Lecture 7

System Function and Frequency Response
(Language – math and keywords; Deduction)

(Ref: Chapter 3 O&W)

I. Decomposition, Inner Product and Projection

II. Decomposition Examples

III. Eigenfunction, System Function, and Frequency Response

IV. Summary and Application Examples

I. Decomposition, Inner Product and Projection

- We now begin the second part of ELEC2100 which is on ***Fourier analysis***.
- ***Analysis*** is to understand a subject by ***decomposing*** it into simpler parts. For a signal/function, decomposition means viewing it as a superposition of some basic functions: $x(t) = \sum_k a_k \phi_k(t)$
  - In Chapter 2, we derived the ***convolution sum/integral*** for LTI systems by decomposing signals into a superposition of shifted delta functions:
    $x[n] = \sum_k x[k]\delta[n-k]$; $\quad\quad\quad x(t) = \int x(\tau)\,\delta(t-\tau)d\tau$
  - In Lecture 1, we mentioned the Taylor series expansion which is a decomposition into power functions
- Fourier analysis is the decomposition of signals into complex or real sinusoids and was pioneered by Joseph Fourier (1768-1830). Fourier analysis is the most fundamental mathematical tool for the study of many engineering problems, particularly vibrations and waves.
- Next, we will introduce "***inner product***" as the basic math technique for decomposition. We start by reviewing the decomposition of vectors.

Vector as Weighted Sum of Basis Vectors

- You can think of a real *N*-vector $\vec{x} = [x_1\ x_2\ \ldots x_k \ldots x_N\ ]$ as a point in a space with x_1, $x_2, \ldots x_k, \ldots x_N$ being its coordinates.

- You can also think of $\vec{x}$ as a weighted sum (superposition) of a set of basis vectors $\{\vec{u}_k\}$;

  how much $\vec{x}$ is in $\vec{u}_1$ axis?

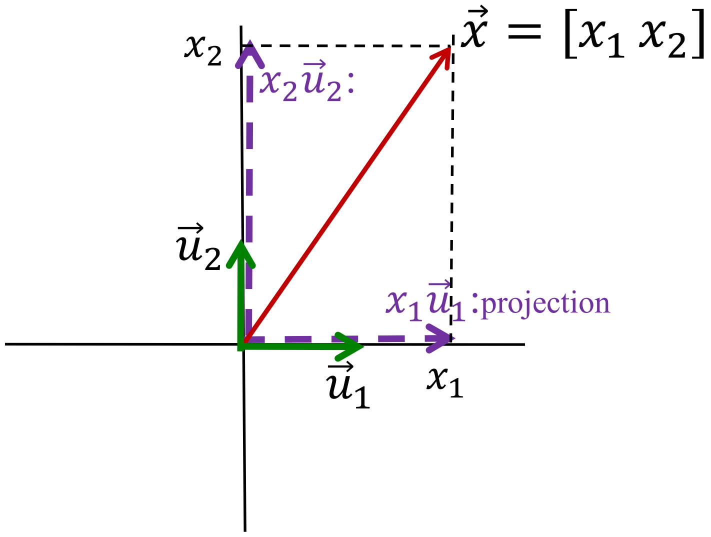
  that is: $\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots x_k\vec{u}_k \ldots + x_N\vec{u}_N$

  where

  $$\vec{u}_1 = [1\ 0\ 0\ \ldots .0]$$
  $$\vec{u}_2 = [0\ 1\ 0\ \ldots .0]$$
  $$\vdots$$
  $$\vec{u}_N = [0\ 0\ 0\ \ldots .1]$$

  Such an interpretation of a vector as a weighted sum of unit vectors is identical to our regarding a DT signal as a weighted sum of shifted impulses in Chapter 2

Example of a 2-vector



- $x_k\vec{u}_k$ is the ***projection vector*** of $\vec{x}$ onto unit basis vectors $\vec{u}_k$. It represents the part of $\vec{x}$ that is in the direction of $\vec{u}_k$.

- $\vec{x}$ is a sum of the projection vectors.

Length of a Vector

- The *length* (also called *magnitude* or *2-norm*) of the vector $\vec{x}$ is: $|\vec{x}| = \left(\sum_{k=1}^{N} {x_k}^2\right)^{\frac{1}{2}}$

  The length is zero only if all elements are zero.

  The basis vectors $\{\vec{u}_k\}$ in previous slide all have unit length: $|\vec{u}_k| = 1$.

Inner Product

- The ***inner product (dot product)*** between two vectors $\vec{x}$ and $\vec{y}$ can be interpreted as the product of the lengths of the two vectors times the cosine of the angle between them.

notation for inner product

$$<\vec{x},\vec{y}>= |\vec{x}|\,|\vec{y}|\,\cos\angle\vec{x}\vec{y}$$

- *For real* $\vec{x}$, $\vec{y}$, *their inner product is simply* the sum of their element–wise products:

$$<\vec{x},\vec{y}> \overset{\vec{x},\vec{y}\text{ real}}{=} \sum_k x_k y_k = \vec{x}\vec{y}^{\,T} \quad (T\text{: Transpose})$$

$$\vec{x}\vec{y}^T = [x_1\ x_2\ \ldots .\, x_N]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

- The magnitude (length) square of a vector is given by its self–inner product:

$$|\vec{x}|^2 =<\vec{x},\vec{x}>;\ |\vec{y}|^2 =<\vec{y},\vec{y}>$$

- The inner product is the most important mathematical operation in your engineering study. Convolution can also be viewed as an inner product: output at time n is inner product of input with convolution kernel anchored at n.

Vector Projection

- The ***projection vector*** $\vec{p}_{\vec{x}\vec{y}}$ of a vector $\vec{x}$ onto another vector $\vec{y}$ is the part of $\vec{x}$ that is in the direction of $\vec{y}$.

- Projection is computed as the inner product of $\vec{x}$ and $\vec{y}$ divided by the self–inner product of $\vec{y}$, which is the length squared of $\vec{y}$:

Projection coefficient

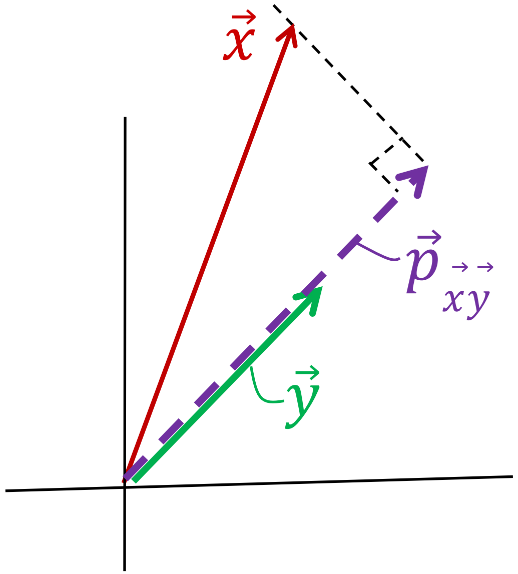
$$\vec{p}_{\vec{x}\vec{y}} = \frac{<\vec{x},\vec{y}>}{<\vec{y},\vec{y}>}\vec{y} = \frac{<\vec{x},\vec{y}>}{|\vec{y}|^2}\vec{y}$$

need to normalize, only depends on the length of $\vec{y}$

- $\frac{<\vec{x},\vec{y}>}{|\vec{y}|^2}$ is the projection coefficient.

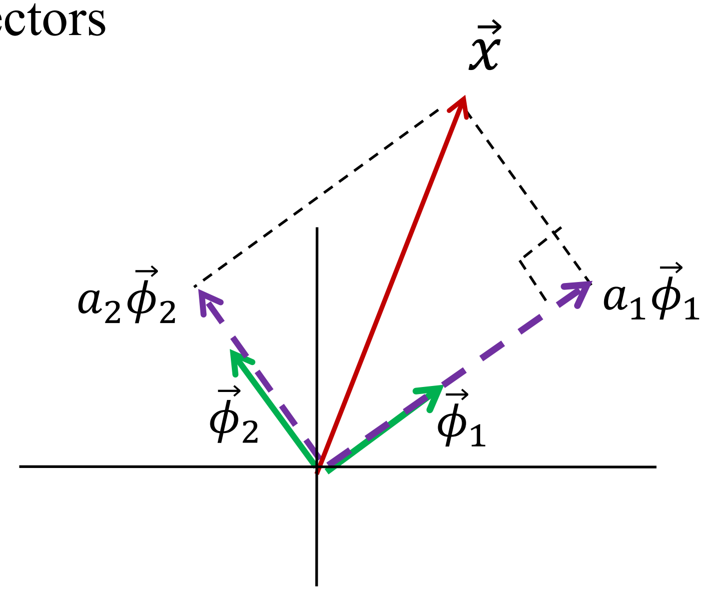
- The projection coefficient tells how much of $\vec{y}$ there is in $\vec{x}$.

- We need to normalize by $<\vec{y},\vec{y}> = |\vec{y}|^2$ in the denominator because we can define $\vec{y}$ with arbitrary lengths but the projection $\vec{p}_{\vec{x}\vec{y}}$ is unchanged.



Orthogonal Decomposition

- Two vectors are ***orthogonal*** if their inner product is zero: $<\vec{x},\vec{y}>=0$; i.e., their projection onto each other is 0.
- The unit vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$ on Slide #4 are all orthogonal.
- We can ***decompose*** vector $\vec{x}$ into a different set of orthogonal basis vectors $\{\vec{\phi}_k, <\vec{\phi}_m, \vec{\phi}_l>=0 \ \forall m \neq l\}$ by regarding $\vec{x}$ as a sum of projections:



$$\vec{x} = a_1\vec{\phi}_1 + a_2\vec{\phi}_2 + \cdots + a_N\vec{\phi}_N$$

where: $$a_k = \frac{<\vec{x},\vec{\phi}_k>}{\left|\vec{\phi}_k\right|^2}\vec{\phi}_k$$

is the projection coefficient that tells how much of $\vec{\phi}_k$ there is in $\vec{x}$.

II. Decomposition Examples

Example 1 Decompose $\vec{x} = [5, 2]$ using $\vec{\phi}_1 = [1\,, 1]$ and $\vec{\phi}_2 = [1, -1]$ as basis vectors.

- We check that $\vec{\phi}_1$ and $\vec{\phi}_2$ are orthogonal: $< \vec{\phi}_1, \vec{\phi}_2 >= 1 \times 1 + 1 \times (-1) = 0$

Projection coefficient of $\vec{x}$ onto $\vec{\phi}_1$: $\frac{<\vec{x},\vec{\phi}_1>}{|\vec{\phi}_1|^2} = \frac{5\times1+2\times1}{1\times1+1\times1} = 3.5$

$\vec{x}_1 = \frac{5+2}{1^2+1^2}\vec{\phi}_1 = \frac{7}{2}\vec{\phi}_1$

Projection coefficient of $\vec{x}$ onto $\vec{\phi}_2$: $\frac{<\vec{x},\vec{\phi}_2>}{|\vec{\phi}_2|^2} = \frac{5\times1+2\times(-1)}{1\times1+(-1)\times(-1)} = 1.5$

$\vec{x}_2 = \frac{5-2}{1^2+1^2}\vec{\phi}_1 = \frac{3}{2}\vec{\phi}_1$

Hence $\vec{x} = 3.5[1, 1]$+$1.5[1, -1]$

- ***We have decompose $\vec{x}$ into one part that does not change*** **([1,1])** ***and one part that changes at frequency ½ Hz or π rad/time unit*** **([1,-1]) !**

- Note that $|\vec{\phi}_1| = |\vec{\phi}_2| = \sqrt{2}$, so we need to divide by 2 in the denominator.

We could use $\vec{\gamma}_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ and $\vec{\gamma}_2 = \left[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$ which have unit length, but then having $\sqrt{2}$ in the denominator everywhere would be very messy.

Inner Product for Complex-Valued Vectors

- If $\vec{x}$ and $\vec{y}$ are complex, to compute $< \vec{x}, \vec{y} >$, we needs to ***conjugate*** $\vec{y}$ before we multiply:

$$< \vec{x}, \vec{y} > = \sum_n x_n y_n^* \longleftarrow \text{conjugation}$$

- The above conjugate-multiplication rule ensures that the self-inner product of a complex vector $\vec{x}$ is non-negative, and is equal to zero only if $\vec{x}$ is the zero vector:

magnitude always ≥ 0

$$< \vec{x}, \vec{x} > = \sum_n x_n x_n^* = \sum_n |x_n|^2$$

- For complex vectors, changing the order of the two operands results in conjugation of the inner product:

$$< \vec{x}, \vec{y} > = < \vec{y}, \vec{x} >^*$$

Example 2

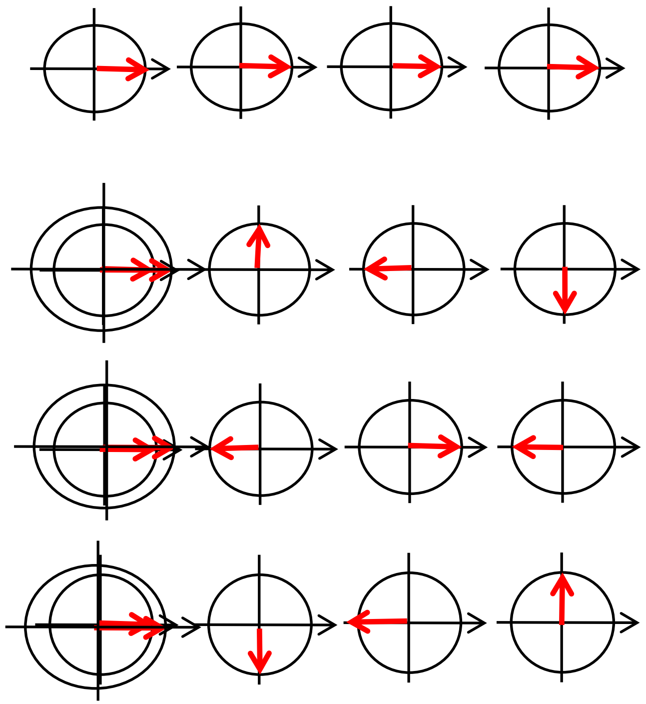
- Decompose the 4-vector $\vec{x} = [5\ \ 2\ \ \text{-}1\ \ 3]$ into the following basis set of complex 4-vectors:

$\vec{\phi}_0 = [\,1\ \ \ 1\ \ \ 1\ \ \ 1\,]$ $\quad \dfrac{5+2-1+3}{1^2+1^2+1^2+1^2}\vec{\phi}_1 = \dfrac{9}{4}\vec{\phi}_1$

$\vec{\phi}_1 = \left[1\ \ e^{j\frac{\pi}{2}}\ \ e^{j\pi}\ \ e^{j\frac{3\pi}{2}}\right] = [\,1\ \ j\ \ \text{-}1\ \ \text{-}j\,]$

$\vec{\phi}_2 = \left[1\ \ e^{j\pi}\ \ e^{j2\pi}\ \ e^{j3\pi}\right] = [\,1\ \ \text{-}1\ \ \ 1\ \ \text{-}1\,]$

$\vec{\phi}_3 = \left[1\ \ e^{j\frac{3\pi}{2}}\ \ e^{j\pi}\ \ e^{j\frac{\pi}{2}}\right] = [\,1\ \ -j\ \ \text{-}1\ \ \ j\,]$



rotations at angular frequency 0

rotations at angular frequency $\frac{\pi}{2}$

rotations at angular frequency π

rotations at angular frequency $\frac{3\pi}{2}$

$|\cdot|$ of all entries =1; $\angle$ changes at different rates. What are these $\vec{\phi}_k$'s?

$\vec{\phi}_0, \vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ are complex sinusoids represent rotations at angular frequencies 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ respectively (or ordinary frequencies 0, ¼, ½, ¾).

$\vec{\phi}_0, \vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ represent the complete set of DT complex sinusoids with period $N = 4$.

You can check that these $\vec{\phi}_k$'s are orthogonal ($\langle \vec{\phi}_k, \vec{\phi}_l \rangle = 0 \ \forall k \neq l$) and all have self-inner products of 4:

$$|\vec{\phi}_0|^2 = |\vec{\phi}_1|^2 = |\vec{\phi}_2|^2 = |\vec{\phi}_3|^2 = 4$$

We compute the ***projection coefficients*** as follows:

conjugation

$$a_0 = \underbrace{\frac{1}{4}}_{|\vec{\phi}_0|^2}\overbrace{[5 \ \ 2 \ \ -1 \ \ 3]}^{\vec{x}}\overbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}^*}^{\vec{\phi}_0^{T*}} = \frac{1}{4}(5+2-1+3) = \frac{9}{4};$$

conjugation

$$a_1 = \underbrace{\frac{1}{4}}_{|\vec{\phi}_1|^2}\overbrace{[5 \ \ 2 \ \ -1 \ \ 3]}^{\vec{x}}\overbrace{\begin{bmatrix}1\\j\\-1\\-j\end{bmatrix}^*}^{\vec{\phi}_1^{T*}} = \frac{1}{4}(5-2j+1+3j) = \frac{6+j}{4};$$

$$a_2 = \frac{1}{4}[5 \ \ 2 \ \ -1 \ \ 3]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}^* = \frac{1}{4}(5-2-1-3) = \frac{-1}{4};$$

$$a_3 = \frac{1}{4}[5 \ \ 2 \ \ -1 \ \ 3]\begin{bmatrix}1\\-j\\-1\\j\end{bmatrix}^* = \frac{1}{4}(5+2j+1-3j) = \frac{6-j}{4}$$

$$\Longrightarrow \vec{x} = \overbrace{\frac{9}{4}}^{a_0}\vec{\phi}_0 + \overbrace{\frac{6+j}{4}}^{a_1}\vec{\phi}_1 - \overbrace{\frac{1}{4}}^{a_2}\vec{\phi}_2 + \overbrace{\frac{6-j}{4}}^{a_3}\vec{\phi}_3$$

We have decomposed a real vector/signal $\vec{x}$ into a set of complex-valued vectors/signals that represent revolutions at different frequencies. We have done so by computing a set of ***projection coefficients***

Extending Inner Product from Vector to DT and CT Signals

- A finite duration DT signal $x[n]$ of duration N can be viewed an N-vector. An infinite duration DT signal can be viewed as an N-vector in the limit $N \to \infty$.

- To find the inner product of two DT signals, we ***conjugate-multiply*** and ***sum***:

$$< x[n], g[n] >= \sum_n x[n]g * [n]$$

Over N terms if finite duration;
Infinite number of terms if infinite duration

- For CT signals, to compute inner product, we ***conjugate-multiply*** and ***integrate***.

$$< x(t), g(t) >= \int x(t)g * (t)\, dt$$

Over an interval T if finite duration;
over the infinite interval if infinite duration

- The interval of summation/integration depends simply on whether we are working with finite duration or infinite duration signals.

III. Eigenfunction, System Function, and Frequency Response

- In ELEC2100 we focus on decomposing signals into ***complex sinusoids***. In some cases we can decompose into ***real sinusoids*** instead, but working with complex sinusoids is more broadly applicable and mathematically more concise.

- Complex sinusoids are special because they are *eigenfunctions* of all LTI systems. An *eigenfunction* is a function that if it is the input to a system, the output is the same function except for the scaling by a constant.

- Regarding signals as superposition of complex sinusoids will allow us to understand systems from a different perspective and to solve many problems much more efficiently.

Complex Exponentials as Eigenfunctions – CT Case

- In fact, we can show more generally that all *complex exponentials* are eigenfunctions of LTI system:

$$e^{st} \longrightarrow \boxed{\text{CT LTI System } h(t)} \longrightarrow \text{Output} = ?$$

- For CT, output is the convolution of $h(t)$ with e^{st}:

$e^{s(t-\tau)} = e^{st}e^{-s\tau}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}$$

$x(t-\tau) = e^{s(t-\tau)}$

If the integral converges

- For values of s such that $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ ***converges*** (i.e., $< \infty$), the output is $H(s)e^{st}$, the same complex exponential scaled by $H(s)$, which is a constant that depends on the complex frequency s but not on t.
- $H(s)$ is the *eigenvalue* for e^{st}.

Complex Exponentials as Eigenfunctions – DT Case

- For DT LTI systems, what is the output when the input is a DT complex exponential z^n ?

$$x[n] = z^n \longrightarrow \boxed{\text{DT LTI system } h[n]} \longrightarrow y[n] = ?$$

Recall that in DT we can more conveniently represent the complex exponential e^{sn} as z^n, with $z = e^s$

- Now we apply the convolution sum to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} = H(z) z^n$$

$x[n-k] = z^{n-k}$

If the sum converges

Hence, $z^n = e^{sn}$ is an ***eigenfunction*** of DT LTI systems. $H(z)$ is the ***eigenvalue*** that depends on the complex frequency z.

System Function of CT and DT LTI Systems

- $H(s)$ and $H(z)$ are called the ***system function*** of CT and DT LTI systems.

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt}$$

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}}$$

We have changed the variables of integration and summation from τ to t and from k to n. They are just dummy variables.

- $H(s)$ is a (2-sided) ***Laplace Transform*** of the CT impulse response $h(t)$.
- $H(z)$ is a (2-sided) ***z-transform*** of the DT impulse response $h[n]$.
- In Chapter 9, we will learn how we can understand LTI systems by considering their system functions.

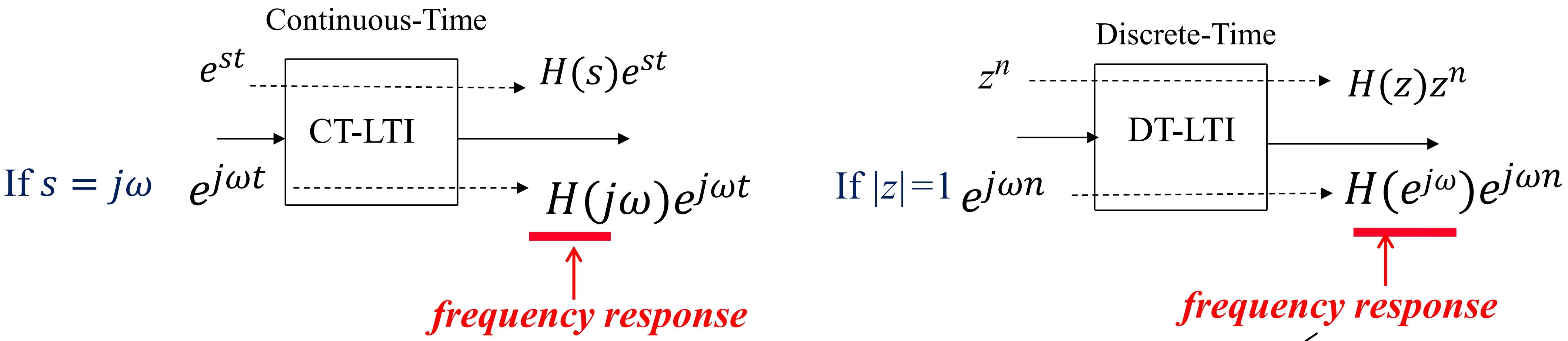
$\int_0^{\infty} h(t)e^{-st}dt$ is the unilateral Laplace transform. You might have already learnt it for solving differential equations.

We can view the z-transform as an approximation by numerical integration of the Laplace transform integral:

$$H(s) = \lim_{\Delta\to 0} \sum_{n=-\infty}^{\infty} h(n\Delta)\ e^{-s(n\Delta)}\Delta$$

Complex Sinusoids and Frequency Response

- Since complex sinusoids are a special case of complex exponential, they are also eigenfunctions.
- They are the eigenfunctions that are physically meaningful. Complex exponentials that have been growing or decaying forever <u>do not represent anything that is physically meaningful</u>! But complex sinusoids represent oscillations that have been turned on for a long time.
- For an LTI system, the eigenvalue at different oscillation frequency ω is called its ***<u>frequency response</u>***.



$H(j\omega) = H(s)|_{s=j\omega}$

<u>The notations look scary</u>, but both are simply function of ω

$H(e^{j\omega}) = H(z)|_{|z|=1; z=e^{j\omega}}$

Frequency Response as Fourier Transform of Impulse Response

- The *frequency response* is given by the *Fourier transform* of the impulse response. The Fourier transform is a cross-section of the Laplace/z-transform.

Frequency Response of CT LTI System:

$$H(j\omega) = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

and is the ***Continuous-Time Fourier transform* (CTFT)** of the CT impulse response (Chapter 4)

Frequency Response of DT LTI System:

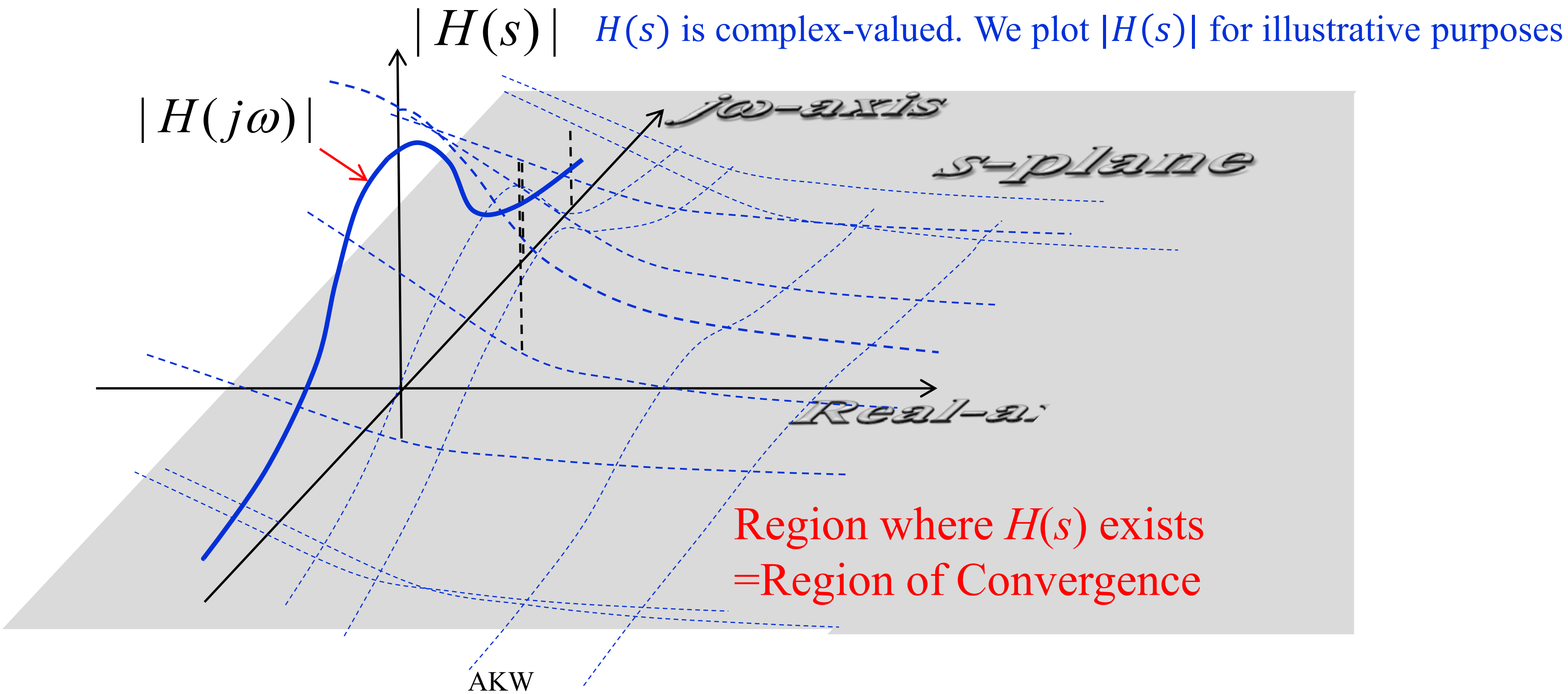
$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

and is the ***Discrete-Time Fourier transform* (DTFT)** of the DT impulse response (Chapter 5)

- We will learn that the Fourier transform exists only if the system is stable, i.e., $h(t)/h[n]$ is absolute integrable/summable.

Frequency Response as Cross-Section of System Function - CT

- $H(s)$ is a function over the 2-dimensional s-plane.
- The region of s where the integral $\int_{-\infty}^{\infty} h(t)e^{-st}dt$ converges is called the *Region of Convergence* (*ROC*).
- $H(j\omega)$ is the cross-section of $H(s)$ along the $j\omega$ axis. It is a 1-dimensional function of ω.
- $H(j\omega)$ exists only if ROC includes the $j\omega$ axis, or equivalently, system is stable.



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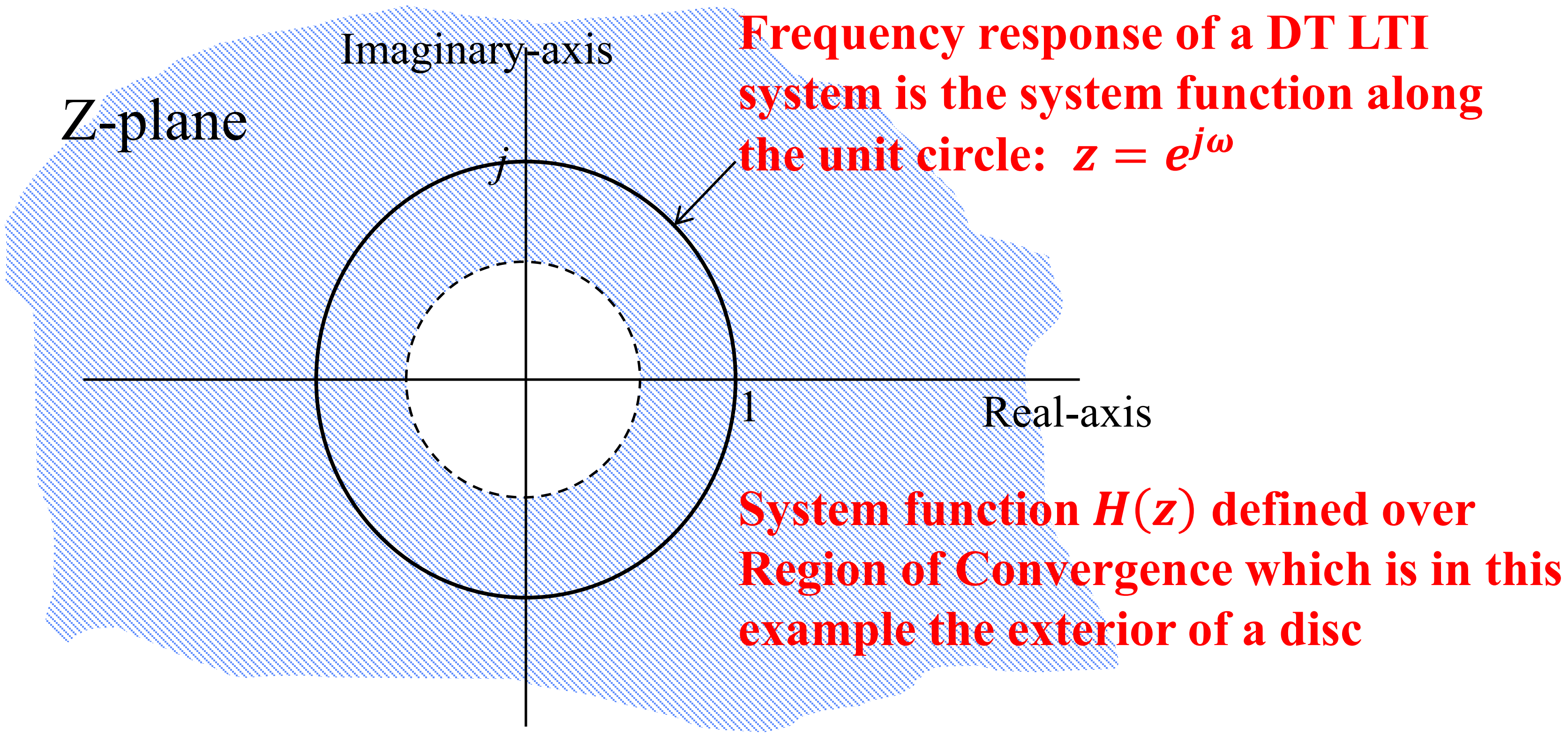
Frequency Response as Cross-Section of System Function - DT

- In DT, the frequency response is the value of the system function along the unit circle.



Impulse Response, Frequency Response, and System Function

- In ELEC2100, we will have three ways of specifying an LTI system:
  1. ***Impulse response*** $h(t), h[n]$
  2. ***Frequency Response*** $H(j\omega), H(e^{j\omega})$
  3. ***System Function*** $H(s), H(z)$

- Since Lecture 1, we have described CT and DT signals as $x(t)$, $x[n]$, and LTI systems by their impulse responses. This is the *time domain representation* of signals and systems.

- In the next 10 lectures, we will introduce Fourier analysis which describes signals by their frequency content (i.e., spectrum) and systems by their frequency response. This is called the *frequency domain representation*. Fourier analysis is applicable only if the LTI system is stable. Also, unless stated otherwise, we assume that we are dealing with causal systems.

- In the final 4 lectures, we will learn how the *system function* (for CT only) allows us to address issues of stability and causality (whether signal/system is right-sided or left-sided). We will also learn how we can "design" the frequency response of an LTI system by designing its system function, and how we can use feedback to manipulate a system to make it stable.

IV. Examples

Example 1

Let $H(s) = \frac{1}{1+2s}$ be the system function of a CT LTI system.

(a) Assume the system is stable. What is its frequency response?

System stable means frequency response exists, and $H(j\omega) = H(s)|_{s=j\omega} = \frac{1}{1+2j\omega}$

(b) Determine the output when the input is:

(i) e^{-3t} $\rightarrow H(-3)e^{-3t} = \dfrac{1}{1-6}e^{-3t} = -\dfrac{1}{5}e^{-3t}$

Eigenfunction with $s = -3$

Is (i) a physically meaningful input?

(ii) $e^{j0.5t}$ $\xrightarrow{s=0.5j} H(0.5j)e^{j0.5t} = \dfrac{1}{1+j}e^{j0.5t}$

(iii) $\cos(0.5t)$

$$\cos(0.5t) = \frac{1}{2}e^{j0.5t} + \frac{1}{2}e^{-j0.5t} \rightarrow \frac{1}{2}\left(\frac{1}{1+j}e^{j0.5t} + \frac{1}{1-j}e^{-j0.5t}\right) = Re\left\{\frac{1}{1+j}e^{j0.5t}\right\} =? \frac{1}{\sqrt{2}}\cos\left(0.5t - \frac{\pi}{4}\right)$$

Conjugate pair

$$\left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}};\ \measuredangle\frac{1}{1+j} = -\frac{\pi}{4}$$

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Example 1 Continue

(c) The system has the same system function above but its region of convergence does not include the $j\omega$ axis. What is the system's frequency response?

Then system is unstable and there is no frequency response

Example 2

Let $H(z) = \dfrac{1}{1+2z}$ be the system function of a DT LTI system.

(a) Assume the system is stable. What is the frequency response? $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \dfrac{1}{1+2e^{j\omega}}$

(b) Determine the output when the input is:

(i) $e^{j\frac{\pi}{4}n}$ $\xrightarrow{z=e^{j\frac{\pi}{4}};\ \omega=\frac{\pi}{4}}$ $H\left(e^{j\frac{\pi}{4}}\right)e^{j\frac{\pi}{4}n} = \dfrac{1}{1+2e^{j\frac{\pi}{4}}}e^{j\frac{\pi}{4}n}$

DT complex sinusoid: eigenfunction

(ii) $\cos\frac{\pi}{4}n$ $= \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n}$

$\omega = \frac{\pi}{4}$ $\omega = -\frac{\pi}{4}$

$\left|H\left(e^{j\frac{\pi}{4}}\right)\right|$; magnitude response $\measuredangle H\left(e^{j\frac{\pi}{4}}\right)$; phase response

Output: $\dfrac{1}{2}\dfrac{1}{1+2e^{j\frac{\pi}{4}}}e^{j\frac{\pi}{4}n} + \dfrac{1}{2}\dfrac{1}{1+2e^{-j\frac{\pi}{4}}}e^{-j\frac{\pi}{4}n} = Re\left\{\dfrac{1}{1+2e^{j\frac{\pi}{4}}}e^{j\frac{\pi}{4}n}\right\} = \left|\dfrac{1}{1+2e^{j\frac{\pi}{4}}}\right|\cos\left(\dfrac{\pi}{4}n + \measuredangle\left(\dfrac{1}{1+2e^{j\frac{\pi}{4}}}\right)\right)$

Conjugate pair

Notice that in Examples 1 & 2, the frequency of real sinusoidal input are not changed by the LTI system. Only their amplitude and phase are changed.

For $H(z) = \dfrac{1}{1+2z}$ in the previous slide

Determine the output when the input is:

(iii) $\cos\frac{41\pi}{4}n$

$$\cos\frac{41\pi}{4}n = \cos\left(\frac{\pi}{4}n + 1\cancel{0}\pi n\right) = \cos\frac{\pi}{4}n$$

$$10\pi n = 5n \times 2\pi$$

Same answer as in part (ii)!

DT frequencies of $\frac{\pi}{4}$ and $\frac{41\pi}{4}$ are equivalent! DT frequency is 2π-periodic!

A Summary

- ***Analysis*** is to decompose something into basic parts. Decomposition can be done by computing a set of ***projection coefficients*** using the ***inner product***.
- ***Complex sinusoids***, and ***complex exponential*** in general, are ***eigenfunctions*** of LTI systems.
- The ***system function***, which is the Laplace transform (for CT) or z-transform (for DT) of the impulse response, provides the ***eigenvalue*** for each eigenfunction.
- The ***frequency response*** is a cross-section of the system function and is given the CT/DT Fourier transform of the impulse response. The frequency response specify the eigenvalues for complex sinusoids. It exists only if the system is stable.
- ***Fourier analysis*** is to decompose signals into complex sinusoids, which are eigenfunctions and represent oscillations that have been turned on for a long time.

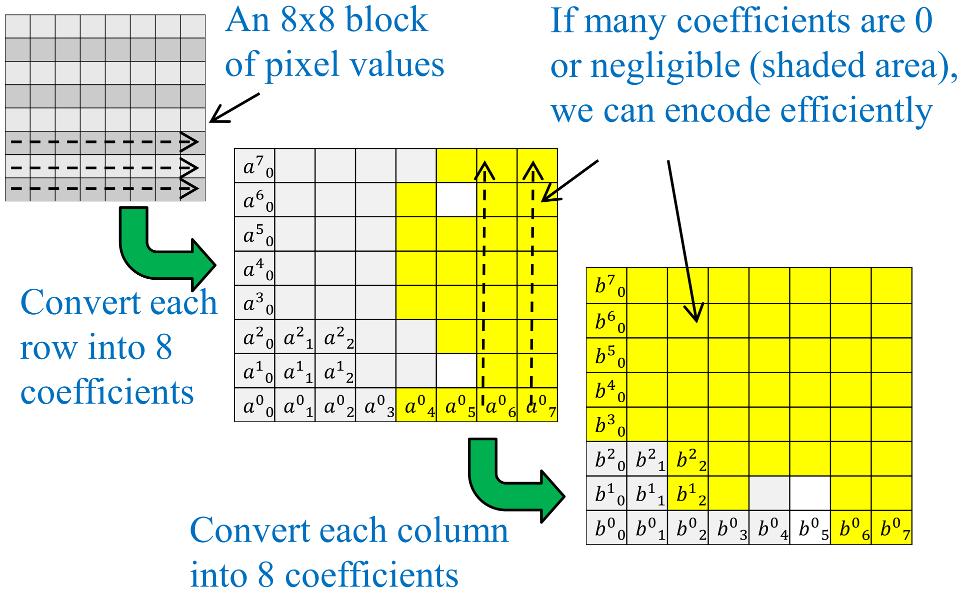
Application Examples – For Reference Only

Reference Example 1 - JPEG Image Compression by Decomposition into Cosine Functions

- A bitmap file stores the pixel values of an image directly. JPEG uses *Discrete Cosine Transform* (DCT). We divide an image into blocks of 8×8 pixels. Each row of a block is a 8-vector and we compute its inner product with the basis vectors $\vec{\phi}_k = [\phi_{k,0}, \phi_{k,1}, \dots \phi_{k,7}]$, $k = 0,..7$, where $\phi_{k,n} = \cos\left(k\frac{2\pi}{16}\left(n+\frac{1}{2}\right)\right)$, without normalizing by $1/|\vec{\phi}_k|^2$. The result is 8 *transform* coefficients which describe how the pixel values vary along a row. Usually many of these coefficients are negligible. Then, we use the same set of basis to determine how the coefficients varies in the vertical direction along the 8 columns. The end result is a set of 64 coefficients, among which many are 0 or negligible so that we can encode them efficiently. DCT is also the basis for MPEG video compression.

Bitmap file 800KB

JPEG file 120KB



Reference Example 2 – Correlation as Inner Product

- The dot/inner product is the most important operator in engineering. It allows us to compute projection and measures *similarity*, *correlation*, or *dependency*.

Example 1: HKUST students have an average height of $\bar{h}$=170cm and average GPA of $\bar{g}$=2.8. You have the height h_i and GPA g_i of a large group of HKUST students, and want to determine if height is a good predictor of GPA. How?

⇒We compute the inner product of the sequence of mean-adjusted heights and mean-adjusted GPAs

Inner product $\sum_i (h_k - \bar{h})(g_k - \bar{g})$ gives correlation between height and GPA.

Normalization by standard deviations of h and g to gives correlation coefficient which is always between -1 and 1:

$$\frac{\sum_i (h_k - \bar{h})(g_k - \bar{g})}{\sqrt{\sum_i (h_k - \bar{h})^2}\ \sqrt{\sum_i (g_k - \bar{g})^2}}$$

Self-inner products of mean-adjusted heights and mean-adjusted GPAs gives variance of heights and of GPA

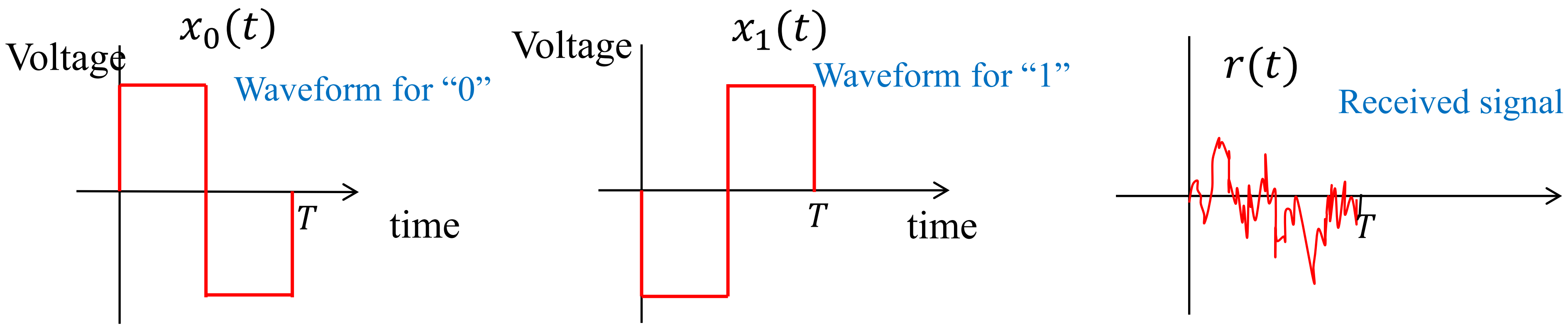
Square root of variance gives standard deviation

- With the following set of data of the heights and GPAs of a group of 6 students, calculate the correlation coefficient and determine if there is a strong relationship between a student's height and GPA (do it in Excel or Python code).

Student	Column1	A	B	C	D	E	F	Column2	Mean
Height (cm)	h	170	163	177	166	182	154	$\bar{h}$	
GPA	g	2.86	3.35	3.12	2.45	3.45	3.75	$\bar{g}$	
a	$h_i-\bar{h}$								
b	$g_i-\bar{g}$								
									Sum
a*b	$(h_i-\bar{h})*(gi-\bar{g})$							$\Sigma(hi-\bar{h})*(gi-\bar{g})$	
a^2	$(h_i-\bar{h})^2$							$\Sigma(hi-\bar{h})^2$	
b^2	$(gi-\bar{g})^2$							$\Sigma(gi-\bar{g})^2$	
Correlation	$\sigma_{h,g}$								
Correlation Coefficient	$\rho_{h,g}$								

Reference Example 3 – Signal Detection in Digital Communication

Example 2: Digital transmission systems often transmit two different waveforms to transmit a “1” or “0” bit. In Manchester coding, the two different waveforms are shown below:



The communication channel may attenuate the signal and add noise to it, so that the signal $r(t)$ is received.

The signal detection problem for the receiver is to decide if a “1” or a “0” bit has been transmitted. How?

$\Rightarrow$We compute $< r(t), x_{0(t)} >$ and $< r(t), x_1(t) >$ and compare

- We will do inner product all the time!