Lecture 1

Course Introduction and Complex Numbers

I. ELEC2100 Course Introduction

II. Definition and Examples of CT and DT Signals

III. Math Fundamentals and Complex Numbers

IV. Working with Complex Numbers

I. ELEC2100 Course Introduction

Learning Activities	Grading Rules	Total Percentage
Homework	Three homework assignments	18 %
Lab (Matlab simulation)	Four lab assignments with prelab exercise and explanatory session	12 %
Online Quiz	Bi-weekly (Five online quizzes)	10 %
Midterm	Tentatively scheduled on 9 April (Wednesday), 7pm – 9pm	25 %
Final	TBA	35 %

- **Requirements for getting a “PASS”**

  You will automatically **FAIL** if you do not attend the midterm or final!!!

  Total score : at least 40 % out of 100 %

- **No textbook**.

You may use Signals and Systems, Oppenheim, Willsky with Nawab (OWN), 1997, as a reference. But our course will follow a much more engineering-oriented approach. And, you can use Andrew D. Lewis, A Mathematical Introduction to Signals and Systems OR E.A. Lee and P. Varaiya, Structure and Interpretation of Signals and Systems to improve your math skill.

The ELEC Core Study Center which tentatively will be open every Wednesday and Friday 4:30-6pm at the ECE Student Common. Please stop by to ask any question or to chit chat.

What is ELEC2100 About?

- ELEC2100 is about learning the languages and techniques for modelling and analysis of a wide range of physical, engineering, and information processing systems.
- It is one of the four core courses for ELEC

Modelling and Analysis

ELEC2100 Signals and Systems
General concepts, modeling and analysis for a broad range of systems: circuits, control, communications, multimedia

ELEC2400 Electronic Circuits
Modeling & analysis for circuit systems

Design and Implementation

COMP2011 Programming with C++
High level design and programming

ELEC2350 Intro. Computer Organization
Low level hardware/software design and programming

What will you learn in ELEC2100?

Week 1 Introduction and Complex Number
Week 2 Basic Characterization and Manipulation of Signals
Week 3 The Impulse and the Complex Exponential
Week 4 Linear Time-Invariant Systems
Week 5 System Function and Frequency Response
Week 6 CTFS and DTFS
Week 7 CTFT and DTFT
Week 8 Wireless Communication
Week 9 Sampling and Digital Processing of Signals
Week 10 Modern Application Examples – Spectrum Analyzer, FFT and OFDM
Week 11 Differential Equations as LTI Systems
Week 12 Laplace Transform and System Characterization
Week 13 Laplace Transform Applications – Feedback Control, Filter Design and Implementation

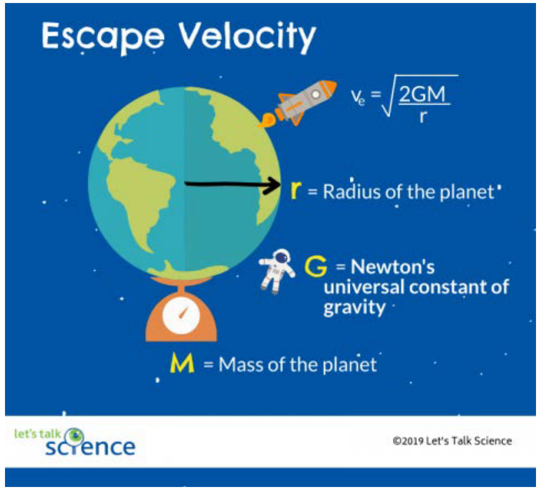
In **ELEC2100**, you will explore the fundamentals of **signal** and **system analysis**, from understanding **complex numbers** and **signal characterization** to mastering techniques like **Fourier transforms** and **Laplace transforms**. The course delves into real-world applications such as **wireless communication**, **spectrum analysis**, and **digital signal processing**, providing a robust foundation for advanced topics like **feedback control** and **filter design**.

What is Modeling?

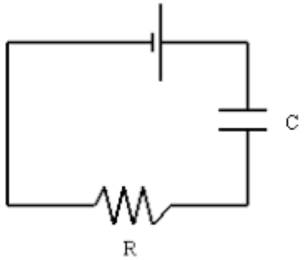
- **Modelling** is the use of mathematical relations, block diagrams, flow charts, etc. to describe the essential relationships, properties, and behaviours of real-world systems.
  - It is the first step in addressing any problem
  - Often, some simplifying assumptions are needed
  - Allows us to deduce additional useful properties and make predictions

Example Models

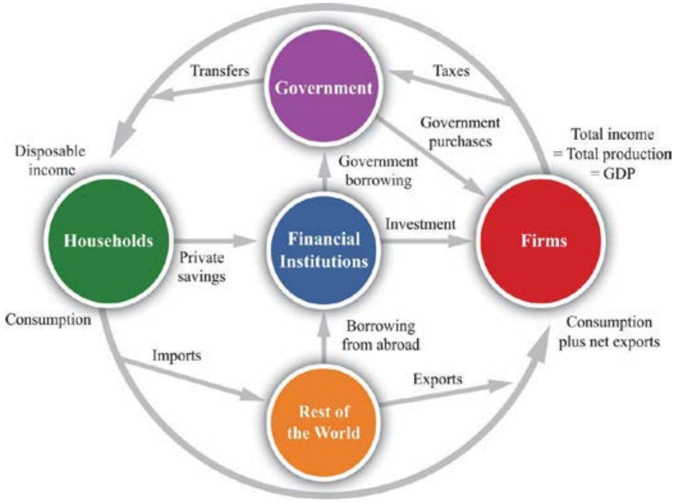
- ***Newtonian model***: $F = Ma$, $F = \frac{M_1 M_2}{r^2} G$
  - Cause and effect: Force and acceleration
  - Assumption: true regardless of mass and velocity
  - Deduction: escape velocity



- ***Circuit model***: $v(t) = i(t)R; i(t) = C\frac{dv(t)}{dt}$



- ***Money flow model***:



What is Analysis?

- **Analysis** is the understanding of something by identifying the simpler elements that something is made up of.
- What do we mean by a “simpler element”?
  - It means something that we can easily model
- In engineering, modelling and analysis enable us to *understand*, *deduct*, *predict*, and *control*

Languages and Techniques for Modelling and Analysis

- In ELEC2100, you will learn the mathematical relationships and English keywords as languages for describing real world relationships, properties, and behaviours.
- You will also learn the techniques for decomposing systems and signals into simpler elements and for putting these elements back together.
- You will learn about these basic elements – the proto-signals and proto-systems from which we can construct complex signals and systems.

Models in ELEC2100

- In ELEC2100 we will focus on one class of systems called *linear and time-invariant* (LTI) systems. LTI systems are widely applicable and thoroughly analysable. From the basic LTI assumptions we will deduce a full range of useful properties.
- From the properties we can predict and control the behaviour of many engineering systems: mechanical systems, signal propagation over communication channels, information transmission, audio and image processing, feedback control for robots – ELEC2100 covers everything in electrical engineering!
- Hence, there are four elements of learning in ELEC2100:
  1. The mathematical and English language for modelling signals and systems
  2. The deduction of properties from simple assumptions
  3. The analysis technique for decomposing signals and systems into simpler signals and systems
  4. Understanding of example engineering applications

In each part of a lecture, I will indicate which element of learning we are focusing on.

Mathematics for ELEC2100

- ELEC2100 is not about mathematical problem solving. It is about getting familiar with the mathematical symbols and relationships needed to describe models.
- The basic mathematical relationships needed in our models are summarized in **<u>Reference 1</u>** - “Key Mathematical Notations and Relationships”.
- The keywords for our models are summarized in **<u>Reference 2</u>** - “30 Key Terms in Signals and Systems with Key Concepts”.
- Most importantly, you must become thoroughly familiar with complex numbers.

Mathematical Problem Solving

Understanding of the languages and math relationships for modelling

Lecture 1

Course Introduction and Complex Numbers

I. ELEC2100 Course Introduction

II. Definition and Examples of CT and DT Signals (Language – keywords)

III. Math Fundamentals and Complex Numbers (Language - math)

IV. Working with Complex Numbers (Language - math)

II. Definitions and Examples of CT and DT Signals

What is signal?

- A signal is some measurable *quantity* that *varies over time and/or space*.
- Conveys information for decision making and control.
- Can be natural or man-made

WIFI Signal

Example 1: Acoustic Speech signal

Acoustic pressure variations as a function of time that leads to our perception of sound and speech.

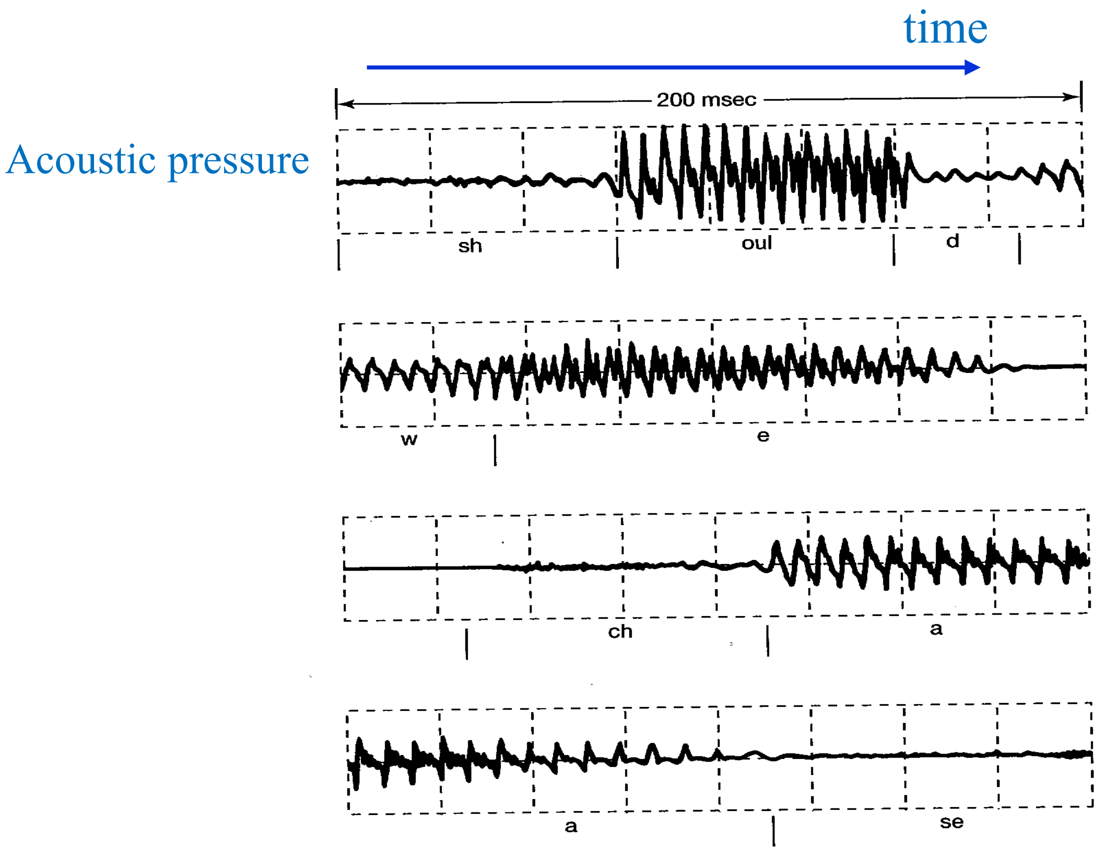


Figure 1.3 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two lines to the word "chase." (We have indicated the approximate beginnings and endings of each successive sound in each word.)

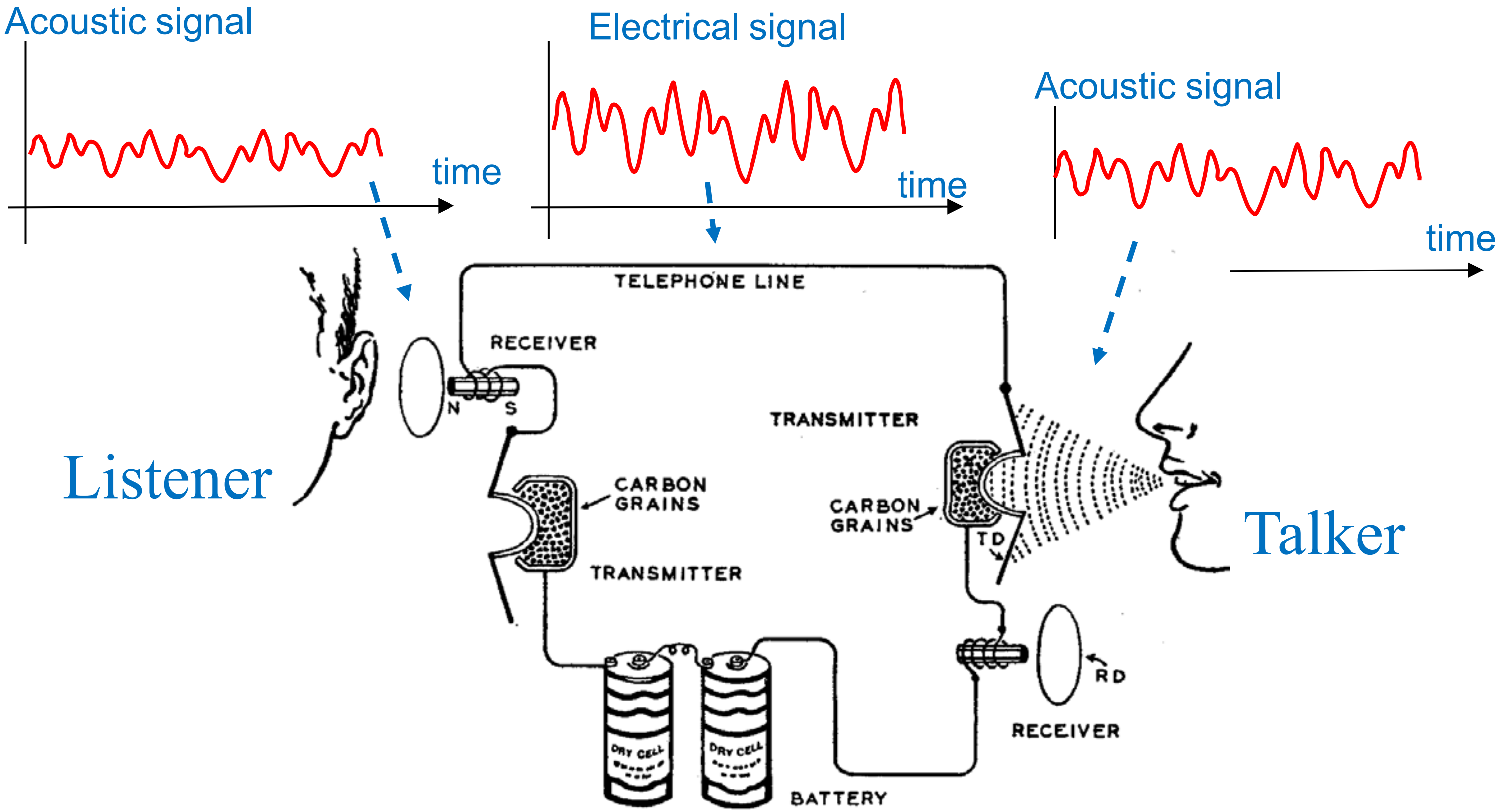
Example 2: Electrical Signal

Variation in electric current/voltage as a function of time.

We began to use electrical signals for communications since the invention of the telegraph in 1842.

In 1876, Alexander Graham Bell invented the telephone, which worked by converting the acoustic speech signal into a varying electrical current that could be transmitted over much greater distances through copper wires.

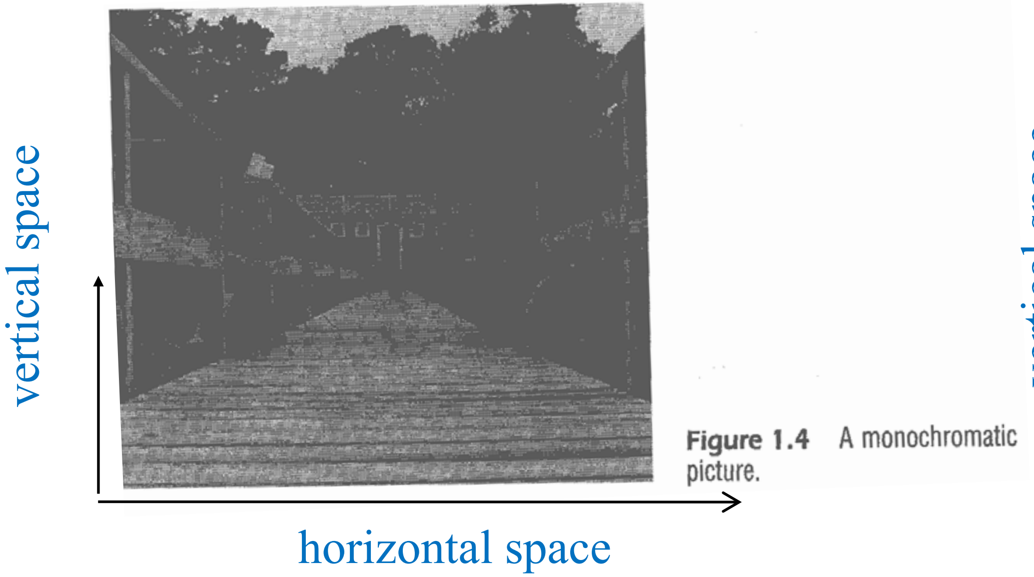
Bell and his Telephone

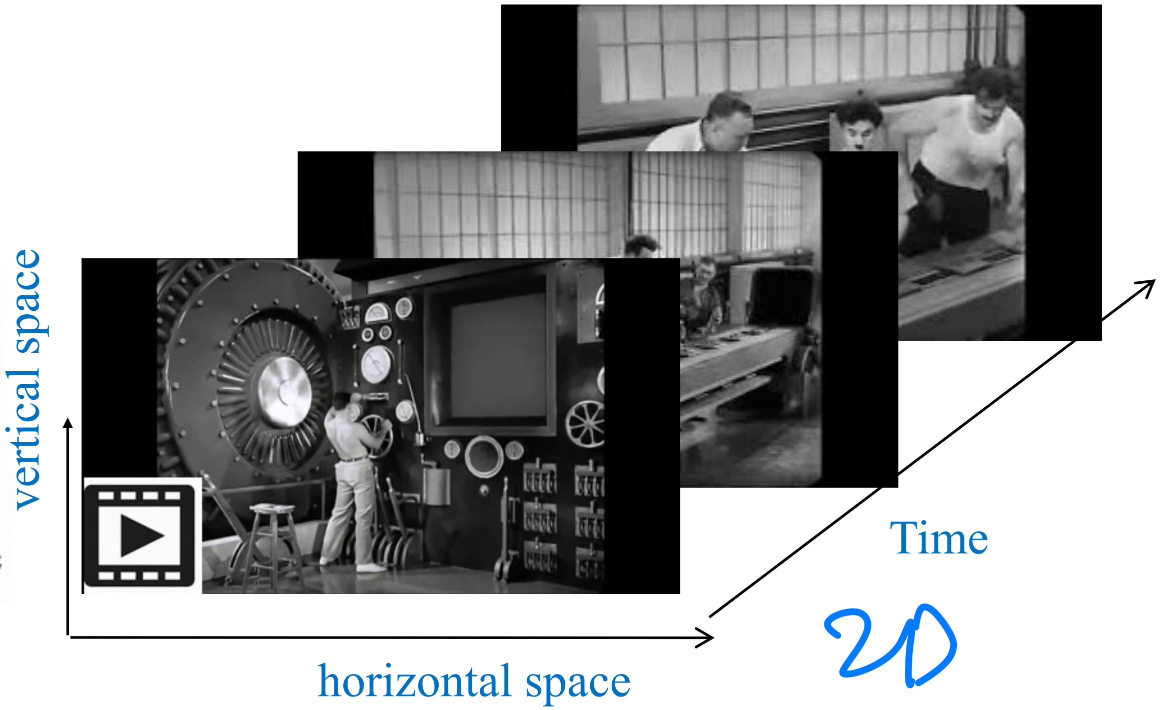


Example 3: Black-and-White Pictures and Videos

- Variation in brightness as a function of 2-dimensional space.
- Called a 2-dimensional signal because there are two “independent variables” – horizontal and vertical space.
- Video: Sequence of pictures in time. 3-dimensional – two dimensions of space and one dimension of time.

Black-and-white Picture



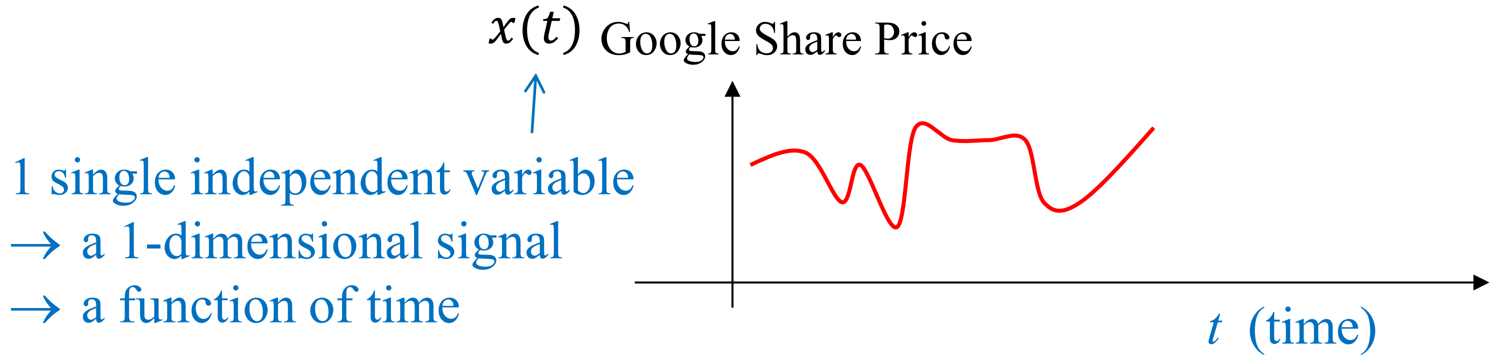


2D

ELEC2100: Signal as Function/Mapping

- We represent signal mathematically as a *function* or *mapping* $x(a_1, a_2, \dots a_n)$
- Function value $x(\cdot)$ is the quantity of interest – pressure, voltage, brightness, etc., and $a_1, a_2, \dots a_n$ are the *independent variables* - time, space, etc. - that the quantity depends on.
- In ELEC2100, we focus on 1-dimensional signal, where there is only one independent variable. To keep discussion simple, we assume this independent variable is "***time***".
- *Hence, in ELEC2100, signals are simply* ***functions of time***.

Example: signal $x(t)$ = share price of Google as a function of time



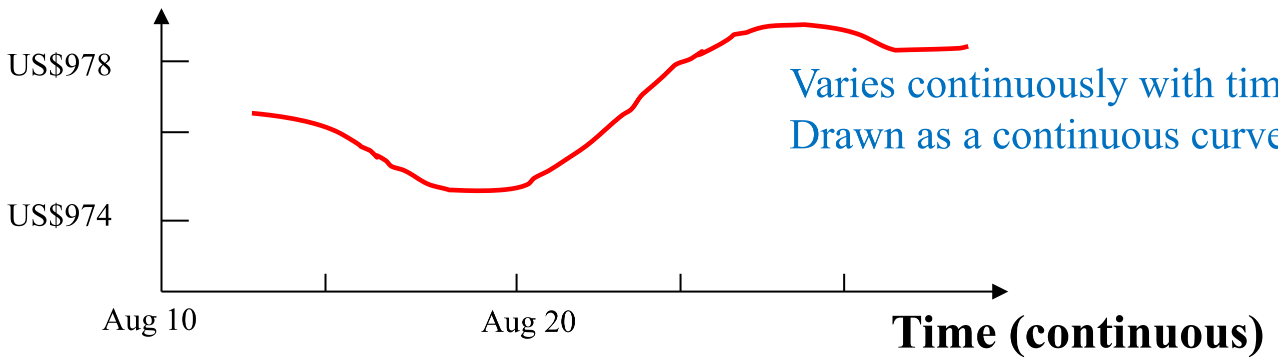
Continuous-Time (CT) and Discrete-Time (DT)

- In the physical world, "time" is typically regarded as a continuous variable.
- Today, however, signals are often processed by computers, which perform computations only at discrete instances of time.
- Therefore, we need both a ***continuous-time*** (**CT**) and a ***discrete-time*** (**DT**) model
- For example, we may model Google's share price as either a CT or DT signal.
  - In the CT model, the share price *varies continuously* with time.
  - In DT, the share price is a *discrete sequence of values* sampled at some given time interval.

CT and DT Model of a Signal

CT Signal

Google share price as a continuous function of time

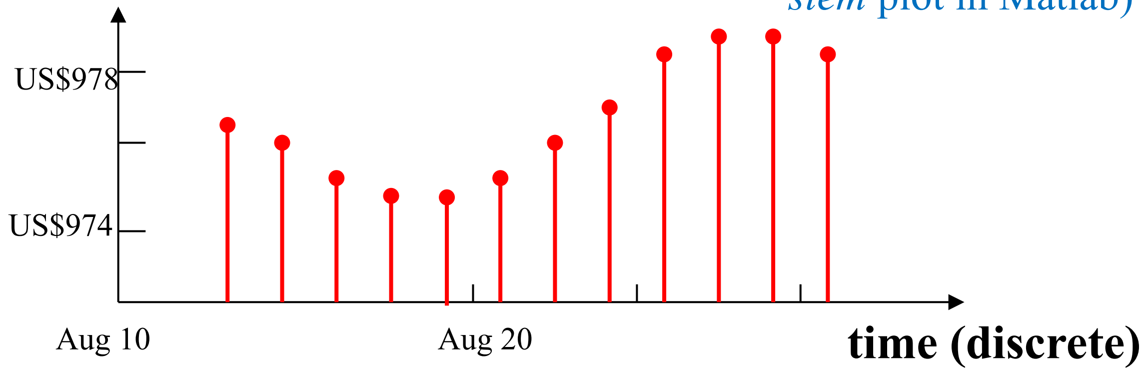


Varies continuously with time: Drawn as a continuous curve

A discrete sequence of values: Plotted as a stick diagram (A *stem* plot in Matlab)

DT Signal

Google share price as a sequence of values



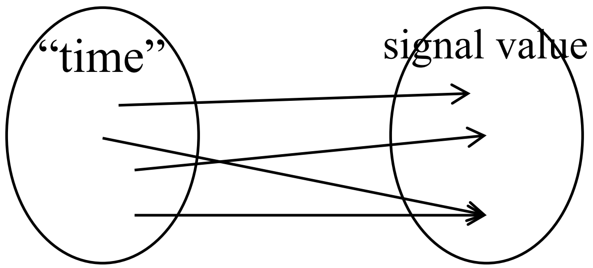
- We need to understand the linkages between the CT and DT models and be able to convert back-and-forth between them.

Signals as Complex-Valued Functions

- In advanced science and engineering, treating physical quantities as ***complex-valued*** will make our mathematics much more ***concise*** and ***powerful***.
- Therefore, we will treat 1-dimensional signal as a ***function/mapping*** of "time", which is either a real number (CT) or an integer (DT), to a complex number.

Signal as mapping/function

Domain:
A set of real numbers (CT)
or integers (DT)



Range:
The set of complex numbers

In ELEC2100, we will use the terms "*signal*", "*function*", "*mapping*" interchangeably.

CT and DT Signal as Function/Mapping

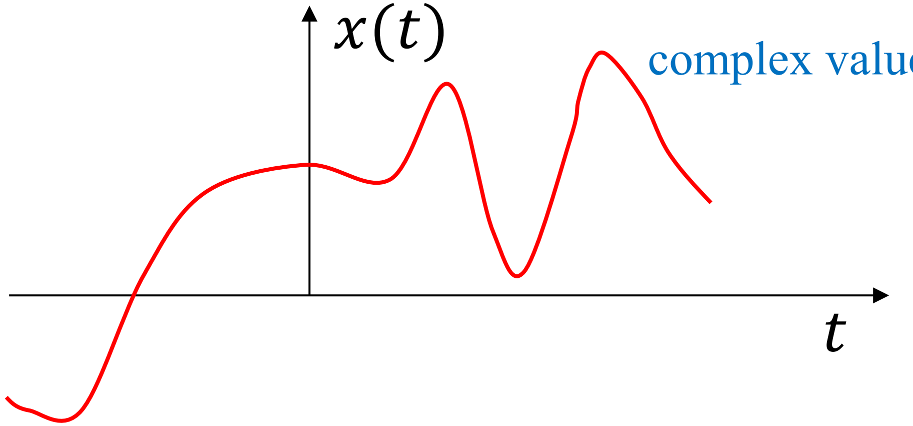
- We will use $x(t)$, $x[n]$ to refer to CT and DT signals respectively.

CT Signal: Mapping of a real number to a complex number – a complex-valued function of continuous time

t: real number

$x(t)$

Parenthesis to mean Continuous-Time

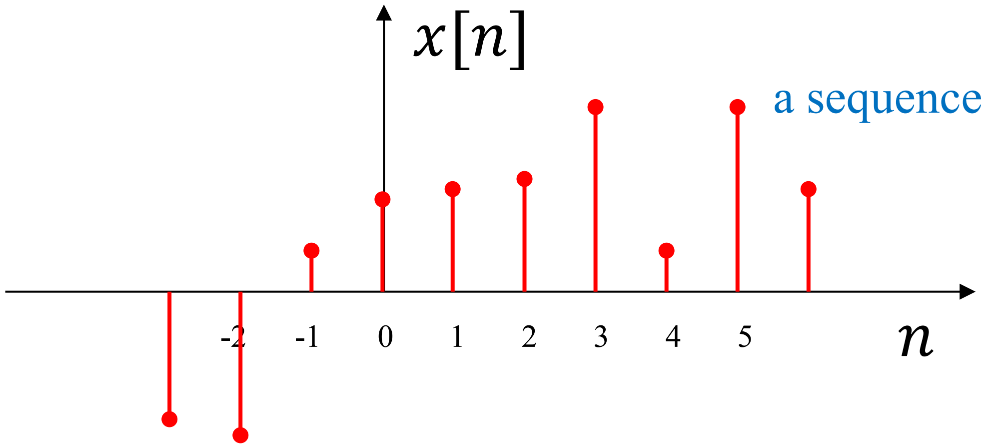


DT Signal: Mapping of an integer to a complex number – a complex-valued function of discrete time

n: integer

$x[n]$

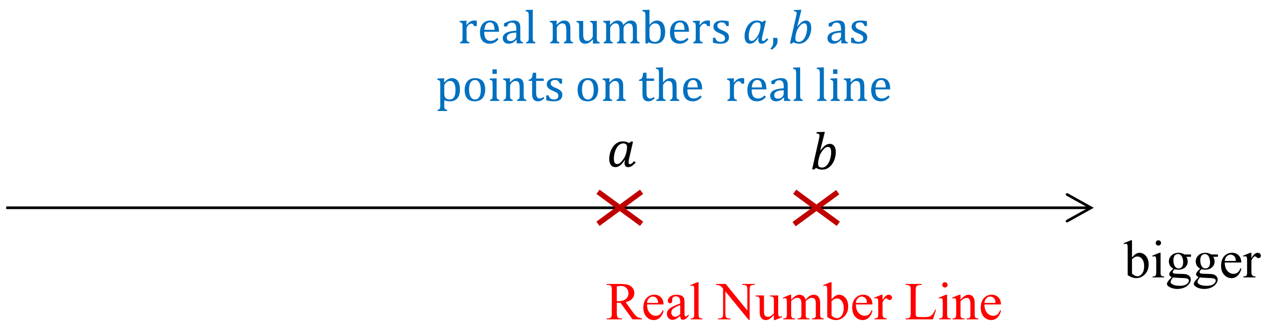
Square bracket to mean Discrete-Time. For further clarity, we use n instead of t to notate integer time



III. Math Fundamentals and Complex Numbers

First, let us review the concept of ***real numbers***, a concept that was not well-developed until the 18th century.

- Today, the set of real numbers, $\mathbb{R}$, can be viewed as all points alone an infinite line called the ***real number line***, which allows us to order all real numbers and measure "bigness".



Subsets of Real Numbers

- The set of real numbers contains these subsets:

- ***Natural numbers***: non-negative (or positive) integers used for counting $0, 1, 2, 3, 4 \ldots \ldots.$ or $1, 2, 3, 4 \ldots \ldots.$

- ***Integers***: non-negative & negative integers $0, \pm 1, \pm 2, \pm 3, \pm 4 \ldots \ldots.$

- ***Rational numbers***: that can be represented as a fraction of two integers $\frac{n}{m}$: n, m integers

- ***Irrational numbers***: cannot be represented as a fraction of two integers. Numbers such as $\sqrt{2}$, π and e have been proven to be irrational. There are infinitely more irrational numbers than rational numbers.

Addition, Multiplication, Exponentiation, and their Inverses

- The set of real numbers is ***closed*** under addition, additive inverse (i.e., subtraction), multiplication, multiplicative inverse (i.e., division):

if $a, b \in \mathbb{R}$, then

$$a + b \in \mathbb{R};$$
$$a - b \in \mathbb{R};$$
$$a \times b \in \mathbb{R};$$
$$a \div b \in \mathbb{R};$$

A set being "closed" means if you apply a given mathematical operation on elements of the set, the result always falls within the same set

- However, exponentiation a^b creates some problems. There is no answer for $(-4)^{\frac{1}{2}}$ in $\mathbb{R}$.

The Quadratic Formula

- The quadratic formula, which provides the roots to a 2nd order polynomial, was known in India and the Muslim world by the middle ages:

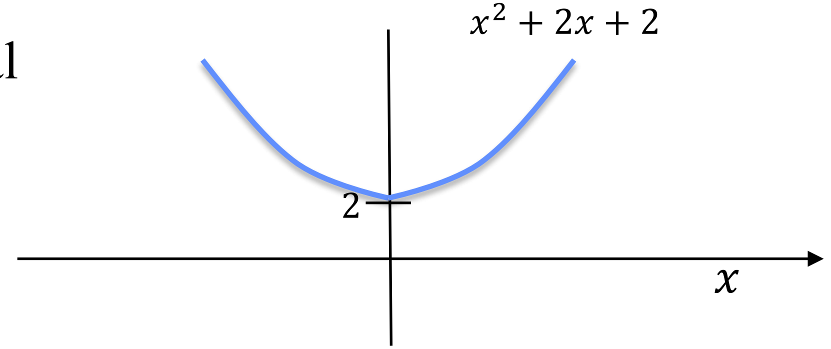
  The roots for $ax^2 + bx + c = 0$ are $\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- We can divide both sides of the equation to obtain $x^2 + a_1 x + a_0 = 0$ and express the roots as: $\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_0}}{2}$

- In secondary school, you learned that if $a_0 > {a_1}^2/4$, the quadratic equation has no root.

  For example, if $x^2 + 2x + 2 = 0$, then $\alpha_1, \alpha_2 = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \sqrt{-1}$

  and there is no solution if you insist that the solution must be real



The Unit Imaginary Number $j = \sqrt{-1}$

- In the 1500's, mathematics worked on solving 3^{rd} and 4^{th} order polynomials in great secrecy because of military implications (e.g., cannon ball trajectory)
- They discovered that if they allowed the ***unit imaginary number*** $\sqrt{-1}$ and ***complex numbers*** in the form of $a + b(\sqrt{-1}\,)$ to exist in some steps, following the rules of algebra, some 3^{rd} and 4^{th} order polynomials could be solved (e.g., $x^3 - 15x - 4 = 0$)
- Still, for the next 100 years, mathematicians remained uncomfortable with imaginary numbers and were not sure what they meant.
- We will use the letter j to represent $\sqrt{-1}$.

SAMPLE OF j

Example 1: Basic Multiplication with j
Simplify j^2.
Solution:

Example 2: Solving a Simple Quadratic
Solve $x^2 + 1 = 0$.
Solution:

$x = \pm\sqrt{-1}$

Example 3: Multiplication of Complex Numbers
Multiply $(2 + j) \times (1 - j)$.
Solution:

Mathematical Series

- In the 1600's, calculus (differentiation, integration) and various mathematical series and limits were developed.
- The Taylor series expansion of a function $f(\cdot)$ around a fixed point a is:

$$f(\alpha) = \sum_{n=0}^{\infty} \left.\frac{d^n f(\alpha)}{d\alpha^n}\right|_{\alpha=a} \frac{(\alpha - a)^n}{n!}$$

- If we set a=0, the Taylor series become a special case called a Maclaurin series:

here we use another notation for the n-th derivative

$$\Longrightarrow \qquad f(\alpha) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{\alpha^n}{n!}$$

Maclaurin series

- Basically our calculators compute function values using the Taylor series

The Number e

- In 1683, Jacob Bernoulli, working on the compound interest problem, discovered the fundamental number e, later called the *Euler Number*, as the value of the following limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

- It was soon found that the derivative of an exponential function with base e is the same exponential function: $$\frac{de^\alpha}{d\alpha} = e^\alpha$$

- Thus, all derivatives of e^α is e^α, and evaluates to 1 at $\alpha = 0$: $\left.\frac{d^n e^\alpha}{d\alpha^n}\right|_{\alpha=0} = e^\alpha|_{\alpha=0} = 1$

- Hence the Taylor series expansion of the exponential function is:

$$e^\alpha = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}$$

The Number e in real world

1. Increasing the Frequency of Interest Calculation

- Imagine the bank calculates interest twice a year (semi-annually). After six months, your money grows to:

$$100 \times (1 + 0.05) = 105 \text{ dollars}$$

Then, for the next six months, interest is calculated on this new amount:

$$105 \times (1 + 0.05) = 110.25 \text{ dollars}$$

Now, it's slightly more than 110 dollars! This is because your money is being compounded more frequently.

2. Further Increasing the Frequency

- If the bank calculates interest quarterly (every three months), you would get:

$$100 \times \left(1 + \frac{0.1}{4}\right)^4 = 110.38 \text{ dollars}$$

- If interest is calculated monthly:

$$100 \times \left(1 + \frac{0.1}{12}\right)^{12} = 110.47 \text{ dollars}$$

You can see that the more frequently the bank compounds interest, the more money you'll have at the end. However, this increase is not unlimited.

3. In the Extreme Case: Infinite Frequency

- Suppose the bank calculates interest every second or even at an infinitely small interval. In this scenario, we reach the mathematical formula:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

As n approaches infinity, the result stabilizes at a specific value, which is e, approximately equal to 2.71828 .

In practical terms, if the bank compounds your interest at infinite frequency, your initial 100 dollars after one year will become:

$$100 \times e^{0.1} \approx 110.52 \text{ dollars}$$

Imagine this: you deposit $100 into a bank that offers a 10% interest rate. If the bank calculates interest only once at the end of the year, after one year you'll have:

$$100 \times (1 + 0.1) = 110 \text{ dollars}$$

Simple, right? But what if the bank is willing to calculate the interest more frequently?

Euler's Formula

- In the 1740's, Leonard Euler decided to play with the Taylor series of e^{α} by making α imaginary.
- He discovered the ***Euler's formula*** which brings together algebra and geometry:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

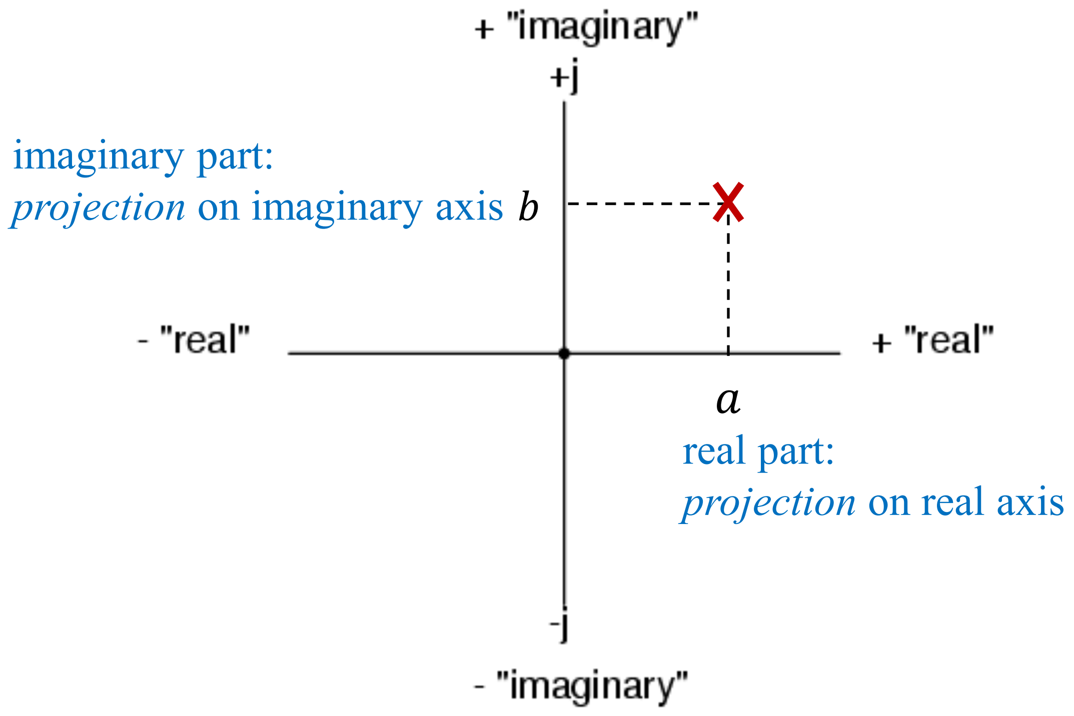
geometric functions

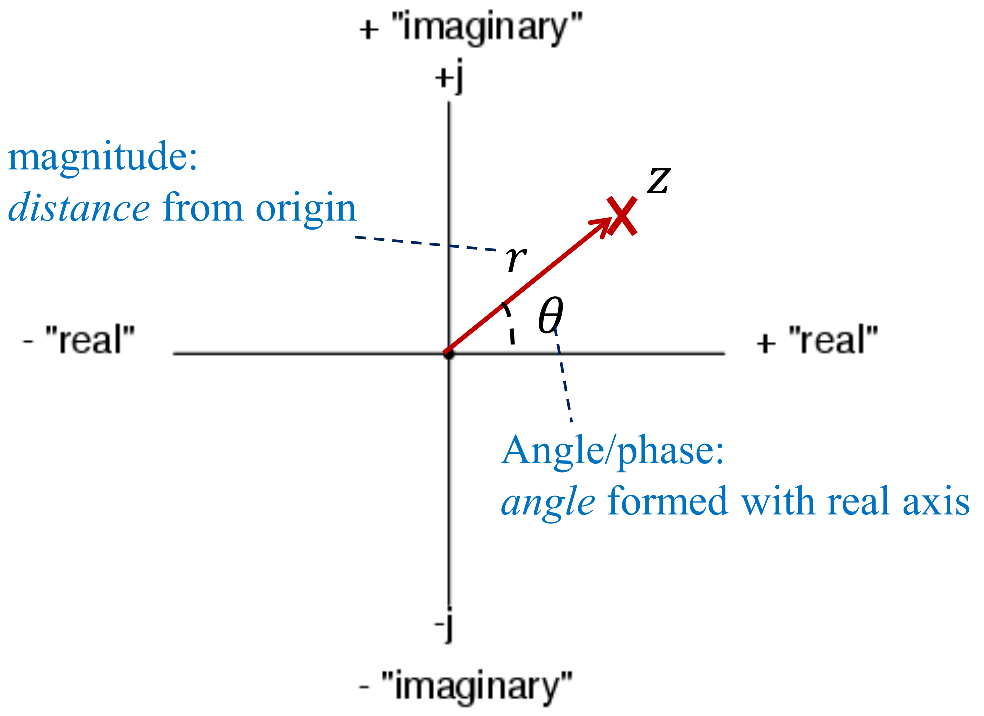
where $j = \sqrt{-1}$ and θ is real. (Proof in Appendix 1.1)

- Since $\cos\theta$ and $\sin\theta$ are the lengths of the two perpendicular sides of a right triangle, now we can think of j as being "perpendicular" to the real number line and complex numbers as points on a 2-D ***complex plane***, with the real number line and imaginary line as the two perpendicular axes.
- With the complex number, the mathematic operations of "+", "×" , exponentiation and their inverses ("−", "÷", root) are now all defined and can be thought of as how we move points on the complex plane!
- *All modern communications, signal processing, circuits, control, computations depend on complex numbers!*

The Complex Plane

- With Euler's formula, we can now view complex numbers as points on a two-dimensional ***complex plane***, with a vertical "***imaginary axis***" perpendicular to the horizontal "***real axis***".
- We can specify each point on the 2-D complex plane by its real and imaginary parts, (a, b), which is its projection on the real and the imaginary axis.
- Or we can specify each point by its magnitude and phase, (r, θ), which are its distance and angle with respect to the origin





Complex Number in Rectangular and Exponential Forms

- In ELEC2100, we will express complex numbers as $z = a + bj$ or $z = re^{j\theta}$
- $z = a + bj$, where a, b are real, is the **rectangular form**. It is the only form that existed before Euler.
  - a is called the ***real part*** and we notate it as $a = Re\{z\}$
  - b is called the ***imaginary part*** and we notate it as $b = Im\{z\}$
- $z = re^{j\theta}$, where r, θ are real and θ is in radian, is the **polar** or **exponential form**.
  - r is called the ***magnitude*** and we notate it as $r = |z|$.
    Magnitude is always non-negative; i.e., $r \geq 0$
  - θ is called the ***angle*** or ***phase*** and we notate it as $\theta = \sphericalangle z$
    Angle is on a *circular domain*; i.e., θ and $\theta + 2\pi$ are the same angle
    $e^{j\theta}$ describes the *unit circle*, i.e., $|e^{j\theta}| = 1$
- In ELEC2400, $r\angle\theta$ means $re^{j\theta}$

Converting Between Rectangular and Exponential/Polar Forms

- From the real and imaginary parts, we can determine the magnitude and phase:
  - The magnitude of $z = a + bj$ is: $|z| = \sqrt{a^2 + b^2}$

    Pythagoras theorem; real and imaginary axes form right angle
  - The phase of z is: $\sphericalangle z = \theta = \arctan\frac{b}{a} = tan^{-1}\frac{b}{a}$

    Be sure to retain the signs of a & b individually

    E.g., let $z = -3 - 2j$

    Then, let $\sphericalangle z = arctan\frac{(-2)}{(-3)}$ which indicates the angle is in the 3rd quadrant

    $\neq arctan\frac{2}{3}$ which is in the 1st quadrant

$\sphericalangle$: spherical angle

- From the magnitude and phase, we can determine the real part and imaginary part:
  - The real part of z is: $a = Re\{z\} = r\cos(\theta)$
  - The imaginary part of z is: $b = Im\{z\} = r\sin(\theta)$

IV. Working with Complex Numbers

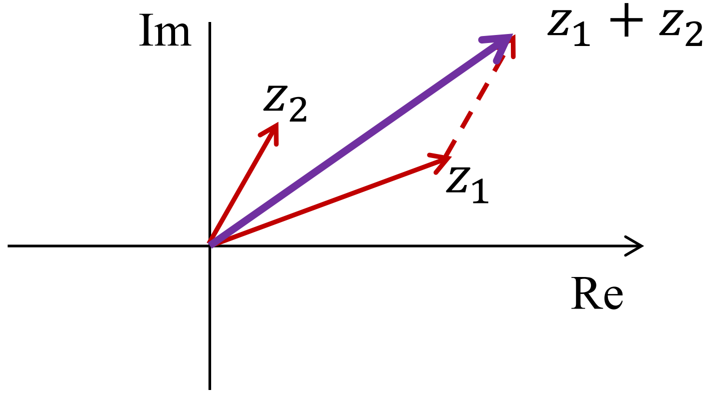
1. Adding Complex Numbers

Just add the real parts and the imaginary parts respectively

$$z_1 = a_1 + b_1 j; \quad z_2 = a_2 + b_2 j$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) j$$

- "+" of two complex numbers is like adding two vectors on the 2-D plane



2. Multiplying Complex Numbers

Multiply their magnitudes and add their phases:

Rotation by θ_2

$$z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

Magnitude scaling by r_2

- "×" of a complex number by another is a scaling in magnitude with rotation in phase

Sometimes we also multiply in rectangular form:

$$z_1 z_2 = (a_1 + b_1 j)(a_2 + b_2 j) = a_1 a_2 - b_1 b_2 + a_1 b_2 j + a_2 b_1 j$$

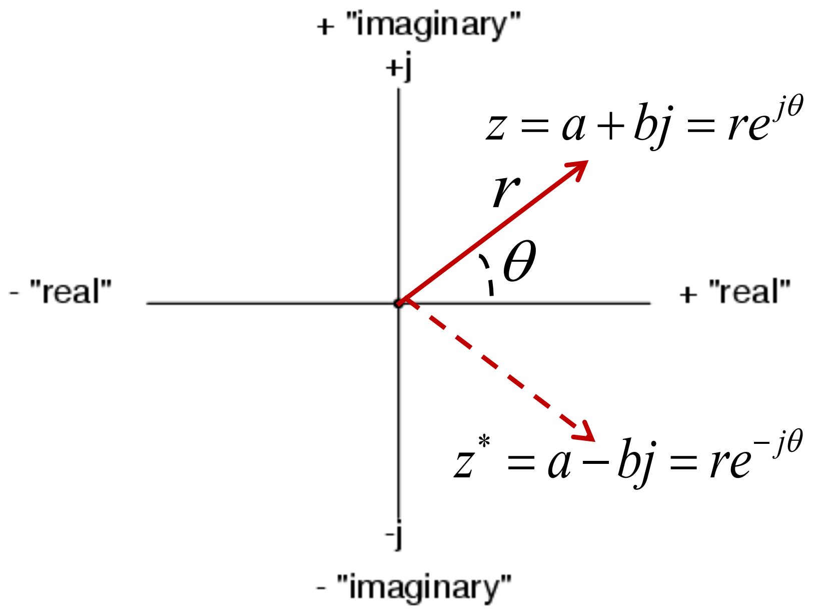
$b_1 j b_2 j = b_1 b_2 j^2 = -b_1 b_2$

Complex Conjugate

- For a complex number $= a + bj = re^{j\theta}$,

  its ***complex conjugate*** is $z^* = a - bj = re^{-j\theta}$

- z and z^* is called a conjugate pair. They have the same real part but negated imaginary parts. Alternatively said, they have the same magnitude but negated phase.



- You must become very familiar with the basic properties of the complex conjugate:

Complex Conjugate - Properties

1. Adding a complex number with its conjugate produces 2 times the real part. In other words, the real part is the ***half sum*** of the conjugate pair.

$$\boxed{z + z^* = 2a = 2\,\mathrm{Re}\{z\} \ \text{ or } \ \mathrm{Re}\{z\} = (z + z^*)/2}$$

2. Subtracting a complex number by its conjugate produces 2*j* times the imaginary part

$$\boxed{z - z^* = 2bj = 2j\,\mathrm{Im}\{z\} \ \text{ or } \ \mathrm{Im}\{z\} = (z - z^*)/2j}$$

3. Multiplying a complex number with its conjugate produces its magnitude square

$$\boxed{zz^* = (a + bj)(a - bj) = a^2 - (bj)^2 = a^2 + b^2 = re^{j\theta} re^{-j\theta} = r^2}$$

or $$\boxed{|z| = \left(zz^*\right)^{1/2}}$$

Complex Conjugate - Properties

4. Conjugate of a sum is the sum of the conjugates and conjugate of a product is the product of the conjugates. That is, for a set of complex numbers z_i's:

$$\boxed{\left(\sum z_i\right)^* = \sum z_i^* \ ; \quad \left(\prod z_i\right)^* = \prod z_i^*}$$

Sigma notation for a sum

Pi notation for a product

5. To conjugate an exponential, we conjugate both the base and the exponent :

$$({z_1}^{z_2})^* = ({z_1}^*)^{z_2^*}$$

In ELEC2100, the base z_1 is typically real; hence we only need to worry about conjugating the exponent:

$$({r_1}^{z_2})^* = {r_1}^{z_2^*} \quad \text{where } r_1 \text{ is real}$$

Try proving properties 4 and 5 yourself?

Review Questions 1

For each complex number below, determine its magnitude and phase and identify its location on the complex plane.

i. $z_1 = 1.5 - j$

$1.5^2 + 1^2$

$= \sqrt{3.25} > 1$

ii. $z_2 = 1.1e^{j\pi/2}$

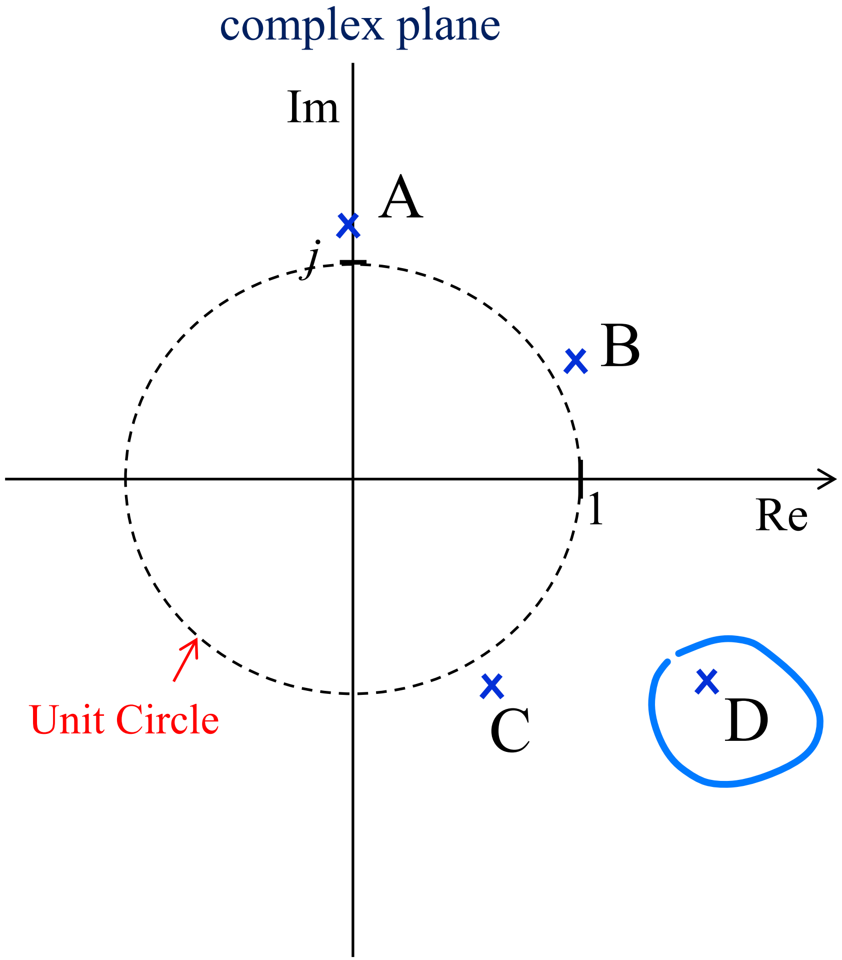
iii. $z_3 = \left(e^{0.1+j}\right)^*$

iv. $z_4 = \left(e^{0.3+j\pi/2}\right)^{1/3}$

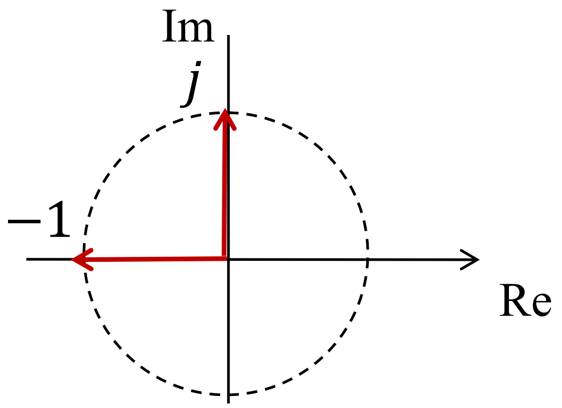
Bonus: $z_1 + z_2 = ?$

$1.5 + 0.1j$

$z_2 z_3 = ?$



Review Question 2



i. Express j in polar form

ii. $\measuredangle -1 =\ ?$

iii. $e^{j\pi} + 1 \ =\ ?$

This is called God's equation.

It relates the five fundamental numbers in mathematics: $0,\ 1,\ \pi,\ e,\ j$

Appendix 1.1 – Derivation of Euler’s Formula
(Covered in Summer Math Review Class)

- This derivation is for reference only.

$\cos(\theta) = \sum_{n=0}^{\infty} \frac{cos^{(n)}(0)\theta^n}{n!} = \sum_{n=0;n\ even}^{\infty} \frac{(-1)^{\frac{n}{2}}\theta^n}{n!} = \sum_{n=0;n\ even}^{\infty} \frac{j^n\theta^n}{n!}$

$\sin(\theta) = \sum_{n=0}^{\infty} \frac{sin^{(n)}(0)\theta^n}{n!} = \sum_{n=1;n\ odd}^{\infty} \frac{(-1)^{\frac{n-1}{2}}\theta^n}{n!} = \sum_{n=1;n\ odd}^{\infty} \frac{j^{n-1}\theta^n}{n!} \Rightarrow j\sin(\theta) = \sum_{n=1;n\ odd}^{\infty} \frac{j^n\theta^n}{n!}$

$$\Rightarrow \cos(\theta) + j\sin(\theta) = \sum_{n=0}^{\infty} \frac{j^n\theta^n}{n!} = e^{j\theta}$$

1st- derivative

- We note that the first derivative of $\cos(\theta)$ is $\cos^{(1)}(\theta) = -\sin(\theta) = \cos(\theta + \frac{\pi}{2})$. Differentiating a sinusoid simply increase the phase of the sinusoid by 90°. Hence:

$$\cos^{(n)}(\theta) = \cos(\theta + \frac{n\pi}{2})$$

Evaluating these derivatives at $\theta = 0$:

$\cos^{(n)}(0) = 1, 0, -1,0,1,0, -1,0 \dots$. for n =0, 1, 2, 3,4…= $(-1)^{\frac{n}{2}}$ for n even and 0 otherwise

$\sin(\theta)$ is identical to $\cos(\theta)$ except for a 90° phase delay: $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$. Hence:

$\sin^{(n)}(\theta) = \sin(\theta + \frac{n\pi}{2})$ and

$\sin^{(n)}(0) = 0,1, 0, -1,0,1,0, -1, \dots$. for n =0, 1, 2, 3,4…= $(-1)^{\frac{n-1}{2}}$ for n odd and 0 otherwise