Exploring Gravity
with Gravitational Waves & Strong-Field Tests

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Appendix A

The signal inner product

We wish to derive an inner product over the space of signals; we shall denote the product of signals g and h as $(g|h)$.

A.1 The Fourier transform

A.1.1 Basic properties

We begin with some basic properties of Fourier transform (Riley *et al.* 2002). We define transforamtions

$$x(t) = \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \,\mathrm{d}f \tag{A.1a}$$

$$\widetilde{x}(f) = \int_{-\infty}^{\infty} x(t) \exp(-2\pi i f t) \,\mathrm{d}t. \tag{A.1b}$$

The Dirac delta-function arises as

$$\delta(f) = \int_{-\infty}^{\infty} \exp(-2\pi i f t) \,\mathrm{d}t \tag{A.2}$$

We shall use Plancherel's theorem which proves the unitarity of the Fourier transfromation

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 \,\mathrm{d}t &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \,\mathrm{d}f \int_{-\infty}^{\infty} \widetilde{x}^*(f') \exp(-2\pi i f' t) \,\mathrm{d}f' \\ &= \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \,\mathrm{d}f. \end{aligned} \tag{A.3}$$

A.1.2 Wiener–Khinchin theorem

We begin by deriving the Wiener–Khinchin theorem (Kittel 1958, chapter 28). For a real signal we have $\widetilde{x}(f) = \widetilde{x}^*(f)$, and since $\widetilde{x}(f) = \widetilde{x}^*(-f)$

$$|\widetilde{x}(f)|^2 = |\widetilde{x}(-f)|^2 . \tag{A.4}$$

We use $\langle \ldots \rangle$ to denote time averaging, then

$$\left\langle x^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [x(t)]^2 \,\mathrm{d}t. \tag{A.5}$$

Applying Plancherel's theorem for our real signal

$$\left\langle x^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \,\mathrm{d}f = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\infty} |\widetilde{x}(f)|^2 \,\mathrm{d}f. \tag{A.6}$$

The power spectrum $G(f)$ is

$$G(f) = \lim_{T\to\infty} \frac{1}{T}\overline{|\widetilde{x}(f)|^2}, \tag{A.7}$$

where an overline represents an ensemble average. Therefore

$$\overline{\langle x^2 \rangle} = \int_0^\infty G(f)\,\mathrm{d}f. \tag{A.8}$$

If $x(t)$ is a randomly varying signal we can use the ergodic principle to equate a time average with an ensemble over multiple realisations. Hence $\overline{\langle x^2 \rangle} = \left\langle x^2 \right\rangle$ and we can drop the overline.

The correlation function for a random process is

$$\begin{aligned} C(\tau) &= \langle x(t)x(t+\tau)\rangle \qquad\qquad (A.9)\\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-\infty}^{\infty}\widetilde{x}(f)\exp(2\pi i f t)\,\mathrm{d}f\int_{-\infty}^{\infty}\widetilde{x}(f')\exp\left[2\pi i f'(t+\tau)\right]\mathrm{d}f'\\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}|\widetilde{x}(f)|^2\exp(2\pi i f\tau)\,\mathrm{d}f. \qquad\qquad (A.10)\end{aligned}$$

We can rewrite this in terms of the power spectrum

$$C(\tau) = \frac{1}{2}\int_{-\infty}^{\infty}G(f)\exp(2\pi i f\tau)\,\mathrm{d}f = \int_0^\infty G(f)\cos(2\pi f\tau)\,\mathrm{d}f. \tag{A.11}$$

Inverting these

$$G(f) = 2\int_{-\infty}^{\infty}C(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau = 4\int_0^\infty C(\tau)\cos(2\pi f\tau)\,\mathrm{d}\tau. \tag{A.12}$$

The power spectrum and correlation function are related to each other by the Fourier transform. This is the Wiener–Khinchin theorem.

A.2 Defining the inner product

A.2.1 Gaussian noise

We consider a normally distributed noise signal $n(t)$ with zero mean and standard deviation σ_n. The variance is

$$\left\langle n^2 \right\rangle = C_n(0) = \sigma_n^2, \tag{A.13}$$

introducing correlation function $C_n(\tau)$. If we have a measured signal $s(t)$ and a true signal $h(t)$, the probability $p(s|h)$ is that of the realisation of noise so

$$s = h + n. \tag{A.14}$$

Let us consider a discrete signal $n_i \equiv n(t_i)$, with $t_i - t_j = (i-j)\Delta t$ $\{i, j = -N, \ldots, N\}$ and $\Delta T = 2T/(2N+1)$. For a single point (Finn 1992)

$$p(s_i|h_i) = \frac{1}{\sqrt{2\pi C_n(0)}}\exp\left[-\frac{1}{2}\frac{n_i^2}{C_n(0)}\right]. \tag{A.15}$$

Expanding this to the entire signal

$$p(s|h) = \frac{1}{\sqrt{(2\pi)^{2N+1}\det C_{n,\,ij}}}\exp\left[-\frac{1}{2}\sum_{k,l}C_{kl}^{-1}n_k n_l\right], \tag{A.16}$$

introducing short-hand $C_{n,\,ij} \equiv C_n(t_i - t_j)$ and defining the inverse matrix C_{kl}^{-1} so

$$\delta_{jl} = \sum_l C_{n,\,jk}C_{kl}^{-1}. \tag{A.17}$$

To transform to the continuum (and infinite duration) limit we identify

$$\lim_{T\to\infty;\Delta t\to 0}\sum_j \Delta t \to \lim_{T\to\infty}\int_{-T}^{T} \mathrm{d}t_j. \tag{A.18}$$

To change between Kronecker and Dirac deltas

$$\sum_j \delta_{jk} = \int_{-T}^{T} \delta(t_j - t_k)\,\mathrm{d}t_j, \tag{A.19}$$

hence

$$\delta(t_j - t_k) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\delta_{jk}. \tag{A.20}$$

Using the inverse matrix definition

$$\begin{aligned}\exp(-2\pi i f t_k) &= \sum_j \exp(-2\pi i f t_j)\delta_{jk}\\ &= \frac{1}{(\Delta t)^2}\sum_j \Delta t\exp(-2\pi i f t_j)\sum_l \Delta t\, C_{n,\,jl}C^{-1}_{lk}.\end{aligned} \tag{A.21}$$

Taking the limit

$$\begin{aligned}\exp(-2\pi i f t_k) &= \lim_{T\to\infty;\Delta t\to 0}\frac{1}{(\Delta t)^2}\int_{-T}^{T}\exp(-2\pi i f t_j)\,\mathrm{d}t_j\int_{-T}^{T} C_n(t_j - t_l)C^{-1}(t_l, t_k)\,\mathrm{d}t_l\\ &= \lim_{\Delta t\to 0}\frac{1}{(\Delta t)^2}\int_{-\infty}^{\infty} C_n(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau\int_{-\infty}^{\infty} C^{-1}(t_l, t_k)\exp(-2\pi i f t_l)\,\mathrm{d}t_l,\end{aligned} \tag{A.22}$$

where $\tau = t_j - t_l$. Defining the transformation

$$\widetilde{C^{-1}}(f, t_k) = \int_{-\infty}^{\infty} C^{-1}(t, t_k)\exp(-2\pi i f t)\,\mathrm{d}t, \tag{A.23}$$

and using the Wiener-Khinchin theorem to define power spectrum (Cutler & Flanagan 1994)

$$S_n(f) = \lim_{T\to\infty}\frac{1}{T}\overline{|\widetilde{n}(f)|^2} \tag{A.24}$$

$$= 2\int_{-\infty}^{\infty} C_n(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau, \tag{A.25}$$

we have

$$\exp(-2\pi i f t_k) = \lim_{\Delta t\to 0}\frac{1}{(\Delta t)^2}\frac{S_n(f)}{2}\widetilde{C^{-1}}(f, t_k). \tag{A.26}$$

This can be rearranged to define $\widetilde{C^{-1}}(f, t_k)$ (Finn 1992).

The term in the exponential in equation (A.16) has the limit

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\lim_{T\to\infty;\Delta t\to 0}\sum_{j,k} C^{-1}_{jk} n_j n_k\\ &= \frac{1}{2}\lim_{T\to\infty;\Delta t\to 0}\frac{1}{(\Delta t)^2}\int_{-T}^{T}\mathrm{d}t_j\int_{-T}^{T}\mathrm{d}t_k\, C^{-1}(t_j, t_k)n(t_j)n(t_k)\\ &= \frac{1}{2}\lim_{\Delta t\to 0}\frac{1}{(\Delta t)^2}\int_{-\infty}^{\infty}\mathrm{d}t_k\int_{-\infty}^{\infty}\mathrm{d}f\,\widetilde{C^{-1}}(f, t_k)\widetilde{n}(-f)n(t_k)\\ &= \int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)}\,\mathrm{d}f\\ &= \frac{1}{2}\left(n|n\right),\end{aligned} \tag{A.27}$$

defining the inner product

$$(g|h) = 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f)}{S_n(f)}\,\mathrm{d}f = 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f) + \widetilde{g}(f)\widetilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f. \tag{A.28}$$

This is a noise-weighted inner product over the space of real signals. The probability of the signal is

$$p(s|h) \propto \exp\left[-\frac{1}{2}\,(n|n)\right]. \tag{A.29}$$

A.2.2 Properties of the inner product

We will consider an ensemble average over multiple noise realisations, which is the same as a time average assuming stationarity of the noise spectrum

$$\begin{aligned}\langle (n|n)\,(n|q)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(n(t+\tau)|p(t))\,(n(t+\tau)|q(t))\,\mathrm{d}\tau \\ &= \lim_{T\to\infty}\frac{2}{T}\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)}\frac{\widetilde{p}(f)\widetilde{q}^*(f)}{S_n(f)}\,\mathrm{d}f\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f')}{S_n(f')}\,\mathrm{d}f' \\ &= (p|q)\,,\end{aligned} \tag{A.30}$$

using the definition of the noise spectrum to obtain the final line.

Appendix B

The loss cone

When considering the orbits of stars about a massive black hole (MBH), the loss cone describes a region of velocity space that is depopulated because of tidal disruption (Frank & Rees 1976; Lightman & Shapiro 1977).

A main sequence star may be disrupted by tidal forces before it is swallowed by a MBH, we will define the tidal disruption radius as r_{T}. We expect any orbit that passes inside r_{T} is depopulated unless stars can successfully escape to another orbit before being disrupted. Stars' velocities change because of gravitational interaction with other stars. Deflections can be modelled as a series of two-body encounters, the cumulative effect of which is a random walk in velocity space (Chandrasekhar 1960, chapter 2). Changes scale with the square-root of time, with the relaxation time-scale τ_{R} setting the scale.

Consider a typical star at a distance r from the MBH. We decompose its motion into radial and tangential components as

$$v_{\mathrm{r}} = v\cos\theta; \quad v_{\perp} = v\sin\theta. \tag{B.1}$$

Over a dynamical time-scale t_{dyn}, we expect that stars change velocity by a typical amount

$$\theta_{\mathrm{D}} \approx \left(\frac{t_{\mathrm{dyn}}}{\tau_{\mathrm{R}}}\right)^{1/2}, \tag{B.2}$$

assuming this change is small. We define the loss cone angle θ_{LC} to describe the range of trajectories that will proceed to pass within a distance r_{T} of the MBH. By comparing the diffusion and loss cone angles we can deduce if we would expect orbits to be depleted: if $\theta_{\mathrm{D}} > \theta_{\mathrm{LC}}$ a star can safely diffuse out of the loss cone before it is destroyed, whereas if $\theta_{\mathrm{D}} < \theta_{\mathrm{LC}}$ a star will be disrupted before it can change its velocity sufficiently, leading to the depopulation of the orbit.

Frank & Rees (1976) first introduced the loss cone. They considered stars on nearly radial orbits. The orbital energy and angular momentum (per unit mass) of an object with eccentricity e and periapse radius r_{p} are

$$\mathcal{E} = -\frac{GM_{\bullet}(1-e)}{2r_{\mathrm{p}}}; \tag{B.3}$$

$$\mathcal{J}^2 = GM_{\bullet}(1+e)r_{\mathrm{p}}, \tag{B.4}$$

where $M_{\bullet}$ is the MBH's mass. The angular momentum can also be defined as

$$\begin{aligned}\mathcal{J}^2 &= v_{\perp}^2 r^2 \\ &\simeq \theta^2 v^2 r^2,\end{aligned} \tag{B.5}$$

using the small angle approximation. Frank & Rees (1976) took the limit $e \to 1$, then setting $r_{\mathrm{p}} = r_{\mathrm{T}}$ to demarcate the limit of the loss cone; we rearrange to find

$$\theta_{\mathrm{LC}} \simeq \frac{2GM_{\bullet}r_{\mathrm{T}}}{v^2 r^2}. \tag{B.6}$$

We need to find the speed at r. Frank & Rees (1976) use a typical value

$$v^2 \simeq 3\sigma^2, \tag{B.7}$$

where σ is the 1D velocity dispersion. They assume the velocity dispersion is Keplerian within the core region, where dynamics are dominated by the MBH, and is a constant outside of this

$$\sigma^2 \simeq \begin{cases} \dfrac{GM_\bullet}{r} & r < r_\mathrm{c} \\ \dfrac{GM_\bullet}{r_\mathrm{c}} & r < r_\mathrm{c} \end{cases}. \tag{B.8}$$

The core radius r_c is

$$r_\mathrm{c} = \frac{GM_\bullet}{\sigma_0^2}, \tag{B.9}$$

where σ_0 is the 1D velocity dispersion far from the MBH. Substituting for v^2 in equation (B.6) gives

$$\theta_\mathrm{LC}^2 \simeq \begin{cases} \dfrac{2r_\mathrm{T}}{3r} & r < r_\mathrm{c} \\ \dfrac{2r_\mathrm{T}r_\mathrm{c}}{3r^2} & r < r_\mathrm{c} \end{cases}. \tag{B.10}$$

Frank & Rees (1976) make one final modification, introducing a gravitational focusing factor f such that

$$\theta_\mathrm{LC} \simeq f \begin{cases} \left(\dfrac{2r_\mathrm{T}}{3r}\right)^{1/2} & r < r_\mathrm{c} \\ \left(\dfrac{2r_\mathrm{T}r_\mathrm{c}}{3r^2}\right)^{1/2} & r < r_\mathrm{c} \end{cases}. \tag{B.11}$$

The focusing factor could be imagined as the correction from assuming that stars travel along straight lines, such that $\tan\theta_\mathrm{LC} = r_\mathrm{T}/r$, to accounting for a Keplerian trajectory about the MBH.

It is unappealing to include an arbitrary, albeit order unitary, factor. Additionally, there are various restrictive approximations in the derivation. Considering the orbital energy for $v^2 = 3\sigma^2$ inside the core

$$\frac{3GM_\bullet}{2r} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{T}} \tag{B.12}$$

$$\implies \frac{r_\mathrm{T}}{r} = e - 1. \tag{B.13}$$

Since the radii must be positive, this enforces that $e \geq 1$: the orbits could be marginally bound at best. As we have taken the limit $e \to 1$, assuming that $r \gg r_\mathrm{T}$ this is still self-consistent. However, it is desirable to relax these conditions.

Let us consider an orbit with $r_\mathrm{p} = r_\mathrm{T}$, which gives the edge of the loss cone. The angular momentum squared is

$$\sin^2\theta_\mathrm{LC} v^2 r^2 = GM_\bullet(1+e)r_\mathrm{T}. \tag{B.14}$$

The energy is

$$\frac{v^2}{2} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{T}}. \tag{B.15}$$

Combining these to eliminate the velocity gives

$$\sin^2\theta_\mathrm{LC} = \frac{(1+e)r_\mathrm{T}^2}{2rr_\mathrm{T} - (1-e)r^2}. \tag{B.16}$$

This has been obtained without making any assumptions about the velocity dispersion or the position of the star. Since we have considered the Keplerian orbit, there should be no need to introduce a focusing factor.

This is similar in form to the classic result. Consider an orbit with eccentricity $e = 1 - \epsilon$, where ϵ is small. Let us choose the star to be at a characteristic distance set by its semimajor axis $a = r_\mathrm{p}/(1-e)$, such that

$$r = \frac{r_\mathrm{T}}{\epsilon}. \tag{B.17}$$

This ensures that $r \gg r_\mathrm{T}$. Therefore we have matched the assumptions of Frank & Rees (1976). Substituting into our loss cone formula

$$\begin{aligned}\sin^2\theta_\mathrm{LC} &= \frac{(2-\epsilon)r_\mathrm{T}^2}{2rr_\mathrm{T} + \epsilon r^2}\\ &\simeq \frac{2r_\mathrm{T}}{3r},\end{aligned} \tag{B.18}$$

retaining terms to first order in ϵ. Since this is small, we can use the small angle approximation to recover the result of equation (B.10).

Appendix C

Semirelativistic fluxes

The semirelativistic approximation for extreme-mass-ratio waveforms uses an exact geodesic of the background for the trajectory of the orbiting body, but only uses the flat-space radiation generation formula (Ruffini & Sasaki 1981). This is at the heart of the numerical kludge approximation. Gair, Kennefick & Larson (2005) derived analytic formulae for the fluxes of energy and angular momentum using the semirelativistic approximation for Schwarzschild geometry. These are useful for checking the accuracy of the numerical kludge waveforms.

The published expressions contain a number of (minor) errors, we rederive the correct forms. We consider an object of mass m orbiting about another of mass M with a trajectory specified by eccentricity e and periapsis r_{p}. For section we use geometric units with $G = c = 1$.

The geodesic equations in Schwarzschild are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2m}{r}\right)^{-1} E, \tag{C.1}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \left(E^2 - 1\right) + \frac{2M}{r}\left(1 + \frac{L_z^2}{r^2}\right) - \frac{L_z^2}{r^2}, \tag{C.2}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{L_z}{r^2}, \tag{C.3}$$

where t, r and ϕ are the usual Schwarzschild coordinates, τ is the proper time, and we have introduced specific energy E and azimuthal angular momentum L_z. Spherical symmetry has been exploited to set $\theta = \pi/2$ without loss of generality. For bound orbits, the radial equation has three roots, and can be written as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -\left(E^2 - 1\right)\frac{(r_{\mathrm{a}} - r)(r - r_{\mathrm{p}})(r - r_3)}{r^3}. \tag{C.4}$$

The turning points are the apoapsis, the periapsis and a third root; the orbit becomes unstable when $r_{\mathrm{p}} = r_3$. An eccentricity can be defined, in analogy to Keplerian orbits, such that

$$r_{\mathrm{a}} = \frac{1+e}{1-e} r_{\mathrm{p}}. \tag{C.5}$$

The third root is then

$$r_3 = \frac{2(1+e)M}{(1+e)r_{\mathrm{p}} - 4M} r_{\mathrm{p}}. \tag{C.6}$$

The last stable orbit with a given eccentricity, has periapse radius

$$r_{\mathrm{p,\,LSO}} = \frac{2(3+e)M}{1+e}. \tag{C.7}$$

Orbits that approach closer than this will plunge into the black hole.

The parameters $\{r_\mathrm{p}, e\}$ can be used to characterise orbits in place of $\{E, L_z\}$. The two are related by

$$E^2 = 1 - \frac{(1-e)\left[(1+e)r_\mathrm{p} - 4M\right]M}{\left[(1+e)r_\mathrm{p} - (3+e^2)\,M\right]r_\mathrm{p}}; \tag{C.8}$$

$$L_z^2 = \frac{(1+e)^2 M r_\mathrm{p}^2}{(1+e)r_\mathrm{p} - (3+e^2)\,M}. \tag{C.9}$$

Following the semirelativistic approximation, the fluxes of energy and angular momentum are derived by inserting the Schwarzschild geodesic into the flat-space radiation formulae, identified the coordinate t with the flat-space time (Misner *et al.* 1973, chapter 36)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{5}\left\langle \frac{\mathrm{d}^3 I\!\!\!-_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 I\!\!\!-^{ij}}{\mathrm{d}t^3}\right\rangle, \tag{C.10}$$

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = -\frac{2}{5}\left\langle \frac{\mathrm{d}^2 I\!\!\!-_{xi}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 I\!\!\!-^{iy}}{\mathrm{d}t^3} - \frac{\mathrm{d}^2 I\!\!\!-_{yi}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 I\!\!\!-^{iz}}{\mathrm{d}t^3}\right\rangle, \tag{C.11}$$

where $I\!\!\!-_{ij} = I_{ij} - (1/3)I\Delta_{ij}$ is the reduced mass quadrupole tensor and $\langle\ldots\rangle$ indicates averaging over several wavelengths (or periods). For a point particle, the mass quadrupole is

$$I^{jk} = \mu x^j x^k, \tag{C.12}$$

for trajectory $x^i(t)$. This is determined from the geodesic equations, and written as a function of r_p, e and r. To calculate the total change over one orbit we integrate r from r_p to r_a and back again. For this purpose it is easier to consider derivatives with respect to r. The integrands are rational functions of r and the square root of a cubic in r; the integrals can thus be written as a combination of elliptic integrals.

The integrals are of a general form

$$\mathscr{I}_n = \int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{M^{n+1}}{r^n\sqrt{(r_\mathrm{a}-r)(r-r_\mathrm{p})(r-r_3)r}}\,\mathrm{d}r. \tag{C.13}$$

By considering the derivative of $r^{-n}\sqrt{(r_\mathrm{a}-r)(r-r_\mathrm{p})(r-r_3)r}$ we may derive a recurrence relationship using integration by parts. After some rearrangement

$$\mathscr{I}_n = \frac{n-1}{2n-1}\mathscr{I}_{n-1} - \frac{2n-3}{2n-1}\frac{(r_\mathrm{a}+r_\mathrm{p}+r_3)M^2}{r_\mathrm{a}r_\mathrm{p}r_3}\mathscr{I}_{n-2} + \frac{2(n-1)}{2n-1}\frac{M^3}{r_\mathrm{a}r_\mathrm{p}r_3}\mathscr{I}_{n-3}. \tag{C.14}$$

Setting $n = 2$, the third term vanishes, hence the integrals $\mathscr{I}_0$ and $\mathscr{I}_1$ are sufficient to specify the series. The zeroth integral can be evaluated using Gradshteyn & Ryzhik (2000, 3.147.6) as

$$\mathscr{I}_0 = \frac{2M}{r_\mathrm{p}}\sqrt{\frac{r_\mathrm{p}}{r_\mathrm{a}-r_3}}K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right], \tag{C.15}$$

where $K(k)$ is the complete elliptic integral of the first kind. The next integral can be evaluated using Gradshteyn & Ryzhik (2000, 3.149.6) as

$$\mathscr{I}_1 = \frac{2M^2}{r_\mathrm{p}r_3\sqrt{r_\mathrm{p}(r_\mathrm{a}-r_3)}}\left\{r_\mathrm{p}K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right] - (r_\mathrm{p}-r_3)\mathit{\Pi}\left[\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}, \sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right]\right\}, \tag{C.16}$$

where $\mathit{\Pi}(n,k)$ is the complete elliptic integral of the third kind. In this instance we may simplify using Olver *et al.* (2010, 19.6.2)

$$\mathit{\Pi}(k^2, k) = \frac{E(k)}{1-k^2} \tag{C.17}$$

to rewrite in terms of the complete elliptic integral of the second kind. Hence

$$\mathscr{I}_1 = \frac{2M^2}{r_3\sqrt{r_\mathrm{p}(r_\mathrm{a}-r_3)}}\left\{K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right] - \frac{r_\mathrm{a}-r_3}{r_\mathrm{a}}E\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right]\right\}. \tag{C.18}$$

Substituting in for the integrals, we find that the energy lost in one orbit is

$$\begin{aligned}\frac{M}{m}\Delta E = {} & -\frac{16M^{11}}{1673196525r_\mathrm{p}^6(1+e)^{19/2}\left\{(r_\mathrm{p}-2M)\left[(1+e)r_\mathrm{p}-2(1-e)M\right]\right\}^{5/2}} \\ & \times\left\{\sqrt{(1+e)\frac{r_\mathrm{p}}{M}-2(3-e)}E\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]f_1\left(\frac{r_\mathrm{p}}{M},e\right)\right. \\ & \left.+\frac{1+e}{\sqrt{(1+e)\left(r_\mathrm{p}/M\right)-2(3-e)}}K\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]f_2\left(\frac{r_\mathrm{p}}{M},e\right)\right\}, \qquad \text{(C.19)}\end{aligned}$$

where we have introduced functions

$$\begin{aligned}f_1(y,e) = {} & 4608(1-e)(1+e)^2\left(3+e^2\right)^2\left(2428691599+313957879e^2+1279504693e^4\right. \\ & \left.+63843717e^6\right)-192(1+e)^2\left(908960573673-155717471796e^2\right. \\ & -88736969547e^4-293676299040e^6-195313674237e^8-26635698156e^{10} \\ & \left.-346799201e^{12}\right)y+384(1+e)^3\left(336063804453-53956775638e^2-33318942522e^4\right. \\ & \left.-92857670352e^6-41764459155e^8-2765710514e^{10}\right)y^2 \\ & -16(1+e)^4\left(3418907055555-580720618635e^2-168432860626e^4\right. \\ & \left.-606890963686e^6-176495184865e^8-3768291999e^{10}\right)y^3 \\ & +32(1+e)^5\left(510454645597-92175635794e^2+26432814256e^4-28250211070e^6\right. \\ & \left.-5713846269e^8\right)y^4-4(1+e)^6\left(1107402703901-174239346926e^2\right. \\ & \left.+100957560852e^4+3707280110e^6-899162673e^8\right)y^5 \\ & +8(1+e)^7\left(143625217397-16032820010e^2+4238287541e^4+275190560e^6\right)y^6 \\ & -(1+e)^8\left(220627324753-14884378223e^2-1210713997e^4+14138955e^6\right)y^7 \\ & +8(1+e)^9\left(2922108518-46504603e^2-2407656e^4\right)y^8 \\ & -3(1+e)^{10}\left(241579935+6314675e^2-149426e^4\right)y^9 \\ & -4(1+e)^{11}\left(8608805-48992e^2\right)y^{10}-2(1+e)^{12}\left(1242083-16320e^2\right)y^{11} \\ & -184320(1+e)^{13}y^{12}-5120(1+e)^{14}y^{13} \qquad \text{(C.20)}\end{aligned}$$

and

$$
\begin{aligned}
f_2(y,e) = {} & 3072(3-e)(3+e)\left(3+e^2\right)\left(7286074797-3299041125e^2+792940362e^4\right.\\
& \left.-1366777698e^6-369698151e^8-5932745e^{10}\right)-384(1+e)\left(2989180413711\right.\\
& -583867932642e^2-131661872359e^4-419423580924e^6-194293515951e^8\\
& \left.-3390301442e^{10}+1353430119e^{12}\right)y+64(1+e)^2\left(14825178681327\right.\\
& -2675442646782e^2-728511901515e^4-1837874368340e^6-591999524567e^8\\
& \left.-1856757710e^{10}+841581651e^{12}\right)y^2-32(1+e)^3\left(14292163934541\right.\\
& -2666166422089e^2-522582885086e^4-1347373382962e^6-307066297439e^8\\
& \left.-1675056789e^{10}\right)y^3+16(1+e)^4\left(9557748374919-1917809903861e^2\right.\\
& \left.-24258045506e^4-511875047746e^6-86779453317e^8-462078345e^{10}\right)y^4\\
& -8(1+e)^5\left(5390797838491-990602472036e^2+161182699002e^4\right.\\
& \left.-89978894004e^6-11363685245e^8\right)y^5+4(1+e)^6\left(2857676457065\right.\\
& \left.-351292910556e^2+79840371470e^4-2670080940e^6-463345647e^8\right)y^6\\
& -2(1+e)^7\left(1249768416047-79903103833e^2+12179840133e^4\right.\\
& \left.+482157413e^6\right)y^7+(1+e)^8\left(363565648057-10040939153e^2-318841465e^4\right.\\
& \left.+14611473e^6\right)y^8-2(1+e)^9\left(13862653487-100645509e^2-11015842e^4\right)y^9\\
& +(1+e)^{10}\left(518128485+16345427e^2-421398e^4\right)y^{10}\\
& +16(1+e)^{11}\left(1220639-13448e^2\right)y^{11}+2(1+e)^{12}\left(689123-18880e^2\right)y^{12}\\
& +153600(1+e)^{13}y^{13}+5120(1+e)^{14}y^{14}. \qquad \text{(C.21)}
\end{aligned}
$$

The angular momentum lost is

$$
\begin{aligned}
\frac{\Delta L_z}{m} = {} & -\frac{16M^{15/2}}{24249225(1+e)^{13/2}r_\mathrm{p}^{7/2}(r_\mathrm{p}-2M)^2\left[(1+e)r_\mathrm{p}-2(1-e)M\right]^2}\\
& \times\left\{\sqrt{(1+e)\frac{r_\mathrm{p}}{M}-2(3-e)}\,E\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]g_1\left(\frac{r_\mathrm{p}}{M},e\right)\right.\\
& \left.+\frac{(1+e)}{\sqrt{(1+e)\left(r_\mathrm{p}/M\right)-2(3-e)}}K\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]g_2\left(\frac{r_p}{M},e\right)\right\} \qquad \text{(C.22)}
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(y,e) = {} & 169728(1-e)(1+e)^2\left(279297+219897e^2+106299e^4+9611e^6\right)\\
& -384(1+e)^2\left(192524061-13847615e^2-36165965e^4-20710173e^6-588532e^8\right)y\\
& +192(1+e)^3\left(235976417+13109547e^2-3369705e^4-3292707e^6\right)y^2\\
& -16(1+e)^4\left(813592799+112906199e^2+53843933e^4+602061e^6\right)y^3\\
& +16(1+e)^5\left(87491089+7247482e^2+4608349e^4\right)y^4+8(1+e)^6\left(9580616\right.\\
& \left.+6179243e^2-92047e^4\right)y^5-4(1+e)^7\left(3760123+272087e^2\right)y^6\\
& -(1+e)^8\left(1168355-35347e^2\right)y^7-71792(1+e)^9y^8-4120(1+e)^{10}y^9 \qquad \text{(C.23)}
\end{aligned}
$$

and

$$\begin{aligned}
g_2(y,e) = {} & 339456(3-e)(3+e)\left(93099-10213e^2-18155e^4-10551e^6-420e^8\right) \\
& - 1536(1+e)\left(319648410-35712133e^2-33099777e^4-11272311e^6+457187e^8\right)y \\
& + 128(1+e)^2\left(2706209781-45415294e^2-103634296e^4-34056010e^6-130293e^8\right)y^2 \\
& - 32(1+e)^3\left(3895435659+212168215e^2+4641265e^4-15197651e^6\right)y^3 \\
& + 16(1+e)^4\left(1396737473+123722895e^2+27602127e^4-465119e^6\right)y^4 \\
& - 16(1+e)^5\left(78148621+3035912e^2+3130827e^4\right)y^5 \\
& - 16(1+e)^6\left(8005570+1485159e^2-47943e^4\right)y^6+2(1+e)^7\left(4015181+601959e^2\right)y^7 \\
& + (1+e)^8\left(737603-39467e^2\right)y^8+47072(1+e)^9y^9+4120(1+e)^{10}y^{10}. \qquad \text{(C.24)}
\end{aligned}$$

Taking limit $r_\mathrm{p} \to \infty$ should recover weak field results. Using series expansions of the elliptic integrals for small arguments

$$\begin{aligned}
\frac{M}{m}\Delta E \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^{7/2} \\
& -\frac{192\pi}{5}\frac{1}{(1+e)^{9/2}}\left(1+\frac{31}{8}e^2+\frac{65}{32}e^4+\frac{1}{6}e^6\right)\left(\frac{M}{r_\mathrm{p}}\right)^{9/2}+\mathcal{O}\left(\frac{M^{11/2}}{r_\mathrm{p}^{11/2}}\right) \qquad \text{(C.25)} \\
\frac{\Delta L_z}{m} \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^2}\left(1+\frac{7}{8}e^2\right)\left(\frac{M}{r_\mathrm{p}}\right)^2 \\
& -\frac{192\pi}{5}\frac{1}{(1+e)^3}\left(1+\frac{35}{24}e^2+\frac{1}{4}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^3+\mathcal{O}\left(\frac{M^4}{r_\mathrm{p}^4}\right). \qquad \text{(C.26)}
\end{aligned}$$

The leading order terms correspond to the Keplerian results of Peters (1964).

For a parabolic orbit with $e=1$, the energy loss reduces to

$$\frac{M}{m}\Delta E = -\frac{2^{7/2}M^{21/2}}{1673196525\left(r_\mathrm{p}-2M\right)^2 r_\mathrm{p}^{17/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)f_1\left(\frac{r_\mathrm{p}}{M}\right)+K\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)f_2\left(\frac{r_\mathrm{p}}{M}\right)\right] \qquad \text{(C.27)}$$

where

$$\begin{aligned}
f_1(y) = {} & -2y\left(27850061568-83550184704y+117662445984y^2-102686941680y^3\right. \\
& +64808064704y^4-33026468872y^5+12784148218y^6-2873196259y^7 \\
& \left.+185808888y^8+17119626y^9+2451526y^{10}+368640y^{11}+20480y^{12}\right) \qquad \text{(C.28)}
\end{aligned}$$

and

$$\begin{aligned}
f_2(y) = {} & -72901570560+274404834816y-424693524096y^2 \\
& +378109481088y^3-249480499840y^4+154011967968y^5 \\
& -84437171728y^6+31689370996y^7-6231594434y^8+321950817y^9 \\
& +27462280y^{10}+4073612y^{11}+696320y^{12}+40960y^{13}. \qquad \text{(C.29)}
\end{aligned}$$

The angular momentum lost is

$$\frac{\Delta L_z}{m} = \frac{64M^7}{24249225{r_\mathrm{p}}^{11/2}\left(r_\mathrm{p}-2M\right)^{3/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)g_1\left(\frac{r_\mathrm{p}}{M}\right)+K\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)g_2\left(\frac{r_\mathrm{p}}{M}\right)\right], \qquad \text{(C.30)}$$

where

$$\begin{aligned}
g_1(y) = {} & 181817664y-363635328y^2-245236248y^3-49673460y^4 \\
& -7833906y^5+2016105y^6+283252y^7+35896y^8+4120y^9 \qquad \text{(C.31)}
\end{aligned}$$

and

$$g_2(y) = 71285760 - 324389184y + 468548880y^2 - 277856496y^3 + 54521424y^4 + 6181872y^5 - 1630457y^6 - 238086y^7 - 31776y^8 - 4120y^9. \tag{C.32}$$