T01

Complex numbers
Complex Plane
Complex functions

Relationship between continuous-time signals (CT) and discrete-time signals (DT)
Time-shift, time-reversal and time scaling
Odd and even parts of a signal
Periodic signals
Poisson sum
Important functions (real exponential and unit step)

Complex Numbers

Mathematical operations of two complex numbers
- Addition, subtraction, multiplication and division
- Complex conjugate

Exponential form
- Magnitude and Phase

⟺

Rectangular form
- Real part and Imaginary part

Euler's Formula

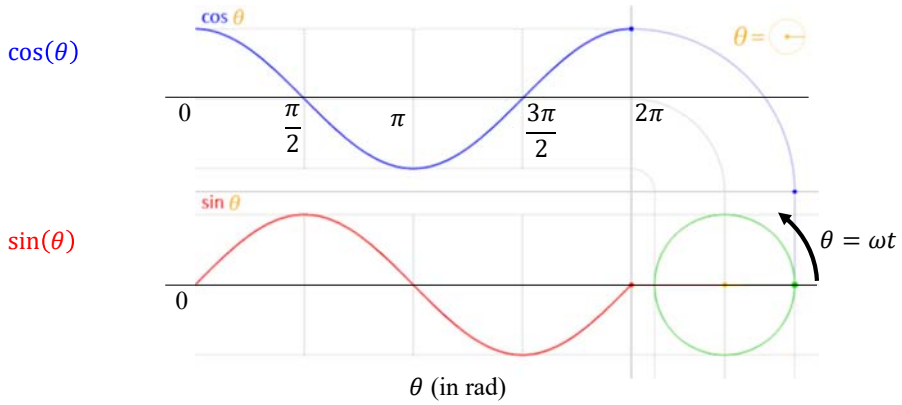
$$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$$

Magnitude, Phase, Real, Imaginary

$$A\cos(\theta) = \frac{A}{2}e^{j\theta} + \frac{A}{2}e^{-j\theta}$$

$$A\sin(\theta) = \frac{A}{2j}e^{j\theta} - \frac{A}{2j}e^{-j\theta}$$

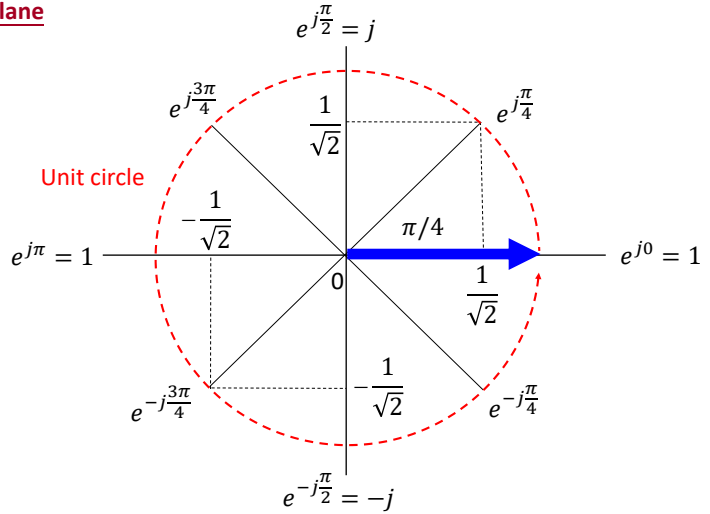
One period ≡ 2π (in rad) ≡ 360 (in degree) ≡ One full circle



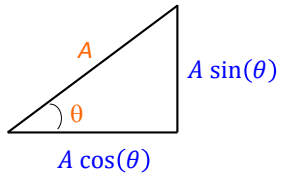
Mathematical expression ?

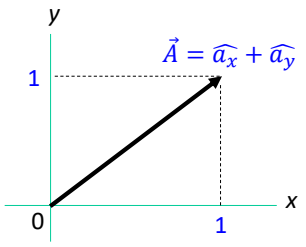
Peter is 4 meters away from Mary at 2 o'clock position.

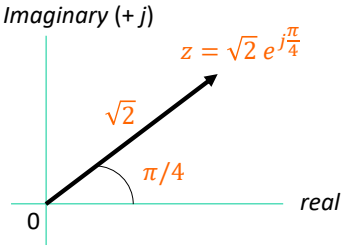
Complex Plane



Trigonometry | Vectors | Complex Numbers



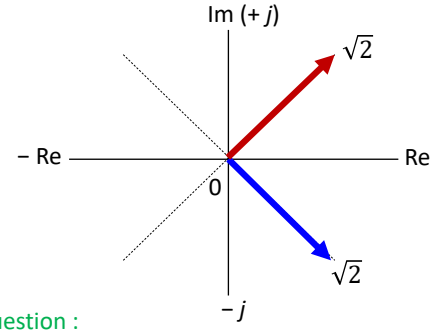




$$Ae^{j\theta} = A\cos(\theta) + j\,A\sin(\theta)$$

$$Magnitude = \sqrt{real^2 + imaginary^2}$$

$$Phase = tan^{-1}\left(\frac{imaginary}{real}\right)$$



$-1 + j$

$$Phase = tan^{-1}\left(\frac{1}{-1}\right)$$
$$= tan^{-1}(-1)$$
$$= -\frac{\pi}{4}$$

$-1 - j$

$$Phase = tan^{-1}\left(\frac{-1}{-1}\right)$$
$$= tan^{-1}(1)$$
$$= \frac{\pi}{4}$$

Question :
- Is the calculation of the phase correct ?
- What is the relationship between these two complex values ?

e.g. Value of Real part ? Imaginary part ? Magnitude ? Phase ?

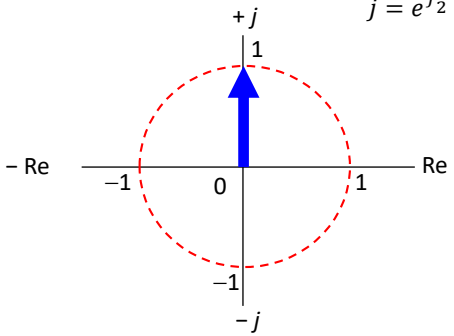
$j = \sqrt{-1}$

$j = e^{j\frac{\pi}{2}}$

j^2 j^3

j^4 j^5

j^6 j^7



Question :
- What is the meaning of multiplication ?
- How to obtain the answer using the complex plane ?

e.g. Value of Real part ? Imaginary part ? Magnitude ? Phase ?

$-e^{j\frac{\pi}{2}}$

$[(-1-j)^*]^2(2+j2)^4$

$e^{0.5+j\frac{\pi}{4}}$

$\frac{(-1-j)^2}{(2+j2)^4}$

$\left(e^{0.5+j\frac{\pi}{4}}\right)^*$

* conjugate

$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$

$-e^{j\frac{\pi}{2}} = (-1)\,e^{j\frac{\pi}{2}} = e^{j\left(\pm\pi+\frac{\pi}{2}\right)} = -j$ $\qquad -1 = e^{j\pi} = e^{-j\pi}$

$e^{0.5+j\frac{\pi}{4}} = e^{0.5}\,e^{j\frac{\pi}{4}} = e^{0.5}\cos\left(\frac{\pi}{4}\right) + je^{0.5}\sin\left(\frac{\pi}{4}\right)$ $\qquad e^{a+b} = e^a e^b$

$\left(e^{0.5+j\frac{\pi}{4}}\right)^* = e^{0.5}\,e^{-j\frac{\pi}{4}} = e^{0.5}\cos\left(\frac{\pi}{4}\right) - je^{0.5}\sin\left(\frac{\pi}{4}\right)$

$[(-1-j)^*]^2(2+2j)^4$

$= \left(\sqrt{2}\,e^{j\frac{3\pi}{4}}\right)^2\left(\sqrt{8}\,e^{j\frac{\pi}{4}}\right)^4$

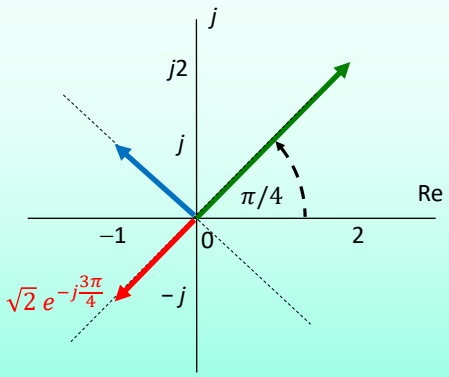
$= (2\,e^{j\frac{3\pi}{2}})\,(2^6\,e^{j\pi})$

$= 2^7 e^{j\frac{\pi}{2}}$

$\frac{(-1-j)^2}{(2+2j)^4}$

$= \frac{\left(\sqrt{2}\,e^{-j\frac{3\pi}{4}}\right)^2}{\left(\sqrt{8}\,e^{j\frac{\pi}{4}}\right)^4} = \frac{(2\,e^{-j\frac{3\pi}{2}})}{(2^6\,e^{j\pi})}$

$= 2^{-5}\,e^{-j\frac{\pi}{2}}$



Question :
- How to solve the problem efficiently ?
- Exponential form or rectangular form ?

Question :
- What is the difference between values and functions ?
- Can we use the concept of complex numbers to deal with complex functions ?

t	$x(t)$
0	0
1	1
2	2
3	3

$x(t) = t$

t	$x(t)$	$\lvert x(t)\rvert$	$\sphericalangle\, x(t)$	$Re\{x(t)\}$	$Im\{x(t)\}$
0	2	2	0	2	0
1	$\sqrt{2}+\sqrt{2}j$	2	$\frac{\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$
2	$2j$	2	$\frac{\pi}{2}$	0	2
3	$-\sqrt{2}+\sqrt{2}j$	2	$\frac{3\pi}{4}$	$-\sqrt{2}$	$\sqrt{2}$

$x(t) = 2e^{j\frac{\pi}{4}t} = 2\cos\left(\frac{\pi}{4}t\right) + j\,2\sin\left(\frac{\pi}{4}t\right)$

e.g. Expression of Real part ? Imaginary part ? Magnitude ? Phase ?

$x_1(t) = 3\cos(2\pi t)$

$x_2(t) = -j3\cos(2\pi t)$

$x_3(t) = (e^t)^*$

$x_4(t) = (3e^{-0.8t}e^{j2\pi t})^*$

$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$

$A(t)\ e^{j\theta(t)} = A(t)\cos(\theta(t)) + jA(t)\sin(\theta(t))$

$x_1(t) = 3\cos(2\pi t)$

$x_2(t) = -j3\sin(2\pi t)$

$x_3(t) = (e^t)^* = e^t$

$x_4(t) = 3e^{-0.8t}\ e^{-j2\pi t} = 3e^{-0.8t}\cos(2\pi t) - j3e^{-0.8t}\sin(2\pi t)$

$x_1(t) = 3\cos(2\pi t)$ $\quad Im\{x_1(t)\} = 0 \ \ \forall\, t$ $\quad A(t)\ e^{j\theta(t)}$

$\mathrm{Re}\{x_1(t)\}$

$3\cos(2\pi t)$

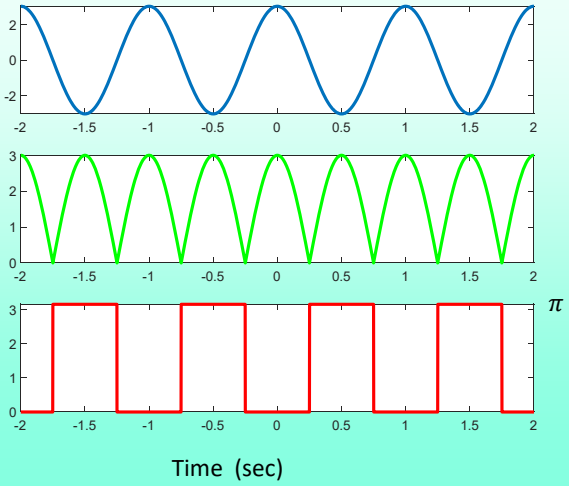
$|x_1(t)|$

$|3\cos(2\pi t)|$

$\measuredangle\, x_1(t)$

$\measuredangle\, x_1(t) = 0 \quad x_1(t) > 0$

$\measuredangle\, x_1(t) = \pi \quad x_1(t) < 0$



$x_2(t) = -j3\sin(2\pi t)$ $\quad \mathrm{Re}\{x_2(t)\} = 0 \ \ \forall\, t$ $\quad A(t)\ e^{j\theta(t)}$

$\mathrm{Im}\{x_2(t)\}$

$-3\sin(2\pi t)$

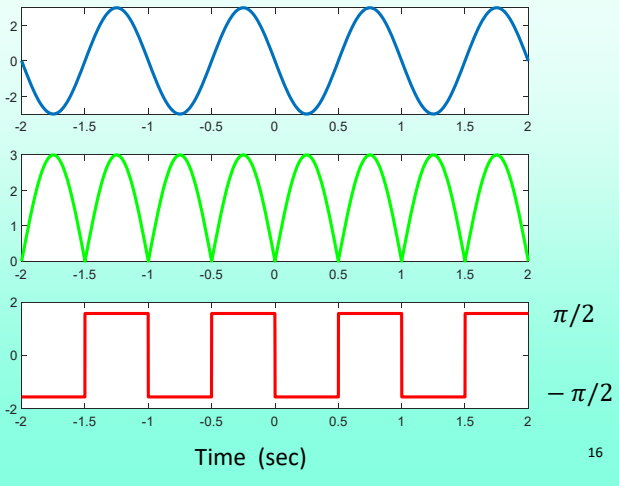
$|x_2(t)|$

$|3\sin(2\pi t)|$

$\measuredangle\, x_2(t)$

$\frac{\pi}{2} \quad x_2(t) > 0$

$-\frac{\pi}{2} \quad x_2(t) < 0$



$x_3(t) = (e^t)^* = e^t$

$\measuredangle\, x_3(t)$ and $\mathrm{Im}\{x_3(t)\} = 0\ \forall\, t$

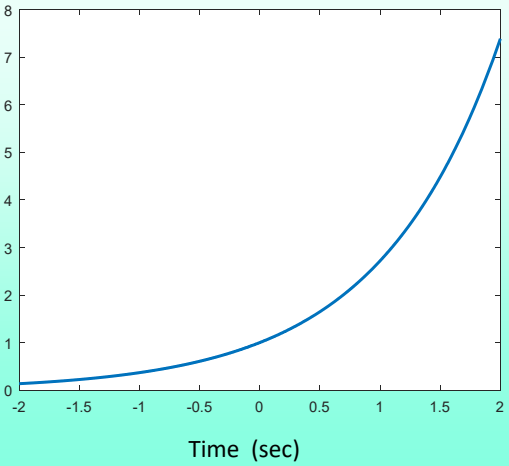
$A(t)\, e^{j\theta(t)}$

$x_3(t)$

$|x_3(t)|$

$\mathrm{Re}\{x_3(t)\}$

e^t



$x_4(t) = 3e^{-0.8t}e^{-j2\pi t} = 3e^{-0.8t}\cos(2\pi t) - j3e^{-0.8t}\sin(2\pi t)$

$A(t)\, e^{j\theta(t)}$

$|x_4(t)|$

$3e^{-0.8t}$

$\measuredangle\, x_4(t)$

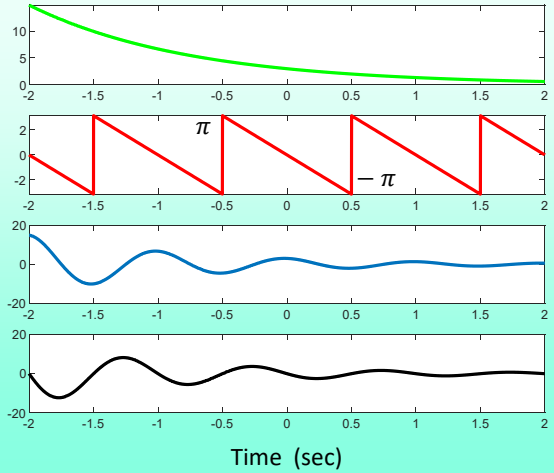
$-2\pi t$

$\mathrm{Re}\{x_4(t)\}$

$3e^{-0.8t}\cos(2\pi t)$

$\mathrm{Im}\{x_4(t)\}$

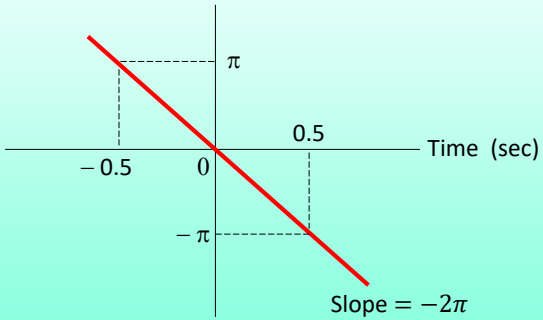
$-3e^{-0.8t}\sin(2\pi t)$



$x_4(t) = 3e^{-0.8t}e^{-j2\pi t}$

$A(t)\, e^{j\theta(t)}$

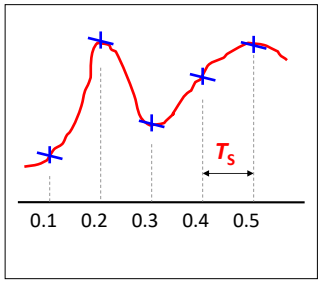
$\measuredangle\, x_4(t) = -2\pi t$



Relationship between continuous-time signals (CT) and discrete-time signals (DT)
Time-shift, time-reversal and time scaling
Odd and even parts of a signal
Periodic signals
Poisson sum
Important functions (real exponential and unit step)

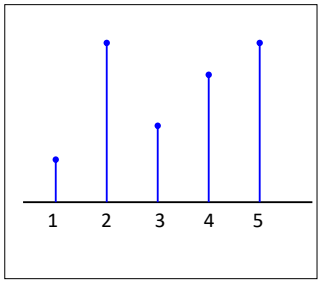
Relationship between CT signals and DT signals

$x(t)$ CT signal



Sampling

$x[n]$ DT signal



Time t

Real values

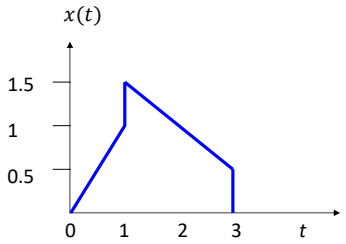
Assume $T_S = 0.1$ s

n

Integer

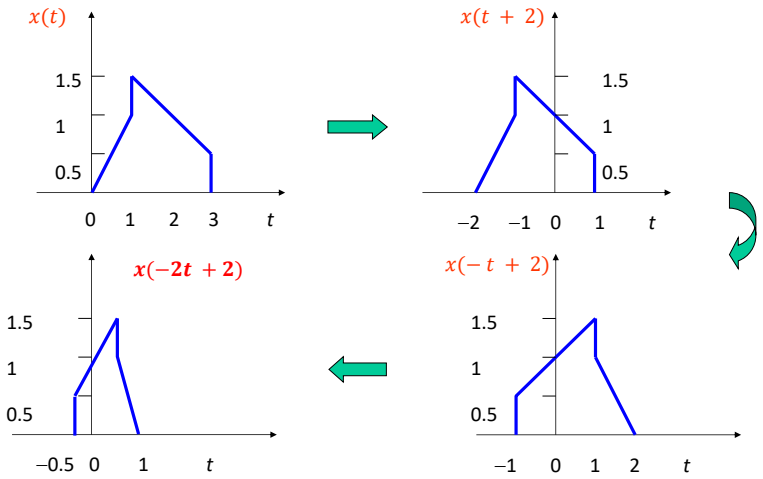
Manipulation (Time shift, Time reversal & Time scaling)

e.g. Given

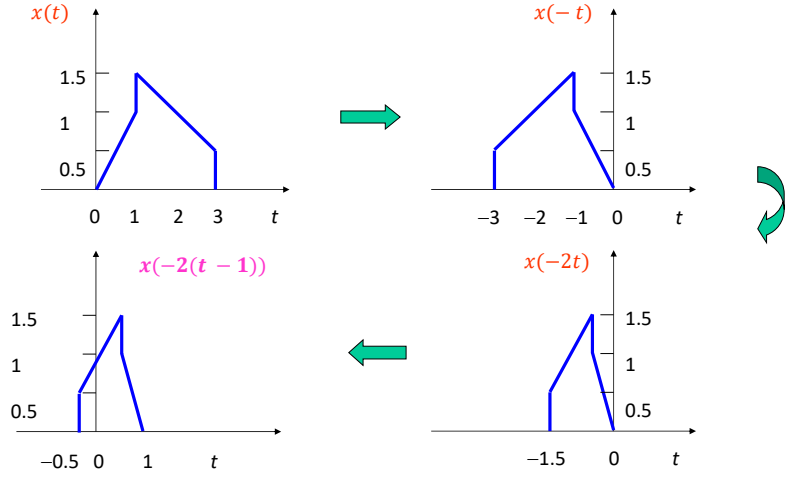


Sketch $x\,(-2t\,+\,2)$

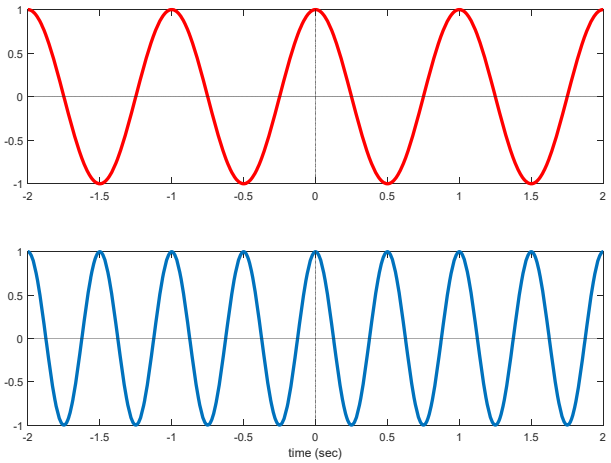
Time shift (FIRST step) → Time reversal & Time scaling



Time reversal & Time scaling → Time shift (LAST step)



e.g.



$\cos(\omega_o t)$

Compression or Expansion?

$\cos(2\omega_o t)$

Question : How about image ?

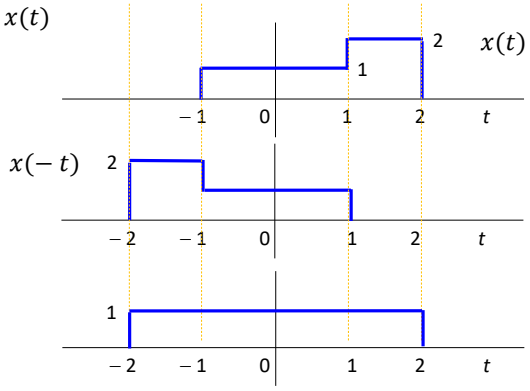
3200×2400

160×120

Meaning of those numbers ?

Even part & Odd part of a Signal

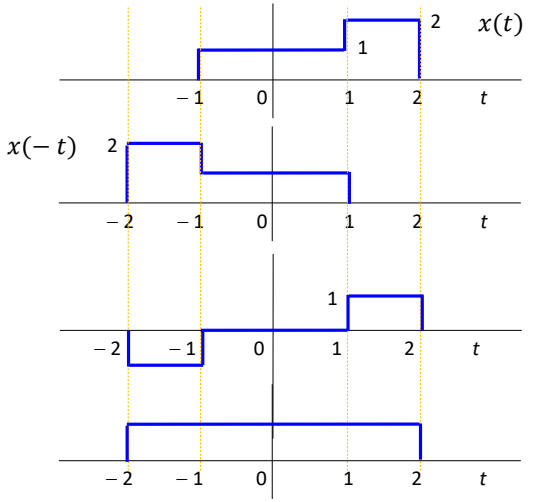
e.g. Sketch the even part of $x(t)$



Even part

$$\frac{x(t)+x(-t)}{2}$$

e.g. Sketch the odd part of $x(t)$



Odd part

$$\frac{x(t)-x(-t)}{2}$$

Periodic Signals (e.g. Real Sinusoid)

$x(t) = x(t \pm T)$ must be fundamental

$\omega_o = 2\pi f = \frac{2\pi}{T}$

rad/s Hz sec

e.g. Find the fundamental frequency and period.

$x_1(t) = \cos\left(\frac{2\pi}{4}t\right)$ $x_2(t) = \sin\left(\frac{2\pi}{3}t\right)$

$\cos(\omega_o t)$

$\sin(\omega_o t)$

e.g. Is $x_3(t) = x_1(t) + x_2(t)$ periodic ?

e.g. Find the fundamental frequency and period.

	$x_1(t) = \cos\left(\frac{2\pi}{4}t\right)$	$x_2(t) = \sin\left(\frac{2\pi}{3}t\right)$
fundamental angular frequency (in rad/s)	$\frac{2\pi}{4}$	$\frac{2\pi}{3}$
fundamental ordinary frequency (in Hz)	$\frac{1}{4}$	$\frac{1}{3}$
fundamental period (in sec)	4	3

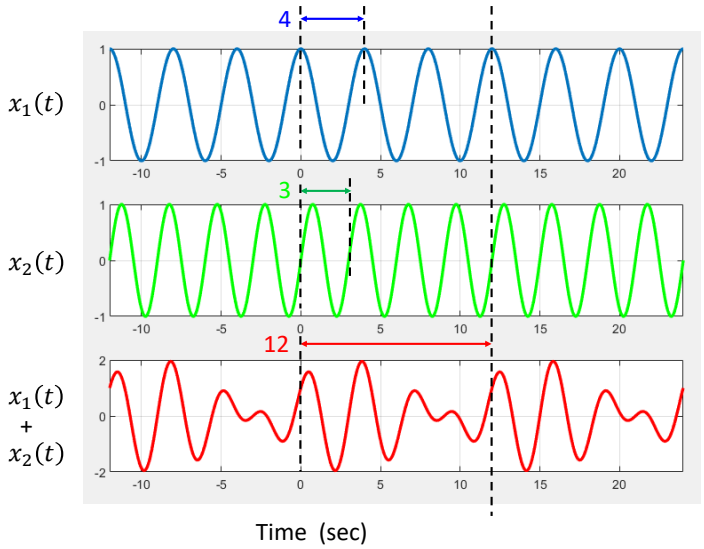
$\omega_o = 2\pi f = \frac{2\pi}{T}$

e.g. Is $x_3(t) = x_1(t) + x_2(t)$ periodic ?

LCM $(4\ ,\ 3\ ) = 12$

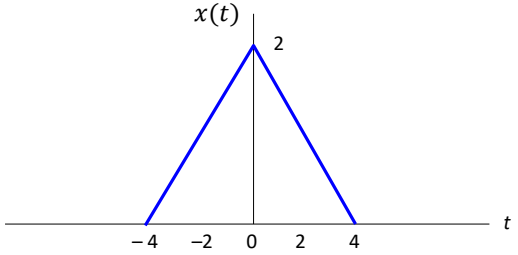
Question : Difference between fundamental period and period ?

shortest time for one period



Poisson Sum

e.g. Given :

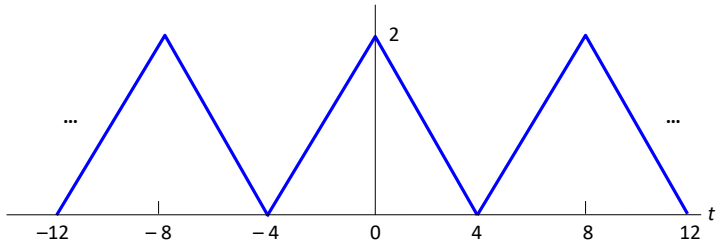


a) Plot $y_1(t) = \sum_{k=-\infty}^{\infty} x(t - 8k)$ Question : Fundamental period ?

b) Plot $y_2(t) = \sum_{k=-\infty}^{\infty} x(t - 6k)$ Question : Fundamental period ?

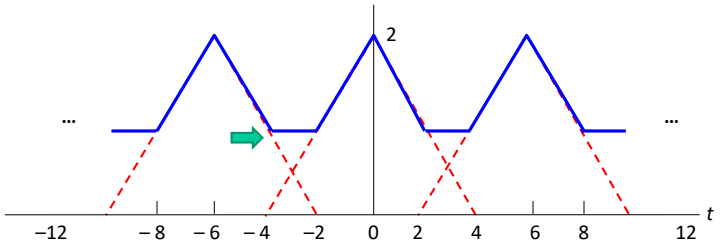
$$y_1(t) = \sum_{k=-\infty}^{\infty} x(t-8k) = x(t+\infty) + \cdots + \boldsymbol{x(t+8) + x(t) + x(t-8)} + \cdots + x(t-\infty)$$

$T = 8$



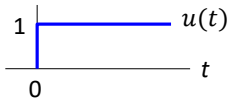
$$y_2(t) = \sum_{k=-\infty}^{\infty} x(t-6k) = x(t+\infty) + \cdots + \boldsymbol{x(t+6) + x(t) + x(t-6)} + \cdots + x(t-\infty)$$

$T = 6$



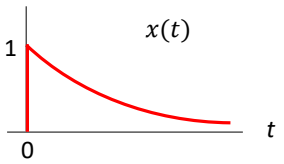
Real Exponential and Unit Step function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



Question : How to create a window function ?

e.g. $x(t) = e^{-2t}\,u(t)$



Question :
- Growing or decaying ?
- Left-sided or right-sided ?
- Finite duration or infinite duration ?

e.g. $x(t) = \begin{cases} 1 & 2 < t < 4 \\ 0 & otherwise \end{cases}$ = ?