In [71]:
```python
import pandas as pd
import numpy as np
import sympy as sp
import math as m
from sympy import collect, simplify, expand, fraction, latex, diff, cancel, nsimplify
from IPython.display import display, Markdown, Math
from scipy.integrate import odeint
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = [20, 10]
```

In [72]:
```python
class numden_coeff:
    def __init__(self, expr, symb):
        self.num, self.denum = fraction(expr)
        self.symb = symb
        self.common_factor = None
        self.lst_denum_coeff = self.build_lst(self.denum)
        self.lst_num_coeff = self.build_lst(self.num)

    def build_lst(self, poly):
        order = sp.Poly(poly, self.symb).degree()
        lst = [expand(poly).coeff(self.symb**i) for i in range((order), 0, -1)]
        lst.append(poly.subs(self.symb,0))
        if (self.common_factor == None):
            self.common_factor = lst[0]

        lst = [simplify(lst[i]/self.common_factor) for i in range(order + 1)]
        return lst

    def disp(self):
        display(Markdown(r"Numerator coefficients (\beta)"), self.lst_num_coeff)
        display(Markdown(r"Denominator coefficients (alpha)"), self.lst_denum_coeff)
```

Problem 1

Part 1

In [73]:
```python
a, b, p, e = sp.symbols("a b p e")
zeta, omega, gamma_prime, gamma, theta1, theta2 = sp.symbols("zeta omega \\gamma^{'} gamma theta_1 theta_2")
y, u, uc, ym = sp.symbols("y(t) u(t) u_{c}(t) y_m")

y_eq = sp.solve(sp.Eq(y*p**2, (-a*p*y + b*u)), y)[0]

u_eq = sp.solve(sp.Eq(u, (theta1*(y - uc) - theta2*p*y)), u)[0]
y_eq = sp.solve(sp.Eq(y, y_eq.subs(u, u_eq)), y)[0]
display(Math("y = "+latex(y_eq)))
```

$$y = -\frac{b\theta_1 u_c(t)}{ap + bp\theta_2 - b\theta_1 + p^2}$$

The above equation is y in which $-p\theta_2 y(t) - \theta_1 u_c(t) + \theta_1 y(t)$ has been subbed in for u

In [74]:
```python
bm0, am1, am0 = sp.symbols("b_{m0} a_{m1} a_{m0}")

b_m0 = omega**2
a_m1 = 2*zeta*omega
a_m0 = b_m0

# Bm = bm0
# Am = (p**2 + am1*p + bm0)

Bm = omega**2
Am = (p**2 + 2*zeta*omega*p + omega**2)

Gm = Bm/Am
Gm
display(Math("G_{m} = "+latex(Gm)))
```

$$G_m = \frac{\omega^2}{\omega^2 + 2\omega p\zeta + p^2}$$

Next, the assumption that the plant y will follow exactly the reference model y_m is made to derive θ_1 and θ_2. This yeilds

In [75]:
```python
num, den = fraction(y_eq)
num_m, den_m = fraction(Gm*uc)

theta_1 = sp.solve(sp.Eq(num, num_m), theta1)[0]
theta_2 = sp.solve(sp.Eq(den.subs(theta1, theta_1), den_m), theta2)[0]

display(Math("\\theta_1 =\;"+latex(theta_1)))
display(Math("\\theta_2 =\;"+latex(theta_2)))
```

$$\theta_1 = -\frac{\omega^2}{b}$$

$$\theta_2 = \frac{-a + 2\omega\zeta}{b}$$

Next, the sensistivity of the error to θ_1 and θ_2 was derived. This will be used to derive equations for $\dot{\theta}_1$ and $\dot{\theta}_2$. The sensitivities $\frac{\partial e}{\partial \theta_1}$ and $\frac{\partial e}{\partial \theta_2}$ can be seen below

In [76]:
```python
del_e_theta1 = collect(simplify(diff(y_eq, theta1)), p)
del_e_theta2 = collect(simplify(diff(y_eq, theta2)), p)

display(Math("\\frac{\\partial e}{\\partial \\theta_1} =\;"+latex(del_e_theta1)))
display(Math("\\frac{\\partial e}{\\partial \\theta_2} =\;"+latex(del_e_theta2)))
```

$$\frac{\partial e}{\partial \theta_1} = -\frac{bpu_c(t)\left(a + b\theta_2 + p\right)}{\left(-b\theta_1 + p^2 + p\left(a + b\theta_2\right)\right)^2}$$

$$\frac{\partial e}{\partial \theta_2} = \frac{b^2 p\theta_1 u_c(t)}{\left(-b\theta_1 + p^2 + p\left(a + b\theta_2\right)\right)^2}$$

These equations can be further simplified by deriving an equation for u_c in terms of y_m. This way, the sensistivitives can be expressed in terms of y_m, a variable in which we have an equation. The equation for u_c and the new equations for $\frac{\partial e}{\partial \theta_1}$ and $\frac{\partial e}{\partial \theta_2}$ in terms of y_m can be seen below

In [77]:
```python
u_c = sp.solve(sp.Eq(ym, Gm*uc), uc)[0]

del_e_theta1_subd = del_e_theta1.subs([(uc,u_c),(theta1, theta_1), (theta2, theta_2)])
del_e_theta2_subd = del_e_theta2.subs([(uc,u_c),(theta1, theta_1), (theta2, theta_2)])

display(Math("u_c =\;"+latex(u_c)))
display(Math("\\frac{\\partial e}{\\partial \\theta_1} =\;"+latex(del_e_theta1_subd)))
display(Math("\\frac{\\partial e}{\\partial \\theta_2} =\;"+latex(del_e_theta2_subd)))
```

$$u_c = \frac{y_m\left(\omega^2 + 2\omega p\zeta + p^2\right)}{\omega^2}$$

$$\frac{\partial e}{\partial \theta_1} = -\frac{bpy_m\left(2\omega\zeta + p\right)}{\omega^2\left(\omega^2 + 2\omega p\zeta + p^2\right)}$$

$$\frac{\partial e}{\partial \theta_2} = -\frac{bpy_m}{\omega^2 + 2\omega p\zeta + p^2}$$

Next, equations for $theta_1$ and $theta_2$ were derived using the equation $\dot{\theta} = -\gamma' e\frac{\partial e}{\partial \theta}$, the results of which can be seen below

In [78]:
```python
theta1_dot = -gamma_prime*e*del_e_theta1_subd
theta2_dot = -gamma_prime*e*del_e_theta2_subd

display(Math("\\dot{\\theta}_1 =\;"+latex(theta1_dot)))
display(Math("\\dot{\\theta}_2 =\;"+latex(theta2_dot)))
```

$$\dot{\theta}_1 = \frac{\gamma' bepy_m\left(2\omega\zeta + p\right)}{\omega^2\left(\omega^2 + 2\omega p\zeta + p^2\right)}$$

$$\dot{\theta}_2 = \frac{\gamma' bepy_m}{\omega^2 + 2\omega p\zeta + p^2}$$

Letting $\gamma = \gamma' b$ gives

In [79]:
```python
theta1_dot_subd = theta1_dot*gamma/(gamma_prime*b)
theta2_dot_subd = theta2_dot*gamma/(gamma_prime*b)

display(Math("\\dot{\\theta}_1 =\;"+latex(theta1_dot_subd)))
display(Math("\\dot{\\theta}_2 =\;"+latex(theta2_dot_subd)))
```

$$\dot{\theta}_1 = \frac{e\gamma py_m\left(2\omega\zeta + p\right)}{\omega^2\left(\omega^2 + 2\omega p\zeta + p^2\right)}$$

$$\dot{\theta}_2 = \frac{e\gamma py_m}{\omega^2 + 2\omega p\zeta + p^2}$$

These equations were develloped into ODEs in which the ODE solver could digest

In [80]:
```python
obj_theta1_dot = numden_coeff(theta1_dot_subd/ym, p)
obj_theta2_dot = numden_coeff(theta2_dot_subd/ym, p)

atheta1 = obj_theta1_dot.lst_denum_coeff[::-1]
btheta1 = obj_theta1_dot.lst_num_coeff[::-1]

atheta2 = obj_theta2_dot.lst_denum_coeff[::-1]
btheta2 = obj_theta2_dot.lst_num_coeff[::-1]

display(Math("\\alpha\\dot{\\theta}_1 =\;"+latex(atheta1)))
display(Math("\\beta\\dot{\\theta}_1 =\;"+latex(btheta1)))

display(Math("\\alpha\\dot{\\theta}_2 =\;"+latex(atheta2)))
display(Math("\\beta\\dot{\\theta}_2 =\;"+latex(btheta2)))
```

$$\alpha\dot{\theta}_1 = \left[\omega^2,\ 2\omega\zeta,\ 1\right]$$

$$\beta\dot{\theta}_1 = \left[0,\ \frac{2e\gamma\zeta}{\omega},\ \frac{e\gamma}{\omega^2}\right]$$

$$\alpha\dot{\theta}_2 = \left[\omega^2,\ 2\omega\zeta,\ 1\right]$$

$$\beta\dot{\theta}_2 = \left[0,\ e\gamma\right]$$

In [81]:
```python
gamma_val = 7.5
omega_val = 1.5
zeta_val = 0.6

ym_d, ym_dd = sp.symbols("\\dot{y}_{m} \\ddot{y}_{m}")
theta1d, theta1dd = sp.symbols("\\dot{\\theta}_{1} \\ddot{\\theta}_{1}")
theta2d, theta2dd = sp.symbols("\\dot{\\theta}_{2} \\ddot{\\theta}_{2}")

theta1_ddd = -atheta1[0]*theta1d - atheta1[1]*theta1dd + btheta1[1]*ym_d + btheta1[2]*ym_dd
theta2_ddd = -atheta2[0]*theta2d - atheta2[1]*theta2dd + btheta2[1]*ym_d

theta1_ddd_subd = theta1_ddd.subs([(gamma, gamma_val),(omega,omega_val), (zeta, zeta_val)])
theta2_ddd_subd = theta2_ddd.subs([(gamma, gamma_val),(omega,omega_val), (zeta, zeta_val)])

theta1_ddd_func = sp.lambdify([theta1d, theta1dd, ym_d, ym_dd, e], theta1_ddd_subd)
theta2_ddd_func = sp.lambdify([theta2d, theta2dd, ym_d, e], theta2_ddd_subd)

display(Math("\\dddot{\\theta}_1 =\;"+latex(theta1_ddd)+"\;=\;"+latex(theta1_ddd_subd)))
display(Math("\\dddot{\\theta}_1 =\;"+latex(theta2_ddd)+"\;=\;"+latex(theta2_ddd_subd)))
```

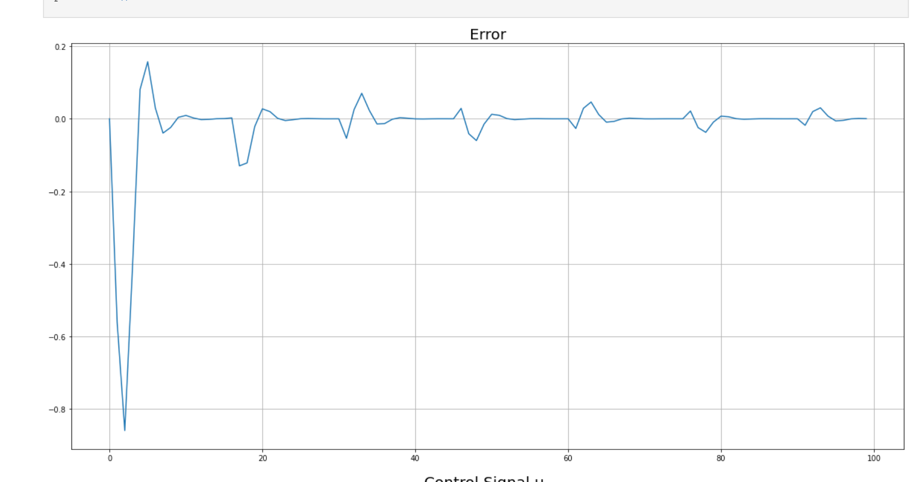
$$\dddot{\theta}_1 = -2\ddot{\theta}_1\omega\zeta + \frac{\ddot{y}_m e\gamma}{\omega^2} - \dot{\theta}_1\omega^2 + \frac{2\dot{y}_m e\gamma\zeta}{\omega} = -1.8\ddot{\theta}_1 + 3.33333333333333\ddot{y}_m e - 2.25\dot{\theta}_1 + 6.0\dot{y}_m e$$

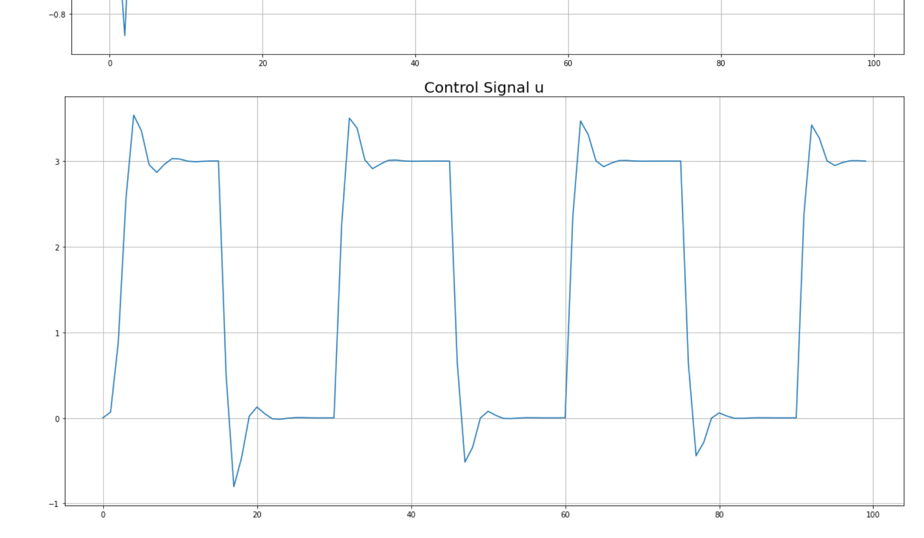
$$\dddot{\theta}_1 = -2\ddot{\theta}_2\omega\zeta - \dot{\theta}_2\omega^2 + \dot{y}_m e\gamma = -1.8\ddot{\theta}_2 - 2.25\dot{\theta}_2 + 7.5\dot{y}_m e$$

Part 2.1 (MIT)

In [82]:
```python
def ode_solver(y0, t, a, b, omega, zeta, gamma):

    ym, ym_dot = y0[0], y0[1]
    y, y_dot = y0[2], y0[3]
    theta1, theta1_dot, theta1_dotdot = y0[4], y0[5], y0[6]
    theta2, theta2_dot, theta2_dotdot = y0[7], y0[8], y0[9]
    u = y0[10]
    u_c_ode = m.sin(m.pi*t/15) >= 0

    ym_dotdot = -2*omega*zeta*ym_dot - omega**2*ym + omega**2*u_c_ode
    y_dotdot = -(a + b*theta2)*y_dot + b*theta1*(y - u_c_ode)
    e = y_dot - ym_dot

    theta1_dotdotdot = theta1_ddd_func(theta1_dot, theta1_dotdot, ym_dot, ym_dotdot, e)
    theta2_dotdotdot = theta2_ddd_func(theta2_dot, theta2_dotdot, ym_dot, e)

    u = theta1*(y - u_c_ode) - theta2*y_dot

    return [ym_dot, ym_dotdot,
            y_dot, y_dotdot,
            theta1_dot, theta1_dotdot, theta1_dotdotdot,
            theta2_dot, theta2_dotdot, theta2_dotdotdot,
            u]
```

In [83]:
```python
T_val = 0.1
sample_depth = int(10/T_val) # 1000 samples totalling 100 seconds (since sample time T is 0.1 secons)
sample_range = range(sample_depth)
starting_samples = 3

gamma_val = 5
omega_val = 1.5
zeta_val = 0.6
a_val = 3
b_val = 1

# calculation of input signal
t = [i for i in sample_range]
u_c = np.ones(sample_depth)
u_c[np.where([m.sin(t[i]*m.pi*T_val/15)<=0 for i in sample_range])] = 0

y0 = [0]*11

ode_res = odeint(ode_solver, y0, t, args=(a_val, b_val,
                                         omega_val,
                                         zeta_val,
                                         gamma_val))
plt.title("Error", fontsize=20)
plt.plot(t, ode_res[:,2] - ode_res[:,0])
plt.grid()
plt.show()

plt.title("Control Signal u", fontsize=20)
plt.plot(t, ode_res[:,10])
plt.grid()
plt.show()
```





Part 2.2 (Normalized MIT)

For the noramlized MIT rule, the equation for theta needed to be updated. The equations derived for $\dot{\theta}_i$ are the sensistivity equations multiplied by γe therfore, the same procedure for building hte $\dot{\theta}_i$ equations can be reused by simply divinding everything by γe at the beggining. These new equations will keep the same names as before in the code as to not make many updates to the existing code. the only difference will be setting the actual $\dot{\theta}_i$ equal to $\frac{\gamma\psi\frac{\partial e}{\partial \theta}}{\alpha + \psi^2}$. Additionally, the fact that $\psi = -\frac{\partial e}{\partial \theta}$ will have to be taken into account.

In [84]:
```python
obj_theta1_dot = numden_coeff(theta1_dot_subd/(ym*e*gamma), p)
obj_theta2_dot = numden_coeff(theta2_dot_subd/(ym*e*gamma), p)

atheta1 = obj_theta1_dot.lst_denum_coeff[::-1]
btheta1 = obj_theta1_dot.lst_num_coeff[::-1]

atheta2 = obj_theta2_dot.lst_denum_coeff[::-1]
btheta2 = obj_theta2_dot.lst_num_coeff[::-1]

display(Math("\\alpha \\frac{\\partial e}{\\partial \\theta_1} =\;"+latex(atheta1)))
display(Math("\\beta\\frac{\\partial e}{\\partial \\theta_1} =\;"+latex(btheta1)))

display(Math("\\alpha\\frac{\\partial e}{\\partial \\theta_2} =\;"+latex(atheta2)))
display(Math("\\beta\\frac{\\partial e}{\\partial \\theta_2} =\;"+latex(btheta2)))
```

$$\alpha\frac{\partial e}{\partial \theta_1} = \left[\omega^2,\ 2\omega\zeta,\ 1\right]$$

$$\beta\frac{\partial e}{\partial \theta_1} = \left[0,\ \frac{2\zeta}{\omega},\ \frac{1}{\omega^2}\right]$$

$$\alpha\frac{\partial e}{\partial \theta_2} = \left[\omega^2,\ 2\omega\zeta,\ 1\right]$$

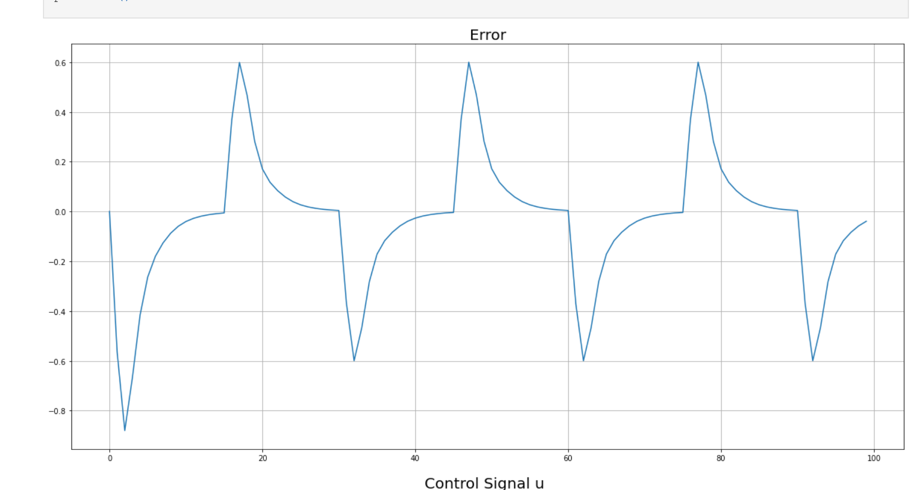
$$\beta\frac{\partial e}{\partial \theta_2} = \left[0,\ 1\right]$$

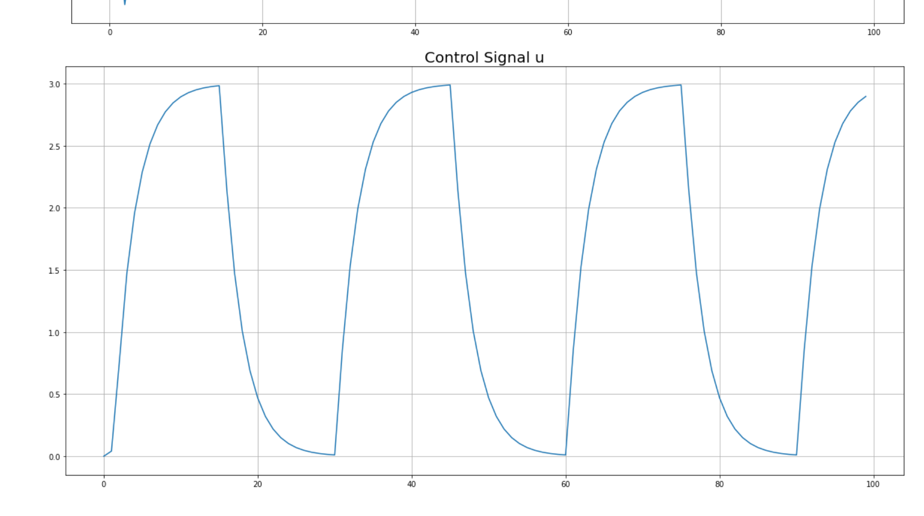
The below equations are named θ_i simply becasue the code template derived above was reused to minimise the updates needed to the code. These equations are actualy for $\frac{\partial e}{\partial \theta_i}$

In [85]:
```python
gamma_val = 7.5
omega_val = 1.5
zeta_val = 0.6

ym_d, ym_dd = sp.symbols("\\dot{y}_{m} \\ddot{y}_{m}")
theta1d, theta1dd = sp.symbols("\\dot{\\theta}_{1} \\ddot{\\theta}_{1}")
theta2d, theta2dd = sp.symbols("\\dot{\\theta}_{2} \\ddot{\\theta}_{2}")

theta1_ddd = atheta1[0]*theta1d + atheta1[1]*theta1dd + btheta1[1]*ym_d + btheta1[2]*ym_dd
theta2_ddd = atheta2[0]*theta2d + atheta2[1]*theta2dd + btheta2[1]*ym_d

theta1_ddd_subd = theta1_ddd.subs([(gamma, gamma_val),(omega,omega_val), (zeta, zeta_val)])
theta2_ddd_subd = theta2_ddd.subs([(gamma, gamma_val),(omega,omega_val), (zeta, zeta_val)])

theta1_ddd_func = sp.lambdify([theta1d, theta1dd, ym_d, ym_dd], theta1_ddd_subd)
theta2_ddd_func = sp.lambdify([theta2d, theta2dd, ym_d], theta2_ddd_subd)
```

In [86]:
```python
def ode_solver(y0, t, a, b, omega, zeta, gamma, alpha):

    ym, ym_dot = y0[0], y0[1]
    y, y_dot = y0[2], y0[3]
    theta1, theta1_dot, theta1_dotdot = y0[4], y0[5], y0[6]
    theta2, theta2_dot, theta2_dotdot = y0[7], y0[8], y0[9]
    theta1_norm = y0[10]
    theta2_norm = y0[11]
    u = y0[12]
    u_c_ode = m.sin(m.pi*t/15) >= 0

    ym_dotdot = -2*omega*zeta*ym_dot - omega**2*ym + omega**2*u_c_ode
    y_dotdot = -(a + b*theta2_norm)*y_dot + b*theta1_norm*(y - u_c_ode)
    e = y_dot - ym_dot

    theta1_dotdotdot = theta1_ddd_func(theta1_dot, theta1_dotdot, ym_dot, ym_dotdot)
    theta2_dotdotdot = theta2_ddd_func(theta2_dot, theta2_dotdot, ym_dot)

    theta1_n = theta1*e*gamma/(alpha + (theta1)**2)
    theta2_n = theta2*e*gamma/(alpha + (theta2)**2)

    u = theta1_norm*(y - u_c_ode) - theta2_norm*y_dot

    return [ym_dot, ym_dotdot,
            y_dot, y_dotdot,
            theta1_dot, theta1_dotdot, theta1_dotdotdot,
            theta2_dot, theta2_dotdot, theta2_dotdotdot,
            theta1_n,
            theta2_n,
            u]
```

In [87]:
```python
T_val = 0.1
sample_depth = int(10/T_val) # 1000 samples totalling 100 seconds (since sample time T is 0.1 secons)
sample_range = range(sample_depth)
starting_samples = 3

alpha_val = 1
gamma_val = 5
omega_val = 1.5
zeta_val = 0.6
a_val = 3
b_val = 1

# calculation of input signal
t = [i for i in sample_range]
u_c = np.ones(sample_depth)
u_c[np.where([m.sin(t[i]*m.pi*T_val/15)<=0 for i in sample_range])] = 0

y0 = [0]*13

ode_res = odeint(ode_solver, y0, t, args=(a_val, b_val,
                                         omega_val,
                                         zeta_val,
                                         gamma_val,
                                         alpha_val))


plt.title("Error", fontsize=20)
plt.plot(t, ode_res[:,2] - ode_res[:,0])
plt.grid()
plt.show()

plt.title("Control Signal u", fontsize=20)
plt.plot(t, ode_res[:,12])
plt.grid()
plt.show()
```





Problem 2

Part 1

First, an equation for y was derived in terms of u_c. This was done by subbing $u = \theta_1 u_c - \theta_2 y$ into $y = \frac{bu}{p}$. This yeilds

In [88]:
```python
y, u, uc, ym, e = sp.symbols("y(t) u(t) u_{c}(t) y_m e")
alpha, beta, gamma, b, theta1, theta2, p = sp.symbols("alpha beta gamma b theta_1 theta_2 p")

V1 = 0.5*e**2
V2 = 1/(b*gamma*2)*(alpha - b*theta2)**2
V3 = 1/(b*gamma*2)*(beta - b*theta1)**2
V = V1 + V2 + V3

y_eq = b*u/p
u_eq = theta1*uc - theta2*y

y_eq = sp.solve(sp.Eq(y,y_eq.subs(u, u_eq)),y)[0]
display(Math("y = "+latex(y_eq)))
```

$$y = \frac{b\theta_1 u_c(t)}{b\theta_2 + p}$$

Next, an equation for y_m was derived from G_m. This giave

In [89]:
```python
Bm = beta
Am = p + alpha

Gm = Bm/Am

ym_eq = sp.solve(sp.Eq(ym,Gm*uc), ym)[0]
display(Math("y_{m} = "+latex(ym_eq)))
```

$$y_m = \frac{\beta u_c(t)}{\alpha + p}$$

Much like in question 1, the true values in terms of process/model parameters were derived for θ_1 and θ_2. This was done by equating the numerators and denominators of y and y_m. This gave

In [90]:
```python
num, den = fraction(y_eq)
num_m, den_m = fraction(ym_eq)

theta_1 = sp.solve(sp.Eq(num, num_m), theta1)[0]
theta_2 = sp.solve(sp.Eq(den, den_m), theta2)[0]

display(Math("\\theta_1 =\;"+latex(theta_1)))
display(Math("\\theta_2 =\;"+latex(theta_2)))
```

$$\theta_1 = \frac{\beta}{b}$$

$$\theta_2 = \frac{\alpha}{b}$$

A sensitivity equations was derived for $\dot{e}$. This was done with the equation $\dot{e} = \dot{y} - \dot{y}_m$. The resulting equation was further manipluated by adding and subtracting αy to have an error term (e) in the equation

In [91]:
```python
y, u, uc, ym = sp.symbols("y(t) u(t) u_{c}(t) y_m")
alpha, beta, b, theta1, theta2, p = sp.symbols("alpha beta b theta_1 theta_2 p")

y_dot = b*u_eq
ym_dot = -alpha*ym + beta*uc
e_dot = collect(expand(y_dot - ym_dot), y)
e_dot_poly = sp.Poly(e_dot, [y,uc])
e_dot_poly_subd = e_dot_poly.as_expr().subs([(theta1, theta_1), (theta2, theta_2)])

e_dot_alt = uc*(b*theta1 - beta) - b*theta2*y - alpha*e + alpha*y
e_dot_alt_poly = sp.Poly(e_dot_alt, [e, theta1, theta2])


V_subd = V.subs([(e,0), (theta1, theta_1), (theta2, theta_2)])

display(Math("\\dot{y} =\;"+latex(y_dot)))
display(Math("\\dot{y}_m =\;"+latex(ym_dot)))
display(Math("\\dot{e} =\;"+latex(e_dot_poly.as_expr())+"\;=\;"+
        latex(e_dot_alt_poly.as_expr())))
# display(e_dot_alt_poly.coeffs()[3])
```

$$\dot{y} = b\left(\theta_1 u_c(t) - \theta_2 y(t)\right)$$

$$\dot{y}_m = -\alpha y_m + \beta u_c(t)$$

$$\dot{e} = \alpha y_m - b\theta_2 y(t) + u_c(t)\left(b\theta_1 - \beta\right) = -\alpha e + \alpha y(t) + b\theta_1 u_c(t) - b\theta_2 y(t) - \beta u_c(t)$$

Subing in θ_1 and θ_2 into $\dot{e}$ gives $\dot{e} = -\alpha y(t) + \alpha y_m$. Therefore, if the error is to go to 0, and θ_1 and θ_2 converge to their true values, then y must converge to y_m

Subing these results into the equation provided in the assingment document gives

$$V(e) = 0.5e^2 + \frac{\left(\alpha - b\theta_2\right)^2}{2b\gamma} + \frac{\left(-b\theta_1 + \beta\right)^2}{2b\gamma} = 0$$

Therefore, the first condition for the lyapunov function is satisfied (i.e. V = 0 at the equilibrium point)

In [92]:
```python
eDot, theta1Dot, theta2Dot = sp.symbols("\\dot{e} \\dot{\\theta}_1 \\dot{\\theta}_2")

V_dot = nsimplify(diff(V1, e))*eDot+ diff(V2,theta2)*theta2Dot + diff(V3,theta1)*theta1Dot
V_dot_subd = V_dot.subs([(eDot, e_dot_alt)])
V_dot_subd_alt = V_dot.subs([(eDot, e_dot_alt), (theta1, theta_1), (theta2, theta_2)])
V_dot_subd_poly = sp.Poly(V_dot_subd, [theta1Dot, theta2Dot, y, uc])

display(Math("\\dot{V}(e) =\;"+latex(V_dot)))
display(Math("\\dot{V}(e) =\;"+latex(V_dot_subd_poly.as_expr())))
```

$$\dot{V}(e) = -\frac{\dot{\theta}_1\left(-b\theta_1 + \beta\right)}{\gamma} - \frac{\dot{\theta}_2\left(\alpha - b\theta_2\right)}{\gamma} + \dot{e}e$$

$$\dot{V}(e) = \frac{\dot{\theta}_1\left(b\theta_1 - \beta\right)}{\gamma} + \frac{\dot{\theta}_2\left(-\alpha + b\theta_2\right)}{\gamma} - \alpha e^2 + u_c(t)\left(be\theta_1 - \beta e\right) + y(t)\left(\alpha e - be\theta_2\right)$$

Subbing the true values for θ_1 and θ_2 into $\dot{V}(e)$ gives

$$\dot{V}(e) = -\alpha e^2$$

Which means that $\dot{V}(e)$ is decreasing thus satisfying Lyanupov's second criteria.

Derivation of Control Parameters

In [93]:
```python
lst_coeffs = V_dot_subd_poly.coeffs()
equ_1 = lst_coeffs[0]*theta1Dot + lst_coeffs[3]*uc
equ_2 = lst_coeffs[1]*theta2Dot + lst_coeffs[2]*y


theta1_dot = sp.solve(sp.Eq(equ_1, 0), theta1Dot)[0]
theta2_dot = sp.solve(sp.Eq(equ_2, 0), theta2Dot)[0]
```

The equations for updating the control parameters can be obtained from

$$\frac{\dot{\theta}_1\left(b\theta_1 - \beta\right)}{\gamma} + u_c(t)\left(be\theta_1 - \beta e\right) = 0$$

$$\Rightarrow\ \dot{\theta}_1 = -e\gamma u_c(t)$$

and

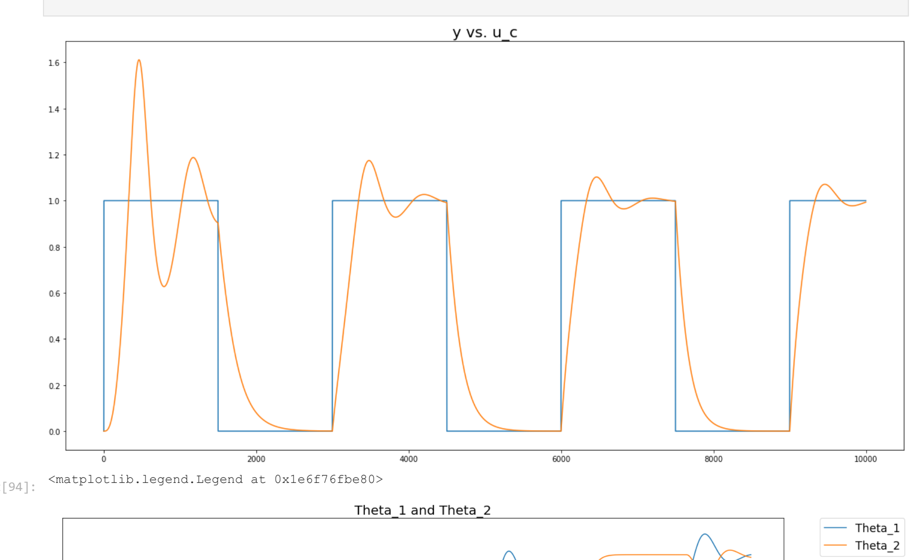
$$\frac{\dot{\theta}_2\left(-\alpha + b\theta_2\right)}{\gamma} + y(t)\left(\alpha e - be\theta_2\right) = 0$$

$$\Rightarrow\ \dot{\theta}_2 = e\gamma y(t)$$

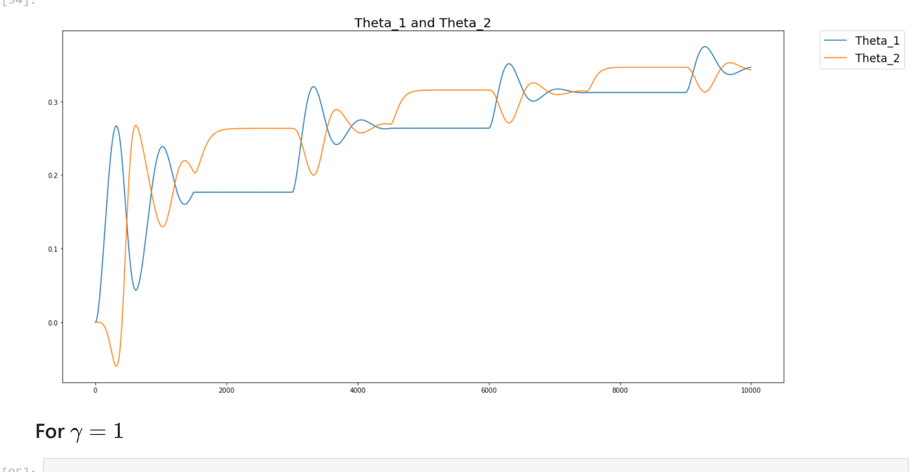
Part 2

For $\gamma = 0.2$

In [94]:
```python
T = 0.01
sample_depth = int(100/T) # 1000 samples totalling 100 seconds (since sample time T is 0.1 secons)
sample_range = range(sample_depth)

t = [i for i in sample_range]
uc = np.ones(sample_depth)
uc[np.where([m.sin(t[i]*m.pi*T/15)<=0 for i in sample_range])] = 0

# actual parameters
b = 2
beta = 1
alpha = 1
gamma = 0.2

y = [0]
ym = [0]
u = [0]
e = 0

theta_1 = [0]
theta_2 = [0]

for i in range(sample_depth):
    theta_1.append(theta_1[i] - T*e*gamma*uc[i])
    theta_2.append(theta_2[i] + T*e*gamma*y[i])

    y.append(y[i] + T*b*(theta_1[-1]*uc[i] - theta_2[-1]*y[i]))
    ym.append(ym[i] + T*(-alpha*ym[i] + beta*uc[i]))

    e = y[i] - ym[i]
y.pop(-1)
theta_1.pop(-1)
theta_2.pop(-1)

plt.title("y vs. u_c", fontsize=20)
plt.plot(t,uc)
plt.plot(t,y)
plt.show()

plt.title("Theta_1 and Theta_2", fontsize=20)
plt.plot(t, theta_1)
plt.plot(t, theta_2)
plt.legend(bbox_to_anchor=(1.05, 1),
                loc=2,
                borderaxespad=0,
                labels=["Theta_1", "Theta_2"],
                fontsize='xx-large')
# plt.legend("theta_1", "Theta_2")
```

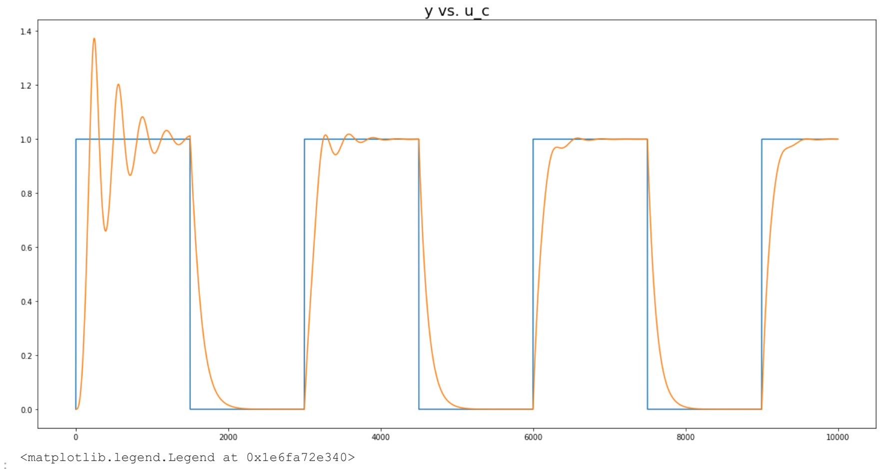


Out[94]:
```
<matplotlib.legend.Legend at 0x1e6f76fbe80>
```



For $\gamma = 1$

In [95]:
```python
T = 0.01
sample_depth = int(100/T) # 1000 samples totalling 100 seconds (since sample time T is 0.1 secons)
sample_range = range(sample_depth)

t = [i for i in sample_range]
uc = np.ones(sample_depth)
uc[np.where([m.sin(t[i]*m.pi*T/15)<=0 for i in sample_range])] = 0

# actual parameters
b = 2
beta = 1
alpha = 1
gamma = 1

y = [0]
ym = [0]
u = [0]
e = 0

theta_1 = [0]
theta_2 = [0]

for i in range(sample_depth):
    theta_1.append(theta_1[i] - T*e*gamma*uc[i])
    theta_2.append(theta_2[i] + T*e*gamma*y[i])

    y.append(y[i] + T*b*(theta_1[-1]*uc[i] - theta_2[-1]*y[i]))
    ym.append(ym[i] + T*(-alpha*ym[i] + beta*uc[i]))

    e = y[i] - ym[i]
y.pop(-1)
theta_1.pop(-1)
theta_2.pop(-1)

plt.title("y vs. u_c", fontsize=20)
plt.plot(t,uc)
plt.plot(t,y)
plt.show()

plt.title("Theta_1 and Theta_2", fontsize=20)
plt.plot(t, theta_1)
plt.plot(t, theta_2)
plt.legend(bbox_to_anchor=(1.05, 1),
                loc=2,
                borderaxespad=0,
                labels=["Theta_1", "Theta_2"],
                fontsize='xx-large')
# plt.legend("theta_1", "Theta_2")
```



Out[95]: <matplotlib.legend.Legend at 0x1e6fa72e340>