T07

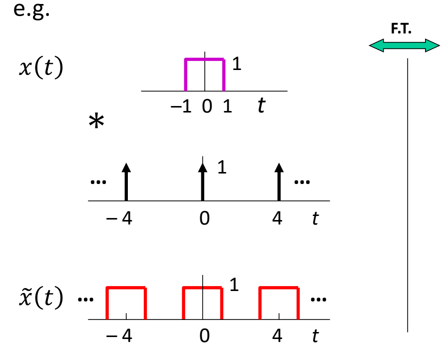
CT Fourier transform for periodic signal
Relation between CTFS and CTFT
Tables 4.1 and 4.2
Duality property

DT Fourier series to DT Fourier transform
DT Fourier transform for aperiodic signal
DT Fourier transform for periodic signal
Tables 5.1 and 5.2

CT Fourier Transform for Periodic Signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} x(t-kT)$$ Poisson sum

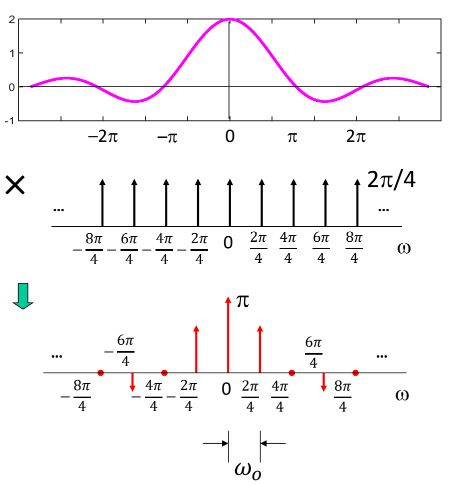
e.g.



$$X(j\omega) = \frac{2\sin(\omega)}{\omega}$$

$$\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{4}\right) \qquad T = 4$$

$$X(j\omega)\left[\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{4}\right)\right]$$



$$X(j\omega) = \frac{2\sin(\omega)}{\omega}$$

$$\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{4}\right)$$

$$\frac{2\pi}{4}\sum_{k=-\infty}^{\infty} X\left(\frac{j\pi k}{2}\right)\delta\left(\omega - \frac{\pi k}{2}\right)$$

Sampling property

Question : Difference on F.T. between aperiodic and periodic ?

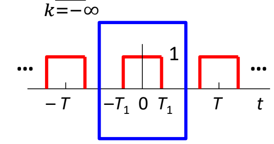
Relation of CTFS and CTFT

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_o t}$$

F.S.

$$a_k = \frac{1}{T}\int_0^T \tilde{x}(t)\, e^{-jk\omega_o t}\, dt \qquad \textbf{(1)}$$

Periodic



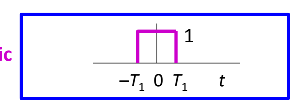
$$x(t) = \begin{cases} \tilde{x}(t) & -\dfrac{T}{2} \le t < T/2 \\ 0 & otherwise \end{cases}$$

F.T.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-j\omega t}\, dt \qquad \textbf{(2)}$$

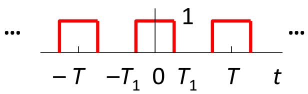
Aperiodic

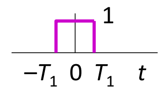


Compare **(1)** and **(2)**

$$a_k = \frac{1}{T} X(j\omega)|_{\omega = k\omega_o}$$

e.g. Periodic square wave and Aperiodic square pulse





$$a_k = \frac{2\sin(k\omega_o T_1)}{Tk\omega_o} = \frac{2\sin(k\omega_o T_1)}{Tk\left(\frac{2\pi}{T}\right)} = \frac{\sin(k\omega_o T_1)}{k\pi}$$

$$X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

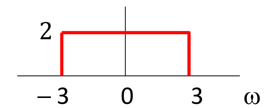
$$a_k \Leftrightarrow \frac{1}{T} X(j\omega)|_{\omega = k\omega_o} = \frac{2\sin(k\omega_o T_1)}{k\left(\frac{2\pi}{T}\right)}$$

Question : Difference between its FS and its FT for a periodic signal ?

Tables 4.1 and 4.2

e.g. Given : $|X(j\omega)| = 2\,u(\omega+3) - u(\omega-3)]$ Obtain $x(t)$

$\angle X(j\omega) = -2\omega + \pi$



$\xleftrightarrow{\text{F.T.}}$ $\frac{2\sin(3t)}{\pi t}$

Linearity & Time shifting

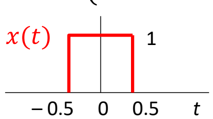
$$X(j\omega) = \begin{cases} 2\,e^{j(-2\omega+\pi)} & |\omega| < 3 \\ 0 & otherwise \end{cases} = (-1)\; e^{-j2\omega} \begin{cases} 2 & |\omega| < 3 \\ 0 & otherwise \end{cases}$$

$$x(t) = -\frac{2\sin(3(t-2))}{\pi(t-2)} = -\frac{2\sin(3t)}{\pi t} * \delta(t-2)$$

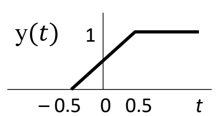
e.g. Given :

$$y(t) = \begin{cases} 0 & t < -0.5 \\ t + 0.5 & -0.5 \le t < 0.5 \\ 1 & 0.5 \le t \end{cases}$$

Obtain the F.T. of $y(t)$



Integration
Differentiation

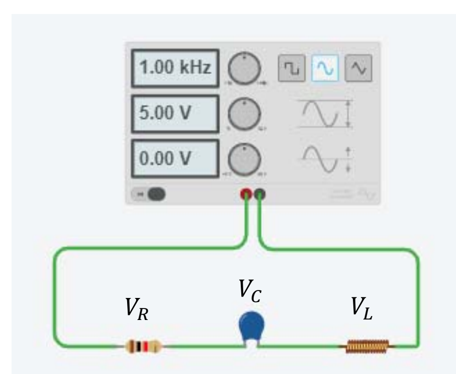


$$x(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & otherwise \end{cases} \xleftrightarrow{\text{F.T.}} X(j\omega) = \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} \qquad X(j0) = \,?$$

$$y(t) = \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{\text{F.T.}} Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

$$\frac{d}{dt} y(t) = x(t) \qquad X(0) = \,?$$

e.g.



R $\quad \frac{1}{j\omega C}$ $\quad j\omega L$

$v_R(t) = i(t)\,R$

$V_R(j\omega) = I(j\omega)\boxed{R}$ $\qquad R = \frac{V}{I}$

$v_C(t) = \frac{1}{C}\int i(t)dt$

$V_C(j\omega) = \boxed{\frac{1}{C}\frac{1}{j\omega}}\, I(j\omega)$

$v_L(t) = L\frac{d}{dt} i(t)$

$V_L(j\omega) = \boxed{Lj\omega}\, I(j\omega)$

e.g. Find $x(t)$ for the following LTI system.

$$x(t) \longrightarrow \boxed{\boldsymbol{H(j\omega)} = \frac{1}{3+j\omega}} \longrightarrow y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

Convolution

$$y(t) = x(t) * h(t) \quad \overset{\text{F.T.}}{\longleftrightarrow} \quad \boldsymbol{Y(j\omega)} = \boldsymbol{X(j\omega)}\boldsymbol{H(j\omega)}$$

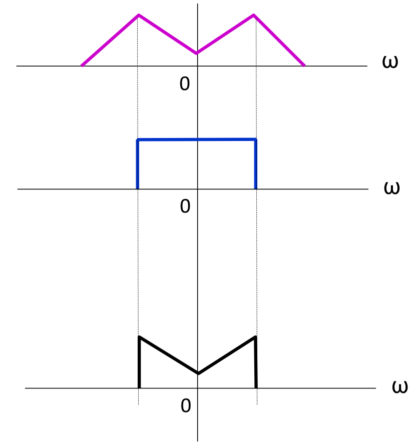
$$\boldsymbol{X(j\omega)} = \frac{\boldsymbol{Y(j\omega)}}{\boldsymbol{H(j\omega)}} \qquad \boldsymbol{Y(j\omega)} = \frac{1}{3+j\omega} - \frac{1}{4+j\omega}$$

$$= \left(\frac{1}{3+j\omega} - \frac{1}{4+j\omega}\right)(3+j\omega)$$

$$= \frac{1}{4+j\omega} \quad \xrightarrow{\text{Inverse F.T.}} \quad x(t) = e^{-4t}u(t)$$

Question : Is it a causal system ? $h(t) =$ **?**

e.g. Given :



Input spectrum

Frequency response

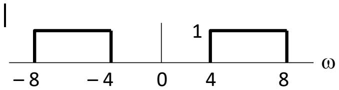
Sketch the output spectrum

$$\boldsymbol{Y(j\omega)} = \boldsymbol{X(j\omega)}\boldsymbol{H(j\omega)}$$

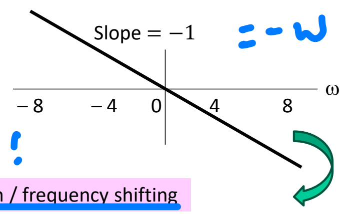
$\delta(t-1)$

e.g. The magnitude response and phase response of an LTI system are given below.

$|H(j\omega)|$



$\sphericalangle H(j\omega)$

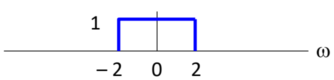


$= -\omega$

a) What is the type of this filter ? bandpass!

b) Find the impulse response $h(t)$.

Multiplication / frequency shifting



$\longleftrightarrow \quad \dfrac{\sin(2t)}{\pi t}$

What does it mean in time domain?

$*$

$\longleftrightarrow \quad \dfrac{1}{\pi}\cos(6t)$

$h(t) = ?$

$$(2\pi)\frac{\sin(2t)}{\pi t}\frac{1}{\pi}\cos(6t) = \frac{2\sin(2t)}{\pi t}\cos(6t) \qquad H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)}$$

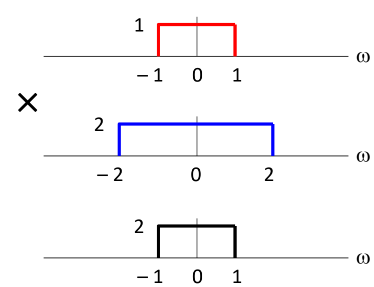
$\boxed{e^{-j\omega}} \rightarrow \delta(t-1)$

e.g. Given : $x(t) = \left(\dfrac{\sin(t)}{\pi t}\right) * \left(\dfrac{2\sin(2t)}{\pi t}\right)$

a) Sketch the F.T. of $x(t)$ Convolution

b) Find the energy of $x(t)$ Parseval's Relation

$$\boldsymbol{X(j\omega)} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$\text{Energy} =$ **?**

$\dfrac{2^2 \times 2}{2\pi}$

$= \dfrac{4}{\pi}$

Duality Property

$$x(t) \qquad\qquad X(j\omega)$$

$$y(t) = X(j\omega)|_{\omega=t} \qquad\qquad Y(j\omega) = 2\pi\, x(t)|_{t=-\omega}$$

Proof : $Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} X(jt) e^{-j\omega t} dt$ **(1)**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

(2)

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-jt) e^{j\omega t} dt \qquad x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jt) e^{-j\omega t} dt$$

(1) → (2) ➡ $Y(j\omega) = 2\pi\, x(-\omega) = 2\pi\, x(t)|_{t=-\omega}$

7,

$$x(t) \qquad\qquad X(j\omega)$$

$$y(t) = X(j\omega)|_{\omega=t} \qquad\qquad Y(j\omega) = 2\pi\, x(t)|_{t=-\omega}$$

e.g. Given : $te^{-|t|} \overset{F.T.}{\Longleftrightarrow} \dfrac{-j4\omega}{(1+\omega^2)^2}$

$$Y(j\omega) = \int_{-\infty}^{\infty} \frac{4t}{(1+t^2)^2} e^{-j\omega t}\, dt$$

Obtain the FT of $\dfrac{4t}{(1+t^2)^2}$

$$F\left\{\frac{4t}{(1+t^2)^2}\right\} = \frac{2\pi(-\omega)e^{-|-\omega|}}{-j}$$

$$= -j2\pi\omega e^{-|\omega|}$$

$$te^{-|t|} \qquad \frac{-j4\omega}{(1+\omega^2)^2}$$

$$\frac{-j4t}{(1+t^2)^2} \qquad 2\pi(-\omega)e^{-|-\omega|}$$

DT Fourier series to DT Fourier transform
DT Fourier transform for aperiodic signal
DT Fourier transform for periodic signal
Tables 5.1 and 5.2

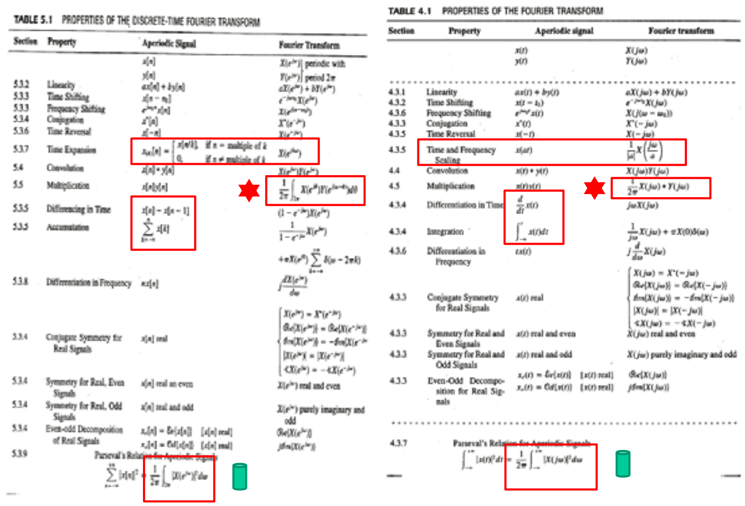
DT Fourier Series ➡ **DT Fourier Transform**

DT Fourier Series	DT Fourier Transform
$\omega_o = \frac{2\pi}{N}$ N = finite	$\omega_o = \frac{2\pi}{N}$ $N \to -\infty$ to ∞
$\omega = k\omega_o$ ω = finite	$k\omega_o \to \omega$
$x[n] = \sum_{k=<N>} a_k\, e^{jk\frac{2\pi}{N}n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$
$a_k = \frac{1}{N}\sum_{n=<N>} x[n]\, e^{-jk\frac{2\pi}{N}n}$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$
Periodic Signals	**Aperiodic** Signals

Question : What is the difference between CTFT and DTFT ?

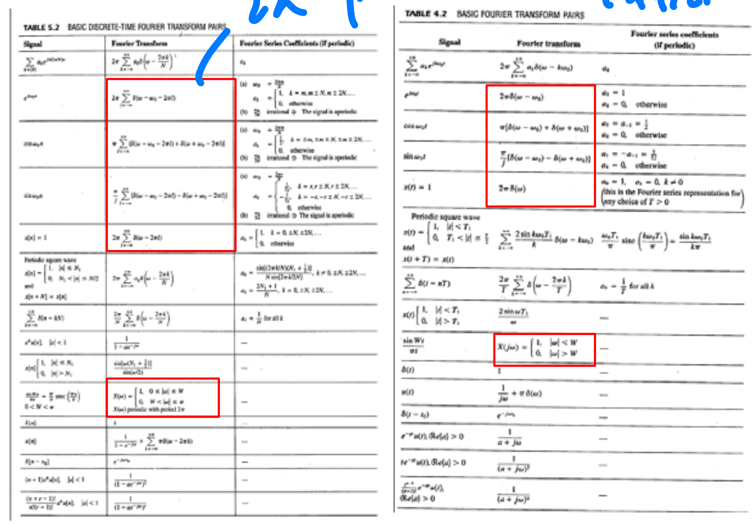
Table 5.1 and 5.2

5.1 4.1



5.2 4.2

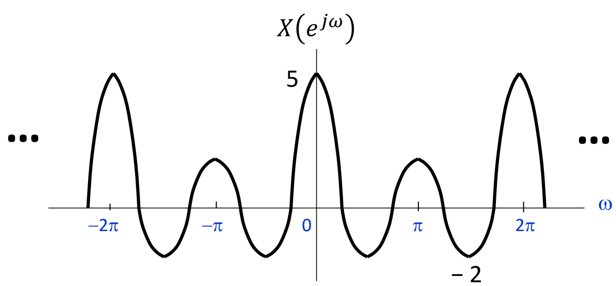
2π-periodic, like F.S. coefficients!

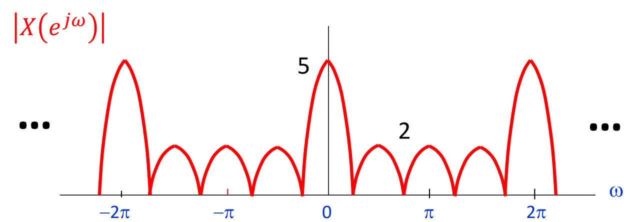


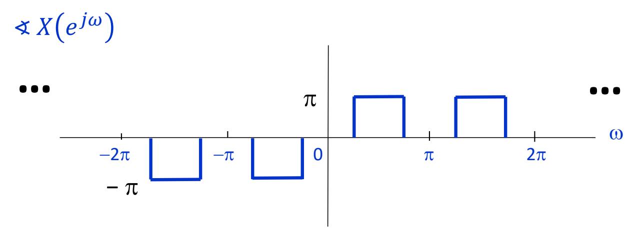
DT Fourier Transform for Aperiodic Signal

$$X(e^{j\omega}) = |X(e^{j\omega})|\, e^{\,j \sphericalangle X(e^{j\omega})}$$

e.g. Magnitude spectrum ? Phase spectrum ? Is $x[n]$ real ?

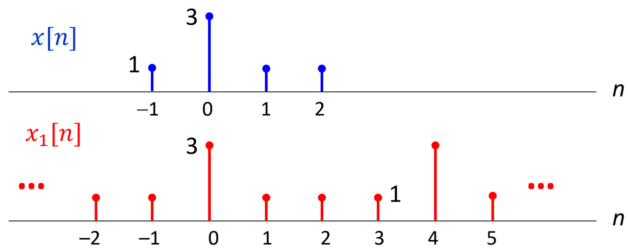






DT Fourier Transform for Periodic Signal

e.g.



a) Represent $x_1[n]$ in terms of $x[n]$.

b) Find the Fourier transform of $x[n]$.

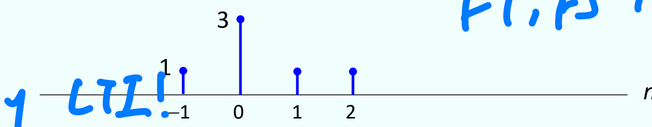
c) Find the Fourier transform of $x_1[n]$.

d) Find the FS of $x_1[n]$.

$$x_1[n] = x[n] * \sum_{k=-\infty}^{\infty} \delta[n-4k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-4k] \qquad N = 4$$

FT, FS is just 2π!



Apply LTI!

$$x[n] = \delta[n+1] + 3\delta[n] + \delta[n-1] + \delta[n-2]$$

$$X(e^{j\omega}) = e^{j\omega} + 3 + e^{-j\omega} + e^{-j2\omega} = 3 + 2\cos(\omega) + e^{-j2\omega}$$

$$X_1(e^{j\omega}) = X(e^{j\omega})\, \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right)$$

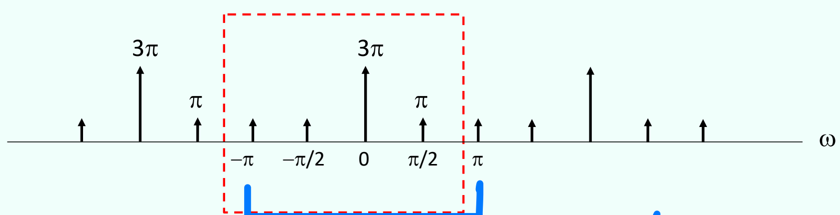
$\omega \to \omega - \frac{\pi k}{2}$

$$= \sum_{k=-\infty}^{\infty} \frac{\pi}{2}\left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k}\right) \delta\left(\omega - \frac{\pi k}{2}\right)$$

$$\boldsymbol{a_k = \frac{1}{2\pi} X_1(e^{j\omega}) = \frac{1}{4}\left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k}\right)}$$

$$X_1(e^{j\omega}) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k}\right) \delta\left(\omega - \frac{\pi k}{2}\right)$$

for $-\pi \le \omega < \pi$



冇 −π 到 π 全包!!!

Question : Why is the frequency component at π excluded ?

A DT LTI system has frequency response as shown below.

$$H(e^{j\omega}) = \begin{cases} 4\, e^{-j\omega} & |\omega| < \frac{2\pi}{3} \\ 0 & \frac{2\pi}{3} < |\omega| < \pi \end{cases} \quad \text{for } |\omega| \le \pi \qquad H(e^{j\omega}) \text{ is } 2\pi\text{-periodic}$$

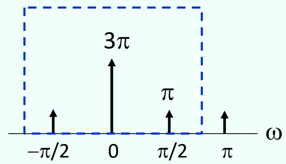
e) What is the type of this filter ? low pass!

If $x_1[n]$ is applied to the above system,

f) Find the Fourier transform of the output.

g) Write down the expression of the output as the sum of real cosine signal(s).

$$H(e^{j\omega}) = \begin{cases} 4\,e^{-j\omega} & |\omega| < \frac{2\pi}{3} \\ 0 & \frac{2\pi}{3} < |\omega| < \pi \end{cases} \quad \text{for } |\omega| \le \pi$$



$$X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{\pi}{2}\left(3 + 2\cos\left(\frac{\pi k}{2}\right) + e^{-j\pi k}\right)\delta\left(\omega - \frac{\pi k}{2}\right)$$

$$Y_1(e^{j\omega}) = X_1(e^{j\omega})\,H(e^{j\omega})$$

$$= 12\pi\delta(\omega) + 4\,e^{-j\frac{\pi}{2}}\,\pi\,\delta\left(\omega - \frac{\pi}{2}\right) + 4\,e^{j\frac{\pi}{2}}\,\pi\,\delta\left(\omega + \frac{\pi}{2}\right) \quad \text{for } |\omega| \le \pi$$

$$y_1[n] = 6 + 4\cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) = 6 + 4\sin\left(\frac{\pi}{2}n\right)$$

e.g. A causal and stable LTI system S has the property that

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \longrightarrow \boxed{\begin{matrix}\text{System S} \\ h[n]\end{matrix}} \longrightarrow y[n] = n\left(\frac{4}{5}\right)^n u[n] \;= x[n] * h[n]$$

Determine the frequency response for the system S.

Differentiation in frequency

$$X(e^{j\omega}) = \frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}} \qquad Y(e^{j\omega}) = j\frac{d}{d\omega}\left[\frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}\right]$$

Convolution

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5}e^{-j\omega}}{1 - \left(\frac{4}{5}\right)e^{-j\omega}} \qquad = \frac{\frac{4}{5}e^{-j\omega}}{\left[1 - \left(\frac{4}{5}\right)e^{-j\omega}\right]^2}$$

Question : What is the impulse response ? Is it a causal system ?

$$H(e^{j\omega}) = \frac{\frac{4}{5}e^{-j\omega}}{1 - \left(\frac{4}{5}\right)e^{-j\omega}} = \frac{4}{5}e^{-j\omega}\frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}$$

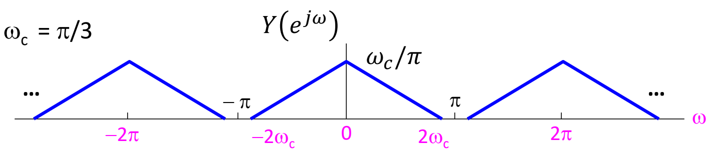
$$\boxed{\left(\frac{4}{5}\right)^n u[n] \;\Longleftrightarrow\; \frac{1}{1 - \left(\frac{4}{5}\right)e^{-j\omega}}}$$

$$h[n] = \frac{4}{5}\left(\frac{4}{5}\right)^{n-1} u[n-1]$$

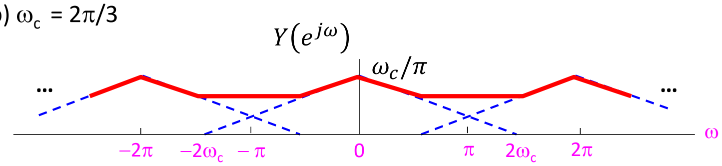
e.g. Given : $y[n] = \left(\frac{\sin(\omega_c n)}{\pi n}\right)^2$ where $0 < \omega_c < \pi$. Multiplication

Sketch the spectrum for the following cases.

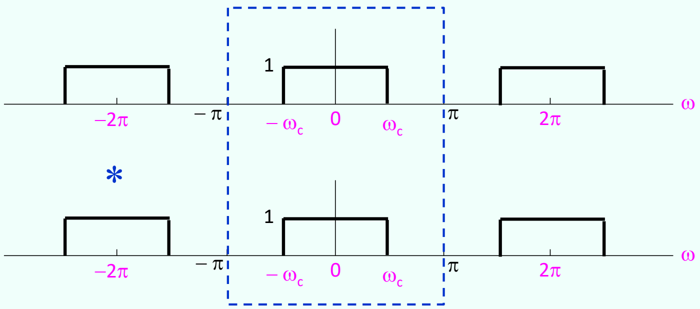
a) $\omega_c = \pi/3$



b) $\omega_c = 2\pi/3$



$$y[n] = \left(\frac{\sin(\omega_c n)}{\pi n}\right)^2 = \frac{\sin(\omega_c n)}{\pi n} \times \frac{\sin(\omega_c n)}{\pi n}$$



$$\frac{1}{2\pi}\,(1)(1)(2\omega_c) = \frac{\omega_c}{\pi}$$

e.g. The following four facts are given about a real signal $x[n]$ with Fourier transform. Determine $x[n]$.

(1) $x[n] = 0$ for $n > 0$

(2) $x[0] > 0$

(3) $Im\{X(e^{j\omega})\} = \sin(\omega) - \sin(2\omega)$

(4) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega = 3$

According to (3) :

Real & odd

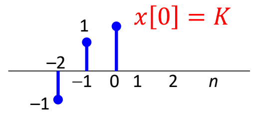
$$j\, Im\{X(e^{j\omega})\} = j\sin(\omega) - j\sin(2\omega)$$

$$\frac{x[n] - x[-n]}{2} = F^{-1}\{j\sin(\omega) - j\sin(2\omega)\}$$

$$\frac{x[n] - x[-n]}{2} = F^{-1}\left\{\frac{1}{2}\left(e^{j\omega} - e^{-j\omega}\right) - \frac{1}{2}\left(e^{j2\omega} - e^{-j2\omega}\right)\right\}$$

$$x[n] - x[-n] = \delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2]$$

According (1), (2) and (3) :



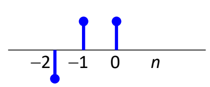
According to (4) and (2) :

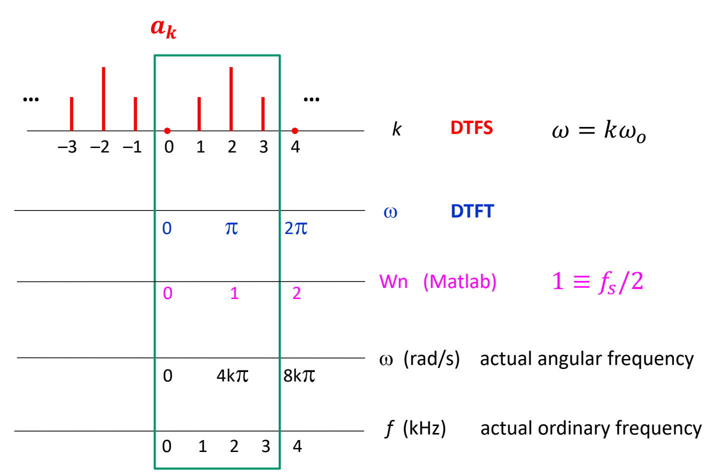
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 3$$

Parseval's Relation

$$(-1)^2 + (1)^2 + (K)^2 = 3$$

$$K = 1$$





e.g. f_s = 4 kHz

f_o = 1 kHz

N = 4