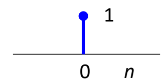
T02

DT unit impulse and unit step
CT unit impulse
Sampling property and Sifting property
CT complex exponential (Damped oscillation) and complex sinusoid
DT complex exponential (Damped oscillation) and complex sinusoid

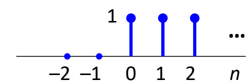
Input-Output relationship
- Memoryless
- Invertibility
- Causality
- Stability
- Time-invariant
- Linearity

DT Unit Impulse and Unit Step

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



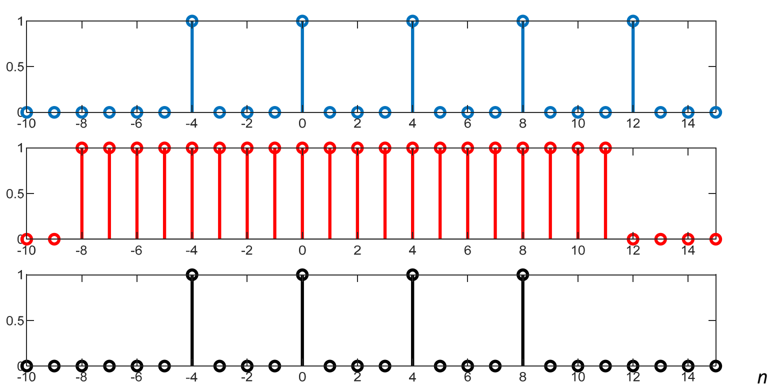
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m]$$

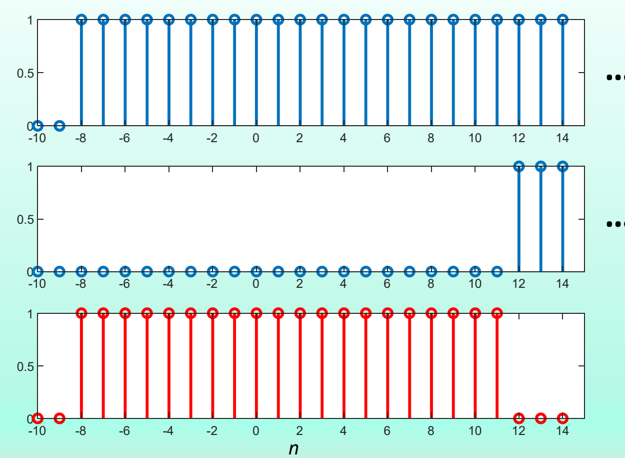
e.g. Sketch $x[n] = \left(\sum_{k=-1}^{3} \delta[n-4k]\right) (u[n+8] - u[n-12])$



Question : Represent $x[n]$ as a sum of impulses ?

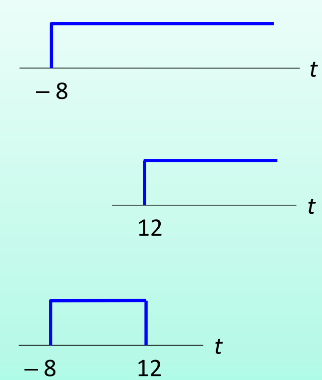
DT

$u[n+8] - u[n-12]$



CT

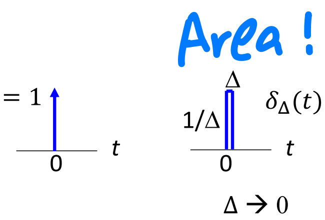
$u(t+8) - u(t-12)$



CT Unit Impulse

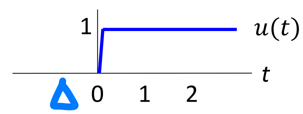
$\delta(t) = 0 \quad for \ \ t \neq 0$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

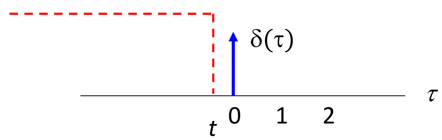


Relationship between $u(t)$ and $\delta(t)$

$$\delta(t) = \frac{d}{dt}\, u(t)$$

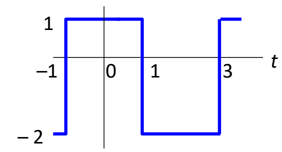


$$u(t) = \boxed{\int_{-\infty}^{t}} \delta(\tau)\, d\tau$$

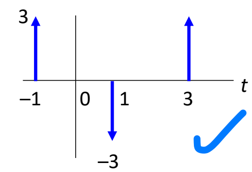


e.g.

$x(t)$

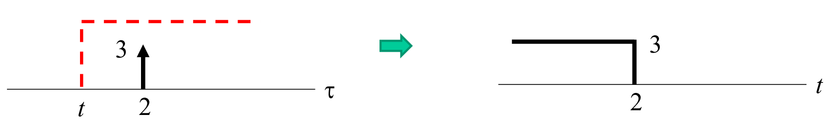


$\frac{d\,x(t)}{dt}$



e.g. $x(t) = \int_t^{\infty} 3\delta(\tau - 2)\, d\tau$

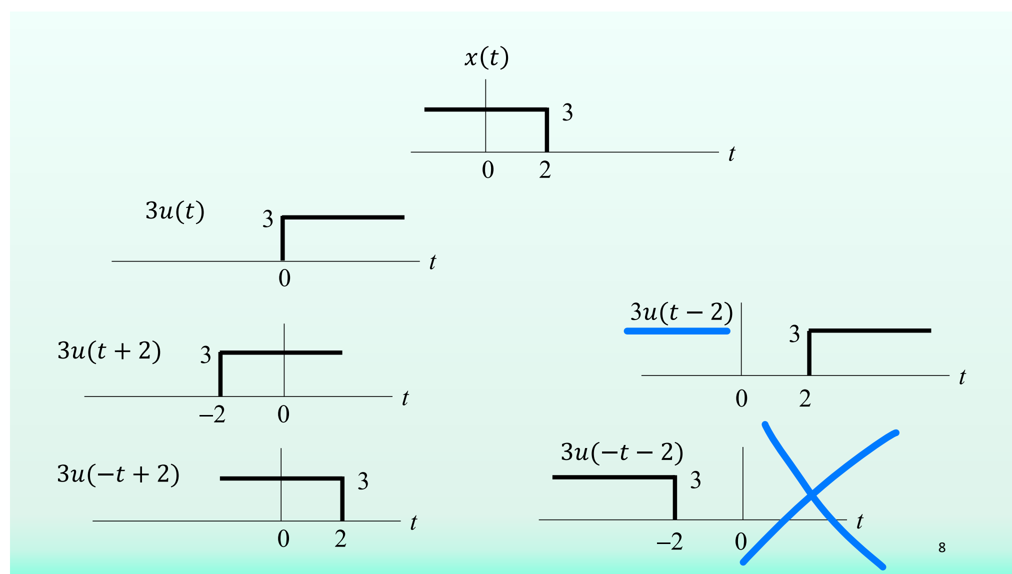
a) Plot $x(t)$



b) Which of the following expressions is correct ?

$x(t) = 3u(-t+2)$ ✓

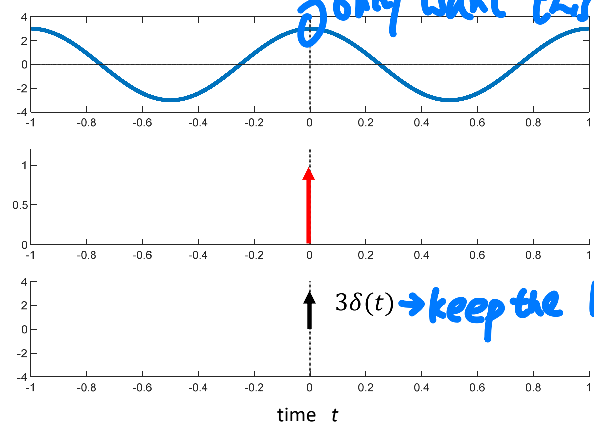
$x(t) = 3u(-t-2)$ ✗



Sampling Property

e.g. Plot $x(t) = 3\cos(2\pi t)\,\delta(t)$

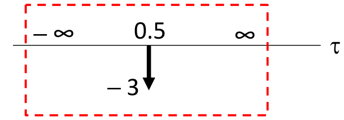
only want this value

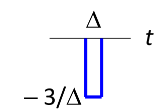


Sifting Property

e.g. Find $\int_{-\infty}^{\infty} 3\cos(2\pi\tau)\,\delta(\tau-0.5)\,d\tau$

$$\int_{-\infty}^{\infty} 3\cos(2\pi\tau)\,\delta(\tau-0.5)\,d\tau = \int_{-\infty}^{\infty} -3\,\delta(\tau-0.5)\,d\tau = -3$$





CT Complex Exponential

e^{st} $\quad s = \sigma + j\omega$

$$e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t)$$

Exponential form

$|\, e^{(\sigma+j\omega)t}\,| = e^{\sigma t}$

$\measuredangle\, e^{(\sigma+j\omega)t} = \omega t$

Rectangular form

$\mathrm{Re}\left\{e^{(\sigma+j\omega)t}\right\} = e^{\sigma t}\cos(\omega t)$

$\mathrm{Im}\left\{e^{(\sigma+j\omega)t}\right\} = e^{\sigma t}\sin(\omega t)$

e.g. $\quad e^{st} \quad s = \sigma + j\omega$

$x_1(t) = e^{(-0.2+j)t}$

$x_2(t) = e^{(0.2-j)t + j\frac{\pi}{2}}$ st, initial phrase shift!

Question : What is the complex frequency s ?

$e^{-0.2t} e^{jt}$ $\qquad e^{0.2t} e^{j\left(-t+\frac{\pi}{2}\right)}$

e.g. Sketch the following signals.

$x_3(t) = Re\{x_1(t)\}\,u(t)$

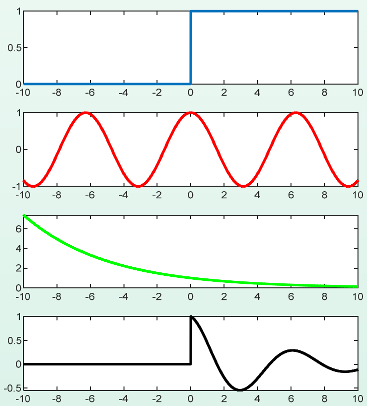
$\quad = e^{-0.2t}\cos(t)\,u(t)$

$x_4(t) = Im\{x_2(t)\}\,u(-t)$

$\quad = e^{0.2t}\sin\left(-t+\frac{\pi}{2}\right)u(-t)$

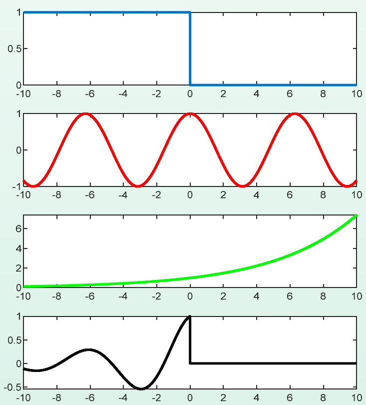
Question :
- Decaying or growing damped oscillation?
- Left-sided or right-sided ?

$x_3(t) = e^{-0.2t} \cos(t)\, u(t)$



Time (sec)

$x_4(t) = e^{0.2t} \sin\left(-t + \frac{\pi}{2}\right) u(-t)$



Time (sec)

CT Complex Sinusoid

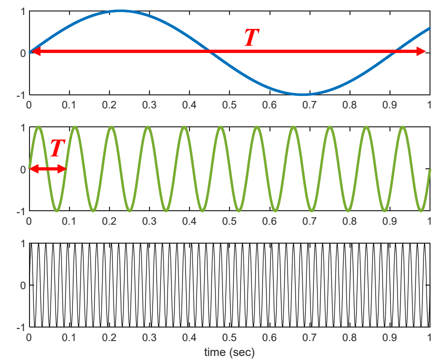
$e^{j\omega t}$

$0 \sim \infty$

$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

- Different ω → Different waveform

The **imaginary part** of $e^{j\omega_o t}$ is plotted below



$\omega = \frac{2\pi}{T}$

$\omega = \frac{22\pi}{10} = \frac{2\pi}{10/11}$

$\omega = 22\pi = \frac{2\pi}{1/11}$

$\omega = 110\pi = \frac{2\pi}{1/55}$

Question : How many distinct CT complex sinusoids ?

Difference between complex exponential and complex sinusoid ?

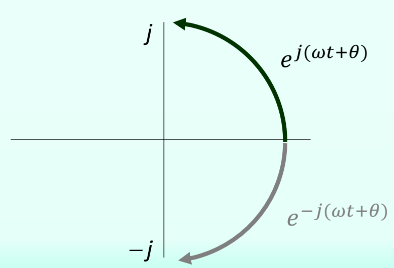
$$e^{\sigma t} e^{j\omega t} = e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t)$$

$$A e^{j\omega t} = A\cos(\omega t) + j\, A\sin(\omega t)$$

$$A e^{j(\omega t+\theta)} = A\cos(\omega t + \theta) + j\, A\sin(\omega t + \theta)$$

$$A\sin(\omega t + \theta) = \frac{A}{2} e^{j(\omega t+\theta)} + \frac{A}{2} e^{-j(\omega t+\theta)}$$

$$A\sin(\omega t + \theta) = \frac{A}{2j} e^{j(\omega t+\theta)} - \frac{A}{2j} e^{-j(\omega t+\theta)}$$



$S = \sigma + j\omega$

DT Complex Exponential

z^n $\quad z = e^s = e^{\sigma + j\omega} = e^{\sigma} e^{j\omega} = |z| e^{j\angle z}$

e.g. $x_1[n] = 0.8^n\, e^{j\frac{2\pi}{5}n}$ $\quad x_2[n] = 2^n\, e^{j\frac{2\pi}{5}n}$ $\quad a^n b^n = (ab)^n$

$= \left(0.8\, e^{j\frac{2\pi}{5}}\right)^n$ $\quad = \left(2\, e^{j\frac{2\pi}{5}}\right)^n$

complex frequency

Question : What is the complex frequency z ?

e.g. Sketch the following two signals.

$x_3[n] = Re\{x_1[n]\}\, u[n]$

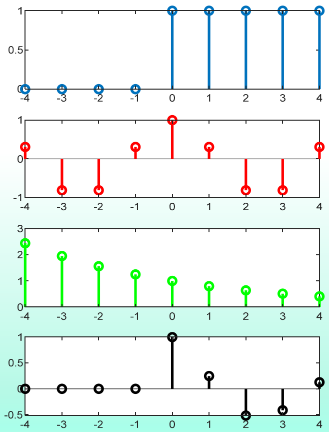
$= 0.8^n \cos\left(\frac{2\pi}{5} n\right) u[n]$

$x_4[n] = Im\{x_2[n]\}\, u[-n]$

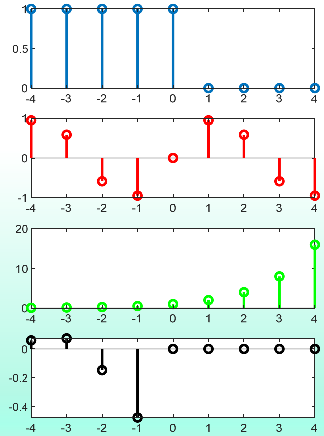
$= 2^n \sin\left(\frac{2\pi}{5} n\right) u[-n]$

Question :
- Decaying or growing damped oscillation?
- Left-sided or right-sided ?

$$x_1[n] = 0.8^n \cos\left(\frac{2\pi}{5} n\right) u[n]$$



$$x_2[n] = 2^n \sin\left(\frac{2\pi}{5} n\right) u[-n]$$



DT Complex Sinusoid $\quad e^{j\omega n}$

$$e^{j\omega n} = \cos(\omega n) + j \sin(\omega n)$$

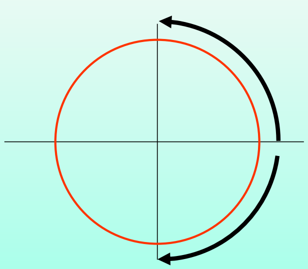
$$\omega = \frac{2\pi\, m}{N}$$ where m and N are integers

Question : Difference between CT complex sinusoid and DT complex sinusoid ?

$e^{j\frac{22\pi}{10}t} =$? $\qquad e^{j\frac{22\pi}{10}n} =$?

$$e^{j\frac{22\pi}{10}t} = e^{j\frac{20\pi}{10}t} e^{j\frac{2\pi}{10}t} = e^{j2\pi t} e^{j\frac{2\pi}{10}t} = e^{j\frac{22\pi}{10}t}$$

$$e^{j\frac{22\pi}{10}n} = e^{j\frac{20\pi}{10}n} e^{j\frac{2\pi}{10}n} = e^{j2\pi n} e^{j\frac{2\pi}{10}n} = e^{j\frac{2\pi}{10}n}$$

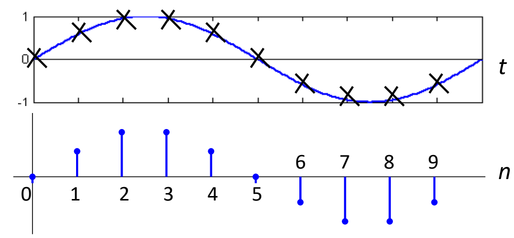


$$e^{j2\pi(0.9)n} = e^{j2\pi(1.9)n} = e^{j2\pi(-0.1)n}$$

$$e^{j2\pi(1.9)n} = e^{j2\pi(1+0.9)n} = e^{j2\pi n} e^{j2\pi(0.9)n}$$

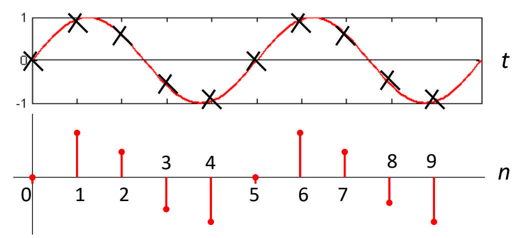
$$e^{j2\pi(-0.1)n} = e^{j2\pi(0.9-1)n} = e^{-j2\pi n} e^{j2\pi(0.9)n}$$

The **imaginary part** of $e^{j\omega n}$ is plotted below. $\qquad$ CT → DT

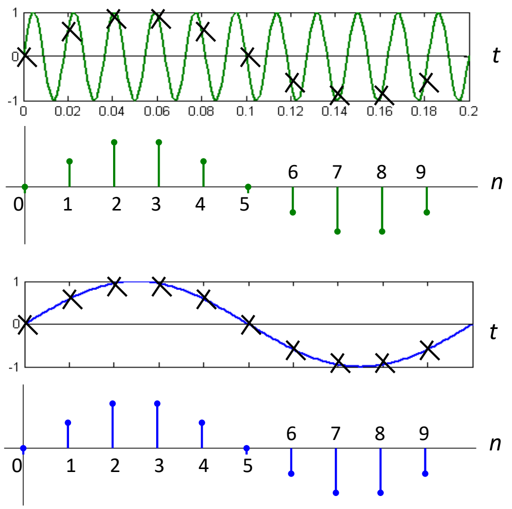


$$\omega = \frac{2\pi\,(1)}{10}$$

Meaning of m ?



$$\omega = \frac{2\pi\,(2)}{10}$$



$$\omega = \frac{2\pi\,(11)}{10}$$

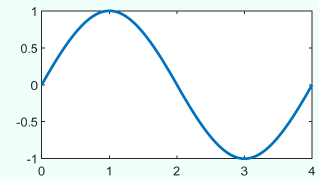
Same sequence !

$$e^{j\frac{2\pi}{10}n} = e^{j\frac{2\pi(11)}{10}n}$$

$$\omega = \frac{2\pi}{10}$$

Question : How many distinct DT complex sinusoids if N = 10 ?

Question : The highest **fundamental angular frequency** for CT complex sinusoid ?



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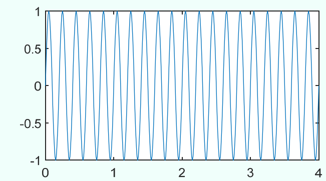

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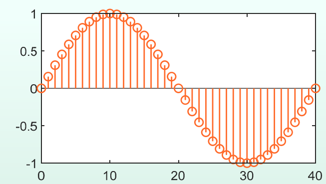
Question : The highest **fundamental angular frequency** for DT complex sinusoid ?



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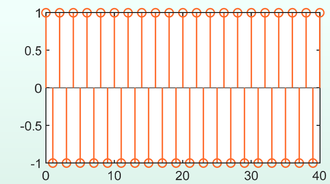
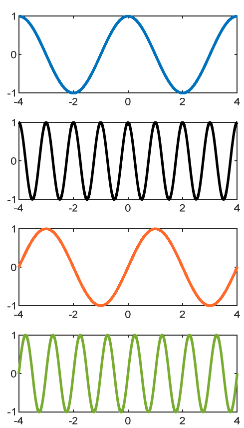
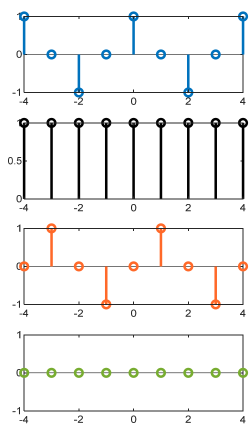


$\frac{2\pi}{N}$ ↗

$N = 2$ (minimum)

e.g. Sketch the following CT and DT signals.





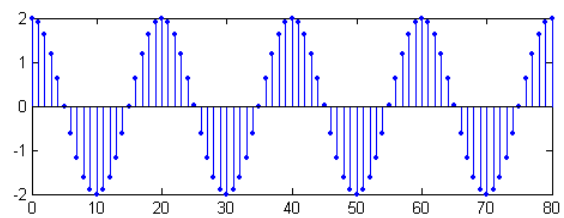
$\cos(\pi t/2)$ | $\cos(\pi n/2)$

$\cos(2\pi t)$ | $\cos(2\pi n)$

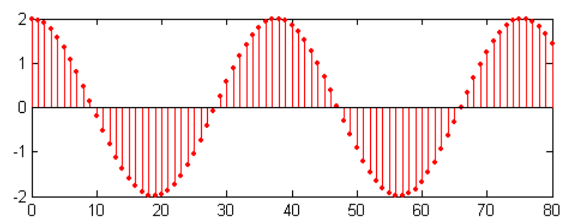
$\sin(\pi t/2)$ | $\sin(\pi n/2)$

$\sin(2\pi t)$ | $\sin(2\pi n)$

e.g. Periodic ? Fundamental period N ?



$$x_1[n] = 2\cos\left(\frac{2\pi}{20}n\right)$$



$$x_2[n] = 2\cos\left(\frac{n}{6}\right)$$

$N = 12\pi$

$\neq$ integer

e.g. Match the following signals with Figure 1, Figure 2 and Figure 3.

$x_1[n] = Re\left\{\left(0.8\, e^{j\frac{2\pi}{5}}\right)^n\right\}$ 2

$x_2[n] = Re\left\{e^{j\frac{2\pi}{5}n}\right\}$ 3

$x_3[n] = Re\left\{e^{j\frac{2}{5}n}\right\}$ 1 non-periodic!

Figure 1

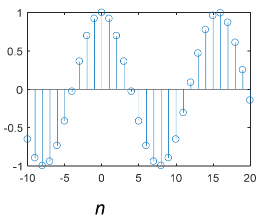


Figure 2

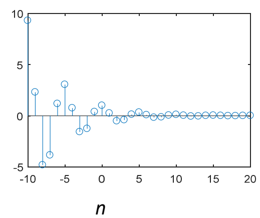
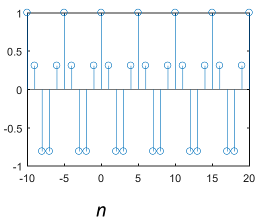
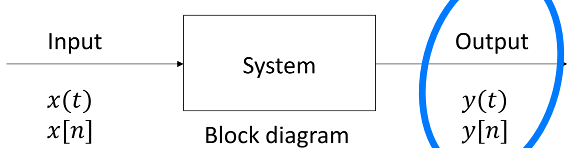


Figure 3

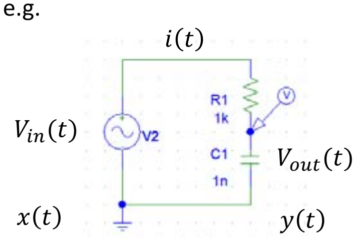


Input-Output relationship
- Memoryless
- Invertibility
- Causality
- Stability
- Time-invariant
- Linearity

Input and Output Relationship



e.g.



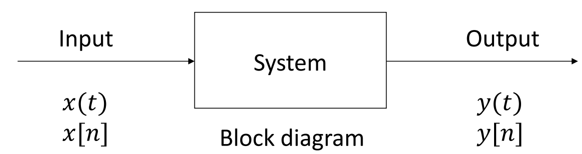
$$i(t) = C\frac{d}{dt}V_{out}(t)$$

$$V_{in}(t) = V_R(t) + V_{out}(t) = i(t)R + V_{out}(t)$$

$$= RC\frac{d}{dt}V_{out}(t) + V_{out}(t)$$

$$x(t) = RC\frac{d}{dt}y(t) + y(t)$$

System



System Properties ⟸ **Input-Output Relationship**

Memoryless The output **only** depends on the **current** input

Causality The output **does not** depend on the **future** input

Invertibility Distinct inputs leads to distinct outputs (i.e. **One-to-one** mapping)

Stability Bounded input results in bounded output (**BIBO**)

$|x(t)| < \infty \ \forall \ t$ → System → $|y(t)| < \infty \ \forall \ t$

Time-invariant The behavior of a system **does not** change over time

$x(t)$ → System → $y(t)$

$x_1(t) = x(t - t_o)$ → $y_1(t) = y(t - t_o)$

Linearity Superposition and Scaling (**zero-input zero-output**)

$x_1(t)$, $x_2(t)$, $x_3(t) = ax_1(t) + bx_2(t)$ → System → $y_1(t)$, $y_2(t)$, $y_3(t) = ay_1(t) + by_2(t)$

Plug a positive and negative

e.g. $y(t) = \cos(x(t)) + 1$

Memoryless ?

Causal ?

Invertible ?

Stable ?

Time-invariant ?

Linear ?

$y(t) = Ev\{x(t)\}$

Memoryless ?

Causal ?

Invertible ?

Stable ?

Time-invariant ?

Linear ?

e.g. $y(t) = \cos(x(t)) + 1$

Memoryless ?

Causal ?

Stable ?

Time-invariant ?

$y(1) = \cos(x(1)) + 1$

$y(-1) = \cos(x(-1)) + 1$

What does this system do ?

Linear ?

$x_3(t) = ax_1(t) + bx_2(t)$

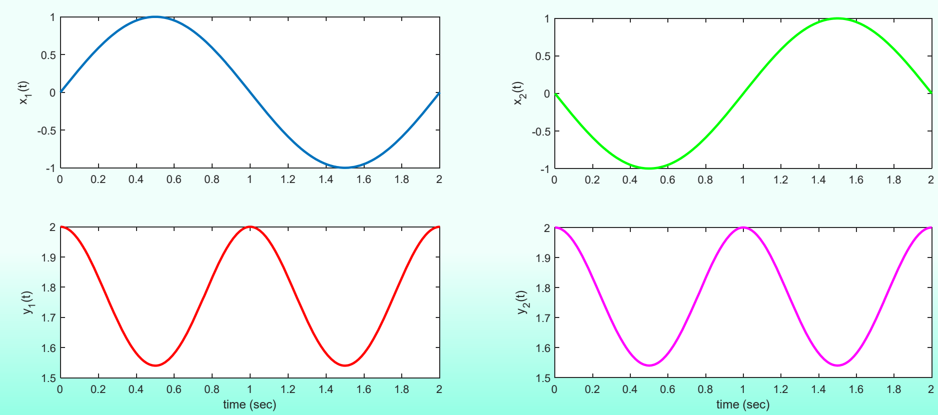
$y_3(t) = \cos(x_3(t)) + 1 = \cos(ax_1(t) + bx_2(t)) + 1$

$\neq ay_1(t) + by_2(t)$

If it is true, $\boxed{y_3(t) = ay_1(t) + by_2(t)} = a\cos(x_1(t)) + 1 + b\cos(x_2(t)) + 1$

Invertible ? X $\cos(-\theta) = \cos(\theta)$

$y(t) = \cos(x(t)) + 1$



e.g. $y(t) = Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$

past input!

Memoryless ? ✗ $y(1) = \frac{1}{2}[x(1) + x(-1)]$ ✓

Causal ? ✗ $y(-1) = \frac{1}{2}[x(-1) + x(1)]$ ✗

future!

Invertible ? ✗

Stable ? ✓

What does this system do ?

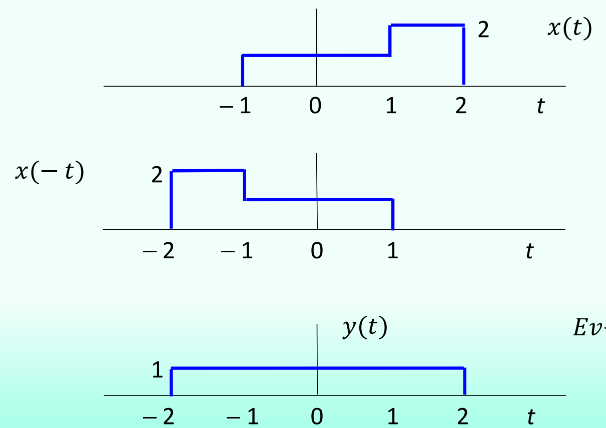
Linear ? ✓

$x_3(t) = ax_1(t) + bx_2(t)$

$$y_3(t) = \frac{1}{2}[ax_1(t) + bx_2(t) + ax_1(-t) + bx_2(-t)]$$
$$= ay_1(t) + by_2(t)$$

If it is true, $\boxed{y_3(t) = ay_1(t) + by_2(t)} = \frac{a}{2}[x_1(t) + x_1(-t)] + \frac{b}{2}[x_2(t) + x_2(-t)]$

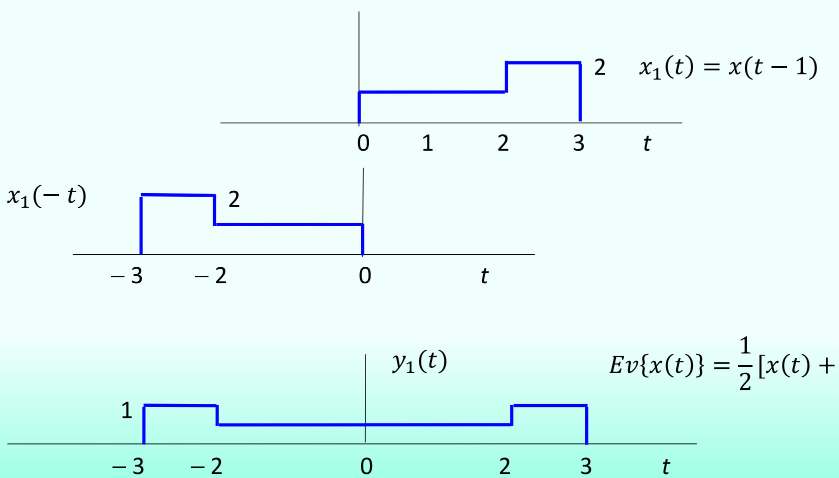
Time-invariant ?



$Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$

-1 a>1

1 a<1



$Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$

e.g. $y[n] = \max\{x[n], x[n-1]\}$

Memoryless ? ✗

Causal ? ✓

Invertible ?

Stable ? ✓

Time-invariant ? ✓

Linear ? ✗

$y[n] = \sum_{k=-\infty}^{n+1} x[k]$

Memoryless ? ✗

Causal ? ✗

Invertible ? ✗

Stable ? ✗

Time-invariant ? ✗

Linear ? ✓

e.g. $y[n] = \max\{x[n], x[n-1]\}$

Memoryless ? ✗

Causal ? ✓

$y[1] = \max\{x[1], x[0]\}$

$y[-1] = \max\{x[-1], x[-2]\}$

Invertible ? ✗

Stable ? ✓

Time-invariant ? ✓

What does this system do ?

$y[n] = \max\{x[n], x[n-1]\}$

Linear ? ✗

$x_1[n] = [3, 2, 1]$ $\qquad$ $y_1[n] = [3, 3, 2]$

$x_2[n] = [1, 8, 6]$ $\qquad$ $y_2[n] = [1, 8, 8]$

$x_3[n] = x_1[n] + x_2[n] = [4, 10, 7]$ $\qquad$ $y_3[n] = [4, 10, 10]$

$\neq y_1[n] + y_2[n]$

If it is true, $\boxed{y_3[n] = y_1[n] + y_2[n]} = [4, 11, 10]$

e.g. $y[n] = \sum_{k=0}^{n+1} x[k]$

Memoryless ? ✗

Causal ? ✗

$$y[1] = \sum_{k=0}^{2} x[k] \qquad y[-2] = \sum_{k=0}^{-1} x[k]$$

Stable ?

Linear ?

What does this system do ?

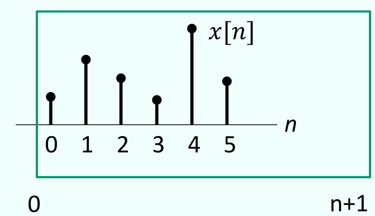
Time-invariant ?

Invertible ?

$y[-1] = x[0]$

$y[0] = x[0] + x[1]$

$y[1] = x[0] + x[1] + x[2]$



$$y[n] = \sum_{k=0}^{n+1} x[k]$$