The gravitational wave energy spectrum of a parabolic encounter

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We derive an analytic expression for the energy spectrum of gravitational waves generated by a parabolic Keplerian binary by taking the limit of the Peters and Matthews spectrum for eccentric orbits. This demonstrates that the location of the peak of the energy spectrum depends primarily on the orbital periapse rather than the eccentricity. We compare this weak-field result to strong-field calculations and find it is reasonably accurate ($\sim 10\%$) provided that the azimuthal and radial orbital frequencies do not differ by more than $\sim 10\%$. For equatorial orbits in the Kerr spacetime, this corresponds to periapse radii of $r_p \gtrsim 20M$. These results will be of use to model radiation bursts from compact objects on highly eccentric orbits about massive black holes in the local Universe, which could be detected by LISA.

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I. INTRODUCTION

An important source of gravitational waves for the proposed space-based gravitational wave detector, the Laser Interferometer Space Antenna (LISA) [? ?], are the inspirals of stellar-mass compact objects into massive black holes in the centres of galaxies. During the last few years of inspiral these systems generate continuous gravitational waves in the LISA band, which will allow the detection of as many as several hundred systems out to redshift $z \sim 1$ [?] **Rather self-promoting but the best reference for this is probably J Gair, Class. Quantum Grav. 26 094034 (2009)**. However, prior to this phase, the inspiral object spends many years on a highly eccentric orbit, generating bursts of gravitational radiation at each periapse passage. LISA could resolve individual bursts from sources in the nearby Universe. Initial estimates [?] suggested a LISA event rate of XXyr^{-1}, including 15yr^{-1} from the centre of the Milky Way, although this was subsequently revised downwards to $YY/1\text{yr}^{-1}$ [?] **NB You were only quoting results for the MW, but I think it is relevant to consider all the events they considered, so can you fill in XX and YY here?**. If even a single burst from the Galactic centre is detected during the LISA mission, this will provide an unparalleled probe of the structure of spacetime there.

The spectrum of radiation from these bursts will be well approximated by the spectrum of a parabolic orbit [?]. In this note we derive an analytic approximation to this spectrum by taking the limit of the Peters and Matthews [? ?] (PM) energy spectrum for eccentric Keplerian binaries. We show that the peak of the spectrum depends primarily on the orbital periapse and only weakly on the eccentricity. We also estimate the range of validity of the approximation (in Section section **??**) by comparing to numerical Teukolsky data, finding that it is a good approximation for equatorial orbits in Kerr with periapse $r_p \gtrsim 20M$. The parabolic spectrum takes a nice analytic form and has not previously appeared in the literature. We therefore hope this note will be a useful resource for future work on gravitational radiation from parabolic orbits.

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II. PARABOLIC LIMIT

A. Energy Spectrum

For an orbit of eccentricity e with periapse radius r_p, Peters and Matthews [?] give the power radiated into the nth harmonic of the orbital angular frequency as

$$P(n) = \frac{32}{5}\frac{G^4}{c^5}\frac{M_1^2 M_2^2 (M_1+M_2)(1-e)^5}{r_p^5} g(n,e) \quad (1)$$

where the function $g(n,e)$ is defined in terms of Bessel functions of the first kind

$$\begin{aligned} g(n,e) \;=\; & \frac{n^4}{32}\Bigg\{\Bigg[J_{n-2}(ne) - 2eJ_{n-1}(ne) \\ & + \frac{2}{n}J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne)\Bigg]^2 \\ & + \left(1-e^2\right)\left[J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)\right]^2 \\ & + \frac{4}{3n^2}\left[J_n(ne)\right]^2\Bigg\}. \end{aligned} \quad (2)$$

The Keplerian orbital frequency is

$$\omega_0{}^2 \;=\; \frac{G(M_1+M_2)(1-e)^3}{r_p^3} = (1-e)^3\omega_c{}^2, \quad (3)$$

where we define ω_c as the angular frequency of a circular orbit of radius r_p. The energy radiated per orbit into the nth harmonic, that is at frequency $\omega_n = n\omega_0$, is

$$E(n) = \frac{2\pi}{\omega_0}P(n); \quad (4)$$

as $e \to 1$ for a parabolic orbit, $\omega_0 \to 0$ as the orbital period becomes infinite. The energy radiated per orbit is the total energy radiated. The spacing of harmonics is $\Delta\omega = \omega_0$, giving the energy spectrum

$$\left.\frac{\mathrm{d}E}{\mathrm{d}\omega}\right|_{\omega_n} \omega_0 = E(n). \tag{5}$$

Changing variable to the linear frequency $2\pi f = \omega$,

$$\begin{aligned} \left.\frac{\mathrm{d}E}{\mathrm{d}f}\right|_{f_n} &= \frac{128\pi^2}{5}\frac{G^3}{c^5}\frac{M_1^2 M_2^2}{r_p^2}(1-e)^2 g(n,e) & (6)\\ &= \frac{4\pi^2}{5}\frac{G^3}{c^5}\frac{M_1^2 M_2^2}{r_p^2}\ell(n,e), & (7)\end{aligned}$$

where the function $\ell(n,e)$ is defined in the last line. For a parabolic orbit, we must take the limit of $\ell(n,e)$ as $e \to 1$.

We simplify $\ell(n,e)$ using the recurrence formulae (Watson [?] 2.12)

$$\begin{aligned} J_{\nu-1}(z) + J_{\nu+1}(z) &= \frac{2\nu}{z}J_\nu(z) & (8)\\ J_{\nu-1}(z) - J_{\nu+1}(z) &= 2J'_\nu(z), & (9)\end{aligned}$$

and eliminate n using

$$n = \frac{\omega_n}{\omega_0} = (1-e)^{-3/2}\tilde{f},$$

where $\tilde{f} = \omega_n/\omega_c = f_n/f_c$ is a dimensionless frequency. To find the limit we define two new functions

$$A(\tilde{f}) = \lim_{e\to 1}\left\{\frac{J_n(ne)}{(1-e)^{1/2}}\right\}; \quad B(\tilde{f}) = \lim_{e\to 1}\left\{\frac{J'_n(ne)}{1-e}\right\}. \tag{10}$$

To give a well defined energy spectrum, both of these must be finite.

The Bessel function has an integral representation

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\vartheta - z\sin\vartheta)\,\mathrm{d}\vartheta, \tag{11}$$

we want the limit of this for $\nu \to \infty$, $z \to \infty$, with $z \le \nu$. Using the stationary phase approximation, the dominant contribution to the integral comes from the regime in which the argument of the cosine is approximately zero (Watson [?] 8.2, 8.43), for small ϑ

$$\begin{aligned} J_\nu(z) &\sim \frac{1}{\pi}\int_0^\pi \cos\left(\nu\vartheta - z\vartheta + \frac{z}{6}\vartheta^3\right)\mathrm{d}\vartheta & (12)\\ &\sim \frac{1}{\pi}\int_0^\infty \cos\left(\nu\vartheta - z\vartheta + \frac{z}{6}\vartheta^3\right)\mathrm{d}\vartheta; & (13)\end{aligned}$$

this last expression is an Airy integral and has a standard form (Watson[?] 6.4)

$$\int_0^\infty \cos(t^3 + xt)\,\mathrm{d}t = \frac{\sqrt{x}}{3}K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right), \tag{14}$$

where $K_\nu(z)$ is a modified Bessel function of the second kind. Using this to evaluate our limit gives

$$J_\nu(z) \sim \frac{1}{\pi}\sqrt{\frac{2(\nu-z)}{3z}}K_{1/3}\left(\frac{2^{3/2}}{3}\sqrt{\frac{(\nu-z)^3}{z}}\right). \tag{15}$$

For our case

$$J_n(ne) \sim \frac{1}{\pi}\sqrt{\frac{2}{3}}(1-e)^{1/2}K_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right), \tag{16}$$

and the limiting function is well defined,

$$A(\tilde{f}) = \frac{1}{\pi}\sqrt{\frac{2}{3}}K_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right). \tag{17}$$

To find the derivative we combine (??) and (??), and expand to lowest order to find

$$\begin{aligned} J'_n(ne) \sim & -\frac{1}{2\pi}\sqrt{\frac{2}{3}}(1-e)\left[2^{3/2}K'_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right.\\ & \left. + \frac{1}{\tilde{f}}K_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right]. \end{aligned} \tag{18}$$

We may re-express the derivative using the recurrence formula (Watson [?] 3.71)

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -2K'_\nu(z) \tag{19}$$

to give

$$\begin{aligned} J'_n(ne) \sim & \frac{1-e}{\sqrt{3}\pi}\left[K_{-2/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right) + K_{4/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right.\\ & \left. - \frac{1}{\sqrt{2}\tilde{f}}K_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right]. \end{aligned} \tag{20}$$

And so finally,

$$\begin{aligned} B(\tilde{f}) = & \frac{1}{\sqrt{3}\pi}\left[K_{-2/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right) + K_{4/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right.\\ & \left. - \frac{1}{\sqrt{2}\tilde{f}}K_{1/3}\left(\frac{2^{3/2}\tilde{f}}{3}\right)\right], \end{aligned} \tag{21}$$

which is also well defined.

Having obtained expressions for $A(\tilde{f})$ and $B(\tilde{f})$ in terms of standard functions, we can now calculate the energy spectrum for a parabolic orbit. From (??)

$$\frac{\mathrm{d}E}{\mathrm{d}f} = \frac{4\pi^2}{5}\frac{G^3}{c^5}\frac{M_1^2 M_2^2}{r_p^2}\ell\left(\frac{f}{f_c}\right), \tag{22}$$

where we have used the limit

$$\begin{aligned} \ell(\tilde{f}) = & \left[8\tilde{f}^2 B(\tilde{f}) - 2\tilde{f}A(\tilde{f})\right]^2\\ & + \left(128\tilde{f}^4 + \frac{4\tilde{f}^2}{3}\right)\left[A(\tilde{f})\right]^2. \end{aligned} \tag{23}$$

B. Total Energy

To check the validity of this limit we can calculate the total energy radiated by integrating (??) over all frequencies, or by summing the energy radiated into each harmonic. These must yield the same result. Summing over harmonics

$$E_{sum} = \frac{64\pi}{5}\frac{G^3}{c^5}\frac{M_1^2M_2^2}{r_p^2}\omega_c(1-e)^{7/2}\sum_n g(n,e), \quad (24)$$

where we have used equations (??), (??) and (??). Peters and Matthews [?] provide the result

$$\sum_n g(n,e) = \left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1-e^2)^{-7/2}. \quad (25)$$

Using this,

$$E_{sum} = \frac{64\pi}{5}\frac{G^3}{c^5}\frac{M_1^2M_2^2}{r_p^2}\omega_c\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1+e)^{-\frac{7}{2}}, \quad (26)$$

which is perfectly well behaved as $e \to 1$,

$$E_{sum} = \frac{85\pi}{2^{5/2}3}\frac{G^3}{c^5}\frac{M_1^2M_2^2}{r_p^2}\omega_c. \quad (27)$$

Integrating the energy spectrum, (??), gives

$$E_{int} = \frac{2\pi}{5}\frac{G^3}{c^5}\frac{M_1^2M_2^2}{r_p^2}\omega_c\int_0^\infty \ell(\tilde{f})\,\mathrm{d}\tilde{f}. \quad (28)$$

The integral can be evaluated numerically as

$$\int_0^\infty \ell(\tilde{f})\,\mathrm{d}\tilde{f} = 12.5216858\ldots = \frac{425}{2^{7/2}3}. \quad (29)$$

The two total energies are consistent, $E_{int} = E_{sum}$.

III. APPLICABILITY

A. Limit Of Approximation

The approach of Peters and Matthews assumes Keplerian orbits in flat spacetime. This should be a valid approximation in the weak-field regime far from a massive body. To find the limit of the approximation, we can compare the PM results with those from more accurate techniques. Energy spectra for parabolic orbits do not seem to be available in the literature yet, so we will make do with the total energy fluxes calculated by Martel[?], who used time-domain black hole perturbation theory for a Schwarzschild black hole of mass M. Fig. ?? shows the ratio of the two energies as a function of periapsis distance. As expected the PM result is more accurate for larger periapses. The agreement worsens as the periapsis decreases. At $r_p = 4M$, corresponding to the radius of the innermost stable circular orbit (ISCO), the energy flux calculated by Martel diverges, so the ratio tends to zero. This divergence is because in a Schwarzschild (or Kerr) spacetime a parabolic orbit may have a zoom-whirl structure where it undergoes a number of near circular orbits (whirls) about the black hole. As the radius of the ISCO is approached, the number of whirls tends to infinity (in the absence of radiation reaction), so an infinite amount of energy is radiated. Fig. ?? shows how the ratio of energies correlates with the number of rotations, defined as $N = \Delta\phi/2\pi$, where $\Delta\phi$ is the total change in the azimuthal angle over an orbit. As N increases the PM approximation worsens since a Keplerian orbit does not include this extra rotation. The accuracy of the PM result deteriorates rapidly once the orbit transitions to a zoom-whirl trajectory and is therefore far from parabolic in shape.

The PM result is accurate to $\sim 10\%$ for orbits with $N \lesssim 1.1$. We will adopt this as a cut-off point. For an equatorial orbit in Kerr spacetime, N is

$$\begin{aligned} N &= \frac{1}{\pi}\int_{r_p}^\infty \frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}r \\ &= \frac{L_z}{\pi\sqrt{2M}}\int_{r_p}^\infty \frac{r^2-2M(1-a/L_z)r}{(r^2-2Mr+a^2)w}\,\mathrm{d}r, \quad (30)\\ \text{where} \quad & w^2 = r^3-(L_z^2/2M)r^2+(L_z-a)^2r; \quad (31)\end{aligned}$$

L_z is the angular momentum about the z-axis; a is the spin parameter, and we have adopted units with $G = c = 1$. We will find it useful to define

$$r_\pm = M \pm \sqrt{M^2-a^2}, \quad (32)$$

and the two non-zero roots of the cubic w^2

$$r_{p,1} = \frac{L_z^2}{2M} \pm \sqrt{\frac{L_z^4}{16M^2}-(L_z-a)^2}, \quad (33)$$

the periapsis is the larger root $r_p > r_1$. The integral may be rewritten as

$$\begin{aligned} N &= \frac{L_z}{\pi\sqrt{2M}}\int_{r_p}^\infty \frac{1}{w}\left[1+\frac{\alpha_+}{r-r_+}+\frac{\alpha_-}{r-r_-}\right]\mathrm{d}r, \\ \text{where} \quad & \alpha_\pm = \pm\frac{2Mar_\pm - a^2L_z}{2L_z\sqrt{M^2-a^2}}. \quad (34)\end{aligned}$$

This may be evaluated using elliptic integrals as (Gradshteyn & Ryzhik [?] 3.131.8)

$$\begin{aligned} N = \frac{L_z}{\pi}\sqrt{\frac{2}{r_pM(M^2-a^2)}}\Bigg[&\left(\frac{Ma}{L_z}-\frac{a^2}{2r_+}\right)\Pi\left(\frac{r_+}{r_p}\middle|\frac{r_1}{r_p}\right) \\ &-\left(\frac{Ma}{L_z}-\frac{a^2}{2r_-}\right)\Pi\left(\frac{r_-}{r_p}\middle|\frac{r_1}{r_p}\right)\Bigg], \quad (35)\end{aligned}$$

where $\Pi(n|m) = \int_0^{\pi/2}\mathrm{d}\vartheta/(1-n\sin^2\vartheta)\sqrt{1-m\sin^2\vartheta}$ is the complete elliptic integral of the third kind. In the limit of $a \to 0$ we recover the Schwarzschild result [?]

$$N = \frac{L_z}{\pi}\sqrt{\frac{2}{r_pM}}K\left(\frac{r_1}{r_p}\right), \quad (36)$$

(a) Ratio of energies verses periapse radius r_p.

(b) Ratio of energies verses the reciprocal of the number of rotations $1/N$. The Kep-le-rian limit corresponds to $N = 1$.

FIG. 1: Ratio of the total energy radiated as calculated using the Peters and Matthews [**?**] approach to that calculated by Martel [**?**] using black hole perturbation theory. The latter approach should give more accurate results.

FIG. 2: Periapsis radius corresponding to $N = 1.1$ as a function of spin parameter a.

where $K(m) = \int_0^{\pi/2} \mathrm{d}\vartheta/\sqrt{1 - m\sin^2\vartheta}$ is the complete elliptic integral of the first kind. Fig. **??** shows the periapsis for which $N = 1.1$ for a range of spins. Equatorial orbits with larger periapses should be reasonably approximated by the PM result.

Non-equatorial orbits are more complicated because of the additional precession of the orbital plane. However, this should be sub-dominant to the perihelion precession effect and so the cut-off periapse should not be much different form the equatorial case. **A bold statement perhaps, but at 20M the effect of the spin-orbit coupling should be quite small so I think this is a reasonable statement to make.**

B. Astrophysical Implications

Considering bursts from the Galactic centre, orbits with periapses of $r_p \lesssim 120M$ could generate bursts that would be detectable with LISA[**? ?**]. It is therefore likely that any such burst that was detected would be in the regime or validity of the Peters and Matthews approach, $r_p \gtrsim 20M$ for equatorial orbits. The results described in this note will therefore have application in that context and it should be possible to explore the interesting region of parameter space using this approximation. The most interesting orbits, those which come deep within the strong-field region of the MBH's spacetime, will be beyond the range of validity of this approximation, but these will be a small subset of all the events. **What about bursts from outside the galactic centre? Does the P and M model apply to those when they are detectable or are they too relativistic? We should make a statement - see my earlier note.**

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[1] We use "parabolic" to refer to marginally bound orbits. Marginally bound Keplerian orbits are parabola; in curved spacetimes they do not retain such a simple shape.