Lecture 22

Chapter 9: Laplace transform Applications (Applications)

I. Feedback System Example – Inverted Pendulum

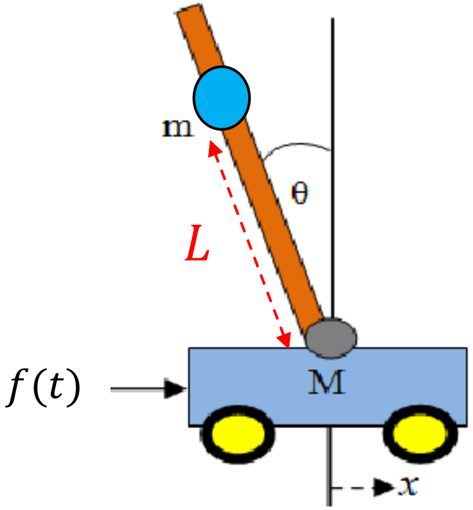
II. Filter Design by Poles Placement

III. Butterworth Filter

IV. Block Diagram Implementation of Systems

I. Feedback System Example – Inverted Pendulum

- The Segway is an example of an inverted pendulum.
- We assume the base of the Segway has mass M, and we model the rider as a rigid rod with mass m at a distance of L from the base
- We want to control $\theta(t)$, the incline angle of the passenger. We let the horizontal position of the base be $x(t)$. Understanding that they are functions of time, we simply use θ and x.
- When θ is small, changes (derivatives) in θ and x are directly related when mass m remains stationary: $\dot{\theta} = \frac{\dot{x}}{L}$; $\ddot{\theta} = \frac{\ddot{x}}{L}$ where we use $\dot{\theta}$ and $\ddot{\theta}$ to mean first and second derivative.



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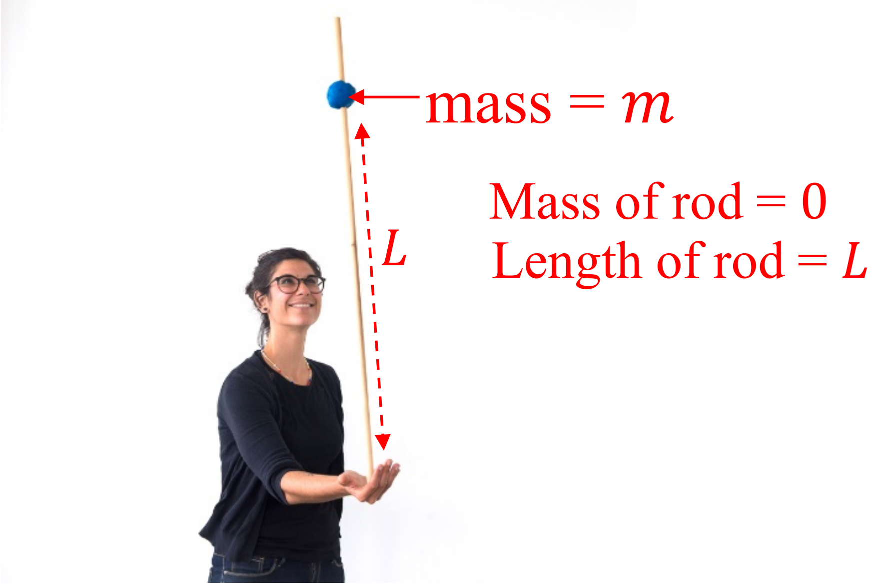
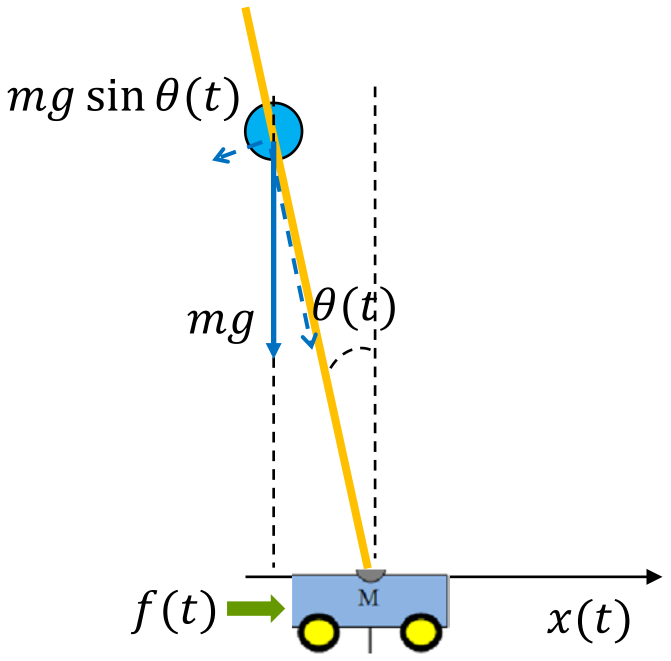


Model for Simple Inverted Pendulum



- The tangential component of the gravitational force (the component perpendicular to the rod) on the mass is $mg \sin \theta \cong mg\ \theta$ for small θ. This force causes acceleration in θ: $\ddot{\theta} = \frac{g}{L}\theta$
- Horizontal disturbance force applied to the base causes horizontal acceleration of the base: $f = M\ddot{x} = ML\ddot{\theta}$ (mass m applies a horizontal force of $mg\ \cos\theta \sin \theta$ through the rod but this force is negligible when θ is small.
- Therefore, for small θ, we have:

$$\ddot{\theta} = \frac{g}{L}\theta + \frac{f}{ML} \Rightarrow \ddot{\theta} - \frac{g}{L}\theta = \frac{f}{ML}$$

which is a 2nd order LCCDE

and the system function is: $P(s) = \dfrac{1/ML}{s^2 - \frac{g}{L}}$

unstable

$\Theta\left(s^2 - \frac{g}{L}\right) = \frac{1}{ML} F$

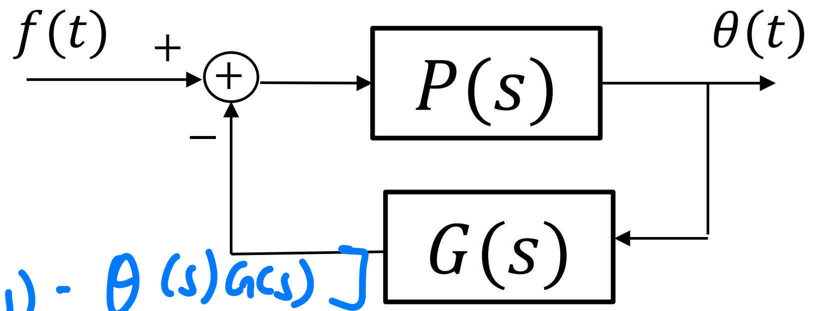
$H(s) = \dfrac{1/ML}{s^2 - \frac{g}{L}}$

The system is unstable. Why?

Stabilizing the Inverted Pendulum by Feedback

- Now, if we monitor the output $\theta(t)$, pass $\theta(t)$ through a linear system $G(s)$, and feed the output as negative feedback as shown, we can stabilize the system.

$F(s)$ $\Theta(s)$



- We have

$$\Theta(s) = (F(s) - G(s)\,\Theta(s))P(s)$$

$$\Rightarrow \Theta(1 + GP) = PF \qquad \Rightarrow \Theta = \frac{P}{1+GP}F$$

$P(s)[F(s) - \Theta(s)G(s)]$
$= \Theta(s)$
$= \Theta(s)[1 + P(s)G(s)]$

$P(s)F(s)$

$\frac{\Theta(s)}{F(s)} = \frac{P(s)}{1+P(s)G(s)}$

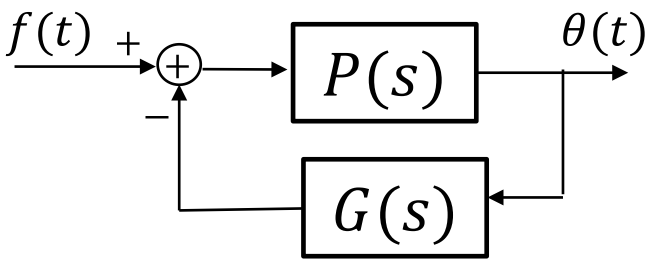
and the overall system function is:

$$H(s) = \frac{\frac{1/ML}{s^2 - \frac{g}{L}}}{1 + G(s)\frac{1/ML}{s^2 - \frac{g}{L}}} = \frac{\frac{1}{ML}}{s^2 - \frac{g}{L} + \frac{1}{ML}G(s)}$$

So what type of $G(s)$ do we need to stabilize the system??

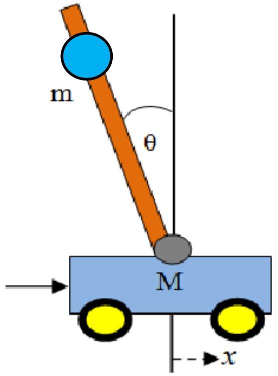
Proportional + Derivative Feedback

- In Lecture 4, Slide #11, we suggested using a combination of proportional and derivative feedback to control the inverted pendulum: $K_1\theta + K_2\frac{d\theta}{dt}$
- That means $G(s) = K_1 + K_2 s$, and:



$$H(s) = \frac{1/ML}{s^2 - \frac{g}{L} + \frac{1}{ML}(K_1 + K_2 s)}$$

$$\Rightarrow H(s) = \frac{1/ML}{s^2 + \frac{K_2}{ML}s + (\frac{K_1}{ML} - \frac{g}{L})}$$



For the system to be stable, we need all coefficients in the denominator polynomial to be of the same sign. Therefore, we need: $K_2 > 0$ and $K_1 > Mg$.

How should we choose $K_1,\ K_2$ smartly to have "smooth" control?

Demo on YouTube:
https://www.youtube.com/watch?v=t7skvD6v2TI

II. Filter Design by Poles Placement

- One of the major ways to build filters is to build them as LCCDE as explained before.
- System function $H(s)$ for LCCDE is rational. In product form:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_{N-1} \prod_{i=1}^{N-1} (s - \beta_i)}{\prod_{k=1}^{N} (s - \alpha_k)}$$

and frequency response $H(j\omega)$ is $H(s)$ evaluated along the $j\omega$ axis:

$$\Rightarrow H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{b_{N-1} \prod_{i=1}^{N-1} (j\omega - \beta_i)}{\prod_{k=1}^{N} (j\omega - \alpha_k)}$$

Each term in the product is a complex number that can be viewed as a 2-vector

- Each factored term in $N(j\omega)$ and $D(j\omega)$ is a complex number which can be thought of geometrically as a vector on the 2-D plane.
- We can think of the magnitude response and phase response in terms of the locations of poles and zeros, and the magnitudes and phases of the individual vectors.

Frequency Response from Pole and Zero Locations

- The magnitude response is a product of magnitudes divided by another product of magnitudes:

$$|H(j\omega)| = \frac{|b_{N-1}| \prod_{i=1}^{N-1} |j\omega - \beta_i|}{\prod_{k=1}^{N} |j\omega - \alpha_k|}$$

- The phase response is a sum of angles minus another sum of angles:

$$\measuredangle H(j\omega) = \underbrace{\measuredangle b_{N-1}}_{\text{Coefficients are real}}^{0} + \sum_{i=1}^{N-1} \measuredangle(j\omega - \beta_i) - \sum_{k=1}^{N-1} \measuredangle(j\omega - \alpha_k)$$

- Instead of exact calculation of the magnitude and phase responses, we can make rough estimation by considering the geometry of the poles and zeros locations.

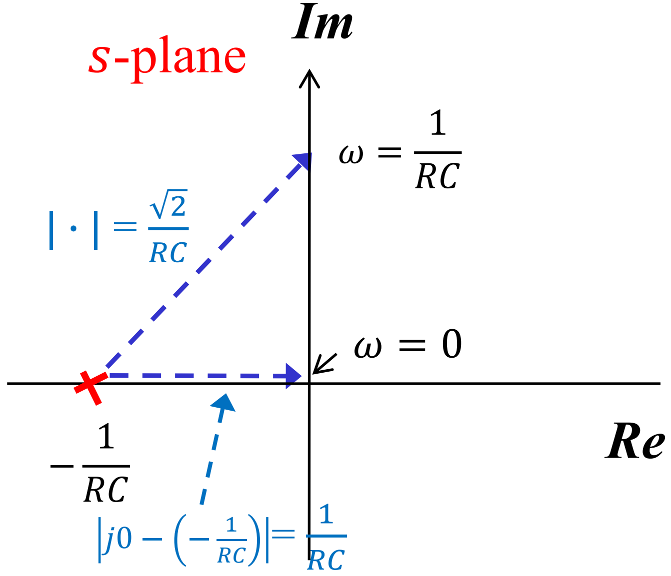
Geometric Evaluation of Frequency Response – 1st Order Filter

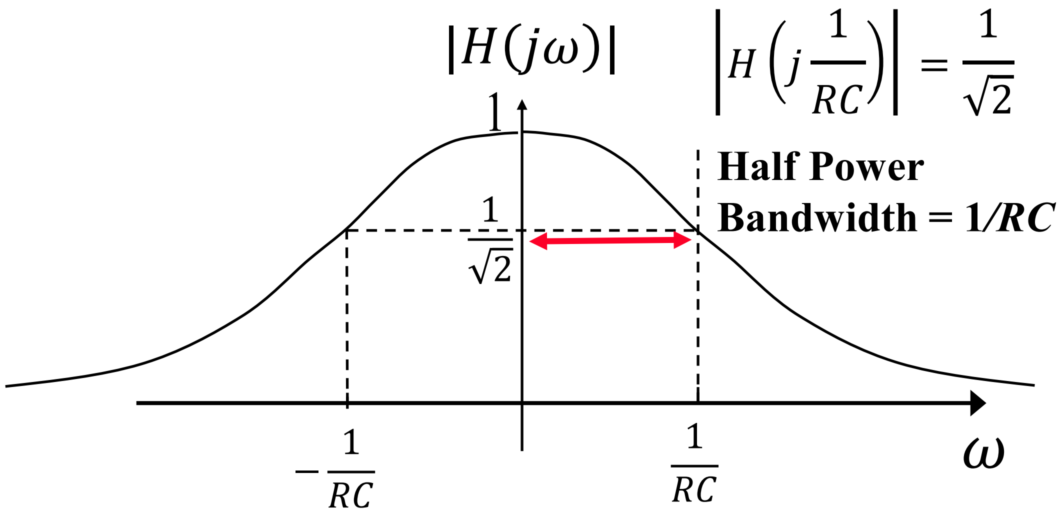
Consider the simple RC circuit with 1st order LCCDE $RC\frac{dy(t)}{dt} + y(t) = x(t)$ again.

- Its system function is $H(s) = \frac{1}{RCs+1} = \frac{1/RC}{s-(-\frac{1}{RC})}$, with pole at $\alpha_1 = -\frac{1}{RC}$ and $H(0) = 1$
- Now consider the magnitude response $|H(j\omega)| = \frac{1/RC}{\left|j\omega-\left(-\frac{1}{RC}\right)\right|}$;

the magnitude $\left|j\omega - \left(-\frac{1}{RC}\right)\right|$ increases with $|\omega|$ and by a factor of $\sqrt{2}$ when ω changes from 0 to $1/RC$.

- Therefore, the system is a non-ideal LPF and $1/RC$ is the half power frequency.





Geometric Evaluation of Frequency Response – 2nd-Order Filter

Non-Ideal Band-pass

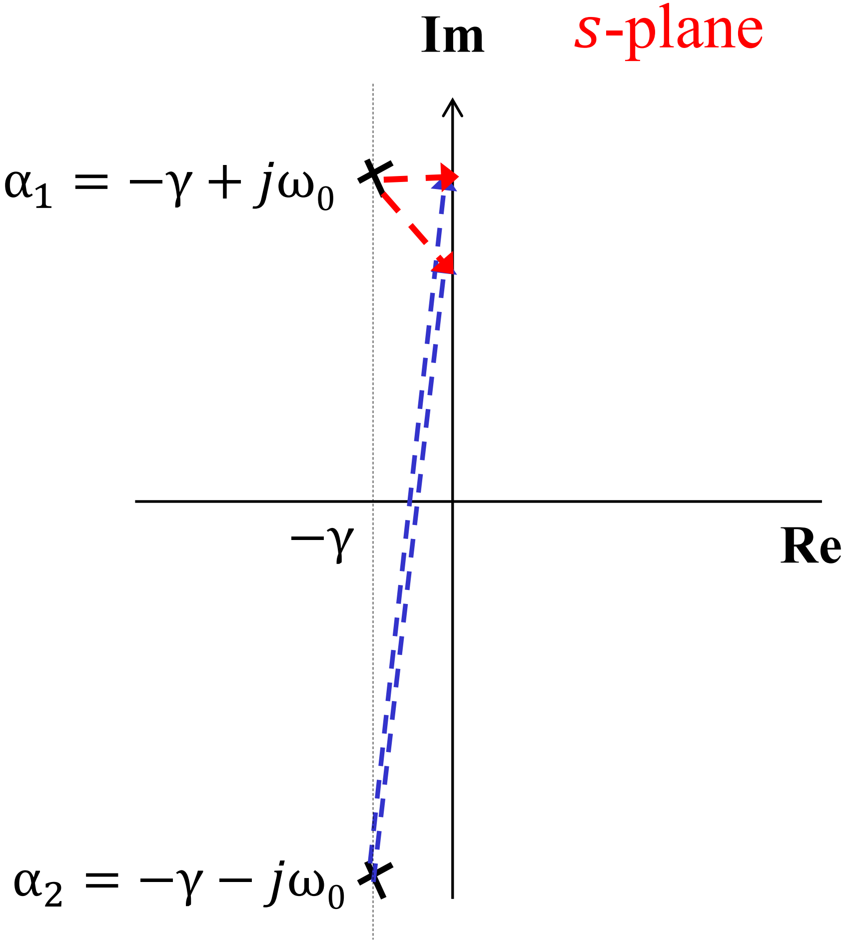
Re{·} Im{·}

Assume we build a 2nd-order system with two poles at $-\gamma \pm j\omega_0$ such that $\omega_0 \gg |\gamma|$.

- The resulting magnitude response is:

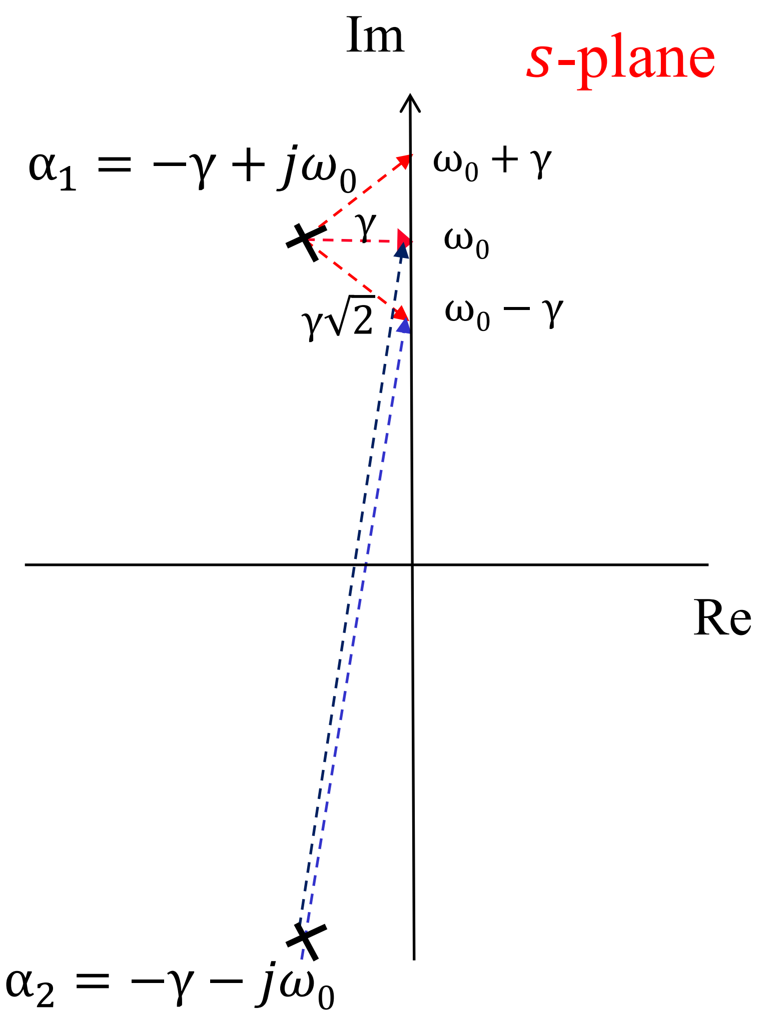
$$|H(j\omega)| = \frac{A}{|j\omega - \underbrace{(-\gamma + j\omega_0)}_{\alpha_1}||j\omega - \underbrace{(-\gamma - j\omega_0)}_{\alpha_2}|}$$

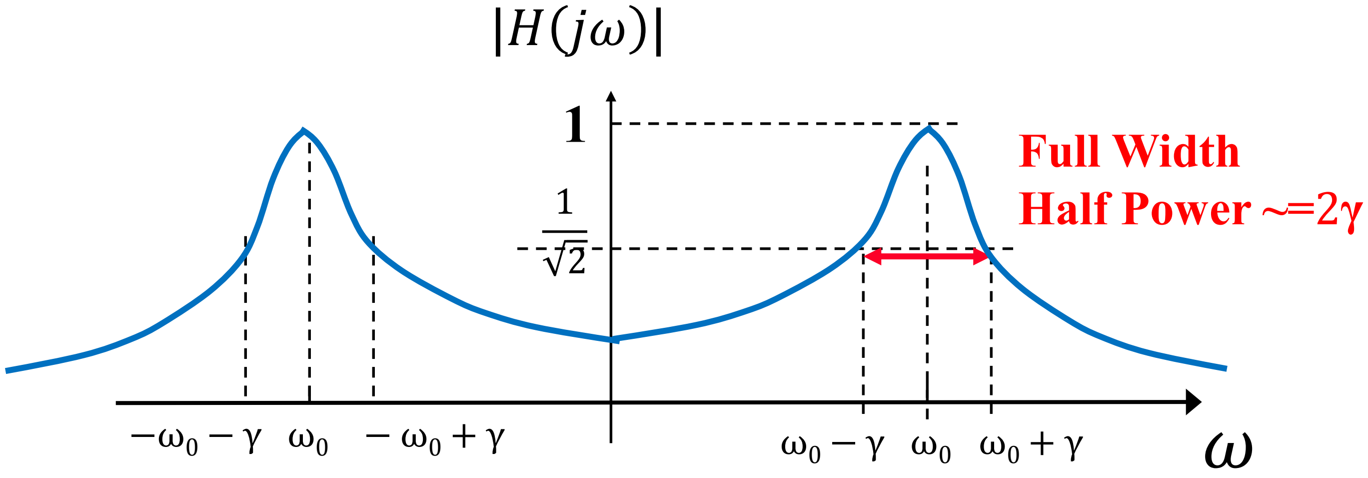
- $|H(j\omega)|$ peaks near $\pm\omega_0$ because the distance to one pole is minimized at $\omega = \pm\omega_0$.
- When ω varies around $\pm\omega_0$, the distance to the nearby pole changes quickly proportional-wise while the distance to the second pole stays close to $2\omega_0$.



- Therefore $|H(j\omega)|$ is reduced by $\sqrt{2}$ at $\omega_0 \pm \gamma$ and $-\omega_0 \pm \gamma$ (half-power frequencies) ;

  $\Rightarrow$ the system is an approximate band-pass filter and the width of the passband between the two half-power frequencies is 2γ.





- As in the 1st-order system, the width of the passband is controlled by the distance of the poles from the $j\omega$-axis.

Coefficients for LCCDE

- To have poles at $-\gamma \pm j\omega_0$, we set the coefficients of our 2nd order LCCDE as follow:

$$\frac{d^2y(t)}{dt^2} + 2\gamma\frac{dy(t)}{dt} + (\gamma^2+\omega_0{}^2)y(t) = Ax(t) \quad \Leftrightarrow H(s) = \frac{A}{s^2 + 2\gamma s + (\gamma^2 + \omega_0{}^2)}$$

roots complex
$\Rightarrow \alpha_1 = \alpha_2{}^*$

$a_1 = -(\alpha_1 + \alpha_2)$
$= -2\mathrm{Re}\{\alpha_1\} = 2\gamma$

$a_0 = \alpha_1\alpha_2 = |\alpha_1|^2 = \omega_n{}^2$
Natural frequency

Quadratic formula

Indeed we can verify that the roots are: $\alpha_1, \alpha_1{}^* = -\gamma \pm \sqrt{\gamma^2 - (\gamma^2 + \omega_0{}^2)} = -\gamma \pm j\omega_0$

- A is for normalizing the peak magnitude response if desired:

$$|H(j\omega_0)| = \frac{A}{\gamma|\gamma - 2j\omega_0|} \cong \frac{A}{2\gamma\omega_0}; \qquad |H(0)| = \frac{A}{|\gamma - j\omega_0||\gamma + j\omega_0|} = \frac{A}{\gamma^2+\omega_0{}^2} \cong \frac{A}{\omega_0{}^2} \ll |H(j\omega_0)|$$

distance to α_1 distance to α_2 $\omega_0 \gg \gamma$

- $\omega_0 \pm \gamma$ is half-power frequency: $|H(j(\omega_0 \pm \gamma))| = \dfrac{A}{\sqrt{2} \times \gamma|\gamma - 2j(\omega_0 \pm \gamma)|} \cong \dfrac{A}{\sqrt{2} \times 2\gamma\omega_0}$

distance to α_1 increased by a factor of $\sqrt{2}$

Example: Sketch the magnitude response $|H(j\omega)|$ for the 2nd-order differential equation below. Label key dimensions.

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 101y(t) = 101x(t)$$

$$H(s) = \frac{101}{s^2+2s+101}$$

$$|H(j\omega)| = \frac{101}{|s^2+2s+101|}$$

$$= \frac{101}{\sqrt{(101-\omega^2)^2+4\omega^2}}$$

$\gamma \pm \omega_0 j$

From previous slide, $\alpha_1, \alpha_2 = -1 \pm j10$

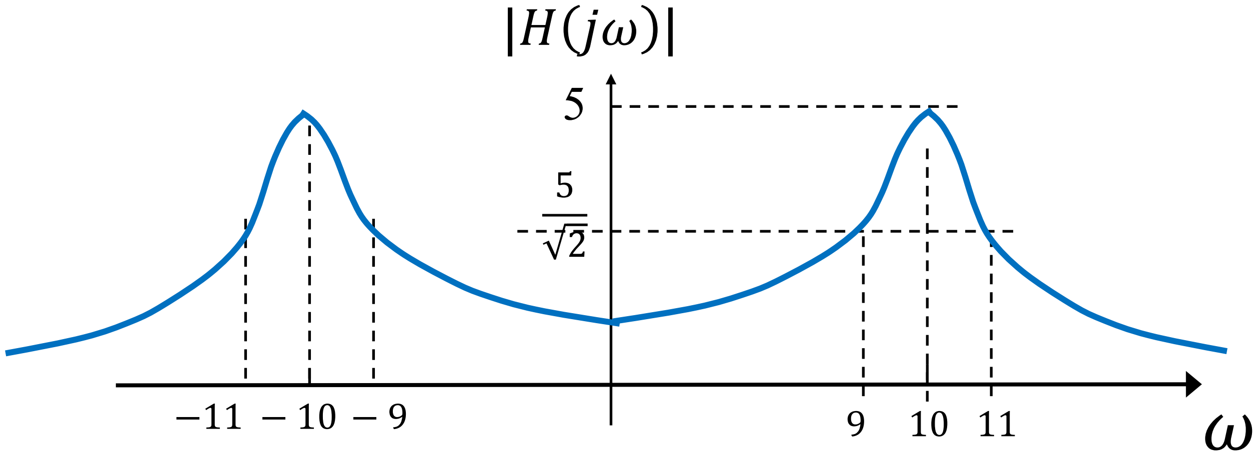
Since $|\omega_0| >> |\gamma|$, magnitude response $|H(j\omega)|$ peaks at approximately $\omega = 10$:

$$|H(j10)| = \frac{101}{|j10 - (-1 + j10)||j10 - (-1 - j10)|} = \frac{101}{|1 + j20|} \cong 5$$

decaying，不关frequency事!

$\omega_0 \pm \gamma = 11, 9$ are half-power frequencies

$|H(0)| = \frac{101}{101} = 1$ from previous slide



III. Higher Order Filter – Butterworth Filter Example

- With the 1st order non-ideal LPF and 2nd-order approximate BPF, we do not have sharp transition from pass-band to cut-off.
- For the 1st-order LPF, $|H(j\omega)|$ decays only as $1/\omega$.
- We can design filters with sharper cut-off using more poles. For an N-th order filter with N poles, we can make $|H(j\omega)|$ decay as $1/\omega^N$ for large ω.
- In a *Butterworth filter*, we place N poles *evenly* on a semi-circle with radius ω_c around the origin left of the $j\omega$-axis, as shown in Fig. 9.29.

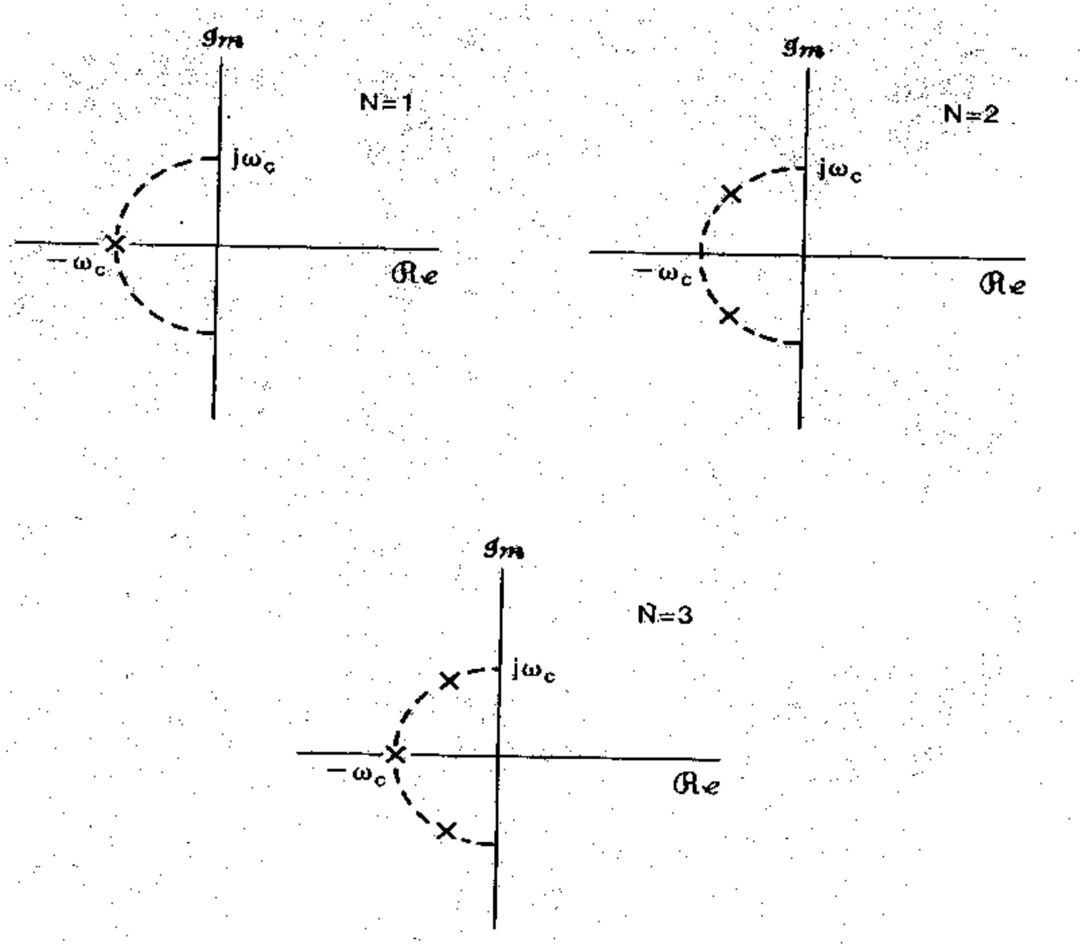


Figure 9.29 Position of the poles of $B(s)$ for $N = 1$, 2, and 3.

Fig. 9.29 Butterworth Filter Poles Placement

Poles Placement for the Butterworth Filter

- The locations of the N poles of the Butterworth filter are specified mathematically as:

$$\alpha_k = e^{j\frac{k2\pi-\pi}{2N}}\, j\, \omega_c, \quad k = 1,2,3 \ldots N$$

$|\cdot| = 1;$ specifies the phase

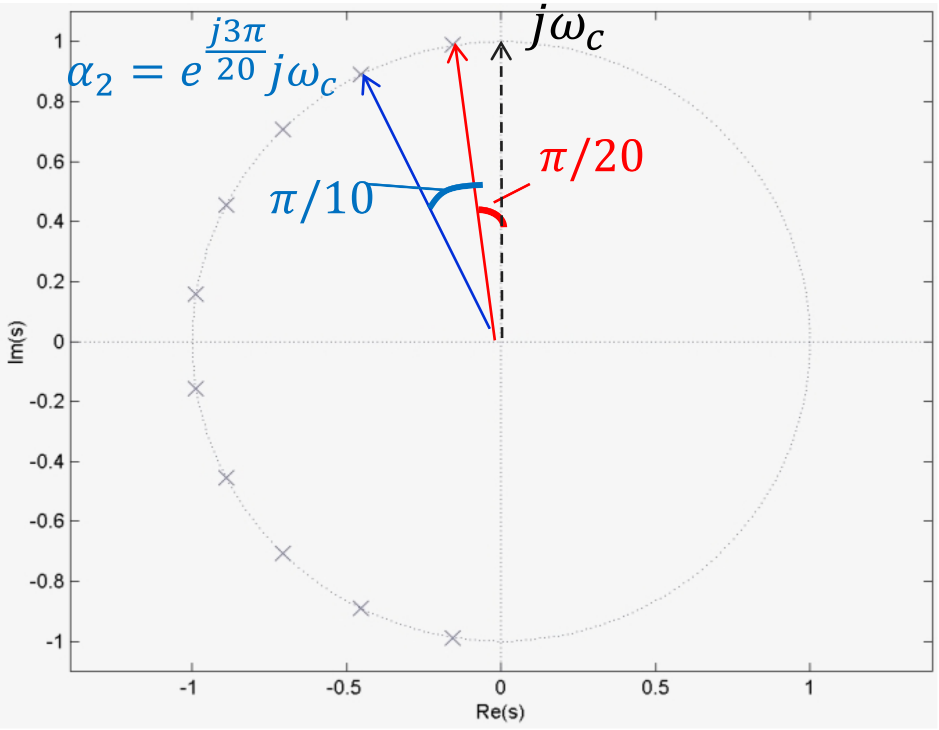
$90°$ phase offset

Frequency scaling

N: Number of poles = Order of filter

Example: $N=10$

$\alpha_1 = e^{\frac{j\pi}{20}} j\omega_c$



Which means:

- All poles are on semi-circle with radius ω_c:

$$|\alpha_k| = \omega_c\, , k = 1,2 \ldots N.$$

- First pole is off $j\omega$-axis by angle $\pi/2N$:

$$\measuredangle\alpha_1 = \frac{\pi}{2} + \frac{\pi}{2N}$$

- Successive poles are separated by angle π/N:

$$\measuredangle\alpha_k = \frac{\pi}{2} + \frac{\pi}{2N} + \frac{(k-1)\pi}{N} = \frac{\pi}{2} + \frac{k2\pi - \pi}{2N}$$

Magnitude Response of the Butterworth Filter

- With the given poles, we can express the system function $B(s)$ of the Butterworth filter in the alternative product form which makes obvious that $B(0) = 1$:

$$B(s) = \frac{1}{\prod_{k=1}^{N}\left(1-\frac{s}{\alpha_k}\right)} = \frac{1}{\prod_{k=1}^{N}\left(1-\frac{s}{e^{j(k2\pi-\pi)/2N}j\omega_c}\right)}$$

- We state without proof that the magnitude response is:

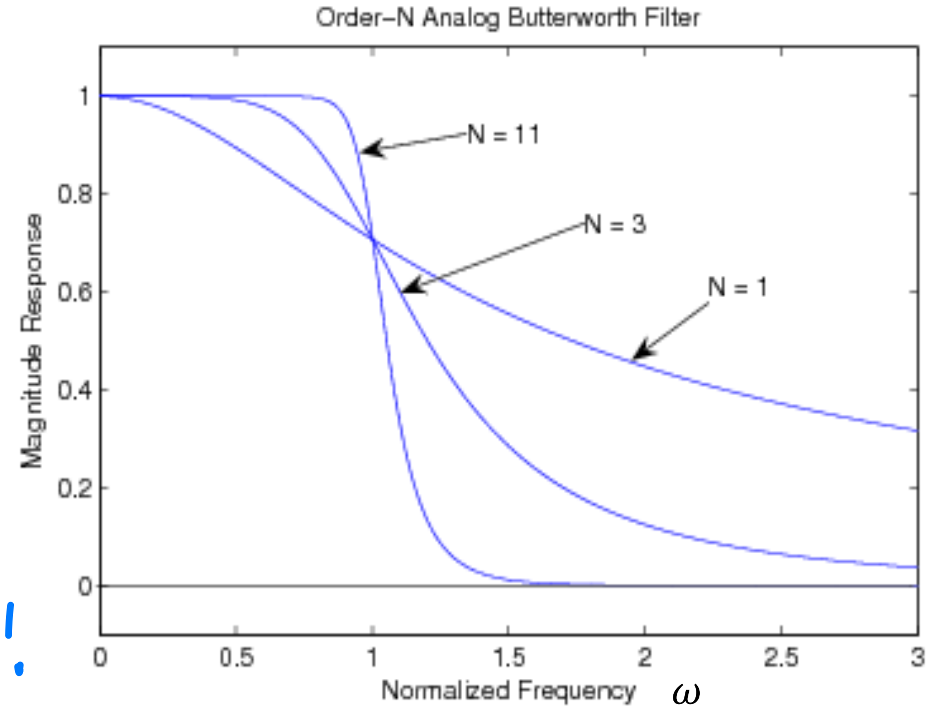
$$|B(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2N}}}$$

attenuate high frequency signal!

So, for large N,

when $|\omega| < \omega_c$, $\left(\frac{\omega}{\omega_c}\right)^{2N} \ll 1$, and $|B(j\omega)| \to 1$

when $|\omega| > \omega_c$, $\left(\frac{\omega}{\omega_c}\right)^{2N} \gg 1$, and $|B(j\omega)| \to \frac{1}{(\omega/\omega_c)^N}$



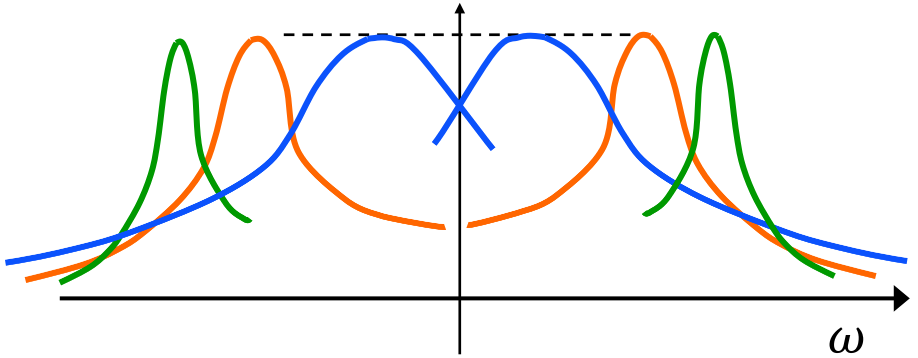
$\frac{\omega}{\omega_c}$

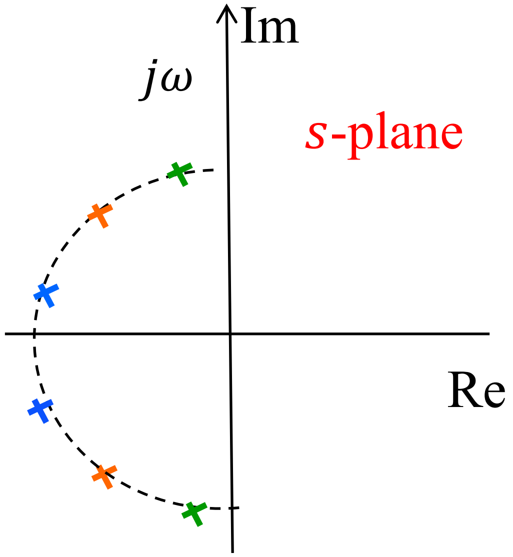
With a larger N, we can achieve a flatter passband and sharper roll-off

- Another way to think of the Butterworth filter is, for N even, that it is simply $N/2$ 2nd-order systems in cascade

Butterworth filter for $N = 6$ as cascade of three 2-nd order systems:

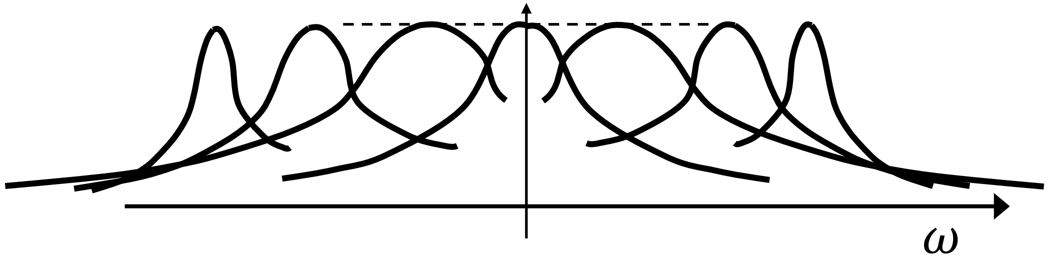
$$|B(j\omega)| = |H_1(j\omega)||H_2(j\omega)||H_3(j\omega)|$$





- For N odd, there is an extra 1st order low-pass filter:

$$|B(j\omega)|$$

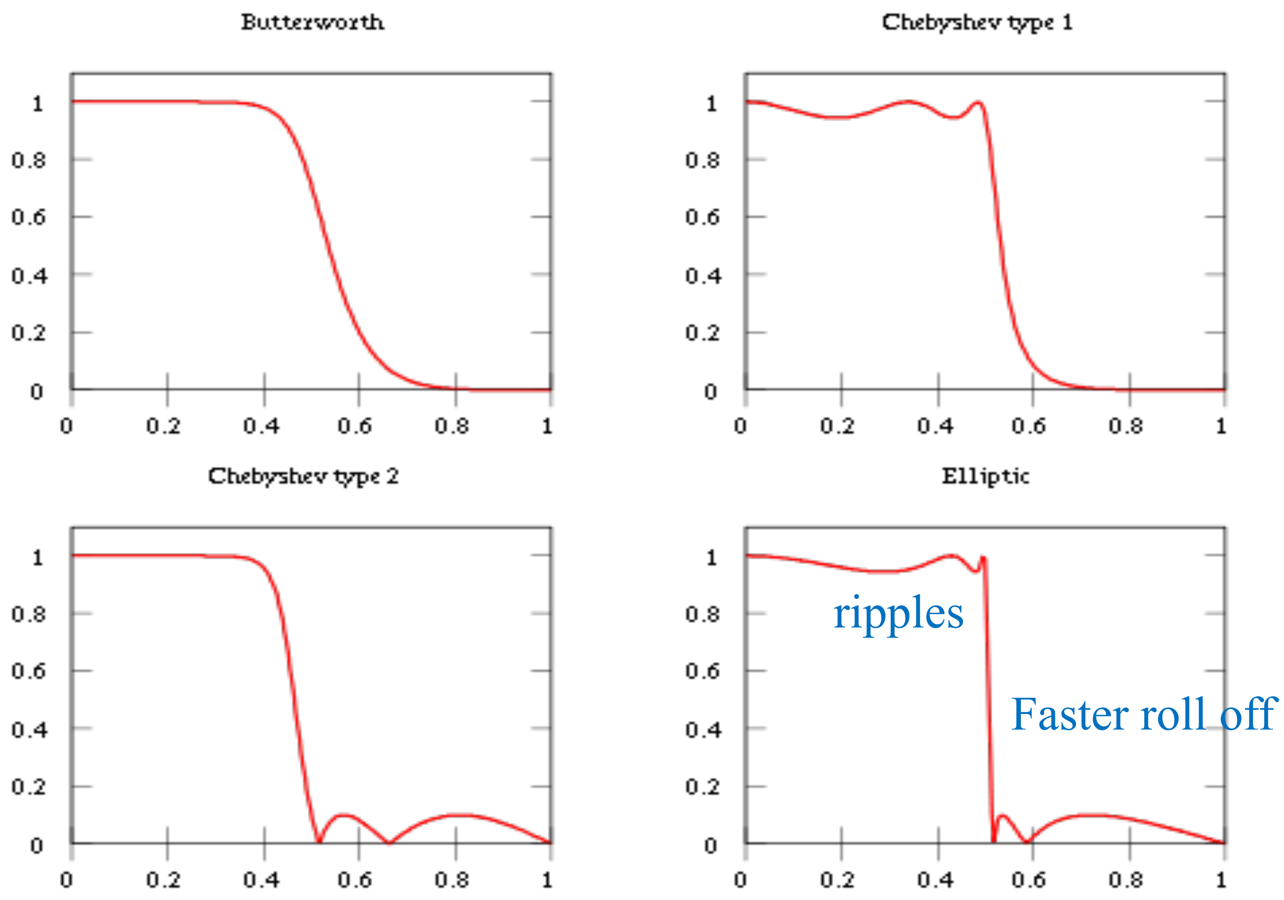


So, a high order system is just a combination of 1st and 2nd-order systems!

Other High Order Filters

Chebyshev and elliptic filters use different strategies to place poles.

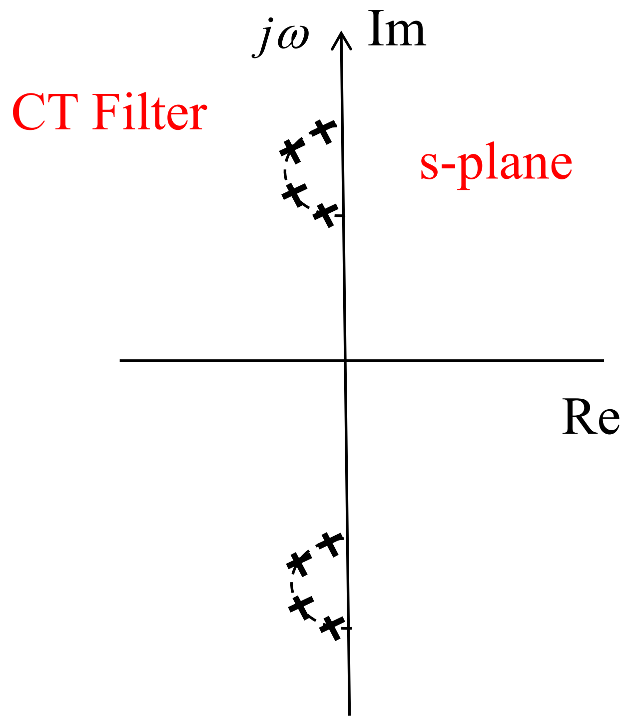
They achieve faster roll-off but have ripples in magnitude response that may be undesirable

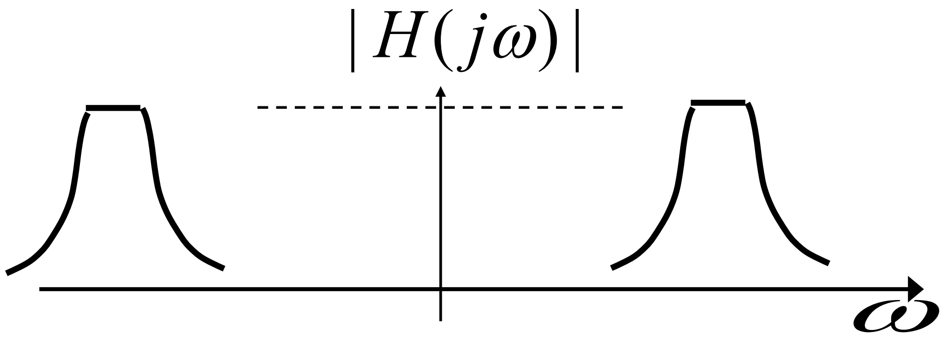


http://cnyack.homestead.com/files/afilt/afilt-buttchebypoles.htm

Band-pass Filter

- Using the notion of poles placement, we can also design band-pass filters using a larger number of poles to achieve flatter pass-band and faster roll-off:



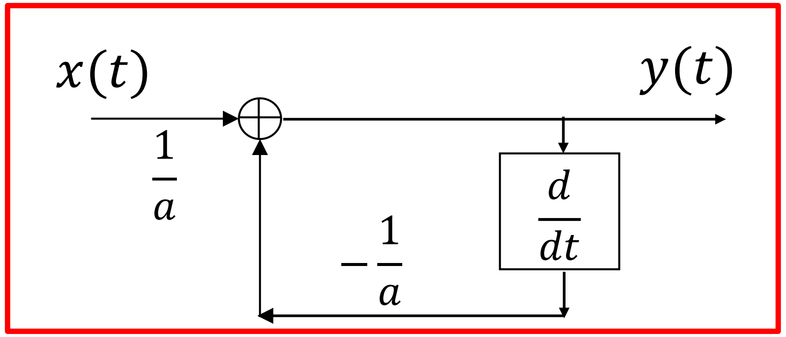


IV. Block Diagram Implementation of Systems

- Now we have learned to design filters by poles placement. The next question is, how do we implement the corresponding LCCDE?
- Let's start with the simplest case, the 1st-order system: $\dfrac{dy(t)}{dt} + ay(t) = x(t)$

rearranging terms, we have $y(t) = \dfrac{1}{a}x(t) - \dfrac{1}{a}\dfrac{dy(t)}{dt}$

and we can implement using the feedback circuit below:

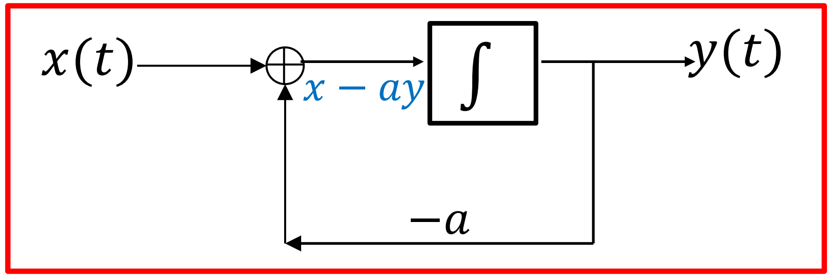


Block diagram – 1st-Order CT System Using Integrator

- Differentiation is sensitive to noise, so usually we implement systems using integrators instead. Integrating both sides of our LCCDE:

$$\frac{dy(t)}{dt} + ay(t) = x(t) \quad \Rightarrow \quad y(t) = \int_{-\infty}^{t} (x(\tau) - ay(\tau))d\tau$$

and we implement using an integrator as shown on the right



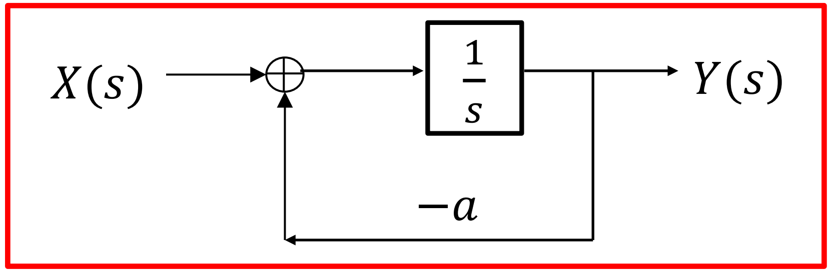
Three Ways for Representing an Integrator

- An integrator can be represented in the following ways:

$$\boxed{\int} \qquad \boxed{\frac{1}{j\omega}} \qquad \boxed{\frac{1}{s}}$$

All these blocks mean an integrator because integration corresponds to $1/j\omega$ for FT and $1/s$ for LT

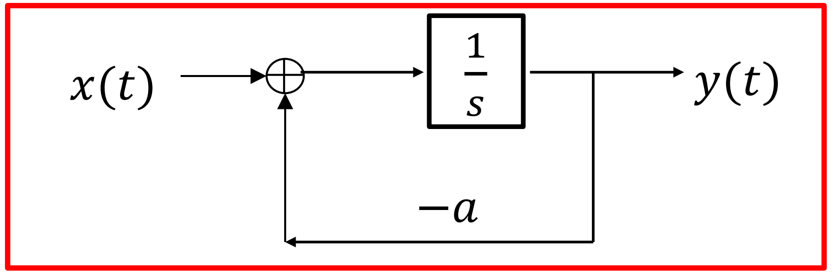
- It means that we can also represent the 1st-order system as:



$$Y(s) = \frac{1}{s}\left(X(s) - aY(s)\right)$$

$$Y(j\omega) = \frac{1}{j\omega}\left(X(j\omega) - aY(j\omega)\right)$$

- Sometimes we get sloppy and mix time and frequency domain symbols:



Building Higher Order Systems – All-Poles System

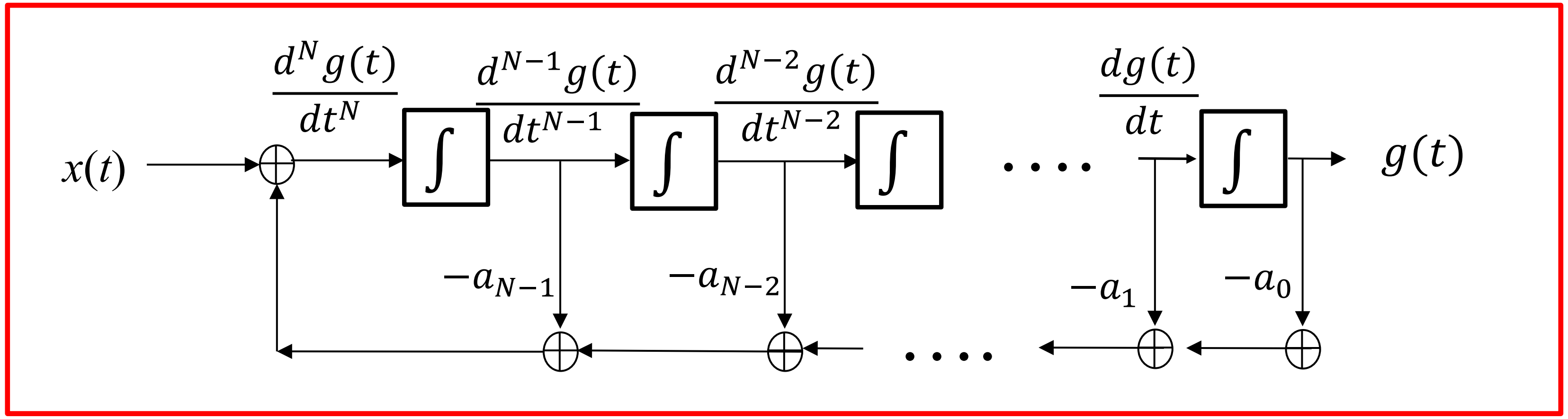
- How do we build a higher order system?
- First, consider an *all-poles system* where the numerator polynomial is a constant:

$$\frac{d^N g(t)}{dt^N} + a_{N-1}\frac{d^{N-1} g(t)}{dt^{N-1}} + \cdots + +a_0 g(t) = x(t) \Rightarrow H_g\,(j\omega) = \frac{1}{(j\omega)^N + a_{N-1}(j\omega)^{N+1} + \cdots + a_0}$$

- Rearranging terms,

$$\frac{d^N g(t)}{dt^N} = x(t) - a_{N-1}\frac{d^{N-1} g(t)}{dt^{N-1}} - \cdots - a_1\frac{dg(t)}{dt} - a_0 g(t)$$

and we can implement the all-poles system as:



System as cascade of All-Poles and All-Zeros Systems

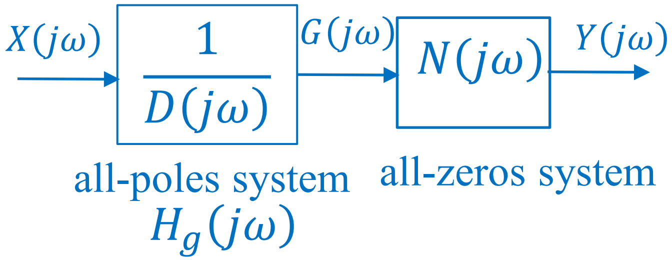
- Now, consider a general system with poles and zeros:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Rightarrow H(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

We can regard it as a cascade of an all-poles system and an all-zeros system:

$$H(j\omega) = \left(\frac{1}{D(j\omega)}\right) N(j\omega) \;\; = H_g\,(j\omega) N(j\omega)$$

$$\Rightarrow Y(j\omega) = G(j\omega) N(j\omega)$$



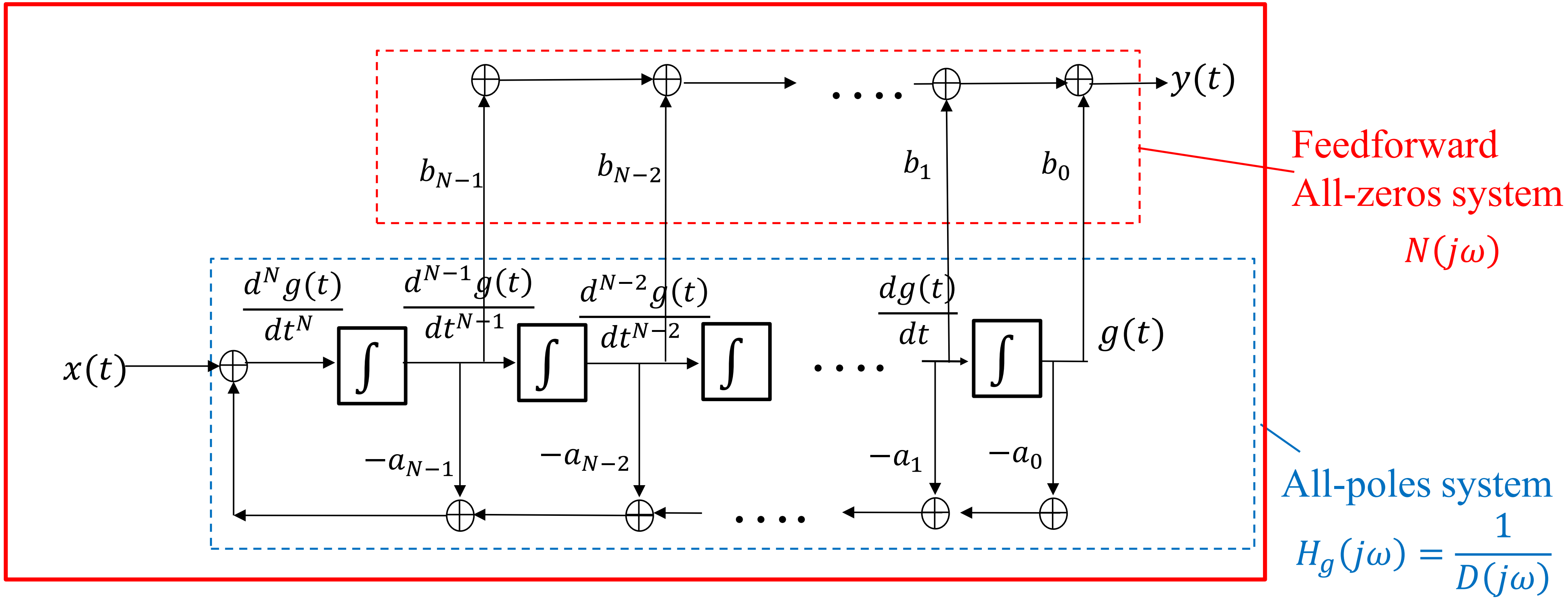
Which means that $y(t)$ of the overall system is related to $g(t)$, output of the all-pole system, by:

$$y(t) = b_{N-1} \frac{d^{N-1} g(t)}{dt^{N-1}} + b_{N-2} \frac{d^{N-2} g(t)}{dt^{N-2}} + \cdots + b_1 \frac{dg(t)}{dt} + b_0 g(t)$$

This part is an *all-zeros system*, which is straightly feed-forward only with no feedback of the output $y(t)$.

N-th order System in Direct Form

- Therefore, we can implement a rational LTI system with poles and zeros in ***direct form*** as follow:



Block Diagram of CT System in Product Form

- We can also express a frequency response in product form and implement as a cascade of systems:

zeros

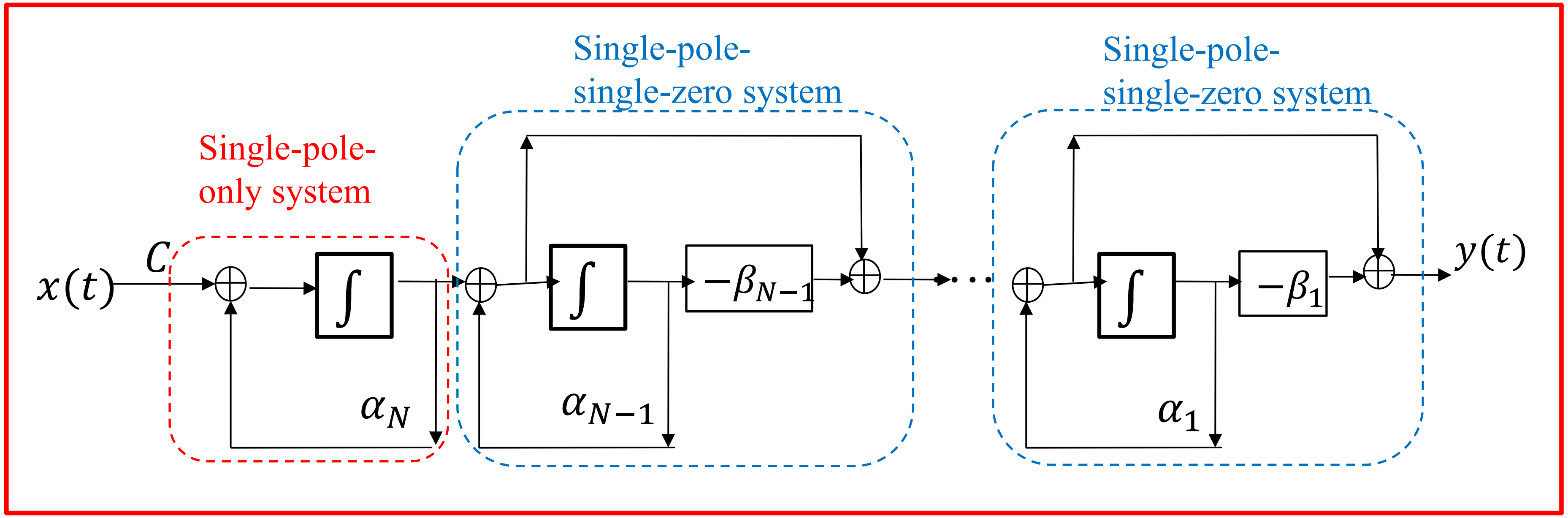
$$H(j\omega) = \frac{C\prod_{i=1}^{N-1}(j\omega - \beta_i)}{\prod_{k=1}^{N}(j\omega - \alpha_k)} = \frac{C}{(j\omega - \alpha_N)} \prod_{i=1}^{N-1} \frac{(j\omega - \beta_i)}{(j\omega - \alpha_i)}$$

poles

Single-pole-only system

Cascade of $N-1$ single-pole-single-zero system

Therefore, we can implement it as:



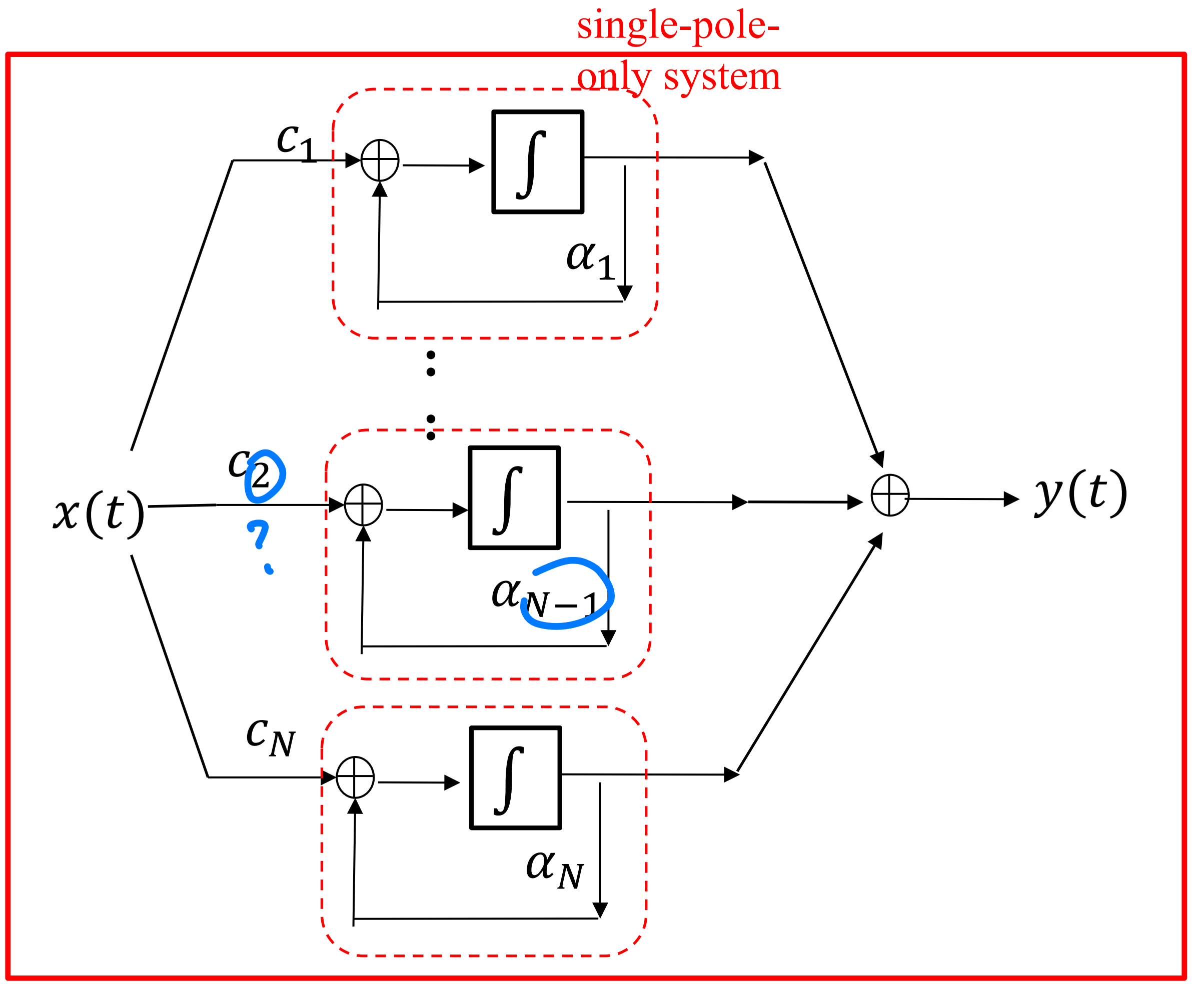
Parallel Form Block Diagram of CT System

- Finally, we can express the frequency response in partial fraction form and view it as multiple single-pole systems in parallel:

$$H(j\omega) = \sum_{k=1}^{N} \frac{c_k}{j\omega - \alpha_k}$$

Sum of N single-pole-only system

Hence, we can implement the system in parallel form as:



Why these Different Forms for Implementation?

- When we implement actual systems, we are concerned with complexity, or the number of elements needed for the system.
- Sometimes we are concerned about speed. In the cascade form, for example, the input signal has to travel through many processing elements before reaching the output. In the parallel form, the delay is the smallest as the input signal does not need to be cascaded through many elements.
- In the future, you will learn about these trade-offs, as well as other ways to implement systems (example: Fast Fourier transform as discussed earlier).

Course Final Words

- Now we come to the end of ELEC2100
- The course has been about four aspects: languages, deduction, analysis, and applications.
- The language, the deduction, and the analysis techniques are the foundation to further study in signal processing, communications, and control systems. The application examples show all the exciting works that you can do in ECE.