# The full spectrum of gravitational wave sensitivity curves

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**Abstract.** *When discussing the sensitivity of gravitational wave detectors there are several common conventions. These are frequently confused. We outline the merits of and differences between the various quantities used for parameterizing noise curves and characterizing gravitational wave amplitudes. We conclude by producing plots that consistently compare different detectors.*

## 1. Introduction

There are many ways of describing both the *sensitivity* of a gravitational wave *(GW)* detector and the strength of a $GW$ source as a function of frequency. It is desirable to have a consistent convention between detectors and sources which is applicable across all frequencies and which allows both to be plotted on the same graph. Another desirable feature of such a plot is that the detectors and sources are *characterised* in such a way that their relative heights gives a measure of *sources'* detectability.

*In this work we tackle the differing conventions common in GW astronomy. The amplitude of a GW is a strain, a dimensionless quantity. When discussing the loudness of sources and the sensitivity of detectors there are three commonly used parameters based upon the starin: the characteristic strain, the power spectral density and the energy spectrum. We aim to disambiguate these three and give a concrete comparison of different detectors. It is hoped that this will provide a useful reference to the new and old alike.*

In the *following* section we present a summary of the various conventions and the relationships between them. A *(near)* exhaustive list of different $GW$ sources is given in section 3 and a list of detectors (past, present and future) is given in section 4. In **??** *we present example sensitivity curves. These* are also made available online.

## 2. Characteristic amplitudes

A source of gravitational waves radiates in two polarisation states with amplitudes $h_+$ and $h_\times$, the sensitivity of our detector to each of these states will depend upon the relative orientations of the source and detector. The most obvious quantity related to the detectability of a GW is the average amplitude,

$$h_0 = \sqrt{\langle h_+^2 + h_\times^2 \rangle} = \frac{1}{2} \langle \text{ r.m.s. amplitude } \rangle . \tag{1}$$

The angled brackets define averaging over all directions and over a wave period (the factor of 1/2 comes from the time averaging). This average strain amplitude does *not* meet the criteria described above*:* for an inspiralling source the instantaneous amplitude can be well below the noise level in the detector even when the source is still detectable. This apparent discrepancy is because the *inspiral continues over an extended period of time which allows the signal-to-noise ratio* $\varrho$ *to be integrated up to a detectable level. To account for this* Finn & Thorne (2000) define the characteristic strain $h_c$ from

$$\varrho^2_{\text{r.m.s.}} = \int \frac{\mathrm{d}f}{f} \left( \frac{h_c(f)}{h_n(f)} \right)^2 = \int \mathrm{d}\left(\ln f\right) \left( \frac{h_c(f)}{h_n(f)} \right)^2 , \tag{2}$$

where $h_n$ is the detectors sky-averaged r.m.s. noise in a bandwidth equal to $f$. *This* is related to the *sky-averaged* one-sided *noise* power spectral density *by*

$$h_n(f) = \sqrt{f S_n(f)}. \tag{3}$$

An alternative convention is to use the *two-sided poser spectral density* $S_n^{(2)}(f) = \frac{1}{2} S_n(f)$.

### *2.1. Characteristic strain*

*To* complete the picture it is necessary to relate the two $h_c$ and $h_0$ *to* satisfy (2). The *means* to do this depends upon the source. *The strain amplitudes* $h_0$, $h_c$ and $h_n$ are all dimensionless, while $S_n$ has units of inverse frequency. When plotting sources on the same diagram it is necessary to have a consistent convention, so the relationship between $h_c$ and $h_0$ is important*;* to emphasise this we give it a suitably pompous name*:* the vociferosity relation.

*Inspiralling binaries source spends a variable amount of time in each frequency band..* If $\phi$ is the orbital phase then the length of time spent at frequency $f$ is given

$$\frac{1}{2\pi} \frac{\mathrm{d}\phi}{\mathrm{d}\left(\log f\right)} = \frac{f}{2\pi} \frac{\mathrm{d}\phi}{\mathrm{d}f} = \frac{f^2}{\dot{f}} .\text{Divide by phi dot?} \tag{4}$$

This leads to the definition of characteristic strain for inspirals of (Finn & Thorne, 2000)

$$h_c(f) = h_0(f) \sqrt{\frac{2f^2}{\dot{f}}}, \tag{5}$$

where the factor of two is inserted to cancel the two arising in (1). *This is the vociferosity relation for inspiralling sources.*

*What about burst sources?*

So one sensible choice of quantities to plot on a sensitivity curve would be $h_n$ for the detector and $h_c$ for the source, see figure **??**. Using this convention, if the frequency is plotted on a logarithmic scale and the strain on a linear scale then the area between the source and detector line represents the signal to noise ratio. This convention allows the reader to "Integrate by eye" for a given detector to see how detectable a given source is. An *additional* advantage of this convention is that the values on the strain axis for the detector curve correspond to the noise amplitude in the detector. The one downside to plotting characteristic strain is that the values on the strain axis do not directly relate to the amplitude of the waves from the source.

### *2.2. Power spectral density*

Another common quantity to plot on sensitivity curves for both detectors and sources is the square root of the power spectral density (see figure **??**), which from (3) is given by,

$$\begin{aligned} \sqrt{S_n(f)} &= h_n(f) f^{-1/2} \quad \text{for detectors,} \\ \sqrt{S_h(f)} &= h_c(f) f^{-1/2} \quad \text{for sources.} \end{aligned} \tag{6}$$

This has one advantage over characteristic strain*:* integrating $S_n$ (*multiplied by the* detector response function, which is *typically* of order unity) over all frequencies gives the mean square noise strain in the detector. If the instantaneous noise strain in the detector is $n(t)$, due to a background say, then *its* Fourier transform is defined via

$$\tilde{n}(f) = \int_{-\infty}^{\infty} \mathrm{d}t \; n(t) \exp\left(-2\pi i f t\right). \tag{7}$$

Since $n(t)$ is dimensionless, $\tilde{n}(f)$ has units of inverse frequency. The Fourier transform of the measured noise in the detector $N(t)$ is given by the noise times a frequency response function

$$\tilde{N}(f) = \mathcal{R}(f)\tilde{n}(f). \tag{8}$$

The mean square noise amplitude in the detector is then given by

$$\left\langle N(t)^2 \right\rangle = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} \mathrm{d}t \; n(t)^2 = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\infty} \mathrm{d}f \; \left|\tilde{n}(f)\right|^2, \tag{9}$$

*using* the Wiener-Kinchin theorem. It then follows that

$$\left\langle N(t)^2 \right\rangle = \int_0^{\infty} \mathrm{d}f \; S_n(f) \mathcal{R}(f)^2, \tag{10}$$

where the power spectral density is given by

$$S_n(f) = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \left|\tilde{n}(f)\right|^2 . \tag{11}$$

It is straightforward to show that (11) is consistent with the previous definition in (3). Let $\tilde{n}(f_i) = h_n(f)\delta(f_i - f)$ and substitute this into (11),

$$S_n(f) = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \left|h_n(f)\delta(f_i - f)\right|^2 \tag{12}$$

$$S_n(f) = \frac{h_n(f)^2}{f} \tag{13}$$

$$h_n(f) = \sqrt{f S_n(f)} \quad . \tag{14}$$

Equation 10 is *the* reason *that* $\sqrt{S_n}$ is sometimes called the strain noise (Phinney, 2001). This is probably the most commonly used quantity on sensitivity curves*;* however, in several ways it is much less appealing than characteristic strain. *First*, the height of the source above the detector curve is no longer is directly related to the *signal-to-noise* ratio. Second, although the integral of $\sqrt{S_n}/f$ is a strain, we are plotting spectral quantities and $\sqrt{S_n}$ is not straightforwardly related to the strain *measured by* the detector.

### *2.3. Energy density*

A third quantity which is sometimes used is the energy density in GWs $S_h$. This is not to be confused with the quantity $S_{\mathrm{E}}$ defined by Hellings & Downs (1983, e.g.). This is the spectral energy density, which is the energy per unit volume per unit frequency and is related to $S_h$ via

$$S_{\mathrm{E}}(f) = \frac{\pi c^2}{4G} f^2 S_h(f). \tag{15}$$

*This needs to be completely rewritten:* So the total energy density of gravitational waves is given by the integral in (17), where the quantity $\Omega_{\mathrm{GW}}$ has been defined as the spectral energy density per unit logarithmic frequency interval normalised by the critical energy density of the universe to make it dimensionless.‡

$$\Omega_{\mathrm{GW}}(f) = \frac{f S_{\mathrm{E}}(f)}{\rho_c c^2} = \frac{\pi}{4G\rho_c} f^2 h_c(f)^2 = \frac{\pi}{4G\rho_c} f^3 S_h(f) \tag{16}$$

$$\text{energy density} = \int \mathrm{d}\left(\log f\right)\ \Omega_{\mathrm{GW}}(f)\rho_c c^2 \tag{17}$$

‡ Just to add to the potential confusion there is another power spectrum quantity widely used in the pulsar timing community, usually denoted $P(f)$, which is the power spectra of timing residuals. This is related to the quantities described here via $P(f) = h_c(f)^2/(12\pi^2 f^3)$ (Jenet et al., 2006).

The critical density of the universe is $\rho_c = 3H^2/(8\pi G)$, where $H$ is the Hubble constant, which is commonly parametrised as $H =_{100} \times 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. This $h$ has nothing to do with strain. The most common quatity related to energy density to be plotted on sensitivity curves is $\Omega_{\mathrm{GW}}h^2$, see figure **??**. This quantity has one *aesthetic* advantage over the *others:* it automatically accounts for there *being* less energy in *lower* frequency waves of the same amplitude, and *it does not* place the sensitivity curves of puslar timing arrays much higher than ground based detectors. However, the area between the source and detector curves is not simply related to the *signal-to-noise* ratio *and* the scale of the *ordinate* axis is not related in any simple way to *either* the amplitude of the GWs or the noise in the detector.

### *2.4. Stochastic backgrounds from binaries*

*Probably needs restructuring*

Aside from inspirals another important detectable form of gravitational waves is a stochastic background due to a population of individually unresolvable binaries. A background is best described in terms of the energy density in gravitational waves. The population of sources will in general be at cosmological distances, in which case we need to distinguish the frequency in the source rest frame, $f_r$, from the measured frequency, $f$, via the redshift, $f_r = (1+z)f$. The comoving number density of sources producing the background will also be a function of redshift, $n(z)$. If the sources producing the stochastic background are all in the local universe, as is the case for the background of unresolvable white dwarf galactic binaries, then simply set $n(z) = \delta(z)$ in all that follows. Equation 16 gives an expression for the energy density per logarithmic frequency interval,

$$f S_{\mathrm{E}}(f) = \frac{\pi c^2}{4G} f^2 h_c^2 \quad . \tag{18}$$

Equation 18 relates the characteristic strain to the energy density which in turn depends on the gravitational wave amplitude, hence (18) is the vociferosity relation for a stochastic background. The fact that $h_c$ is given in terms of an energy density averaged over some region of space and not as a simple function of $h_0$ reflects the stochastic nature of the source. Let the total outgoing energy emitted in gravitation waves between frequencies $f_r$ and $f_r + \mathrm{d}f_r$ by a single binary in our population be $\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}f_r}\mathrm{d}f_r$, then the energy density may be written as,

$$f S_{\mathrm{E}}(f) = \int_0^\infty dz\, \frac{\mathrm{d}n}{\mathrm{d}z} \frac{1}{1+z} \frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\left(\log f\right)} \quad . \tag{19}$$

For simplicity consider all the binaries comprising our background to be in circular orbits with frequencies $\nu = f_r/2$, and to be far from their last stable orbit so the quadrapole approximation holds. The chirp mass is defined as $\mathcal{M} = \mu^{3/5} M^{2/5}$, where $\mu$ is the reduced mass and $M$ the total mass. Thorne (1987) gives the energy in gravitational waves from a single binary per logarithmic frequency interval,

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\left(\log f\right)} = \frac{G^{2/3}\pi^{2/3}}{3}\mathcal{M}^{5/3} f_r^{2/3} \quad . \tag{20}$$

An expression for characteristic strain can now be found (see, for example, Sesana et al. (2008)).

$$h_c(f)^2 = \frac{4G}{\pi c^2 f^2}\int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}\mathcal{M}\, \frac{\mathrm{d}^2 n}{\mathrm{d}z\,\mathrm{d}\mathcal{M}} \frac{1}{1+z}\frac{G^{2/3}\pi^{2/3}}{3}\mathcal{M}^{5/3} f_r^{2/3} \tag{21}$$

$$h_c(f)^2 = \frac{4G^{5/3}}{3\pi^{1/3}c^2} f^{-4/3}\int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}\mathcal{M}\, \frac{\mathrm{d}^2 n}{\mathrm{d}z\,\mathrm{d}\mathcal{M}} \left(\frac{\mathcal{M}^5}{1+z}\right)^{1/3} \tag{22}$$

From (22) it can be seen that the characteristic strain due to a stochastic background of binaries is a power law with index $\alpha = -2/3$. The amplitude of the background depends on the population statistics for the binaries under consideration and depends on the double integral in (22). The power law is usually parametrised as,

$$h_c(f) = A\left(\frac{f}{\mathrm{yr}^{-1}}\right)^{\alpha} \tag{23}$$

and constraints are then placed on $A$. In practice this power law will also have upper and lower frequency cut-offs related to the population of objects causing the spectra. The stochastic background due to other sources, such as cosmic strings or the reheating process, are also usually written in the same form as equation 23, however they will have a different spectral indices, $\alpha$.

### *2.5. Burst sources*

A signal is burst like if it's duration at a detectable amplitude is of order the wave period. If this is the case then the signal does not have time to accumulate signal to noise ratio in each frequency band in the way inspirals do. Hence the vociferosity relation is simply,

$$h_c(f) = h_0(f) \quad . \tag{24}$$

## 3. Sources

All the sources described here are plotted drawn as boxes in figures **??**, **??** and **??**. The boxes are drawn in such a way that there is a reasonable event rate for sources at a detectable signal to noise ratio. However both of these criteria are somewhat vague and detector specific so we describe the exact box we plot for each source in the corresponding section. We draw sources with short duration (i.e. burst sources) and sources which evolve in time over much longer timescales than our observations with flat topped boxes. Inspiralling sources which change there frequency over observable timescales are drawn with slopped tops. The slope chosen for the top of the box is a gradient of $-2/3$ which is also the gradient for the stochastic backgrounds of binaries.

### 3.1. *Sources for ground based detectors*

Another potential source of gravitational waves in this frequency range not discussed here is rotating, non-spherical neutron stars (Sturani, 2013).

*3.1.1. Supernova* Simulations of gravitational waves from a core collapse supernova event produce radiation from approximately $10^2$–$10^3$ Hz (Dimmelmeier et al., 2002). Because the signal is burst like, and only undergoes $\mathcal{O}(1)$ oscillation the signal does not accumulate signal to noise in the same way as an inspiral event does, so we use the vociferosity relation in (24). Dimmelmeier et al. (2002) calculate the average maximum amplitude of gravitational waves for a supernova at distance $r$ as,

$$h_0 = 8.9 \times 10^{-21} \left( \frac{10\,\mathrm{kpc}}{r} \right). \tag{25}$$

Adopting an expected event rate for supernova (Sturani, 2013, e.g.).

$$\rho_{\mathrm{SN}} = 5 \times 10^{-4} \mathrm{Mpc}^{-3} \mathrm{yr}^{-1} \tag{26}$$

Choosing a distance, $r \approx 3\,\mathrm{Mpc}$, such that we expect one supernova per 10 years, gives an amplitude of $h_c = 10^{-22}$. Hence what is plotted in figures **??**, **??** and **??** is a box between $10^2\,\mathrm{Hz}$ and $10^3\,\mathrm{Hz}$, with a height corresponding to $h_c = 10^{-22}$.

*3.1.2. Neutron star binaries* The inspiral and merger of a pair of neutron stars is the prime candidate source for ground based detectors. The last few orbits and merger, which produces the largest amplitude waves, occurs over $\mathcal{O}(10)$ orbits, so as with the supernova the vociferosity relation in (24) is used. The typical strain generated by an event that released $E_{\mathrm{GW}}$ at a distance $r$, centred around frequncy $f$ and with a duration $\tau$ is given by Sturani (2013).

$$h_0 \approx 5 \times 10^{-21} \left( \frac{E_{\mathrm{GW}}}{10^{-7} M_\odot c^2} \right)^{1/2} \left( \frac{\tau}{1\,\mathrm{ms}} \right)^{-1/2} \left( \frac{f}{1\,\mathrm{kHz}} \right)^{-1} \left( \frac{r}{10\,\mathrm{kpc}} \right)^{-1} \tag{27}$$

The expected event rate for this type of event is uncertain, but estimates centre around (see, for example, Andersson (2011)),

$$\rho_{\text{NS-NS}} = 10^{-6}\ \mathrm{Mpc}^{-3} \mathrm{yr}^{-1}. \tag{28}$$

Plotted in figures **??**, **??** and **??** is a box with an amplitude equal to an event releasing $10^{-4} M_\odot c^2$ of energy, centred at a frequency of $100\,\mathrm{Hz}$, over a period of $10\,\mathrm{ms}$ and at a distance of $10\,\mathrm{Mpc}$.

### *3.2. Sources for space based detectors*

Space based detectors are sensitive to lower frequency gravitational waves than their ground based counterparts, typically $10^{-4}$–$10^{-2}$ Hz. This is partly because space based detectors can have much longer arms and partly because they are unaffected by seismic noise which limits the low frequency performance of ground based detectors. The loudest predicted source in this frequency range is the merger of supermassive black holes associated with galaxy mergers Gair et al. (2012). The classic space based detector is LISA, all the sources discussed here lie within LISA's sensitivity curve.

*3.2.1. Massive black hole coalescences* The majority of galaxies contain a supermassive BH in their centres, space based detectors will be sensitive to equal mass mergers in the mass range $10^4 - 10^7\, M_\odot$ Gair et al. (2012). Predictions of the event rate for these megers range from $\mathcal{O}(1\text{–}100)$ yr$^{-1}$ for LISA with signal to noise ratios of up to 1000. The uncertainty in this rate reflects our uncertainty in the growth mechanisms of the supermassive black hole population. Plotted in figures **??**, **??** and **??** is a box with signal to noise ratio of 100 for eLISA at it's peak sensitivity. The range of frequencies plotted is $3\times10^{-4}$ Hz and $3\times10^{-1}$ Hz, this corresponds to circular binaries in the mass range quoted above.

*3.2.2. Galactic binaries* These divide into two classes, the unresolvable galactic binaries and the resolvable galactic binaries which lie at higher frequencies and amplitudes. The distinction between resolvable and unresolvable is detector specific, here we choose LISA. This boundary will not be too different for eLISA but would move substantially for either of the decihertz detectors, BBO or DECIGO. For the *unresolvable* binaries the box is plotted in the knee of the LISA noise curve, this unfortunately makes them look *undetectable*, however the entire population will be detectable but only as a stochastic background.

For the resolvable binaries the event rates range from $\mathcal{O}(10^4)$ for LISA to $\mathcal{O}(10^3)$ for eLISA, see Gair et al. (2012). Signal to noise ratios for these events observed with eLISA will extend to above 50, see Amaro-Seoane et al. (2012). The box plotted in figures **??**, **??** and **??** has an signal to noise ratio of 50 for eLISA at *its* peak sensitivity, the frequency range of the box is based on visual inspection of Monte *Carlo* population simulation results presented in Amaro-Seoane et al. (2012).

*3.2.3. Extreme mass ratio inspirals* Extreme mass ratio inspiral (EMRI) events occur when a compact stellar mass object, i.e. a black hole or neutron star, inspirals into a supermassive black hole in the centre of a galaxy. There is extreme uncertainty in the event rate for EMRIs due to the poorly constrained astrophysics in galactic centres; event rates estimates per galaxy range from $\mathcal{O}(1\text{–}10^3)$ Gyr$^{-1}$ for black hole EMRIs and $\mathcal{O}(10\text{–}5000)$ Gyr$^{-1}$ for white dwarfs and neutron star EMRIs. The box plotted in figures **??**, **??** and **??** has a characteristic strain of $h_c = 3 \times 10^{-20}$ at $10^{-2}$ Hz which corresponds to a $10M_\odot$ BH inspiralling into a $10^6 M_\odot$ supermassive black hole with a last stable orbital eccentricity of $e = 0.3$ at a distance of 1 Gpc. The frequency width of the box is somewhat unknown, EMRI events can occur into a black hole of any mass, and hence EMRIs can occur at any frequency. However there is substantial uncertainty in the population statistics of supermassive black holes, particularly of intermediate mass, $(10^2 - 10^4)\, M_\odot$.

### *3.3. Sources for pulsar timing arrays*

Pulsar timing arrays (PTAs) can be thought of a naturally occuring interferometers with galactic scale arms, hence PTAs are sensitive to much lower frequency gravitational waves the the detectors considered so far. There are several extremely interesting potential cosmological sources of gravitational waves in this frequency band, for example cosmic strings networks and the reheating process, see for example van Haasteren et al. (2011). However, here we ristrict ourselves to sources which are somewhat more certain, the redshifted mergers of $\mathcal{O}(10^7 - 10^{10}) M_\odot$ black holes associated with galaxy mergers at cosmological distances.

*3.3.1. Stochastic background of supermassive black hole inspirals* The current best published limit for the amplitude of the stochastic background is $h_c = 6 \times 10^{-15}$ at frequency of $\mathrm{yr}^{-1}$, see van Haasteren et al. (2011). There is strong theoretical evidence that the actual background lies close to the current limit (McWilliams et al., 2012; Sesana, 2012). As the frequency increases, the sources become less redshifted and hence closer and louder. Hence at a certain frequency they will cease to be a background and become individually resolvable. The exact point at which a source become individually resolvable is not exactly defined, and will also depend upon the detector used; plotted in figures **??**, **??** and **??** is a third of the limit due to van Haasteren et al. (2011) with a cut off frequency of $\mathrm{yr}^{-1}$ which is suggested by Monte Carlo population studies (Sesana et al., 2008).

*3.3.2.* $10^9$ *solar mass binaries* If an inspiral between two supermassive black holes is close and loud enough it will become individually resolvable. These sources would be expected to occur at higher frequencies (i.e. closer and less redshifted) than the stochastic background. Plotted in figures **??**, **??** and **??** is the current EPTA limit (van Haasteren et al., 2011).

## 4. Detectors

### *4.1. Ground based detectors*

The ground based detectors here all show the same characteristic shape. All these detectors have multiple narrow spikes in their sensitivity curves due, in part, to resonant frequencies in the suspension systems used to isolate the mirrors, these have all been removed in figures **??**, **??** and **??** for clarity. The detectors fall broadly into three categories*: first, second and third* generation detectors. The first generation detectors include GEO600, TAMA, (initial) LIGO and (initial) VIRGO, the second generation detectors include (advanced) LIGO, KAGRA and (advanced) VIRGO and the Einstein Telescope is the only third generation detector discussed.

| **Detector** | **Country** | **Arm length** | **Date** | **Reference** |
|---|---|---|---|---|
| GEO600 | Germany | 600 m | 2001-present | Sathyaprakash & Schutz (2009) |
| TAMA300 | Japan | 300 m | 1995-present | Sathyaprakash & Schutz (2009) |
| iLIGO | US | 4 km | 2004-2010 | Sathyaprakash & Schutz (2009) |
| iVIRGO | Italy | 3 km | 2007-?? | Sathyaprakash & Schutz (2009) |
| aLIGO | US | 4 km | *est.* 2015 | Sathyaprakash & Schutz (2009) |
| KAGRA | Japan | 3 km | *est.* 2018 | KAGRA website |
| aVIRGO | Italy | 3 km | *est.* 2015 | VIRGO website |
| ET | Italy | 10 km | Unknown | Sathyaprakash & Schutz (2009) |

Table 1: A list of ground based detectors. For 6 of the detectors simple analytic fits to the sensitivity curves due to Sathyaprakash & Schutz (2009) were used (these fits do not include the resonance spikes). For KAGRA an interpolation to the data for version D of the detector published on http://gwcenter.icrr.u tokyo.ac.jp/en/researcher/parameter (2013) was used with the resonance spikes smoothed. For aVIRGO an interpolation to the data published on.

### *4.2. Space based detectors*

Launching interferometers into space allows the arms to be longer and removes the problem of seismic noise at low frequencies. Various detectors with differing number of satellites, numbers of laser arms, arms lengths etc have been proposed. The most studied is the classic LISA mission. Here we divide the proposed mission into two classes; the milli-Hz detectors LISA and eLISA, and the deci-Hz detectors DECIGO and BBO.

*4.2.1. LISA and eLISA* The laser interferometer space antenna (LISA) consists of three satellites in an equilateral triangle configuration. We use an analytic fit to the instrumental noise curve given by Sathyaprakash & Schutz (2009). As described in Sathyaprakash & Schutz (2009) for observing individual sources there is an additional contribution to the noise from a background of unresolvable binaries, this is not included in for consistency with the other detectors plotted. eLISA is a re-scoped version of the classic LISA mission with slightly reduced sensitivity and peak sensitivity shifted to higher frequencies. We use an analytic fit to the instrumental noise curve given by Amaro-Seoane et al. (2012).

*4.2.2. DECIGO and BBO* These missions are designed to probe the decihertz region of the gravitational wave spectrum, both are considerably more ambitious than the LISA or eLISA mission and are likely to be launched much further into the future. Here simple analytic fits to the sensitivity curved given by Yagi & Seto (2011) are used.

### *4.3. Pulsar timing arrays*

Pulsar timing arrays can be thought of a naturally occuring interferometers with galactic scale arm lengths, hence they are sensitive to much lower frequencies than either the ground or space based detectors already considered. Each pulsar is a very regular clock and the measured arrival time can be compared against a prediction leaving a residual which includes the effects of passing gravitational waves. Sesana et al. (2008) describe the shape of the sensitivity curves obtained by correlating the timing residuals from each of the $N_p$ pulsars in the array. Let the total timing residual, $\delta t$, be the sum of the residuals due to noise, $\delta t_n$, and due to gravitational waves, $\delta t_h$.

$$\delta t = \delta t_n + \delta t_h \tag{29}$$

An upper limit to any background may be placed by assuming the residuals are due entirely to gravitational waves. First considering the simple case of two pulsars at a distance $d$ and with $\delta t_h \approx h_0 d/c \approx h_0/f$ gives,

$$h^2\Omega_{\mathrm{GW}}(f) \propto \frac{\delta t^2_{\mathrm{r.m.s.}} f^4}{\sqrt{T\Delta f}} \tag{30}$$

With larger values of $N_p$ each sperate pair of pulsars forms an independent detector, so the number of detectors scales as $N_p^2$. The optimum signal to noise ratio is given by the combination of all the detectors, the individual signal to noise ratios add in quadrature. So assuming each pair of pulsars in our array is an identical detector gives,

$$h^2\Omega_{\mathrm{GW}}(f) \propto \frac{\delta t^2_{\mathrm{r.m.s.}} f^4}{N_p\sqrt{T\Delta f}} \quad . \tag{31}$$

This can be related to the characteristic strain using (16),

$$h_c(f) \propto \frac{\delta t_{\rm r.m.s.} f}{N_p^{1/2} \left(T \Delta f\right)^{1/4}} \quad . \tag{32}$$

The characteristic strain the detector is sensitive to scales linearly with $f$ down to a minimum frequency of $T^{-1}$ (there is also an upper cut-off determined by the frequency of the pulsar observations). This gives the wedge shaped curves plotted in figures **??**, **??** and **??**. The absolute values of the sensitivity is fixed by normalising (32) to agree with a limit at a given frequency for each array.

*4.3.1. EPTA/PPTA/nanograv* The currently operating pulsar timing arrays are the European pulsar timing array (EPTA), the Parkes pulsar timing array (PPTA) and the North American nanohertz observatory for gravitational waves (nanograv). There are published limits on the amplitude of the stochastic background from all three detectors, the lowest currently is from the EPTA. For the EPTA limit see van Haasteren et al. (2011), for the PPTA limit see Hobbs (2005) and for the nanograv limit see Demorest et al. (2013). The EPTA curves in figures **??**, **??** and **??** shows the limit published in van Haasteren et al. (2011) based on an analysis of 7 pulsars over approximately 10 years.

*4.3.2. IPTA* The international pulsar timing array (IPTA) is a planned combination of the three existing pulsar timing arrays using approximately 30 pulsars. The curves plotted in figures **??**, **??** and **??** are based on 3 times as many pulsars as the EPTA curves and timed for twice as long. Mock data challenges for the IPTA have already been undertaken, see van Haasteren et al. (2013).

*4.3.3. SKA* Following Sesana et al. (2008) it is assumed that the SKA is able to measure $\delta t^2_{\rm r.m.s.}$ a factor of 10 better than the current pulsar timing arrays, the curves drawn in figures **??**, **??** and **??** are based on 3 times as many pulsars timed for 5 times as long as the current EPTA curves.


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