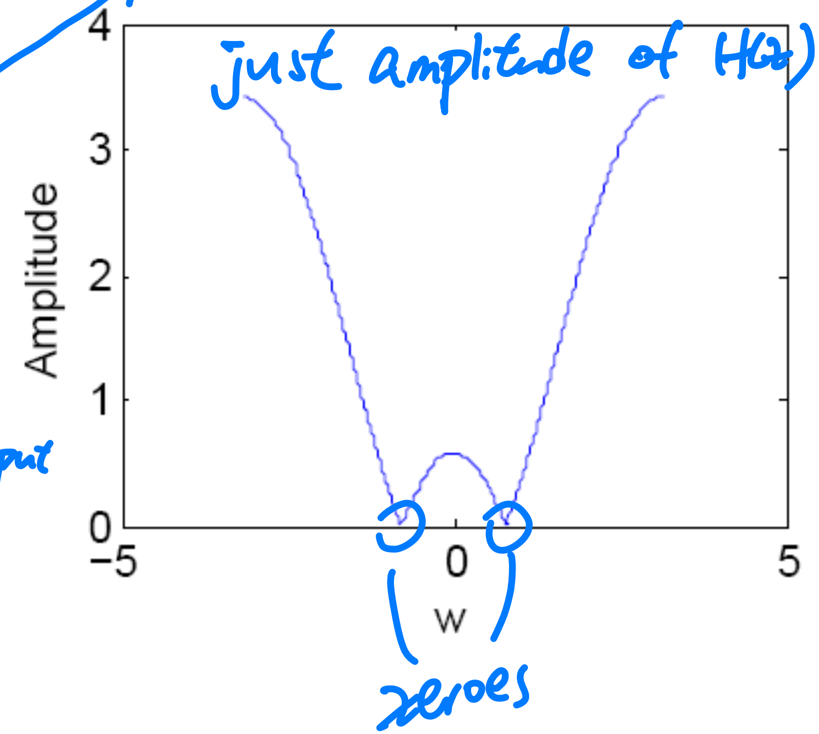
- Z-transform provides analysis of signal and design of filters

Use of z-Transform: Filter Design

- Can you figure out what kind of filter has the following z-transform and magnitude spectrum?

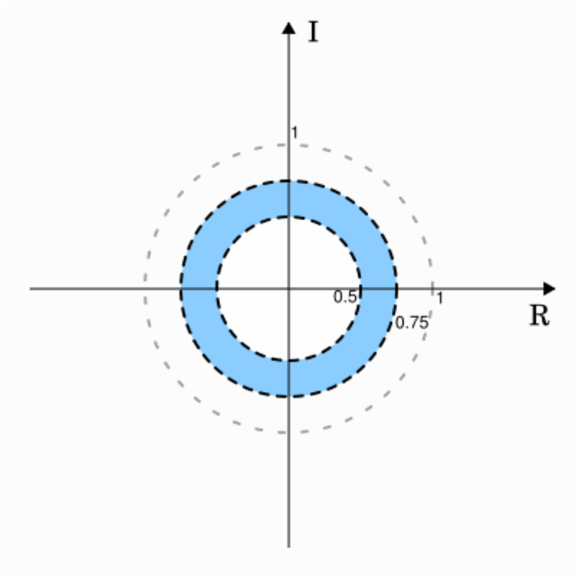
can design the control





Ch3.4: z-Transform

- Definition of z-Transform
- **Region of Convergence**
- Inverse z-Transform
- Z-Transform Properties
- LTI System in Transform Domain



Region of Convergence (ROC)

- Since *z* can be any point on the *z*-plane, generally there exists some *z* which makes *X(z)* not converge

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| \rightarrow \infty$$

- The set of *z* values for which *X*(*z*) converges

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| < \infty$$

need to specify

不可只 write down expression

is called the region of convergence (ROC).

- **<u>The ROC must be specified along with *X*(*z*) in order for the *z*-transform to be completely defined.</u>**

The ROC is important because different sequences can have the same z-transform, i.e., the z-transform is not unique without its ROC.

When we specify the Z-transform of a sequence, we also must specify its ROC (except for certain special cases):

$$x[n] \overset{z}{\longleftrightarrow} X(z) \qquad \mathrm{ROC} : \mathcal{S}$$

Myth 1: z-transform is unique

DTFT X exist!

Solved with example

Example 1: $x[n] = u[n]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |z|^{-n} < \infty$. We know this sum is finite only if $|z| > 1$. Hence the ROC of $x[n] = u[n]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| > 1\}$. For $z \in \mathcal{S}$, we have

2

$|z^{-1}| < 1$ $|z| > 1$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1 - z^{-1}}.$$

1

Example 2: $x[n] = -u[-n-1]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=-\infty}^{-1} |z|^{-n} = \sum_{n=1}^{\infty} |z|^n < \infty$. We know this sum is finite only if $|z| < 1$. Hence the ROC of $x[n] = -u[-n-1]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| < 1\}$. For $z \in \mathcal{S}$, we have

$|z| < 1$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \overset{*}{=} -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n} = -\frac{z}{1-z} = \frac{1}{1 - z^{-1}}.$$

Different shape

Same $X(z)$ but different ROC.

Two different signals

Myth 2: One can always obtain DTFT by substituting z = exp(jw) in X(z)

The DTFT converges if and only if the ROC of $X(z)$ includes the ring $|z| = 1$. It is incorrect to just substitute $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$ if the ROC of $X(z)$ does not include the unit circle.

Example: We saw earlier that, given $x[n] = u[n]$, we can compute $X(z) = \frac{1}{1-z^{-1}}$. Does $X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$? X

$$X(z) = \frac{b_0}{a_0} \frac{\prod_k (z - c_k)}{\prod_k (z - d_k)}$$

c_k : zero $\Rightarrow X(c_k) = 0$

d_k : pole $\Rightarrow X(d_k) = \infty$

maybe the same (repeat)

e.g. $X(z) = z^{-2}$

$$= \frac{1}{z^2} = \frac{1}{(z-0)(z-0)}$$

2 poles at $z = 0$

$X_1(z) = z^3 = (z-0)^3$

3 zeros at $z = 0$

: order of zeros

$X_2(z) = z^2$

Just compare with the $X(z)$

Poles and Zeros in *z*-Transform

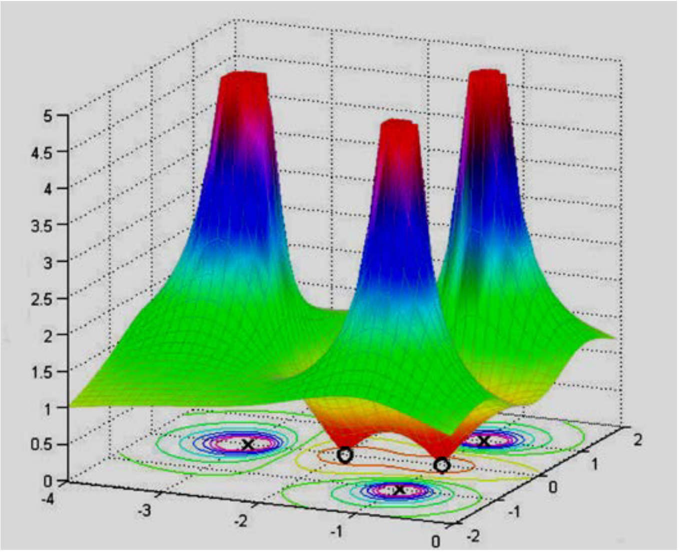
- In many applications, $X(z)$ can be expressed as a rational function:

zeroes

$$X(z)=\frac{P(z)}{Q(z)}=\frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k}=\frac{b_0}{a_0}\frac{(z-c_1)(z-c_2)\cdots(z-c_M)}{(z-d_1)(z-d_2)\cdots(z-d_N)}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)} \qquad a_0\neq 0, b_0\neq 0$$

pole

- $P(z)$ and $Q(z)$ are numerator and denominator polynomials in z
- The values of z for which $X(z)=0$, i.e. roots of $P(z)$, are called *zeros* of $X(z)$
- The values of z for which $X(z)=\infty$, i.e. roots of $Q(z)$, are called *poles* of $X(z)$

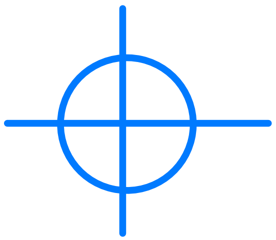


Properties of the ROC

- The ROC is ring or disk in the z-plane centered at the origin, i.e.

$$0 \le \boldsymbol{r}_R \le |z| \le \boldsymbol{r}_L \le \infty$$

- The DTFT of $x[n]$ converges absolutely iff the ROC of the z-transform includes the **unit circle**.



$r_R = 0$, $r_L = \infty$] possible

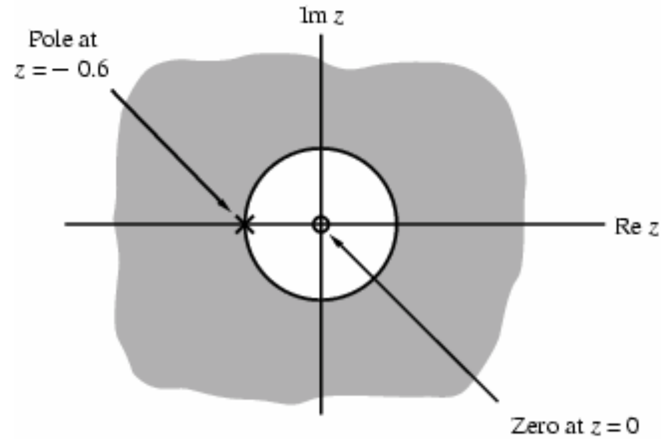
- **The ROC cannot contain any poles.**

- **Example:** The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n u[n]$ is given by

$$H(z) = \frac{1}{1+0.6z^{-1}}, |z| > 0.6$$

Here, the ROC is just outside the circle going through the point $z = -0.6$.

ROC 주의 contains poles!



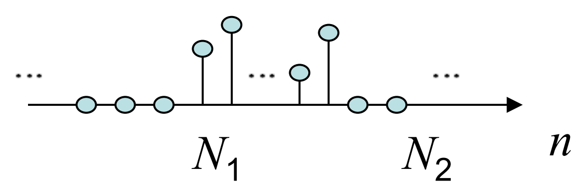
ROC of Finite-Length Sequence

- If $x[n]$ is a *finite-duration sequence*, i.e. a sequence that is zero except over a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the **entire z-plane, except z = 0**. (it is non-zero
- **Example:** Consider a finite-length sequence $g[n]$ defined for $-M \le n \le N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n] z^{-n}$$

highest order!

- **Note:** $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$.
- Thus, the ROC is the entire z-plane except possibly at $z = 0$ **and/or** at $z = \infty$



$x[n] \qquad N_1 \le n \le N_2$

1) $x[1] = 1 \qquad x[2] = 3$

$X(z) = z^{-1} + 3z^{-2}$ (pole at 0)

$$= \frac{z}{z^2} + \frac{3}{z^2}$$

$$= \frac{z+3}{z^2}$$

zero : -3

poles : 0

ROC: Remove the poles!

ROC $= \{z \in \mathbb{C}, z \neq 0\}$

2) $x[-3] = 2 \quad x[-1] = 4$

$X(z) = 2z^3 + 4z^1$ (pole at infinite)

$ROC = \{z \in C, |z| < \infty\}$

3) $x[-2] = -1 \quad x[1] = 3$

$X(z) = -z^2 + 3z^{-1}$ pole at 0 & ∞

$ROC = \{z \in C, z \neq 0 \ |z| < \infty\}$

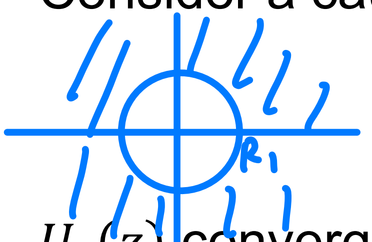
ROC of Right-Sided Sequence

- A right-sided sequence is a sequence with nonzero sample values for $n > N$. A right-sided sequence with nonzero sample values for $n \geq 0$ is called a **causal sequence**.

large z^{-n}

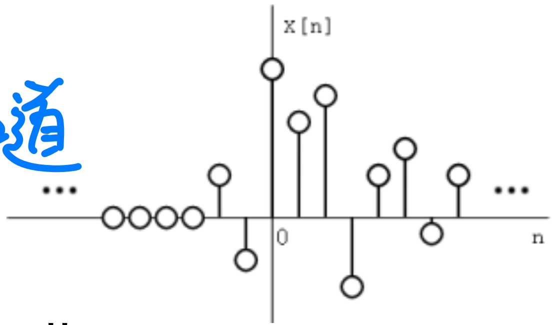
$z\uparrow \Rightarrow$ no problem $z\downarrow \Rightarrow$ may have trouble ?!

- Consider a causal sequence $\mu_1[n]$, whose z-transform is given by



$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

不知道



- $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$.
- On the other hand, a right-sided sequence $\mu_2[n]$ with nonzero sample values only for $n > -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$.
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$.

Right sided sequence

Intuition

causal $x[n]$ $n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

if $|z|$ is large, $|z^{-n}|$ is small $\Rightarrow$ Good for convergence

$$\text{RoC} = \{z \in \mathbb{C},\ |z| > R_1\}$$

Non-causal $x[n]$ $n \geq -M$ (M is positive)

$$X(z) = \sum_{n=-M}^{\infty} x[n] z^{-n}$$

$$= x[-M] z^{M} + x[-M+1] z^{M-1} + \ldots$$

poles at $z = \infty$

$$\text{RoC} = \{z \in \mathbb{C} : |z| > R_1,\ |z| < \infty\}$$

By intuition!

Example 5.1 Right-Sided Sequence

exp. sequence

- Determine the z-transform of $x[n] = a^n u[n]$ ← causal !

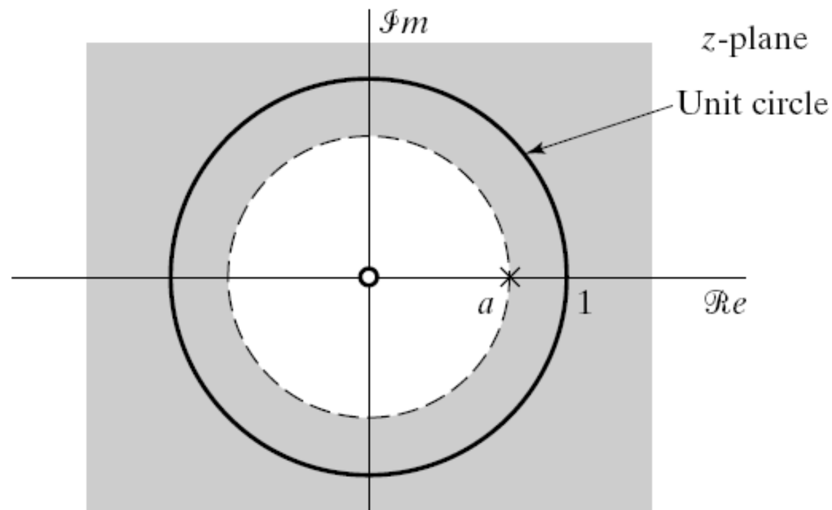
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- $X(z)$ converges if $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. This requires

$|az^{-1}| < 1 \Rightarrow \boxed{|z| > |a|}$, and gives

$$X(z) = \boxed{\frac{1}{1-az^{-1}}} = \frac{z}{z-a}$$

$z \neq a$



Shaded area = ROC

x denotes as the pole of $X(z)$

o denotes as the zero of $X(z)$

Note: If |a| < 1, the ROC contains the unit circle and hence the DTFT of this sequence $x[n]$ exists. ✓

- Remark: $u[n]$ has z-transform, but not DTFT

ROC of Left-Sided Sequence

- A left-sided sequence is a sequence with nonzero sample values for $n \leq N$. A left-sided sequence with nonzero sample values for $n \leq 0$ is called an **anticausal sequence**.

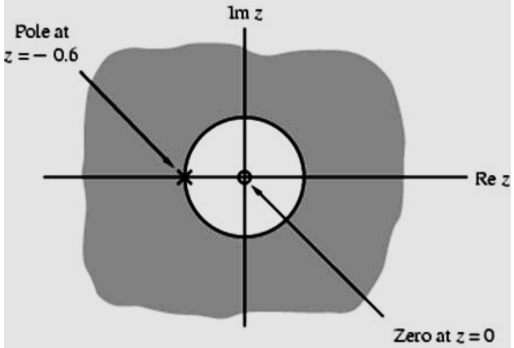
Within circle!

- Consider an anticausal sequence $u_1[n]$, whose z-transform is given by

$|z^{-n}|$ small!

want z to be small

$$V_1(z) = \sum_{n=-\infty}^{0} u_1[n] z^{-n}$$



- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$.

$ROC = \{z \in \mathbb{C}, |z| < R_3\}$

- On the other hand, a left-sided sequence $\mu_2[n]$ with nonzero sample values only for $n \leq N$ with *N* nonnegative has a z-transform $V_2(z)$ with N poles at $z = 0$.
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$.

$ROC = \{z \in \mathbb{C}, 0 < |z| < R_4\}$

Example 5.2 Left-Sided Sequence

anticausal

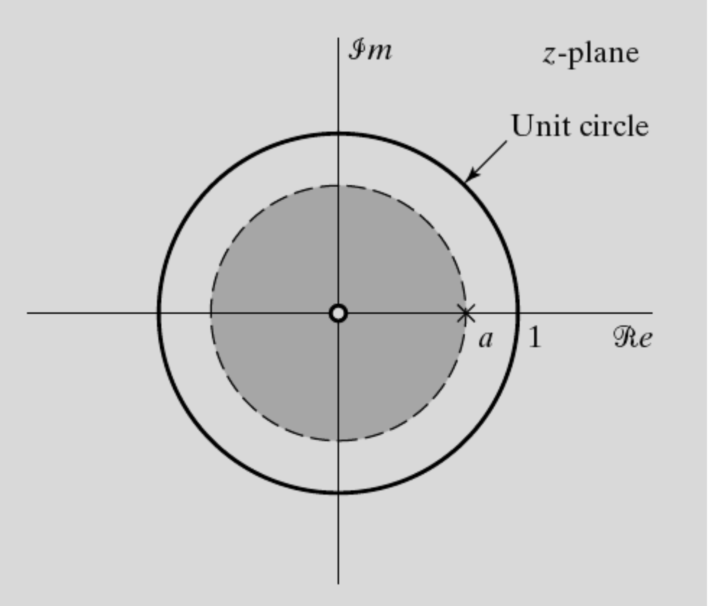
- Determine the z-transform of another signal $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

- $X(z)$ converges if $\sum_{m=1}^{\infty} |az^{-1}|^m < \infty$. This requires

$|a^{-1}z| < 1 \Rightarrow \boxed{|z| < |a|}$, and gives Same as Example 5.1

$$X(z) = \frac{1}{1-az^{-1}} = \boxed{\frac{z}{z-a}}$$



Note: If |a| < 1, the ROC does not contain the unit circle and the DTFT of this sequence $x[n]$ does not exist.

x denotes as the pole of $X(z)$
o denotes as the zero of $X(z)$

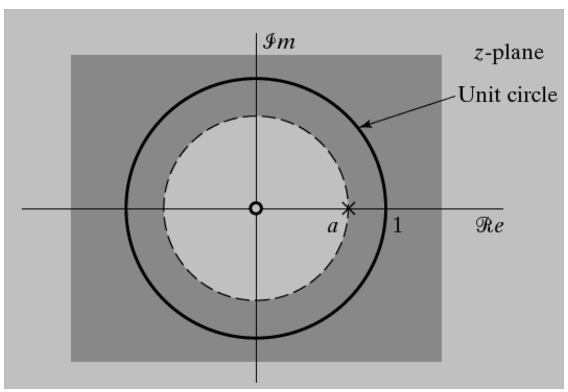
Observation from Example 5.1 and 5.2

- Different signals can give same *z*-transform, although the ROCs will differ

$$x[n] = a^n u[n]$$

$$\Updownarrow$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{if} \quad |z| > |a|$$



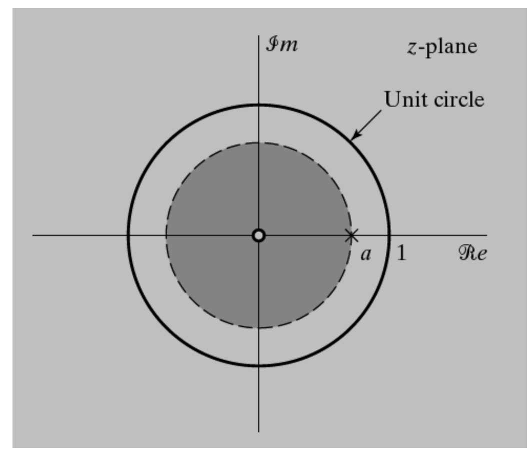
$X(e^{j\omega})$ exists if $|a| < 1$

$X(e^{j\omega})$ does not exist if $|a| > 1$

$$x[n] = -a^n u[-n-1]$$

$$\Updownarrow$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{if} \quad |z| < |a|$$



Same z-transform

$X(e^{j\omega})$ exists if $|a| > 1$

$X(e^{j\omega})$ does not exist if $|a| < 1$

ROC of Two-Sided Sequence

causal anti-causal

- The z-transform of a two-sided sequence $w[n]$ can be expressed as

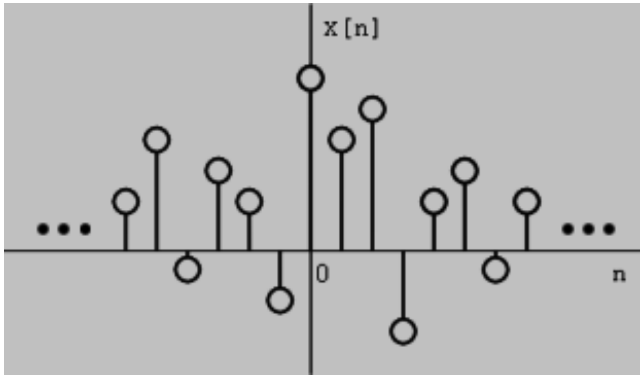
$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \boxed{\sum_{n=0}^{\infty} w[n]z^{-n}} + \boxed{\sum_{n=-\infty}^{-1} w[n]z^{-n}}$$

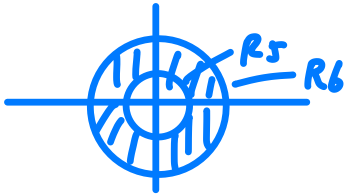
exterior interior

- The first term on the RHS can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$.

- The second term on the RHS can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$.

- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$. Otherwise, the z-transform does not exist.





Example 5.3 Two-Sided Sequence

5.1 and 5.2

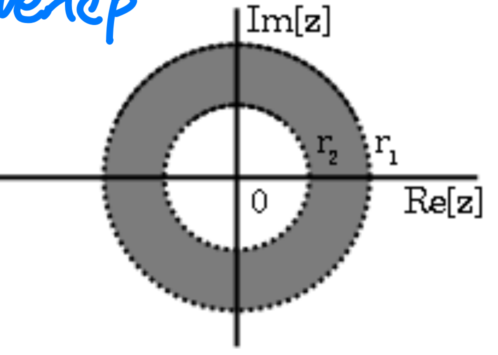
- Consider the two-sided sequence $u[n] = \alpha^n$, whose z-transform is given by

$$W(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > \alpha$ whereas the second term converges for $|z| < \alpha$.

no-overlap

- There is no overlap between two regions.



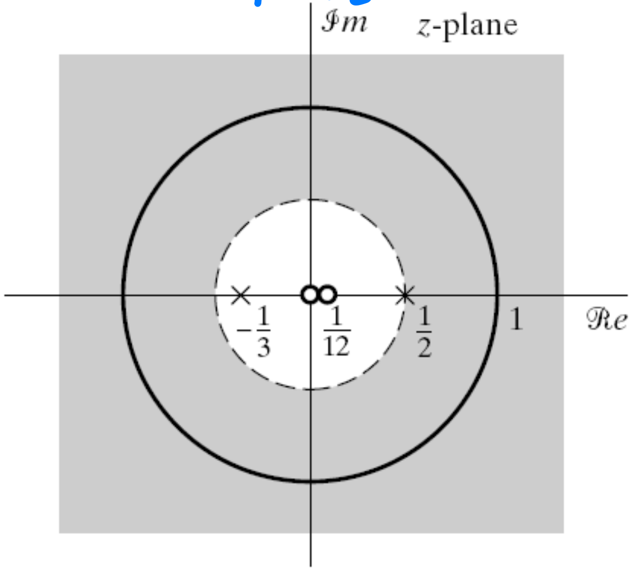
- Hence, the z-transform does not exist.

Example 5.4 Sum of Two Sequences

- Consider a signal that is the sum of two real sequences:

$a^n u[n] \rightarrow \frac{1}{1 - a z^{-1}}$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$$



- Using the result in Example 5.1, $a^n u[n]$

$a = \frac{1}{2}$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} X_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad if \quad |z| > \frac{1}{2}$$

$a = -\frac{1}{3}$

$$x_2[n] = \left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z} X_2(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} \quad if \quad |z| > \frac{1}{3}$$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} \quad if \quad \left(|z| > \frac{1}{3}\right) \cap \left(|z| > \frac{1}{2}\right) \Rightarrow |z| > \frac{1}{2}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

two poles
two zeros

The overall ROC is the region of overlap of the ROC for $X_1(z)$ and the ROC for $X_2(z)$; unless there is pole-zero cancellation.

Causal !

Table 3.4.1: Common z-transform pairs

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Ch3.4: z-Transform

- Definition of z-Transform
- Region of Convergence

Most difficult!!!

- **Inverse z-Transform**

Make filter in z-domain

- Z-Transform Properties

need conversion

- LTI System in Transform Domain

Inverse z-Transform

The inverse z-transform is based on a special case of the Cauchy integral theorem

Complex analysis

$$\boxed{\frac{1}{2\pi j}\oint_C z^{-\ell}\,dz = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1 \end{cases}}$$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

where C is a counterclockwise contour that encircles the origin. If we multiply $X(z)$ by z^{n-1} and compute

$X(z) z^{n-1}$

$$\begin{aligned}
\frac{1}{2\pi j}\oint_C \boxed{X(z) z^{n-1}}\,dz &= \frac{1}{2\pi j}\oint_C \boxed{\sum_{m=-\infty}^{\infty} x[m] z^{-m+n-1}}\,dz \\
&= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\frac{1}{2\pi j}\oint_C z^{-(m-n+1)}\,dz}_{=1 \text{ only when } m-n+1=1} \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta(m-n) \\
&= x[n]
\end{aligned}$$

proved for correctness!

$\Rightarrow m = n$

Hence, the inverse z-transform of $X(z)$ is defined as $x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\,dz$ where C is a counterclockwise closed contour in the ROC of $X(z)$ encircling the origin.

Inverse z-Transform via Cauchy's Residue Theorem

$X(z) = \frac{1}{(z-1)^2}$ $\lambda_1 = 1, m_1 = 2$ identify!

Denote the unique poles of $X(z)$ as $\lambda_1, \ldots, \lambda_R$ and their algebraic multiplicities as $m_1, \ldots, m_R$. As long as R is finite (which is the case if $X(z)$ is rational) we can evaluate the inverse z-transform via Cauchy's residue theorem which states

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz = \sum_{\lambda_k \text{ inside } C} \mathrm{Res}(X(z) z^{n-1}, \lambda_k, m_k)$$

check

where $\mathrm{Res}(F(z), \lambda_k, m_k)$ is the "residue" of $F(z) = X(z) z^{n-1}$ at the pole λ_k with algebraic multiplicity m_k, defined as

$$\mathrm{Res}(F(z), \lambda_k, m_k) = \frac{1}{(m_k - 1)!} \left[\frac{d^{m_k - 1}}{dz^{m_k - 1}} \left\{ (z - \lambda_k)^{m_k} F(z) \right\} \right]_{z = \lambda_k}$$

constant!

In other words, Cauchy's residue theorem allows us to compute the contour integral by computing derivatives.

Cauchy's residue theorem works, but it can be tedious and there are lots of ways to make mistakes. The Matlab function `residuez` (discrete-time residue calculator) can be useful to check your results.

Other (typically easier) options for computing inverse z-transforms:

1. Inspection (table lookup).
2. Partial fraction expansion (only for rational z-transforms).
3. Power series expansion (can be used for non-rational z-transforms).

more general, but tedious!

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

e.g. $X(z) = 2z^{-1} + 4z^{-3}$

$$X(z) = x[1] z^{-1} + x[3] z^{-3}$$

$$x[n] = 2\delta[n-1] + 4\delta[n-3]$$

$$= \begin{cases} 2 & n=1 \\ 4 & n=3 \\ 0 & \text{otherwise} \end{cases}$$

The Inverse z-Transform

- There are 4 commonly used techniques to evaluate the inverse *z*-transform. They are
  - – **1. Inspection Method** Search table
  - – **2. Partial Fraction Expansion** More general
  - – **3. Power Series Expansion**
  - – 4. Cauchy Integral Theorem
    - This is the formal inverse *z*-transform expression based on Contour integral (an integration techniques together with right-hand rule) onto z-plane.
    - **We will not cover this method in this course.**

Inverse z-transform: Inspection Method

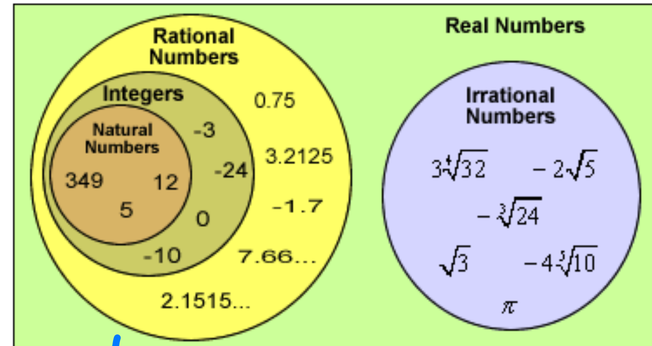
- **Inspection Method:** Familiar with, or recognizing "by inspection", certain transform pairs.

May not have unique solution

- **Example:**
  - Find the inverse z-transform of $X(z) = \dfrac{z}{z - \left(\frac{1}{2}\right)} \quad |z| > \left|\dfrac{1}{2}\right|$
  - This gives $X(z) = \dfrac{1}{1 - \left(\frac{1}{2}\right) z^{-1}} \quad |z| > \left|\dfrac{1}{2}\right|$
  - By making use of the transform pair in Table 3.4.1,

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Partial Fraction Expansion Method



- A rational $G(z)$ can be expressed as

$$G(z) = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$, then $G(z)$ can be re-expressed as

$$G(z) = \sum_{l=0}^{M-N} \eta_l \, z^{-l} + \frac{P_1(z)}{D(z)}$$

degree < N

degree = N

where the degree of $P_1(z)$ is less than *N*. The rational function $\frac{P_1(z)}{D(z)}$ is called a proper fraction.

degree ≥ N

proper fraction

- <u>Example</u> - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

$$G(z) = \sum_{l=0}^{M-N} r_l z^{-l} + \frac{P_1(z)}{D(z)}$$

$$g[n] = \sum_{l=0}^{M-n} r_l \delta[n-l] + \cdots$$

Proper fraction:

$$G(z) = \frac{\cdots}{\prod_{l=1}^{N} (z-\lambda_l)} \quad \leftarrow \text{degree} < N$$

$$\Rightarrow \quad G(z) = \sum_{l=1}^{N} \frac{\rho_l}{1-\lambda_l z^{-1}}$$

$$\rho_k = \,?$$

$$\Rightarrow \quad G(z)\cdot(1-\lambda_l z^{-1}) = \sum_{p=1}^{N} \frac{\rho_l}{1-\lambda_l z^{-1}}(1-\lambda_k z^{-1})$$

$$= \rho_k + \sum_{l\neq k} \frac{\rho_l}{1-\lambda_l z^{-1}}(1-\lambda_k z^{-1})$$

$$G(z)(1-\lambda_k z^{-1})\Big|_{z=\lambda_k} = p_k$$

$$\frac{p_\ell}{1-\lambda_\ell z^{-1}} \xrightarrow{\text{by table}} p_\ell (\lambda_\ell)^n u[n]$$

$|z| > e$

Inverse Transform: Partial-Fraction Expansion

$$\frac{1}{(z-\lambda_1)(z-\lambda_2)\cdots}$$

- **Simple Poles:** In most cases, the rational *z*-transform of interest $G(z)$ is a proper fraction with simple poles. Let the poles of $G(z)$ be at $z = \lambda_k, 1 \le k \le N$. A partial-fraction expansion of $G(z)$ is then of the form

$$\frac{x^2-3x+5}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

$$G(z) = \sum_{l=1}^{N} \left(\frac{\rho_l}{1-\lambda_l z^{-1}} \right)$$

- The constants $\{\rho_l\}$ are called the residues and given by

$$\rho_l = (1-\lambda_l z^{-1})G(z)|_{z=\lambda_l}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $|z| > \lambda_l$ and, thus has an inverse transform of the form $\rho_l(\lambda_l)^n u[n]$. Therefore, the inverse transform of *G*(*z*) is given by

$$g[n] = \sum_{l=1}^{N} \rho_l \, (\lambda_l)^n u[n].$$

- **Reminder: There are cases with multiple poles.**

Inverse Transform: Partial-Fraction Expansion

a causal sequence

- **Example 5.5:** Determine $h[n]$ whose z-transform is given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

proper fraction

filter designed

λ_1 λ_2

- **Solution:** A partial-fraction expansion of $H(z)$ is of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

- Given

$$\rho_1 = (1-0.2z^{-1})H(z)\big|_{z=0.2} = \frac{1+2z^{-1}}{1+0.6z^{-1}}\bigg|_{z=0.2} = 2.75$$

$$\rho_2 = (1+0.6z^{-1})H(z)\big|_{z=-0.6} = \frac{1+2z^{-1}}{1-0.2z^{-1}}\bigg|_{z=-0.6} = -1.75$$

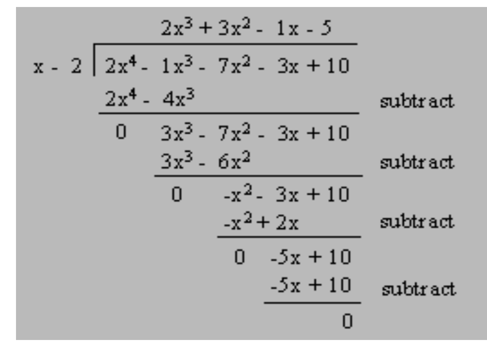
we have $H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}}$. Thus,

$$h[n] = 2.75(0.2)^n\mu[n] - 1.75(-0.6)^n\mu[n]$$

Inverse Transform by Long Division

- The z-transform *G*(*z*) of a sequence $g[n]$ can be expanded in a **power series** in z^{-1}. In the series expansion, the coefficient multiplying the term z^{-n} is then the *n*-th sample of $g[n]$.

- For a rational *z*-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division.



- **Example –** Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

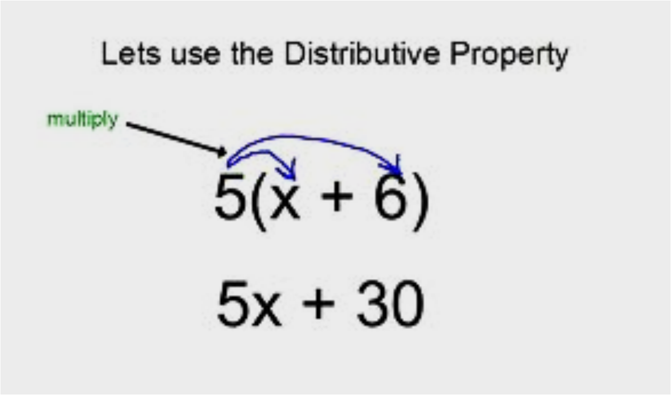
- Long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result: $h[n] = \{1 \;\; 1.6 \;\; -0.52 \;\; 0.4 \;\; -0.2224 \;\; \ldots\}$. Compare this result with that of Example 5.5.

Ch3.4: z-Transform

- Definition of z-Transform
- Region of Convergence
- Inverse z-Transform
- **Z-Transform Properties**
- LTI System in Transform Domain



z-Transform Properties

- **Linearity**

  If $x_1[n] \xleftrightarrow{Z} X_1(z), \quad \text{ROC} = R_{x_1}$ and $x_2[n] \xleftrightarrow{Z} X_2(z), \quad \text{ROC} = R_{x_2}$

  then $ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$

- **Example:** Consider the two-sided sequence

$$v[n] = \alpha^n u[n] - \beta^n u[-n-1].$$

- Two sequences $x[n] = \alpha^n u[n]$ and $y[n] = -\beta^n u[-n-1]$ have z-transforms $X(z) = \frac{1}{1-\alpha z^{-1}}, |z| > |\alpha|$ and $Y(z) = \frac{1}{1-\beta z^{-1}}, |z| < |\beta|$
- With linearity property, we arrives at

ROC: $|\alpha| < |z| < |\beta|$

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- ROC is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$.
- **Question: What happens if $|\alpha| > |\beta|$?** $\rightarrow$ ROC $= \phi$

z-Transform Properties

- Time Shifting

$\rightarrow \boxed{z^{-1}} \rightarrow$

$$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z), \qquad \mathrm{ROC} = R_x$$

ROC is unchanged, except for the possible addition or deletion of the points z = 0 or z = ∞. It is because the factor z^{-n_0} can alter the number of poles at z = 0 or z =∞.

- Multiplication by an Exponential Sequence

Prove!

$$z_0^n x[n] \overset{Z}{\longleftrightarrow} X(z/z_0), \qquad \mathrm{ROC} = |z_0| R_x$$

where $\mathrm{ROC} = |z_0|R_x$ denotes that the ROC R_x is scaled by $|z_0|$; ie. if R_x is the set of values of z such that $r_R < |z| < r_L$, then the new ROC is the set of values of z such that $|z_0|r_R < |z| < |z_0|\, r_L$

z-Transform Properties

- Differentiation $\quad nx[n] \overset{Z}{\longleftrightarrow} -\boxed{z}\dfrac{dX(z)}{dz}, \qquad \text{ROC} = R_x$

  ROC is unchanged except for the possible addition or deletion of $z = 0$.

- Conjugation of a Complex Sequence

$$\underline{x^*[n]} \overset{Z}{\longleftrightarrow} \underline{X^*(z^*)}, \qquad \text{ROC} = R_x$$

- Time Reversal $\quad x[-n] \overset{Z}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = 1/R_x$

  where $\text{ROC} = 1/R_x$ denotes that the ROC R_x is inverted; i.e. if R_x is the set of values of z such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $\underline{1/r_L < |z| < 1/r_R}$

Example 5.6: Time-Reversal

anti-causal!

- Find the *z*-transform of $x[n] = a^{-n}u[-n]$

- **Solution:**

causal

$$a^n u[n] \xleftrightarrow{Z} \boxed{\frac{1}{1-az^{-1}},} \qquad |z| > |a|$$

$z \to \frac{1}{z}$!

$$a^{-n}u[-n] \xleftrightarrow{Z} \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad \underline{\underline{|z| < |a^{-1}|}}$$

z-Transform Properties

☆ Most useful!

- Convolution of Sequences

If $x_1[n] \xleftrightarrow{Z} X_1(z), \quad \text{ROC} = R_{x_1}$ and $x_2[n] \xleftrightarrow{Z} X_2(z), \quad \text{ROC} = R_{x_2}$

then $\underline{x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z)}, \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$

The resulting ROC contains at least the intersection of R_{x_1} and R_{x_2}

- – If there is no pole-zero cancellation in the linear combination, the ROC is exactly equal to $R_{x_1} \cap R_{x_2}$.
- – If there is pole-zero cancellation in the linear combination, the ROC may be larger. ✓

?

- Initial-Value Theorem

initial!

If $x[n]$ is zero for $n < 0$ (i.e. if $x[n]$ is causal), then $\underline{x[0] = \lim_{z \to \infty} X(z)}$

Quick test!

More general!

Example 5.7: Convolution

linear convolution!

- Find the convolution sum of $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$, where $|a| < 1$.

$X(e^{j\omega})$ DTFT ?!

- **Solution:**
  - Method 1: direct convolution
  - Method 2: $Y(z) = X_1(z)X_2(z) \xrightarrow{z^{-1}} y[n] = x_1[n] * x_2[n]$

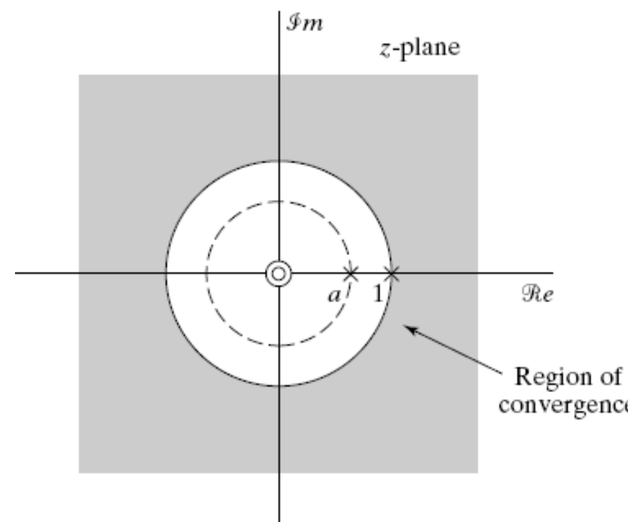
$$X_1(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a| \quad \text{and} \quad X_2(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

This gives

$$Y(z) = \frac{1}{1-az^{-1}} \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$= \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right)$$

$$y[n] = \frac{1}{1-a}\left(u[n] - a^{n+1}u[n]\right)$$



Prove !!!

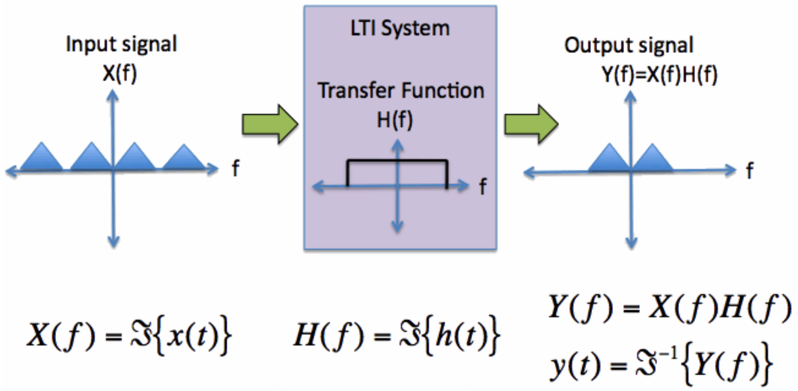
Table 3.4.2: z-transform properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert \mathcal{R}_g$
Differentiation of $G(z)$	$n g[n]$	$-z \dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g \mathcal{R}_h$ denotes the region $R_{g^-} R_{h^-} < |z| < R_{g^+} R_{h^+}$.

Ch3.4: z-Transform

- Definition of z-Transform
- Region of Convergence
- Inverse z-Transform
- Z-Transform Properties
- **LTI System in Transform Domain**



LTI Systems in the Transform Domain

- An LTI discrete-time system is completely characterized in the time-domain by its **impulse response sequence** $h[n]$.

- Each LTI discrete-time system has a **transform-domain representation**.

★ 好设计，只需写 zeros/poles 即可!

- Such transform-domain representations provide additional insight into the behavior of such systems. It is easier to design and implement these systems in the transform-domain for certain applications.

filter

- We consider now the use of the **DTFT** and the ***z*-transform** in developing the transform domain representation of an LTI system.

The Transfer Function

- **Transfer Function:** A generalization of the frequency response function.

- The convolution sum description of an LTI discrete-time system with an impulse response $h(n)$ is given by $y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$.

  $h[n] \circledast x[n]$

- Taking the z-transform of both sides, we get

$$
\begin{aligned}
Y(z) &= \textstyle\sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= \textstyle\sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right) z^{-n} \\
&= \textstyle\sum_{k=-\infty}^{\infty} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k] z^{-n}\right) \\
&= \textstyle\sum_{k=-\infty}^{\infty} h[k] \left(\sum_{l=-\infty}^{\infty} x[l] z^{-(l+k)}\right) \\
&= \textstyle\sum_{k=-\infty}^{\infty} h[k] z^{-k} \left(\sum_{l=-\infty}^{\infty} x[l] z^{-l}\right) \\
&= H(z) X(z)
\end{aligned}
$$

Multiplication!

- Here, $H(z) = Y(z)/X(z)$ is the z-transform of $h(n)$ and is called the **transfer function** or the **system function**.

FIR vs. IIR Filter

- A finite-duration impulse response (**FIR**) filter has an impulse response

$$h(n)=\begin{cases} b_n & 0\le n\le M-1 \\ 0 & else \end{cases}$$

- The difference equation representation is

h[n] * x[n]

$$y[n]=b_0x[n]+b_1x[n-1]+\cdots+b_{M-1}x[n-M+1]$$

which is a linear convolution of finite support.

$$h[n]=\begin{cases} C_n & n\ge 0 \\ 0 & else \end{cases}$$

- **IIR** filters are characterized by infinite-duration impulse response.
- The difference equation representation of an IIR filter is expressed as

Difference equation

$$y[n]=\sum_{m=0}^{M-1} b_m x[n-m]-\sum_{m=1}^{N-1} a_m y[n-m]$$

Recursive!

$y[n]=b_0x[n]-a_1y[n-1]$

$=b_0x[n]-a_1(b_0x[n-1]-a_1\cdots)$

Much more compact

Depends on previous input!

minus infinity!

LTI Systems in the Transform Domain

- Consider an LTI discrete-time system characterized by linear constant coefficient difference equations of the form:

$$\sum_{k=0}^{N} d_k\, y[n-k] = \sum_{k=0}^{M} p_k\, x[n-k].$$

- Applying the *z*-transform to both sides of the difference equation, we arrive at

$$\sum_{k=0}^{N} d_k\, z^{-k} Y(z) = \sum_{k=0}^{M} p_k\, z^{-k} X(z)$$

where $Y(z)$ and $X(z)$ denote the z-transforms of $y[n]$ and $x[n]$ with associated ROCs, respectively.

- A more convenient form of the **z-domain representation** of the difference equation is given by

$$\left(\sum_{k=0}^{N} d_k\, z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k\, z^{-k}\right) X(z)$$

LTI Systems in the Transform Domain

- Thus,

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} p_k\, z^{-k}}{\sum_{k=0}^{N} d_k\, z^{-k}}$$

$$= z^{(N-M)} \frac{\sum_{k=0}^{M} p_k\, z^{M-k}}{\sum_{k=0}^{N} d_k\, z^{N-k}}$$

$N>M$

$$= \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

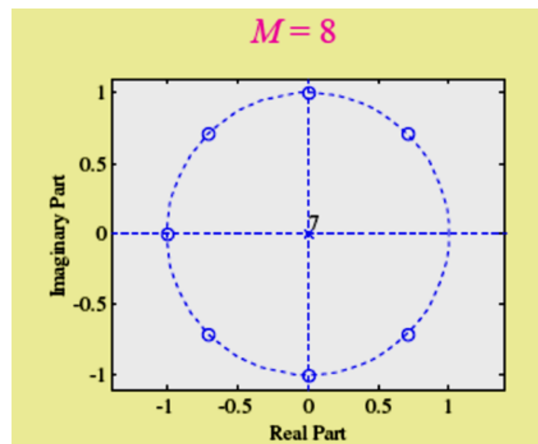
- $\xi_1, \xi_2, \dots, \xi_M$ are finite zeros, and $\lambda_1, \lambda_2, \dots, \lambda_N$ are finite poles.
- If $N > M$, there are additional $(N-M)$ zeros at $z = 0$.
- If $N < M$, there are additional $(M-N)$ poles at $z = 0$.
- For a causal digital filter, the impulse response is a causal sequence. The ROC of the **causal transfer function** is exterior to a circle going through the pole furthest from the origin with $|z| > \max|\lambda_k|$

Example 5.8: Moving Average Filter

- Consider the *M*-point moving-average FIR filter with an impulse response, $\mathrm{h}(n) = \begin{cases} \frac{1}{M}, 0 \le n \le M-1 \\ 0, \mathrm{otherwise} \end{cases}$
- Its transfer function is given by

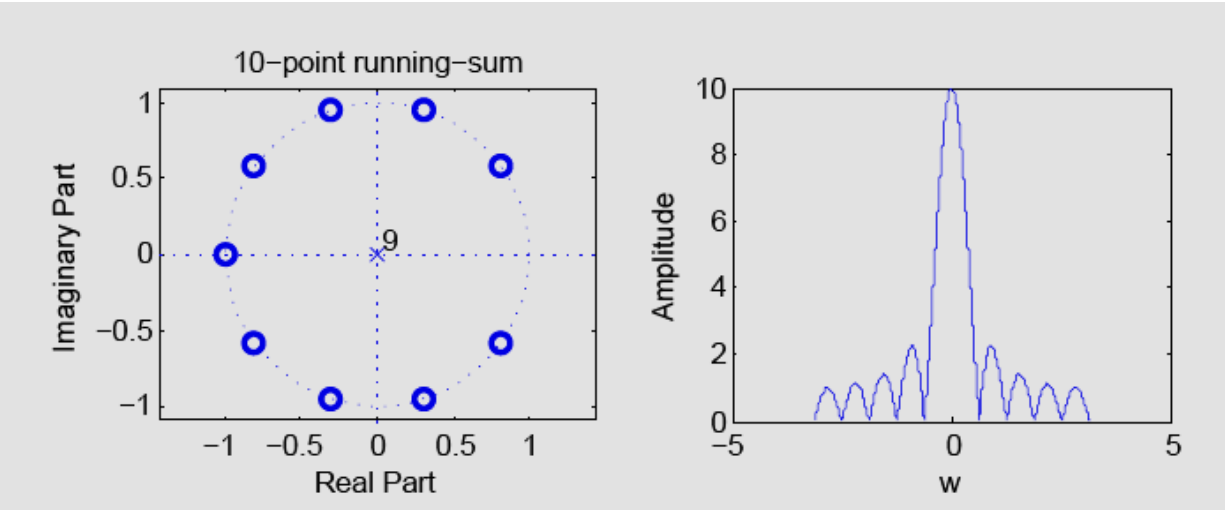
$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^M - 1}{M[z^{M-1}(z-1)]}$$

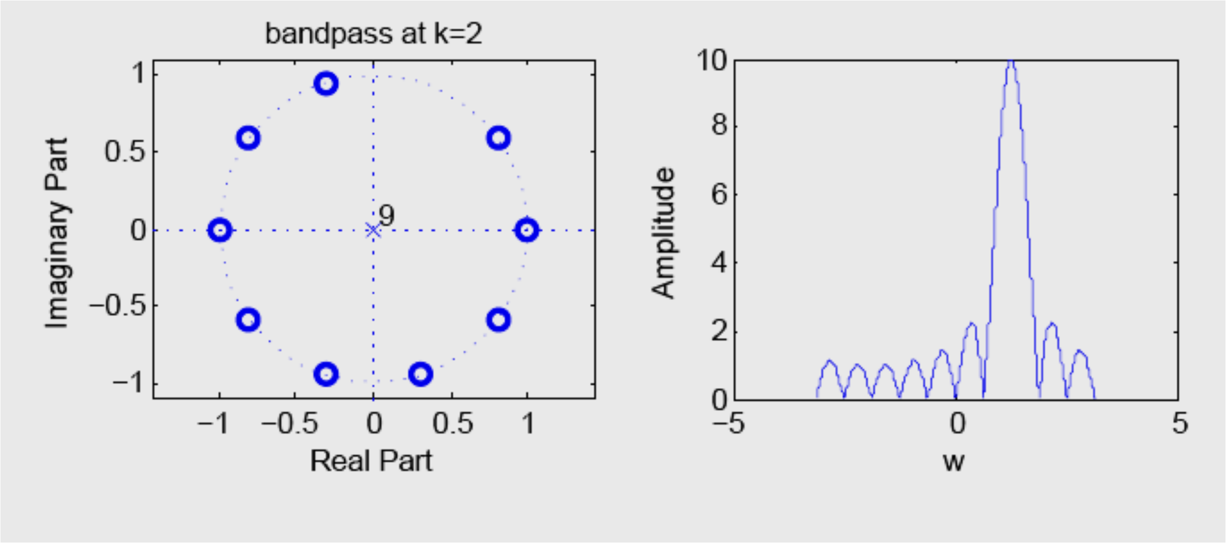
- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}, 0 \le k \le M-1.$
- There is an $(M-1)$th order pole at $z = 0$ and a single pole at $z = 1.$
- The ROC is the entire z-plane except $z = 0$



Example 5.8: Moving Average Filter

$M=10$





Frequency Response from Transfer Function

- If the ROC of the transfer function *H*(*z*) includes the unit circle, then the frequency response of the LTI digital filter can be obtained simply as follows: $H\left(e^{j\omega}\right) = H(z)|_{z=e^{j\omega}}$

$z \rightarrow e^{jw}$

- For a **stable rational transfer function** in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z - \xi_k)}{\prod_{k=1}^{N}(z - \lambda_k)}$$

the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - \xi_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - \lambda_k\right)}$$

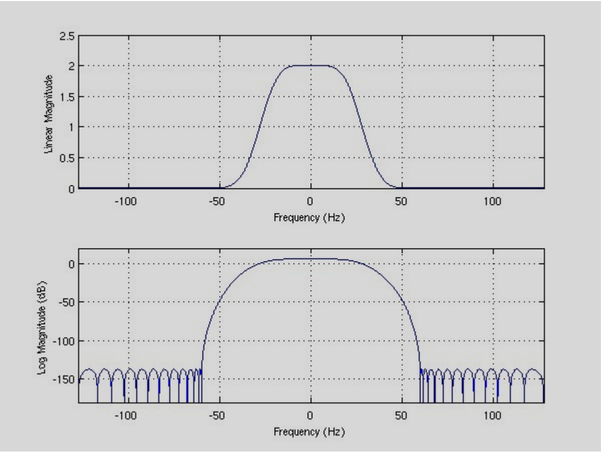
- It is convenient to visualize the contributions of the **zero factor** and the **pole factor** from the factored form of the frequency response.

Frequency Response from Transfer Function

- The **magnitude response function** is given by:

$$|H(e^{j\omega})| = |\frac{p_0}{d_0}||e^{j\omega(N-M)}|\frac{\prod_{k=1}^{M}|e^{j\omega}-\xi_k|}{\prod_{k=1}^{N}|e^{j\omega}-\lambda_k|}$$

$$= |\frac{p_0}{d_0}|\frac{\prod_{k=1}^{M}|e^{j\omega}-\xi_k|}{\prod_{k=1}^{N}|e^{j\omega}-\lambda_k|}$$



- The **phase response** is of the form

$$arg H(e^{j\omega}) = \arg\left(\frac{p_0}{d_0}\right) + \omega(N-M)$$
$$+\sum_{k=1}^{M}\arg(e^{j\omega}-\xi_k) - \sum_{k=1}^{M}\arg(e^{j\omega}-\lambda_k)$$

- **How can we use this?**

Geometric Interpretation

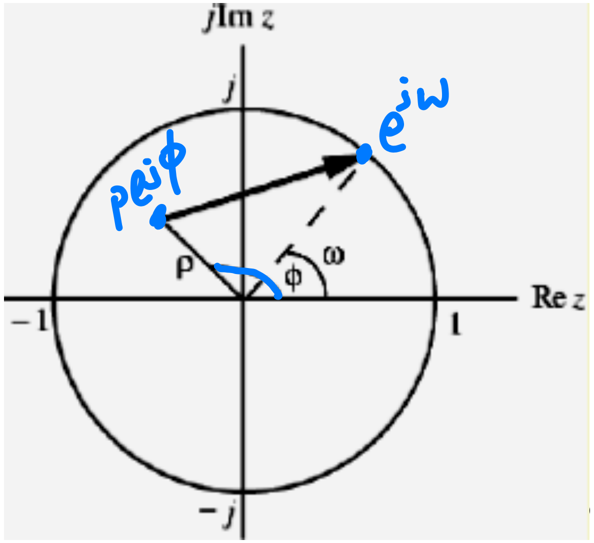
- The factored form of frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

  is convenient to develop a **geometric interpretation** of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle.

- A **typical factor** in the factored form of the frequency response is given by $\left(e^{j\omega} - \rho e^{j\phi}\right)$ where $\rho e^{j\phi}$ is a zero or pole.

- It can be observed from the right figure that $\left(e^{j\omega} - \rho e^{j\phi}\right)$ represents a vector starting at $z = \rho e^{j\phi}$ and ending on the unit circle $z = e^{j\omega}$.



Geometric Interpretation

input: $a_0 e^{j\omega} \longrightarrow$ output: $a_0 e^{j\omega} H(e^{j\omega})$ if $|H(e^{j\omega})|$ is small $\Rightarrow$ output is small!

- As indicated by $|H(e^{j\omega})| = |\frac{p_0}{d_0}| \frac{\prod_{k=1}^{M} |e^{j\omega} - \xi_k|}{\prod_{k=1}^{N} |e^{j\omega} - \lambda_k|}$, the magnitude response at a specific frequency ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors.

numerator: $0 \rightarrow \omega = \phi$

- **Question: When can we obtain the smallest magnitude?**

A zero (pole) vector has the smallest magnitude when $\omega = \phi$

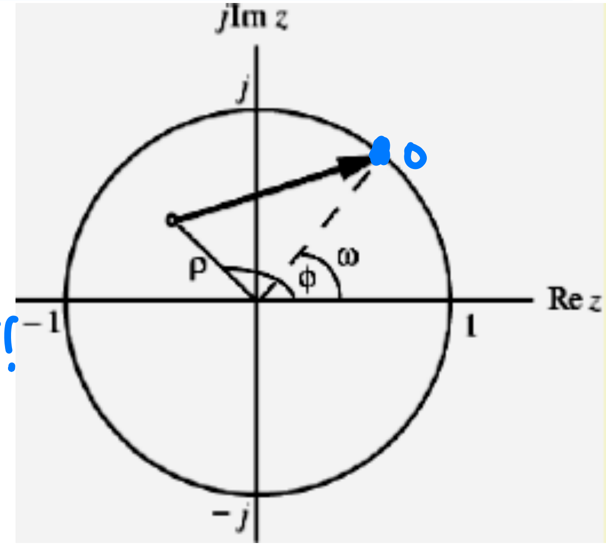
zero

- To **attenuate** signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range.

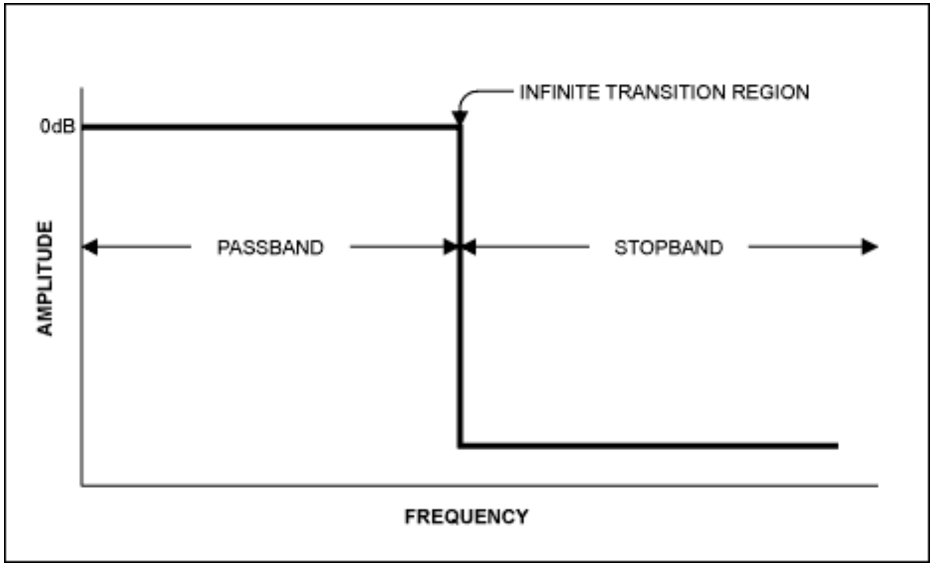
pole

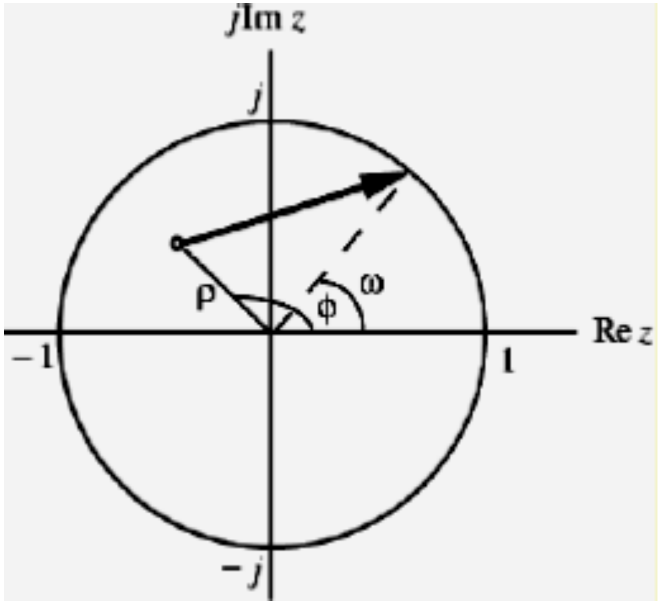
- To **emphasize** signal components in a specified frequency range, we need to place poles very close to $\boxed{}$ the unit circle in this range.

否则没有DTFT!



Can we build an ideal bandpass filter?





No!

Can we build an ideal bandpass filter?

- The z-transform of an FIR impulse response can be expressed as a simple polynomial $P(z)$ of degree L - 1 where L is the number of nonzero taps of the filter.

- The fundamental theorem of algebra states that such a polynomial has at most L - 1 roots. Thus, the frequency response of an FIR filter can never be identically zero over a frequency interval since, if it were, its z-transform would have an infinite number of roots.

- The argument can be easily extended to rational transfer functions, confirming the impossibility of a realizable filter whose characteristic is piecewise perfectly flat.

- Then, how can we design a bandpass filter? (To be continued…)

Stability Condition

- A causal LTI digital filter is BIBO stable if and only if its impulse response *h*[*n*] is absolutely summable, i.e., $S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- Thus, an FIR digital filter with bounded impulse response is always stable.

Write down!

- **Question: What is the condition in terms of pole locations of $H(z)$?**

Tell some freq. is stable or not!

- The ROC of the $H(z)$ for the impulse response sequence $h[n]$ is defined by values of $|z| = r$ where $\sum_{n=-\infty}^{\infty} |h[n]r^{-n}| < \infty$. Thus, if the digital filter is stable, the ROC includes the unit circle.

$r = 1$

Stability Condition in Terms of Pole Position

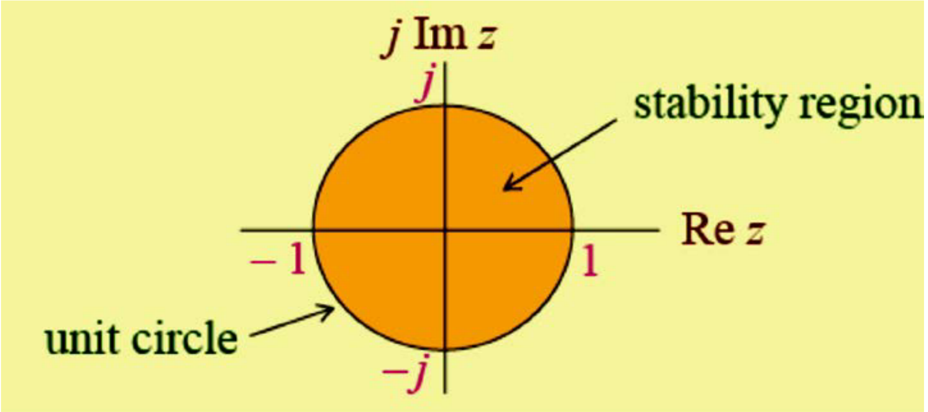
- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by $H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$. Its impulse response is right-sided sequence.

  $h(n)$

- The ROC of the $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$. But stability requires ROC includes the unit cycle.

  $|z| > \max|\lambda_k| \rightarrow$ must be less than 1!

- **Conclusion:** All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle.
- The stability region (shown shaded) in the z-plane is shown below



Example 5.9: Stability Condition IIR Filter

- Example - The factored form of

$$H(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$H(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since both poles are not inside the unit circle, $H(z)$ is unstable