

T07

CT Fourier transform for periodic signal
Relation between CTFS and CTFT
Tables 4.1 and 4.2
Duality property

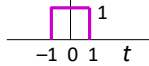
DT Fourier series to DT Fourier transform
DT Fourier transform for aperiodic signal
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Tables 5.1 and 5.2

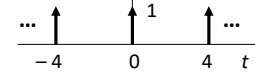
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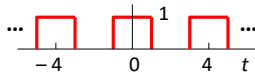
CT Fourier Transform for Periodic Signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(t - kT) \quad \text{Poisson sum}$$

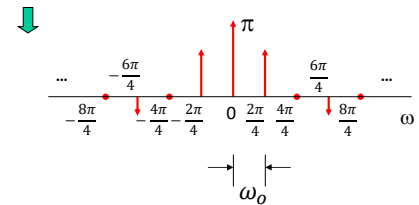
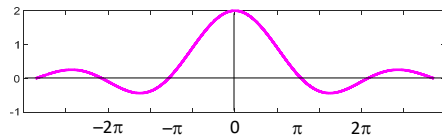
e.g.

$x(t)$ 
 $\xrightarrow{\text{F.T.}}$
 $X(j\omega) = \frac{2 \sin(\omega)}{\omega}$

$*$ 
 $\xrightarrow{\text{F.T.}}$
 $\frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right) \quad T=4$

$\tilde{x}(t)$ 
 $\xrightarrow{\text{F.T.}}$
 $X(j\omega) \left[\frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right) \right]$

2



$$X(j\omega) = \frac{2 \sin(\omega)}{\omega}$$

$$\frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right)$$

$$\frac{2\pi}{4} \sum_{k=-\infty}^{\infty} X\left(\frac{j\pi k}{2}\right) \delta\left(\omega - \frac{\pi k}{2}\right)$$

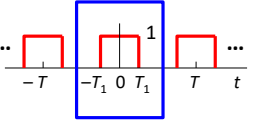
Sampling property

Question : Difference on F.T. between aperiodic and periodic ?

3

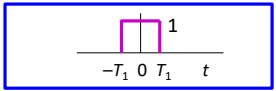
Relation of CTFS and CTFT

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Periodic 

$\xleftrightarrow{\text{F.S.}}$
 $a_k = \frac{1}{T} \int_0^T \tilde{x}(t) e^{-jk\omega_o t} dt \quad (1)$

$$x(t) = \begin{cases} \tilde{x}(t) & -T/2 \leq t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

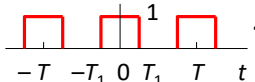
Aperiodic 

$\xleftrightarrow{\text{F.T.}}$
 $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
 $= \int_{-T/2}^{T/2} \tilde{x}(t) e^{-j\omega t} dt \quad (2)$

Compare (1) and (2) $a_k = \frac{1}{T} X(j\omega)|_{\omega=k\omega_o}$

4

e.g. **Periodic** square wave and **Aperiodic** square pulse

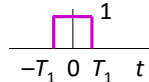
...  ...

$$a_k = \frac{2 \sin(k\omega_0 T_1)}{Tk\omega_0}$$

$$= \frac{2 \sin(k\omega_0 T_1)}{Tk \left(\frac{2\pi}{T}\right)}$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi}$$

\Leftrightarrow



$$X(j\omega) = \frac{2 \sin(\omega T_1)}{\omega}$$

$$= \frac{2 \sin(k\omega_0 T_1)}{k \left(\frac{2\pi}{T}\right)}$$

$\frac{1}{T} X(j\omega)|_{\omega=k\omega_0}$

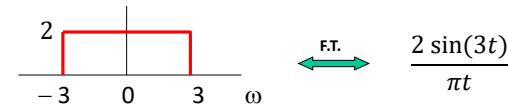
Question : Difference between its FS and its FT for a periodic signal ?

5

Tables 4.1 and 4.2

e.g. Given : $|X(j\omega)| = 2u(\omega + 3) - u(\omega - 3)$ Obtain $x(t)$

$$\angle X(j\omega) = -2\omega + \pi$$



F.T.

$$\frac{2 \sin(3t)}{\pi t}$$

Linearity & Time shifting

$$X(j\omega) = \begin{cases} 2 e^{j(-2\omega + \pi)} & |\omega| < 3 \\ 0 & \text{otherwise} \end{cases} = (-1) e^{-j2\omega} \begin{cases} 2 & |\omega| < 3 \\ 0 & \text{otherwise} \end{cases}$$

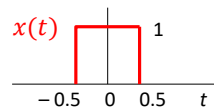
$$x(t) = -\frac{2 \sin(3(t-2))}{\pi(t-2)} = -\frac{2 \sin(3t)}{\pi t} * \delta(t-2)$$

6

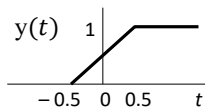
e.g. Given :

$$y(t) = \begin{cases} 0 & t < -0.5 \\ t + 0.5 & -0.5 \leq t < 0.5 \\ 1 & 0.5 \leq t \end{cases}$$

Obtain the F.T. of $y(t)$



Integration
Differentiation



$$x(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{\text{F.T.}} X(j\omega) = \frac{2 \sin(\frac{\omega}{2})}{\omega} \quad X(j0) = ?$$

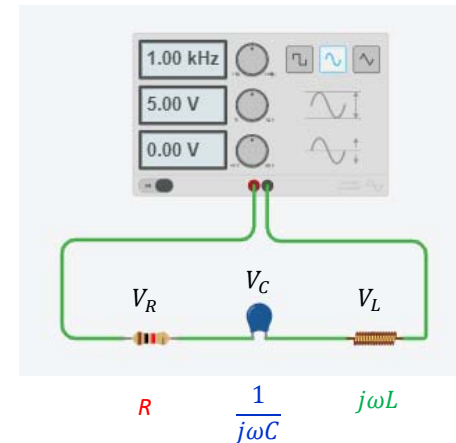
$$y(t) = \int_{-\infty}^t x(t) dt \xleftrightarrow{\text{F.T.}} Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$$

$$\frac{d}{dt} y(t) = x(t)$$

$$X(0) = ?$$

7

e.g.



$$v_R(t) = i(t) R$$

$$V_R(j\omega) = I(j\omega) R \quad R = \frac{V}{I}$$

$$v_C(t) = \frac{1}{C} \int i(t) dt$$

$$V_C(j\omega) = \frac{1}{C} \frac{1}{j\omega} I(j\omega)$$

$$v_L(t) = L \frac{d}{dt} i(t)$$

$$V_L(j\omega) = L j\omega I(j\omega)$$

8

e.g. Find $x(t)$ for the following LTI system.

$$x(t) \longrightarrow \boxed{H(j\omega) = \frac{1}{3+j\omega}} \longrightarrow y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

Convolution

$$y(t) = x(t) * h(t) \quad \xleftrightarrow{\text{F.T.}} \quad Y(j\omega) = X(j\omega)H(j\omega)$$

$$\begin{aligned} X(j\omega) &= \frac{Y(j\omega)}{H(j\omega)} \\ &= \left(\frac{1}{3+j\omega} - \frac{1}{4+j\omega} \right) (3+j\omega) \\ &= \frac{1}{4+j\omega} \end{aligned}$$

Inverse F.T.

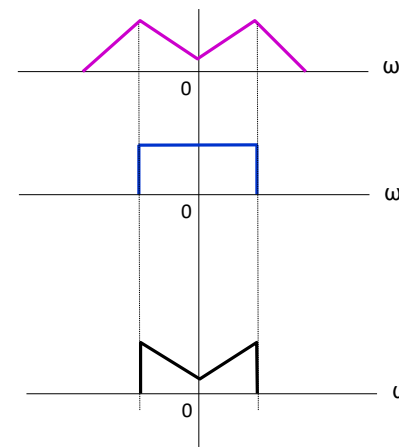
$$x(t) = e^{-4t}u(t)$$

Question : Is it a causal system ?

$$h(t) = ?$$

9

e.g. Given :



Input spectrum

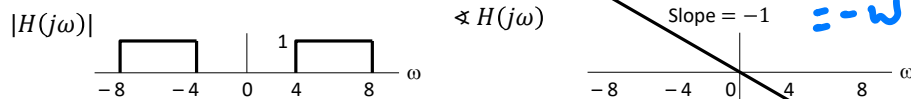
Frequency response

Sketch the output spectrum

$$Y(j\omega) = X(j\omega)H(j\omega)$$

10

e.g. The magnitude response and phase response of an LTI system are given below.

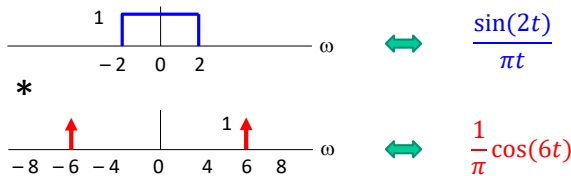


a) What is the type of this filter ?

b) Find the impulse response $h(t)$.

bandpass!

Multiplication / frequency shifting



What does it mean in time domain?

$$h(t) = ?$$

$$(2\pi) \frac{\sin(2t)}{\pi t} \frac{1}{\pi} \cos(6t) = \frac{2 \sin(2t)}{\pi t} \cos(6t)$$

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

$$\boxed{e^{j\omega t}} \rightarrow \delta(t-1)$$

11

$$\text{e.g. Given : } x(t) = \left(\frac{\sin(t)}{\pi t} \right) * \left(\frac{2 \sin(2t)}{\pi t} \right)$$

a) Sketch the F.T. of $x(t)$

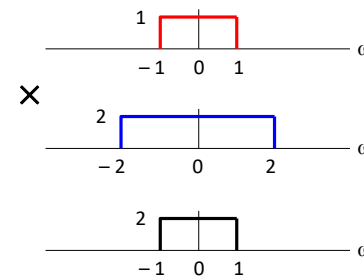
Convolution

b) Find the energy of $x(t)$

Parseval's Relation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



$$\text{Energy} =$$

$$\frac{2 \times 2}{2\pi} = \frac{4}{\pi}$$

12

Duality Property

$$\begin{array}{ccc} x(t) & & X(j\omega) \\ & \swarrow \quad \searrow & \\ y(t) = X(j\omega)|_{\omega=t} & & Y(j\omega) = 2\pi x(t)|_{t=-\omega} \end{array}$$

Proof: $Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} X(jt) e^{-j\omega t} dt \quad (1)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-jt) e^{j\omega t} dt \quad x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jt) e^{-j\omega t} dt \quad (2)$$

$(1) \rightarrow (2) \quad \Rightarrow \quad Y(j\omega) = 2\pi x(-\omega) = 2\pi x(t)|_{t=-\omega}$

13

?

$$\begin{array}{ccc} x(t) & & X(j\omega) \\ & \swarrow \quad \searrow & \\ y(t) = X(j\omega)|_{\omega=t} & & Y(j\omega) = 2\pi x(t)|_{t=-\omega} \end{array}$$

e.g. Given: $t e^{-|t|} \xleftrightarrow{F.T.} \frac{-j4\omega}{(1+\omega^2)^2}$ $Y(j\omega) = \int_{-\infty}^{\infty} \frac{4t}{(1+t^2)^2} e^{-j\omega t} dt$

Obtain the FT of $\frac{4t}{(1+t^2)^2}$

$$\begin{array}{ccc} & & \frac{-j4\omega}{(1+\omega^2)^2} \\ & \swarrow \quad \searrow & \\ F\left\{\frac{4t}{(1+t^2)^2}\right\} = \frac{2\pi(-\omega)e^{-|-\omega|}}{-j} & & \\ = -j2\pi\omega e^{-|\omega|} & & \end{array}$$

14

DT Fourier series to DT Fourier transform
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Tables 5.1 and 5.2

15

DT Fourier Series



DT Fourier Transform

$$\omega_o = \frac{2\pi}{N} \quad N = \text{finite}$$

$$\omega_o = \frac{2\pi}{N} \quad N \rightarrow -\infty \text{ to } \infty$$

$$\omega = k\omega_o \quad \omega = \text{finite}$$

$$k\omega_o \rightarrow \omega$$

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{N}n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Periodic Signals

Aperiodic Signals

Question : What is the difference between CTFT and DTFT ?

16

Table 5.1 and 5.2

5.1

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with 2π
5.3.2	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_1[n] = \begin{cases} x[n] & \text{if } n \text{ is multiple of } k \\ 0 & \text{if } n \text{ is not multiple of } k \end{cases}$	$X(e^{j\omega/k})$
5.4	Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
5.5	Multiplication	$x_1[n]x_2[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega - \theta)})d\theta$
5.5.5	Differentiation in Time	$n x[n]$	$j \frac{d}{d\omega} X(e^{j\omega})$
5.5.5	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$
5.5.8	Differentiation in Frequency	$x[n]n$	$j \frac{d}{d\omega} X(e^{j\omega})$
5.5.8	Conjugate Symmetry for Real Signals	$x[n]$ real	$X(e^{j\omega}) = X^*(e^{-j\omega})$
5.5.8	Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
5.5.8	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.5.8	Even-Odd Decomposition of Real Signals	$x[n] = \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n])$	$X(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X(e^{-j\omega})) + \frac{1}{2}(X(e^{j\omega}) - X(e^{-j\omega}))$
5.5.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

4.1

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
4.3.1	Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(j\omega) + a_2X_2(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.4	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.5	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)X_2(j\omega)$
4.5	Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\theta)X_2(j(\omega - \theta))d\theta$
4.5.4	Differentiation in Time	$t x(t)$	$j \frac{d}{d\omega} X(j\omega)$
4.5.4	Integration	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
4.5.6	Differentiation in Frequency	$x(t)t$	$j \frac{d}{d\omega} X(j\omega)$
4.5.6	Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega) = X^*(-j\omega)$
4.5.6	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.5.6	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.5.6	Even-Odd Decomposition for Real Signals	$x(t) = \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t))$	$X(j\omega) = \frac{1}{2}(X(j\omega) + X(-j\omega)) + \frac{1}{2}(X(j\omega) - X(-j\omega))$
4.5.7	Parseval's Relation for Aperiodic Signals	$\int_{-\infty}^{\infty} x(t)y^*(t)dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$

17

5.2

2 π -periodic, like T.S. coefficients!

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - 2\pi k)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$	$a_k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$
$\cos \omega_0 n$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)]$	$a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$
$\sin \omega_0 n$	$j\pi \sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k)]$	$a_k = \begin{cases} \frac{j}{2} & k = 1 \\ -\frac{j}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases}$
$\delta[n]$	1	$a_k = 1$
$x[n]$	$X(e^{j\omega})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega - \omega_0)})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]n$	$j \frac{d}{d\omega} X(e^{j\omega})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]n$	$j \frac{d}{d\omega} X(e^{j\omega})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]$ real	$X(e^{j\omega}) = X^*(e^{-j\omega})$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]$ real and even	$X(e^{j\omega})$ real and even	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$x[n] = \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n])$	$X(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X(e^{-j\omega})) + \frac{1}{2}(X(e^{j\omega}) - X(e^{-j\omega}))$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$
$\sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega k} d\omega$

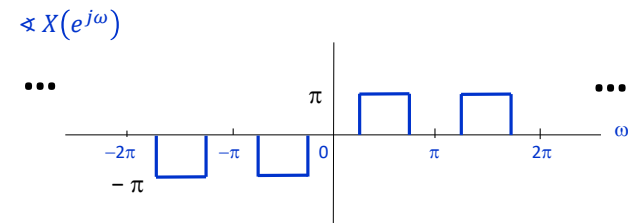
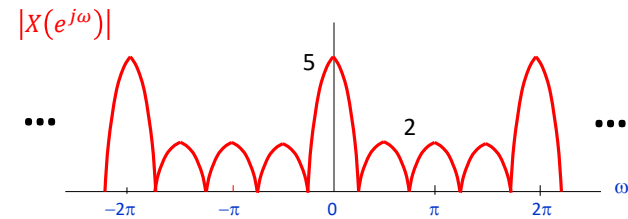
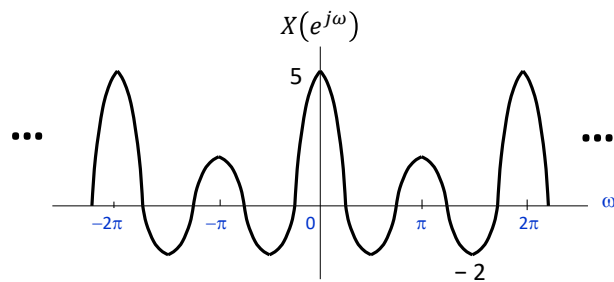
Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - 2\pi k)$	a_k
$e^{j\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$	$a_k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$
$\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$
$\sin \omega_0 t$	$j\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_k = \begin{cases} \frac{j}{2} & k = 1 \\ -\frac{j}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases}$
$\delta(t)$	1	$a_k = 1$
$x(t)$	$X(j\omega)$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)t$	$j \frac{d}{d\omega} X(j\omega)$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)t$	$j \frac{d}{d\omega} X(j\omega)$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)$ real	$X(j\omega) = X^*(-j\omega)$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)$ real and even	$X(j\omega)$ real and even	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$x(t) = \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t))$	$X(j\omega) = \frac{1}{2}(X(j\omega) + X(-j\omega)) + \frac{1}{2}(X(j\omega) - X(-j\omega))$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$
$\int_{-\infty}^{\infty} x(t)y^*(t)dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$	$a_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega k} d\omega$

18

DT Fourier Transform for Aperiodic Signal

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$$

e.g. Magnitude spectrum? Phase spectrum? Is $x[n]$ real?

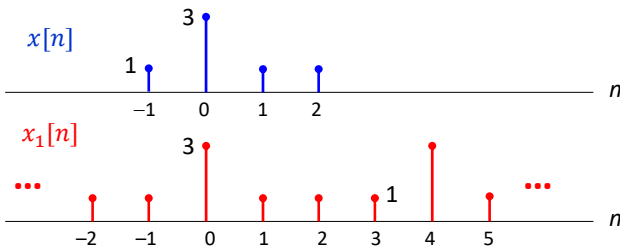


19

20

DT Fourier Transform for Periodic Signal

e.g.



- Represent $x_1[n]$ in terms of $x[n]$.
- Find the Fourier transform of $x[n]$.
- Find the Fourier transform of $x_1[n]$.
- Find the FS of $x_1[n]$.

$$x_1[n] = x[n] * \sum_{k=-\infty}^{\infty} \delta[n - 4k]$$

$$= \sum_{k=-\infty}^{\infty} x[n - 4k] \quad N = 4$$

21

FT, FS is just 2π!

Apply LTI!

$$x[n] = \delta[n + 1] + 3\delta[n] + \delta[n - 1] + \delta[n - 2]$$

$$X(e^{j\omega}) = e^{j\omega} + 3 + e^{-j\omega} + e^{-j2\omega} = 3 + 2\cos(\omega) + e^{-j2\omega}$$

$$X_1(e^{j\omega}) = X(e^{j\omega}) \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right)$$

$$= \sum_{k=-\infty}^{\infty} \frac{\pi}{2} \left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k} \right) \delta\left(\omega - \frac{\pi k}{2}\right)$$

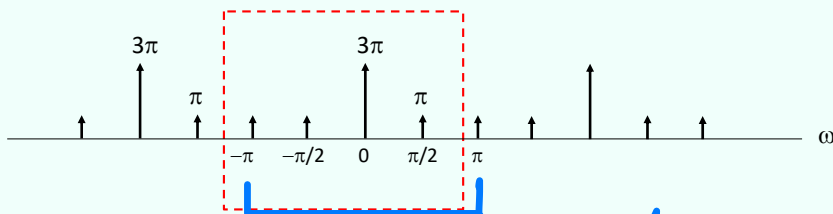
$\omega \rightarrow \omega \cdot \frac{\pi k}{2}$

$$a_k = \frac{1}{2\pi} X_1(e^{j\omega}) = \frac{1}{4} \left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k} \right)$$

22

$$X_1(e^{j\omega}) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k} \right) \delta\left(\omega - \frac{\pi k}{2}\right)$$

for $-\pi \leq \omega < \pi$



低通!!!

Question : Why is the frequency component at π excluded ?

23

A DT LTI system has frequency response as shown below.

$$H(e^{j\omega}) = \begin{cases} 4e^{-j\omega} & |\omega| < \frac{2\pi}{3} \\ 0 & \frac{2\pi}{3} < |\omega| < \pi \end{cases} \quad \text{for } |\omega| \leq \pi \quad \underline{H(e^{j\omega}) \text{ is } 2\pi\text{-periodic}}$$

e) What is the type of this filter ? **low pass!**

If $x_1[n]$ is applied to the above system,

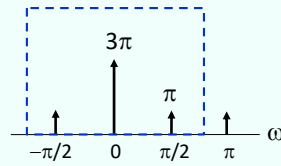
f) Find the Fourier transform of the output.

g) Write down the expression of the output as the sum of real cosine signal(s).

24

$$H(e^{j\omega}) = \begin{cases} 4e^{-j\omega} & |\omega| < \frac{2\pi}{3} \\ 0 & \frac{2\pi}{3} < |\omega| < \pi \end{cases} \quad \text{for } |\omega| \leq \pi$$

$$X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{\pi}{2} \left(3 + 2 \cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k} \right) \delta\left(\omega - \frac{\pi k}{2}\right)$$



$$Y_1(e^{j\omega}) = X_1(e^{j\omega}) H(e^{j\omega})$$

$$= 12\pi\delta(\omega) + 4e^{-j\frac{\pi}{2}}\pi\delta\left(\omega - \frac{\pi}{2}\right) + 4e^{j\frac{\pi}{2}}\pi\delta\left(\omega + \frac{\pi}{2}\right) \quad \text{for } |\omega| \leq \pi$$

$$y_1[n] = 6 + 4\cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) = 6 + 4\sin\left(\frac{\pi}{2}n\right)$$

25

e.g. A causal and stable LTI system S has the property that

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \longrightarrow \boxed{\text{System S}} \longrightarrow y[n] = n\left(\frac{4}{5}\right)^n u[n] = x[n] * h[n]$$

Determine the frequency response for the system S.

Differentiation in frequency

$$X(e^{j\omega}) = \frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}$$

$$Y(e^{j\omega}) = j \frac{d}{d\omega} \left[\frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}} \right]$$

Convolution

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5}e^{-j\omega}}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}$$

$$= \frac{\frac{4}{5}e^{-j\omega}}{\left[1 - \left(\frac{4}{5}\right)e^{-j\omega}\right]^2}$$

Question : What is the impulse response ? Is it a causal system ?

26

$$H(e^{j\omega}) = \frac{\frac{4}{5}e^{-j\omega}}{1 - \left(\frac{4}{5}\right)e^{-j\omega}} = \frac{4}{5}e^{-j\omega} \frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}$$

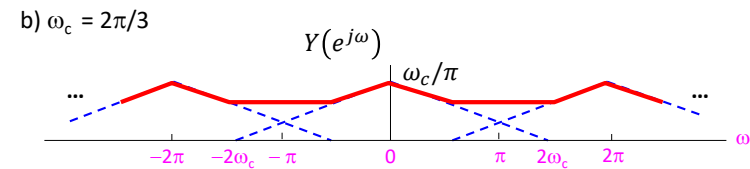
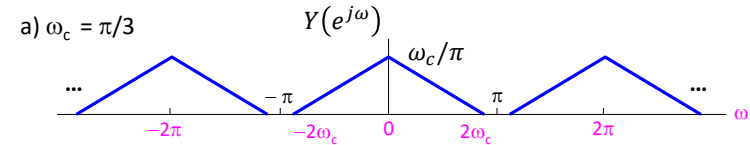
$$\left(\frac{4}{5}\right)^n u[n] \longleftrightarrow \frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}$$

$$h[n] = \frac{4}{5} \left(\frac{4}{5}\right)^{n-1} u[n-1]$$

27

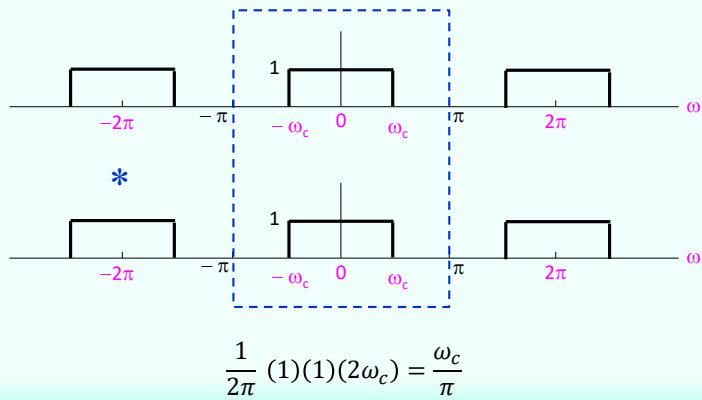
e.g. Given : $y[n] = \left(\frac{\sin(\omega_c n)}{\pi n}\right)^2$ where $0 < \omega_c < \pi$. Multiplication

Sketch the spectrum for the following cases.



28

$$y[n] = \left(\frac{\sin(\omega_c n)}{\pi n} \right)^2 = \frac{\sin(\omega_c n)}{\pi n} \times \frac{\sin(\omega_c n)}{\pi n}$$



29

e.g. The following four facts are given about a real signal $x[n]$ with Fourier transform. Determine $x[n]$.

- (1) $x[n] = 0$ for $n > 0$
- (2) $x[0] > 0$
- (3) $\text{Im}\{X(e^{j\omega})\} = \sin(\omega) - \sin(2\omega)$
- (4) $\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 3$

According to (3) :

Real
&
odd

$$j \text{Im}\{X(e^{j\omega})\} = j \sin(\omega) - j \sin(2\omega)$$

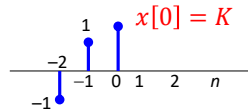
$$\frac{x[n] - x[-n]}{2} = F^{-1}\{j \sin(\omega) - j \sin(2\omega)\}$$

$$\frac{x[n] - x[-n]}{2} = F^{-1}\left\{\frac{1}{2}(e^{j\omega} - e^{-j\omega}) - \frac{1}{2}(e^{j2\omega} - e^{-j2\omega})\right\}$$

$$x[n] - x[-n] = \delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2]$$

30

According to (1), (2) and (3) :

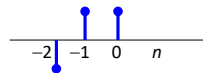


According to (4) and (2) :

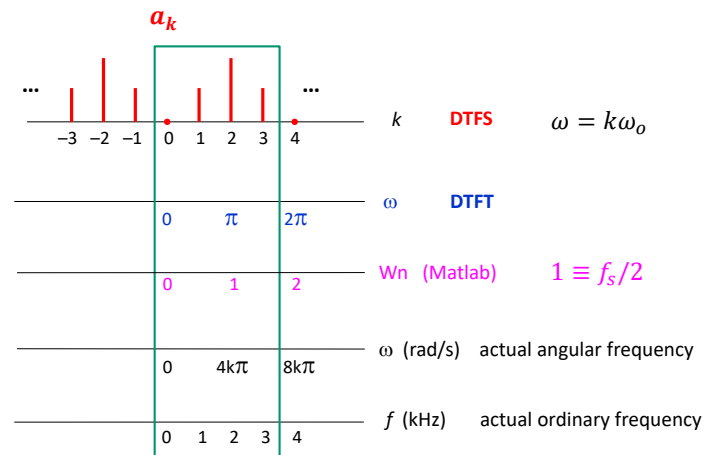
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 3 \quad \text{Parseval's Relation}$$

$$(-1)^2 + (1)^2 + (K)^2 = 3$$

$$K = 1$$



31



e.g. $f_s = 4$ kHz

$f_o = 1$ kHz

$N = 4$

$$1 \equiv f_s/2$$

32