Lecture Outline

Digital TM

Save energy

All signals are orthogonal!

- **Have previously considered**
  - Optimum detector structure.
  - The optimum receiver (in the sense of minimizing P_e) for general M-ary signaling in the presence of AWGN
  - Graphical interpretation of decision regions

» **Will now consider**
  - » Probability of error expressions
    - » Case study: Orthogonal signaling
  - » Union bound on Pe for generic M-ary modulations
  - » Orthogonal signaling & its variations

Prelude

Performance evaluation of M-ary modulation:

- Exact error probability computation is quite complicated
- Engineers typically look for good approximation that makes system analysis and design less complicated

Union Bound will be developed

- A common tool used by communication engineers
- Very easy to derive
- Can give accurate probability estimates

Orthogonal Signals are important

- Will show a method which can generate orthogonal signals based on binary pulses (Walsh functions) second architecture:

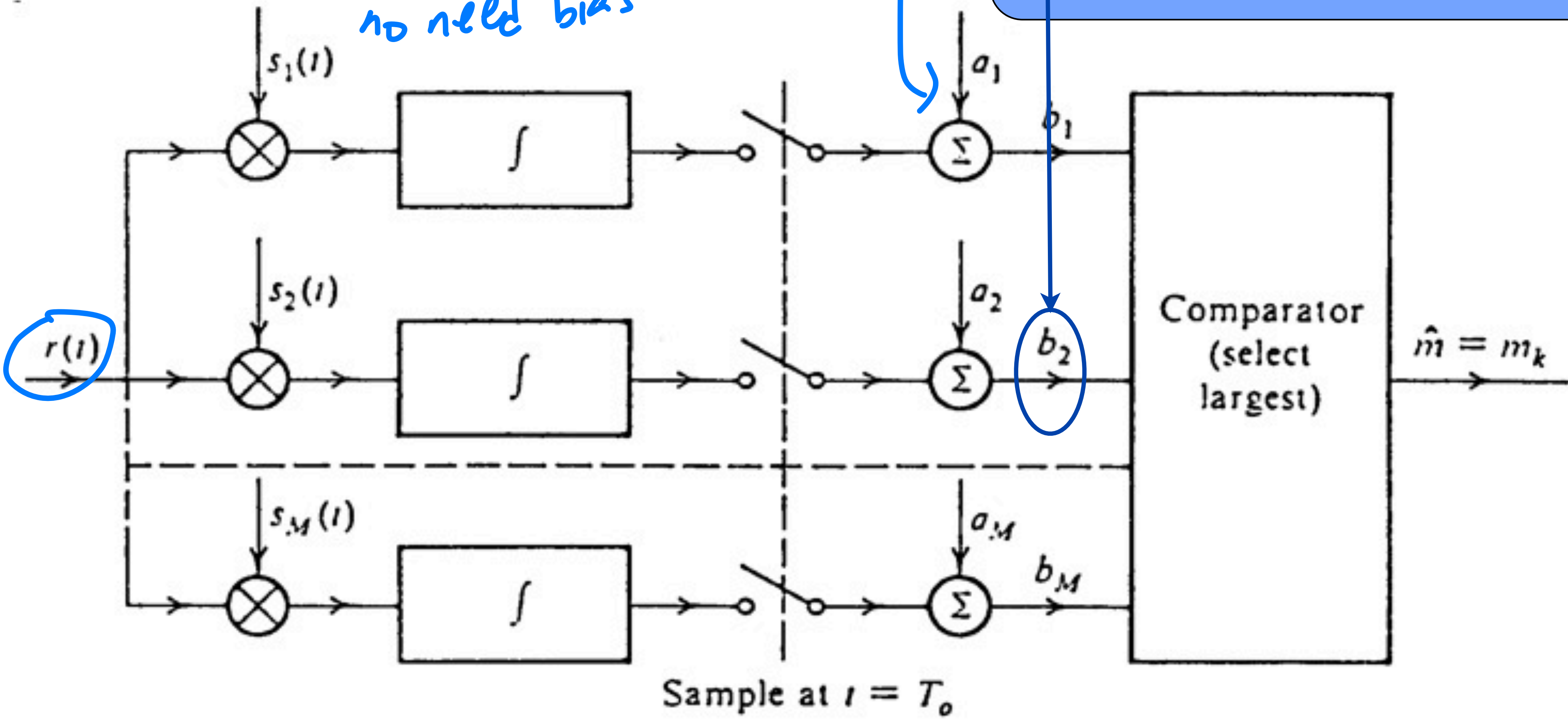
randomness

Recall the Optimum M-ary Receiver

bias

$$b_k = \int_0^{T_s} r(t)s_k(t)dt - E_k/2$$

if equal distance remove this!
no need bias



$r_1 = s_1(t) + n_1(t)$

$$b_1 = \int_0^{T_s} r(t)\, s_1(t)\, dt - \frac{E_1}{2} = \int_0^{T_s} s_1(t)^2\, dt + \int_0^{T_s} \langle s_1(t), n(t) \rangle\, dt - \frac{E_1}{2}$$

$= s_1(t)^2 + n_1$

$$b_2 = \int r(t)\, s_2(t)\, dt$$

$$= \int_0^{T_s} s_1(t)\, s_2(t)\, dt + \int_0^{T_s} n(t)\, s_2(t)\, dt$$

$\downarrow$

0

General Expression for P_e

If m₁ is sent, → correct decision is made only if $b_1 > b_2, b_3, \ldots, b_M$

$$P(c/m_1) = P(b_1 > b_2, b_3, \ldots, b_M / m_1)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{b_1} \ldots \int_{-\infty}^{b_1} f(b_1, b_2, \ldots, b_M / m_1)\, db_M\, db_{M-1} \ldots db_1$$

b_1 .... b_M

union bound!

→ $$P(c) = \sum_{j=1}^{M} P(c/m_j) P(m_j)$$

total probability!

and $$P_e = P_{eM} = 1 - P(c)$$

$$\int\int\int_A f(x_1 \cdots x_m)\, dx_1 \cdots x_m$$

① $f(x_1, x_2 \cdots x_m) = f(x_1) f(x_2) \cdots f(x_m)$

② $A = A_1 \times A_2 \times \cdots A_m$

$$I = \prod_{i=1}^{m} \left(\int f(x_i)\, dx_i \right)$$

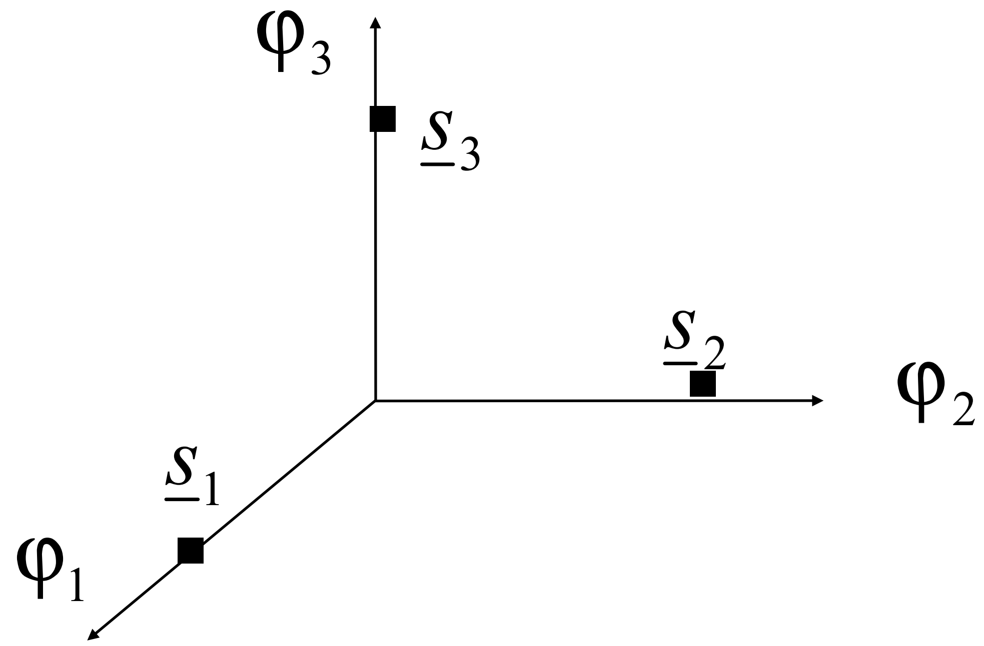
P_e for Orthogonal Signals

$$s_k(t) = \sqrt{E}\varphi_k(t) \qquad \underline{s}_j \cdot \underline{s}_k = \begin{cases} 0 & j \neq k \\ E & j = k \end{cases}$$

Assume equiprobable messages

$$P(m_k) = \frac{1}{M} \qquad \forall k = 1,2,\ldots,M$$

$E_s = E$



P_e for Orthogonal Signals

Now, since $\{s_k\}$ is an orthogonal set

$$b_k = \left\{ \begin{array}{cc} E + n_1 & \text{if k=1} \\ n_k & \text{otherwise} \end{array} \right\} \quad \text{where } n_k = \int_0^{T_s} n(t) s_k(t) dt$$

$\{n_k\}$ are i.i.d. Gaussian r.v.'s $\qquad \overline{n}_k = 0 \qquad E[n_k^2] = E[N_0/2] = \frac{N_0}{2}$

$$\Rightarrow \quad f(b_1, b_2, \ldots, b_M / m_1) = \frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_1 - E)^2}{N_0 E}} \prod_{k=2}^{M} \left[\frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_k)^2}{N_0 E}} \right]$$

$$\Rightarrow \quad P(c / m_1) = \frac{1}{\sqrt{\pi N_0 E}} \int_{-\infty}^{\infty} e^{-\frac{(b_1 - E)^2}{N_0 E}} \left\{ \prod_{k=2}^{M} \int_{-\infty}^{b_1} \frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_k)^2}{N_0 E}} db_k \right\} db_1$$

P_e for Orthogonal Signals

Now note that since signal set is geometrically symmetric:

$$\Rightarrow \quad P(c/m_1) = P(c/m_2) = \ldots = P(c/m_M)$$

$$\Rightarrow \quad P(c) = P(c/m_1)$$

and

$$P_e = P_{eM} = 1 - P(c)$$

P_e for Orthogonal Signals

Let $\lambda = \dfrac{E_s}{N_0}$ and $y = x + \sqrt{\dfrac{2E_s}{N_0}}$

Correct expression!

$$\Rightarrow \quad P_{eM} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{1 - Q[y]\}^{M-1} e^{-\frac{(y-\sqrt{2\lambda})^2}{2}} dy$$

very sensitive to one constellation diagram

Overall Probability of Symbol Error.

P_e for Orthogonal Signals

Let $$x = \frac{b_1 - E - a}{\sqrt{\frac{N_0 E}{2}}}$$ ➡ cannot simplify more!

$$P(c/m_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ 1 - Q\left[x + \sqrt{\frac{2E}{N_0}} \right] \right\}^{M-1} e^{-\frac{x^2}{2}} dx$$

Symbol energy vs bit energy

Next recall that

- # bits that can be represented by a set of M signals:

$$k = log_2(M)$$

difference symbol energy and bit energy:

- **Energy per bit**

$$E_b = \frac{E_s}{\log_2(M)} = \frac{E_s}{k}$$

$$\Rightarrow \quad \frac{2E_s}{N_0} = \frac{2kE_b}{N_0}$$

$$\frac{\bar{E}_s}{N_0} = \frac{k\bar{E}_b}{N_0}$$

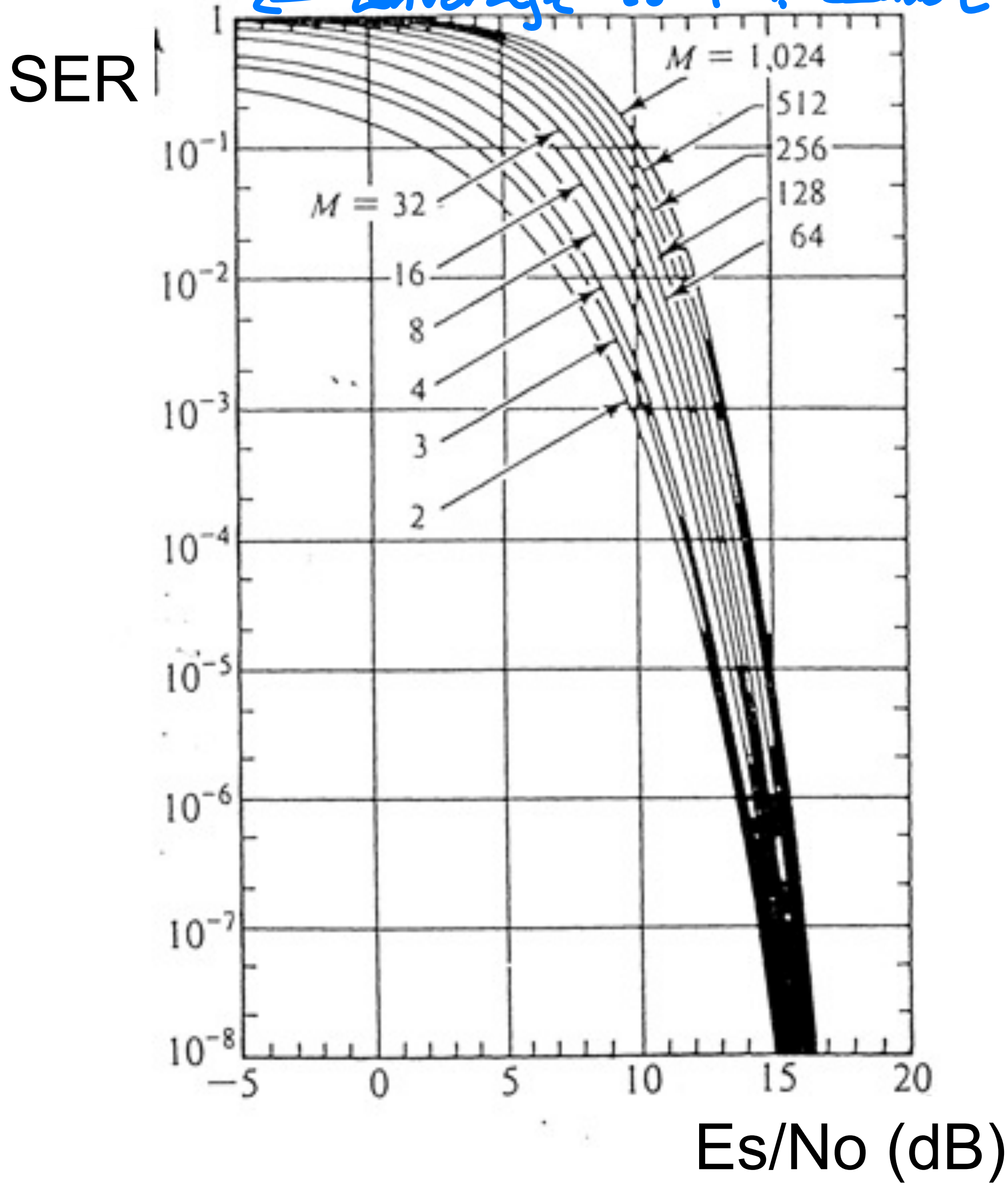
P_e = symbol error
≠
P_b = bit error!

↓

what the decoded bits
different from original bits!

Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Es/No

← converge to 1 ! cannot flip the bits!



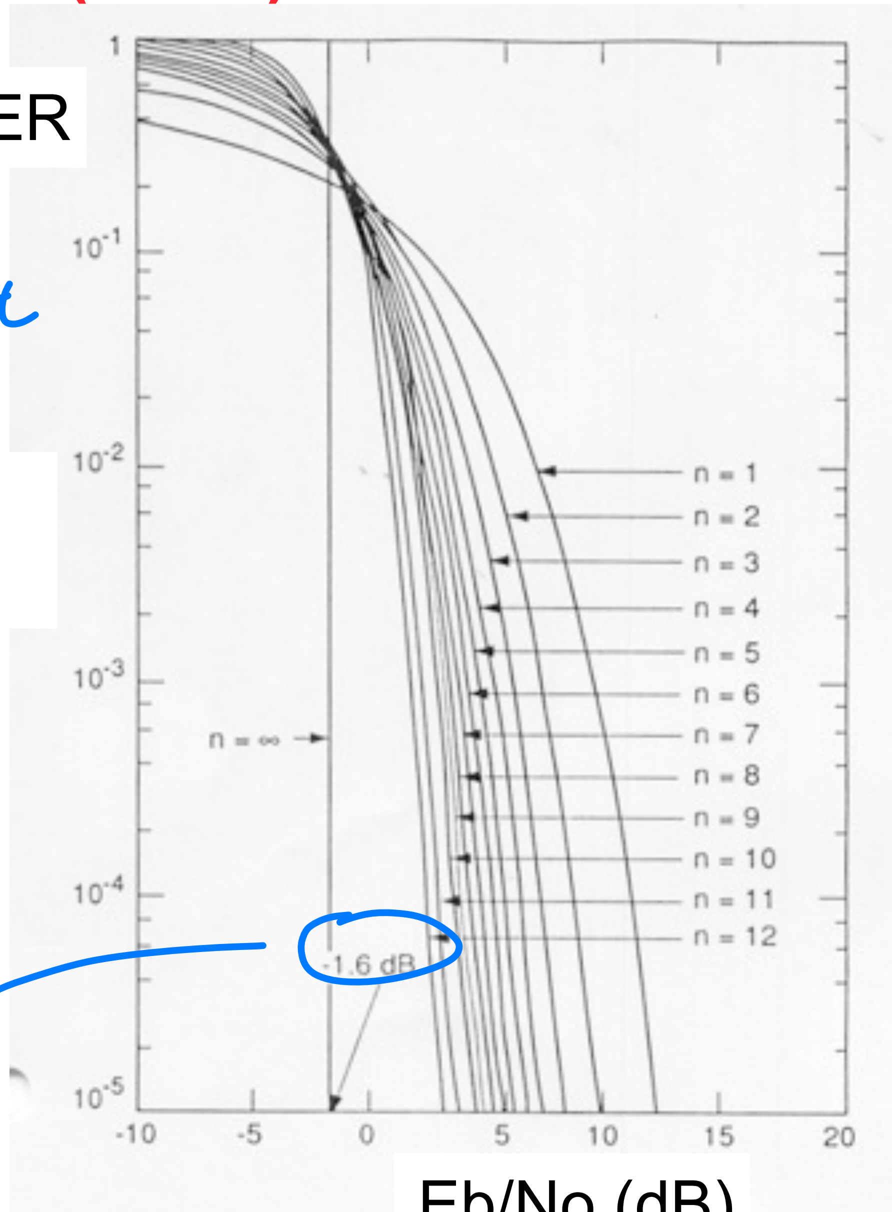
M↑ ⇒ SER worse!

Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No

MFSK is the only way to achieve this limit

like ber
if ber is
vertical line?

fundamental
limit!



$P_e \neq P_b$
↑
symbol error, ≥ 1
bits are incorrect!

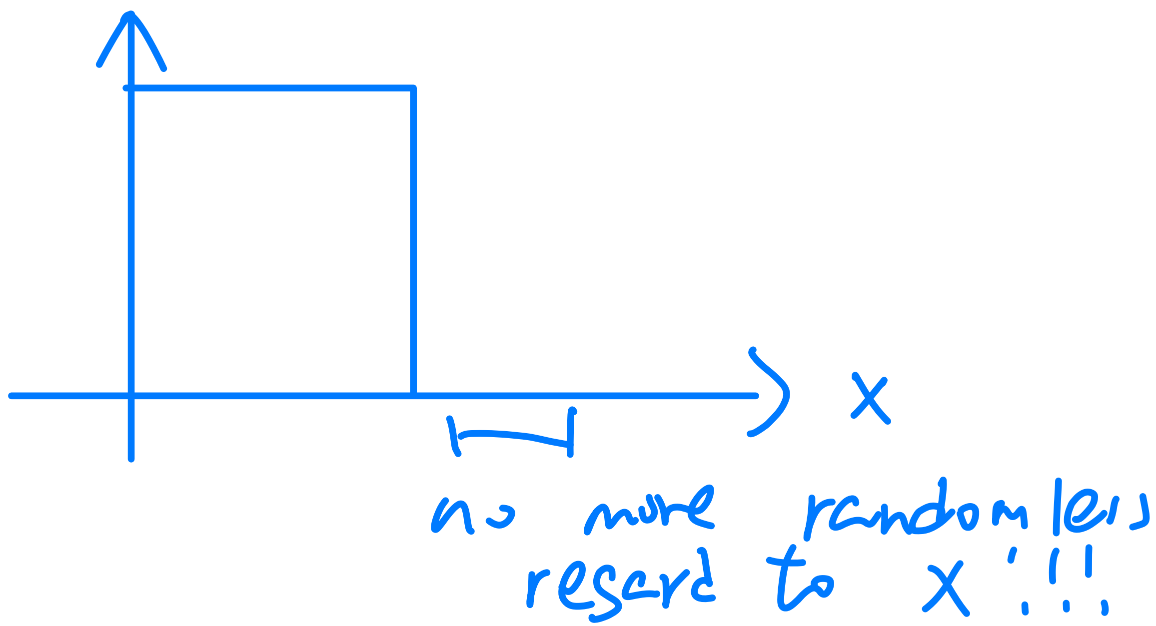
All symbols
will be incorrect!
if < -1.6 dB

less and less
Eb/No to
achieve same P_e
if n↑!

if $E_b/N_0 < -1.6\text{dB}$

$\Rightarrow$ symbol error rate $= 100\%$

$Pr(x \le x_0)$



omnibus passengers!

Symbol Errors to Bit Errors

MFSK

Symbol errors are different from bit errors.

When a symbol error occurs all $k=\log_2(M)$ bits could be in error

For orthogonal modulation when an error occurs anyone of the other M-1 symbols may result equally likely

On average therefore half the bits will be incorrect

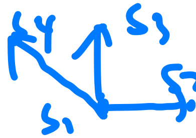
That is k/2 bits in error every k bits will on average be in error when there is a symbol error

Therefore for a particular bit the probability of error is half the symbol error

$$P_e \cong \frac{1}{2} P_{eM}$$

nice symmetric structure!

$= \frac{P_e}{M-1}$



Exact derivation

only for MFSK!

In orthogonal modulation when there is an error it will lead to any one of the other M-1=2^k-1 possible symbols equally. That is, when there is an error event the probability of a particular symbol getting that error is $P_{eM}/(M-1)$

$\bar{E}(n)$ = Av. # of bits error per symbol

For this given symbol error assume there are n bits in error

on average!

There are (k,n) combinations in which this may happen and therefore (k,n) symbols in total with a possible n bit errors.

$\binom{k}{n}\frac{P_{eM}}{k}$

Therefore the probability of a n bit errors occurring is $\boxed{\binom{k}{n}\frac{P_{eM}}{(M-1)}}$

Thus for every k bits there will be on average $\sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{(M-1)}$ bit errors

at most $\quad k = \log_2 M$

$E(n) = \sum_{n=1}^{k} n\,p(n)$

Therefore
$$P_b = P_e = \frac{1}{k}\sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{(M-1)} = \frac{M}{2(M-1)}P_{eM}$$
and for large M

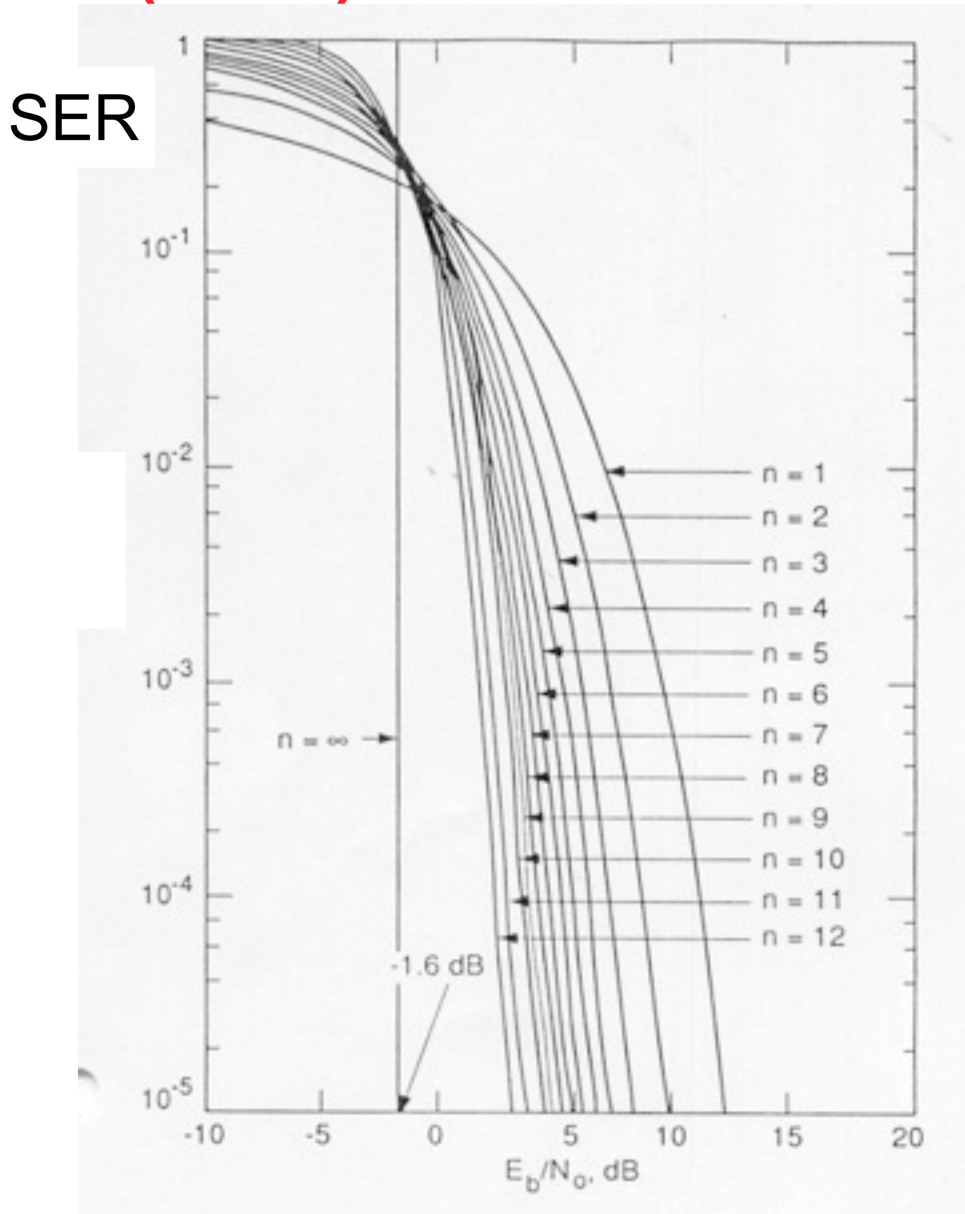
if $M \to \infty$

$$P_e \cong \frac{1}{2}P_{eM}$$

$$E(n) = \sum_{n=1}^{k} n\,p(n)$$

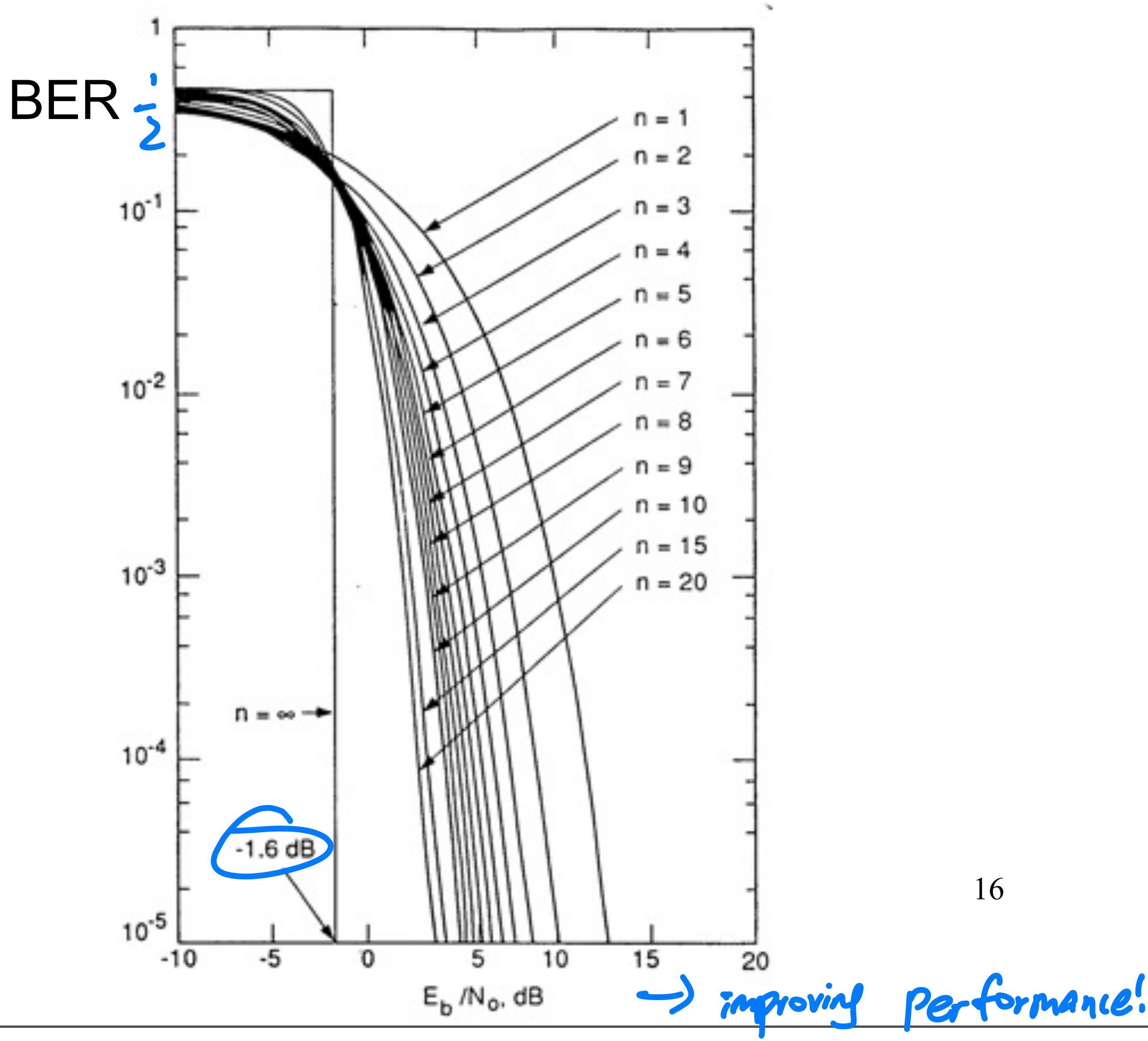
$$P_b = \frac{E(n)}{k}$$

Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No



performance improves!

Bit error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No



M-ary Communication
Performance Evaluation

So far, we have considered the performance evaluation of M-ary digital modulation.

- Typically involve the computation of error probability.
- Such computation is often very complicated.
- Derived the symbol error probability of orthogonal signals.
- Derived the bit error probability of orthogonal signals.

Bit error probability and **Symbol** error probability **ARE DIFFERENT**.

Bit errors

Signal errors

union bound is "generic"

Union Bound

$b_1 > b_2$

$b_1 > b_3 \cdots b_1 > b_M$

Multi-dimension integral and quite difficult to evaluate

$$P(error / m_j) = 1 - P(\bigcap (b_j > b_k / m_j \quad \forall k \neq j))$$

$$OR$$

$$P(error / m_j) = P(\bigcup (b_j \leq b_k / m_j \quad \forall k \neq j))$$

Now we simplify Pe calculation using an ***approximation*** known as the **union upper bound**

But note that

$$P(error / m_j) \leq \sum_{\substack{k=1 \\ k \neq j}}^{M} P(b_j \leq b_k / m_j)$$

$$P(\bigcup_i A_i) \leq \sum_i P(A_i)$$

$$P_{eM} = \sum_{j=1}^{M} \underbrace{p(s_j)}_{\frac{1}{M}} \, p(\text{error} \mid s_j) = \frac{1}{M} \sum_{j=1}^{M} p(\text{error} \mid s_j)$$

$$\le \frac{(M-1)}{\cancel{M}} \cancel{M} Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$\le (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$= p(\text{error} \mid s_j) = \Pr\left(\bigcup_{b \ne j} (b_j \le b_k \mid s_j)\right)$$

$$\le \Pr_{k \ne j} (b_j \le b_k \mid s_j)$$

if sit on s_j, can see $M-1$ neighbours!

$$Q\left(Q\sqrt{\frac{E_s}{N_0}}\right)$$

no bottomneck!

$$= (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Union Bound

The key approximation is that there may be several pairwise comparisons that imply the same symbol error

The union bound does not subtract out this intersecting possibility- therefore it is an upper bound

Now for equally likely symbols,

$$P_{eM} = \frac{1}{M} \sum_{j=1}^{M} P(error / m_j)$$

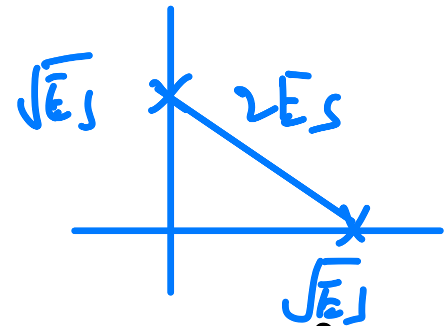
$$P_{eM} \le \frac{1}{M} \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k \ne j}}^{M} P(b_j \le b_k / m_j)$$

Pairwise error probability

Union Bound

Let $P_e(j,k)$ = Pairwise error probability for signals j and k

$$P_e(j,k) = Q\left[\sqrt{\frac{d_{kj}^2}{2N_0}}\right]$$



where $$d_{ij}^2 = \int_0^T \left[s_i(t) - s_j(t)\right]^2 dt = \left\|\underline{s}_i - \underline{s}_j\right\|^2 = 2E$$

$$P_{eM} \leq (M-1)Q\left[\sqrt{\frac{E}{N_0}}\right]$$

$$P_b = \frac{M}{2(M-1)} P_{eM}$$

$$\boxed{P_b \leq \frac{M}{2} Q\left[\sqrt{\frac{E}{N_0}}\right]}$$

Union Bound for Orthogonal signals

Also we can upper bound the Q function with (different from previous approximation)

intuition!

$$Q[x] \leq \frac{1}{2} e^{-\frac{x^2}{2}}$$ → Quite accurate for $x \geq 3$

exponentially by $\frac{E_b}{N_0}$

$$P_b \leq \frac{M}{4} e^{-\frac{E}{2N_0}}$$ → Widely used

bit error rate!

where

$$k = log_2(M)$$

$$E_b = \frac{E}{\log_2(M)} = \frac{E}{k}$$

$$P_b \approx \frac{M}{4} e^{-\frac{kE_b}{2N_0}}$$

Comparison of union bound with the exact P_e

Comparison of union bound with exact result for orthogonal signals

if want 精 ⇒ 不可用 union bound!

-1.6 dB [illegible] 其他数字

M	Exact $P(\varepsilon)$ ($E/N_0 = 18.2$)	Union bound ($E/N_0 = 18.2$)
2	10^{-5}	10^{-5}
4	2.9×10^{-5}	3×10^{-5}
8	6.9×10^{-5}	7×10^{-5}
16	1.45×10^{-4}	1.5×10^{-4}
32	2.70×10^{-4}	3.1×10^{-4}
64	5.1×10^{-4}	6.3×10^{-4}
128	1.1×10^{-3}	1.27×10^{-3}

Union Bound

- Have shown that the Union bound is a nice and good approximation.
- Must keep in mind that one does not know a priori (before hand) when such approximation is accurate or not.
- In general, SIMULATION is the practical alternative.
- Typical bit error rate is small. Hence, simulation can also take a very long time.