Lecture 17 Application of Sampling Theorem (Application)

I. Equivalency of a bandlimited signal and its Sampled Sequence

II. Discrete-time processing of CT signals

III. Example – Digital AAF by Over-Sampling

IV. Discrete Approximation of Filters

I. Equivalency of a bandlimited signal and its Sampled Sequence

- Last lecture, we proved the sampling theorem which states that a bandlimited signal can be recovered from its sampled values.
- We also discussed the advantages of DT/digital signal processing systems.
- In this lecture, we seek to further understand the relationship between a CT signal $x_c(t)$ and its sampled DT representation $x_d[n] = x_c(nT)$. Understanding of this relationship is the key to understanding how digital signal processing works.
- We will show that if the sampling frequency is above the Nyquist rate, the sampled sequence $x_d[n]$ is "the same" as $x_c(t)$ because the spectrum of the two signals are "the same", so we can use filters that are "the same" in DT as in CT.

The Sampled Impulse Train Again

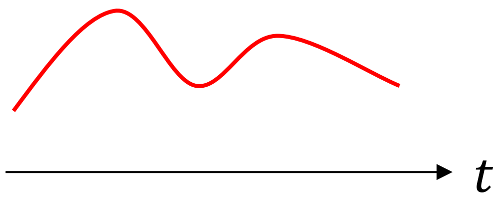
- Recall that we prove the sampling theorem by using the sampled impulse train $x_p(t) = x_c(t)p(t)$.
- Recall also that the spectrum of $x_p(t)$ is a Poisson sum of the spectrum of $x_c(t)$ with scaling by $\frac{1}{T}$:

$$X_p(j\omega) = \underbrace{\frac{1}{T}}_{\text{Scaling constant}} \underbrace{\sum_{k=-\infty}^{\infty} X_c\left(j\left(\omega - k\frac{2\pi}{T}\right)\right)}_{\text{Poisson sum}}$$

- We will use $X_p(j\omega)$ to show the connection between $X_d(e^{j\Omega})$, the spectrum of $x_d[n]$, and $X_c(j\omega)$, the spectrum of $x_c(t)$

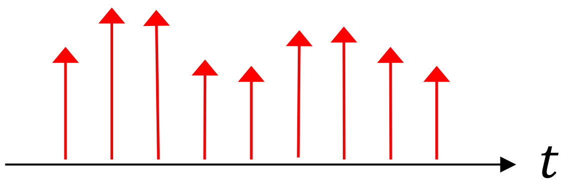
CT Signal

$x_c(t)$



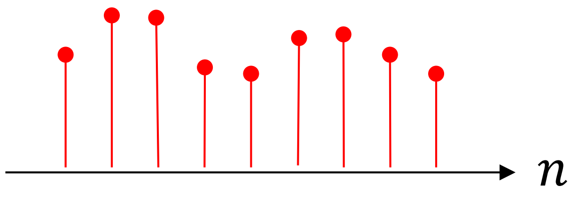
Sampled Impulse Train

$x_p(t) = x_c(t)p(t)$



Sampled Sequence

$x_d[n] = x_c(nT)$



Spectrally Equivalent!

Spectral Equivalence of Sampled Impulse Train and Sampled Sequence

- The spectrum of $x_d[n]$ is given by its DTFT:

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \overset{x_d[n] = x_c(nT)}{\overset{\downarrow}{x_c(nT)}}e^{-j\Omega n} \qquad \ldots.(1)$$

- Next, we consider $x_p(t)$ as a weighted sum of shifted impulses:

sampling property of impulse

$$x_p(t) = x_c(t)p(t) = x_c(t)\sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \qquad \ldots(2)$$

- Recall that the FT of $\delta(t-nT)$ is $e^{-j\omega nT}$

Taking FT of (2) yields:

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} \qquad \ldots.(3)$$

We recognize that (1) and (3) are "the same" except for a scaling in frequency:

$$\boxed{X_d\left(e^{j\Omega}\right) = X_p\left(j\frac{\Omega}{T}\right) \quad \text{or } X_p(j\omega) = X_d\left(e^{j\omega T}\right)}$$

Eq (7.21)

- Again $X_p(j\omega)$ is a scaled Poisson sum of $X_c(j\omega)$: $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\omega - k\frac{2\pi}{T}\right)\right)$ $\omega_s = \frac{2\pi}{T}$
- If $x_c(t)$ is bandlimited to $\omega_M < \frac{\pi}{T}$ so that there is no aliasing, then:

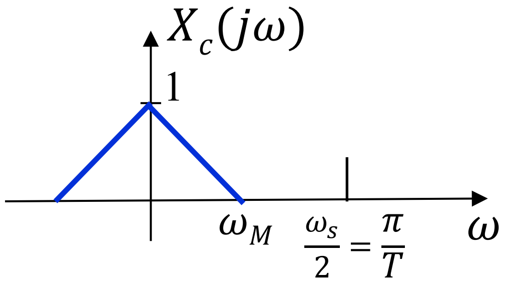
Eq.7.21 of preceding slide

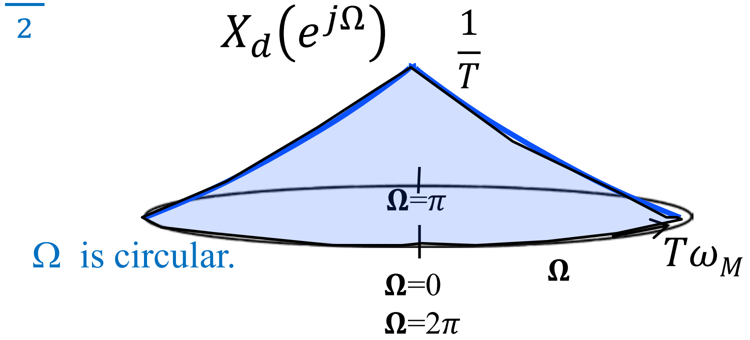
$$X_c(j\omega) = \begin{cases} TX_p(j\omega) = TX_d\left(e^{j\omega T}\right) & -\frac{\pi}{T} \le \omega \le \frac{\pi}{T} \quad -\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$

- Ω is circular. We can regard the DTFT $X_d\left(e^{j\Omega}\right)$ as a frequency-scaled and circular version of $X_c(j\omega)$;

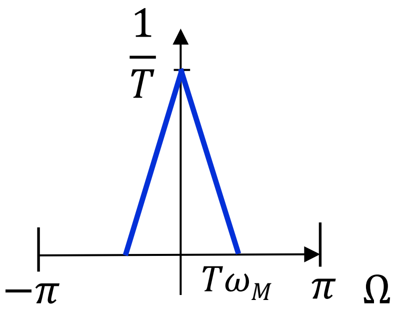
i.e., $\boxed{X_d\left(e^{j\Omega}\right) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right) \quad \text{for } -\pi < \Omega \le \pi}$, as shown below:

Bandlimited to $\omega_M < \frac{\pi}{T} = \frac{\omega_s}{2}$



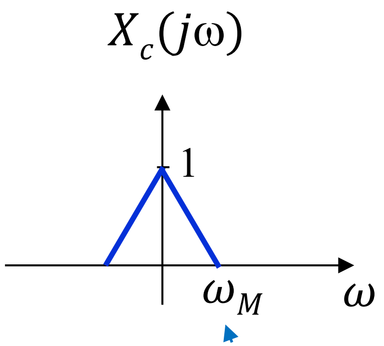


$X_d\left(e^{j\Omega}\right)$, $-\pi < \Omega \le \pi$



Summary: Relationship among $X_c(j\omega)$, $X_p(j\omega)$ and $X_d(e^{j\Omega})$

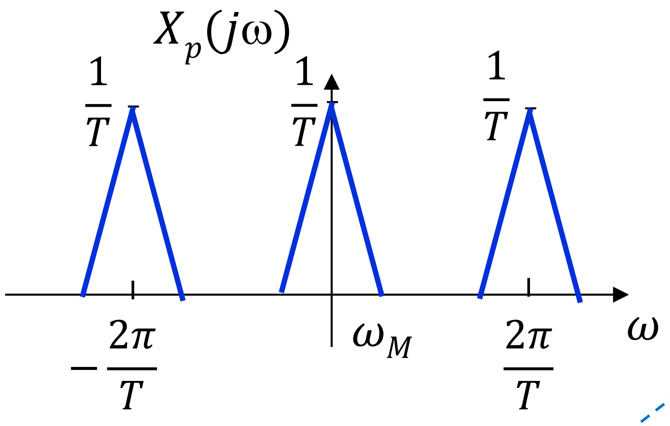
$X_c(j\omega)$: CTFT of bandlimited CT signal $x_c(t)$



$X_p(j\omega)$: CTFT of sampled impulse train $x_p(t) = x_c(t)\,p(t)$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\omega - k\frac{2\pi}{T}\right)\right)$$

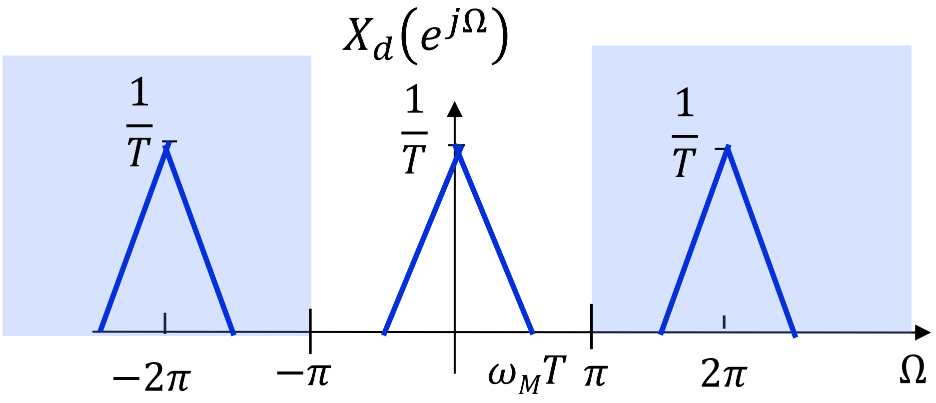
Poisson sum



$X_d(e^{j\Omega})$: DTFT of sampled DT signal $x_d[n] = x_c(nT)$

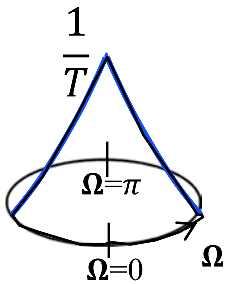
$$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right)$$

Frequency scaling

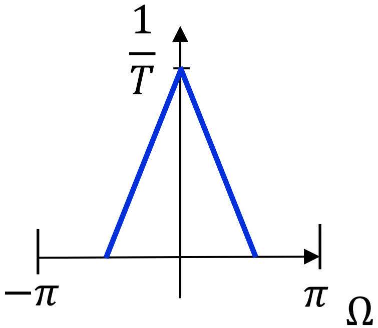


When T is small enough, $X_d(e^{j\Omega})$ is equivalent to $X_c(j\omega)$ within the DT operating frequency range:

$$\boxed{X_d(e^{j\Omega}) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right),\quad -\pi < \Omega \le \pi}$$



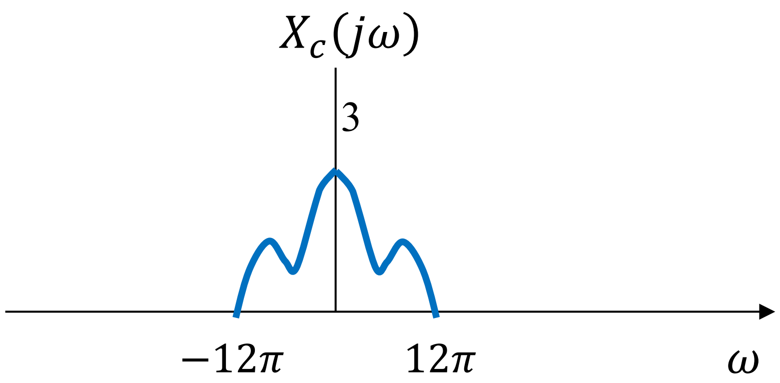
DT frequency is circular



DT frequency is from $-\pi$ to π only

Example

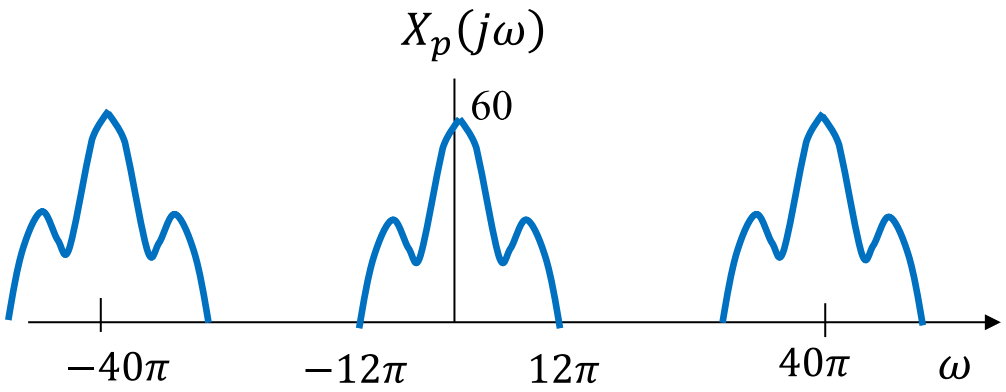
The spectrum of CT signal $x_c(t)$ is as shown:



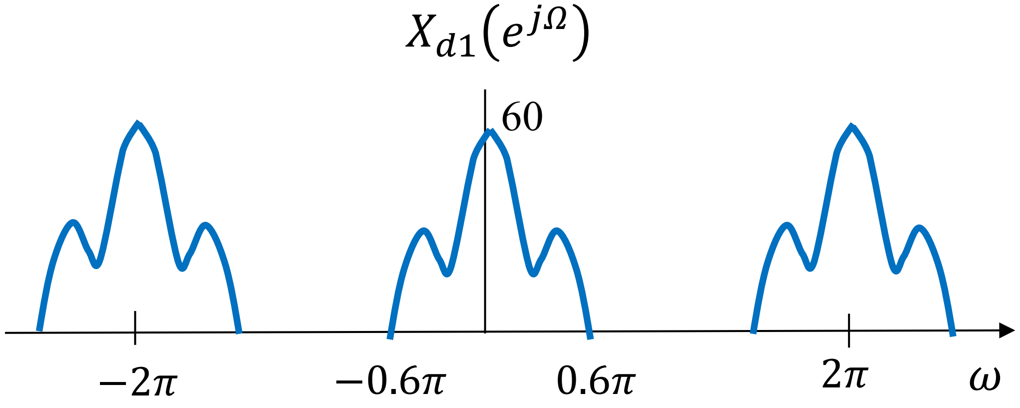
1. Sketch the spectrum of $x_{d1}[n] = x_c(0.05n)$

$T = 0.05 \Rightarrow f_s = 20 \Rightarrow \omega_s = 40\pi$

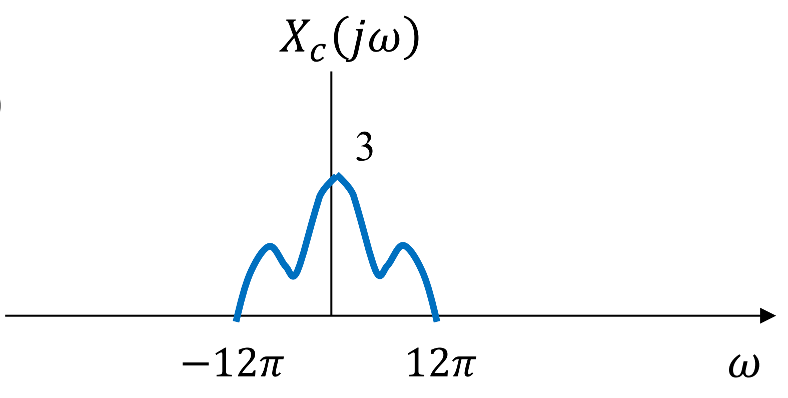
Spectrum of sampled impulse train is Poisson sum of $X_c(j\omega)$ with scaling by $1/T$



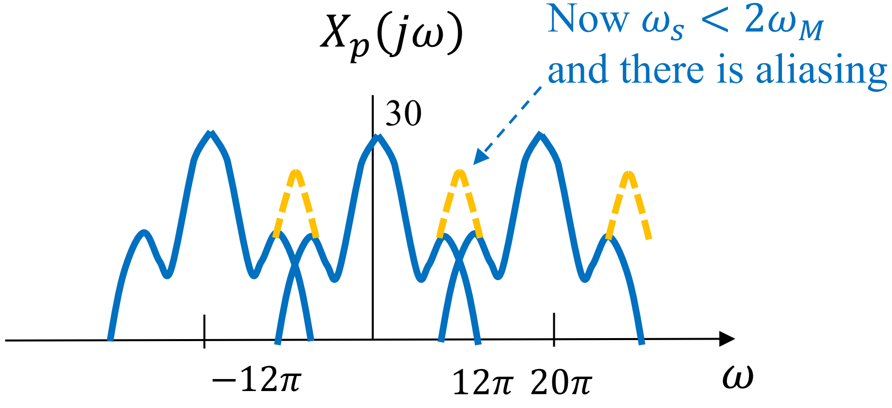
Spectrum of sampled DT signal is frequency-scaled version of $X_p(j\omega)$. Frequency scaling is by $f_s = 20$.



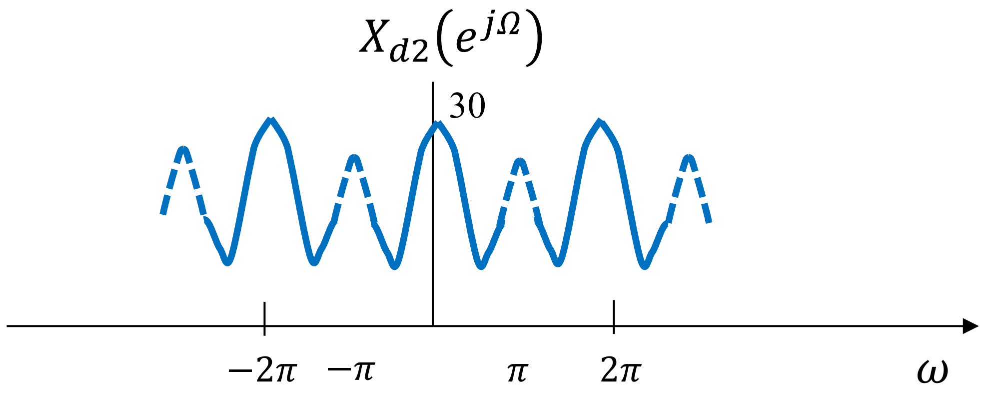
2. Sketch the spectrum of $x_{d2}[n] = x_c(0.1n)$



$T = 0.1 \Rightarrow f_s = 10 \Rightarrow \omega_s = 20\pi$



Spectrum of sampled DT signal is again a frequency-scaled version of $X_p(j\omega)$. Frequency scaling is by $f_s = 10$.



Lecture 17

Chapter 7: Application of Sampling Theorem

I. Equivalency of a bandlimited signal and its Sampled Sequence

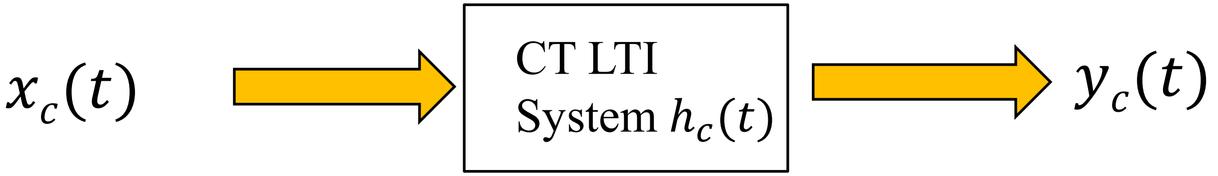
II. Discrete-time processing of CT signals

III. Example – Digital AAF by Over-Sampling

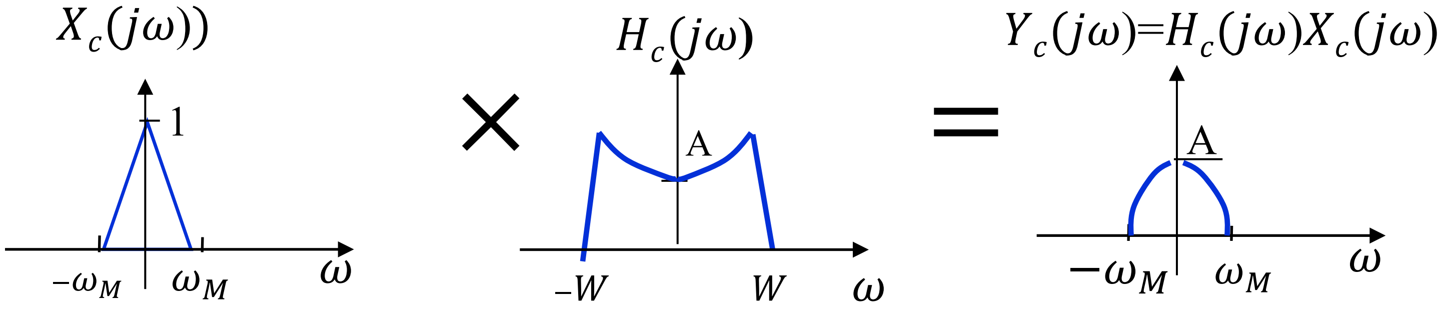
IV. Discrete Approximation of Filters

II. Discrete-Time processing of CT signals

- Suppose we want to filter a bandlimited CT signal $x_c(t)$ using an LTI system to produce the output $y_c(t)$.



- In frequency domain, $Y_c(j\omega) = H_c(j\omega)X_c(j\omega)$.



- Given the equivalency of a bandlimited CT signal and its sampled DT signal, we can produce the same output by converting $x_c(t)$ to DT and applying a DT filter $H_d(e^{j\Omega})$ that is "equivalent" to $H_c(j\omega)$.

- To process CT signals in DT, conceptually we need to take the following steps:

  1. Convert CT input to its DT sampled sequence: $x_d[n] = x_c(nT)$
  2. Apply an equivalent DT filter $H_d(e^{j\Omega})$
  3. Convert DT output to CT by interpolation/ILP with scaling

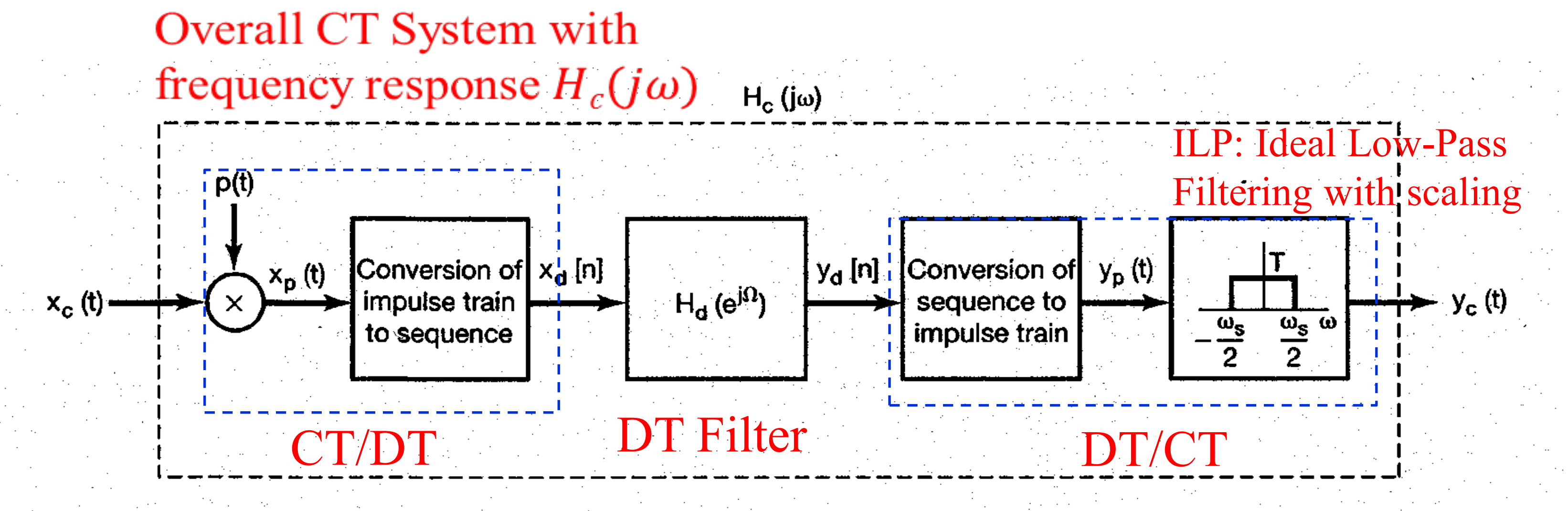


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.

Again, we use the symbols Ω and ω to refer to DT and CT frequency to avoid confusion.

The Equivalent DT Filter

- The frequency response of the DT filter simply needs to be the same frequency scaled version of the CT filter:

$$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \leq \pi}$$

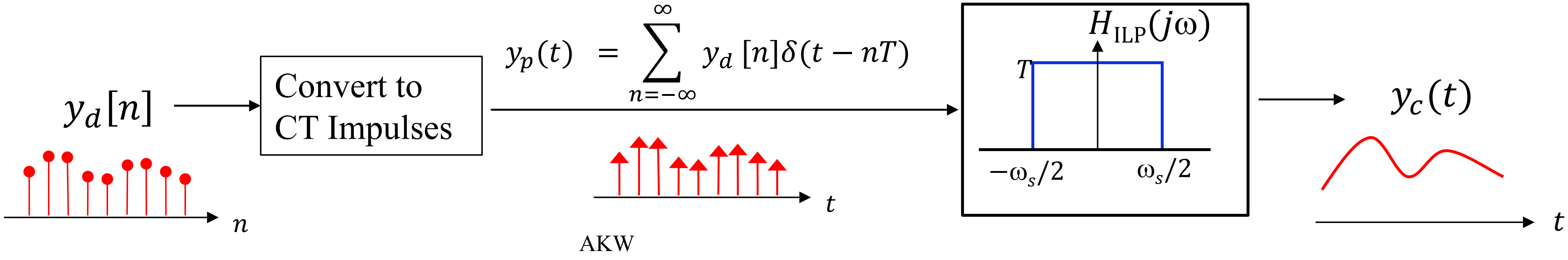
- Then in the frequency range $|\Omega| \leq \pi$, the output

$$Y_d(e^{j\Omega}) = X_d(e^{j\Omega})H_d(e^{j\Omega}) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right)H_c\left(j\frac{\Omega}{T}\right) = \frac{1}{T}Y_c\left(j\frac{\Omega}{T}\right) \quad \text{for } |\Omega| \leq \pi$$

- This means $y_d[n]$ is the sample sequence of the CT output $y_c(t)$:

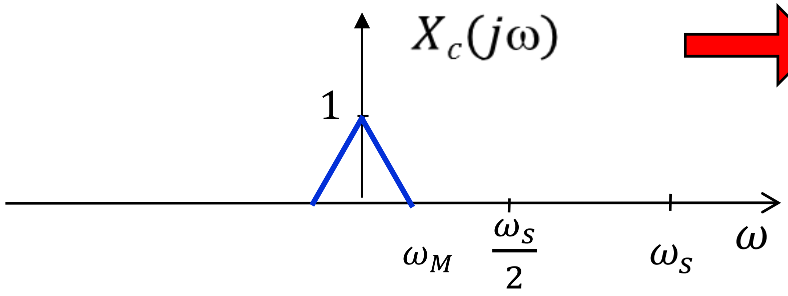
$$y_d[n] = y_c(nT)$$

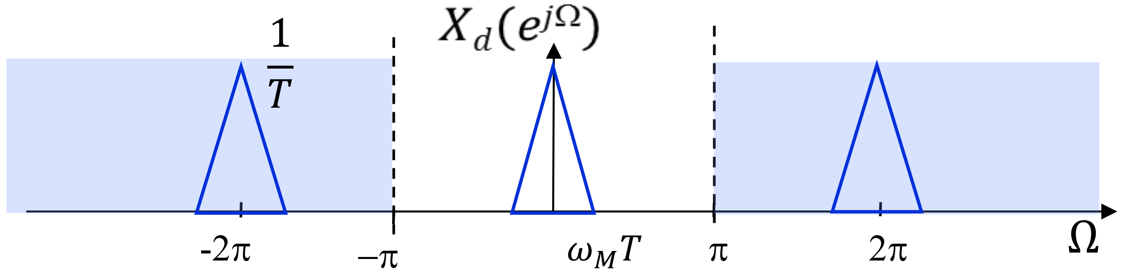
and we can convert $y_d[n]$ to $y_c(t)$ by converting $y_d[n]$ to a sampled impulse train and pass it through an ILP with scaling:



C/D Sampling

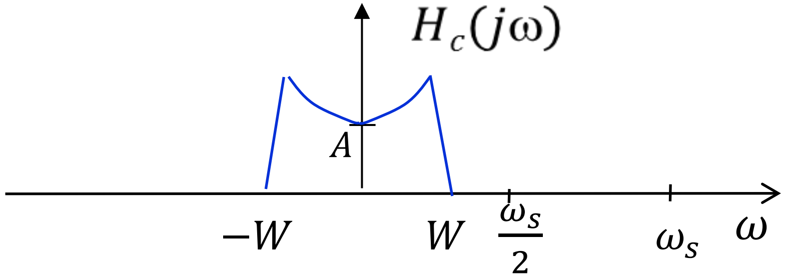
$x_d[n] = x_c(nT)$

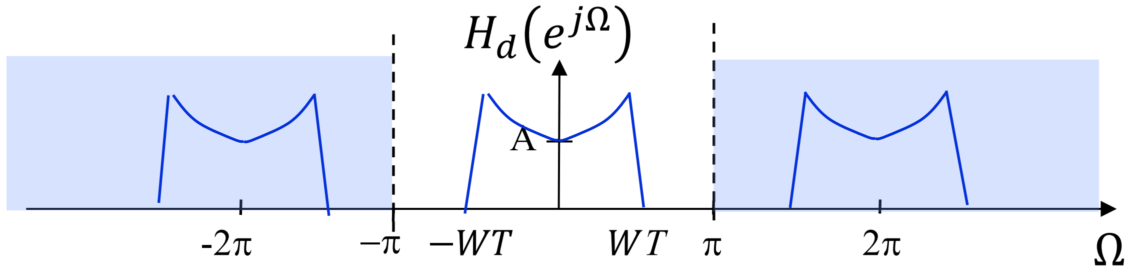




Apply DT filter w/ frequency-scaled frequency response

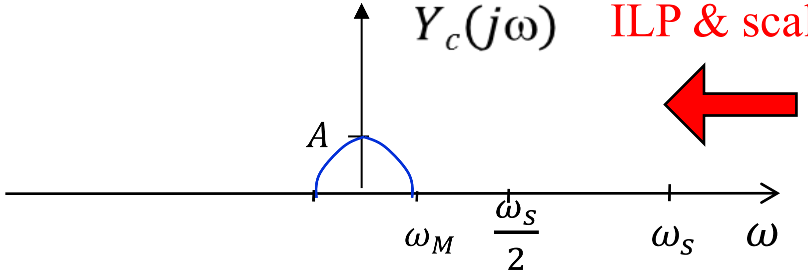
$$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad -\pi \leq \Omega < \pi}$$



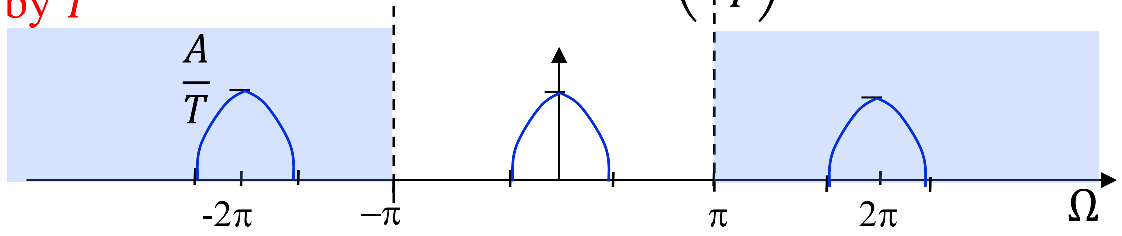


D/C Extrapolation

ILP & scale by T



$Y_d(e^{j\Omega}) = Y_p\left(j\frac{\Omega}{T}\right)$



AKW

Impulse Response of the Equivalent DT Filter

- The frequency response of the equivalent DT filter is a frequency-scaled version of the CT filter:

DTFT of equivalent DT filter CTFT of $h_c(t)$

$$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \leq \pi}$$

- Can we also say that the impulse response $h_d[n]$ of the equivalent DT filter are the sample values (with scaling) of the impulse response $h_c(t)$ of the CT filter?

The answer is **yes** *if* $H_c(j\omega)$ is also $\frac{\omega_s}{2}$-bandlimited by the same argument we went through for signals.

- All these constant scaling by T and $\frac{1}{T}$ seem very confusing. In real implementation we can simply do one scaling once and for all at the end. Sometimes we do not care about the scaling at all (we need to apply amplification anyway or we simply want to compare results)

Summary - Discrete-time processing of continuous-time signals

$x_c(t)$ is band-limited and we want to filter it with a CT filter $H_c(j\omega)$.

1. Sample $x(t)$ at above Nyquist rate to obtain DT signal $x_d[n] = x_c(nT)$
2. Apply DT filter $H_d(e^{j\Omega})$ such that

$$\boxed{H_d\left(e^{j\Omega}\right) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \leq \pi}$$

3. DT output will be sampled value of intend CT output; i.e., $y_d[n] = y_c(nT)$, and we can recover $y_c(t)$ by converting $y_d[n]$ to a sampled impulse train $y_p(t)$, low-pass filtering, and scaling by T.
4. If the impulse response $h_c(t)$ of the CT filter is band-limited, we can also say that the impulse response of the needed DT filter is the sampled value of $h_c(t)$ with scaling by T. That is, $h_d[n] = Th(nT)$.

Today, most signal processing is done in DT using chips and computers!

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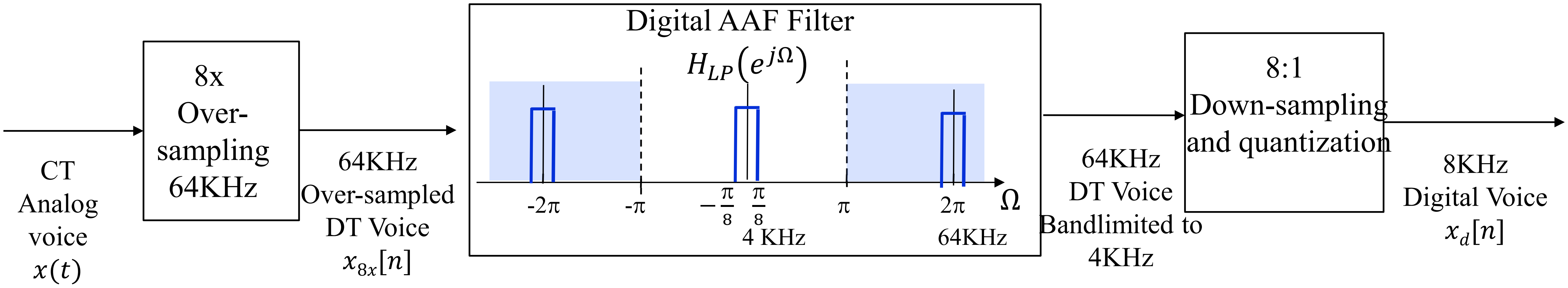
III. Example – AAF by Over-Sampling

IV. Discrete Approximation of Filters

III. Over-Sampling and Digital AAF Filter

- Last lecture, we explained that the telephone network must apply AAF before sampling our voice signal at 8KHz. *But how do we do this low pass filtering*?
- In the past, we added inductor coils to the subscriber loop as LPF. In modern days, we use digital LPF in the line-card of telephone switches. Digital LPF is better and cheaper.
- How do we apply digital filter while the signal is analog?
- Here is the idea of over-sampling: we sample the analog signal at many times (say 8x) the intended sampling rate to ensure no aliasing, filter the over-sampled DT signal using DT filter, and then down-sample to the intended sampling rate.



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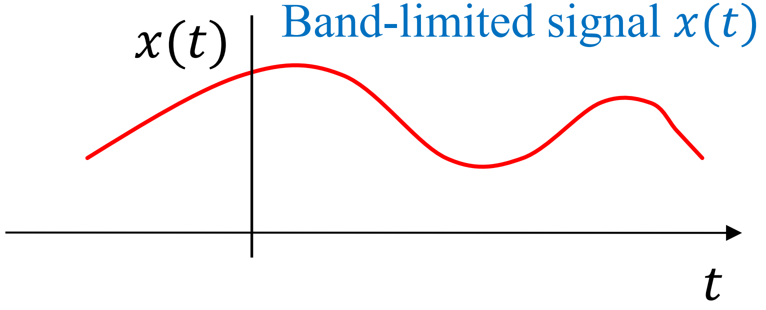
IV. Discrete Approximation of Filters

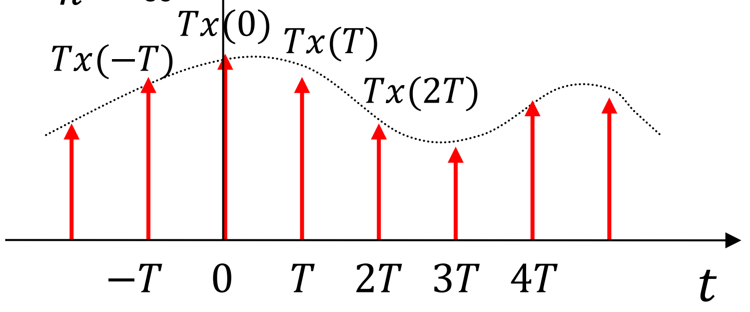
In some cases, CT filters can also be very effective. One way to build CT filter is by discrete approximation.

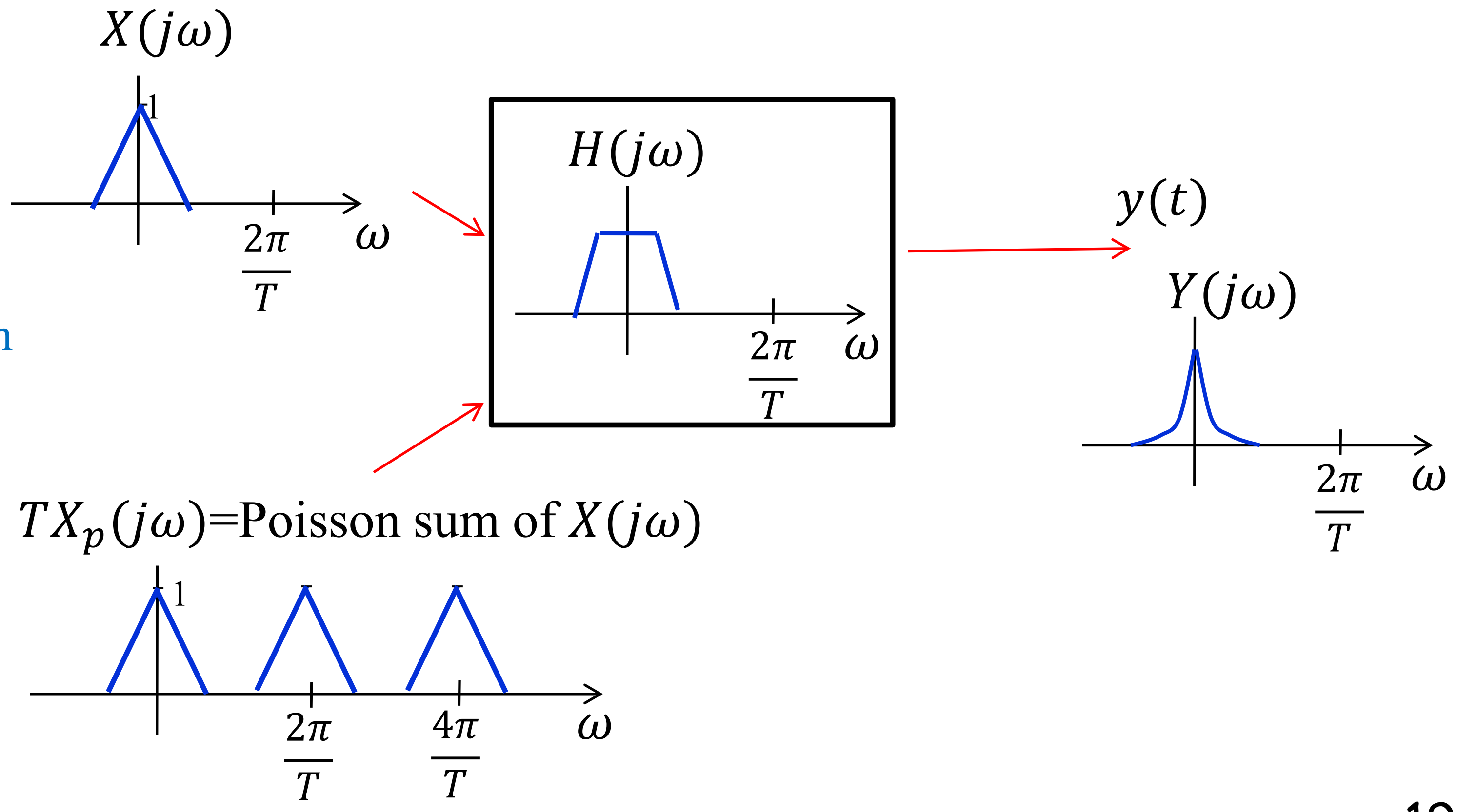
From study of the sampling theorem, we see that a bandlimited signal $x(t)$ and its scaled sampled impulse train version $Tx_p(t)$ lead to the same output when passing through the same bandlimited system $H(j\omega)$.



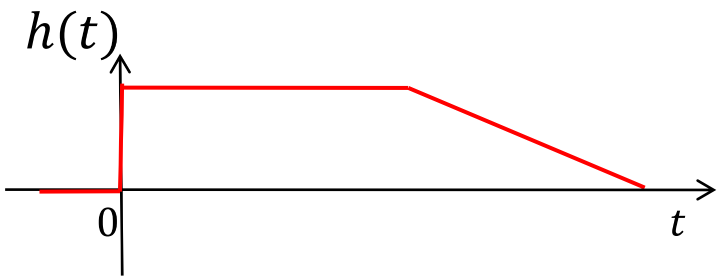
Scaled sampled impulse train

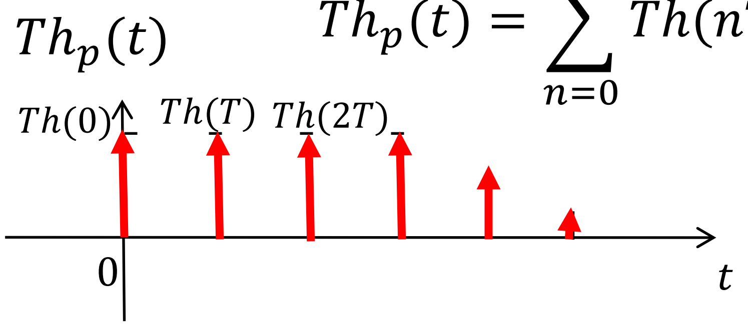
$$Tx_p(t) = \sum_{n=-\infty}^{\infty} Tx(nT)\delta(t-nT)$$



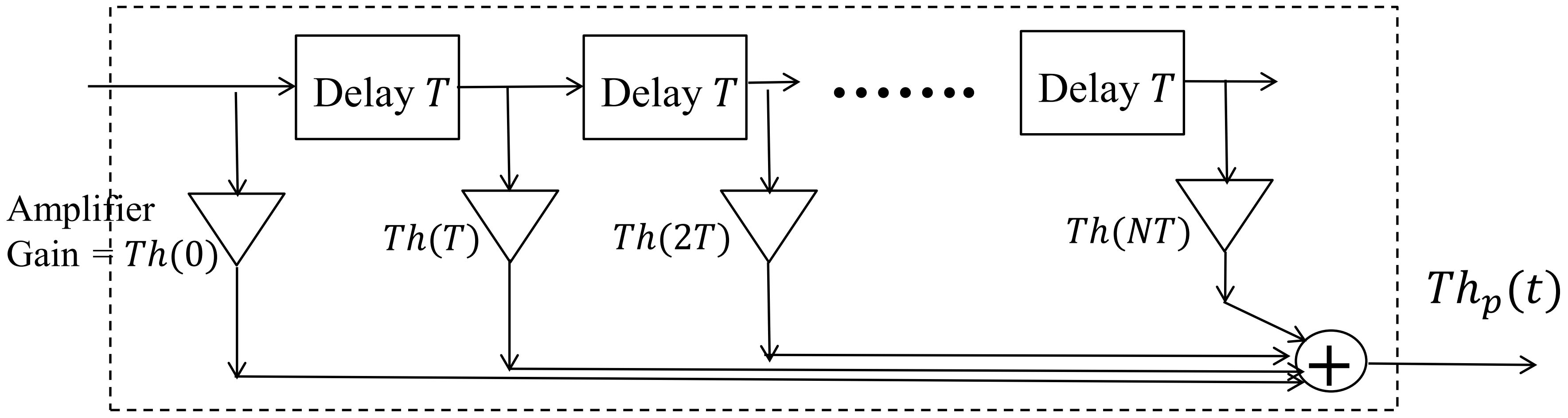


Because LTI systems are commutative, we can apply the same insight to systems. Given impulse response $h(t)$, we can implement a discretized system $Th_p(t)$ using a network of delay lines and amplifiers as shown below:

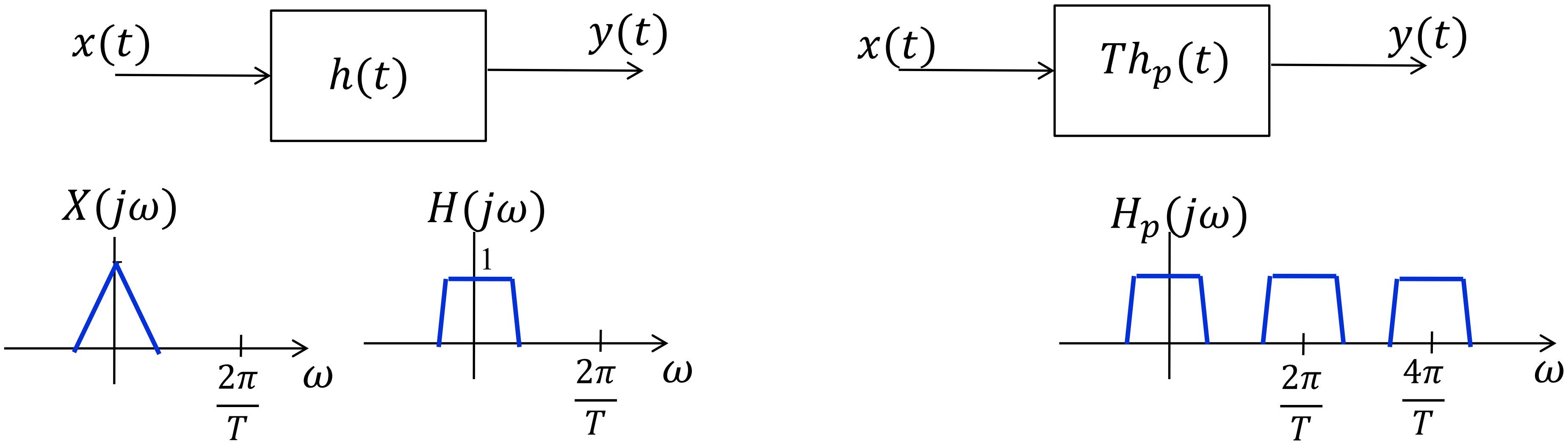




$$Th_p(t) = \sum_{n=0}^{N} Th(nT)\delta(t-nT)$$



If $h(t)$ is used to process a signal $x(t)$, and both $h(t)$ and $x(t)$ are bandlimited, then $h(t)$ and $Th_p(t)$ are equivalent



- In every 2G/3G cell phone and there are multiple **SAW** (Surface Acoustic Wave) filters, which are based on discrete approximation to build continuous-time band-pass and other filters. In SAW, electrical signals are turned into acoustic waves which travel across a substrate, and then back into electrical signals. The distance for the acoustic wave to travel across the substrate serves as the delay.

- In newer phones, SAW filters are replaced by Film Bulk Acoustic Resonator (FBAR).

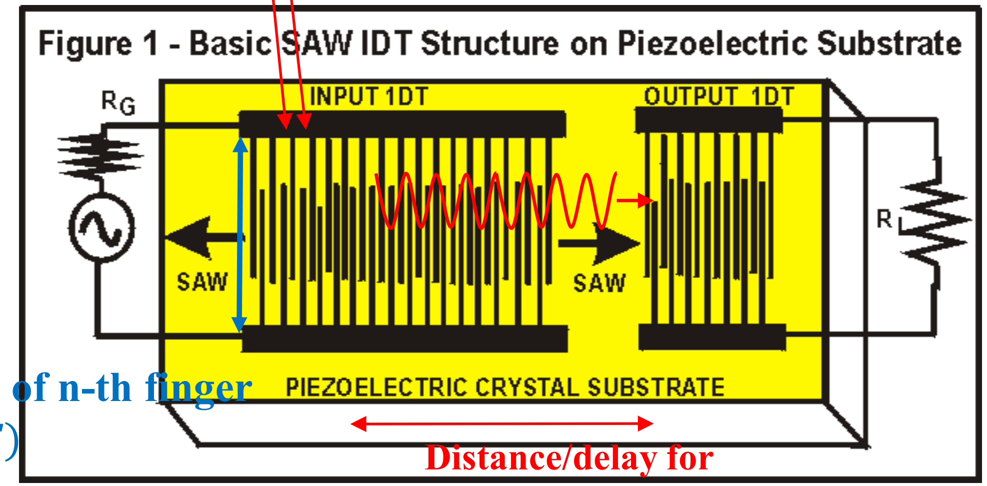
Example - Working Principle of SAW Filters

- Input IDT (Inducer Transmitter) made up of electrode "fingers" laid on top of a piezoelectric substrate at spacing Δ.
- When input electrical signal is applied, each finger induces an acoustic wave with strength proportional to the finger's length. The acoustic wave travels at velocity v to the output IDT and is converted back into an electrical signal.
- Propagation delay of acoustic wave from the n-th finger is $\tau_{offset} + nT$ where $T = \Delta/v$.
- Hence, ignoring the offset delay, the impulse response of the filter is:

$$h_{SAW}(t) = \sum_{n=0}^{N} h(nT)\delta(t - nT)$$

Where $h(nT)$ is determined by the length of the n-th finger

Electrode Fingers at spacing Δ

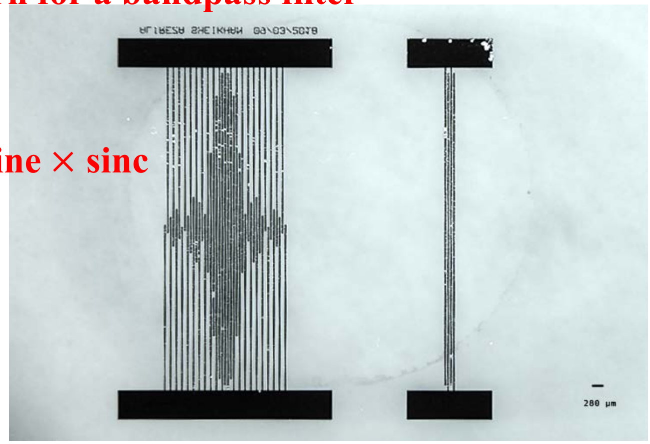


Length of n-th finger $= h(nT)$

Distance/delay for n-th finger

Finger pattern for a bandpass filter

cosine × sinc

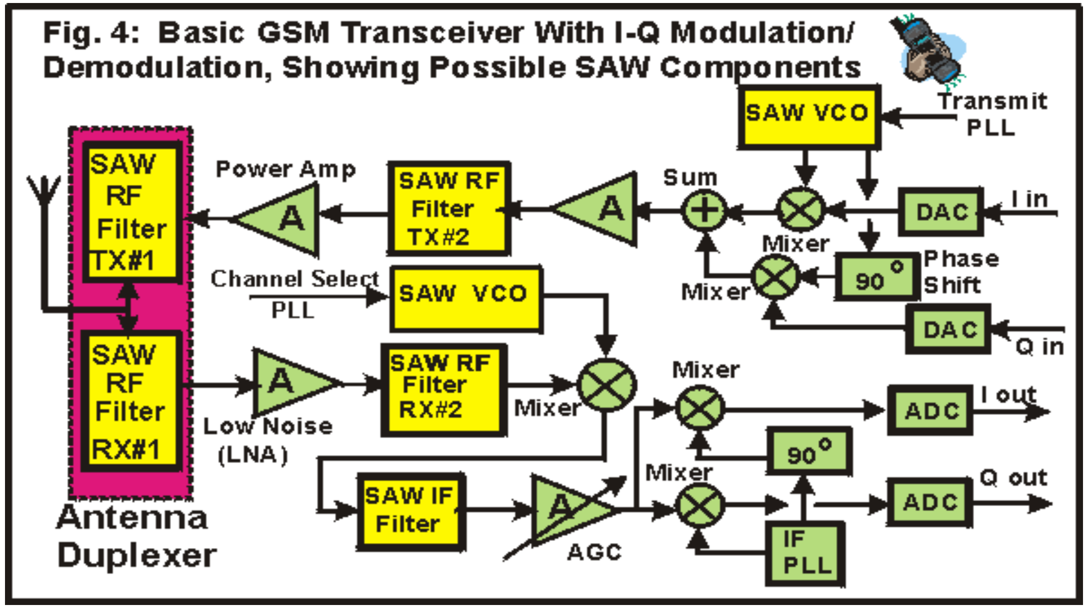


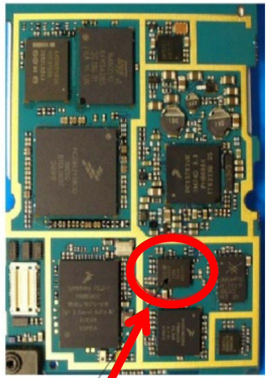
Assume that the maximum frequency in $x(t)$ and $h(t)$ is 1 GHz, and v is 4000 meters/sec (13x of velocity of sound in air). To avoid aliasing, we need:

$$T < \frac{1}{2 \times 1GHz} = 0.5ns$$

Hence spacing between fingers should be <0.5ns × 4000 m/s = 2μm

Assume there are 1,000 fingers, the total width of the input IDT is less than 2 mm.
Modern SAW filters are implemented as VLSI "chips".





SAW filter chip