Gravitational-wave sensitivity curves

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Abstract. There are several common conventions use by the gravitational-wave community to describe the amplitude of sources and the sensitivity of detectors. These are frequently confused. We outline the merits of and differences between the various quantities used for parameterizing noise curves and characterizing gravitational-wave amplitudes. We conclude by producing plots that consistently compare different detectors. Similar figures can be generated on-line for general use at `http://www.ast.cam.ac.uk/~rhc26/sources/`.

1. Introduction

The next few years promises to deliver the first direct detection of gravitational waves (GWs). This will most likely be achieved by the advanced Laser Interferometer Gravitational-wave Observatory (LIGO; Harry, 2010) and Virgo (Accadia et al., 2011) detectors operating in the frequency range $(10–10^3)$ Hz. By the end of the decade, it is expected that pulsar timing arrays (PTAs; Foster & Backer, 1990) shall detect very low frequency GWs around 10^{-8} Hz. Further into the future, space-based detectors, such as the evolved Laser Interferometer Space Antenna (eLISA; Amaro-Seoane et al., 2012b), will probe GWs in the millihertz regime. These advances shall herald the beginning of multi-wavelength GW astronomy as a means of observing the Universe.

There already exists an extensive literature assessing the potential of all of these detectors to probe the astrophysics of various sources. There are several different methods commonly used to describe the sensitivity of a GW detector and the strength of a GW source. It is common practice to summarise this information on a sensitivity-curve plot. When producing these plots, it is desirable to have a consistent convention between detectors and sources which allows information about both to be plotted on the same graph. Ideally, the detectors and sources are represented in such a way that the relative detectability of the signals is immediately apparent from the plot.

In this work, we discuss the differing conventions commonly used in GW astronomy. The amplitude of a GW is a strain, a dimensionless quantity h. This gives a fractional change in length, or equivalently light travel time, across a detector. The strain is small, making it a challenge to measure: we are yet to obtain a direct detection of a GW. To calibrate our expectations for detection, it is necessary to quantify the sensitivity of our instruments and the strength of their target signals.

When discussing the loudness of sources and the sensitivity of detectors there are three commonly used parametrizations based upon the strain: the characteristic strain, the power spectral density and the spectral energy density. We aim to disambiguate these three and give a concrete comparison of different detectors. It is hoped that this will provide a useful reference for new and experienced researchers in this field alike.

We begin by expounding the various conventions and the relationships between the conventions in sections 2 and 3. A review of GW detectors (both current and proposed) is given in section 4 and a review of GW sources is given in section 5. In Appendix A several example sensitivity curves are presented. A website where similar figures can be generated is available at `www.ast.cam.ac.uk/~rhc26/sources/`. Here the user may select which sources and detectors to include to tailor the figure to their specific requirements.

2. Signal parametrization

2.1. Signal analysis preliminaries

Gravitational radiation has two independent polarization states denoted $+$ and $\times$; a general signal can be described as a linear combination of the two polarization states, $h = A_+ h_+ + A_\times h_\times$. The sensitivity of a detector to these depends upon the relative orientations of the source and detector. The output of a gravitational wave detector $s(t)$ contains a superposition of noise $n(t)$ and (possibly) the signal $h(t)$,

$$s(t) = n(t) + h(t) \ . \tag{1}$$

We will work with the Fourier transform of the signal, using the conventions that

$$\tilde{x}(f) = \mathcal{F}\left\{x(t)\right\} \quad = \int_{-\infty}^{\infty} \mathrm{d}t \, x(t) \exp(-2\pi \mathrm{i} f t) \ , \tag{2}$$

$$x(t) = \mathcal{F}^{-1}\left\{\tilde{x}(f)\right\} = \int_{-\infty}^{\infty} \mathrm{d}f \, \tilde{x}(f) \exp(2\pi \mathrm{i} f t) \ . \tag{3}$$

For simplicity, it is assumed that the noise in the GW detector is stationary and Gaussian (with zero mean); under these assumptions the noise is fully characterised via the one-sided noise power spectral density $S_n(f)$,

$$\langle \tilde{n}(f)\tilde{n}^*(f') \rangle = \frac{1}{2}\delta(f - f')S_n(f) \ , \tag{4}$$

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where angle brackets $\langle \ldots \rangle$ denote an ensemble average over many noise realisations (Cutler & Flanagan, 1994). In reality, we have only a realisation to work with, however the ensemble average can replaced by a time average for stationary stochastic noise. The procedure is to measure the noise over a sufficiently long duration T and then compute the Fourier transform $\tilde{n}(f)$ with a frequency resolution $\Delta f = T^{-1}$; this is repeated many times to give an average. The noise power spectral density $S_n(f)$ has units of inverse frequency.

Since the GW signal and detector output are both real it follows that $\tilde{h}(-f) = \tilde{h}^*(f)$ and $\tilde{n}(-f) = \tilde{n}^*(f)$; therefore, $S_n(f) = S_n(-f)$. The fact that $S_n(f)$ is an even function means that Fourier integrals over all frequencies can instead be written as integrals over positive frequencies only, e.g., (7) and (12); it is for this reason that $S_n(f)$ is called the *one-sided* power spectral density.‡

When integrated over all positive frequencies the power spectral density gives the mean square noise amplitude. Starting with taking the average of the square of the detector noise:

$$\overline{n(t)^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t \; |n(t)|^2$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \; \tilde{n}(f)\tilde{n}^*(f') \exp\left(2\pi\mathrm{i}ft\right)\exp\left(-2\pi\mathrm{i}f't\right) \quad (5)$$

Using the Fourier property $\mathcal{F}\{n(t+\tau)\} = \tilde{n}(f)\exp(-2\pi\mathrm{i}f\tau)$, the exponential factors may be absorbed as a time-translation of the noise by an amount t. The noise is stationary and is invariant under this time translation. We are thus taking the time average... I give up here. This derivation makes no sense in it's current state.

$$\overline{n(t)^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \; \mathcal{F}\{n(\tau-t)\}(f)\mathcal{F}\{n(\tau-t)\}(f')^*$$

$$= \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \; \langle\tilde{n}(f)\tilde{n}^*(f')\rangle$$

$$= \int_{0}^{\infty} \mathrm{d}f \; S_n(f) \quad (6)$$

Using the property of Fourier transforms, $\mathcal{F}\{n(t+\tau)\}(f) = \tilde{n}(f)\exp(-2\pi\mathrm{i}f\tau)$, the time average can be turned into an ensemble average and hence the definition of power spectral density in (4) may be used.

$$\overline{n(t)^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \int_{-T}^{T} \mathrm{d}t \; \tilde{n}(f)\tilde{n}^*(f')$$

$$= \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \; \langle\tilde{n}(f)\tilde{n}^*(f')\rangle$$

$$= \int_{0}^{\infty} \mathrm{d}f \; S_n(f) \quad (7)$$

Try someting based on Wiener–Khinchin theorem?

‡ An alternative convention is to use the two-sided power spectral density $S_n^{(2)}(f) = S_n(f)/2$.

Given a detector output, the challenge is to extract the signal. There is a well known solution to this problem which involves constructing a Wiener optimal filter [citation needed]. Let $K(t)$ be the filter function with Fourier transform $\tilde{K}(f)$. Convolving this with the detector output gives a contribution from the signal and a contribution from the noise

$$(s * K)(\tau) = \int_{-\infty}^{\infty} \mathrm{d}t \ (n(t) + h(t))K(t - \tau) = \mathcal{S} + \mathcal{N} \ . \tag{8}$$

The signal contribution $\mathcal{S}$ is defined as the expectation of the convolution in (8) when a signal is present, maximised by varying the offset τ. I think this needs expanding. Since the expectation of pure noise is zero it, follows that

$$\mathcal{S} = \int_{-\infty}^{\infty} \mathrm{d}t \ h(t)K(t) = \int_{-\infty}^{\infty} \mathrm{d}t \ h(t)K^*(t) = \int_{-\infty}^{\infty} \mathrm{d}f \ \tilde{h}(f)\tilde{K}^*(f) \ . \tag{9}$$

The squared contribution from noise $\mathcal{N}^2$ is defined as the mean square of the convolution in (8) when no signal is present,

$$\begin{aligned} \mathcal{N}^2 &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \ K(t)K(t') \langle n(t)n(t') \rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \ K(t)K^*(t') \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \ \langle \tilde{n}(f)\tilde{n}^*(f') \rangle \exp\left[2\pi \mathrm{i}(ft - f't')\right] \\ &= \int_{-\infty}^{\infty} \mathrm{d}f \ \frac{1}{2}S_n(f)\tilde{K}(f)\tilde{K}^*(f) \ , \end{aligned} \tag{10}$$

using the definition of $S_n(f)$ from (4). Hence the optimum signal-to-noise ratio (SNR) ϱ is given by Why is this optimal?

$$\varrho^2 = \frac{\mathcal{S}^2}{\mathcal{N}^2} = \frac{\left(\frac{1}{2}S_n(f)\tilde{K}(f) \Big| \tilde{h}\right)^2}{\left(\frac{1}{2}S_n(f)\tilde{K}(f) \Big| \frac{1}{2}S_n(f)\tilde{K}(f)\right)} \ , \tag{11}$$

Inner product should be between signals: you don't specify the Fourier transform where we have introduced the inner product between signal A and B as (Finn, 1992)

$$(A|B) = 4\Re \left\{ \int_0^{\infty} \mathrm{d}f \ \frac{\tilde{A}^*(f)\tilde{B}(f)}{S_n(f)} \right\} \ . \tag{12}$$

The optimum filter maximises the SNR in (11). From the Cauchy–Schwarz inequality, it follows that the optimum filter is

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)} \ . \tag{13}$$

This is the Wiener filter, it may be multiplied by an arbitrary constant since this does not change the SNR. Using this form for $\tilde{K}(f)$, the squared SNR is

$$\varrho^2 = \int_0^\infty \mathrm{d}f \frac{4|\tilde{h}(f)|^2}{S_n(f)} = (h|h) \,. \tag{14}$$

In order to construct the Wiener filter, it is necessary to know *a priori* the form of the signal, $\tilde{h}(f)$.

Whilst the magnitude of the Fourier transform of the signal $|\tilde{h}(f)|$ provides a simple quantification of the GW amplitude as a function of frequency, it has one main deficiency. For an inspiralling source the instantaneous amplitude can be orders of magnitude below the noise level in the detector; however, as the signal continues over many orbits, the SNR can be integrated up to a detectable level. It is useful to have a quantification of the GW amplitude that accounts for this effect; three such methods are described in the following subsections.

2.2. Characteristic strain

The characteristic strain h_c is designed to include the effect of integrating an inspiralling signal. Its counterpart for describing noise is the noise amplitude h_n. These are defined as

$$h_\mathrm{c}(f)^2 = 4f^2 \left|\tilde{h}(f)\right|^2 \,, \tag{15}$$

$$h_n(f)^2 = fS_n(f), \tag{16}$$

such that the SNR in (14) may be written

$$\varrho^2 = \int_0^\infty \mathrm{d}\left(\log f\right) \left[\frac{h_\mathrm{c}(f)}{h_n(f)}\right]^2 \,. \tag{17}$$

The strain amplitudes $h_\mathrm{c}(f)$ and $h_c(f)$ are dimensionless. Using this convention, when plotting on a log–log scale, the area between the source and detector curves is related to the SNR via (17). This convention allows the reader to integrate by eye to assess the detectability of a given source (see figure A1).

An additional advantage of this convention is that the values on the strain axis for the detector curve $h_n(F)$ have a simple physical interpretation: they correspond to the root-mean-square noise in a bandwidth f. One downside to plotting characteristic strain is that the values on the strain axis $h_\mathrm{c}(f)$ do not directly relate to the amplitude of the waves from the source. Another disadvantage is that the characteristic strain for a monochromatic source is infinite.

2.3. Power spectral density

A second commonly used quantity for sensitivity curves is the square root of the power spectral density (see figure A2). When discussing a detector this is

$$\sqrt{S_n(f)} = h_n(f)f^{-1/2} \,, \tag{18}$$

where we have used (16); by analogy, we can define an equivalent for for source amplitudes

$$\sqrt{S_h(f)} = h_{\rm c}(f)f^{-1/2} = 2f^{1/2}\left|\tilde{h}(f)\right| \,, \tag{19}$$

where we have used (15). Both $\sqrt{S_n(f)}$ and $\sqrt{S_h(f)}$ have units of $\mathrm{Hz}^{-1/2}$. The root power spectral density is the most frequently plotted quantity in the literature.

The power spectral density has the nice property, demonstrated in (7), that integrated over all positive frequencies it gives the mean square amplitude of the signal in the detector. However, in one important regard it is less appealing than characteristic strain: the height of the source above the detector curve is no longer is directly related to the SNR.

2.4. Energy density

A third way of describing the amplitude of a GW is through the energy carried by the waves. This has the advantage of having a clear physical significance. The energy density is most commonly used in sensitivity curves showing stochastic backgrounds of GWs (see section 3.2).

The energy in GWs is described by the Isaacson stress–energy tensor (Misner et al., 1973, section 35.15)

$$T_{\mu\nu} = \frac{c^2}{32\pi G}\left\langle \partial_\mu \bar{h}_{\alpha\beta}\partial_\nu \bar{h}^{\alpha\beta}\right\rangle \,, \tag{20}$$

where the anle brackets denote averaging over several wavelengths or periods. The energy density ρc^2 is given by the T_{00} component of this tensor. Consequently

$$\rho c^2 = \frac{c^2}{16\pi G}\int_{-\infty}^{\infty} \mathrm{d}f \,\, (2\pi f)^2 \,\tilde{h}(f)\tilde{h}^*(f) \tag{21}$$

$$= \int_0^{\infty} \mathrm{d}f \, \frac{\pi c^2}{4G}f^2 S_h(f) \,, \tag{22}$$

where the definition (19) have been used. Equation (21) gives the definition of the spectral energy density, the energy per unit volume of space, per unit frequency (Hellings & Downs, 1983)

$$S_{\rm E}(f) = \frac{\pi c^2}{4G}f^2 S_h(f) \,; \tag{23}$$

a corresponding expression for the noise can be formulated by replacing $S_h(F)$ with $S_n(f)$.

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Cosmological studies often work in terms of the dimensionless quantity Ω_{GW}, the energy density per logarithmic frequency interval normalised to the critical density of the universe:

$$\Omega_{\mathrm{GW}}(f) = \frac{f S_{\mathrm{E}}(f)}{\rho_{\mathrm{c}} c^2} \ . \tag{24}$$

The critical density of the universe is

$$\rho_{\mathrm{c}} = 3 H_0^2 / 8\pi G \ , \tag{25}$$

where H_0 is the Hubble constant which is commonly parametrized as

$$H_0 = h_{100} \times 100 \ \mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{26}$$

The reduced Hubble parameter h_{100} has nothing to do with strain. The most common quantity related to energy density to be plotted on sensitivity curves is $\Omega_{\mathrm{GW}} h_{100}^2$ (figure A3) as this removes sensitivity to the measured value of the Hubble constant.

This quantity has one aesthetic advantage over the others: it automatically accounts for the fact that there is less energy in low frequency waves of the same amplitude. However, unlike characteristic strain, the area between the source and detector curves is no longer simply related to the SNR.

2.5. Relating the different descriptions

The dimensionless energy density in GWs Ω_{GW}, spectral energy density S_{E}, one-sided power spectral density S_h, characteristic strain h_{c} and frequency-domain strain $\tilde{h}(f)$ are related via

$$H_0^2 \Omega_{\mathrm{GW}}(f) = \frac{8\pi G}{3c^2} f S_{\mathrm{E}}(f) = \frac{2\pi^2}{3} f^3 S_h(f) = \frac{2\pi^2}{3} f^2 h_{\mathrm{c}}^2(f) = \frac{8\pi^2}{3} f^4 \left| \tilde{h}(f) \right|^2 \ , \tag{27}$$

using (15), (19), (23), (24) and (25). Corresponding expressions for the noise are obtained by substituting $S_n(f)$ for $S_h(f)$, $h_n(f)$ for $h_{\mathrm{c}}(f)$ and $\tilde{n}(f)$ for $\tilde{h}(f)$.

3. Types of source

Sources of GWs can be broadly split into three categories: well modelled sources, for which we have a description of the expected waveform; stochastic backgrounds, for which we can describe the statistical behaviour; and unmodelled (or poorly-modelled) transient sources. The classic example of a well modelled source is the inspiral of two compact objects, this is discussed in section 3.1. Stochastic backgrounds can either be formed from many overlapping sources, which are each individually susceptible to being modelled, or from some intrinsically random process, these are discussed in section 3.2. An example of an unmodelled (or poorly-modelled) transient source is a supernova; searches of this type are often called burst searches and are discussed in section 3.3.

3.1. Inspirals

Inspiralling binaries spend a variable amount of time in each frequency band. If ϕ is the orbital phase then the number of cycles generated at frequency f can be estimated as

$$\mathcal{N}_{\text{cycles}} = \frac{f}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}f} = \frac{f^2}{\dot{f}} \; , \tag{28}$$

where an overdot represents the time derivative and $f = \dot{\phi}$. The squared SNR will scale with $\mathcal{N}_{\text{cycles}}$, so it would be expected that $h_c(f) \approx \sqrt{\mathcal{N}_{\text{cycles}}}\tilde{h}(f)$. The exact conversion can be derived by considering the Fourier transform in the stationary phase approximation. Consider a source signal with approximately constant amplitude, h, and central frequency f'.

$$h(t) = h \exp\left(\mathrm{i}\phi(t)\right) \tag{29}$$

$$\Rightarrow \tilde{h}(f) = \int_{-\infty}^{\infty} \mathrm{d}t \; h \exp\left(2\pi\mathrm{i}\left(\frac{\phi(t)}{2\pi t} - f\right)t\right) \tag{30}$$

Without loss of generality assume an initial phase of zero and expand the exponent in powers of t,

$$\begin{aligned}
\tilde{h}(f) &\approx \int_{-\infty}^{\infty} \mathrm{d}t \; h \exp\left[2\pi\mathrm{i}\dot{f}'\left(t^2 + \frac{f'-f}{\dot{f}'}t\right)\right] \\
&\approx h \exp\left[-\pi\mathrm{i}\frac{(f'-f)^2}{2\dot{f}'}\right]\int_{-\infty}^{\infty} \mathrm{d}t \; \exp\left[2\pi\mathrm{i}\dot{f}'\left(t + \frac{f'-f}{2\dot{f}'}\right)^2\right] \\
&\approx \frac{h}{\sqrt{2i\dot{f}'}} \exp\left[-\pi\mathrm{i}\frac{(f'-f)^2}{2\dot{f}'}\right] \; .
\end{aligned} \tag{31}$$

From (**??**) and (31) the characteristic strain and the Fourier transform can be related for inspiralling sources (Finn & Thorne, 2000)

$$h_c(f) = \sqrt{\frac{2f^2}{\dot{f}}}h \; . \tag{32}$$

Equation (32) is the relation between $h_c(f)$ and the instantaneous amplitude h for an inspiralling source; for other types of source a new relation which satisfies (17) will have to be found. Equation (17) should be considered as the definition of characteristic strain and (32) a consequence of it for inspirals.

3.2. Stochastic backgrounds

Another important source of GWs arises from stochastic backgrounds due to large populations of individually unresolvable sources. The population of sources will in general

be at cosmological distances and it is necessary to distinguish the frequency in the source rest frame, f_{r}, from the measured frequency, f; these are related through the redshift, z, via $f_{\mathrm{r}} = (1 + z)f$. The comoving number density of sources, n, producing the background will also be a function of redshift; if the sources producing the stochastic background are all in the local universe then simply set $n(z) = \delta(z)$ and replace $d_{\mathrm{L}}(z)$ with d in all that follows (where $d_{\mathrm{L}}(z)$ and d are respectively the luminosity and comoving distances to the source, $d_{\mathrm{L}}(z) = (1 + z)d_m$). Equation (24) gives an expression for the energy density in GWs per logarithmic frequency interval,

$$f S_{\mathrm{E}}(f) = \frac{\pi c^2}{4G} f^2 h_c(f)^2 . \tag{33}$$

The total energy emitted in the logarithmic frequency interval $\mathrm{d}\left(\log(f_{\mathrm{r}})\right)$ by a single binary in the population is $(\mathrm{d}E_{\mathrm{GW}}/\mathrm{d}(\log f_{\mathrm{r}}))\,\mathrm{d}(\log f_{\mathrm{r}})$; the energy density may be written as

$$f S_{\mathrm{E}}(f) = \int_0^\infty dz\, \frac{\mathrm{d}n}{\mathrm{d}z} \frac{1}{(1 + z)} \frac{1}{d_{\mathrm{L}}(z)^2} \frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\left(\log f_{\mathrm{r}}\right)} , \tag{34}$$

where the factor of $(1 + z)^{-1}$ accounts for the redshift of the energy.

For simplicity consider all the binaries comprising the background to be in circular orbits with frequencies $\nu = f_{\mathrm{r}}/2$, and to be far from their last stable orbit; the energy radiated may then be calculated using the quadrupole approximation, Peters & Mathews (1963). The chirp mass is defined as $\mathcal{M} = \mu^{3/5} M^{2/5}$, where μ is the reduced mass and M is the total mass of the binary. The number density of sources will also be a function of chirp mass, $n(z, \mathcal{M})$. The energy in GWs from a single binary per logarithmic frequency interval is,

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\left(\log f_{\mathrm{r}}\right)} = \frac{G^{2/3} \pi^{2/3}}{3} \mathcal{M}^{5/3} f_{\mathrm{r}}^{2/3} , \tag{35}$$

between an minimum and maximum frequency set by the initial and final radius of the binary orbit. Here we assume that the maximum and minimum frequencies are outside of the range of frequencies probed by our detector and hence may be neglected. Using (33), (34) and (35) an expression for characteristic strain can now be found, Sesana et al. (2008)

$$h_c(f)^2 = \frac{4 G^{5/3}}{3 \pi^{1/3} c^2} f^{-4/3} \int_0^\infty dz \int_0^\infty d\mathcal{M} \frac{\mathrm{d}^2 n}{\mathrm{d}z\,\mathrm{d}\mathcal{M}} \frac{1}{d_{\mathrm{L}}(z)^2} \left(\frac{\mathcal{M}^5}{1 + z}\right)^{1/3} . \tag{36}$$

From (36) it can be seen that the characteristic strain due to a stochastic background of binaries is a power law in frequency with spectral index $\alpha = -2/3$. The amplitude of the background depends on the population statistics of the binaries under consideration via $n(z, \mathcal{M})$. The power law is often parametrised as

$$h_c(f) = A \left(\frac{f}{f_0}\right)^\alpha , \tag{37}$$

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and constraints are then placed on A. In practice this power law will also have upper and lower frequency cut-offs related to the population of objects creating the GW spectrum. The stochastic background due to other sources, such as cosmic strings or relic GWs from the early Universe, can also be written in the same form as (37) but with different spectral indices: $\alpha = -7/6$ for cosmic strings or α in the range -1 to -0.8 for relic GWs, Jenet et al. (2006).

3.3. Burst sources

Some sources of GWs can produce signals with large amplitudes, greater than the detector noise. The duration of such a signal is typically very short, of the order of a few wave periods, and so there is not time to accumulate SNR in each frequency band in the way inspirals do. As a consequence, waveform models are not required to produce a detection; we simply rely on the "excess power" produced by these so-called burst sources. Typically, we may be looking for signals from core collapse supernovae (Ott, 2009), the late stages of merging compact binaries, or more generally, signals from any unexpected or poorly modelled sources.

Burst searches are often carried out using time-frequency techniques. The data stream from a detector is temporally split into segments, the length of which can be tuned to give greater sensitivity to particular sources. Each segment is then transformed into the frequency domain, whitened and normalised to the noise spectrum to produce a time-frequency plot. Potential GW signals are identified by searching for clusters of pixels that contain an excess of power (see e.g. Abadie et al. (2012a)).

The presence of excess power across a number of pixels eliminates modelled noise sources, but such a cluster may also be caused by atypical noise within a detector. We can improve our confidence of a GW signal by making use of information obtained from other GW detectors. Signals across a network of detectors should have compatible arrival times (given the sky direction) as well as consistent amplitudes, frequencies and shapes of the waveform. Different pipelines are currently in use that analyse the signal consistency in different ways: both coincidence searches (Chatterji et al., 2004) and fully coherent methods (Klimenko et al., 2008) are used.

An important aspect of burst search algorithms is to accurately estimate the noise properties within each time segment. To this extent, null data streams can be constructed that should be insensitive to real GW signals. In order to estimate the false alarm rate, the data from different detectors can be shifted in time to remove any genuine coincident GW signals. These time-shifts are then analysed to simulate the potential occurence of coincident noise events. The algorithms are tuned using time-shifted data to ensure there is no bias in the final search.

As discussed in 3.1, the expected relation between $h_c(f)$ and a typical waveform $\tilde{h}(f)$

is simply

$$h_c(f) = \tilde{h}(f)\sqrt{\mathcal{N}_{\text{cycles}}} \,, \tag{38}$$

where $\mathcal{N}_{\text{cycles}}$ is the number of cycles of radiation generated by the source. Here, $\mathcal{N}_{\text{cycles}}$ is of order unity and $\tilde{h}(f)$ is strongly peaked around some central frequency.

4. Detectors

4.1. Operating principle of an interferometric detector

All of the man-made detectors discussed in this section work on the principle of interferometry. Such detectors work by taking a beam of monochromatic light and splitting it into two beams travelling at some angle to each other. Each beam is passed in to an optical cavity where it undergoes a number of round trips before being recombined to form an interference pattern. The ends of the cavity are, in the ideal case, freely floating test masses which move in response to a passing GW, this effect is measured by observing the changing interference pattern.

The response of a detector to an incident plane-fronted GW depends upon the relative orientations of the detector and the incoming wave. Let us choose the origin of our coordinate system to be the beam-splitter of the interferometer, and l_1^i and l_2^i to be unit 3-vectors pointing along the two arms. In the absence of noise the output of the detector is the difference in strain between the two arms (Thorne, 1987) I think you mean this rather than MTW

$$h(t) = \frac{1}{2}h_{ij}\left(l_1^i l_1^j - l_2^i l_2^j\right) \,, \tag{39}$$

where h_{ij} are the spatial components of the GW metric perturbation. Let $\hat{r}^i$ be the unit 3-vector pointing towards the source of the GWs, with spherical polar angles (θ, ϕ) relative to some axes fixed to the detector, and let p^i and q^i be unit vectors orthogonal to $\hat{r}^i$. We can now define the basis tensors

$$H_{ij}^+ = p_i p_j - q_i q_j$$
$$H_{ij}^\times = p_i q_j + q_i p_j \,. \tag{40}$$

There remains a freedom in the coordinates described, a rotation of p^i and q^i through an angle ψ about $\hat{r}^i$ known as the polarization angle. For a single frequency component, the strain induced by GW may be written as

$$h_{ij} = A_+ H_{ij}^+ \cos\left(2\pi f t\right) + A_\times H_{ij}^\times \cos\left(2\pi f t + \Delta\phi\right) \,, \tag{41}$$

where A_+ and $A_\times$ are the amplitudes of the two polarisation states. Combining (39) and (41) allows the detector output to be written as

$$h(t) = F^+(\theta, \phi, \psi)A_+ \cos\left(2\pi f t\right) + F^\times(\theta, \phi, \psi)A_\times \cos\left(2\pi f t + \Delta\phi\right) \,, \tag{42}$$

Figure 1: The angular response function of an interferometric detector shown both as a surface plot and in an Aitoff–Hammer projection [citation needed]. The two detector arms lie in the x–y plane either side of one of the zeros in the response. Can you label the colour bar/explain what the surface indicates?

where the response functions inherit their angular dependence from the choice of coordinates,

$$F^+(\theta, \phi, \psi) = \frac{1}{2} H^+_{ij} \left(l^i_1 l^j_1 - l^i_2 l^j_2 \right) \ ,$$
(43)

$$F^\times(\theta, \phi, \psi) = \frac{1}{2} H^\times_{ij} \left(l^i_1 l^j_1 - l^i_2 l^j_2 \right) \ .$$
(44)

The response function of a two-arm interferometric detector is quadrupolar, an example is plotted in figure 1.

Throughout this paper detector sensitivity refers to the polarisation and sky averaged sensitivity F, where

$$F^2 = \int_0^{2\pi} \frac{\mathrm{d}\psi}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\pi} \frac{\sin\theta \, \mathrm{d}\theta}{2} \left[\frac{F^+(\theta, \phi, \psi)^2 + F^\times(\theta, \phi, \psi)^2}{2} \right] \ .$$
(45)

For a single 90°-interferometer, such as LIGO, the sky and polarisation averaged response is $F = \sqrt{1/5} \approx 0.447$.

A detector may consist of several interferometers. Let F_a be the averaged response of the a-th interferometer, the average response of a network of k detectors is obtained by adding in quadrature,

$$F^2_{\text{Total}} = \frac{1}{k)} \sum_{a=1}^{k} F^2_a \ .$$
(46)

The averaging in (45) assumes a uniform distribution of polarisation angles ψ. This is the case for a stochastic background; however, for a non-inspiralling circular binary, the

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polarisation is a function of the two spherical polar angles (ι, ξ) specifying the orientation of the binary's orbital angular momentum. Here, ι is the inclination angle, the polar angle between the orbital angular momentum and the line joining the source to the detector $(-\hat{r})$ and ξ is the azimuthal angle around the same line. In this case, to characterise the detector sensitivity we average over all four angles $(\theta, \phi, \iota, \xi)$. If the binary is inspiralling, then the polarisation depends on still more parameters which need to be averaged over. These more complicated averages all have the property that they depend on both the detector and the source, hence they are unhelpful for our present purpose of .... Additionally, the different averages do not work out to be so different from each other: Finn & Chernoff (1993) calculated the sensitivity for a detector with the LIGO geometry averaged over the four angles $(\theta, \phi, \iota, \xi)$ as $\sqrt{4/25} = 0.4$ times peak sensitivity, which should be compared with the value $\sqrt{1/5} \approx 0.447$ above. For the remainder of this paper the three-angle average defined in (45) is used.

4.2. Ground-based detectors

Ground-based detectors are the most numerous. A collection of interferometric detectors are listed in table 1, these are sensitive to GWs in the frequency range $\mathcal{O}(10\text{--}10^3)$ Hz. They all simulate free-floating test masses by suspending a mass from a pendulum system with natural frequency much greater than that of the GW. Their sensitivity curves include narrow lines, which arise from noise sources in the instrument including resonances in the suspension system and electrical noise at multiples of 60 Hz: these have been removed in the Appendix A figures for clarity. The detectors fall broadly into three categories: first-generation detectors, which have already operated; second-generation detectors currently under construction, and third-generation detectors at the planning stage.

The most notable ground-based detectors are LIGO and Virgo (Abramovici et al., 1992; Acernese et al., 2008). These work in collaboration, supported by GEO600 (Willke et al., 2002). LIGO and Virgo have completed science runs as first-generation detectors, now referred to as Initial LIGO (iLIGO) and Initial Virgo (iVirgo). They are now being upgraded to second-generation detectors, advanced LIGO (aLIGO) and advanced Virgo (AdV). I think AdV is the official acronym, although it does break the pattern. LIGO has two observatories, one at Hanford, Washington, which has two detectors, and another at Livingston, Louisiana. There is an agreement to move one of the upgraded Hanford detector systems to a location in India (Iyer et al., 2011; Unnikrishnan, 2013). The advanced detectors should start operation in the next couple of years, with LIGO-India following further in the future.

TAMA300 is a Japanese first-generation detector. Its successor is the Kamioka Gravitational Wave Detector (KAGRA), formerly the Large-scale Cryogenic Gravitational wave Telescope (LCGT), which is located underground in the Kamioka mine. It employs more sophisticated noise-reduction techniques than LIGO or Virgo, such as cryogenic cooling

Table 1: Summary of ground-based laser interferometers.

Detector	Country	Arm length	Approximate date	Generation
GEO600[a]	Germany	600 m	2001–present	First
TAMA300[b]	Japan	300 m	1995–present	First
iLIGO[c]	USA	4 km	2004–2010	First
Virgo[d]	Italy	3 km	2007–2010	First
aLIGO[e]	USA	4 km	*est.* 2015	Second
AdV[f]	Italy	3 km	*est.* 2016	Second
KAGRA[g]	Japan	3 km	*est.* 2018	Second
ET[h]	—	10 km	*est.* 2025	Third

[a]Grote (2008), [b]Ando (2002), [c]Abbott et al. (2009), [d]Accadia et al. (2012), [e]Harry (2010), [f]Acernese et al. (2009), [g]Somiya (2012), [h]Hild et al. (2011).

(Somiya, 2012).

The Einstein Telescope (ET) is an ambitious proposal to construct an underground third-generation detector (Punturo et al., 2010; Sathyaprakash et al., 2012). Its location would provide shielding from seismic noise, allowing it to observe frequencies of 10–10^4 Hz.

In Appendix A, we use an interpolation to the data published on `https://wwwcascina.virgo.infn.it/advirgo/` (2013) for the AdV sensitivity curve, an interpolation to the data for version D of the detector published on `http://gwcenter.icrr.u-tokyo.ac.jp/en/researcher/parameter` (2013) for KAGRA and analytic fits to the sensitivity curves from Sathyaprakash & Schutz (2009) for the remaining detectors.

4.3. Space-based detectors

Space-based detectors work on similar principles to ground based detectors, but with the test masses residing inside of independent, widely separated satellites. Space-based detectors are sensitive to lower frequency GWs than their ground-based counterparts; this is partly because space based detectors can have much longer arms, and partly because they are unaffected by seismic noise which limits the low frequency performance of ground-based detectors.

The canonical design for a space-based detector is the aser Interferometer Space Antenna (LISA), which is sensitive to millihertz GWs. LISA would consist of three satellites flying in a triangular constellation with arms of length 5×10^9 m in a 1 AU orbit around the sun, trailing the Earth by $20°$. The laser arms in a LISA like detector are not a cavity, the light only travels once along each arm. eLISA is a simpler version of LISA designed to probe the same frequency range, while proposals such as the Advanced Laser Interferometer Antenna (ALIA), Big Bang Observer (BBO) and Deci-hertz Interferometer GW Observatory (DECIGO) are designed to probe decihertz GWs.

4.3.1. LISA and eLISA   The instrumental noise curves for LISA by the analytic fit given by Sathyaprakash & Schutz (2009), which we use for the plots in Appendix A. When observing individual sources with LISA there is an additional contribution to the noise from a background of unresolvable binaries. This is not included here as we consider the background as a source of GWs (see section 5.2.2). eLISA is a re-scoped version of the classic LISA mission, the main differences are shorter arms (10^9 m instead of 5×10^9 m), two laser arms instead of three, and a different orbit (drifting away from Earth instead of 20° Earth trailing). The effect of these changes is a slightly reduced peak sensitivity and a shift to higher frequencies. In Appendix A, we use an analytic fit to the instrumental noise curve given by Amaro-Seoane et al. (2012a).

4.3.2. DECIGO, ALIA and BBO   These missions are designed to probe the decihertz region of the GW spectrum; they are considerably more ambitious than the LISA or eLISA mission and their launches will be further into the future. In Appendix A we use a simple analytic fit to the sensitivity curve for ALIA [citation needed], while for DECIGO and BBO, fits to the sensitivity curves given by Yagi & Seto (2011) are used.

4.4. Pulsar timing arrays

PTAs can be thought of as naturally occurring interferometers with galactic scale arm lengths, hence they are sensitive to much lower frequencies than the detectors considered so far. Each pulsar is a very regular clock and the measured pulse arrival time (in practice, arrival times are corrected to arrival times at the solar-system barycenter, SSB) can be compared against a prediction leaving a residual which includes the effects of passing GWs. Using an array of these pulsars spread across the sky allows us to correlate residuals between different pulsars and exploit the fact that the GWs influence all pulsars where as intrinsic pulsar noise clearly does not. The correlation between different pulsars depends only on their angular separation on the sky and has a distinctive shape, known as the Hellings and Downs curve (Hellings & Downs, 1983), which we will now derive.

The redshift of the rate of arrival of pulses for a pulsar at a distance L from Earth, in the direction of the unit spatial vector $\vec{p}$ induced by a GW travelling in the direction of the unit vector $\hat{\Omega}$ is given by (Anholm et al., 2009)

$$z(t, \hat{\Omega}) = \frac{1}{2} \frac{\hat{p}^j \hat{p}^i}{1 + \hat{\Omega} \cdot \hat{p}} \left(h_{ij}^{\text{Pulsar}}(t - L, \hat{\Omega}) - h_{ij}^{\text{Earth}}(t, \hat{\Omega}) \right) = \frac{1}{2} \frac{\hat{p}^j \hat{p}^i}{1 + \hat{\Omega} \cdot \hat{p}} \Delta h_{ij}(t, \hat{\Omega}) . \qquad (47)$$

The redshift includes two terms, the pulsar term and the Earth term. The pulsar term is often neglected in PTA analysis as it can be considered as an extra noise term which averages to zero. The experimentally measured quantity is not the redshift but the timing residual,

the two are related via

$$R(t, \hat{\Omega}) = \int_0^t \mathrm{d}t' \, z(t', \hat{\Omega}) \,. \tag{48}$$

All of the pulsars, and the Earth, reside in the same metric perturbation field which may be expressed in terms of its Fourier transform

$$h_{ij}(t, \hat{x}) = \sum_{A=+,\times} \int \mathrm{d}f \iint_{S_2} \mathrm{d}\hat{\Omega} \, \tilde{h}_A(f, \hat{\Omega}) e_{ij}^A(\hat{\Omega}) e^{2\pi i f(t - \hat{\Omega} \cdot \vec{x})} \,, \tag{49}$$

where $e_{ij}^A(\hat{\Omega})$ is the A polarisation basis tensor for direction $\hat{\Omega}$. Choosing the SSB as the origin of our coordinate system, so the pulsar is at position $L\hat{p}$, gives

$$\Delta h_{ij}(t, \hat{\Omega}) = \sum_{A=+,\times} \int \mathrm{d}f \, \tilde{h}_A(f, \hat{\Omega}) e_{ij}^A(\hat{\Omega}) e^{2\pi i f t} \left(e^{-2\pi i f L(1 + \hat{p} \cdot \hat{\Omega})} - 1 \right) \,. \tag{50}$$

From (47) and (50) the Fourier transform of the redshift, $\tilde{z}(f, \hat{\Omega})$, may be read off as

$$\tilde{z}(f, \hat{\Omega}) = \left(e^{-2\pi i f L(1 + \hat{p} \cdot \hat{\Omega})} - 1 \right) \sum_{A=+,\times} \tilde{h}_A(f, \hat{\Omega}) F^A(\hat{\Omega}) \,, \text{where } F^A(\hat{\Omega}) = \frac{e_{ij}^A(\hat{\Omega}) \hat{p}^j \hat{p}^i}{2(1 + \hat{\Omega} \cdot \hat{p})} \,. \tag{51}$$

The function $F^A(\hat{\Omega})$ may be regarded as the PTA equivalent of the detector response functions in (43). The stochastic background of GWs is fully characterised by the one-sided power spectral density via the following expectation value

$$\left\langle h_A^*(f, \hat{\Omega}) h_{A'}(f', \hat{\Omega}') \right\rangle = \frac{1}{2} S_h(f) \delta^{(2)}(\hat{\Omega}, \hat{\Omega}') \delta_{AA'} \delta(f - f') \,, \tag{52}$$

where $\delta^{(2)}(\hat{\Omega}, \hat{\Omega}')$ is the delta function on the sphere. From (52) and (51) the expectation of the product of signals from two different pulsars in directions $\hat{p}_1$ and $\hat{p}_2$ may be evaluated

$$\langle z_1(f) z_2^*(f') \rangle = \frac{1}{2} S_h(f) \delta(f - f') \Gamma(f) \quad \text{where,} \tag{53}$$

$$\Gamma(f) = \sum_{A=+,\times} \iint_{\mathcal{S}^2} \mathrm{d}\hat{\Omega} \left(e^{2\pi i f L_1(1 + \hat{\Omega} \cdot \hat{p}_1)} - 1 \right) \left(e^{-2\pi i f L_2(1 + \hat{\Omega} \cdot \hat{p}_2)} - 1 \right) F_1^A(\hat{\Omega}) F_2^A(\hat{\Omega}) \,.$$

The overlap function, $\Gamma(f)$, tends to a constant value in the limit that the distances to the pulsars are large compared to the wavelength of GWs, PTAs operate in this limit, so the overlap may be approximated as a constant

$$\Gamma(f) \approx \Gamma_0 = \sum_{A=+,\times} \iint_{\mathcal{S}^2} \mathrm{d}\hat{\Omega} \, F_1^A(\hat{\Omega}) F_2^A(\hat{\Omega}) \,. \tag{54}$$

Neglecting the exponential terms in the overlap is the frequency domain equivalent of neglecting the pulsar term in (47). The integral may be evaluated to give an expression
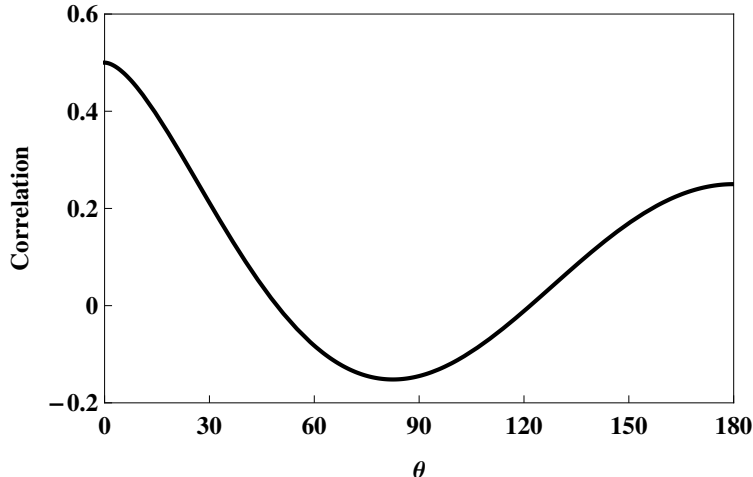
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Figure 2: The Hellings and Downs curve, the correlation between two pulsars seperated on the sky by an angle θ.

depending only on the angle, θ, between the two pulsars; this is the famous Hellings and Downs curve, see figure 2,

$$\Gamma_0 = \frac{1}{2} + \frac{3x}{2}\left(\log(x) - \frac{1}{6}\right) \quad \text{where} \quad x = \frac{1 - \cos\theta}{2} \ . \tag{55}$$

The sensitivity bandwidth of a PTA is set by the sampling properties of the data set. If measurements are spaced in time by δt and taken for a total length of time T then the PTA is sensitive to frequencies in the range $(1/T) < f < (1/\delta t)$. The characteristic strain that the PTA is sensitive to scales linearly with f in this range. This gives the wedge shaped curves plotted in Appendix A. The absolute value of the sensitivity is fixed by normalising to a calculated limit at a given frequency for each PTA. For a discussion of the sensitivities of PTA to both individual sources and stochastic background see Moore et al. (2014).

There is a discrepancy between the treatment of PTA sensitivity curves here and the higher frequency detectors discussed in sections 4.2 and 4.3. When observing a long lived source, such as an inspiral, with a high frequency detector the convention was to define a *characteristic* strain to satisfy (17). Here the convention is to leave the strain untouched and instead adjust the PTA sensitivity curve with observation time, again to satisfy (17). This discrepancy is an unfortunate result of the conventions in use by the different GW communities; however it is also natural given the sources under observation. When observing a transient source, such as a burst or inspiral, which changes within the lifetime of the detector it is natural to consider the detector as performing constantly while the signal changes. However when observing a monochromatic source or a stochastic background which is unchanging over the detector lifetime it is more natural to consider the source as being fixed and the sensitivity of the detector gradually improving. All that is required by the

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definition in (17) is that the ratio $h_c(f)/h_n(f)$ is constant.

4.4.1. EPTA/PPTA/NANOGrav The PTAs currently in operation are the EPTA (European Pulsar Timing Array), PPTA (Parkes Pulsar Timing Array, based in Australia) and NANOGrav (North American Nanohertz Observatory for GWs). There are published limits on the amplitude of the stochastic background from all three detectors: currently the lowest is from the EPTA, van Haasteren et al. (2011). The EPTA curves in the figures show the published limit based on an analysis of 5 pulsars over approximately 10 years.

4.4.2. IPTA Combining the existing arrays would yield a single PTA using approximately 4 times as many pulsars pulsars. The IPTA curves plotted in the figures are based on 20 pulsars timed for twice as long as the current EPTA.

4.4.3. SKA The sensitivity curves plotted in Appendix A for the SKA (Square Kilometer Array) assume values of $\delta t^2_{\mathrm{r.m.s.}}$ a factor of 10 better than current PTAs and 20 pulsars timed for twice as long as the current EPTA.

5. Astrophysical sources

All the sources described here are represented by shaded boxes in Appendix A. The boxes are drawn in such a way that there is a reasonable event rate for sources at a detectable SNR. Descriptions of each source are given in the following sections. Sources with short durations (i.e., burst sources) and sources which evolve in time over much longer timescales than our observations are drawn with flat-topped boxes for $h_c(f)$. Inspiralling binaries, or stochastic backgrounds of binaries, are drawn with a sloping top proportional to $f^{-2/3}$ for $h_c(f)$, which is result derived in section 3.2. The width of the box gives the range of frequencies sources of a given type can have a while remaining at a detectable amplitude.

5.1. Sources for ground based detectors

5.1.1. Neutron star binaries The inspiral and merger of a pair of neutron stars is the primary target for ground based detectors. The expected event rate for this type of source is uncertain, but estimates centre around $\rho_{\mathrm{NS-NS}} = 1.3 \times 10^{-4}$ Mpc^{-3} yr^{-1} (Abadie et al., 2012b). Plotted in Appendix A is a box with an amplitude such that it produces a ratio $h_c/h_n = 16$ for Advanced LIGO at peak sensitivity and a width between (3–300) Hz.

5.1.2. Supernovae Simulations of core-collapse supernovae show that GWs between $(10^2$–$10^3)$ Hz can be produced (Kotake et al., 2006). The GW signal undergoes $\mathcal{O}(1)$ oscillation

and is hence burst-like. Dimmelmeier et al. (2002) calculate the average maximum amplitude of GWs for a supernova at distance r as

$$h = 8.9 \times 10^{-21} \left(\frac{10 \text{ kpc}}{r} \right) . \tag{56}$$

The event rate for supernovae is approximately $\rho_{SN} = 5 \times 10^{-4} \text{ Mpc}^{-3} \text{yr}^{-1}$. The boxes plotted in Appendix A correspond to a distance $r = 3$ Mpc with the frequency range quoted above. The LIGO and Virgo detectors have already placed bounds on the event rate for these sources (Abadie et al., 2012a).

5.1.3. Continuous waves from rotating neutron stars   Rotating neutron stars are a source of continuous GWs if they possess some degree of axial asymmetry (Abbott et al., 2007; Prix, 2009; Aasi et al., 2013). The signals are near monochromatic with a frequency twice the rotation frequency of the neutron star, and are a potential source for ground-based detectors. The amplitude of the GWs depends upon the deformation of the neutron star, which itself depends upon the neutron star equation of state. This is currently uncertain (Lattimer, 2012). Several known pulsars could be sources for the advanced detectors and upper limits from the initial detectors help to constrain the deformations (Aasi et al., 2014b). The boxes plotted in Appendix A is the same as for supernovae for reasons I do not entirely understand.

5.2. Sources for space based detectors

For a review of the GW sources for space based missions see, for example, Amaro-Seoane et al. (2012a), Gair et al. (2012) or Jennrich et al. (2011).

5.2.1. Massive black hole binaries   Space-based detectors shall be sensitive to equal-mass mergers in the range $(10^4–10^7) \, M_\odot$. Predictions of the event rate for these mergers range from $\mathcal{O}(3–300) \text{ yr}^{-1}$ for LISA with SNRs of up to 10^4 [citation needed]. The large range in the rate reflects our uncertainty in the growth mechanisms of the supermassive black hole population (Volonteri, 2010). Plotted in Appendix A are boxes with a ratio $h_c/h_n = 100$ for eLISA at its peak sensitivity. The range of frequencies plotted is $(3 \times 10^{-4}–3 \times 10^{-1})$ Hz, this corresponds to circular binaries in the mass range quoted above.

5.2.2.   Galactic white dwarf binaries For space-based detectors, these are the most numerous GW sources; they are also the only guaranteed source since several detectable systems (known as verification binaries) have already been identified by electromagnetic observations (Stroeer & Vecchio, 2006).

Galactic binaries divide into two classes: the unresolvable and the resolvable galactic binaries. The unresolvable binaries overlap to form a stochastic background as discussed in

section 3.2. The distinction between resolvable and unresolvable is detector specific; here we choose LISA. This boundary will not be too different for eLISA but would move substantially for either of the decihertz detectors. Plotted in Appendix A is the estimate of this background due to Nelemans et al. (2001) where an observation time of one year has been assumed,

$$h_{\mathrm{c}}(f) = 5 \times 10^{-21} \left(\frac{f}{10^{-3} \ \mathrm{Hz}} \right)^{-2/3} . \tag{57}$$

Estimates for the event rate of resolvable binaries centre around $\mathcal{O}(10^3)$ events for eLISA. The boxes plotted in Appendix A have a ratio $h_{\mathrm{c}}/h_n = 50$ for eLISA at its peak sensitivity. The frequency range of the box is $(3 \times 10^{-4}$–$10^{-2})$ Hz, estimated from Monte Carlo population simulation results presented in Amaro-Seoane et al. (2012a).

5.2.3. Extreme mass-ratio inspirals EMRIs occur when a compact stellar mass object inspirals into a supermassive black hole. There is extreme uncertainty in the event rate for EMRIs due to the poorly constrained astrophysics in galactic centres (e.g., Merritt et al., 2011); the best guess estimate is $\mathcal{O}(10)$ events per year with eLISA with SNR ≥ 20 [citation needed]. The boxes plotted in Appendix A have a characteristic strain of $h_c = 3 \times 10^{-20}$ at 10^{-2} Hz, which corresponds to a $10 M_\odot$ BH inspiralling into a $10^6 M_\odot$ black hole at a luminosity distance of 1 Gpc. The frequency width of the box is somewhat unknown; EMRI events can occur into a black hole of any mass, and hence EMRIs can occur at any frequency. However, there is substantial uncertainty in the black hole mass function [citation needed], so the boxes in Appendix A are drawn with a width comparable to that of the LISA sensitivity curve.

5.3. Sources for PTAs

5.3.1. Supermassive black hole binaries The main target for PTAs is a stochastic background of GWs produced by a population of supermassive black hole binaries at cosmological distances (Sesana et al., 2008). Supermassive black holes are known to lie at the centres of most galaxies and the black hole mergers are associated with the mergers of the host galaxies (Volonteri et al., 2003; Ferrarese & Ford, 2005). The current best published limit for the amplitude of the stochastic background is $h_c = 6 \times 10^{-15}$ at a frequency of $f_0 = 1 \ \mathrm{yr}^{-1}$ (van Haasteren et al., 2011). There is strong theoretical evidence that the actual background lies close to the current limit (Sesana, 2012).

Supermassive black hole binaries at higher frequencies are inspiralling faster and hence there are fewer of them per frequency bin. At a certain frequency, these sources will cease to be a background and become individually resolvable. It is currently unclear whether PTAs will detect an individual binary or a stochastic background first. Plotted in Appendix A for the unresolvable background is a third of the current limit with a cut off frequency of

$f = 1 \ \mathrm{yr}^{-1}$ which is suggested by Monte Carlo population studies (Sesana et al., 2008). For the resolvable sources the amplitude of the current limit is plotted between $(3 \times 10^{-9} - 3 \times 10^{-7})$ Hz.

5.4. Cosmological sources

In addition to the sources above, early Universe processes, such as inflation (Grishchuk, 2005) or a first-order phase transition (Binétruy et al., 2012), could have created GWs. More speculatively, it has been hypothesised that cosmic strings could also be a potential source (Damour & Vilenkin, 2005; Binétruy et al., 2012; Aasi et al., 2014a). These relic GWs allow us to explore energy scales far beyond those accessible by other means, providing insight into new and exotic physics. The excitement surrounding the tentative discovery by BICEP2 of the imprint of primordial GWs (generated during inflation) in the cosmic microwave background (Ade et al., 2014), and the subsequent flurry of activity, has shown the scientific potential of such cosmological GWs. These GW signals are so alluring because they probe unknown physics; this also makes them difficult to predict. Cosmological stochastic backgrounds have been predicted across a range of frequencies with considerable variation in amplitude. As a consequence of this uncertainty, although we could learn much from measuring these signals, we have not included them amongst the sources shown in Appendix A.

6. Concluding remarks

When quantifying the sensitivity of a GW detector and the loudness of a GW source, there are three commonly used quantities: the characteristic strain, the power spectral density, and the spectral energy density. We have carefully defined these quantities and derived the relationships between them. The characteristic strain (section 2.2) is most directly related to the SNR, the power spectral density is mostly closely related to the mean square amplitude in the detector, and the energy density has a clear physical interpretation. We have produced example plots using each of these quantities for a wide range of detectors and sources. The predicted source amplitudes are based on astrophysical estimates of the event rates and are subject to varying degrees of uncertainty.§ Interactive versions of these plots, with user-specified detectors and sources, are available on-line at `http://www.ast.cam.ac.uk/~rhc26/sources/`. Trying to summarise an entire field of astronomy on one plot is an impossible task; however, we hope that the figures and analysis presented here provide useful insight.

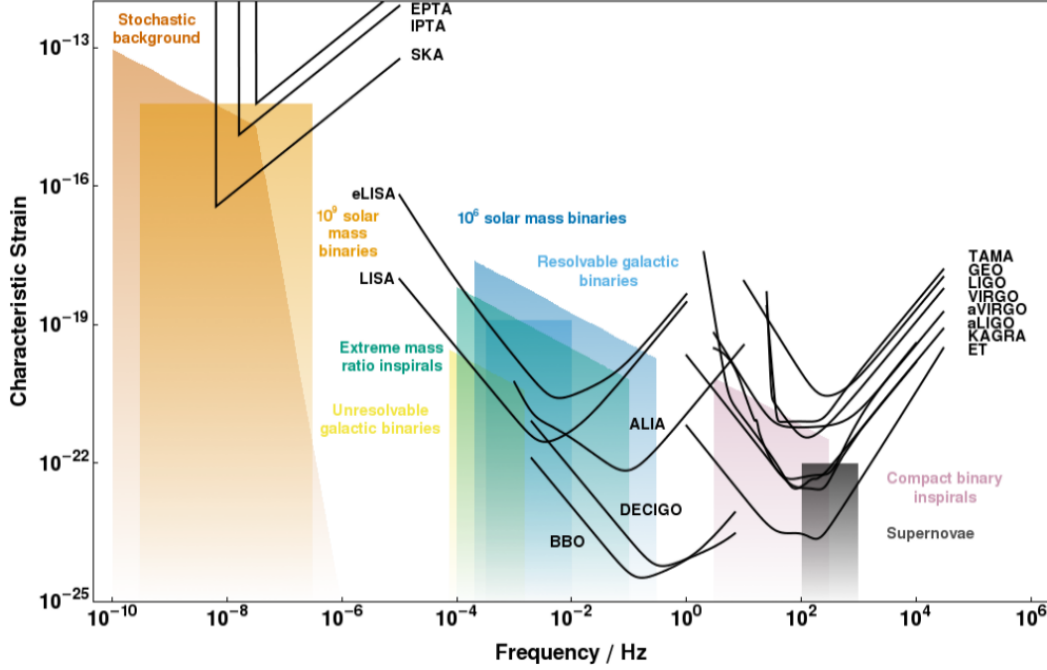§ There is also the exciting possibility of sources yet to be considered.

Figure A1: A plot of characteristic strain against frequency for a variety of detectors and sources.

Acknowledgments

Appendix A. Sensitivity curves

The plots in this section show all of the detectors and sources described in the main text. Interactive versions of these plots may be viewed on-line, `http://www.ast.cam.ac.uk/~rhc26/sources/`. The detector noise curves all have their resonance spikes removed for clarity.

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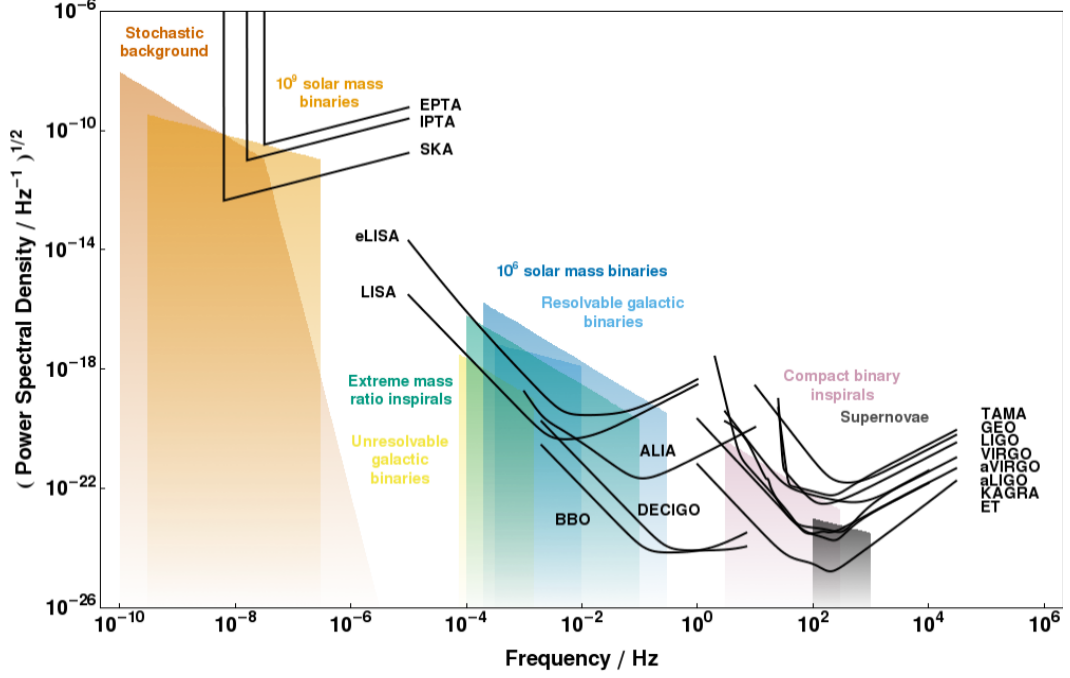Abadie, J., Abbott, B. P., Abbott, R., et al. 2012a, prd, 85, 122007

Figure A2: A plot of the square root of power spectral density against frequency for a variety of detectors and sources.
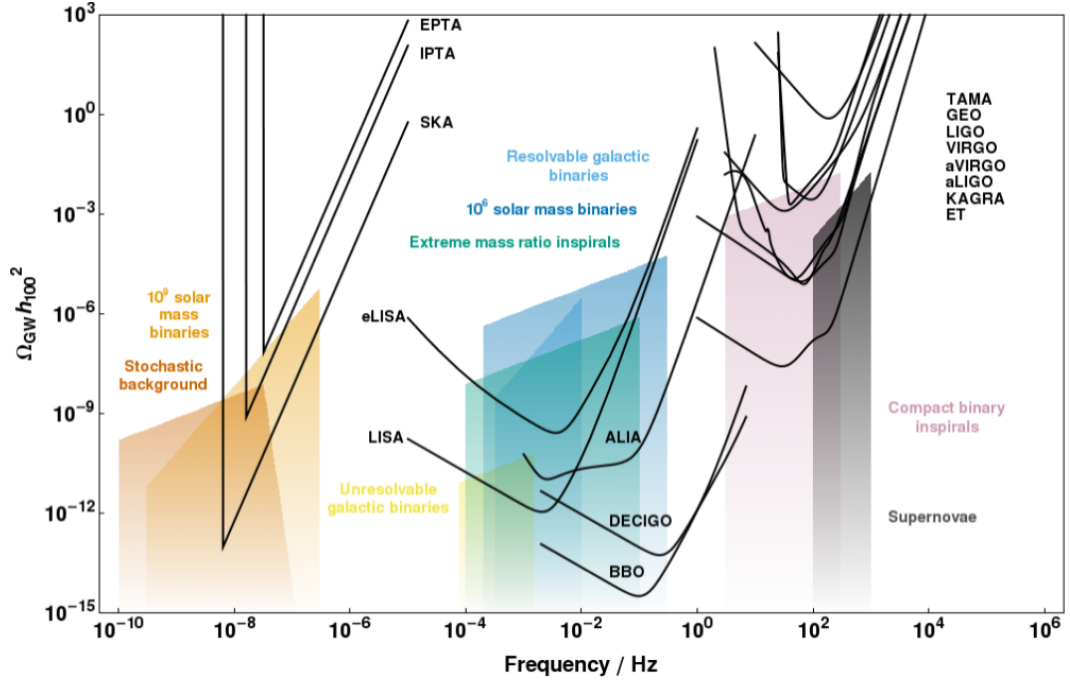

Figure A3: A plot of the dimensionless energy density in GWs against frequency for a variety of detectors and sources.

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