Galactic Dynamics

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ABSTRACT
I thought I would try out the MNRAS LaTeX style.

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1 EVENT RATES

1.1 The distribution function

We wish to calculate the probability that there is an encounter between a compact object on an orbital trajectory described by eccentricity e and periapse radius r_p and the massive black hole (MBH) at the galactic centre. To do so we must assume a particular distribution of stars. We begin by following the work of Bahcall & Wolf (1976, 1977) and assuming that the distribution function (DF) within the galactic core is just a function of the orbital energy; we define the energy per unit mass of the orbit as

$$\mathcal{E} = \frac{v^2}{2} - \frac{GM_\bullet}{r} \tag{1}$$

where $M_\bullet$ is the mass of the MBH. The number of stars is given by

$$N = \int \mathrm{d}^3 r \int \mathrm{d}^3 v f(\mathcal{E}). \tag{2}$$

Close to the centre of the galactic core dynamics will be dominated by the influence of the MBH as it is significantly more massive than the surrounding stars. We will define its radius of influence as (Frank & Rees 1976)

$$r_\mathrm{c} = \frac{GM_\bullet}{\sigma^2} \tag{3}$$

where σ^2 is the line-of-sight velocity dispersion. We will assume that the mass of stars enclosed within the radius is greater than the black hole mass, which is much greater than the mass of a typical star $M_\star$ (Bahcall & Wolf 1976). We will define a reference number density from the enclosed mass as

$$m_\star(r_\mathrm{c}) = \frac{4\pi r_\mathrm{c}^3}{3} n_\star M_\star. \tag{4}$$

Within the core, the distribution function can be calculated using the approximation of Fokker-Planck formalism. The population of bound stars is evolved numerically until a

steady state is reached: the unbound stars form a reservoir with an assumed Maxwellian distribution. Denoting a species of star by its mass M:

$$f_M(\mathcal{E}) = \frac{C_M n_\star}{(2\pi\sigma_M^2)^{3/2}} \exp\left(-\frac{\mathcal{E}}{\sigma_M^2}\right), \quad \mathcal{E} > 0, \tag{5}$$

where C_M is a normalisation constant.[1] If different stellar species are in equipartition (as was assumed by Bahcall & Wolf 1976, 1977) then we expect

$$M\sigma_M^2 = M_\star \sigma_\star^2. \tag{6}$$

However, if the unbound stellar population has reached equilibrium by violent relaxation, then all mass groups are expected to have similar velocity dispersions:

$$\sigma_M = \sigma_\star = \sigma, \tag{7}$$

and we have equipartition of energy per unit mass (Lynden-Bell 1967). This will be assumed here following Alexander & Hopman (2009); O'Leary et al. (2009). The steady-state distribution function is largely insensitive to this choice (Bahcall & Wolf 1977; Alexander & Hopman 2009).

For bound orbits the DF can be approximated as a power law (Peebles 1972)

$$f_M(\mathcal{E}) = \frac{k_M n_\star}{(2\pi\sigma^2)^{3/2}} \left(-\frac{\mathcal{E}}{\sigma^2}\right)^{p_M}, \quad \mathcal{E} < 0. \tag{8}$$

The exponent p_M varies depending upon the mass of the object, determining mass segregation. For a system with a single mass component, Bahcall & Wolf (1976) find that $p = 1/4$. The normalisation constant k_M reflects the relative abundances of the different species.[2]

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[1] C_M determines the population ratio of species M far from the black hole (Alexander & Hopman 2009).
[2] For a single mass population ($p = 1/4$) $k = 2C$ gives a fit correct to within a factor a two (Bahcall & Wolf 1976; Keshet et al. 2009), we will assume this holds for the dominant species of stars as, although it will change slightly with p, variation is small compared to errors introduced by fitting a simple power law (Hopman & Alexander 2006a; Alexander & Hopman 2009).

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Table 1. Stellar model parameters for the galactic centre using the results of Alexander & Hopman (2009) We use the main sequence star as our reference. The number fractions for unbound stars are estimates corresponding to a model of continuous star formation (Alexander 2005); O'Leary et al. (2009) arrive at the same proportions.

Star	$M/M_\odot$	$C_M/C_\star$	p_M	$k_M/k_\star$ [a]
MS	1.0	1	−0.1	1
WD	0.6	0.1	−0.1	0.09
NS	1.4	0.01	0.0	0.01
BH	10	0.001	0.5	0.008

[a] Toonen et al. (2009)

1.2 Model parameters

We will be using the Fokker-Planck model of Hopman & Alexander (2006a,b); Alexander & Hopman (2009). This includes four stellar species: main sequence stars (MS), white dwarfs (WD), neutron stars (NS), and black holes (BH). Their properties are summarised in table 1. The behaviour of the Fokker-Planck model has been verified by N-body simulations (Preto & Amaro-Seoane 2010). The steeper power law for black holes means that they segregate about the MBH: they dominate in place of main sequence stars for radii $r < 10^{-4} r_c$.

Binaries may form in the galactic centre, encouraged by its high stellar density (O'Leary et al. 2009). However the binary fraction is still expected to be small (Hopman 2009). Consequently, and because binaries will be disrupted by the MBH for periapses smaller than

$$r_B \simeq \left(\frac{M_\bullet}{M_1 + M_2} \right)^{1/3} a_B, \tag{9}$$

where M_1 and M_2 are the masses of the binary's components, and a_B is the binary's semi-major axis [cf. (33) below], we shall ignore the possible presence of binaries.

We assume a black hole mass of $M_\bullet = (4.31 \pm 0.36) \times 10^6 M_\odot$ (Gillessen et al. 2009) and a velocity dispersion of $\sigma = (103 \pm 20) \ \text{km s}^{-1}$ (Tremaine et al. 2002). This gives a core radius of $r_c = (1.7 \pm 0.7)$ pc. Using the results of Ghez et al. (2008) we would expect the total mass of stars core to be $m_\star(r_c) = 6.4 \times 10^6 M_\odot$ (which is within 5% of the value obtained similarly from Genzel et al. 2003). This gives a reference stellar density of $n_\star = 2.8 \times 10^5 \ \text{pc}^{-3}$.

1.3 Parameterising in terms of eccentricity & periapsis

We will characterise orbits by their eccentricity e and periapse radius r_p. The latter, unlike the semimajor axis, is always well defined regardless of eccentricity. For Keplerian orbits, the energy $\mathcal{E}$ and angular momentum J per unit mass are entirely characterised by these parameters

$$\mathcal{E} = -\frac{GM_\bullet(1-e)}{2r_p}, \tag{10}$$

$$J^2 = GM_\bullet(1+e)r_p. \tag{11}$$

The distribution function, however, is defined per element of phase space: it is necessary to change variables from position and velocity to eccentricity and periapsis. We start

by decomposing the velocity into three orthogonal components: radial v_r, azimuthal v_ϕ and polar v_θ. We will assume that the galactic core is spherically symmetric (Genzel et al. 2003; Schödel et al. 2007), therefore we are only interested in the combination

$$v_\perp^2 = v_\phi^2 + v_\theta^2 = v^2 - v_r^2. \tag{12}$$

Under this change of variables

$$\mathrm{d}^3 v = \mathrm{d}v_r \mathrm{d}v_\phi \mathrm{d}v_\theta \to 2\pi v_\perp \, \mathrm{d}v_r \, \mathrm{d}v_\perp. \tag{13}$$

The specific energy and angular momentum are given by

$$\mathcal{E} = \frac{v_r^2 + v_\perp^2}{2} - \frac{GM_\bullet}{r}, \tag{14}$$

$$J^2 = r^2 v_\perp^2. \tag{15}$$

If we combine these with our earlier expressions in terms of e and r_p we find

$$v_\perp^2 = \frac{GM_\bullet(1+e)r_p}{r^2},$$

$$v_r^2 = GM \left[\frac{2}{r} - \frac{(1-e)}{r_p} - \frac{(1+e)r_p}{r^2} \right]. \tag{16}$$

From the latter we can verify that the turning points of an orbit occur at

$$r = r_p, \ \frac{1+e}{1-e} r_p; \tag{17}$$

the periapse is the only turning point for orbits with $e > 1$. Since we now have expressions for $\{v_r, v_\perp\}$ in terms of $\{e, r_p\}$ we can calculate the Jacobian

$$\left| \frac{\partial(v_r, v_\perp)}{\partial(e, r_p)} \right| = \frac{\partial v_r}{\partial e} \frac{\partial v_\perp}{\partial r_p} - \frac{\partial v_r}{\partial r_p} \frac{\partial v_\perp}{\partial e} \tag{18}$$

$$= \frac{1}{2v_r v_\perp} \frac{e}{r_p} \left(\frac{GM}{r} \right)^2. \tag{19}$$

Using this, we may rewrite our velocity element as

$$\mathrm{d}^3 v \to \frac{\pi e}{v_r r_p} \left(\frac{GM}{r} \right)^2 \mathrm{d}e \, \mathrm{d}r_p. \tag{20}$$

As a consequence of our assumed spherical symmetry, the volume element can be expressed as

$$\mathrm{d}^3 r = 4\pi r^2 \, \mathrm{d}r. \tag{21}$$

Thus, the phase space volume element can be expressed as

$$\mathrm{d}^3 r \mathrm{d}^3 v = \frac{4\pi^2 (GM)^2 e}{v_r r_p} \, \mathrm{d}r \, \mathrm{d}e \, \mathrm{d}r_p. \tag{22}$$

The number of stars in an element $\mathrm{d}r \, \mathrm{d}e \, \mathrm{d}r_p$ is

$$n(r, e, r_p) = \frac{4\pi^2 (GM)^2 e}{v_r r_p} f(\mathcal{E}). \tag{23}$$

From this, we can construct the expected number of stars to be on orbits defined by $\{e, r_p\}$. We will define this locally, allowing it to vary with position. The number of stars found in a small radius range δr with given orbital properties can be calculated by multiplying the total number of stars with these properties by the relative amount of time they spend in that range

$$n(r, e, r_p)\delta r = N(e, r_p; r) \frac{\delta t}{P(e, r_p)} \tag{24}$$

where $N(e, r_p; r)$ is the total number of stars with orbits

given by $\{e, r_{\mathrm{p}}\}$ defined at r, δt is the time spent in δr and $P(e, r_{\mathrm{p}})$ is the period of the orbit. We will defer the definition of this time for unbound orbits for now. The time spent in the radius range is

$$\delta t = 2 \frac{\delta r}{v_r}, \tag{25}$$

where the factor of 2 is included to account for both inwards and outwards motion. Hence

$$N(e, r_{\mathrm{p}}; r) = \frac{1}{2} v_r P n(r, e, r_{\mathrm{p}}) \tag{26}$$

$$= \frac{2\pi^2 (GM)^2 e P}{r_{\mathrm{p}}} f(\mathcal{E}). \tag{27}$$

The right hand side of this equation is independent of position, subject to the constraint that the radius is in the allowed range for the orbit $r_{\mathrm{p}} \leqslant r \leqslant (1+e)r_{\mathrm{p}}/(1-e)$, and so we may define $N(e, r_{\mathrm{p}}) \equiv N(e, r_{\mathrm{p}}; r)$. This is a consequence of the distribution function being dependent only upon a constant of the motion.

If a burst of radiation is emitted each time a star passes through periapse, then the event rate for burst emission from orbits with parameters $\{e, r_{\mathrm{p}}\}$, is given by

$$\Gamma(e, r_{\mathrm{p}}) = \frac{N(e, r_{\mathrm{p}})}{P(e, r_{\mathrm{p}})} \tag{28}$$

$$= \frac{2\pi^2 (GM)^2 e}{r_{\mathrm{p}}} f(\mathcal{E}). \tag{29}$$

The orbital period drops out from the calculation, so we do not have to worry about an appropriate definition for unbound orbits.

From the event rate we may define a probability of seeing a given number of events subject to the assumption that they are uncorrelated: it is given by the Poisson distribution. The probability of there being r events is

$$\Pr(r | \Gamma(e, r_{\mathrm{p}})) = \frac{\Gamma^r \exp(-\Gamma)}{r!}. \tag{30}$$

The probability of there being a burst from an orbit with periapse r_{p} and eccentricity e is hence

$$\Pr(r \neq 0 | \Gamma(e, r_{\mathrm{p}})) = 1 - \Pr(r = 0 | \Gamma(e, r_{\mathrm{p}})). \tag{31}$$

To estimate the expectation of a quantity across all orbits we use

$$\langle X \rangle = \sum_{\mathrm{R}} \int_0^\infty \mathrm{d}e \int_0^\infty \mathrm{d}r_{\mathrm{p}} X(r; r_{\mathrm{p}}, e) \Pr(r | \Gamma(e, r_{\mathrm{p}})). \tag{32}$$

Since the probability decays rapidly for large r, we may truncate the sum to give the required level of accuracy,

To generate a representative sample for the orbital parameters e and r_{p}, we use $\Gamma(e, r_{\mathrm{p}})$ as an unnormalised probability distribution and draw from it appropriately.

1.4 The inner cut-off

From (29) we see that the event rate is highly sensitive to the smallest value of the periapsis. The inner cut-off for r_{p} could result from a number of different physical causes. Ultimately the orbits cannot encroach closer to the black hole than its last stable orbit. This depends upon the spin of the black hole, but is of the order of its Schwarzschild radius. Before we reach this point, however, there are other processes that

may intervene to deplete the orbiting stars. Our treatment of these is approximate, however should hopefully produce reasonable estimates. We will consider three processes: tidal disruption, gravitational wave inspiral and collisional disruption. Tidal disruption imposes a definite cut-off, while the others use statistical arguments; they are therefore true for a typical star, and it is unlikely that a star would be found beyond the imposed limits. For these methods we will need to define a reference time-scale for relaxation, over which two-body interactions lead to significant changes in the orbital properties of the star. This is done in Sec. 1.4.2, with further details found in Appendix A.

1.4.1 Tidal disruption

Tidal forces from the black hole can disrupt stars. This occurs at the tidal radius

$$r_{\mathrm{T}} \simeq \left(\frac{M_\bullet}{M} \right)^{1/3} R_M \tag{33}$$

where R_M is the radius of the star (Hills 1975; Rees 1988; Kobayashi et al. 2004). Any star on an orbit with $r_{\mathrm{p}} < r_{\mathrm{T}}$ will be disrupted in the course of its orbit. Parameterising orbits by their periapsis allows us to easily determine which stars should be disrupted. We do not include the full effects of the loss cone (Frank & Rees 1976; Lightman & Shapiro 1977) as these were not incorporated into the Fokker-Planck calculations.[3] The effect of the loss cone should be small, only modifying the DF by a logarithmic term (Lightman & Shapiro 1977; Bahcall & Wolf 1977). Its effects are diluted by resonant relaxation.

Tidal disruption is most significant for MS stars since they are least dense; calculated in this way, only MS stars would be tidally disrupted outside of the MBH's event horizon (Sigurdsson & Rees 1997). The tidal radius defines the cut-off for periapsis of high eccentricity ($e \gtrsim 1$) orbits (Lightman & Shapiro 1977).

1.4.2 Relaxation time-scale

The motion of a star is determined not only by the dominant influence of the central MBH, but also by the other stars. The potential of the stars may be split into two components: the smooth background potential representing the average distribution of stars, and statistical fluctuations from random deviations in the stellar distribution. The former only contributes to the stars' orbits: we neglect this since we are more interested in the influence of the MBH. The latter may be approximated as a series of two-body encounters. These lead to scattering, in a manner much like Brownian motion.

The two-body interactions mostly lead to small deflections. Over time, these may accumulate into a significant change in the dynamics. The relaxation time-scale characterises the time taken for this to happen (Binney & Tremaine 1987). It therefore quantifies the time over which an orbit may be repopulated by scattering. There are a variety of

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[3] The loss cone includes a region in angular momentum space that is depleted of stars because two-body scattering would allow diffusion over an orbital period to an orbit subject to tidal disruption.

definitions for the relaxation time-scale. For a system with a purely Maxwellian distribution, the time-scale has form

$$\tau_R \simeq \kappa \frac{\sigma^3}{GM_\star \rho_\star \ln \Lambda}, \tag{34}$$

where $\rho_\star$ is the mass density, the Coulomb logarithm is $\ln \Lambda = \ln(M_\bullet/M_\star)$, and κ is a dimensionless number. In his pioneering work, Chandrasekhar (1941b, 1960) defined the time-scale as the period over which the squared change in energy was equal to the kinetic energy squared, this gives $\kappa = 9/16\sqrt{\pi} \simeq 0.32$. Subsequently, Chandrasekhar (1941a) described relaxation statistically, describing fluctuations in the gravitational field probabilistically, this gives $\kappa = 9/2(2\pi)^{3/2} \simeq 0.29$. Bahcall & Wolf (1977) define a reference time-scale from their Boltzmann equation with $\kappa = 3/4\sqrt{8\pi} \simeq 0.15$; this is equal to the reference time-scale defined as the reciprocal of the coefficient of dynamical friction by Chandrasekhar (1943). Spitzer & Harm (1958) define a reference time-scale from the gravitational Boltzmann equation of Spitzer & Schwarzschild (1951) where $\kappa = \sqrt{2}/\pi \simeq 0.45$. Following Spitzer & Hart (1971), Binney & Tremaine (1987, section 8.3.4) estimate the time-scale from the velocity diffusion coefficient of the Fokker-Planck equation yielding $\kappa \simeq 0.34$.

All these approaches yield consistent values, suggesting as a first approximation any approach could be valid. We will follow the classic treatment of Chandrasekhar (1960, chapter 2) which is transparent in its assumptions, adapting from a Maxwellian distribution of velocities to one derived from the DFs (5) and (8). Since there is uncertainty in the astrophysical parameters, we will not be concerned by small discrepencies in the numerical prefactor that result from the simplifying approximations of this approach.

1.4.3   Gravitational wave inspiral

Stars orbiting about the black hole will continually emit gravitational radiation, this carries away energy and angular momentum, causing the stars to inspiral. Using the analysis of Peters (1964) for Keplerian binaries it is possible to define a characteristic inspiral time-scale from the rate of change of energy: we will assume that an orbit is depleted of stars if the characteristic inspiral time-scale is shorter than the relaxation time-scale. For consistency with the definition of the relaxation time-scale, we will define the characteristic inspiral timescale as...

For comparison, the total time taken for the inspiral is given in (8). The characteristic time-scale is a better measure of the depletion of an orbit as it only depends upon the parameters of that orbit, and not its future evolution.

Unbound stars only undergo a single periapse passage and only radiate one burst of radiation; we shall therefore neglect any evolution in their orbital parameters.[4]

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[4] Using the analysis of Turner (1977) it is possible to show that the relative changes in eccentricity $\Delta e/e$ and periapsis $\Delta r_p/r_p$ for an extreme mass-ratio binary are less than $\sqrt{e}\,\mathcal{O}(\eta)$, where $\eta = M/M_\bullet$ is small, and so are only important for very high eccentricity orbits (Appendix B2). These are very high energy, and exponentially rare because of the Boltzmann factor in (5).

1.4.4   Collisions

As a consequence of the higher densities in the galactic core, stars may undergo a large number of close encounters with other stars. A star is unlikely to be disrupted by a single collision (Freitag & Benz 2005), rather it takes 20–30 grazing collisions to disrupt a MS star (Freitag et al. 2006). We shall ignore the possibility of disruption for the other species since they are harder to disrupt: as a consequence of smaller cross-sectional areas, none of the other species would undergo 30 collisions with a periapsis greater than the cut-off implied by gravitational wave inspiral for bound orbits, or the MBH's event horizon for unbound orbits. The number of collisions a star will undergo in a time interval δt is

$$\delta K = n(r) A v(r, e, r_p) \delta t, \tag{35}$$

where A is the star's cross-sectional area for a grazing collision. We shall assume that the relative velocity of the colliding stars is much greater than the escape velocity of the star so we may neglect the effects of gravitational focusing, then the cross-sectional area is simply the geometric $A = \pi R_\star^2$.

For circular orbits we can find the radius at which collisions will lead to disruptions by setting $\delta K = 30$ and $\delta t = \tau_R$. We use the relaxation time-scale for the system as this is the time over which stars are replenished from the reservoir. For non-circular orbits we must consider variation with position. Using $\delta r = v_r \delta t$, and then converting to an integral, we have for bound orbits

$$K = 2\pi R_\star^2 \frac{\tau_R}{P(r_p, e)} \int_{r_p}^{(1+e)r_p/(1-e)} n(r) \frac{v(r, e, r_p)}{v_r(r, e, r_p)} \, \mathrm{d}r, \tag{36}$$

where P is the period of the orbit. Again we may set $K = 30$ to find the orbits for which stars will be disrupted within a relaxation time-scale. For unbound stars we are only interested in stars that would become disrupted before their periapse passage, so

$$K = \pi R_\star^2 \int_{r_p}^{r_c} n(r) \frac{v(r, e, r_p)}{v_r(r, e, r_p)} \, \mathrm{d}r, \tag{37}$$

assuming that the stars in the reservoir external to the core are unlikely to undergo close collisions.

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REFERENCES

Alexander T., 2005, Physics Reports, 419, 65

Alexander T., Hopman C., 2009, The Astrophysical Journal, 697, 1861

Bahcall J. N., Wolf R. A., 1976, The Astrophysical Journal, 209, 214

Bahcall J. N., Wolf R. A., 1977, The Astrophysical Journal, 216, 883

Binney J., Tremaine S., 1987, Galactic Dynamics, first edn. Princeton University Press, Princeton

Chandrasekhar S., 1941a, The Astrophysical Journal, 94, 511

Chandrasekhar S., 1941b, The Astrophysical Journal, 93, 285

Chandrasekhar S., 1941c, The Astrophysical Journal, 93, 323

Chandrasekhar S., 1943, The Astrophysical Journal, 98, 54

Chandrasekhar S., 1960, Principles of Stellar Dynamics, enlarged edn. Dover Publications, New York

Frank J., Rees M. J., 1976, Monthly Notices of the Royal Astronomical Society, 176, 633

Freitag M., Amaro-Seoane P., Kalogera V., 2006, The Astrophysical Journal, 649, 91

Freitag M., Benz W., 2005, Monthly Notices of the Royal Astronomical Society, 358, 1133

Genzel R., Schoedel R., Ott T., Eisenhauer F., Hofmann R., Lehnert M., Eckart A., Alexander T., Sternberg A., Lenzen R., Clenet Y., Lacombe F., Rouan D., Renzini A., Tacconi-Garman L. E., 2003, The Astrophysical Journal, 594, 812

Ghez A. M., Salim S., Weinberg N. N., Lu J. R., Do T., Dunn J. K., Matthews K., Morris M. R., Yelda S., Becklin E. E., Kremenek T., Milosavljevic M., Naiman J., 2008, The Astrophysical Journal, 689, 1044

Gillessen S., Eisenhauer F., Trippe S., Alexander T., Genzel R., Martins F., Ott T., 2009, The Astrophysical Journal, 692, 1075

Hills J. G., 1975, Nature, 254, 295

Hopman C., 2009, The Astrophysical Journal, 700, 1933

Hopman C., Alexander T., 2006a, The Astrophysical Journal, 645, 1152

Hopman C., Alexander T., 2006b, The Astrophysical Journal, 645, L133

Keshet U., Hopman C., Alexander T., 2009, The Astrophysical Journal, 698, L64

Kobayashi S., Laguna P., Phinney E. S., Meszaros P., 2004, The Astrophysical Journal, 615, 855

Lightman A. P., Shapiro S. L., 1977, The Astrophysical Journal, 211, 244

Lynden-Bell D., 1967, Monthly Notices of the Royal Astronomical Society, 136, 101

O'Leary R. M., Kocsis B., Loeb A., 2009, Monthly Notices of the Royal Astronomical Society, 395, 2127

Olver F. W. J., Lozier Daniel W., Boisvert R. F., Clark C. W., eds, 2010, NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge

Peebles P. J. E., 1972, The Astrophysical Journal, 178, 371

Peters P. C., 1964, Physical Review, 136, B1224

Preto M., Amaro-Seoane P., 2010, The Astrophysical Journal, 708, L42

Rees M. J., 1988, Nature, 333, 523

Schödel R., Eckart A., Alexander T., Merritt D., Genzel R., Sternberg A., Meyer L., Kul F., Moultaka J., Ott T., Straubmeier C., 2007, Astronomy and Astrophysics, 469, 125

Sigurdsson S., Rees M., 1997, Monthly Notices of the Royal Astronomical Society, 284, 318

Spitzer L., Hart M. H., 1971, The Astrophysical Journal, 164, 399

Spitzer Jr. L., Harm R., 1958, The Astrophysical Journal, 127, 544

Spitzer Jr. L., Schwarzschild M., 1951, The Astrophysical Journal, 114, 385

Toonen S., Hopman C., Freitag M., 2009, Monthly Notices of the Royal Astronomical Society, 398, 1228

Tremaine S., Gebhardt K., Bender R., Bower G., Dressler A., Faber S. M., Filippenko A. V., Green R., Grillmair C., Ho L. C., Kormendy J., Lauer T. R., Magorrian J., Pinkney J., Richstone D., 2002, The Astrophysical Journal, 574, 740

Turner M., 1977, The Astrophysical Journal, 216, 610

APPENDIX A: CHANDRASEKHAR'S RELAXATION TIME-SCALE

Chandrasekhar (1960, chapter 2) defined a relaxation time-scale for a stellar system by approximating the fluctuations in the stellar gravitational potentail as a series of two-body encounters. The time over which the squared change in energy is equal to the (initial) kinetic energy of the star is the time taken for relaxation.

A1   Chandrasekhar's derivation of the change in energy

We will consider the interaction of a field star, denoted by 1, with a test star, 2; the centre of gravity and relative velocities are

$$\boldsymbol{V}_{\mathrm{g}} = \frac{1}{m_1 + m_2} \left(m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 \right); \tag{A1a}$$

$$\boldsymbol{V} = \boldsymbol{v}_1 - \boldsymbol{v}_2. \tag{A1b}$$

Hence

$$v_1^2 = V_{\mathrm{g}}^2 - 2 \frac{m_2}{m_1 + m_2} V_{\mathrm{g}} V \cos \Phi + \left(\frac{m_2}{m_1 + m_2} \right)^2 V^2; \tag{A2a}$$

$$v_2^2 = V_{\mathrm{g}}^2 + 2 \frac{m_1}{m_1 + m_2} V_{\mathrm{g}} V \cos \Phi + \left(\frac{m_1}{m_1 + m_2} \right)^2 V^2, \tag{A2b}$$

where Φ is the angle between $\boldsymbol{V}_{\mathrm{g}}$ and $\boldsymbol{V}$, and

$$V_{\mathrm{g}}^2 = \frac{1}{(m_1 + m_2)^2} \left(m_1 v_1^2 + m_2 v_2^2 + 2 m_1 m_2 v_1 v_2 \cos \theta \right); \tag{A3a}$$

$$V^2 = v_1^2 + v_2 - 2 v_1 v_2 \cos \theta, \tag{A3b}$$

where θ is the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The change in energy of during the interaction is

$$\Delta E = \frac{1}{2} m_2 \left(v_2'^2 - v_2^2 \right) \tag{A4}$$

$$= \frac{m_1 m_2}{m_1 + m_2} V_{\mathrm{g}} V \left(\cos \Phi' - \cos \Phi \right), \tag{A5}$$

using a primed variables for values after the interaction, and unprimed ones for before. If we project the angle out onto the orbital plane

$$\Delta E = \frac{m_1 m_2}{m_1 + m_2} V_{\mathrm{g}} V \left(\cos \phi' - \cos \phi \right) \cos i, \tag{A6}$$

where ϕ is the angle in the plane, and i is the inclination of $\boldsymbol{V}_{\mathrm{g}}$ out of the plane. We define the deflection angle ψ such that

$$\phi' - \phi = \pi - 2\psi, \tag{A7}$$

hence

$$\Delta E = -2 \frac{m_1 m_2}{m_1 + m_2} V_{\mathrm{g}} V \cos(\phi - \psi) \cos \psi \cos i. \tag{A8}$$

We now need to calculate the encounter rate. This requires us to know number of field stars per unit volume and per volume space element. We will make the simplifying assumption that the density of stars is uniform. This is not the case for the Galactic centre; however, we will approximate it as so in order to make the problem tractable. It may seem especially bad for treatment of compact objects very close to the central MBH, inside the radius where main sequence stars would be tidally disrupted; however, it must be remembered that it is small angle deflections, rather than large deflections from close collisions, that are most important for relaxation. Using an averaged value, the number of stars is

$$\mathrm{d}N = n(v_1, \theta, \phi) \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}^3 r, \tag{A9}$$

using spherical polar coordinates for velocity space. Using D as the impact parameter for the encounter and Θ for the angle between the fundamental plane (containing $\boldsymbol{v}_1$ ans $\boldsymbol{v}_2$) and the the orbital plane, the number of events in time interval δt is

$$\mathrm{d}\Gamma = n(v_1, \theta, \varphi) \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \frac{\mathrm{d}\Theta}{2\pi} 2\pi D \, \mathrm{d}D V \, \delta t. \tag{A10}$$

The squared change in energy for these encounters is

$$\Delta E^2(v_1, \theta, \varphi, D, \Theta) = (\Delta E)^2 \, \mathrm{d}\Gamma \tag{A11}$$

$$= 4 n(v_1, \theta, \varphi) V_{\mathrm{g}}^2 V^3 \left(\frac{m_1 m_2}{m_1 + m_2} \right)^2 \cos^2 i \cos^2(\phi - \psi) \cos^2 \psi D \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\Theta \, \mathrm{d}D \, \delta t. \tag{A12}$$

We must integrate out all these dependencies.

Since we have assumed that the stellar density does not depend upon position, we can simply integrate over the impact parameter; this is related to the deflection angle by

$$\frac{1}{\cos^2 \psi} = 1 + \frac{D^2 V^4}{G(m_1 + m_2)}. \tag{A13}$$

Thus

$$\Delta E^2(v_1, \theta, \varphi, \psi, \Theta) = 4n(v_1, \theta, \varphi)\frac{V_{\text{g}}^2}{V}G^2 m_1^2 m_2^2 \cos^2 i \frac{\cos^2(\phi - \psi)\sin \psi}{\cos \psi} \, \mathrm{d}\psi \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\Theta \, \delta t. \tag{A14}$$

The integral over ψ is

$$I(\psi_0) = \int_0^{\psi_0} \frac{\cos^2(\phi - \psi)\sin \psi}{\cos \psi} \, \mathrm{d}\psi \tag{A15}$$

$$= \frac{\sin 2\phi}{2}\left(\psi_0 - \frac{\sin 2\psi_0}{2}\right) - \frac{\cos 2\phi}{2}\left(\frac{\cos 2\psi_0}{2}\right) - \sin^2 \phi \ln(\cos \psi_0). \tag{A16}$$

Naively we would think that the upper limit for the deflection limit should be $\psi_0 = \pi/2$; however, this would introduce a logarithmic divergence. In actuality there will be a physical cut-off, reflecting a finite bound for the maximum impact parameter D_0. This will be set by the scale of the system, beyond which scattering is negligible. While the logarithmic term is finite, it will still be large, dominating the other terms which are $\mathcal{O}(1)$; we shall therefore neglect the subdominant terms

$$\Delta E^2(v_1, \theta, \varphi, \Theta) \simeq 4n(v_1, \theta, \varphi)\frac{V_{\text{g}}^2}{V}G^2 m_1^2 m_2^2 \cos^2 i \sin^2 \phi \ln\left(\frac{1}{\cos \psi_0}\right) \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\Theta \, \delta t. \tag{A17}$$

Next, we integrate over the orbital plane inclination using

$$\cos i \sin \phi = \sin \Phi \cos \Theta, \tag{A18}$$

so that

$$\Delta E^2(v_1, \theta, \varphi) \simeq 4\pi n(v_1, \theta, \varphi)\frac{V_{\text{g}}^2}{V}G^2 m_1^2 m_2^2 \sin^2 \Phi \ln\left(\frac{1}{\cos \psi_0}\right) \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \delta t. \tag{A19}$$

We are now left with just the velocity variables.

An expression for $\sin^2 \Phi$ can be obtained from (A1) and (A3), after some rearrangement

$$\frac{V_g^2}{V}\sin^2 \Phi = \frac{v_1^2 v_2^2 \sin^2 \theta}{(v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta)^{3/2}}. \tag{A20}$$

To proceed further we must specify the form of $n(v_1, \theta, \varphi)$, if we assume isotropy

$$n(v_1, \theta, \varphi) = n(v_1)\frac{1}{4\pi}\sin \theta. \tag{A21}$$

The integral over φ is then trivial,

$$\Delta E^2(v_1, \theta) \simeq \pi n(v_1)\frac{G^2 m_1^2 m_2^2 v_1^2 v_2^2 \sin^3 \theta}{(v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta)^{3/2}}\ln\left[1 + \frac{D_0(v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta)^2}{G^2(m_1 + m_2)^2}\right] \, \mathrm{d}v_1 \, \mathrm{d}\theta \, \delta t. \tag{A22}$$

To integrate over θ it is easier to recast in terms of V; the integral is

$$J = v_1^2 v_2^2 \int_0^\pi \frac{\sin^3 \theta}{(v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta)^{3/2}}\ln\left[1 + \frac{D_0(v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta)^2}{G^2(m_1 + m_2)^2}\right] \, \mathrm{d}\theta \tag{A23}$$

$$= v_1 v_2 \int_{V_-}^{V_+} \frac{\sin^2 \theta}{V^2}\ln\left(1 + q^2 V^4\right) \, \mathrm{d}V, \tag{A24}$$

where the limits are

$$V_+ = v_1 + v_2; \quad V_- = |v_1 - v_2|, \tag{A25}$$

and we have introduced

$$q = \frac{D_0}{G(m_1 + m_2)}. \tag{A26}$$

Using (A3) to rearrange, and then integrating by parts gives

$$J = \frac{1}{4v_1 v_2}\int_{V_-}^{V_+} \frac{(V_+^2 - V^2)(V^2 - V_-^2)}{V^2}\ln\left(1 + q^2 V^4\right) \, \mathrm{d}V \tag{A27}$$

$$= \frac{1}{4v_1 v_2}\left\{\left[\frac{3V_+^2 V_-^2 + (V_+^2 + V_-^2)V^2 - V^4}{3V}\right]_{V_-}^{V_+} - \int_{V_-}^{V_+} \frac{3V_+^2 V_-^2 + (V_+^2 + V_-^2)V^2 - V^4}{3V}\frac{4q^2 V^3}{1 + q^2 V^4} \, \mathrm{d}V\right\}; \tag{A28}$$

the former piece still contains the logarithmic term which we know must be large. It is therefore the dominant piece of the integral and we neglect the latter (Chandrasekhar 1941b),

$$J \simeq \frac{1}{6v_1 v_2} \left[\left(3V_-^2 + V_+^2\right) V_+ \ln\left(1 + q^2 V_+^4\right) - \left(3V_+^2 + V_-^2\right) V_- \ln\left(1 + q^2 V_-^4\right) \right]. \tag{A29}$$

This may be further simplified, reusing the limit of large q (Chandrasekhar 1941b,c),

$$J \simeq \frac{1}{3v_1 v_2} \left[\left(3V_-^2 + V_+^2\right) V_+ \ln\left(qV_+^2\right) - \left(3V_+^2 + V_-^2\right) V_- \ln\left(qV_-^2\right) \right] \tag{A30}$$

$$\simeq \frac{4}{3v_1 v_2} \begin{cases} \left(v_1^3 + v_2^3\right) \ln\left[q\left(v_1 + v_2\right)^2\right] - \left(v_2^3 - v_1^3\right) \ln\left[q\left(v_1 - v_2\right)^2\right] & v_1 \leqslant v_2 \\ \left(v_1^3 + v_2^3\right) \ln\left[q\left(v_1 + v_2\right)^2\right] - \left(v_1^3 - v_2^3\right) \ln\left[q\left(v_1 - v_2\right)^2\right] & v_1 \geqslant v_2 \end{cases} \tag{A31}$$

$$\simeq \frac{8}{3} \begin{cases} \dfrac{v_2^2}{v_1} \ln\left(\dfrac{v_1 + v_2}{v_2 - v_1}\right) + \dfrac{v_1^2}{v_2} \left[\ln qv_2^2 + \ln\left(1 - \dfrac{v_1^2}{v_2^2}\right)\right] & v_1 < v_2 \\ v_2 \left(\ln qv_2^2 + \ln 4\right) & v_1 = v_2 \\ \dfrac{v_1^2}{v_2} \ln\left(\dfrac{v_1 + v_2}{v_1 - v_2}\right) + \dfrac{v_2^2}{v_1} \left[\ln qv_2^2 + \ln\left(\dfrac{v_1^2}{v_2^2} - 1\right)\right] & v_1 > v_2 \end{cases} \tag{A32}$$

$$\approx \frac{8}{3} \begin{cases} \dfrac{v_1^2}{v_2} \ln qv_2^2 & v_1 \leqslant v_2 \\ \dfrac{v_2^2}{v_1} \ln qv_2^2 & v_1 \geqslant v_2 \end{cases}. \tag{A33}$$

This form maintains the correct limit for $v_2 \to 0$. We are left with

$$\Delta E^2(v_1) \approx \frac{8\pi}{3} n(v_1) G^2 m_1^2 m_2^2 \ln qv_2^2 \begin{Bmatrix} \dfrac{v_1^2}{v_2} & v_1 \leqslant v_2 \\ \dfrac{v_2^2}{v_1} & v_1 \geqslant v_2 \end{Bmatrix} \mathrm{d}v_1 \delta t. \tag{A34}$$

The final integral requires a specific form for the velocity distribution.

A2   Velocity distributions

The velocity space distribution function can be obtained by integrating out the spatial dependence in the full DF

$$f(v) = \int \mathrm{d}^3 r f(\mathcal{E}). \tag{A35}$$

As we are restricting our attention to the Galactic centre and assuming spherical symmetry

$$f(v) = 4\pi \int_0^{r_c} r^2 f(\mathcal{E}) \, \mathrm{d}r, \tag{A36}$$

where r_c is defined by (3). It is useful to work in terms of dimensionless variables

$$x = \frac{\mathcal{E}}{\sigma^2}; \tag{A37}$$

$$w = \frac{v^2}{2\sigma^2}. \tag{A38}$$

Changing the integral to be over dimensionless energy

$$f(v) = 4\pi r_c^3 \int_\infty^{w-1} \frac{f(x)}{(w-x)^4} \, \mathrm{d}x, \tag{A39}$$

keeping w as a function of v.

The DF for unbound stars is assumed to be Maxwellian; it is defined in (5) and is applicable for $x > 0$, implying $w > 1$:

$$f_{\mathrm{u},M}(v) = \frac{N_*}{(2\pi\sigma^2)^{3/2}} C_M \int_0^{w-1} \frac{\exp(-x)}{(w-x)^4} \, \mathrm{d}x \tag{A40}$$

$$= \frac{N_*}{(2\pi\sigma^2)^{3/2}} C_M \epsilon\left(\frac{v^2}{2\sigma^2}\right), \tag{A41}$$

introducing

$$\epsilon(w) = \frac{1}{2} \left\{ \exp(-w) \left[4\exp(1) + \mathrm{Ei}(w) - \mathrm{Ei}(1)\right] - \frac{2 + w + w^2}{w^3} \right\}, \tag{A42}$$

where $\mathrm{Ei}(x)$ is the exponential integral.

The DF for bound stars is approximated as a simple power law as defined in (8), it is applicable for $x < 0$:

$$f_{\mathrm{b},\,M}(v) = \frac{N_*}{(2\pi\sigma^2)^{3/2}} k_M \begin{cases} \displaystyle\int_{-\infty}^{w-1} \frac{(-x)^{p_M}}{(w-x)^4}\,\mathrm{d}x & w \leqslant 1 \\ \displaystyle\int_{-\infty}^{0} \frac{(-x)^{p_M}}{(w-x)^4}\,\mathrm{d}x & w \geqslant 1 \end{cases}. \tag{A43}$$

The integral may be evaluated in terms of the hypergeometric function $_2F_1(a,b;c;x)$, but for the limits needed here the results simplify to

$$f_{\mathrm{b},\,M}(v) = \frac{N_*}{(2\pi\sigma^2)^{3/2}} k_M \left(\frac{v^2}{2\sigma^2}\right) \begin{cases} 3\,\mathrm{B}\left(\dfrac{v^2}{2\sigma^2}; 3-p_M, 1+p_M\right) & \dfrac{v^2}{2\sigma^2} \leqslant 1 \\ 3\,\mathrm{B}\left(3-p_M, 1+p_M\right) & \dfrac{v^2}{2\sigma^2} \geqslant 1 \end{cases}, \tag{A44}$$

where $\mathrm{B}(x;a,b)$ is the incomplete beta function (Olver et al. 2010, 8.17), $\mathrm{B}(a,b)$ is the complete beta function, and $\Gamma(x)$ is the gamma function.

The velocity space density is related to the distribution by

$$\frac{4\pi r_{\mathrm{c}}^3}{3} n_M(v_1) = 4\pi v_1^2 \left[f_{\mathrm{u},\,M}(v_1) + f_{\mathrm{b},\,M}(v_1)\right]. \tag{A45}$$

A3 Defining the relaxation time-scale

Using the specific forms for the velocity space density, we can calculate the squared change in energy. The functional form depends upon the velocity of the test star. If $v_2^2/2\sigma^2 < 1$, then

$$\Delta E^2 \approx \frac{8}{3}\sqrt{2\pi} \frac{G^2 m_1^2 m_2^2 n_*}{\sigma^3} \ln q v_2^2 \left\{ \int_0^{v_2} 3k \frac{v_1^4}{v_2} \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(\frac{v_1^2}{2\sigma^2}; 3-p, 1+p\right) \mathrm{d}v_1 \right.$$

$$+ \int_{v_2}^{\sqrt{2}\sigma} 3k v_1 v_2^2 \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(\frac{v_1^2}{2\sigma^2}; 3-p, 1+p\right) \mathrm{d}v_1$$

$$\left. + \int_{\sqrt{2}\sigma}^{\infty} v_1 v_2^2 \left[3k \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(3-p_M, 1+p_M\right) + C\epsilon\left(\frac{v_1^2}{2\sigma^2}\right)\right] \mathrm{d}v_1 \right\} \delta t, \tag{A46}$$

where we have suppressed the subscript M for brevity; it would be necessary to sum over all the species to get the total value. Computing the integrals yields

$$\Delta E^2 \approx \frac{16}{3}\sqrt{2\pi} \frac{G^2 m_1^2 m_2^2 n_*}{\sigma^3} \ln q v_2^2 \frac{v_2^2}{2\sigma^2} \left\{ k\left[\frac{12}{5(1-2p)} \frac{v_2^2}{2\sigma^2} {}_2F_1\left(\frac{5}{2}, -p; \frac{7}{2}; \frac{v_2^2}{2\sigma^2}\right) \right.\right.$$

$$\left.\left. - \frac{9}{(2-p)(1-2p)} \left(\frac{v_2^2}{2\sigma^2}\right)^{p-2} \mathrm{B}\left(\frac{v_2^2}{2\sigma^2}; 3-p, 1-p\right) + \frac{3}{(2-p)(1+p)} \left(1 - \frac{v_2^2}{2\sigma^2}\right)^{1+p} \right] + C \right\} \delta t, \tag{A47}$$

assuming that $p < 1/2$. The contribution from bound and unbound stars can be identified by the coefficients k and C respectively.

If $v_2^2/2\sigma^2 > 1$,

$$\Delta E^2 \approx \frac{8}{3}\sqrt{2\pi} \frac{G^2 m_1^2 m_2^2 n_*}{\sigma^3} \ln q v_2^2 \left\{ \int_0^{\sqrt{2}\sigma} 3k \frac{v_1^4}{v_2} \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(\frac{v_1^2}{2\sigma^2}; 3-p, 1+p\right) \mathrm{d}v_1 \right.$$

$$+ \int_{\sqrt{2}\sigma}^{v_2} \frac{v_1^4}{v_2} \left[3k \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(3-p, 1+p\right) + C\epsilon\left(\frac{v_1^2}{2\sigma^2}\right)\right] \mathrm{d}v_1$$

$$\left. + \int_{v_2}^{\infty} v_1 v_2^2 \left[3k \left(\frac{v_1^2}{2\sigma^2}\right)^{p-3} \mathrm{B}\left(3-p_M, 1+p_M\right) + C\epsilon\left(\frac{v_1^2}{2\sigma^2}\right)\right] \mathrm{d}v_1 \right\} \delta t. \tag{A48}$$

Computing this gives

$$\Delta E^2 \approx \frac{16}{3}\sqrt{2\pi} \frac{G^2 m_1^2 m_2^2 n_*}{\sigma^3} \ln q v_2^2 \left(\frac{v_2^2}{2\sigma^2}\right)^{-1/2} \left\{ k\left[\frac{9\sqrt{\pi}\Gamma(1+p)}{2(1-2p)\Gamma(p+7/2)} - \frac{9\,\mathrm{B}\left(3-p_M, 1+p_M\right)}{(1-2p)(2-p)} \left(\frac{v_2^2}{2\sigma^2}\right)^{p-1/2}\right] + C\alpha\left(\frac{v_2^2}{2\sigma^2}\right) \right\} \delta t, \tag{A49}$$

where

$$\alpha(w) = \frac{1}{2}\left\{ 3w^{-1/2} + 5 + [4\exp(1) - \mathrm{Ei}(1) + \mathrm{Ei}(w)]\left[\frac{3\sqrt{\pi}}{4}\mathrm{erf}\left(w^{1/2}\right) - \frac{3}{2}w^{1/2}\exp(-w)\right] - 3\sqrt{\pi}\exp(1)\mathrm{erf}(1) \right.$$

$$\left. + 3\left[{}_2F_2\left(\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; 1\right) - w^{1/2}{}_2F_2\left(\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; w\right)\right] \right\}. \tag{A50}$$

The hypergeometric function originates from the integral

$$\int^w \frac{\exp(w')\,\mathrm{erf}\left(w'^{1/2}\right)}{w'}\,\mathrm{d}w' = \frac{4w^{1/2}}{\sqrt{\pi}}{}_2F_2\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};w\right). \tag{A51}$$

Combining the two regimes gives some quite complicated expressions. It is possible to simplify these while maintaining reasonable accuracy, for this purpose we will separate the contributions from bound and unbound stars. The bound contribution is relatively straightforward for larger v_2, it is just the sum of two power laws, the relative importance of each depends upon the value of p. For smaller v_2 we may use a series expansion in $v_2^2/2\sigma^2 < 1$ to approximate the special functions:

$$\Delta E_b^2 \approx 16\sqrt{2\pi}G^2 m_1^2 m_2^2 n_* \sigma \ln q v_2^2 k \left\{ \begin{array}{ll} \frac{3}{(1+p)(2-p)} - \frac{9}{5(3-p)}\frac{v_2^2}{2\sigma^2} + \frac{9p}{14(7-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^2 + \frac{p(1-p)}{6(5-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^3 & \frac{v_2^2}{2\sigma^2} < 1 \\ \frac{1}{1-2p}\left(\frac{v_2^2}{2\sigma^2}\right)^{-1/2}\left[\frac{9\sqrt{\pi}\,\Gamma(1+p)}{2\Gamma(p+7/2)} - \frac{9\,\mathrm{B}\,(3-p,1+p)}{(2-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^{p-1/2}\right] & \frac{v_2^2}{2\sigma^2} > 1 \end{array} \right\} \delta t. \tag{A52}$$

This can be rolled into one continuous function

$$\Delta E_b^2 \approx 16\sqrt{2\pi}G^2 m_1^2 m_2^2 n_* \sigma \ln q v_2^2 k \left\{ \left[1 + \left(\frac{v_2^2}{2\sigma^2}\right)^4\right]^{-1}\left[\frac{3}{(1+p)(2-p)} - \frac{9}{5(3-p)}\frac{v_2^2}{2\sigma^2} + \frac{9p}{14(7-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^2 \right. \right.$$

$$\left. \left. + \frac{p(1-p)}{6(5-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^3\right] + \frac{1}{1-2p}\left(\frac{v_2^2}{2\sigma^2}\right)^{7/2}\left[1 + \left(\frac{v_2^2}{2\sigma^2}\right)^4\right]^{-1}\left[\frac{9\sqrt{\pi}\,\Gamma(1+p)}{2\Gamma(p+7/2)} - \frac{9\,\mathrm{B}\,(3-p,1+p)}{(2-p)}\left(\frac{v_2^2}{2\sigma^2}\right)^{p-1/2}\right]\right\}\delta t \tag{A53}$$

$$\approx 16\sqrt{2\pi}G^2 m_1^2 m_2^2 n_* \sigma \ln q v_2^2 k\beta\left(\frac{v_2^2}{2\sigma^2};p\right)\delta t, \tag{A54}$$

defining the function $\beta(w;p)$ in the last line. The resulting error (ignoring variation from $\ln q v_2^2$) is less than 3%.

The unbound contribution is very simple for small v_2, but much more complicated for large v_2. In the limit of $v_2 \to \infty$, it decays as v_2^{-1}:

$$\lim_{w\to\infty}\{\alpha(w)\} = \frac{1}{2}\left\{5 + 3\sqrt{\pi}\left[\exp(1) - \frac{\mathrm{Ei}(1)}{4} - \exp(1)\,\mathrm{erf}(1)\right] + 3{}_2F_2\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};1\right)\right\} \tag{A55}$$

$$= \Xi. \tag{A56}$$

Using the two limiting forms, the function

$$\Delta E_u^2 \approx 16\sqrt{2\pi}G^2 m_1^2 m_2^2 n_* \sigma \ln q v_2^2 C\Xi\frac{v_2^2}{2\sigma^2}\left[\Xi^2 + \left(\frac{v_2^2}{2\sigma^2}\right)^3\right]^{-1/2}\delta t, \tag{A57}$$

reproduces the full function to better than 5% (ignoring variation from $\ln q v_2^2$).

Making explicit the sum over the different species, the total change in energy squared is approximately

$$\Delta E^2 \approx \sum_M 16\sqrt{2\pi}G^2 m_M^2 m_2^2 n_* \sigma \ln q v_2^2 \left\{k_M\beta\left(\frac{v_2^2}{2\sigma^2};p_M\right) + C_M\Xi\frac{v_2^2}{2\sigma^2}\left[\Xi^2 + \left(\frac{v_2^2}{2\sigma^2}\right)^3\right]^{-1/2}\right\}\delta t. \tag{A58}$$

The relaxation time-scale is the time interval δt over which the squared change in energy becomes equal to the kinetic energy of the test star squared

$$\tau_{\mathrm{R}} = \left(\frac{m_2 v_2^2}{2}\right)^2 \frac{\delta t}{\Delta E^2} \tag{A59}$$

$$\approx \frac{v_2^4}{16\sqrt{2\pi}G^2 n_* \sigma \ln q v_2^2}\left(\sum_M m_M^2\left\{k_M\beta\left(\frac{v_2^2}{2\sigma^2};p_M\right) + C_M\Xi\frac{v_2^2}{2\sigma^2}\left[\Xi^2 + \left(\frac{v_2^2}{2\sigma^2}\right)^3\right]^{-1/2}\right\}\right)^{-1}. \tag{A60}$$

This is the time required for stellar encounters to become effective in changing the energy of an orbit in the smooth background potential.

A4   Averaged time-scale

The relaxation time-scale defined above is appropriate for a particular initial test star velocity v_2. This is not of much use to describe the Galactic centre or even a (non-circular) orbit where there is a range in velocity. It is necessary to calculate an averaged time-scale. Both the change in energy squared and the kinetic energy are averaged; comparing these gives the

appropriate timescale. We will use two averages: over the distribution of bound velocities to give the relaxation time-scale for the system, and over a single orbit.

The total number of bound stars in the core is

$$N_{\mathrm{b}, M} = \frac{3}{3/2 - p_M} \frac{\Gamma(p_M + 1)}{\Gamma(p_M + 7/2)} N_* k_M. \tag{A61}$$

Using this as a normalisation constant, the probability of a bound star having a velocity in the range $v \to v + \mathrm{d}v$ is given by

$$4\pi v^2 p_{\mathrm{b}, M} \, \mathrm{d}v = 4\pi v^2 \frac{f_{\mathrm{b}, M}}{N_{\mathrm{b}, M}} \, \mathrm{d}v \tag{A62}$$

$$= \sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} \frac{(3/2 - p_M)\,\Gamma(p_M + 7/2)}{\Gamma(p_M + 1)} k_M \left(\frac{v^2}{2\sigma^2}\right) \left\{ \begin{array}{ll} \mathrm{B}\left(\dfrac{v^2}{2\sigma^2}; 3 - p_M, 1 + p_M\right) & \dfrac{v^2}{2\sigma^2} \leqslant 1 \\[2mm] \mathrm{B}\left(3 - p_M, 1 + p_M\right) & \dfrac{v^2}{2\sigma^2} \geqslant 1 \end{array} \right\} \, \mathrm{d}v. \tag{A63}$$

The mean squared velocity for bound stars in the core is then

$$\overline{v^2} = 4\pi \int_0^\infty v^4 p_{\mathrm{b}, M} \, \mathrm{d}v \tag{A64}$$

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \frac{3/2 - p_M}{1/2 - p_M} \frac{\Gamma(p_M + 7/2)}{\Gamma(1 + p_M)} \mathrm{B}\left(\frac{5}{2}, 1 + p_M\right). \tag{A65}$$

APPENDIX B: EVOLUTION OF ORBITAL PARAMETERS FROM GRAVITATIONAL WAVE EMISSION

B1 Bound orbits

For bound orbits it is possible to define a gravitational wave inspiral time from the orbit-averaged change in the orbital parameters. Using the analysis of Peters (1964) for Keplerian binaries, the averaged rates of change of the periapsis and eccentricity are

$$\left\langle \frac{\mathrm{d}r_{\mathrm{p}}}{\mathrm{d}t} \right\rangle = -\frac{64}{5} \frac{\zeta}{r_{\mathrm{p}}^3} \frac{(1 - e)^{3/2}}{(1 + e)^{7/2}} \left(1 - \frac{7}{12}e + \frac{7}{8}e^2 + \frac{47}{192}e^3\right) \tag{B1}$$

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{304}{5} \frac{\zeta}{r_{\mathrm{p}}^4} \frac{e(1 - e)^{3/2}}{(1 + e)^{5/2}} \left(1 + \frac{121}{304}e^2\right), \tag{B2}$$

where we have introduced

$$\zeta = \frac{G^3 M_\bullet M(M_\bullet + M)}{c^5}. \tag{B3}$$

For a circular orbit the inspiral time from initial periapsis $r_{\mathrm{p}0}$ is

$$\tau_{\mathrm{c}}(r_{\mathrm{p}0}) = \frac{5}{256} \frac{r_{\mathrm{p}0}^4}{\Theta}. \tag{B4}$$

For an orbit of finite eccentricity $(0 < e < 1)$, we can solve for the periapsis as a function of eccentricity

$$r_{\mathrm{p}}(e) = \chi(1 + e)^{-1} \left(1 + \frac{121}{304}e^2\right)^{870/2299} e^{12/19}, \tag{B5}$$

where Λ is a constant fixed by the initial conditions: for an orbit with initial eccentricity e_0

$$\chi = (1 + e_0) \left(1 + \frac{121}{304}e_0^2\right)^{-870/2299} e_0^{-12/19} r_{\mathrm{p}0}. \tag{B6}$$

The inspiral is complete when the eccentricity has decayed to zero. Consequently the inspiral time is (Peters 1964)

$$\tau(r_{\mathrm{p}0}, e_0) = \int_0^{e_0} \frac{15}{304} \frac{\chi^4}{\zeta} \frac{e^{29/19}}{(1 - e^2)^{3/2}} \left(1 + \frac{121}{304}e^2\right)^{1181/2299} \mathrm{d}e. \tag{B7}$$

This is best evaluated numerically, however it may be written in closed form as

$$\tau(r_{\mathrm{p}0}, e_0) = \tau_{\mathrm{c}}(r_{\mathrm{p}0})(1 + e_0)^4 \left(1 + \frac{121}{304}e_0^2\right)^{-3480/2299}$$

$$\times F_1\left(\frac{24}{19}; \frac{3}{2}, -\frac{1181}{2299}; \frac{43}{19}; e_0^2, -\frac{121}{304}e_0^2\right), \tag{B8}$$

using the Appell hypergeometric function of the first kind $F_1(\alpha; \beta, \beta'; \gamma; x, y)$ (Olver et al. 2010, 16.15.1).[5]

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[5] For small eccentricities $\tau(r_{\mathrm{p}0}, e_0) \simeq \tau_{\mathrm{c}}(r_{\mathrm{p}0})[1 + 4e_0 + (273/43)e_0^2 + \mathcal{O}(e_0^3)]$.

B2   Unbound orbits

Unbound orbits only pass through periapsis once. We therefore expect that their evolution through the loss of energy and angular momentum carried away by gravitational waves to be small. Following the approach of Turner (1977) we can calculate the change in the eccentricity and periapse of an unbound Keplerian binary. The change in fractional eccentricity over an orbit, approximating the orbital parameters remain constant throughout, is

$$
\frac{\Delta e}{e} = -\frac{608}{15}\Sigma \left[\frac{1}{(1+e)^{5/2}} \left(1 + \frac{121}{304}e^2 \right) \cos^{-1}\left(-\frac{1}{e} \right) \right.
$$
$$
\left. + \frac{(e-1)^{1/2}}{e^2(1+e)^2} \left(\frac{67}{456} + \frac{1069}{912}e^2 + \frac{3}{38}e^4 \right) \right], \tag{B9}
$$

introducing dimensionless parameter

$$
\Sigma = \frac{G^{5/2}M_\bullet M(M_\bullet + M)}{c^5 r_{\mathrm{p}}^{5/2}}. \tag{B10}
$$

Similarly, the fractional change in periapsis is

$$
\frac{\Delta r_{\mathrm{p}}}{r_{\mathrm{p}}} = -\frac{128}{5}\Sigma \left[\frac{1}{(1+e)^{7/2}} \left(1 - \frac{7}{12}e + \frac{7}{8}e^2 + \frac{47}{192}e^3 \right) \cos^{-1}\left(-\frac{1}{e} \right) \right.
$$
$$
\left. - \frac{(e-1)^{1/2}}{e(1+e)^3} \left(\frac{67}{288} - \frac{13}{8}e + \frac{133}{576}e^2 - \frac{1}{4}e^3 - \frac{1}{8}e^4 \right) \right]. \tag{B11}
$$

Both of these changes obtain their greatest magnitudes for large eccentricities, then

$$
\frac{\Delta e}{e} \simeq \frac{\Delta r_{\mathrm{p}}}{r_{\mathrm{p}}} \simeq -\frac{16}{5}\Sigma e^{1/2}. \tag{B12}
$$

For extreme mass-ratio binaries, as is the case here, the mass-ratio is a small quantity

$$
\eta = M/M_\bullet \ll 1. \tag{B13}
$$

The smallest possible periapsis is of order of the Schwarzschild radius of the MBH, such that

$$
r_{\mathrm{p}} = \alpha\frac{GM_\bullet}{c^2}; \quad \alpha > 1. \tag{B14}
$$

These give

$$
\Sigma = \frac{\eta}{\alpha^{5/2}} < \eta \ll 1. \tag{B15}
$$

Hence the changes in orbital parameters will be significant for

$$
e \sim \frac{25}{256}\frac{\alpha^5}{\eta^2} > \frac{25}{256}\frac{1}{\eta^2}. \tag{B16}
$$

Such orbits should be exceedingly rare, and so it is safe to neglect inspiral for unbound orbits.

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