Exploring Strong Field Gravity

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Abstract

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Chapter 1

Introduction

1.1 Introduction

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1.2   Conventions

Throughout this work we will use the the timelike sign convention of Landau and Lifshitz[1]:

1. The metric has signature $(+, -, -, -)$.

2. The Reinann tensor is defined as $R^{\mu}{}_{\nu\sigma\rho} = \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} - \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\mu}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\sigma\nu}$.

3. The Ricci tensor is defined as the contraction $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\mu}$.

Greek indicies are used to represent spacetime indicies $\mu = \{0, 1, 2, 3\}$ and Latin indices are for spatial indicies $i = \{1, 2, 3\}$. Geometric units with $G = c = 1$ will be used where noted, but in general factors of G and c will be retained.

Chapter 2

Gravitational Radiation In $f(R)$ Theory

2.1 The Action & Field Equations

In $f(R)$ we modify the standard Einstein-Hilbert action

$$S[g] = \frac{c^4}{16\pi G} \int R\sqrt{-g}\, \mathrm{d}^4 x, \qquad (2.1)$$

to include an arbitrary function of the Ricci scalar R such that

$$S[g] = \frac{c^4}{16\pi G} \int f(R)\sqrt{-g}\, \mathrm{d}^4 x. \qquad (2.2)$$

We will assume that $f(R)$ is analytic about $R = 0$ so that it may be expressed as a power series

$$f(R) = a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \dots \qquad (2.3)$$

Note that since the dimensions of $f(R)$ must be the same as of R, $[a_n] = [R]^{(1-n)}$. To link to GR we will set $a_1 = 1$, any rescaling may be absorbed into the definition of G.

We may calculate the field equations in $f(R)$ theory by varying the action with respect to the metric $g^{\mu\nu}$,

$$\delta S = \frac{c^4}{16\pi G} \int \frac{\delta}{\delta g^{\mu\nu}}(f(R)\sqrt{-g})\delta g^{\mu\nu}\, \mathrm{d}^4 x$$

$$= \frac{c^4}{16\pi G} \int \left\{ f'(R)\frac{\delta R}{\delta g^{\mu\nu}}\sqrt{-g} - f(R)\frac{1}{2\sqrt{-g}}\frac{\delta g}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu}\, \mathrm{d}^4 x, \qquad (2.4)$$

where we have used the Liebniz and chain rules, and a prime indicates differentiation with respect to R. We will consider each term individually.

The variation of the determinant gives, by consideration of the cofactor of $g^{\mu\nu}$,

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}$$

$$= -g g_{\mu\nu}\delta g^{\mu\nu}. \qquad (2.5)$$

The second line follows from

$$\delta\delta^\mu_\nu = 0 \tag{2.6}$$

where the Kronecker represents $\delta^\mu_\nu = g^{\mu\lambda}g_{\lambda\nu}$.

The variation of the Ricci scalar is more involved. We start with variation in the Riemann tensor

$$\begin{aligned}
\delta R^\rho{}_{\sigma\mu\nu} &= \partial_\mu\delta\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\delta\Gamma^\rho{}_{\mu\sigma} + \delta\Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\sigma} + \Gamma^\rho{}_{\lambda\mu}\delta\Gamma^\lambda{}_{\nu\sigma} \\
&\quad - \delta\Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\mu\sigma} - \Gamma^\rho{}_{\lambda\nu}\delta\Gamma^\lambda{}_{\mu\sigma} \\
&= \nabla_\mu\delta\Gamma^\rho{}_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma}.
\end{aligned} \tag{2.7}$$

Thus, contracting over indices, the variation in the Ricci tensor is

$$\begin{aligned}
\delta R_{\mu\nu} &= \delta R^\rho{}_{\mu\rho\nu} \\
&= \nabla_\rho\delta\Gamma^\rho{}_{\nu\mu} - \nabla_\rho\delta\Gamma^\nu{}_{\rho\mu}.
\end{aligned} \tag{2.8}$$

Using this, the variation in the Ricci scalar is

$$\begin{aligned}
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\
&= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\rho\delta\Gamma^\rho{}_{\nu\mu} - \nabla_\rho\delta\Gamma^\nu{}_{\rho\mu}\right).
\end{aligned} \tag{2.9}$$

We just need the variation in the connection coefficients

$$\begin{aligned}
\delta\Gamma^\rho{}_{\mu\nu} &= \frac{1}{2}\left[\delta g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\right. \\
&\quad \left. + g^{\rho\lambda}(\partial_\mu\delta g_{\lambda\nu} + \partial_\nu\delta g_{\lambda\mu} - \partial_\lambda\delta g_{\mu\nu})\right] \\
&= \frac{1}{2}\left[\delta g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\right. \\
&\quad + g^{\rho\lambda}(\nabla_\mu\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\lambda\mu} - \nabla_\lambda\delta g_{\mu\nu}) \\
&\quad \left. + g^{\rho\lambda}g^{\sigma\tau}(\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau\delta g_{\mu\nu})g_{\lambda\sigma}\right].
\end{aligned} \tag{2.10}$$

Identifying $g^{\rho\lambda}g^{\sigma\tau}\delta g_{\lambda\sigma} = -\delta g^{\rho\tau}$, this simplifies to

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\lambda\mu} - \nabla_\lambda\delta g_{\mu\nu}\right) \tag{2.11}$$

Note that this, as the difference between two connections, is a tensor. Armed with this, we find that

$$\begin{aligned}
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu} \\
&= R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu},
\end{aligned} \tag{2.12}$$

where the d'Alembertian operator is $\Box = g_{\mu\nu}\nabla^\mu\nabla^\nu$.

Inserting equations (2.5) and (2.12) into (2.4) yields

$$\delta S = \frac{c^4}{16\pi G}\int\left\{f'(R)\sqrt{-g}\left[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right] - f(R)\frac{1}{2\sqrt{-g}}gg_{\mu\nu}\right\}\delta g^{\mu\nu}\,\mathrm{d}^4 x. \tag{2.13}$$

Integrating by parts and assuming that the variation $\delta g^{\mu\nu}$ vanishes at infinity allows this to be rewritten as

$$\delta S = \frac{c^4}{16\pi G} \int \left\{ f'(R)R_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) - f(R)\frac{1}{2}g_{\mu\nu} \right\} \delta g^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x. \tag{2.14}$$

For the action to be extremised for, $\delta S = 0$, for any variation $\delta g^{\mu\nu}$, the term in braces must vanish. Thus we obtain the vacuum field equation

$$f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu} = 0. \tag{2.15}$$

For standard GR, when $f(R) = R$, this reduces to the familiar

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 0. \tag{2.16}$$

Taking the trace of our field equation gives

$$f'R + 3\Box f' - 2f = 0. \tag{2.17}$$

Note if we consider a uniform flat spacetime $R = 0$, this equation gives

$$a_0 = 0. \tag{2.18}$$

If we introduce matter with a stress-energy tensor $T_{\mu\nu}$, the field equation becomes

$$f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.19}$$

If we act upon this with the covariant derivative we obtain

$$\frac{8\pi G}{c^4}\nabla^\mu T_{\mu\nu} = (\nabla^\mu f')R_{\mu\nu} + f'\nabla^\mu R_{\mu\nu} - \Box\nabla_\nu f' + \nabla_\nu\Box f' - \frac{f'}{2}\nabla^\mu R g_{\mu\nu}$$

$$= R_{\mu\nu}\nabla^\mu f' + f'\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) - (\Box\nabla_\nu - \nabla_\nu\Box)\,f'. \tag{2.20}$$

The second term contains the covariant derivative of the Einstein tensor and so is zero. Looking at the final term

$$\begin{aligned}
(\Box\nabla_\nu - \nabla_\nu\Box)\,f' &= g^{\mu\sigma}\left[\nabla_\mu\nabla_\sigma\nabla_\nu - \nabla_\nu\nabla_\mu\nabla_\sigma\right]f' \\
&= g^{\mu\sigma}\left[\partial_\mu\nabla_\sigma\partial_\nu - \Gamma^\rho{}_{\nu\mu}\nabla_\sigma\partial_\rho - \Gamma^\rho{}_{\sigma\mu}\nabla_\rho\partial_\nu\right. \\
&\qquad\left. -\partial_\nu\nabla_\mu\partial_\sigma + \Gamma^\rho{}_{\sigma\nu}\nabla_\mu\partial_\rho + \Gamma^\rho{}_{\mu\nu}\nabla_\rho\partial_\sigma\right]f' \\
&= g^{\mu\sigma}\left[\partial_\mu\left(\partial_\sigma\partial_\nu - \Gamma^\tau{}_{\nu\sigma}\partial_\tau\right) - \Gamma^\rho{}_{\sigma\mu}\left(\partial_\rho\partial_\nu - \Gamma^\tau{}_{\nu\rho}\partial_\tau\right)\right. \\
&\qquad\left. -\partial_\nu\left(\partial_\mu\partial_\sigma - \Gamma^\tau{}_{\sigma\mu}\partial_\tau\right) + \Gamma^\rho{}_{\mu\nu}\left(\partial_\rho\partial_\sigma - \Gamma^\tau{}_{\sigma\rho}\partial_\tau\right)\right]f' \\
&= g^{\mu\sigma}\left[\partial_\nu\Gamma^\tau{}_{\sigma\mu} - \partial_\mu\Gamma^\tau{}_{\nu\sigma} + \Gamma^\rho{}_{\sigma\mu}\Gamma^\tau{}_{\nu\rho} - \Gamma^\rho{}_{\mu\nu}\Gamma^\tau{}_{\sigma\rho}\right]\partial_\tau f' \\
&= g^{\mu\sigma}R^\tau{}_{\sigma\nu\mu}\partial_\tau f' \\
&= R_{\tau\nu}\nabla^\tau f', \tag{2.21}
\end{aligned}$$

which is a nice geometric identity. Using this we find that

$$\frac{8\pi G}{c^4}\nabla^\mu T_{\mu\nu} = R_{\mu\nu}\nabla^\mu f' - R_{\mu\nu}\nabla^\mu f'$$
$$= 0. \tag{2.22}$$

Consequently we see that energy-momentum is a conserved quantity in the same way as in GR as may be expected from the symmetries of the action.

2.2   Linear Perturbations

We will now consider the case that the metric is perturbed slightly from flat Minkowski such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.23}$$

where more formally we mean that $h_{\mu\nu} = \epsilon H_{\mu\nu}$ for small parameter ϵ. We will consider terms only to $\mathcal{O}(\epsilon)$. Thus, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2.24}$$

where we have used the Minkowski metric to raise the indices, effectively defining

$$h^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}. \tag{2.25}$$

Similarly, the trace h is given by

$$h = \eta^{\mu\nu}h_{\mu\nu}. \tag{2.26}$$

This means that all quantities denoted by "h" are strictly $\mathcal{O}(\epsilon)$. We will have to careful later on to distinguish between quantities where the Minkowski metric has been used and those where the full metric has been used.

The linearized ($\mathcal{O}(\epsilon)$) connection coefficient is

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \tag{2.27}$$

The covariant derivative of any perturbed quantity will be the same as the partial derivative. The Riemann tensor is

$$R^\lambda{}_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu\partial_\nu h^\lambda_\rho + \partial^\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\rho h^\lambda_\nu - \partial^\lambda\partial_\nu h_{\mu\rho}), \tag{2.28}$$

where we have raised the index on the differential operator with the background Minkowski metric. Contracting gives the Ricci tensor

$$R_{\mu\nu} = \frac{1}{2}(\partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h), \tag{2.29}$$

where the d'Alembertian operator is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Contracting this with $\eta^{\mu\nu}$ we find the first order Ricci scalar

$$R = \partial_\mu\partial_\rho h^{\rho\mu} - \Box h. \tag{2.30}$$

Since R is $\mathcal{O}(\epsilon)$ we may write $f(R)$ as a Maclaurin series to first order such that

$$f(R) = a_0 + R \qquad (2.31)$$
$$f'(R) = 1 + a_2 R. \qquad (2.32)$$

As we are perturbing from a flat Minkowski background where the Ricci scalar vanishes, we may use equation (2.18) to set $a_0 = 0$. Inserting these into equation (2.15) and retaining terms to first order we obtain

$$R_{\mu\nu} - \partial_\mu \partial_\nu (a_2 R) + \eta_{\mu\nu} \Box (a_2 R) - \frac{R}{2} \eta_{\mu\nu} = 0. \qquad (2.33)$$

We see that we need to find a relation between R and its derivatives. Let us consider the linearized equation (2.17)

$$R + 3\Box(a_2 R) - 2R = 0$$
$$3a_2 \Box R - R = 0 \qquad (2.34)$$

This is the massive Klein-Gordon equation

$$\Box R + m^2 R = 0, \qquad (2.35)$$

if we define mass

$$m^2 = -\frac{1}{3a_2}. \qquad (2.36)$$

For a physically meaningful solution we require $m^2 > 0$, thus we constrain $f(R)$ such that $a_2 < 0$. Now substituting for $\Box R$ in the field equation gives

$$R_{\mu\nu} - a_2 \partial_\mu \partial_\nu R + \frac{R}{3} \eta_{\mu\nu} - \frac{R}{2} \eta_{\mu\nu} = 0$$
$$R_{\mu\nu} - a_2 \partial_\mu \partial_\nu R - \frac{R}{6} \eta_{\mu\nu} = 0. \qquad (2.37)$$

The next step is to substitute in $h_{\mu\nu}$ to try to find wave solutions. We hope to find a quantity $\overline{h}_{\mu\nu}$ that will satisfy a wave equation, where it is related to $h_{\mu\nu}$ by

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + A_{\mu\nu}. \qquad (2.38)$$

In GR we use the trace reversed form where $A_{\mu\nu} = -(h/2)\eta_{\mu\nu}$. This will not suffice here as we have additional terms, but let us look for a similar solution

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2} \eta_{\mu\nu} + B_{\mu\nu}. \qquad (2.39)$$

The only rank two tensors in our theory are: $h_{\mu\nu}$, $\eta_{\mu\nu}$, $R_{\mu\nu}$, and $\partial_\mu \partial_\nu$; $h_{\mu\nu}$ has been used already, and we wish to eliminate $R_{\mu\nu}$, so we will try the simpler option based around $\eta_{\mu\nu}$. We want $B_{\mu\nu}$ to be $\mathcal{O}(\epsilon)$, there are three scalar quantities that satisfy this: h, R and $\Box R$; h is used already and $\Box R$ is related to R by equation (2.34). Therefore, we may construct an ansatz

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + \left(ba_2 R - \frac{h}{2} \right) \eta_{\mu\nu} \qquad (2.40)$$

where a_2 has been included to ensure dimensional consistency, and b is a dimensionless number. Contracting with the background metric yields

$$\overline{h} = 4ba_2R - h, \tag{2.41}$$

so we may eliminate h in our definition of $\overline{h}_{\mu\nu}$ to give

$$h_{\mu\nu} = \overline{h}_{\mu\nu} + \left(ba_2R - \frac{\overline{h}}{2} \right) \eta_{\mu\nu}. \tag{2.42}$$

We will assume a Lorenz, or de Donder, gauge choice such that

$$\nabla^\mu \overline{h}_{\mu\nu} = 0, \tag{2.43}$$

to first order this gives

$$\partial^\mu \overline{h}_{\mu\nu} = 0. \tag{2.44}$$

Subject to this, the Ricci tensor is, from equation (2.29),

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{2} \left\{ \partial_\nu \partial_\mu \left(ba_2R - \frac{\overline{h}}{2} \right) + \partial_\mu \partial_\nu \left(ba_2R - \frac{\overline{h}}{2} \right) \right.\\
&\quad \left. - \Box \left[\overline{h}_{\mu\nu} + \left(ba_2R - \frac{\overline{h}}{2} \right) \eta_{\mu\nu} \right] - \partial_\mu \partial_\nu (4ba_2R - \overline{h}) \right\}\\
&= -\frac{1}{2} \left\{ 2ba_2 \partial_\mu \partial_\nu R + \Box \left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2}\eta_{\mu\nu} \right) + \frac{b}{3} R\eta_{\mu\nu} \right\}.
\end{aligned}
\tag{2.45}
$$

Using this in our field equation (2.37) gives

$$\frac{1}{2}\Box \left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2} \right) + (b+1)\left(a_2 \partial_\mu \partial_\nu R + \frac{R}{6}\eta_{\mu\nu} \right) = 0. \tag{2.46}$$

If we set $b = -1$, then the second term vanishes. Let us now consider the Ricci scalar in the case $b = -1$, then from equation (2.30)

$$
\begin{aligned}
R &= \Box \left(a_2 R - \frac{\overline{h}}{2} \right) - \Box(-4a_2R - \overline{h})\\
&= 3a_2 \Box R + \frac{1}{2}\Box \overline{h}.
\end{aligned}
\tag{2.47}
$$

For consistency with equation (2.34), we see that

$$\Box \overline{h} = 0. \tag{2.48}$$

This means finally that

$$\Box \overline{h}_{\mu\nu} = 0, \tag{2.49}$$

we have our wave equation and it is consistent.

Should a_2 be sufficiently small that it may be regarded an $\mathcal{O}(\epsilon)$ quantity then we recover GR to leading within our analysis.

2.3 Gravitational Radiation

Having established two wave equations, (2.34) and (2.49), we may now investigate their solutions. We will use natural units $c = \hbar = 1$. Using a standard Fourier decomposition

$$\overline{h}_{\mu\nu} = \Re \left\{ \hat{\overline{h}}_{\mu\nu}(k_\rho) \exp\left(ik_\rho x^\rho\right) \right\}, \tag{2.50}$$

$$R = \Re \left\{ \hat{R}(q_\rho) \exp\left(iq_\rho x^\rho\right) \right\}, \tag{2.51}$$

where k_μ and q_μ are the 4-wavevectors, or 4-momenta, of the waves. From equation (2.49) we know that k_μ is a null vector, so for a wave traveling along the z-axis

$$k^\mu = \omega(1, 0, 0, 1), \tag{2.52}$$

where ω is the frequency. Similarly, from equation (2.34)

$$q^\mu = (\Omega, 0, 0, \sqrt{\Omega^2 - m^2}), \tag{2.53}$$

for frequency Ω. This means that these waves do not travel at c, but have a group velocity

$$v = \frac{\sqrt{\Omega^2 - m^2}}{\Omega}. \tag{2.54}$$

Provided that $m^2 > 0$, $v < c = 1$, but we do not get propagating modes for $\Omega^2 < m^2$.

From the condition (2.43) we find that k^μ is orthogonal to $\hat{\overline{h}}_{\mu\nu}$,

$$k^\mu \hat{\overline{h}}_{\mu\nu} = 0, \tag{2.55}$$

thus in this case

$$\hat{\overline{h}}_{0\nu} + \hat{\overline{h}}_{3\nu} = 0. \tag{2.56}$$

Let us now consider the implications of equation (2.48) using equations (2.41) and (2.34),

$$\Box \left(4a_2 R + h\right) = 0$$
$$\Box h = -\frac{4}{3} R. \tag{2.57}$$

For non-zero R (as required for the Ricci mode) there is no way we can make such that the trace h will vanish. This is distinct from the case in GR. It is possible, however, to make a gauge choice such that the trace $\overline{h}$ will vanish. Consider a gauge transformation generated by ξ_μ which satisfies $\Box \xi_\mu = 0$, and so has a Fourier decomposition

$$\xi_\mu = \hat{\xi}_\mu \exp\left(ik_\rho x^\rho\right). \tag{2.58}$$

We see that a transformation

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\rho \xi_\rho, \tag{2.59}$$

would ensure both conditions (2.43) and (2.49) are satisfied. Under such a transformation

$$\hat{\bar{h}}_{\mu\nu} \to \hat{\bar{h}}_{\mu\nu} + i\left(k_\mu\hat{\xi}_\nu + k_\nu\hat{\xi}_\mu - \eta_{\mu\nu}k^\rho\hat{\xi}_\rho\right). \tag{2.60}$$

We may therefore impose four further constraints (one for each $\hat{\xi}_\mu$) upon $\hat{\bar{h}}_{\mu\nu}$, and we may take these to be

$$\hat{\bar{h}}_{0\nu} = 0, \quad \hat{\bar{h}} = 0. \tag{2.61}$$

This may appear to be five constraints, however we have already imposed (2.56), and so setting $\hat{\bar{h}}_{00} = 0$ automatically implies $\hat{\bar{h}}_{03} = 0$.

We see that $\bar{h}_{\mu\nu}$ behaves just as its counterpart in GR so we may define

$$\left[\hat{\bar{h}}_{\mu\nu}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.62}$$

where h_+ and $h_\times$ are appropriate constants representing the amplitudes of the two transverse polarizations of gravitational radiation.

Using this choice of gauge we have

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - a_2 R \eta_{\mu\nu}, \tag{2.63}$$

$$h = -4a_2 R. \tag{2.64}$$

Combining both wave modes ($\bar{h}_{\mu\nu}$ and R) the perturbation to the flat background associated with our wave solutions looks like

$$[h_{\mu\nu}(z,t)] = \begin{bmatrix} -a_2\hat{R}(t-vz) & 0 & 0 & 0 \\ 0 & h_+(t-z) + a_2\hat{R}(t-vz) & h_\times(t-z) & 0 \\ 0 & h_\times(t-z) & -h_+ + a_2\hat{R}(t-vz) & 0 \\ 0 & 0 & 0 & a_2\hat{R}(t-vz) \end{bmatrix}.$$
$$\tag{2.65}$$

If we now assume that both are excited by the same source so that the frequencies will be equal $\omega = \Omega$.

It is important that our solutions reduce to those of GR in the event that $f(R) = R$. In this linearized approach this corresponds to $a_2 \to 0$, $m^2 \to \infty$. We see from equation (2.53) that in this limit it would take an infinite frequency to excite a propagating Ricci mode, and evanescent waves would decay away infinitely quickly. Therefore there would not be any detectable Ricci modes and we would only observe the two polarizations found in the analysis of GR. Additionally $\bar{h}_{\mu\nu}$ would simplify to its usual trace-reversed form.

2.4   Energy-momentum Tensor

We expect that the gravitational field would carry energy-momentum. Unfortunately it is difficult to define a proper energy-momentum tensor for a gravitational field: as

a consequence of the equivalence principle it is possible to transform to a freely falling frame, eliminating the gravitational field and any associated energy density for a given event, although we may still define curvature in the neighbourhood of this point. We see that the full field equation, equation (2.15), has no energy-momentum tensor for the gravitational field on the right-hand side. However, by expanding beyond the linear terms we may find a suitable energy-momentum pseudotensor for gravitational radiation. Let us first define, in analogy to the Einstein tensor

$$\mathcal{G}_{\mu\nu} = f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu}, \tag{2.66}$$

so that in a vacuum

$$\mathcal{G}_{\mu\nu} = 0. \tag{2.67}$$

We may expand $\mathcal{G}_{\mu\nu}$ in orders of $h_{\mu\nu}$

$$\mathcal{G}_{\mu\nu} = \mathcal{G}^{(B)}{}_{\mu\nu} + \mathcal{G}^{(1)}{}_{\mu\nu} + \mathcal{G}^{(2)}{}_{\mu\nu} + \dots \tag{2.68}$$

We use (B) for the background term instead of (0) to avoid confusion regarding its order in ϵ. Our linearised equation would then read

$$\mathcal{G}^{(1)}{}_{\mu\nu} = 0. \tag{2.69}$$

So far we have assumed that our background is flat, however we can imagine that should the gravitational radiation carry energy-momentum then this would act as a source of curvature for the background. This is a second-order effect that may be encoded, to accuracy of $\mathcal{O}(\epsilon^2)$, as

$$\mathcal{G}^{(B)}{}_{\mu\nu} = -\mathcal{G}^{(2)}{}_{\mu\nu}. \tag{2.70}$$

By shifting $\mathcal{G}^{(2)}{}_{\mu\nu}$ to the right-hand side we effectively create an energy-momentum tensor. As in GR we will average over several wavelengths, assuming that the background curvature is on a larger scale,

$$\mathcal{G}^{(B)}{}_{\mu\nu} = -\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle. \tag{2.71}$$

By doing this, we may probe the curvature in a macroscopic region about a given point in spacetime. This gives a gauge invariant measure of the gravitational field strength. The averaging can be thought of as smoothing out the rapidly varying ripples of the radiation, leaving only the coarse-grained component that acts as a source for the background curvature. The energy-momentum pseudotensor for the radiation may be identified as

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle. \tag{2.72}$$

Having made this provisional identification, we must now set about carefully evaluating the various terms in equation (2.68). We begin as in section 2.2 by defining a total metric

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}, \tag{2.73}$$

where $\gamma_{\mu\nu}$ is our background metric. This is changing slightly our definition for $h_{\mu\nu}$: instead of it being the total perturbation from flat Minkowski, it is instead the dynamical part of the metric with which we associate radiative effects. Since we know that $\mathcal{G}^{(B)}{}_{\mu\nu}$ is $\mathcal{O}(\epsilon^2)$, we may decompose our background metric as

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}, \tag{2.74}$$

where $j_{\mu\nu}$ is $\mathcal{O}(\epsilon^2)$ to ensure that $R^{(B)\lambda}{}_{\mu\nu\rho}$ is also $\mathcal{O}(\epsilon^2)$. Therefore its introduction will make no difference to the linearized theory.

We will consider terms only to $\mathcal{O}(\epsilon^2)$. The connection coefficient is

$$\Gamma^\rho{}_{\mu\nu} = \Gamma^{(B)\rho}{}_{\mu\nu} + \Gamma^{(1)\rho}{}_{\mu\nu} + \Gamma^{(2)\rho}{}_{\mu\nu} + \ldots \tag{2.75}$$

We identify $\Gamma^{(1)\rho}{}_{\mu\nu}$ from equation (2.27) to the accuracy of our analysis. There is one small subtlety: whether we use the background metric $\gamma^{\mu\nu}$ or $\eta^{\mu\nu}$ to raise indices of ∂_μ and $h_{\mu\nu}$. Fortunately to the accuracy considered here it does not make a difference, however we will consider the indices to be changed with the background metric. This is more appropriate for considering the effect of curvature on gravitational radiation. We will not distinguish between ∂_μ and $\nabla^{(B)}{}_\mu$, the covariant derivative for the background metric: note that to the order of accuracy considered here covariant derivatives would commute and $\nabla^{(B)}{}_\mu$ behaves just like ∂_μ. The connection coefficient has

$$\Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}\gamma^{\rho\lambda}\left[\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2 R^{(1)}\gamma_{\lambda\nu}\right) + \partial_\nu\left(\overline{h}_{\lambda\mu} - a_2 R^{(1)}\gamma_{\lambda\mu}\right) \right.$$
$$\left. - \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2 R^{(1)}\gamma_{\mu\nu}\right)\right], \tag{2.76}$$

where $R^{(1)}$ is the first order Ricci scalar, and

$$\Gamma^{(2)\rho}{}_{\mu\nu} = -\frac{1}{2}h^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})$$
$$= -\frac{1}{2}\left(\overline{h}^{\rho\lambda} - a_2 R^{(1)}\gamma^{\rho\lambda}\right)\left[\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2 R^{(1)}\gamma_{\lambda\nu}\right) + \partial_\nu\left(\overline{h}_{\lambda\mu} - a_2 R^{(1)}\gamma_{\lambda\mu}\right)\right.$$
$$\left. - \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2 R^{(1)}\gamma_{\mu\nu}\right)\right]. \tag{2.77}$$

The Riemann tensor is

$$R^\lambda{}_{\mu\nu\rho} = R^{(B)\lambda}{}_{\mu\nu\rho} + R^{(1)\lambda}{}_{\mu\nu\rho} + R^{(2)\lambda}{}_{\mu\nu\rho} + \ldots \tag{2.78}$$

We may use our expression from equation (2.28) for $R^{(1)\lambda}{}_{\mu\nu\rho}$. Contracting gives the Ricci tensor

$$R_{\mu\nu} = R^{(B)}{}_{\mu\nu} + R^{(1)}{}_{\mu\nu} + R^{(2)}{}_{\mu\nu} + \ldots \tag{2.79}$$

We may again use our linearized expression, equation (2.29), for the first order term.

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\partial_\rho\left(\overline{h}^\rho_\nu - a_2 R^{(1)}\gamma^\rho_\nu\right) + \partial_\nu\partial_\rho\left(\overline{h}^\rho_\mu - a_2 R^{(1)}\gamma^\rho_\mu\right)\right.$$
$$\left. - \Box^{(B)}\left(\overline{h}_{\mu\nu} - a_2 R^{(1)}\gamma_{\mu\nu}\right) - \partial_\mu\partial_\nu\left(-4a_2 R\right)\right] \tag{2.80}$$
$$= 2a_2\partial_\mu\partial_\nu R^{(1)} + \frac{1}{6}R^{(1)}\gamma_{\mu\nu},$$

Note that the d'Almebertian is now $\Box^{(\mathrm{B})} = \gamma^{\mu\nu}\partial_\mu\partial_\nu$. The next term is

$$
\begin{aligned}
R^{(2)}{}_{\mu\nu} &= \partial_\rho\Gamma^{(2)\rho}{}_{\mu\nu} - \partial_\nu\Gamma^{(2)\rho}{}_{\mu\rho} + \Gamma^{(1)\rho}{}_{\mu\nu}\Gamma^{(1)\sigma}{}_{\rho\sigma} - \Gamma^{(1)\rho}{}_{\mu\sigma}\Gamma^{(1)\sigma}{}_{\rho\nu} \\
&= \frac{1}{2}\left[\frac{1}{2}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + h^{\sigma\rho}\left(\partial_\mu\partial_\nu h_{\rho\sigma} + \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\sigma\partial_\mu h_{\rho\nu}\right)\right. \\
&\quad \left. - \left(\partial_\sigma h^{\sigma\rho} - \frac{1}{2}\partial^\rho h\right)\left(\partial_\mu h_{\rho\mu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right) + \partial^\rho h^\sigma_\nu\left(\partial_\rho h_{\sigma\mu} - \partial_\sigma h_{\rho\mu}\right)\right] \\
&= \frac{1}{2}\left\{\frac{1}{2}\partial_\mu\overline{h}_{\sigma\rho}\partial_\nu\overline{h}^{\sigma\rho} + \overline{h}^{\sigma\rho}\left[\partial_\mu\partial_\nu\overline{h}_{\sigma\rho} + \partial_\sigma\partial_\rho\left(\overline{h}_{\mu\nu} - a_2 R^{(1)}\gamma_{\mu\nu}\right)\right.\right. \\
&\quad \left.\left. - \partial_\nu\partial_\rho\left(\overline{h}_{\sigma\mu} - a_2 R^{(1)}\gamma_{\sigma\mu}\right) - \partial_\mu\partial_\rho\left(\overline{h}_{\sigma\nu} - a_2 R^{(1)}\gamma_{\sigma\nu}\right)\right] \right. \\
&\quad \left. + \partial^\rho\overline{h}^\sigma_\nu\left(\partial_\rho\overline{h}_{\sigma\mu} - \partial_\sigma\overline{h}_{\rho\mu}\right) - a_2\partial^\sigma R^{(1)}\partial_\sigma\overline{h}_{\mu\mu} + a_2{}^2\left[2R^{(1)}\partial_\mu\partial_\nu R^{(1)}\right.\right. \\
&\quad \left.\left. + 3\partial_\mu R^{(1)}\partial_\nu R^{(1)} + R^{(1)}\Box^{(\mathrm{B})}R^{(1)}\gamma_{\mu\nu}\right]\right\}.
\end{aligned}
\tag{2.81}
$$

To find the Ricci scalar we must contract the Ricci tensor, but we must decide which metric to use to do this. It is tempting to use the background metric, as we used this for raising the indices on $h_{\mu\nu}$, however this was just a notational convenience. The physical metric is the full metric, so we must use this to form R. Remembering that we are only considering terms to $\mathcal{O}(\epsilon^2)$, this gives

$$
R^{(\mathrm{B})} = \gamma^{\mu\nu}R^{(\mathrm{B})}{}_{\mu\nu}
\tag{2.82}
$$

$$
R^{(1)} = \gamma^{\mu\nu}R^{(1)}{}_{\mu\nu}
\tag{2.83}
$$

$$
\begin{aligned}
R^{(2)} &= \gamma^{\mu\nu}R^{(2)}{}_{\mu\nu} - h^{\mu\nu}R^{(1)}{}_{\mu\nu} \\
&= \gamma^{\mu\nu}R^{(2)}{}_{\mu\nu} - \overline{h}^{\mu\nu}R^{(1)}{}_{\mu\nu} + a_2 R^{(1)2} \\
&= \frac{3}{4}\partial_\mu\overline{h}_{\sigma\rho}\partial^\mu\overline{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho\overline{h}^{\sigma\mu}\partial_\sigma\overline{h}_{\rho\mu} - 2a_2\overline{h}^{\mu\nu}\partial_\mu\partial_\nu R^{(1)} \\
&\quad + a_2 R^{(1)2} + \frac{3a_2}{2}\partial_\mu R^{(1)}\partial^\mu R^{(1)}.
\end{aligned}
\tag{2.84}
$$

Using these

$$
f^{(\mathrm{B})} = R^{(\mathrm{B})}
\tag{2.85}
$$

$$
f^{(1)} = R^{(1)}
\tag{2.86}
$$

$$
f^{(2)} = R^{(2)} + \frac{a_2}{2}R^{(1)2},
\tag{2.87}
$$

and

$$
f'^{(\mathrm{B})} = a_2 R^{(\mathrm{B})}
\tag{2.88}
$$

$$
f'^{(0)} = 1
\tag{2.89}
$$

$$
f'^{(1)} = a_2 R^{(1)}
\tag{2.90}
$$

$$
f'^{(2)} = a_2 R^{(2)} + \frac{a_3}{2}R^{(1)2}.
\tag{2.91}
$$

We list a zeroth order term here for clarity.

Using all of these

$$
\begin{aligned}
\mathcal{G}^{(2)}{}_{\mu\nu} &= R^{(2)}{}_{\mu\nu} + f'^{(1)} R^{(1)}{}_{\mu\nu} - \partial_\mu\partial_\nu f'^{(2)} + \Gamma^{(1)\rho}{}_{\nu\mu}\partial_\rho f'^{(1)} + \gamma_{\mu\nu}\gamma^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(2)} \\
&\quad - \gamma_{\mu\nu}\gamma^{\rho\sigma}\Gamma^{(1)\lambda}{}_{\sigma\rho}\partial_\lambda f'^{(1)} - \gamma_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(1)} + h_{\mu\nu}\gamma^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(1)} \\
&\quad - \frac{1}{2}f^{(2)}\gamma_{\mu\nu} - \frac{1}{2}f^{(1)}h_{\mu\nu} \\
&= R^{(2)}{}_{\mu\nu} + a_2 R^{(1)} R^{(1)}{}_{\mu\nu} - \partial_\mu\partial_\nu\left(a_2 R^{(2)} + \frac{a_3}{2}R^{(1)2}\right) + \Gamma^{(1)\rho}{}_{\nu\mu}\partial_\rho\left(a_2 R^{(1)}\right) \\
&\quad + \gamma_{\mu\nu}\Box^{(\mathrm{B})}\left(a_2 R^{(2)} + \frac{a_3}{2}R^{(1)2}\right) - \gamma_{\mu\nu}\gamma^{\rho\sigma}\Gamma^{(1)\lambda}{}_{\sigma\rho}\partial_\lambda\left(a_2 R^{(1)}\right) \\
&\quad - \gamma_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\left(a_2 R^{(1)}\right) + h_{\mu\nu}\Box^{(\mathrm{B})}\left(a_2 R^{(1)}\right) - \frac{1}{2}\left(R^{(2)} + \frac{a_2}{2}R^{(1)2}\right)\gamma_{\mu\nu} \\
&\quad - \frac{1}{2}R^{(1)}h_{\mu\nu} \\
&= R^{(2)}{}_{\mu\nu} + a_2\left(\gamma_{\mu\nu}\Box^{(\mathrm{B})} - \partial_\mu\partial_\nu\right)R^{(2)} - \frac{1}{2}R^{(2)}\gamma_{\mu\nu} + a_3\left(\gamma_{\mu\nu}\Box^{(\mathrm{B})} - \partial_\mu\partial_\nu\right)R^{(1)2} \\
&\quad - \frac{1}{6}\overline{h}_{\mu\nu}R^{(1)} - a_2\gamma_{\mu\nu}\overline{h}^{\sigma\rho}\partial_\sigma\partial_\rho R^{(1)} + \frac{a_2}{2}\partial^\rho R^{(1)}\left(\partial_\mu\overline{h}_{\rho\nu} + \partial_\nu\overline{h}_{\rho\mu} - \partial_\rho\overline{h}_{\mu\nu}\right) \\
&\quad + a_2\left(R^{(1)}R^{(1)}{}_{\mu\nu} + \frac{1}{4}R^{(1)2}\gamma_{\mu\nu}\right) - a_2{}^2\left(\partial_\mu R^{(1)}\partial_\nu R^{(1)}\right. \\
&\quad \left. + \frac{1}{2}\gamma_{\mu\nu}\partial^\rho R^{(1)}\partial_\rho R^{(1)}\right) \tag{2.92}
\end{aligned}
$$

It is simplest to split this up for the purposes of averaging. Since we average over over all directions at each point gradients average to zero

$$
\langle\partial_\mu V\rangle = 0. \tag{2.93}
$$

As a corollary of this we have the relation

$$
\langle U\partial_\mu V\rangle = -\langle\partial_\mu U V\rangle. \tag{2.94}
$$

Repeated application of this, together with our gauge condition, equation (2.43), and wave equations, (2.34) and (2.49), allows us to eliminate many terms. Considering terms that do not trivially average to zero

$$
\left\langle R^{(2)}{}_{\mu\nu}\right\rangle = \left\langle -\frac{1}{4}\partial_\mu\overline{h}_{\sigma\rho}\partial^\mu\overline{h}^{\rho\sigma} + \frac{a_2{}^2}{2}\partial_\mu R^{(1)}\partial_\nu R^{(1)} + \frac{a_2}{6}\gamma_{\mu\nu}R^{(1)}\right\rangle; \tag{2.95}
$$

$$
\left\langle R^{(2)}\right\rangle = \left\langle\frac{3a_2}{2}R^{(1)2}\right\rangle; \tag{2.96}
$$

$$
\left\langle\overline{h}_{\mu\nu}R^{(1)}\right\rangle = 0; \tag{2.97}
$$

$$
\left\langle R^{(1)}R^{(1)}{}_{\mu\nu}\right\rangle = \left\langle a_2 R^{(1)}\partial_\mu\partial_\nu R^{(1)} + \frac{1}{6}\gamma_{\mu\nu}R^{(1)2}\right\rangle. \tag{2.98}
$$

Combining these gives

$$\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle = \left\langle -\frac{1}{4} \partial_\mu \overline{h}_{\sigma\rho} \partial^\mu \overline{h}^{\rho\sigma} - \frac{3a_2{}^2}{2} \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle. \tag{2.99}$$

Thus we obtain the result

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu \overline{h}_{\sigma\rho} \partial^\mu \overline{h}^{\rho\sigma} + 6a_2{}^2 \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle. \tag{2.100}$$

In the limit of $a_2 \to 0$ we obtain the standard GR result as required. Note that the GR result is also recovered if $R^{(1)} = 0$, as would be the case if the Ricci mode was not excited, for example if the frequency was below the cut off frequency m. Rewriting the pseudotensor in terms of metric perturbation $h_{\mu\nu}$, using equation (2.64), we obtain

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu h_{\sigma\rho} \partial^\mu h^{\rho\sigma} + \frac{1}{8} \partial_\mu h \partial_\nu h \right\rangle. \tag{2.101}$$

2.5 Radiation With A Source

Having consider radiation in a vacuum, we now move on to consider the case with a source term. We want a first order perturbation from our background metric so that the linearised field equation is

$$\mathcal{G}^{(1)}{}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.102}$$

Again assuming a Minkowski background, our first order field equations are

$$\mathcal{G}^{(1)}{}_{\mu\nu} = R^{(1)}{}_{\mu\nu} - \partial_\mu \partial_\nu (a_2 R^{(1)}) + \eta_{\mu\nu} \Box (a_2 R^{(1)}) - \frac{R^{(1)}}{2} \eta_{\mu\nu} \tag{2.103}$$

$$\mathcal{G}^{(1)} = 3a_2 \Box R^{(1)} - R^{(1)} \tag{2.104}$$

where $\mathcal{G}^{(1)} = \eta^{\mu\nu} \mathcal{G}^{(1)}{}_{\mu\nu}$. Will we again use our ansatz $\overline{h}_{\mu\nu}$ from equation (2.40), with $b = -1$. The Ricci tensor is

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2} \left[2a_2 \partial_\mu \partial_\nu R^{(1)} - \Box \left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2} \eta_{\mu\nu} \right) + \frac{a_2}{2} \Box R^{(1)} \eta_{\mu\nu} \right]. \tag{2.105}$$

Taking the traces gives

$$R^{(1)} = 3a_2 \Box R^{(1)} + \frac{1}{2} \Box \overline{h} \tag{2.106}$$

just as in equation (2.47). Comparing this with the trace of the field equation yields

$$-\frac{1}{2} \Box \overline{h} = \mathcal{G}^{(1)} \tag{2.107}$$

Using this with our expression for the Ricci tensor in the field equation gives

$$\mathcal{G}^{(1)}{}_{\mu\nu} = a_2 \partial_\mu \partial_\nu R^{(1)} - \frac{1}{2} \Box \overline{h}_{\mu\nu} + \frac{1}{2} \Box \left(\frac{\overline{h}}{2} + a_2 R^{(1)} \right) \eta_{\mu\nu} - a_2 \partial_\mu \partial_\nu R^{(1)}$$

$$+ a_2 \Box R^{(1)} \eta_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu}$$

$$= -\frac{1}{2} \Box \overline{h}_{\mu\nu}. \tag{2.108}$$

We see that our field equations are consistent.

Chapter 3

Parabolic Encounters Of A Massive Black Hole

3.1 Background & Introduction

Currently it is understood that many, if not all at some point, galactic nuclei harbour a black hole[2, 3]. The best opportunity to study these objects comes from the compact object in our own galactic centre, which is coincident with Sagittarius A*. This is identified as a massive black hole (MBH) of mass $M_\bullet = 4M_\odot$ and a distance of only $R_0 = 8\,\mathrm{kpc}$. According to the no-hair theorem the MBH should be described completely by its mass $M_\bullet$ and spin a (since we do not expect an astrophysical black hole to be charged)[4, 5, 6, 7, 8, 9]. Consequently, measuring the spin is necessary to fully understand the MBH and its rôle in the evolution of the Galaxy. It has been suggested that the spin could be inferred from careful observation of the orbits of stars within a few millparsecs of the Galactic centre[10], although this is complicated because of perturbations due to other stars, or from observations of quasi-periodic oscillations (QPOs) of radio emissions[11]: this latter method has produced a value of for dimensionless spin, $a_* = Jc/GM_\bullet^2$ where J is the MBH's angular momentum, of $a_* = 0.44 \pm 0.08$, though this also uses observations of Galactic X-ray sources, containing solar mass BHs, to find a best-fit unique spin parameter for all BHs.

An exciting means of inferring information about the Galactic MBH is through gravitational waves (GWs) emitted when compact objects (COs), such as smaller BHs, neutron stars (NSs), white dwarfs (WDs) or low mass main sequence (MS) stars, pass close by. The planned NASA/ESA Laser Interferometer Space Antenna (LISA) is designed to be able to detect GWs in the frequency range of interest for these encounters. The identification of waves requires a set of accurate waveform templates covering parameter space. Much work has already been done on the waveforms generated when companion objects inspiral towards an MBH; as they orbit, the GWs carry away energy and angular momentum, causing the orbit to shrink until eventually the object plunges into the MBH. The initial orbits may be highly elliptical and a burst of radiation is emitted each close encounter. These are known as extreme mass ratio bursts (EMRBs)[12]. Assuming that the companion is not scattered from its orbit, and does not

plunge straight into the MBH, its orbit will evolve, becoming more circular, and it will begin to continuously emit significant gravitational radiation. The resulting waveforms are known as extreme mass ratio inspirals (EMRIs).

Studies of these systems have usually focused upon when the orbit completes multiple cycles, thus allowing a high signal-to-noise ratio to be accumulated. Here, we will investigate what can be learnt from high eccentricity orbits. These are the initial orbits we expect that COs may be scattered onto by interactions with other bodies. We will make the simplifying assumption that all these orbits are marginally bound, or parabolic, since highly eccentric orbits will appear almost indistinguishable from an appropriate parabolic orbit[13]. Here "parabolic" and "eccentricity" refer to the energy of the geodesic and not to the geometric shape of the orbit.[1] Following such a trajectory a nobject may make just one pass of the MBH or, if the periapsis distance is small enough, it may complete a number or rotations. Such an orbit is referred to as zoom-whirl.

In order to compute the gravitational waveform produced in such as case, we integrate the geodesic equations for a parabolic orbit in Kerr spacetime. We assume that the orbitting body is a test particle, such that it does not influence the underlying spacetime, and that the orbital parameters evolve neglibly during the orbit so that they may be held constant. We use this to construct an approximate numerical kludge waveform[14].

3.2 Parabolic Orbits in Kerr Spacetime

3.2.1 The Metric & Geodesic Equations

Astrophysical BHs are described by the Kerr metric[15]. In standard Boyer-Lindquist coordinates the line element is[16, 17]

$$\mathrm{d}s^2 = \frac{\rho^2 \Delta}{\Sigma^2} c^2 \mathrm{d}t^2 - \frac{\Sigma \sin^2 \theta}{\rho^2} (\mathrm{d}\phi - \omega \mathrm{d}t)^2 - \frac{\rho^2}{\Delta} \mathrm{d}r^2 - \rho^2 \mathrm{d}\theta^2, \tag{3.1}$$

where we have introduced functions

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \tag{3.2}$$

$$\Delta = r^2 - \frac{2GM_\bullet r}{c^2} + a^2, \tag{3.3}$$

$$\Sigma = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2 \theta, \tag{3.4}$$

$$\omega = \frac{2GM_\bullet ar}{c\Sigma}. \tag{3.5}$$

$$\tag{3.6}$$

The spin parameter is related to the BH's angular momentum by

$$J = M_\bullet ac. \tag{3.7}$$

For the remainder of this section we shall work in natural units with $G = c = 1$.

---

[1]Marginally bound Keplerian orbits in flat spacetime are parabolic in both senses.

Geodesics are parameterized by three conserved quantities (aside from the particle's mass μ): the energy (per unit mass) E, the specific angular momentum about the symmetry axis (the z-axis) L_z, and Carter's constant Q[18, 9]. The geodesic equations are

$$\rho^2 \frac{\mathrm{d}t}{\mathrm{d}\tau} = a \left(L_z - aEz\sin^2\theta \right) + \frac{r^2 + a^2}{\Delta} T, \tag{3.8}$$

$$\rho^2 \frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm \sqrt{V_r}, \tag{3.9}$$

$$\rho^2 \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \pm \sqrt{V_\theta}, \tag{3.10}$$

$$\rho^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{L_z}{\sin^2\theta} - aE + \frac{a}{\Delta} T, \tag{3.11}$$

$$\tag{3.12}$$

where we have introduced potentials

$$T = E \left(r^2 + a^2 \right) - aL_z, \tag{3.13}$$

$$V_r = T^2 - Delta \left[r^2 + (L_z - aE)^2 + Q \right], \tag{3.14}$$

$$V_\theta = Q - \cos^2\theta \left[a^2 \left(1 - E^2 \right) + \frac{L_z{}^2}{\sin^2\theta} \right], \tag{3.15}$$

and τ is proper time. The signs of the r and θ equations may be chosen independently.

For a parabolic orbit $E = 1$, thus the particle is at rest at infinity. This simplifies the geodesic equations. It also allows us to give a simple interpretation for Carter's constant Q: this is defined such that

$$Q = L_\theta{}^2 + \cos^2\theta \left[a^2 \left(1 - E^2 \right) + \frac{L_z{}^2}{\sin^2\theta} \right], \tag{3.16}$$

where L_θ is the (non-conserved) specific angular momentum in the θ direction (note $L_\theta{}^2 = V_\theta$). For $E = 1$ we have

$$\begin{aligned} Q &= L_\theta{}^2 + \cot^2\theta L_z{}^2 \\ &= L_\infty{}^2 - L_z{}^2 \end{aligned} \tag{3.17}$$

where L_∞ is the total specific angular momentum at infinity, where the metric is asymptotically flat[19].[2] This is just as in the Schwarzschild case.

3.2.2 Integration Variables & Turning Points

In integrating the geodesic equations difficulties can arise because of the presence of turning points in the motion, when the sign of the r or θ geodesic equation will change

---

[2]See Rosquist, Bylund & Samuelsson[20] for an interesting discussion of the interpretation of Q in the limit $G \to 0$ corresponding to a flat spacetime.

sign. The radial turning points are at the periapsis r_p and at infinity. We may find the location of the periapsis by finding the roots of

$$V_r = 0$$

$$2M_\bullet r^3 - \left(L_z{}^2 + Q\right) r^2 + 2M_\bullet \left[(L_z - a)^2 + Q\right] r - a^2 Q = 0. \tag{3.18}$$

This has three roots, which we shall denote $\{r_1, r_2, r_p\}$; the periapsis r_p is the largest real root. We do not find the apoapsis as a (fourth) root to this equation as we have removed it by taking $E = 1$ before solving: it is simple to show this is a turning point by setting the unconstrained expression for V_r equal to zero and then solving for $E(r)$, taking the limit $r \to \infty$ gives $E \to 1$, so parabolic orbits do have a stationary point at infinity [21]. We may avoid the difficulties of the turning point by introducing an angular variable that always increases with proper time [22]: inspired by Keplerian orbits, we may parameterize our parabolic trajectory by

$$r = \frac{p}{1 - e\cos\phi} \tag{3.19}$$

where $e = 1$ is the eccentricity and $p = 2r_p$ is the semilatus rectum. As ϕ covers its full range from 0 to 2π, r traces out one full orbit from infinity through the periapsis at $\phi = \pi$ back to infinity. The geodesic equation for ϕ is

$$\rho^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \left\{ M_\bullet \left[2r_p - (r_1 + r_2)(1 + \cos\phi) + \frac{r_1 r_2}{2r_p}(1 + \cos\phi)^2 \right] \right\}^{1/2}. \tag{3.20}$$

This may be integrated without problem. Parameterizing an orbit by it periapsis and eccentricity has the additional benefit of allowing easier comparison with its flat-space equivalent than considering energy and angular momentum [23].

The θ motion is (usually) bounded, with $\theta_0 \leq \theta \leq \pi - \theta_0$. The turning points are given by where

$$V_\theta = 0$$

$$Q - \cot^2\theta L_z{}^2 = 0. \tag{3.21}$$

Note that only if $L_z = 0$ may we reach the poles [21]. If we change variable to $\zeta = \cos^2\theta$, we have a maximum value $\zeta_0 = \cos^2\theta_0$ given by

$$\zeta_0 = \frac{Q}{Q + L_z{}^2} \tag{3.22}$$

$$= \frac{Q}{L_\infty{}^2}. \tag{3.23}$$

See figure 3.1 for a geometrical visulaization. Let us now introduce a second angular variable [22]

$$\zeta = \zeta_0 \cos^2\chi. \tag{3.24}$$

Over one 2π period of χ, θ oscillates over its full range, from its minimum value to its maximum and back. The geodesic equation for χ is

$$\rho^2 \frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \sqrt{Q + L_z{}^2}, \tag{3.25}$$

and may be integrated simply.

Figure 3.1: The angular momenta L_∞, L_z and $\sqrt{Q}$ define a right-angled triangle. The accute angles are θ_0, the extremal value of the polar angle, and ι, the orbital inclination[24].

3.3 Waveform Construction

With the geodesic calculated for given angular momenta L_z and Q, and initial starting positions, the orbiting body is assumed to follow this trajectory exactly: we ignore evolution due to the radiation of energy and angular momentum. From this we calculate the gravitational waveform using a semirelativistic approximation[25]: we assume that the particle moves along a geodesic in the Kerr geometry, but radiates as if it were in flat spacetime. This quick-and-dirty technique is known as a numerical kludge[14].

3.3.1 Kludge Approximation

Numerical kludge approximations aim to encapsulate the main characteristics of a waveform by using the exact particle trajectory (ignoring inaccuracies from the evolution of the orbital parameters), whilst saving on computational time by using approximate waveform generation techniques. To start, we build an equivalent flat spacetime trajectory from the Kerr geodesic. This is done by identifying the Boyer-Lindquist coordinates with a set of flat-space coordinates; we consider two choices here:

1. Identify the Boyer-Lindquist coordinates with flat-space spherical polars $\{r_{\mathrm{BL}}, \theta_{\mathrm{BL}}, \phi_{\mathrm{BL}}\} \to \{r_{\mathrm{sph}}, \theta_{\mathrm{sph}}, \phi_{\mathrm{sph}}\}$, then define flat-space Cartesian coordinates[23, 14]

$$x = r_{\mathrm{sph}} \sin\theta_{\mathrm{sph}} \cos\phi_{\mathrm{sph}} \tag{3.26}$$

$$y = r_{\mathrm{sph}} \sin\theta_{\mathrm{sph}} \sin\phi_{\mathrm{sph}} \tag{3.27}$$

$$z = r_{\mathrm{sph}} \cos\theta_{\mathrm{sph}}. \tag{3.28}$$

2. Identify the Boyer-Lindquist coordinates with flat-space oblate-spheroidal coordinates $\{r_{\mathrm{BL}}, \theta_{\mathrm{BL}}, \phi_{\mathrm{BL}}\} \to \{r_{\mathrm{ob}}, \theta_{\mathrm{ob}}, \phi_{\mathrm{ob}}\}$ so that the flat-space Cartesian coordinates are

$$x = \sqrt{r_{\mathrm{ob}}^2 + a^2} \sin\theta_{\mathrm{ob}} \cos\phi_{\mathrm{ob}} \tag{3.29}$$

$$y = \sqrt{r_{\mathrm{ob}}^2 + a^2} \sin\theta_{\mathrm{ob}} \sin\phi_{\mathrm{ob}} \tag{3.30}$$

$$z = r_{\mathrm{ob}} \cos\theta_{\mathrm{ob}}. \tag{3.31}$$

These are appealing since in the limit of $G \to 0$, so that the gravitating mass goes to zero, the Kerr metric in Boyer-Lindquist coordinates reduces to the Minkowski metric in oblate spheroidal coordinates.[3]

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[3]We must take the limit $G \to 0$, rather than $M_\bullet \to 0$ to avoid the problem of an over-extreme BH with $a > M_b ullet$.

In the limit of $a \to 0$, the two coincide, as they do in the limit of large r_{BL}. It must be stressed that there is no well motivated argument that either coordinate system must yield an accurate GW; their use is justified *post facto* by comparison with result obtained from more accurate, and computationally intensive, methods[23, 14]. This ambiguity in assigning flat-space coordinates reflects the inconsistency of the semirelativistic approximation: the geodesic trajectory was calculated for the Kerr geometry, by moving to flat spacetime we lose the reason for its existance. However, this inconsistency should not be regarded as a major problem; it is just an artifact of the basic assumption that the shape of the trajectory is important for determining the character of the radiation, but the curvature of the spacetime in the vicintiy of the source is not. By binding the particle to the exact geodesic, we ensure that the kludge waveform has spectral components at the correct frequencies, but by assuming flat spacetime for generation of GWs they will not have the correct amplitudes.

3.3.2 Quadrupole-Octopole Formula

Now we have a flat-space particle trajectory, we may apply a flat-space wave generation formula. We shall use the quadrupole-octopole formula to calculate the gravitational strain[26, 27]

$$\overline{h}^{jk} = -\frac{2G}{c^6 r} \left[\ddot{I}^{jk} - 2n_i \dddot{S}^{ijk} + n_i \dddot{M}^{ijk} \right]_{t'=t-cr} \tag{3.32}$$

where an overdot represents differentiation with respect to time t, t' is the retarded time, r is the radial distance, $\boldsymbol{n}$ is the radial unit vector, and the mass quadrupole I^{jk}, current quadrupole S^{ijk} and mass octopole M^{ijk} are defined by

$$I^{jk}(t') = \int x'^j x'^k T^{00}(t', \boldsymbol{x'}) \, \mathrm{d}^3 x' \tag{3.33}$$

$$S^{ijk}(t') = \int x'^j x'^k T^{0i}(t', \boldsymbol{x'} \, \mathrm{d}^3 x' \tag{3.34}$$

$$M^{ijk}(t') = \frac{1}{c} \int x'^i x'^j x'^k T^{00}(t', \boldsymbol{x'} \, \mathrm{d}^3 x'. \tag{3.35}$$

This is correct for a slow moving source. It is the familiar quadrupole formula[28, 17], derived from linearized theory, but with the next order term included. For a point mass the energy-momentum tensor $T^{\mu\nu}$ contains a δ-function which allows easy evaluation of the integrals of the various moments to give

$$I^{jk} = c^2 \mu x^j x^k \tag{3.36}$$

$$S^{ijk} = c\mu v^i x^j x^k \tag{3.37}$$

$$M^{ijk} = c\mu x^i x^j x^k. \tag{3.38}$$

The velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$ can be calculated simply from the geodesic equations.

3.4 Energy Spectra

The energy flux from the kludge waveform is larger than the Peters and Matthews result. This behaviour has been seen before for high eccentricity orbits about a non-spinning

BH[23]

3.5   Parameter Extimation & Fisher Matrices

Chapter 4

Further Work

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