Mon. Not. R. Astron. Soc. **000**, 1–5 (2011) Printed 15 March 2011 (MN LaTeX style file v2.2)

# Galactic Dynamics

C. P. L. Berry[1⋆] and J. R. Gair[1]
[1]*Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge, CB3 0HA*

15 March 2011

**ABSTRACT**
I thought I would try out the MNRAS LaTeX style.

**Key words:** black hole physics – celestial mechanics – Galaxy: centre – gravitational waves.

## 1 EVENT RATES

### 1.1 The distribution function

We wish to calculate the probability that there is an encounter between a compact object on an orbital trajectory described by eccentricity $e$ and periapse radius $r_p$ and the massive black hole (MBH) at the galactic centre. To do so we must assume a particular distribution of stars. We begin by following the work of Bahcall & Wolf (1976, 1977) and assuming that the distribution function within the galactic core is just a function of the orbital energy; we define the energy per unit mass of the orbit as

$$\mathcal{E} = \frac{v^2}{2} - \frac{GM_\bullet}{r} \tag{1}$$

where $M_\bullet$ is the mass of the MBH. The number of stars is given by

$$N = \int \mathrm{d}^3r \int \mathrm{d}^3v f(\mathcal{E}). \tag{2}$$

Close to the centre of the galactic core dynamics will be dominated by the influence of the MBH as it is significantly more massive than the surrounding stars. We will define its radius of influence as (Frank & Rees 1976)

$$r_\mathrm{c} = \frac{GM_\bullet}{\sigma^2} \tag{3}$$

where $\sigma^2$ is the line-of-sight velocity dispersion. We will assume that the mass of stars enclosed within the radius is greater than the black hole mass, which is much greater than the mass of a typical star $M_\star$ (Bahcall & Wolf 1976). We will define a reference number density from the enclosed mass as

$$m_\star(r_\mathrm{c}) = \frac{4\pi r_\mathrm{c}^3}{3} n_\star M_\star. \tag{4}$$

Within the core, the distribution function can be calculated using the approximation of Fokker-Planck formalism. The population of bound stars is evolved numerically until a steady state is reached: the unbound stars form a reservoir with an assumed Maxwellian distribution. Denoting a species of star by its mass $M$:

$$f_M(\mathcal{E}) = \frac{C_M n_\star}{(2\pi\sigma_M^2)^{3/2}} \exp\left(-\frac{\mathcal{E}}{\sigma_M^2}\right), \quad \mathcal{E} > 0, \tag{5}$$

where $C_M$ is a normalisation constant.[1] If different stellar species are in equipartition (as was assumed by Bahcall & Wolf 1976, 1977) then we expect

$$M\sigma_M^2 = M_\star\sigma_\star^2. \tag{6}$$

However, if the unbound stellar population has reached equilibrium by violent relaxation, then all mass groups are expected to have similar velocity dispersions:

$$\sigma_M = \sigma_\star = \sigma, \tag{7}$$

and we have equipartition of energy per unit mass (Lynden-Bell, D. 1967). This will be assumed here following Alexander & Hopman (2009); O'Leary et al. (2009). The steady-state distribution function is largely insensitive to this choice (Bahcall & Wolf 1977; Alexander & Hopman 2009).

For bound orbits the distribution function can be approximated as a power law

$$f_M(\mathcal{E}) = \frac{k_M n_\star}{(2\pi\sigma^2)^{3/2}} \left(-\frac{\mathcal{E}}{\sigma^2}\right)^{p_M}, \quad \mathcal{E} < 0. \tag{8}$$

The exponent $p_M$ varies depending upon the mass of the object, determining mass segregation. For a system with a single mass component, Bahcall & Wolf (1976) find that $p = 1/4$. The normalisation constant $k_M$ reflects the relative abundances of the different species.[2]

[1] $C_M$ determines the population ratio of species $M$ far from the black hole Alexander & Hopman (2009).
[2] For a single mass population ($p = 1/4$) $k = 2C$ gives a fit correct to within a factor a two (Bahcall & Wolf 1976; Keshet et al. 2009), we will assume this holds for the dominant species of stars as, although it will vary slightly with $p$, variation is small compared to errors introduced by fitting a simple power law (Hopman & Alexander 2006a; Alexander & Hopman 2009).

⋆ E-mail: cplb2@cam.ac.uk

**Table 1.** Stellar model parameters for the galactic centre using the results of Alexander & Hopman (2009) We use the main sequence star as our reference. The number fractions for unbound stars are estimates corresponding to a model of continuous star formation (Alexander 2005); O'Leary et al. (2009) arrive at the same proportions.

| Star | $M/M_\odot$ | $C_M/C_\star$ | $p_M$ | $k_M/k_\star$ [a] |
|---|---|---|---|---|
| MS | 1.0 | 1 | −0.1 | 1 |
| WD | 0.6 | 0.1 | −0.1 | 0.09 |
| NS | 1.4 | 0.01 | 0.0 | 0.01 |
| BH | 10 | 0.001 | 0.5 | 0.008 |

[a] Toonen et al. (2009)

### 1.2 Model parameters

We will be using the Fokker-Planck model of Hopman & Alexander (2006a,b); Alexander & Hopman (2009). This includes four stellar species: main sequence stars (MS), white dwarfs (WD), neutron stars (NS), and black holes (BH). Their properties are summarised in table 1. The behaviour of the Fokker-Planck model has been verified by $N$-body simulations (Preto & Amaro-Seoane 2010). The steeper power law for black holes means that they segregate about the MBH: they dominate in place of main sequence stars for radii $r < 10^{-4} r_\mathrm{c}$.

Binaries may form in the galactic centre, encouraged by its high stellar density (O'Leary et al. 2009). However the binary fraction is still expected to be small (Hopman 2009). Consequently, and because binaries will be disrupted by the MBH for periapses smaller than

$$r_\mathrm{B} \simeq r_\mathrm{T} \simeq \left(\frac{M_\bullet}{M_1 + M_2}\right)^{1/3} a_\mathrm{B}, \tag{9}$$

where $M_1$ and $M_2$ are the masses of the binary's components, and $a_\mathrm{B}$ is the binary's semi-major axis [cf. (33) below], we shall ignore the possible presence of binaries.

We assume a black hole mass of $M_\bullet = (4.31 \pm 0.36) \times 10^6 M_\odot$ (Gillessen et al. 2009) and a velocity dispersion of $\sigma = (103 \pm 20)\ \mathrm{km\,s^{-1}}$ (Tremaine et al. 2002). This gives a core radius of $r_\mathrm{c} = (1.7 \pm 0.7)$ pc. Using the results of Ghez et al. (2008) we would expect the total mass of stars core to be $m_\star(r_\mathrm{c}) = 6.4 \times 10^6 M_\odot$ (which is within 10% of the value obtained similarly from Genzel et al. 2003). This gives a reference stellar density of $n_\star = 2.8 \times 10^5\ \mathrm{pc^{-3}}$.

### 1.3 Parameterising in terms of eccentricity & periapsis

We will characterise orbits by their eccentricity $e$ and periapse radius $r_\mathrm{p}$. The latter, unlike the semimajor axis, is always well defined regardless of eccentricity. For Keplerian orbits, the energy $\mathcal{E}$ and angular momentum $J$ per unit mass are entirely characterised by these parameters

$$\mathcal{E} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{p}}, \tag{10}$$

$$J^2 = GM_\bullet(1+e)r_\mathrm{p}. \tag{11}$$

The distribution function, however, is defined per element of phase space: it is necessary to change variables from position and velocity to eccentricity and periapsis. We start by decomposing the velocity into three orthogonal components: radial $v_r$, azimuthal $v_\phi$ and polar $v_\theta$. We will assume that the galactic core is spherically symmetric (Genzel et al. 2003; Schödel et al. 2007), therefore we are only interested in the combination

$$v_\perp^2 = v_\phi^2 + v_\theta^2 = v^2 - v_r^2. \tag{12}$$

Under this change of variables

$$\mathrm{d}^3v = \mathrm{d}v_r\mathrm{d}v_\phi\mathrm{d}v_\theta \rightarrow 2\pi v_\perp \mathrm{d}v_r\mathrm{d}v_\perp. \tag{13}$$

The specific energy and angular momentum are given by

$$\mathcal{E} = \frac{v_r^2 + v_\perp^2}{2} - \frac{GM_\bullet}{r}, \tag{14}$$

$$J^2 = r^2 v_\perp^2. \tag{15}$$

If we combine these with our earlier expressions in terms of $e$ and $r_\mathrm{p}$ we find

$$v_\perp^2 = \frac{GM_\bullet(1+e)r_\mathrm{p}}{r^2},$$

$$v_r^2 = GM\left[\frac{2}{r} - \frac{(1-e)}{r_\mathrm{p}} - \frac{(1+e)r_\mathrm{p}}{r^2}\right]. \tag{16}$$

From the latter we can verify that the turning points of an orbit occur at

$$r = r_\mathrm{p},\ \frac{1+e}{1-e}r_\mathrm{p}; \tag{17}$$

the periapse is the only turning point for orbits with $e > 1$. Since we now have expressions for $\{v_r, v_\perp\}$ in terms of $\{e, r_\mathrm{p}\}$ we can calculate the Jacobian

$$\left|\frac{\partial(v_r, v_\perp)}{\partial(e, r_\mathrm{p})}\right| = \frac{\partial v_r}{\partial e}\frac{\partial v_\perp}{\partial r_\mathrm{p}} - \frac{\partial v_r}{\partial r_\mathrm{p}}\frac{\partial v_\perp}{\partial e} \tag{18}$$

$$= \frac{1}{2v_r v_\perp}\frac{e}{r_\mathrm{p}}\left(\frac{GM}{r}\right)^2. \tag{19}$$

Using this, we may rewrite our velocity element as

$$\mathrm{d}^3v \rightarrow \frac{\pi e}{v_r r_\mathrm{p}}\left(\frac{GM}{r}\right)^2 \mathrm{d}e\mathrm{d}r_\mathrm{p}. \tag{20}$$

As a consequence of our assumed spherical symmetry, the volume element can be expressed as

$$\mathrm{d}^3r = 4\pi r^2 \mathrm{d}r. \tag{21}$$

Thus, the phase space volume element can be expressed as

$$\mathrm{d}^3r\mathrm{d}^3v = \frac{4\pi^2(GM)^2e}{v_r r_\mathrm{p}}\mathrm{d}r\mathrm{d}e\mathrm{d}r_\mathrm{p}. \tag{22}$$

The number of stars in an element $\mathrm{d}r\mathrm{d}e\mathrm{d}r_\mathrm{p}$ is

$$n(r, e, r_\mathrm{p}) = \frac{4\pi^2(GM)^2e}{v_r r_\mathrm{p}} f(\mathcal{E}). \tag{23}$$

From this, we can construct the expected number of stars to be on orbits defined by $\{e, r_\mathrm{p}\}$. We will define this locally, allowing it to vary with position. The number of stars found in a small radius range $\delta r$ with given orbital properties can be calculated by multiplying the total number of stars with these properties by the relative amount of time they spend in that range

$$n(r, e, r_\mathrm{p})\delta r = N(e, r_\mathrm{p}; r)\frac{\delta t}{P(e, r_\mathrm{p})} \tag{24}$$

where $N(e, r_\mathrm{p}; r)$ is the total number of stars with orbits given by $\{e, r_\mathrm{p}\}$ defined at $r$, $\delta t$ is the time spent in $\delta r$ and $P(e, r_\mathrm{p})$ is the period of the orbit. We will defer the definition of this time for unbound orbits for now. The time spent in the radius range is

$$\delta t = 2\frac{\delta r}{v_r}, \tag{25}$$

where the factor of 2 is included to account for both inwards and outwards motion. Hence

$$N(e, r_\mathrm{p}; r) = \frac{1}{2} v_r P n(r, e, r_\mathrm{p}) \tag{26}$$
$$= \frac{2\pi^2 (GM)^2 e P}{r_\mathrm{p}} f(\mathcal{E}). \tag{27}$$

The right hand side of this equation is independent of position, subject to the constraint that the radius is in the allowed range for the orbit $r_\mathrm{p} \leqslant r \leqslant (1+e)r_\mathrm{p}/(1-e)$, and so we may define $N(e, r_\mathrm{p}) \equiv N(e, r_\mathrm{p}; r)$. This is a consequence of the distribution function being dependent only upon a constant of the motion.

If a burst of radiation is emitted each time a star passes through periapse, then the event rate for burst emission from orbits with parameters $\{e, r_\mathrm{p}\}$, is given by

$$\Gamma(e, r_\mathrm{p}) = \frac{N(e, r_\mathrm{p})}{P(e, r_\mathrm{p})} \tag{28}$$
$$= \frac{2\pi^2 (GM)^2 e}{r_\mathrm{p}} f(\mathcal{E}). \tag{29}$$

The orbital period drops out from the calculation, so we do not have to worry about an appropriate definition for unbound orbits.

From the event rate we may define a probability of seeing a given number of events subject to the assumption that they are uncorrelated: it is given by the Poisson distribution. The probability of there being $r$ events is

$$\Pr(r|\Gamma(e, r_\mathrm{p})) = \frac{\Gamma^r \exp(-\Gamma)}{r!}. \tag{30}$$

The probability of there being a burst from an orbit with periapse $r_\mathrm{p}$ and eccentricity $e$ is hence

$$\Pr(r \neq 0|\Gamma(e, r_\mathrm{p})) = 1 - \Pr(r = 0|\Gamma(e, r_\mathrm{p})). \tag{31}$$

To estimate the expectation of a quantity across all orbits we use

$$\langle X \rangle = \sum_\mathrm{R} \int_0^\infty \mathrm{d}e \int_0^\infty \mathrm{d}r_\mathrm{p} X(r; r_\mathrm{p}, e) \Pr(r|\Gamma(e, r_\mathrm{p})). \tag{32}$$

Since the probability decays rapidly for large $r$, we may truncate the sum to give the required level of accuracy,

To generate a representative sample for the orbital parameters $e$ and $r_\mathrm{p}$, we use $\Gamma(e, r_\mathrm{p})$ as an unnormalised probability distribution and draw from it appropriately.

### 1.4 The inner cut-off

From (29) we see that the event rate is highly sensitive to the smallest value of the periapsis. The inner cut-off for $r_\mathrm{p}$ could result from a number of different physical causes. Ultimately the orbits cannot encroach closer to the black hole than its last stable orbit. This depends upon the spin of the black hole, but is of the order of its Schwarzschild radius. Before we reach this point, however, there are other processes that may intervene to deplete the orbiting stars. Our treatment of these is approximate, however should hopefully produce reasonable estimates. We will consider three processes: tidal disruption, gravitational wave inspiral and collisional disruption. Tidal disruption imposes a definite cut-off, while the others use statistical arguments; they are therefore true for a typical star, and it is unlikely that a star would be found beyond the imposed limits.

#### *1.4.1 Tidal disruption*

Tidal forces from the black hole can disrupt stars. This occurs at the tidal radius

$$r_\mathrm{T} \simeq \left(\frac{M_\bullet}{M}\right)^{1/3} R_M \tag{33}$$

where $R_M$ is the radius of the star (Kobayashi et al. 2004). Any star on an orbit with $r_\mathrm{p} < r_\mathrm{T}$ will be disrupted in the course of its orbit. Tidal disruption is most significant for MS stars since they are least dense; calculated in this way, only MS stars would be tidally disrupted outside of the MBH's event horizon. The tidal radius defines the cut-off for periapsis of high eccentricity ($e \gtrsim 1$ orbits (Lightman & Shapiro 1977)

#### *1.4.2 Gravitational wave inspiral*

Stars orbiting about the black hole will continually radiate energy and angular momentum causing them to inspiral. Using the analysis of Peters (1964) for Keplerian binaries, the orbit-averaged rate of change of the periapsis and eccentricity for bound orbits are

$$\left\langle \frac{\mathrm{d}r_\mathrm{p}}{\mathrm{d}t} \right\rangle = -\frac{64}{5}\frac{\Theta}{r_\mathrm{p}^3}\frac{(1-e)^{3/2}}{(1+e)^{7/2}}\left(1 - \frac{7}{12}e + \frac{7}{8}e^2 + \frac{47}{192}e^3\right) \tag{34}$$

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{304}{5}\frac{\Theta}{r_\mathrm{p}^4}\frac{e(1-e)^{3/2}}{(1+e)^{5/2}}\left(1 + \frac{121}{304}e^2\right), \tag{35}$$

where we have introduced

$$\Theta = \frac{G^3 M_\bullet M (M_\bullet + M)}{c^5}. \tag{36}$$

For a circular orbit the inspiral time from initial periapsis $r_\mathrm{p0}$ is

$$\tau_\mathrm{c}(r_\mathrm{p0}) = \frac{5}{256}\frac{r_\mathrm{p0}^4}{\Theta}. \tag{37}$$

For an orbit of finite eccentricity ($0 < e < 1$), we can solve for the periapsis as a function of eccentricity

$$r_\mathrm{p}(e) = \Lambda(1+e)^{-1}\left(1 + \frac{121}{304}e^2\right)^{870/2299} e^{12/19}, \tag{38}$$

where $\Lambda$ is a constant fixed by the initial conditions: for an orbit with initial eccentricity $e_0$

$$\Lambda = (1+e_0)\left(1 + \frac{121}{304}e_0^2\right)^{-870/2299} e_0^{-12/19} r_\mathrm{p0}. \tag{39}$$

The inspiral is complete when the eccentricity has decayed to zero. Consequently the inspiral time is (Peters 1964)

$$\tau(r_\mathrm{p0}, e_0) = \int_0^{e_0} \frac{15}{304}\frac{\Lambda^4}{\Theta}\frac{e^{29/19}}{(1-e^2)^{3/2}}\left(1 + \frac{121}{304}e^2\right)^{1181/2299} \mathrm{d}e.$$

(40)

This is best evaluated numerically, however it may be written in closed form as

$$\tau(r_{\mathrm{p}0}, e_0) = \tau_{\mathrm{c}}(r_{\mathrm{p}0})(1+e_0)^4 \left(1 + \frac{121}{304}e_0^2\right)^{-3480/2299} \times F_1\left(\frac{24}{19}; \frac{3}{2}, -\frac{1181}{2299}; \frac{43}{19}; e_0^2, -\frac{121}{304}e_0^2\right), \quad (41)$$

using the Appell hypergeometric function of the first kind $F_1(\alpha; \beta, \beta'; \gamma; x, y)$ (Olver et al. 2010, 16.15.1).[3]

We will assume that an orbit is depleted of stars if the inspiral time-scale is shorter than the relaxation time-scale. This indicates the characteristic time for two-body collisions to change the velocity of the star by order of itself (Binney & Tremaine 1987), and so indicates the time over which scattering may repopulate the orbit. Following (Spitzer & Hart 1971), a relaxation time-scale can be estimated from the diffusion coefficient of the Fokker-Planck equation (Binney & Tremaine 1987, section 8.3.4), for a system with a purely Maxwellian distribution

$$\tau_{\mathrm{R}} \simeq 0.34 \frac{\sigma^3}{G M_\star \rho_\star \ln \Lambda}, \quad (42)$$

where $\rho_\star$ is the mass density and the Coulomb logarithm is $\ln \Lambda = \ln(M_\bullet/M_\star)$. Bahcall & Wolf (1977) find a similar characteristic time-scale, but with numerical prefactor $3/4\sqrt{8\pi} \simeq 0.15$. We shall adopt the former to be conservative. From this we estimate the relaxation time for an orbit as

$$T_{\mathrm{R}}(r_{\mathrm{p}}, e) \approx \left(\frac{\langle v(r)\rangle_{\mathrm{orb}}}{\sigma}\right)^3 \frac{M_\star \rho_\star}{\langle M(r)\rangle_{\mathrm{orb}} \langle \rho(r)\rangle_{\mathrm{orb}}} \tau_{\mathrm{R}} \quad (43)$$

using the velocity $\langle v(r)\rangle_{\mathrm{orb}}$, stellar mass of encountered stars $\langle M(r)\rangle_{\mathrm{orb}}$ and density $\langle \rho(r)\rangle_{\mathrm{orb}}$ averaged over the duration of the orbit as characteristic quantities.[4] The subscript orb indicates that we average along the trajectory of the orbit specified by parameters $\{r_{\mathrm{p}}, e\}$.[5] The average velocity may be written in closed form using the complete elliptic integral of the second kind,

$$\frac{\langle v(r)\rangle_{\mathrm{orb}}}{\sigma} = \frac{2E(e)}{\pi} \sqrt{\frac{r_{\mathrm{c}}(1-e)}{r_{\mathrm{p}}}}, \quad (44)$$

however the other averages are best evaluated numerically.

Unbound stars only undergo a single periapse passage and only radiate one burst of radiation; we shall therefore neglect any evolution in their orbital parameters.[6]

[3] For small eccentricities $\tau(r_{\mathrm{p}0}, e_0) \simeq \tau_{\mathrm{c}}(r_{\mathrm{p}0})[1 + 4e_0 + (273/43)e_0^2 + \mathcal{O}(e_0^3)]$.

[4] For comparison, Hopman & Alexander (2005) approximate the typical relaxation time associated with an orbit with the value at the semimajor axis.

[5] In calculating $\langle \rho(r)\rangle_{\mathrm{orb}}$ and $\langle M(r)\rangle_{\mathrm{orb}}$ we include the tidal disruption of stars but not the possible depopulation of orbits due to inspiral or collisions. This would require solving iteratively for $n(r)$, which is not justifiable at this level of approximation. The relaxation time-scale will be underestimated using this simple approach. The error will be greatest for low eccentricity orbits which spend most of their time in the same region of space.

[6] Using the analysis of Turner (1977) it is possible to show that the relative changes in eccentricity $\Delta e/e$ and periapsis $\Delta r_{\mathrm{p}}/r_{\mathrm{p}}$ for an extreme mass-ratio binary are less than $\mathcal{O}(\eta)\sqrt{e}$, where $\eta = M/M_\bullet$ is small, and so are only important for very high eccentricity orbits (Appendix A). These are very high energy, and exponentially rare because of the Boltzmann factor in (5).

### 1.4.3 Collisions

As a consequence of the higher densities in the galactic core, stars may undergo a large number of close encounters with other stars. A star is unlikely to be disrupted by a single collision (Freitag & Benz 2005), rather it takes 20–30 grazing collisions to disrupt a MS star (Freitag et al. 2006). We shall ignore the possibility of disruption for the other species since they are harder to disrupt: as a consequence of smaller cross-sectional areas, none of the other species would undergo 30 collisions with a periapsis greater than the cut-off implied by gravitational wave inspiral for bound orbits, or the MBH's event horizon for unbound orbits. The number of collisions a star will undergo in a time interval $\delta t$ is

$$\delta K = n(r) A v(r, e, r_{\mathrm{p}}) \delta t, \quad (45)$$

where $A$ is the star's cross-sectional area for a grazing collision. We shall assume that the relative velocity of the colliding stars is much greater than the escape velocity of the star so we may neglect the effects of gravitational focussing, then the cross-sectional area is simply the geometric $A = \pi R_\star^2$.

For circular orbits we can find the radius at which collisions will lead to disruptions by setting $\delta K = 30$ and $\delta t = \tau_{\mathrm{R}}$. We use the relaxation time-scale for the system as this is the time over which stars are replenished from the reservoir. For non-circular orbits we must consider variation with position. Using $\delta r = v_r \delta t$, and then converting to an integral, we have for bound orbits

$$K = 2\pi R_\star^2 \frac{\tau_{\mathrm{R}}}{P(r_{\mathrm{p}}, e)} \int_{r_{\mathrm{p}}}^{(1+e)r_{\mathrm{p}}/(1-e)} n(r) \frac{v(r, e, r_{\mathrm{p}})}{v_r(r, e, r_{\mathrm{p}})}\, \mathrm{d}r, \quad (46)$$

where $P$ is the period of the orbit. Again we may set $K = 30$ to find the orbits for which stars will be disrupted within a relaxation time-scale. For unbound stars we are only interested in stars that would become disrupted before their periapse passage, so

$$K = \pi R_\star^2 \int_{r_{\mathrm{p}}}^{r_{\mathrm{c}}} n(r) \frac{v(r, e, r_{\mathrm{p}})}{v_r(r, e, r_{\mathrm{p}})}\, \mathrm{d}r, \quad (47)$$

assuming that the stars in the reservoir external to the core are unlikely to undergo close collisions.

## ACKNOWLEDGMENTS

CPLB is supported by STFC. JRG is supported by the Royal Society.

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## APPENDIX A: EVOLUTION OF ORBITAL PARAMETERS FOR UNBOUND ORBITS DUE TO GRAVITATIONAL WAVE EMISSION

Following the approach of Turner (1977) we can calculate the evolution of the eccentricity and periapse of a Keplerian binary through the loss of energy and angular momentum carried away by gravitational radiation. The change in fractional eccentricity over an orbit, approximating the orbital parameters remain constant throughout the orbit, is

$$\begin{aligned}\frac{\Delta e}{e} = &-\frac{608}{15}\Sigma\left[\frac{1}{(1+e)^{5/2}}\left(1+\frac{121}{304}e^2\right)\cos^{-1}\left(-\frac{1}{e}\right)\right.\\ &\left.+\frac{(e-1)^{1/2}}{e^2(1+e)^2}\left(\frac{67}{456}+\frac{1069}{912}e^2+\frac{3}{38}e^4\right)\right],\end{aligned} \tag{A1}$$

introducing dimensionless parameter

$$\Sigma = \frac{G^{5/2}M_\bullet M(M_\bullet + M)}{c^5 r_{\mathrm{p}}^{5/2}}. \tag{A2}$$

Similarly, the fractional change in periapsis is

$$\begin{aligned}\frac{\Delta r_{\mathrm{p}}}{r_{\mathrm{p}}} = &-\frac{128}{5}\Sigma\left[\frac{1}{(1+e)^{7/2}}\left(1-\frac{7}{12}e+\frac{7}{8}e^2+\frac{47}{192}e^3\right)\cos^{-1}\left(-\frac{1}{e}\right)\right.\\ &\left.-\frac{(e-1)^{1/2}}{e(1+e)^3}\left(\frac{67}{288}-\frac{13}{8}e+\frac{133}{576}e^2-\frac{1}{4}e^3-\frac{1}{8}e^4\right)\right].\end{aligned} \tag{A3}$$

Both of these changes obtain their greatest magnitudes for large eccentricities, then

$$\frac{\Delta e}{e} \simeq \frac{\Delta r_{\mathrm{p}}}{r_{\mathrm{p}}} \simeq -\frac{16}{5}\Sigma e^{1/2}. \tag{A4}$$

For extreme mass-ratio binaries, as is the case here, the mass-ratio is a small quantity

$$\eta = M/M_\bullet \ll 1. \tag{A5}$$

The smallest possible periapsis is of order of the Schwarzschild radius of the MBH, such that

$$r_{\mathrm{p}} = \alpha\frac{GM_\bullet}{c^2}; \quad \alpha > 1. \tag{A6}$$

These give

$$\Sigma = \frac{\eta}{\alpha^{5/2}} < \eta \ll 1. \tag{A7}$$

Hence the changes in orbital parameters will be significant for

$$e \sim \frac{25}{256}\frac{\alpha^5}{\eta^2} > \frac{25}{256}\frac{1}{\eta^2}. \tag{A8}$$

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