Exploring Gravity
with Gravitational Waves & Strong-Field Tests

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Part I

Astronomical systems

Part II

Understanding gravitation

Chapter 1

Gravitational radiation in $f(R)$-gravity

1.1 Beyond general relativity: $f(R)$ modified gravity

GR is a well tested theory of gravity (Will 2006). However, there are many unanswered questions that remain regarding gravity which motivate the exploration of alternate theories: What are the true natures of dark matter and dark energy? How should we formulate a quantizable theory of gravity? What drove inflation in the early Universe? Is GR the only theory that is consistent with current observations? Moreover, the majority of the tests that have been carried out to date have been in the weak-field, low-energy regime (Will 2006, 1993; Psaltis 2008). It is not unreasonable to suspect there may be higher order corrections that are only discovered in the strong-field regime, where curvature is high or spacetime dynamic.

In this and the following chapter, we shall study a modified theory of gravitation to assess the feasibility of potential corrections to GR. We investigate metric $f(R)$-gravity, in which the Einstein-Hilbert action is modified by replacing the Ricci scalar R with an arbitrary function $f(R)$. This is one of the simplest extensions to standard GR (Sotiriou & Faraoni 2010; De Felice & Tsujikawa 2010). It has attracted significant interest because the flexibility in defining the function $f(R)$ allows a wide range of cosmological phenomena to be described (Nojiri & Odintsov 2007; Capozziello & Francaviglia 2007). For example, Starobinsky (1980) suggested that a quadratic addition to the field equations could drive exponential expansion of the early Universe (Vilenkin 1985): inflation in modern terminology. In this model $f(R) = R - R^2/(6\Upsilon^2)$; the size of the quadratic correction can be tightly constrained by considering the spectrum of curvature perturbations generated during inflation (Starobinskii 1983, 1985). Using the results of the Wilkinson Microwave Anisotropy Probe (WMAP; Bennett *et al.* 2012; Hinshaw *et al.* 2012), the inverse length-scale can be constrained to $\Upsilon \simeq 3 \times 10^{-6}(50/N)\ell_{\mathrm{P}}^{-1}$, where N is the number of e-folds during inflation and ℓ_{P} is the Planck length (Starobinsky 2007; De Felice & Tsujikawa 2010).

We consider simple $f(R)$ corrections within the framework of linearized gravity and discover how GWs are modified. We begin by reviewing the properties of the $f(R)$ field equations. From these, we derive the linearized equations (section 1.3) and use these to determine the properties of GWs (section 1.4). These are largely known in the literature, but are worked out here *ab initio.* We proceed to derive an effective energy-momentum tensor for gravitational radiation in section 1.5, following the short-wavelength approximation of Isaacson (1968a, b).

Following on from the theory developed in this chapter, in chapter 2 we consider observational tests of $f(R)$-gravity. We explore what constraints LISA, Solar System tests and laboratory experiments can place on the form of $f(R)$. We do not consider cosmological implications where terms beyond linear order could play a significant role. The overall conclusion is that LISA could place constraints on $f(R)$-gravity which may be more powerful than those in the Solar System, but are not as powerful as constraints from laboratory experiments. A brief summary of findings from both chapters is found in section 2.4.

Natural units with $c = 1$ are used throughout both chapters, but factors of G are retained.

1.2 Description of $f(R)$-gravity

1.2.1 The action and field equations

General relativity can be derived from the Einstein-Hilbert action (Misner *et al.* 1973, chapter 21; Landau & Lifshitz 1975, section 93; Dirac 1996, section 26)

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x. \tag{1.1}$$

In $f(R)$ theory we make a simple modification of the action to include an arbitrary function of the Ricci scalar R such that (Buchdahl 1970)

$$S_{f(R)}[g] = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\,\mathrm{d}^4x. \tag{1.2}$$

Including the function $f(R)$ gives extra freedom in defining the behaviour of gravity. While this action may not encode the true theory of gravity it could contain sufficient information to act as an effective field theory, correctly describing phenomenological behaviour (Park *et al.* 2010); it may be that as an effective field theory, a particular $f(R)$ shall have a limited region of applicability and shall not be universal. We assume that $f(R)$ is analytic about $R = 0$ so that it can be expressed as a power series (Buchdahl 1970; Capozziello *et al.* 2007; Faulkner *et al.* 2007; Clifton 2008; Psaltis *et al.* 2008)

$$f(R) = a_0 + a_1R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \dots \tag{1.3}$$

Since the dimensions of $f(R)$ must be the same as of R, $[a_n] = [R]^{(1-n)}$. To link to GR we set $a_1 = 1$; any rescaling can be absorbed into the definition of G.

Various models of cosmological interest may be expressed in such a form, for example, the model of Starobinsky (2007)

$$f(R) = R + \lambda R_0\left[\left(1 + \frac{R^2}{R_0^2}\right)^{-n} - 1\right], \tag{1.4}$$

can be expanded as

$$f(R) = R - \frac{\lambda n}{R_0}R^2 + \frac{\lambda n(n+1)}{2R_0^3}R^4 + \dots \tag{1.5}$$

Consequently such a series expansion can constrain model parameters, although we cannot specify the full functional form from only a few terms.

The field equations are obtained by a variational principle; there are several ways of achieving this. To derive the Einstein field equations from the Einstein-Hilbert action one may use the standard metric variation or the Palatini variation (Misner *et al.* 1973, section 21.2). Both approaches can be used for $f(R)$, but they yield different results (Sotiriou & Faraoni 2010; De Felice & Tsujikawa 2010). Following the metric formalism, one varies the action with respect to the metric $g^{\mu\nu}$. Following the Palatini formalism one varies the action with respect to both the metric $g^{\mu\nu}$ and the connection $\Gamma^{\rho}{}_{\mu\nu}$, which are treated as independent quantities: the connection is not the Levi-Civita metric connection.[1]

[1] Imposing the condition that the metric and Palatini formalisms produce the same field equations, assuming an action that only depends on the metric and Riemann tensor, results in Lovelock gravity (Exirifard & Sheik-Jabbari 2008). Lovelock gravities require the field equations to be divergence free and no more than second order; in four dimensions the only possible Lovelock gravity is GR with a potentially non-zero cosmological constant (Lovelock 1970, 1971, 1972).

Finally, there is a third version of $f(R)$-gravity: metric-affine $f(R)$-gravity (Sotiriou & Liberati 2007a, b). This goes beyond the Palatini formalism by supposing that the matter action is dependent on the variational independent connection. Parallel transport and the covariant derivative are divorced from the metric. This theory has its attractions: it allows for a natural introduction of torsion. However, it is not a metric theory of gravity and so cannot satisfy all the postulates of the Einstein equivalence principle (Will 2006): a free particle does not necessarily follow a geodesic and so the effects of gravity might not be locally removed (Exirifard & Sheik-Jabbari 2008). The implications of this have not been fully explored, but for this reason we will not consider the theory further.

We restrict our attention to metric $f(R)$-gravity. This is preferred as the Palatini formalism has undesirable properties: static spherically symmetric objects described by a polytropic equation of state are subject to a curvature singularity (Barausse *et al.* 2008b, a; De Felice & Tsujikawa 2010). Varying the action with respect to the metric $g^{\mu\nu}$ produces

$$\delta S_{f(R)} = \frac{1}{16\pi G}\int\left\{f'(R)\sqrt{-g}\left[R_{\mu\nu} - \nabla_\mu\nabla_\nu \;+ g_{\mu\nu}\Box\right] \;- f(R)\frac{1}{2}\sqrt{-g}g_{\mu\nu}\right\}\delta g^{\mu\nu}\,\mathrm{d}^4x, \quad (1.6)$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian and a prime denotes differentiation with respect to R.

Proceeding from here requires certain assumptions regarding surface terms. In the case of the Einstein-Hilbert action these gather into a total derivative; it is possible to subtract this from the action to obtain a well-defined variational quantity (York 1972; Gibbons & Hawking 1977). This is not the case for general $f(R)$-gravity (Madsen & Barrow 1989). However, since the action includes higher-order derivatives of the metric, we are at liberty to fix more degrees of freedom at the boundary, in so doing eliminating the importance of the surface terms (Dyer & Hinterbichler 2009; Sotiriou & Faraoni 2010). Setting the variation $\delta R = 0$ on the boundary allows us to subtract a term similar to that in GR (Guarnizo *et al.* 2010). We then have a well-defined variational quantity, from which we can obtain the field equations.

The vacuum field equations are

$$f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu} = 0. \quad (1.7)$$

Taking the trace gives

$$f'R + 3\Box f' - 2f = 0. \quad (1.8)$$

If we consider a uniform flat spacetime $R = 0$, this requires (Capozziello *et al.* 2007)

$$a_0 = 0. \quad (1.9)$$

In analogy to the Einstein tensor, we define

$$\mathcal{G}_{\mu\nu} = f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu}, \quad (1.10)$$

so that in a vacuum

$$\mathcal{G}_{\mu\nu} = 0. \quad (1.11)$$

1.2.2 Conservation of energy-momentum

If we introduce matter with a stress-energy tensor $T_{\mu\nu}$, the field equations become

$$\mathcal{G}_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.12)$$

Acting upon this with the covariant derivative

$$\begin{aligned} 8\pi G\nabla^\mu T_{\mu\nu} &= \nabla^\mu\mathcal{G}_{\mu\nu} \\ &= R_{\mu\nu}\nabla^\mu f' + f'\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \left(\Box\nabla_\nu - \nabla_\nu\Box\right)f'. \end{aligned} \quad (1.13)$$

The second term contains the covariant derivative of the Einstein tensor and so is zero. The final term can be shown to be

$$(\Box\nabla_\nu - \nabla_\nu\Box) f' = R_{\mu\nu}\nabla^\mu f', \tag{1.14}$$

which is a useful geometric identity (Koivisto 2006). Using this

$$8\pi G\nabla^\mu T_{\mu\nu} = 0. \tag{1.15}$$

Consequently energy-momentum is a conserved quantity in the same way as in GR, as is expected from the symmetries of the action.

1.3 Linearised theory

We start our investigation of $f(R)$ by looking at linearised theory. This is a weak-field approximation that assumes only small deviations from a flat background, greatly simplifying the field equations. Just as in GR, the linearised framework provides a natural way to study GWs. We shall see that the linearised field equations reduce down to flat-space wave equations: GWs are as much a part of $f(R)$-gravity as of GR.

Consider a perturbation of the metric from flat Minkowski space such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \tag{1.16}$$

where, more formally, $h_{\mu\nu} = \varepsilon H_{\mu\nu}$ for a small parameter ε.[2] We only consider terms to $\mathcal{O}(\varepsilon)$. The inverse metric is then

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{1.17}$$

where we have used the Minkowski metric to raise the indices on the right, defining

$$h^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}. \tag{1.18}$$

Similarly, the trace h is given by

$$h = \eta^{\mu\nu}h_{\mu\nu}. \tag{1.19}$$

All quantities denoted by "h" are strictly $\mathcal{O}(\varepsilon)$.

The linearised connection is

$$\Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \tag{1.20}$$

To $\mathcal{O}(\varepsilon)$ the covariant derivative of any perturbed quantity is the same as the partial derivative. The Riemann tensor is

$$R^{(1)\lambda}{}_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu\partial_\nu h^\lambda_\rho + \partial^\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\rho h^\lambda_\nu - \partial^\lambda\partial_\nu h_{\mu\rho}), \tag{1.21}$$

where we have raised the index on the differential operator with the background Minkowski metric. Contracting gives the Ricci tensor

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}(\partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}), \tag{1.22}$$

where the d'Alembertian operator is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Contracting this with $\eta^{\mu\nu}$ gives the first-order Ricci scalar

$$R^{(1)} = \partial_\mu\partial_\rho h^{\rho\mu} - \Box h. \tag{1.23}$$

To $\mathcal{O}(\varepsilon)$ we can write $f(R)$ as a Maclaurin series

$$f(R) = a_0 + R^{(1)}; \tag{1.24a}$$

$$f'(R) = 1 + a_2 R^{(1)}. \tag{1.24b}$$

[2] It is because we wish to perturb about flat spacetime that we have required $f(R)$ to be analytic about $R = 0$.

As we are perturbing from a Minkowski background where the Ricci scalar vanishes, we use equation (1.9) to set $a_0 = 0$. Inserting these into equation (1.10) and retaining terms to $\mathcal{O}(\varepsilon)$ yields

$$\mathcal{G}^{(1)}{}_{\mu\nu} = R^{(1)}{}_{\mu\nu} - \partial_\mu\partial_\nu(a_2 R^{(1)}) + \eta_{\mu\nu}\Box(a_2 R^{(1)}) - \frac{R^{(1)}}{2}\eta_{\mu\nu}. \tag{1.25}$$

Now consider the linearised trace equation, from equation (1.8)

$$\begin{aligned}\mathcal{G}^{(1)} &= R^{(1)} + 3\Box(a_2 R^{(1)}) - 2R^{(1)} \\ &= 3\Box(a_2 R^{(1)}) - R^{(1)},\end{aligned} \tag{1.26}$$

where $\mathcal{G}^{(1)} = \eta^{\mu\nu}\mathcal{G}^{(1)}{}_{\mu\nu}$. This is the massive inhomogeneous Klein-Gordon equation. Setting $\mathcal{G} = 0$, as for a vacuum, we obtain the standard Klein-Gordon equation

$$\Box R^{(1)} + \Upsilon^2 R^{(1)} = 0, \tag{1.27}$$

defining the reciprocal length (squared)

$$\Upsilon^2 = -\frac{1}{3a_2}. \tag{1.28}$$

For a physically meaningful solution $\Upsilon^2 > 0$: we constrain $f(R)$ such that $a_2 < 0$ (Schmidt 1986; Teyssandier 1990; Olmo 2005; Corda 2008). From Υ we define a reduced Compton wavelength

$$\lambda\!\!\!^{-}_R = \frac{1}{\Upsilon} \tag{1.29}$$

associated with this scalar mode.

The next step is to substitute in $h_{\mu\nu}$ to find wave solutions. We want a quantity $\overline{h}_{\mu\nu}$ that satisfies a wave equation, related to $h_{\mu\nu}$ by

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + A_{\mu\nu}. \tag{1.30}$$

In GR we use the trace-reversed form where $A_{\mu\nu} = -(h/2)\eta_{\mu\nu}$. This does not suffice here, but let us look for a similar solution

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + B_{\mu\nu}. \tag{1.31}$$

The only rank-two tensors in our theory are: $h_{\mu\nu}$, $\eta_{\mu\nu}$, $R^{(1)}{}_{\mu\nu}$, and $\partial_\mu\partial_\nu$; $h_{\mu\nu}$ has been used already, and we wish to eliminate $R^{(1)}{}_{\mu\nu}$, so we can try the simpler option based around $\eta_{\mu\nu}$. We want $B_{\mu\nu}$ to be $\mathcal{O}(\varepsilon)$; since we have already used h, we shall try the other scalar quantity $R^{(1)}$. Therefore, we construct an ansatz

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + \left(ba_2 R^{(1)} - \frac{h}{2}\right)\eta_{\mu\nu}, \tag{1.32}$$

where a_2 has been included to ensure dimensional consistency and b is a dimensionless number. Contracting with the background metric yields

$$\overline{h} = 4ba_2 R^{(1)} - h, \tag{1.33}$$

so we can eliminate h in our definition of $\overline{h}_{\mu\nu}$ to give

$$h_{\mu\nu} = \overline{h}_{\mu\nu} + \left(ba_2 R^{(1)} - \frac{\overline{h}}{2}\right)\eta_{\mu\nu}. \tag{1.34}$$

Just as in GR, we have the freedom to perform a gauge transformation (Misner *et al.* 1973, box 18.2; Hobson *et al.* 2006, section 17.1): the field equations are gauge-invariant since we started with a function of the gauge-invariant Ricci scalar. We shall assume a Lorenz, or de Donder, gauge choice

$$\nabla^{\mu}\overline{h}_{\mu\nu} = 0; \tag{1.35}$$

or for a flat spacetime

$$\partial^{\mu}\overline{h}_{\mu\nu} = 0. \tag{1.36}$$

Subject to this, from equation (1.22), the Ricci tensor is

$$R^{(1)}{}_{\mu\nu} = -\left[b\partial_{\mu}\partial_{\nu}(a_2R^{(1)}) + \frac{1}{2}\Box\left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2}\eta_{\mu\nu}\right) + \frac{b}{6}(R^{(1)} + \mathcal{G}^{(1)})\eta_{\mu\nu}\right]. \tag{1.37}$$

Using this with equation (1.26) in equation (1.25) gives

$$\mathcal{G}^{(1)}{}_{\mu\nu} = \frac{2-b}{6}\mathcal{G}^{(1)}\eta_{\mu\nu} - \frac{1}{2}\Box\left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2}\eta_{\mu\nu}\right) - (b+1)\left[\partial_{\mu}\partial_{\nu}(a_2R^{(1)}) + \frac{1}{6}R^{(1)}\eta_{\mu\nu}\right]. \tag{1.38}$$

Picking $b = -1$ the final term vanishes, thus we set (Will 1993, section 10.3; Corda 2008; Capozziello *et al.* 2008)

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \left(a_2R^{(1)} + \frac{h}{2}\right)\eta_{\mu\nu} \tag{1.39a}$$

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \left(a_2R^{(1)} + \frac{\overline{h}}{2}\right)\eta_{\mu\nu}. \tag{1.39b}$$

From equation (1.23) the Ricci scalar is

$$\begin{aligned} R^{(1)} &= \Box\left(a_2R^{(1)} - \frac{\overline{h}}{2}\right) + \Box(4a_2R^{(1)} + \overline{h}) \\ &= 3\Box(a_2R^{(1)}) + \frac{1}{2}\Box\overline{h}. \end{aligned} \tag{1.40}$$

For consistency with equation (1.26), we require

$$-\frac{1}{2}\Box\overline{h} = \mathcal{G}^{(1)}. \tag{1.41}$$

Inserting this into equation (1.38), with $b = -1$, we see

$$-\frac{1}{2}\Box\overline{h}_{\mu\nu} = \mathcal{G}^{(1)}{}_{\mu\nu}; \tag{1.42}$$

we have our wave equations.

Should a_2 be sufficiently small that it can be regarded an $\mathcal{O}(\varepsilon)$ quantity, we recover the usual GR formulae to leading order within our analysis.

1.4 Gravitational radiation

Having established two wave equations, (1.26) and (1.42), we now investigate their solutions. Consider waves in a vacuum, such that $\mathcal{G}_{\mu\nu} = 0$. Using a standard Fourier decomposition

$$\overline{h}_{\mu\nu} = \widehat{h}_{\mu\nu}(k_{\rho})\exp(ik_{\rho}x^{\rho}), \tag{1.43a}$$

$$R^{(1)} = \widehat{R}(q_{\rho})\exp(iq_{\rho}x^{\rho}), \tag{1.43b}$$

where k_{μ} and q_{μ} are four-wavevectors. From equation (1.42) we know that k_{μ} is a null vector, so for a wave travelling along the z-axis

$$k^{\mu} = \omega(1, 0, 0, 1), \tag{1.44}$$

where ω is the angular frequency. Similarly, from equation (1.26)

$$q^{\mu} = \left(\Omega, 0, 0, \sqrt{\Omega^2 - \Upsilon^2}\right), \tag{1.45}$$

for frequency Ω. These waves do not travel at c, but have a group velocity

$$v(\Omega) = \frac{\sqrt{\Omega^2 - \Upsilon^2}}{\Omega}, \tag{1.46}$$

provided that $\Upsilon^2 > 0$, $v < 1 = c$. For $\Omega < \Upsilon$, we find an evanescently decaying wave. The travelling wave is dispersive; for waves made up of a range of frequency components there shall be a time delay between the arrival of the high-frequency and low-frequency constituents. This may make it difficult to reconstruct a waveform, should the scalar mode be observed with a GW detector (Corda 2009a).

From the gauge condition equation (1.36) we find that k^{μ} is orthogonal to $\widehat{h}_{\mu\nu}$,

$$k^{\mu}\widehat{h}_{\mu\nu} = 0, \tag{1.47}$$

in this case

$$\widehat{h}_{0\nu} + \widehat{h}_{3\nu} = 0. \tag{1.48}$$

Let us consider the implications of equation (1.41) using (1.26) and (1.33),

$$\begin{aligned} \Box\left(4a_2 R^{(1)} + h\right) &= 0 \\ \Box h &= -\frac{4}{3}R^{(1)}. \end{aligned} \tag{1.49}$$

For nonzero $R^{(1)}$ (as required for the Ricci mode) there is no way to make a gauge choice such that the trace h vanishes (Corda 2008; Capozziello *et al.* 2008). This is distinct from in GR. It is possible, however, to make a gauge choice such that the trace $\overline{h}$ vanishes. Consider a gauge transformation generated by ξ_{μ} which satisfies $\Box\xi_{\mu} = 0$, and so has a Fourier decomposition

$$\xi_{\mu} = \widehat{\xi}_{\mu} \exp\left(ik_{\rho}x^{\rho}\right). \tag{1.50}$$

A transformation

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial^{\rho}\xi_{\rho}, \tag{1.51}$$

would ensure both conditions (1.36) and (1.42) are satisfied (Misner *et al.* 1973, section 35.2). Under such a transformation

$$\widehat{h}_{\mu\nu} \to \widehat{h}_{\mu\nu} + i\left(k_{\mu}\widehat{\xi}_{\nu} + k_{\nu}\widehat{\xi}_{\mu} - \eta_{\mu\nu}k^{\rho}\widehat{\xi}_{\rho}\right). \tag{1.52}$$

We may impose four further constraints (one for each $\widehat{\xi}_{\mu}$) upon $\widehat{h}_{\mu\nu}$. We take these to be

$$\widehat{h}_{0\nu} = 0, \qquad \widehat{h} = 0. \tag{1.53}$$

This might appear to be five constraints, but we have already imposed equation (1.48), and so setting $\widehat{h}_{00} = 0$ automatically implies $\widehat{h}_{03} = 0$. In this gauge

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - a_2 R^{(1)}\eta_{\mu\nu}; \quad h = -4a_2 R^{(1)}. \tag{1.54}$$

Thus $\overline{h}_{\mu\nu}$ behaves just as its GR counterpart; we can define

$$\left[\widehat{h}_{\mu\nu}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.55}$$

where h_+ and $h_\times$ are constants representing the amplitudes of the two transverse polarizations of gravitational radiation.

It is important that our solutions reduce to those of GR if $f(R) = R$. In our linearised approach this corresponds to $a_2 \to 0$, $\Upsilon^2 \to \infty$. We see from equation (1.45) that in this limit it would take an infinite frequency to excite a propagating Ricci mode, and evanescent waves would decay away infinitely fast. Therefore, there would be no detectable Ricci modes and we would only observe the two polarizations found in GR. Additionally, $\overline{h}_{\mu\nu}$ would simplify to its usual trace-reversed form.

1.5 Energy-momentum tensor

We expect gravitational radiation to carry energy-momentum. Unfortunately, it is difficult to define a proper energy-momentum tensor for a gravitational field: as a consequence of the equivalence principle it is possible to transform to a freely falling frame, eliminating the gravitational field and any associated energy density at a given point, although we can still define curvature in the neighbourhood of this point (Misner *et al.* 1973, section 20.4; Hobson *et al.* 2006, section 17.11). We do nothing revolutionary, but follow the approach of Isaacson (1968a, b). The full field equations (1.7) have no energy-momentum tensor for the gravitational field on the right-hand side; however, by expanding beyond the linear terms we can find a suitable effective energy-momentum tensor for GWs. Expanding $\mathcal{G}_{\mu\nu}$ in orders of $h_{\mu\nu}$

$$\mathcal{G}_{\mu\nu} = \mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} + \mathcal{G}^{(1)}{}_{\mu\nu} + \mathcal{G}^{(2)}{}_{\mu\nu} + \dots \tag{1.56}$$

We use (B) for the background term instead of (0) to avoid confusion regarding its order in ε. So far we have assumed that our background is flat; however, we can imagine that should the gravitational radiation carry energy-momentum then this would act as a source of curvature for the background (Wald 1984, section 4.4b). This is a second-order effect that may be encoded, to accuracy of $\mathcal{O}\left(\varepsilon^2\right)$, as

$$\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} = -\mathcal{G}^{(2)}{}_{\mu\nu}. \tag{1.57}$$

By shifting $\mathcal{G}^{(2)}{}_{\mu\nu}$ to the right-hand side we create an effective energy-momentum tensor. As in GR we average over several wavelengths, assuming that the background curvature is on a larger scale (Misner *et al.* 1973, section 35.13; Stein & Yunes 2011),

$$\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} = -\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle. \tag{1.58}$$

By averaging we probe the curvature in a macroscopic region about a given point in spacetime, yielding a gauge-invariant measure of the gravitational field strength. The averaging can be thought of as smoothing out the rapidly varying ripples of the radiation, leaving only the coarse-grained component that acts as a source for the background curvature.[3] The effective energy-momentum tensor for the radiation is

$$t_{\mu\nu} = -\frac{1}{8\pi G} \left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle. \tag{1.59}$$

Having made this provisional identification, we must set about carefully evaluating the various terms in equation (1.56). We begin as in section 1.3 by defining a total metric

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}, \tag{1.60}$$

where $\gamma_{\mu\nu}$ is the background metric. This changes our definition for $h_{\mu\nu}$: instead of being the total perturbation from flat Minkowski, it is the dynamical part of the metric with which we associate radiative effects. Since we know that $\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu}$ is $\mathcal{O}\left(\varepsilon^2\right)$, we decompose our background metric as

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}, \tag{1.61}$$

[3]By averaging we do not try to localise the energy of a wave to within a wavelength; for the massive Ricci scalar mode we always consider scales greater than λ_R.

where $j_{\mu\nu}$ is $\mathcal{O}\left(\varepsilon^2\right)$ to ensure that ${R^{(\mathrm{B})}}^{\lambda}{}_{\mu\nu\rho}$ is also $\mathcal{O}\left(\varepsilon^2\right)$. Therefore its introduction makes no difference to the linearised theory.

We consider terms only to $\mathcal{O}\left(\varepsilon^2\right)$. We identify ${\Gamma^{(1)}}^{\rho}{}_{\mu\nu}$ from equation (1.20).[4] We do not distinguish between ∂_μ and ${\nabla^{(\mathrm{B})}}_{\mu}$, the covariant derivative for the background metric: to the order of accuracy required covariant derivatives commute and ${\nabla^{(\mathrm{B})}}_{\mu}$ behaves just like ∂_μ. Thus

$$\begin{aligned}{\Gamma^{(1)}}^{\rho}{}_{\mu\nu} = \frac{1}{2}\gamma^{\rho\lambda}\Big[&\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2R^{(1)}\gamma_{\lambda\nu}\right) + \partial_\nu\left(\overline{h}_{\lambda\mu} - a_2R^{(1)}\gamma_{\lambda\mu}\right) \\ &- \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2R^{(1)}\gamma_{\mu\nu}\right)\Big],\end{aligned} \tag{1.62}$$

and

$$\begin{aligned}{\Gamma^{(2)}}^{\rho}{}_{\mu\nu} = {} & -\frac{1}{2}h^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\ = {} & -\frac{1}{2}\left(\overline{h}^{\rho\lambda} - a_2R^{(1)}\gamma^{\rho\lambda}\right)\Big[\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2R^{(1)}\gamma_{\lambda\nu}\right) + \partial_\nu\left(\overline{h}_{\lambda\mu} - a_2R^{(1)}\gamma_{\lambda\mu}\right) \\ & - \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2R^{(1)}\gamma_{\mu\nu}\right)\Big].\end{aligned} \tag{1.63}$$

For the Ricci tensor we can use our linearised expression, equation (1.37), for the first-order term

$${R^{(1)}}_{\mu\nu} = a_2\partial_\mu\partial_\nu R^{(1)} + \frac{1}{6}R^{(1)}\gamma_{\mu\nu}. \tag{1.64}$$

The next term is

$$\begin{aligned}{R^{(2)}}_{\mu\nu} = {} & \partial_\rho {\Gamma^{(2)}}^{\rho}{}_{\mu\nu} - \partial_\nu {\Gamma^{(2)}}^{\rho}{}_{\mu\rho} + {\Gamma^{(1)}}^{\rho}{}_{\mu\nu}{\Gamma^{(1)}}^{\sigma}{}_{\rho\sigma} - {\Gamma^{(1)}}^{\rho}{}_{\mu\sigma}{\Gamma^{(1)}}^{\sigma}{}_{\rho\nu} \\ = {} & \frac{1}{2}\Big\{\frac{1}{2}\partial_\mu\overline{h}_{\sigma\rho}\partial_\nu\overline{h}^{\sigma\rho} + \overline{h}^{\sigma\rho}\Big[\partial_\mu\partial_\nu\overline{h}_{\sigma\rho} + \partial_\sigma\partial_\rho\left(\overline{h}_{\mu\nu} - a_2R^{(1)}\gamma_{\mu\nu}\right) \\ & - \partial_\nu\partial_\rho\left(\overline{h}_{\sigma\mu} - a_2R^{(1)}\gamma_{\sigma\mu}\right) - \partial_\mu\partial_\rho\left(\overline{h}_{\sigma\nu} - a_2R^{(1)}\gamma_{\sigma\nu}\right)\Big] \\ & + \partial^\rho\overline{h}^{\sigma}_{\nu}\left(\partial_\rho\overline{h}_{\sigma\mu} - \partial_\sigma\overline{h}_{\rho\mu}\right) - a_2\partial^\sigma R^{(1)}\partial_\sigma\overline{h}_{\mu\nu} \\ & + a_2^2\left(2R^{(1)}\partial_\mu\partial_\nu R^{(1)} + 3\partial_\mu R^{(1)}\partial_\nu R^{(1)} + R^{(1)}\Box^{(\mathrm{B})}R^{(1)}\gamma_{\mu\nu}\right)\Big\}.\end{aligned} \tag{1.65}$$

The d'Alembertian is $\Box^{(\mathrm{B})} = \gamma^{\mu\nu}\partial_\mu\partial_\nu$.

To find the Ricci scalar we contract the Ricci tensor with the full metric. To $\mathcal{O}\left(\varepsilon^2\right)$,

$$R^{(\mathrm{B})} = \gamma^{\mu\nu}{R^{(\mathrm{B})}}_{\mu\nu} \tag{1.66a}$$

$$R^{(1)} = \gamma^{\mu\nu}{R^{(1)}}_{\mu\nu} \tag{1.66b}$$

$$\begin{aligned}R^{(2)} = {} & \gamma^{\mu\nu}{R^{(2)}}_{\mu\nu} - h^{\mu\nu}{R^{(1)}}_{\mu\nu} \\ = {} & \frac{3}{4}\partial_\mu\overline{h}_{\sigma\rho}\partial^\mu\overline{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho\overline{h}^{\sigma\mu}\partial_\sigma\overline{h}_{\rho\mu} - 2a_2\overline{h}^{\mu\nu}\partial_\mu\partial_\nu R^{(1)} \\ & + 2a_2{R^{(1)}}^2 + \frac{3a_2^2}{2}\partial_\mu R^{(1)}\partial^\mu R^{(1)}.\end{aligned} \tag{1.66c}$$

Using these

$$f^{(\mathrm{B})} = R^{(\mathrm{B})} \tag{1.67a}$$

$$f^{(1)} = R^{(1)} \tag{1.67b}$$

$$f^{(2)} = R^{(2)} + \frac{a_2}{2}{R^{(1)}}^2, \tag{1.67c}$$

[4] There is one small subtlety: whether we use the background metric $\gamma^{\mu\nu}$ or $\eta^{\mu\nu}$ to raise indices of ∂_μ and $h_{\mu\nu}$. Fortunately, to the accuracy considered here, it does not make a difference, but we shall consider the indices to be changed with $\gamma^{\mu\nu}$.

and

$$f'^{(\mathrm{B})} = a_2 R^{(\mathrm{B})} \tag{1.68a}$$
$$f'^{(0)} = 1 \tag{1.68b}$$
$$f'^{(1)} = a_2 R^{(1)} \tag{1.68c}$$
$$f'^{(2)} = a_2 R^{(2)} + \frac{a_3}{2} {R^{(1)}}^2. \tag{1.68d}$$

We list a zeroth-order term for clarity; $R^{(\mathrm{B})}$ is $\mathcal{O}\left(\varepsilon^2\right)$.

Combining all of these

$$\begin{aligned}
{\mathcal{G}^{(2)}}_{\mu\nu} = {} & {R^{(2)}}_{\mu\nu} + f'^{(1)} {R^{(1)}}_{\mu\nu} - \partial_\mu \partial_\nu f'^{(2)} + {\Gamma^{(1)}}^{\rho}{}_{\nu\mu} \partial_\rho f'^{(1)} + \gamma_{\mu\nu} \gamma^{\rho\sigma} \partial_\rho \partial_\sigma f'^{(2)} \\
& - \gamma_{\mu\nu} \gamma^{\rho\sigma} {\Gamma^{(1)}}^{\lambda}{}_{\sigma\rho} \partial_\lambda f'^{(1)} - \gamma_{\mu\nu} h^{\rho\sigma} \partial_\rho \partial_\sigma f'^{(1)} + h_{\mu\nu} \gamma^{\rho\sigma} \partial_\rho \partial_\sigma f'^{(1)} \\
& - \frac{1}{2} f^{(2)} \gamma_{\mu\nu} - \frac{1}{2} f^{(1)} h_{\mu\nu} \\
= {} & {R^{(2)}}_{\mu\nu} + a_2 \left(\gamma_{\mu\nu} \Box^{(\mathrm{B})} - \partial_\mu \partial_\nu \right) R^{(2)} - \frac{1}{2} R^{(2)} \gamma_{\mu\nu} \\
& + \frac{a_3}{2} \left(\gamma_{\mu\nu} \Box^{(\mathrm{B})} - \partial_\mu \partial_\nu \right) {R^{(1)}}^2 - \frac{1}{6} \overline{h}_{\mu\nu} R^{(1)} - a_2 \gamma_{\mu\nu} \overline{h}^{\sigma\rho} \partial_\sigma \partial_\rho R^{(1)} \\
& + \frac{a_2}{2} \partial^\rho R^{(1)} \left(\partial_\mu \overline{h}_{\rho\nu} + \partial_\nu \overline{h}_{\rho\mu} - \partial_\rho \overline{h}_{\mu\nu} \right) + a_2 \left(R^{(1)} {R^{(1)}}_{\mu\nu} + \frac{1}{4} {R^{(1)}}^2 \gamma_{\mu\nu} \right) \\
& - a_2^2 \left(\partial_\mu R^{(1)} \partial_\nu R^{(1)} + \frac{1}{2} \gamma_{\mu\nu} \partial^\rho R^{(1)} \partial_\rho R^{(1)} \right).
\end{aligned} \tag{1.69}$$

It is simplest to split this up for the purposes of averaging. Since we average over all directions at each point, gradients average to zero (Hobson *et al.* 2006, section 17.11; Stein & Yunes 2011)

$$\langle \partial_\mu V \rangle = 0. \tag{1.70}$$

As a corollary of this

$$\langle U \partial_\mu V \rangle = - \langle V \partial_\mu U \rangle. \tag{1.71}$$

Repeated application of this, together with our gauge condition (1.36), and wave equations (1.26) and (1.42) allows us to eliminate many terms. Those that do not average to zero are the last three terms in equation (1.69) and

$$\left\langle {R^{(2)}}_{\mu\nu} \right\rangle = \left\langle -\frac{1}{4} \partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} + \frac{a_2^2}{2} \partial_\mu R^{(1)} \partial_\nu R^{(1)} + \frac{a_2}{6} \gamma_{\mu\nu} {R^{(1)}}^2 \right\rangle; \tag{1.72a}$$
$$\left\langle R^{(2)} \right\rangle = \left\langle \frac{3a_2}{2} {R^{(1)}}^2 \right\rangle; \tag{1.72b}$$
$$\left\langle R^{(1)} {R^{(1)}}_{\mu\nu} \right\rangle = \left\langle a_2 R^{(1)} \partial_\mu \partial_\nu R^{(1)} + \frac{1}{6} \gamma_{\mu\nu} {R^{(1)}}^2 \right\rangle. \tag{1.72c}$$

Combining terms gives

$$\left\langle {\mathcal{G}^{(2)}}_{\mu\nu} \right\rangle = \left\langle -\frac{1}{4} \partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} - \frac{3a_2^2}{2} \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle. \tag{1.73}$$

Thus we obtain the result

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} + 6a_2^2 \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle. \tag{1.74}$$

In the limit of $a_2 \to 0$ we obtain the familiar GR result as required. The GR result is also recovered if $R^{(1)} = 0$, as would be the case if the Ricci mode was not excited; for example, if the frequency was below the cut-off frequency Υ.

Rewriting the effective energy-momentum tensor in terms of metric perturbation $h_{\mu\nu}$, using equation (1.54),

$$t_{\mu\nu} = \frac{1}{32\pi G}\left\langle \partial_\mu h_{\sigma\rho}\partial_\nu h^{\rho\sigma} + \frac{1}{8}\partial_\mu h \partial_\nu h \right\rangle. \tag{1.75}$$

These results do not depend upon a_3 or higher-order coefficients (Stein & Yunes 2011).

This result has been subsequently confirmed by Näf & Jetzer (2011). They used the Landau-Lifshitz complex (Landau & Lifshitz 1975, section 94) appropriately generalised for $f(R)$-gravity (Nutku 1969), to derive the result.[5] This is equivalent to the approach used by Will (1993, section 10.3) to derive the energy flux for scalar-tensor theories. The consistency between approaches is reassuring.

The effective energy-momentum tensor could be used to constrain the parameter a_2 through observations of the energy and momentum carried away by GWs. Of particular interest would be a system with a frequency that evolved from $\omega < \Upsilon$ to $\omega > \Upsilon$, as then we would witness the switching on of the propagating Ricci mode. If we could accurately identify the cut-off frequency we could accurately measure a_2. However, we shall see in section 2.3.3 that this is unlikely to happen.

[5]The Landau-Lifshitz complex is defined such that the (ordinary) derivative of it plus the energy-momentum vanishes. The sum defines an energy-momentum pseudo-tensor that is conserved in the familiar way: the rate of change of energy-momentum contained in a spacial volume is given by the flux through the surface of the volume. The complex is also symmetric in its indices to ensure conservation of angular momentum.

Chapter 2

Observational constraints for $f(R)$-gravity

In the previous chapter we introduced an extended theory of gravity, metric $f(R)$-gravity, and derived its behaviour in the linearised framework. We now continue, to find what constraints we could place on this theory to quantify deviations from GR. In section 2.1 we look at the effects of introducing a source term and derive the weak-field metrics for a point source, a slowly rotating point source, and a uniform density sphere, recovering some results known for quadratic theories of gravity. These are used in section 2.2 to compute the frequencies of radial and vertical epicyclic oscillations about circular-equatorial orbits in the weak-field, slow-rotation metric, and hence to construct an estimate of the detectability of the $f(R)$ deviations in LISA EMRI observations. For comparison, in section 2.3, we describe the constraints on $f(R)$-gravity that can be obtained from Solar System and laboratory tests. We conclude in section 2.4 with a summary of our findings.

2.1 $f(R)$-gravity with a source

Having considered radiation in a vacuum, we now include a source term. We want a first-order perturbation, so the linearised field equations are

$$\mathcal{G}^{(1)}{}_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.1}$$

We again assume a Minkowski background, considering terms to $\mathcal{O}(\varepsilon)$ only. To solve the wave equations (1.26) and (1.42) with this source term we use a Green's function

$$\left(\Box + \Upsilon^2\right) \mathscr{G}_\Upsilon(x, x') = \delta(x - x'), \tag{2.2}$$

where $\Box$ acts on x. The Green's function is familiar as the Klein-Gordon propagator (up to a factor of $-i$) (Peskin & Schroeder 1995, section 2.4)

$$\mathscr{G}_\Upsilon(x, x') = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp\left[-ip \cdot (x - x')\right]}{\Upsilon^2 - p^2}. \tag{2.3}$$

This can be evaluated by a suitable contour integral to give

$$\mathscr{G}_\Upsilon(x, x') = \begin{cases} \displaystyle\int \frac{\mathrm{d}\omega}{2\pi} \exp\left[-i\omega(t - t')\right] \frac{1}{4\pi r} \exp\left[i\left(\omega^2 - \Upsilon^2\right)^{1/2} r\right] & \omega^2 > \Upsilon^2 \\ \displaystyle\int \frac{\mathrm{d}\omega}{2\pi} \exp\left[-i\omega(t - t')\right] \frac{1}{4\pi r} \exp\left[-\left(\Upsilon^2 - \omega^2\right)^{1/2} r\right] & \omega^2 < \Upsilon^2 \end{cases}, \tag{2.4}$$

where we have introduced $t = x^0$, $t' = x'^0$ and $r = |\boldsymbol{x} - \boldsymbol{x'}|$. For $\Upsilon = 0$,

$$\mathscr{G}_0(x, x') = \frac{\delta(t - t' - r)}{4\pi r}, \tag{2.5}$$

the familiar retarded-time Green's function. We can use this to solve equation (1.42)

$$\begin{aligned}\overline{h}_{\mu\nu}(x) = & -16\pi G \int \mathrm{d}^4x' \, \mathscr{G}_0(x,x') T_{\mu\nu}(x') \\ = & -4G \int \mathrm{d}^3x' \frac{T_{\mu\nu}(t-r,\boldsymbol{x'})}{r}.\end{aligned} \tag{2.6}$$

This is exactly as in GR, so we can use standard results.

Solving for the scalar mode:

$$R^{(1)}(x) = -8\pi G \Upsilon^2 \int \mathrm{d}^4x' \, \mathscr{G}_\Upsilon(x,x') T(x'). \tag{2.7}$$

To proceed further we must know the form of the trace $T(x')$. In general the form of $R^{(1)}(x)$ is complicated.

2.1.1 The Newtonian limit

Let us consider the limiting case of a Newtonian source, such that

$$T_{00} = \rho; \quad |T_{00}| \gg |T_{0i}|; \quad |T_{00}| \gg |T_{ij}|, \tag{2.8}$$

with a mass distribution of a stationary point source

$$\rho = M\delta(\boldsymbol{x'}). \tag{2.9}$$

This source does not produce any radiation. As in GR

$$\overline{h}_{00} = -\frac{4GM}{r}; \qquad \overline{h}_{0i} = \overline{h}_{ij} = 0. \tag{2.10}$$

Solving for the Ricci scalar (Havas 1977)

$$R^{(1)} = -2G\Upsilon^2 M \frac{\exp(-\Upsilon r)}{r}. \tag{2.11}$$

Combining these in equation (1.39b) yields a metric perturbation with nonzero elements

$$h_{00} = -\frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]; \quad h_{ij} = -\frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\delta_{ij}. \tag{2.12}$$

Thus, to first order, the metric for a point mass in $f(R)$-gravity is

$$\mathrm{d}s^2 = \left\{1 - \frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}t^2 - \left\{1 + \frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}\Sigma^2, \tag{2.13}$$

using $\mathrm{d}\Sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ (Capozziello *et al.* 2007, 2009; Näf & Jetzer 2010). This is not the linearised limit of the Schwarzschild metric, although this is recovered as $a_2 \to 0$, $\Upsilon \to \infty$ (Chiba *et al.* 2007). This metric has already been derived for the case of quadratic gravity, which includes terms like R^2 and $R_{\mu\nu}R^{\mu\nu}$ in the Lagrangian (Pechlaner & Sexl 1966; Stelle 1978; Schmidt 1986; Teyssandier 1990). In linearised theory our $f(R)$ reduces to quadratic theory, as to first order $f(R) = R + a_2R^2/2$.

Extending this result to a slowly rotating source with angular momentum J, we then have the additional term (Hobson *et al.* 2006, section 13.20)

$$\overline{h}^{0i} = -\frac{2G}{c^2r^3}\epsilon^{ijk}J_j x_k, \tag{2.14}$$

where ϵ^{ijk} is the Levi-Civita alternating tensor. The metric is

$$\begin{aligned}\mathrm{d}s^2 = &\left\{1 - \frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}t^2 + \frac{4GJ}{r^3}\left(x\mathrm{d}y - y\mathrm{d}x\right)\mathrm{d}t \\ &- \left\{1 + \frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}\Sigma^2, \end{aligned} \tag{2.15}$$

where z is the rotation axis. This is not the first-order limit of the Kerr metric (aside from as $a_2 \to 0$, $\Upsilon \to \infty$).

In $f(R)$-gravity Birkhoff's theorem no longer applies (Pechlaner & Sexl 1966; Stelle 1978; Clifton 2006; Capozziello & Stabile 2009; Stabile 2010): the metric about a spherically symmetric mass does not correspond to the equivalent of the Schwarzschild solution. The distribution of matter influences how the Ricci scalar decays, and consequently Gauss' theorem is not applicable. Repeating our analysis for a (nonrotating) sphere of uniform density and radius L,

$$\overline{h}_{00} = -\frac{4GM}{r}; \qquad \overline{h}_{0i} = \overline{h}_{ij} = 0, \tag{2.16}$$

as in GR, and for the point mass, but

$$\begin{aligned} R^{(1)} = &-6GM\frac{\exp(-\Upsilon r)}{r}\left[\frac{\Upsilon L\cosh(\Upsilon L) - \sinh(\Upsilon L)}{\Upsilon L^3}\right] \\ = &-6GM\frac{\exp(-\Upsilon r)}{r}\Upsilon^2\Xi(\Upsilon L), \end{aligned} \tag{2.17}$$

defining $\Xi(\Upsilon L)$ in the last line.[1] The metric perturbation thus has nonzero first-order elements (Stelle 1978; Capozziello & Stabile 2009; Stabile 2010)

$$h_{00} = -2GM\left[1 + \exp(-\Upsilon r)\Xi(\Upsilon L)\right]; \quad h_{ij} = -2GM\left[1 - \exp(-\Upsilon r)\Xi(\Upsilon L)\right]\delta_{ij}, \tag{2.18}$$

where we have assumed that $r > L$ at all stages.[2]

Solving the full field equations to find the exact metric in $f(R)$ is difficult because of the higher-order derivatives that enter the equations. However, we expect a solution to have the appropriate limiting form as given above.

It has been suggested that since $R = 0$ is a valid solution to the vacuum equations, the BH solutions of GR are also the BH solutions in $f(R)$ (Psaltis *et al.* 2008; Barausse & Sotiriou 2008). We have seen that having a non-zero stress-energy tensor at the origin, because of equation (1.26), forces R to be nonzero in the surrounding vacuum, although it decays to zero at infinity (Olmo 2007). Therefore, it is not obvious that the end-state of gravitational collapse must be a GR solution and that it could not settle to a different solution.[3]

However, a uniqueness theorem exists for the closely related Brans-Dicke theory (Hawking 1972; Bekenstein & Meisels 1978; Thorne & Dykla 1971; Scheel *et al.* 1995), and recently this has been extended to $f(R)$-gravity, assuming only the stationarity of the solution (Sotiriou & Faraoni 2012). Therefore astrophysical BHs in $f(R)$-gravity are also described by the Kerr solution. We can only detect differences in the properties of extended sources.

2.1.2 The weak-field metric

It is useful to transform the weak-field metric, equation (2.15), to the more familiar form

$$\mathrm{d}s^2 = A(\widetilde{r})\mathrm{d}t^2 + \frac{4GJ}{\widetilde{r}}\sin^2\theta\mathrm{d}\phi\mathrm{d}t - B(\widetilde{r})\mathrm{d}\widetilde{r}^2 - \widetilde{r}^2\mathrm{d}\Omega^2. \tag{2.19}$$

[1] $\Xi(0) = 1/3$ is the minimum of $\Xi(\Upsilon L)$.

[2] Inside the source $R^{(1)} = -(6GM/L^3)[1 - (\Upsilon L + 1)\exp(-\Upsilon L)\sinh(\Upsilon r)/\Upsilon r]$.

[3] We cannot simply extrapolate from our δ-function solution, as it is necessary to consider the junction conditions required for a physical solution (Deruelle *et al.* 2008).

The coordinate $\tilde{r}$ is a circumferential measure, as in the Schwarzschild metric, as opposed to r, used in preceding sections, which is a radial distance (an isotropic coordinate) (Misner *et al.* 1973, section 40.1; Olmo 2007). To simplify the algebra we introduce the Schwarzschild radius

$$r_\mathrm{S} = 2GM. \tag{2.20}$$

In the linearised regime, we require that the new radial coordinate satisfies

$$\tilde{r}^2 = \left\{1 + \frac{r_\mathrm{S}}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\right\} r^2 \tag{2.21}$$

$$\tilde{r} = r + \frac{r_\mathrm{S}}{2}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]. \tag{2.22}$$

This can be used as an implicit definition of r in terms of $\tilde{r}$. To first order in r_S/r (Olmo 2007)

$$A(\tilde{r}) = 1 - \frac{r_\mathrm{S}}{\tilde{r}}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]. \tag{2.23}$$

We see that the functional form of g_{00} is almost unchanged upon substituting $\tilde{r}$ for r, but r is left in the exponential.

To find $B(\tilde{r})$ we consider, using equation (2.22),

$$\frac{\mathrm{d}\tilde{r}}{\tilde{r}} = \mathrm{d}\ln\tilde{r} = \left\{\frac{1 + \Upsilon r_\mathrm{S} r \exp(-\Upsilon r)/6\tilde{r}}{1 + (r_\mathrm{S}/2\tilde{r})\left[1 - \exp(-\Upsilon r)/3\right]}\right\}\frac{\mathrm{d}r}{r}. \tag{2.24}$$

Thus

$$\mathrm{d}\tilde{r}^2 = \frac{\tilde{r}^2}{r^2}\left\{\frac{1 + \Upsilon r_\mathrm{S} r \exp(-\Upsilon r)/6\tilde{r}}{1 + (r_\mathrm{S}/2\tilde{r})\left[1 - \exp(-\Upsilon r)/3\right]}\right\}^2 \mathrm{d}r^2. \tag{2.25}$$

The term in braces is $[B(\tilde{r})]^{-1}$. We assume that in the weak-field

$$\varepsilon \sim \frac{r_\mathrm{S}}{r} \tag{2.26}$$

is small; then the metric perturbations from Minkowski are small. Expanding to first order (Olmo 2007)

$$B(\tilde{r}) = 1 + \frac{r_\mathrm{S}}{\tilde{r}}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right] - \frac{\Upsilon r_\mathrm{S}\exp(-\Upsilon r)}{3}. \tag{2.27}$$

In the limit $\Upsilon \to \infty$, where we recover GR, $A(\tilde{r})$ and $B(\tilde{r})$ tend to their Kerr (Schwarzschild) forms.

In the following sections we use these weak-field metrics (in both coordinates) with astrophysical and laboratory tests of gravity to place constraints on $f(R)$.

2.2 Epicyclic frequencies

One means of probing the nature of a spacetime is through observations of orbital motions (Gair *et al.* 2008). We consider the epicyclic motion produced by perturbing a circular orbit. There are two epicyclic frequencies associated with any circular-equatorial orbit, characterising perturbations in the radial and vertical directions respectively (Binney & Tremaine 2008, section 3.2.3). We start by deriving a general result for any metric of the form of equation (2.19), and then specialise to our $f(R)$ solution. We work in the slow-rotation limit, keeping only linear terms in J.

An orbit in a spacetime described by equation (2.19) has as constants of motion: the orbiting particle's rest mass, the energy (per unit mass) of the orbit E and the z-component of the angular momentum (per unit mass) L_z. Using an over-dot to denote differentiation with respect to an affine parameter, which we identify as proper time τ,

$$E = A\dot{t} + \frac{2GJ}{\widetilde{r}} \sin^2\theta\dot{\phi}; \tag{2.28}$$

$$L_z = \widetilde{r}^{\,2}\sin^2\theta\,\dot{\phi} - \frac{2GJ}{\widetilde{r}}\sin^2\theta\dot{t}. \tag{2.29}$$

For circular equatorial orbits $\dot{\widetilde{r}} = \ddot{\widetilde{r}} = \dot{\theta} = 0$ and $\theta = \pi/2$. The time-like geodesic equation can be written in the covariant form

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \frac{1}{2}\left(\partial_\mu g_{\rho\sigma}\right) u^\rho u^\sigma, \tag{2.30}$$

where u^μ is the 4-velocity. For a circular equatorial orbit, setting $\mu = \widetilde{r}$ gives the frequency of the orbit $\omega_0 = \mathrm{d}\phi/\mathrm{d}t$ as

$$\omega_0 = -\frac{GJ}{\widetilde{r}^{\,3}} \pm \frac{1}{2}\sqrt{\frac{2A'}{\widetilde{r}} + \left(\frac{2GJ}{\widetilde{r}^{\,3}}\right)^2}, \tag{2.31}$$

in which a prime denotes $\mathrm{d}/\mathrm{d}\widetilde{r}$ and the $+/-$ sign denotes prograde/retrograde orbits. The definition of proper time gives

$$\dot{t} = \left(A + \frac{4GJ\omega_0}{\widetilde{r}} - \widetilde{r}^{\,2}\omega_0^2\right)^{-1/2}. \tag{2.32}$$

We now have both $\dot{t}$ and $\dot{\phi} = \omega_0\dot{t}$ as functions of $\widetilde{r}$; substitution into equations (2.28) and (2.29) allows us to find the energy and angular momentum in terms of $\widetilde{r}$.

From the Hamiltonian $\mathcal{H} = g_{\mu\nu}u^\mu u^\nu$ we can obtain the general equation of motion for massive particles, using the substitutions

$$\dot{t} = \frac{E}{A} - \frac{2GJ}{A\widetilde{r}^{\,3}}L_z, \tag{2.33}$$

$$\dot{\phi} = \frac{2GJE}{A\widetilde{r}^{\,3}} + \frac{L_z}{\widetilde{r}^{\,2}\sin^2\theta}, \tag{2.34}$$

where we have linearised in J, as appropriate for the slow-rotation limit. With these replacements, the general time-like geodesic equation takes the form

$$\begin{aligned}\dot{\widetilde{r}}^{\,2} + \frac{\widetilde{r}^{\,2}}{B}\dot{\theta}^2 &= \frac{E^2}{AB} - \frac{4GJEL_z}{AB\widetilde{r}^{\,3}} - \frac{1}{B}\left(1 + \frac{L_z^2}{\widetilde{r}^{\,2}\sin^2\theta}\right) \\ &= V(\widetilde{r}, \theta, E, L_z).\end{aligned} \tag{2.35}$$

To compute the epicyclic frequency we imagine the orbit is perturbed by a small amount, while E and L_z are unchanged.[4] For radial perturbations $\widetilde{r} = \overline{r}(1+\delta)$, where $\overline{r}$ is the radius of the unperturbed orbit, the orbit undergoes small oscillations with frequency

$$\dot{t}^2\Omega_{\mathrm{rad}}^2 = -\frac{1}{2}\left.\frac{\partial^2 V}{\partial\widetilde{r}^{\,2}}\right|_{\overline{r},\,\theta=\pi/2}. \tag{2.36}$$

Small vertical perturbations $\theta = \pi/2 + \delta$ oscillate with frequency

$$\dot{t}^2\Omega_{\mathrm{vert}}^2 = -\frac{1}{2}\frac{B(\overline{r})}{\overline{r}^2}\left.\frac{\partial^2 V}{\partial\theta^2}\right|_{\overline{r},\,\theta=\pi/2}. \tag{2.37}$$

[4] It is not possible for the orbit to be perturbed without changing the energy or angular momentum. However, these corrections are quadratic in the amplitude of the perturbation, and so can be ignored them at linear order.

We denote $A(\overline{r}) \equiv \overline{A}$, $B(\overline{r}) \equiv \overline{B}$, $A'(\overline{r}) \equiv \overline{A}'$, etc.; differentiating the potential from equation (2.35) we find

$$\begin{aligned}
\dot{t}^2\Omega_{\mathrm{rad}}^2 = & -\frac{E^2}{\overline{A}\overline{B}}\left(\frac{\overline{A}'^2}{\overline{A}^2} - \frac{\overline{A}''}{2\overline{A}} + \frac{\overline{A}'\overline{B}'}{\overline{A}\,\overline{B}} + \frac{\overline{B}'^2}{\overline{B}^2} - \frac{\overline{B}''}{2\overline{B}}\right) - \frac{\overline{B}''}{2\overline{B}^2} + \frac{\overline{B}'^2}{\overline{B}^3} \\
& - \frac{L_z^2}{\overline{B}\overline{r}^2}\left(\frac{\overline{B}''}{2\overline{B}} - \frac{\overline{B}'^2}{\overline{B}^2} - \frac{2\overline{B}'}{\overline{B}\overline{r}} - \frac{3}{\overline{r}^2}\right) \\
& + \frac{4GJEL_z}{\overline{A}\,\overline{B}\overline{r}^3}\left[\frac{\overline{A}'^2}{\overline{A}^2} - \frac{\overline{A}''}{2\overline{A}} + \frac{\overline{A}'\overline{B}'}{\overline{A}\,\overline{B}} + \frac{\overline{B}'^2}{\overline{B}^2} - \frac{\overline{B}''}{2\overline{B}} + \frac{3}{\overline{r}}\left(\frac{\overline{A}'}{\overline{A}} + \frac{\overline{B}'}{\overline{B}}\right) + \frac{6}{\overline{r}^2}\right] \qquad (2.38) \\
= & \frac{L_z^2}{\overline{r}^3\overline{B}}\left(\frac{\overline{A}''}{\overline{A}'} - \frac{2\overline{A}'}{\overline{A}} + \frac{3}{\overline{r}}\right) + \frac{6GJEL_z}{\overline{A}\,\overline{B}\overline{r}^4}\left(\frac{\overline{A}''}{\overline{A}'} + \frac{4}{\overline{r}}\right); \qquad (2.39) \\
\dot{t}\Omega_{\mathrm{vert}} = & \frac{L_z}{\overline{r}^2}. \qquad (2.40)
\end{aligned}$$

To simplify equation (2.38) we used conditions imposed by setting V and $\partial V/\partial\widetilde{r}$ equal to zero for circular equatorial orbits. These results hold for any metric of the general form equation (2.19), subject to the slow-rotation condition, which we have used to linearise in J at various stages.

2.2.1 Gravitational-wave constraints

We now consider if the deviation arising from the $f(R)$ correction could be observable. This should be possible if the orbit is sufficiently different from its counterpart in Kerr. To quantify the difference, we must identify equivalent orbits in the two spacetimes. For circular equatorial orbits there is a natural way to do this: by identifying orbits with the same frequency ω_0, as this is a gauge invariant observable quantity (Detweiler 2008). The quantity

$$\Delta(\omega_0, \Upsilon) = \Omega(\omega_0, \Upsilon) - \Omega(\omega_0, \Upsilon \to \infty) \qquad (2.41)$$

characterises the rate of increase in the phase difference between the $f(R)$ trajectory and the Kerr trajectory with the same frequency and spin parameter.[5]

Consider the GWs emitted by an object undergoing epicyclic motion. We assume that we are in the extreme-mass-ratio regime, such that we can ignore the gravity of the orbiting body. The correction is detectable if it leads to a significant phase shift in a gravitational waveform over the length of an observation; we adopt the criterion that it is detectable if $T_{\mathrm{obs}}\Delta > 2\pi$ for observation period T_{obs}. This is a significant over-simplification. We have assumed that the orbital frequency has been matched to a Kerr value, but small changes in the other parameters such as the central object's mass or spin, the orbital eccentricity or the inclination, could mimic (or disguise) the effect. However, we are also keeping the orbital frequency fixed whereas we shall observe inspirals. This can break the parameter degeneracies. Since we are interested in extreme-mass-ratio systems, for which the inspiral proceeds slowly, it is likely that we are being over-optimistic, so these results can be considered upper bounds on what could be measurable.

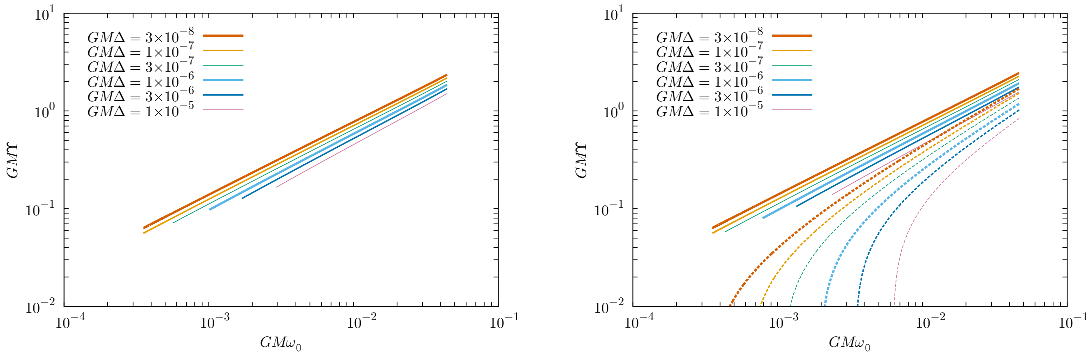
EMRIs are a potential source of observation for LISA. However, these will be for systems about massive BHs. As BH spacetimes are no different in $f(R)$-gravity, we must assume that the central object is an extended body: it must be an exotic object such as a boson star. There is no evidence for the existence of these non-standard objects. Therefore, this analysis is only included as a sketch of what could be potentially achievable if this were the case.

The time-scale of the systems we are considering is set by the central object mass, and the quantities $M\omega_0$ and $M\Delta$ are mass-independent. A duration of a typical EMRI observation with LISA would be of the order of a year, and so the criterion for detectability becomes

$$GM\Delta = 9.8 \times 10^{-7}\left(\frac{M}{10^6 M_\odot}\right)\left(\frac{\mathrm{yr}}{T_{\mathrm{obs}}}\right). \qquad (2.42)$$

[5]By comparing with the $\Upsilon \to \infty$ limit of the trajectory rather than the exact Kerr result ensures that we are taking the same slow rotation limit in both cases, and we need not be concerned with $\mathcal{O}\left(J^2\right)$ corrections.

In figure 2.1 we show the region of Υ-ω_0 parameter space in which corrections could be distinguished from Kerr, as defined by this criterion. Each curve represents a particular choice



(a) For spin $a_* = 0$.

(b) For spin $a_* = 0.5$.

Figure 2.1: Region of parameter space in which $f(R)$ and Kerr trajectories can be distinguished. Curve corresponds to different values of the detectability criterion equation (2.42), given in the key. Dashed lines are measurements of the vertical epicyclic frequency, solid lines are for measurements of the radial epicyclic frequency. The region below a curve could be distinguishable in a LISA observation with that detectability.

for $GM\Delta$: the region below the curve is detectable in an observation characterised by that choice. Equation (2.42) indicates that the curve $GM\Delta = 10^{-6}$ is what would be achieved in a one-year observation for a $10^6 M_\odot$ mass central object. The curves $GM\Delta = 10^{-5}/10^{-7}$ are the corresponding results for a $10^7/10^5 M_\odot$ mass object, while the curve $GM\Delta = 3 \times 10^{-7}$ represents what would be achieved in a three-year observation. We show results for two choices of spin: $a_* = J/GM^2 = 0$ and $a_* = 0.5$. There is not much difference between the two. The vertical epicyclic frequency is only measurable for $a_* \neq 0$ as it coincides with the orbital frequency for $a_* = 0$ as a consequence of the spherical symmetry. Results are shown only for prograde orbits. For $a_* \neq 0$, we can compute results for retrograde orbits; these differ from the prograde results by an amount which is almost indistinguishable on the scale of the plots.

From figure 2.1 we conclude that we could be able to distinguish spacetimes with $GM\Upsilon \lesssim 1$. For a $10^6 M_\odot$ central object this corresponds to $\Upsilon \lesssim 10^{-9}$ m^{-1}. Larger values are accessible at higher frequencies, but the inspiral would pass through that region quickly, and these orbits correspond to relatively small radii at which the weak-field approximations begin to break down, so we must be cautious extrapolating these results. Using this detectability criterion, the radial epicyclic frequency is always a more powerful probe than the vertical epicyclic frequency. This is expected; the latter is generally smaller in magnitude and so accumulates fewer cycles over a typical observation.

2.3 Solar System and laboratory tests

2.3.1 Light bending & the post-Newtonian parameter γ

The parametrized post-Newtonian (PPN) formalism was created to quantify deviations from GR (Will 1993, chapter 4; Will 2006). It is ideal for Solar System tests. The only parameter we need to consider here is γ, which measures the space-curvature produced by unit rest mass. The PPN metric has components

$$g_{00}^{\mathrm{PPN}} = 1 - 2U; \qquad g_{ij}^{\mathrm{PPN}} = -(1 + 2\gamma U)\delta_{ij}, \tag{2.43}$$

where for a point mass

$$U(r) = \frac{GM}{r}. \tag{2.44}$$

The metric must be in isotropic coordinates (Misner *et al.* 1973, section 40.1; Will 1993, section 4.1(c)). The $f(R)$ metric equation (2.13) is of a similar form, but there is not a direct correspondence because of the exponential.[6] It has been suggested that this may be incorporated by changing the definition of the potential U (Olmo 2007; Faulkner *et al.* 2007; Bisabr 2010; De Felice & Tsujikawa 2010), then

$$\gamma = \frac{3 - \exp(-\Upsilon r)}{3 + \exp(-\Upsilon r)}. \tag{2.45}$$

As $\Upsilon \to \infty$, the GR value of $\gamma = 1$ is recovered. However, the experimental bounds for γ are derived assuming that it is a constant (Will 1993, section 6.1). Since this is not the case, we shall rederive the post-Newtonian, or $\mathcal{O}(\varepsilon)$, corrections to photon trajectories for a more general metric. We define

$$\mathrm{d}s^2 = [1 + 2\Psi(r)]\,\mathrm{d}t^2 - [1 - 2\Phi(r)]\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{2.46}$$

To post-Newtonian order, this has nonzero connection coefficients

$$\Gamma^0{}_{0i} = \frac{\Psi'(r)x^i}{r}; \quad \Gamma^i{}_{00} = \frac{\Psi'(r)x^i}{r}; \quad \Gamma^i{}_{jk} = \frac{\Phi'(r)\left(\delta_{jk}x^i - \delta_{ij}x^k - \delta_{ik}x^j\right)}{r}. \tag{2.47}$$

The photon trajectory is described by the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0, \tag{2.48}$$

for affine parameter σ. The time coordinate obeys

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\sigma^2} + \Gamma^0{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0, \tag{2.49}$$

so we can rewrite the spatial components of equation (2.48) using t as an affine parameter (Will 1993, section 6.1)

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \left(\Gamma^i{}_{\nu\rho} - \Gamma^0{}_{\nu\rho}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = 0. \tag{2.50}$$

Since the geodesic is null we also have

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0. \tag{2.51}$$

To post-Newtonian accuracy these become

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\left(\frac{\Psi'}{r} + \frac{\Phi'}{r}\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|^2\right)x^i + 2\frac{\Psi' + \Phi'}{r}\boldsymbol{x}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}, \tag{2.52}$$

$$0 = 1 + 2\Phi - (1 - 2\Phi)\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|^2. \tag{2.53}$$

The Newtonian, or zeroth-order, solution of these is propagation in a straight line at constant speed (Will 1993, section 6.1)

$$x^i_{\mathrm{N}} = n^i t; \qquad |\boldsymbol{n}| = 1. \tag{2.54}$$

The post-Newtonian trajectory can be written as

$$x^i = n^i t + x^i_{\mathrm{pN}} \tag{2.55}$$

[6] Our $f(R)$ theory is equivalent to a Brans-Dicke theory with a potential and parameter $\omega_{\mathrm{BD}} = 0$ (Teyssandier & Tourrenc 1983; Wands 1994). We cannot use the familiar result $\gamma = (1 + \omega_{\mathrm{BD}})/(2 + \omega_{\mathrm{BD}})$ (Will 2006) as this was derived for Brans-Dicke theory without a potential (Will 1993, section 5.3).

where $x^i_{\rm pN}$ is the deviation from the straight line. Substituting this into equations (2.52) and (2.53) gives

$$\frac{\mathrm{d}^2\boldsymbol{x}_{\rm pN}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}(\Psi+\Phi) + 2\boldsymbol{n}\cdot\boldsymbol{\nabla}(\Psi+\Phi)\boldsymbol{n}, \tag{2.56}$$

$$\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{x}_{\rm pN}}{\mathrm{d}t} = \Psi+\Phi. \tag{2.57}$$

The post-Newtonian deviation only depends upon the combination $\Psi+\Phi$. From equation (2.13)

$$\begin{aligned}\Psi(r)+\Phi(r) &= -\frac{2GM}{r}\\ &= -2U(r).\end{aligned} \tag{2.58}$$

This is identical to the form in GR. The result holds not just for a point mass, using equation (1.39b),

$$\begin{aligned}2\left(\Psi+\Phi\right) &= h_{00}+h_{ii} \qquad \text{(no summation)}\\ &= \overline{h}_{00}+\overline{h}_{ii},\end{aligned} \tag{2.59}$$

and since $\overline{h}_{\mu\nu}$ obeys equation (1.42) exactly as in GR, there is no difference (Zhao *et al.* 2011). We conclude that an appropriate definition for the post-Newtonian parameter for light-bending is

$$\gamma = -\frac{g_{00}+g_{ii}}{2U} - 1 \qquad \text{(no summation)}. \tag{2.60}$$

Using this, our $f(R)$ solutions have $\gamma = 1$. This agrees with the result found by Clifton (2008).[7] Consequently, $f(R)$-gravity is indistinguishable from GR in this respect and is entirely consistent with the current observational value of $\gamma = 1 + (2.1\pm2.3)\times10^{-5}$ (Will 2006; Bertotti *et al.* 2003).

This result has important implications. To first order, the gravitational lensing of light in $f(R)$-gravity is identical to that in GR. Therefore, we still need dark matter to explain lensing observations of galaxies and galaxy clusters (Lubini *et al.* 2011). $f(R)$-gravity is not an alternative to dark matter.

There are non-linear signatures of $f(R)$-gravity that may be observable with lensing measurements; however, these are too small to be currently observable (Vanderveld *et al.* 2011). We must use other experiments to put constraints upon $f(R)$.

2.3.2 Planetary precession

The epicyclic frequencies derived in section 2.2 can be used for the classic test of planetary precession in the Solar System.[8] Radial motion perturbs the orbit into an ellipse. The amplitude of our perturbation δ gives the eccentricity e of the ellipse (Kerner *et al.* 2001). Unless $\omega_0 = \Omega_{\rm rad}$ the epicyclic motion is asynchronous with the orbital motion: there is periapse precession. In one revolution the ellipse precesses about the focus by

$$\varpi = 2\pi\left(\frac{\omega_0}{\Omega_{\rm rad}} - 1\right) \tag{2.61}$$

where ω_0 is the frequency of the circular orbit given in equation (2.31). The precession is cumulative, so a small deviation may be measurable over sufficient time. Taking the non-rotating limit, the epicyclic frequency is

$$\Omega^2_{\rm rad} = \omega_0^2\left[1 - \frac{3r_{\rm S}}{\overline{r}} - \zeta(\Upsilon, r_{\rm S}, \overline{r})\right], \tag{2.62}$$

[7] Clifton (2008) also gives PPN parameters $\beta = 1$, $\zeta_1 = 0$, $\zeta_3 = 0$ and $\zeta_4 = 0$, all identical to the values in GR.

[8] Since the Sun is an extended body, we do not have to worry about BH solutions being identical in GR and $f(R)$-gravity here.

defining the function

$$\zeta = r_{\mathrm{S}}\left(\frac{1}{\widetilde{r}} + \Upsilon\right)\frac{\exp(-\Upsilon r)}{3} + \frac{\Upsilon^2\widetilde{r}^2\exp(-\Upsilon r)}{3 + (1+\Upsilon\widetilde{r})\exp(-\Upsilon r)}\left[1 - \frac{r_{\mathrm{S}}}{\widetilde{r}} + r_{\mathrm{S}}\left(\frac{1}{\widetilde{r}} + \Upsilon\right)\frac{\exp(-\Upsilon r)}{3}\right]. \tag{2.63}$$

This characterises the deviation from the Schwarzschild case: the change in the precession per orbit relative to Schwarzschild is

$$\Delta\varpi = \varpi - \varpi_{\mathrm{S}} \tag{2.64}$$

$$= \pi\zeta, \tag{2.65}$$

using the subscript S to denote the Schwarzschild value. To obtain the last line we have expanded to lowest order, assuming that ζ is small.[9] Since $\zeta \geq 0$, the precession rate is enhanced relative to GR.

Table 2.1 shows the orbital properties of the planets. We use the deviation in perihelion precession rate from the GR prediction to constrain the value of ζ, and hence Υ and a_2. All

Planet	Semimajor axis $r/10^{11}$ m	Orbital period $(2\pi/\omega_0)$/yr	Precession rate $\Delta\varpi \pm \sigma_{\Delta\varpi}$/mas yr^{-1}	Eccentricity e
Mercury	0.57909175	0.24084445	-0.040 ± 0.050	0.20563069
Venus	1.0820893	0.61518257	0.24 ± 0.33	0.00677323
Earth	1.4959789	0.99997862	0.06 ± 0.07	0.01671022
Mars	2.2793664	1.88071105	-0.07 ± 0.07	0.09341233
Jupiter	7.7841202	11.85652502	0.67 ± 0.93	0.04839266
Saturn	14.267254	29.42351935	-0.10 ± 0.15	0.05415060
Uranus	28.709722	83.74740682	-38.9 ± 39.0	0.04716771
Neptune	44.982529	163.7232045	-44.4 ± 54.0	0.00858587
Pluto	59.063762	248.0208	28.4 ± 45.1	0.24880766

Table 2.1: Orbital properties of the eight major planets and Pluto. We take the semimajor orbital axis to be the flat-space distance r, not the coordinate $\widetilde{r}$. The eccentricity is not used in calculations, but is given to assess the accuracy of neglecting terms $\mathcal{O}\left(e^2\right)$. Semimajor axis, orbital period and eccentricity are taken from Cox (2000), the precession rate is from Pitjeva (2009)

the precession rates are consistent with GR predictions ($\Delta\varpi = 0$) within their uncertainties. Assuming that these uncertainties constrain the possible deviation from GR we can use them as bounds for $f(R)$ corrections. Table 2.2 shows the constraints for Υ and a_2 obtained by equating the uncertainty in the precession rate $\sigma_{\Delta\varpi}$ with the $f(R)$ correction, and similarly using twice the uncertainty $2\sigma_{\Delta\varpi}$. The tightest constraint is obtained from the orbit of Mercury. Adopting a value of $\Upsilon \geq 5.3 \times 10^{-10}$ m^{-1}, the cut-off frequency for the Ricci mode is ≥ 0.16 s^{-1}. Therefore it could lie in the upper range of the LISA frequency band (Bender *et al.* 1998; Danzmann & Rüdiger 2003) or in the ground-based detector frequency range (Abramovici *et al.* 1992; Abbott *et al.* 2009; Accadia *et al.* 2010). The constraints are not as tight as those from GW observations; however, as we shall see in section 2.3.3, it is possible to place stronger constraints on Υ using laboratory experiments.

2.3.3 Fifth-force tests

From the metric (2.13), a point mass has a Yukawa gravitational potential (Stelle 1978; Capozziello *et al.* 2009; Näf & Jetzer 2010)

$$V(r) = \frac{GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]. \tag{2.66}$$

[9] There is one term in ζ that is not explicitly $\mathcal{O}(\varepsilon)$. Numerical evaluation shows that this is < 0.6 for the applicable range of parameters.

Planet	Using $\sigma_{\Delta\varpi}$ $\Upsilon/10^{-11}$ m^{-1}	$\lvert a_2\rvert/10^{18}$ m^2	Using $2\sigma_{\Delta\varpi}$ $\Upsilon/10^{-11}$ m^{-1}	$\lvert a_2\rvert/10^{18}$ m^2
Mercury	52.6	1.2	51.3	1.3
Venus	25.3	5.2	24.6	5.5
Earth	19.1	9.1	18.6	9.6
Mars	12.2	22	11.9	24
Jupiter	2.96	380	2.87	410
Saturn	1.69	1200	1.63	1200
Uranus	0.58	9800	0.56	11000
Neptune	0.35	28000	0.33	31000
Pluto	0.26	49000	0.25	55000

Table 2.2: Bounds calculated using uncertainties in planetary perihelion precession rates. Υ must be greater than or equal to the tabulated value, $|a_2|$ must be less than or equal to the tabulated value.

Potentials of this form are well studied in fifth-force tests (Will 2006; Adelberger *et al.* 2009, 2003) which consider a potential defined by a coupling constant α and a length-scale $\lambdabar$ such that

$$V(r) = \frac{GM}{r}\left[1 + \alpha \exp\left(-\frac{r}{\lambdabar}\right)\right]. \tag{2.67}$$

We are able to put strict constraints upon our length-scale $\lambdabar_R$, and hence a_2, because our coupling constant $\alpha_R = 1/3$ is relatively large. This can be larger for extended sources: comparison with equation (2.18) shows that for a uniform sphere $\alpha_R = \Xi(\Upsilon L) \geq 1/3$.

The best constraints at short distances come from the Eöt-Wash experiments, which use torsion balances (Kapner *et al.* 2007; Hoyle *et al.* 2004). These constrain $\lambdabar_R \lesssim 8\times10^{-5}$ m. Hence we determine $|a_2| \lesssim 2 \times 10^{-9}$ m^2. Similar results have been obtained by Cembranos (2009), and by Näf & Jetzer (2010). This would mean that the cut-off frequency for a propagating scalar mode would be $\gtrsim 4 \times 10^{12}$ s^{-1} which is much higher than expected for astrophysical objects.

Fifth-force tests also permit $\lambdabar_R$ to be large. This degeneracy can be broken using other tests; from section 2.2 we know that the large range for $\lambdabar_R$ is excluded by planetary precession rates. This is supported by a result of Näf & Jetzer (2010) obtained using the results of Gravity Probe B (Everitt *et al.* 2009).

While the laboratory bound on $\lambdabar_R$ may be strict compared to astronomical length-scales, it is still much greater than the expected characteristic gravitational scale, the Planck length ℓ_P. We might expect for a natural quantum theory that $a_2 \sim \mathcal{O}\left(\ell_\mathrm{P}^2\right)$, but $\ell_\mathrm{P}^2 = 2.612 \times 10^{-70}$ m^2, thus the bound is still about 60 orders of magnitude greater than the natural value. The only other length-scale that we could introduce is defined by the cosmological constant Λ. Using the concordance values $\Lambda = 1.202\times10^{-52}$ m^{-2} (Bennett *et al.* 2012; Hinshaw *et al.* 2012), we see that $\Lambda^{-1} \gg |a_2|$. It is intriguing combining these two length-scales we find $\ell_\mathrm{P}/\Lambda^{1/2} = 1.474\times10^{-9}$ m^2, which is of the order of the current bound. This is coincidence, since there is nothing fundamental about the current level of precision, but it would be interesting to see if the measurements could be improved to rule out a Yukawa interaction around this length-scale.

2.4 Summary of $f(R)$-gravity

Over the course of two chapters, we have examined the possibility of testing $f(R)$-type modifications to gravity using future GW observations and other measurements. We have seen that gravitational radiation is modified in $f(R)$-gravity as the Ricci scalar is no longer constrained to be zero; in linearised theory there is an additional mode of oscillation, that of the Ricci scalar. This is can only propagate above a cut-off frequency, but once excited, does carry additional energy-momentum away from the source. The two transverse GW modes are modified from their GR counterparts to include a contribution from the Ricci scalar, which would allow us to probe the curvature of the strong-field source regions. However, further study is needed in order to understand how GWs behave in a region with background curvature, in particular, when R is nonzero.

From linearised theory we have deduced the weak-field metrics for some simple mass distributions and found they are not the same as in GR. Birkhoff's theorem no longer applies in $f(R)$-gravity, and extended bodies have a different gravitational field than in GR. However, the BH metrics of GR remain solutions in $f(R)$-gravity. This restricts the potential GW observations that could be made to test $f(R)$ theories.

LISA observations of EMRIs are sensitive to small differences in the precession frequencies of orbits: even tiny differences accumulate into a measurable dephasing over the $\sim 10^5$ cycles LISA would observe. However, as BHs are identical in both GR and $f(R)$-gravity there would be no difference in the orbital frequencies. There would be a difference if the compact object was an exotic extended object; in this case deviations would only be detectable when $|a_2| \gtrsim 10^{17}$ m^2, assuming an extreme-mass-ratio binary with a massive object of mass $\sim 10^6 M_\odot$. This is calculated using the weak-field, slow-rotation metric. There could still be differences in the evolution of the inspiral because of a difference in the self-force.

We calculated constraints that can be placed using Solar System observations of planetary precessions and laboratory experiments. While the LISA constraints could beat those from Solar System observations (which presently give $|a_2| \lesssim 1.2 \times 10^{18}$ m^2), considerably stronger constraints have already been placed from fifth-force tests. Using existing results from the Eöt-Wash experiment, we constrain $|a_2| \lesssim 2 \times 10^{-9}$ m^2. For this range of a_2, we do expect the propagating Ricci mode to be excited by astrophysical systems as the cut-off frequency is too high. However, even in the absence of excitation of the Ricci mode, gravitational radiation in $f(R)$-gravity is still modified through the dependence of the transverse polarizations on the Ricci scalar.

Although the constraints from astrophysical observations are much weaker than this laboratory bound, they are still of interest since they probe gravity at a different scale and in a different environment. It is possible that $f(R)$-gravity is not universal, but changes in different regions of space or at different energy scales. The $f(R)$ model could be regarded as an approximate effective theory, and the range of validity of a particular parameterization is limited to a specific scale. For example, the effective theory in the vicinity of a massive BH, where the curvature is large, could be distinct from the appropriate effective theory in the Solar System, where curvature is small; or $f(R)$ could evolve with cosmological epoch such that it varies with redshift. If the laboratory bound is indeed universal there should be no deviation in GW observations: detection of a deviation would prove both that GR is incomplete and that the effective a_2 varies with environment.

One method of obtaining variation in the behaviour of gravity is via the chameleon mechanism. Then $f(R)$-gravity is modified in the presence of matter (Khoury & Weltman 2004b, a; Brax *et al.* 2004). In metric $f(R)$-gravity this is a non-linear effect arising from a large departure of the Ricci scalar from its background value (De Felice & Tsujikawa 2010). The mass of the effective scalar degree of freedom then depends upon the density of its environment (Faulkner *et al.* 2007; Li & Barrow 2007). In a region of high matter density, such as the Earth, the deviations from standard gravity would be exponentially suppressed due to a large effective Υ; while on cosmological scales, where the density is low, the scalar would have a small Υ, perhaps of the order H_0/c (Khoury & Weltman 2004b, a). The chameleon mechanism allows $f(R)$-gravity to pass laboratory, or Solar System, tests while remaining of interest for cosmology.[10] In the context of gravitational radiation, this would mean that the Ricci scalar mode could freely propagate on cosmological scales (Corda 2009b). Unfortunately, because the chameleon mechanism suppresses the effects of $f(R)$ in the presence of matter, this mode would have to be excited by something other than the acceleration of matter. Additionally, since electromagnetic radiation has a traceless energy-momentum tensor it cannot excite the Ricci mode.[11] To be able to detect

[10]The need to reconcile laboratory experiments with a non-trivial $f(R)$ could be regarded as motivation for introducing the chameleon mechanism.

[11]The standard transverse polarizations of gravitational radiation have an energy-momentum tensor that averages to be traceless, although this may not be the case locally (Butcher *et al.* 2010); the contribution to

the Ricci mode we must observe it well away from any matter, which would cause it to become evanescent: a space-borne detector such as LISA could be our only hope.

As the chameleon mechanism is inherently non-linear, it is difficult to discuss in terms of our linearised framework. Treating $f(R)$ as an effective theory, we could incorporate the effects of matter by taking the coefficients $\{a_n\}$ to be functions of the matter stress-energy tensor (or its trace). In this case, the results presented here would hold in the event that the coefficient a_2 is slowly varying, such that it may be treated as approximately constant in the region of interest. The linearised wave equations, (1.26) and (1.42), retain the same form in the case of a variable a_2, the only alteration would be that $a_2 R^{(1)}$ replaces $R^{(1)}$ as subject of the Klein-Gordon equation. In particular, the conclusion that $\gamma = 1$ is unaffected by the possibility of a variable a_2.

An interesting extension to the work presented here would be to consider the case when the constant term in the function $f(R)$, a_0, is nonzero. We would then be able to study perturbations with respect to (anti-)de Sitter space. This is relevant because the current ΛCDM paradigm indicates that we live in a universe with a positive cosmological constant (Jarosik *et al.* 2011; Komatsu *et al.* 2011). Such a study would naturally complement an investigation into the effects of background curvature on propagation (Yang *et al.* 2011).

the gravitational averaged energy-momentum tensor from a propagating Ricci mode does have a nonzero trace, see equation (1.74). In any case it is doubtful that gravitational energy-momentum could act as a source for detectable radiation.

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Appendix A

The signal inner product

We wish to derive an inner product over the space of signals; we shall denote the product of signals g and h as $(g|h)$.

A.1 The Fourier transform

A.1.1 Basic properties

We begin with some basic properties of Fourier transform (Riley *et al.* 2002, section 13.1). We define transformations

$$x(t) = \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \, \mathrm{d}f \tag{A.1a}$$

$$\widetilde{x}(f) = \int_{-\infty}^{\infty} x(t) \exp(-2\pi i f t) \, \mathrm{d}t. \tag{A.1b}$$

The Dirac delta-function arises as

$$\delta(f) = \int_{-\infty}^{\infty} \exp(-2\pi i f t) \, \mathrm{d}t. \tag{A.2}$$

We shall use Plancherel's theorem which proves the unitarity of the Fourier transformation

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \, \mathrm{d}f \int_{-\infty}^{\infty} \widetilde{x}^*(f') \exp(-2\pi i f' t) \, \mathrm{d}f' \\ &= \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \, \mathrm{d}f. \end{aligned} \tag{A.3}$$

A.1.2 Wiener–Khinchin theorem

We begin by deriving the Wiener–Khinchin theorem (Kittel 1958, chapter 28). For a real signal we have $\widetilde{x}(f) = \widetilde{x}^*(f)$, and since $\widetilde{x}(f) = \widetilde{x}^*(-f)$,

$$|\widetilde{x}(f)|^2 = |\widetilde{x}(-f)|^2 . \tag{A.4}$$

We use $\langle \ldots \rangle$ to denote time averaging, then

$$\left\langle x^2 \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [x(t)]^2 \, \mathrm{d}t. \tag{A.5}$$

Applying Plancherel's theorem for our real signal

$$\left\langle x^2 \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \, \mathrm{d}f = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{\infty} |\widetilde{x}(f)|^2 \, \mathrm{d}f. \tag{A.6}$$

The power spectrum $G(f)$ is

$$G(f) = \lim_{T\to\infty} \frac{1}{T}\overline{|\widetilde{x}(f)|^2}, \tag{A.7}$$

where an overline represents an ensemble average. Therefore

$$\overline{\langle x^2 \rangle} = \int_0^\infty G(f)\,\mathrm{d}f. \tag{A.8}$$

If $x(t)$ is a randomly varying signal we can use the ergodic principle to equate a time average with an ensemble over multiple realisations. Hence $\overline{\langle x^2 \rangle} = \left\langle x^2 \right\rangle$ and we can drop the overline.

The correlation function for a random process is

$$\begin{aligned} C(\tau) =& \ \langle x(t)x(t+\tau)\rangle & \text{(A.9)} \\ =& \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-\infty}^{\infty}\widetilde{x}(f)\exp(2\pi i f t)\,\mathrm{d}f\int_{-\infty}^{\infty}\widetilde{x}(f')\exp\left[2\pi i f'(t+\tau)\right]\mathrm{d}f' & \\ =& \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty}\left|\widetilde{x}(f)\right|^2\exp(2\pi i f\tau)\,\mathrm{d}f. & \text{(A.10)} \end{aligned}$$

We can rewrite this in terms of the power spectrum

$$C(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} G(f)\exp(2\pi i f\tau)\,\mathrm{d}f = \int_0^\infty G(f)\cos(2\pi f\tau)\,\mathrm{d}f. \tag{A.11}$$

Inverting these

$$G(f) = 2\int_{-\infty}^{\infty} C(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau = 4\int_0^\infty C(\tau)\cos(2\pi f\tau)\,\mathrm{d}\tau. \tag{A.12}$$

The power spectrum and correlation function are related to each other by the Fourier transform. This is the Wiener–Khinchin theorem.

A.2 Defining the inner product

A.2.1 Gaussian noise

We consider a normally distributed noise signal $n(t)$ with zero mean and standard deviation σ_n. The variance is

$$\left\langle n^2 \right\rangle = C_n(0) = \sigma_n^2, \tag{A.13}$$

introducing correlation function $C_n(\tau)$. If we have a measured signal $s(t)$ and a true signal $h(t)$, the probability $p(s|h)$ is that of the realisation of noise such

$$s = h + n. \tag{A.14}$$

Let us consider a discrete signal $n_i \equiv n(t_i)$, with $t_i - t_j = (i-j)\Delta t$ $\{i, j = -N, \dots, N\}$ and $\Delta T = 2T/(2N+1)$. For a single point (Finn 1992):

$$p(s_i|h_i) = \frac{1}{\sqrt{2\pi C_n(0)}}\exp\left[-\frac{1}{2}\frac{n_i^2}{C_n(0)}\right]. \tag{A.15}$$

Expanding this to the entire signal

$$p(s|h) = \frac{1}{\sqrt{(2\pi)^{2N+1}\det C_{n,\,ij}}}\exp\left[-\frac{1}{2}\sum_{k,l} C_{kl}^{-1} n_k n_l\right], \tag{A.16}$$

introducing short-hand $C_{n,\,ij} \equiv C_n(t_i - t_j)$ and defining the inverse matrix C^{-1}_{kl} such that

$$\delta_{jl} = \sum_{l} C_{n,\,jk} C^{-1}_{kl}. \tag{A.17}$$

To transform to the continuum (and infinite duration) limit we identify

$$\lim_{T\to\infty;\,\Delta t\to 0} \sum_{j} \Delta t \to \lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t_j. \tag{A.18}$$

To change between Kronecker and Dirac deltas

$$\sum_{j} \delta_{jk} = \int_{-T}^{T} \delta(t_j - t_k)\,\mathrm{d}t_j, \tag{A.19}$$

hence

$$\delta(t_j - t_k) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \delta_{jk}. \tag{A.20}$$

Using the inverse matrix definition

$$\begin{aligned} \exp(-2\pi i f t_k) &= \sum_{j} \exp(-2\pi i f t_j) \delta_{jk} \\ &= \frac{1}{(\Delta t)^2} \sum_{j} \Delta t \exp(-2\pi i f t_j) \sum_{l} \Delta t\, C_{n,\,jl} C^{-1}_{lk}. \end{aligned} \tag{A.21}$$

Taking the limit

$$\begin{aligned} \exp(-2\pi i f t_k) &= \lim_{T\to\infty;\,\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-T}^{T} \exp(-2\pi i f t_j)\,\mathrm{d}t_j \int_{-T}^{T} C_n(t_j - t_l) C^{-1}(t_l, t_k)\,\mathrm{d}t_l \\ &= \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{\infty} C_n(\tau) \exp(-2\pi i f \tau)\,\mathrm{d}\tau \int_{-\infty}^{\infty} C^{-1}(t_l, t_k) \exp(-2\pi i f t_l)\,\mathrm{d}t_l, \end{aligned} \tag{A.22}$$

where $\tau = t_j - t_l$. Defining the transformation

$$\widetilde{C^{-1}}(f, t_k) = \int_{-\infty}^{\infty} C^{-1}(t, t_k) \exp(-2\pi i f t)\,\mathrm{d}t, \tag{A.23}$$

and using the Wiener-Khinchin theorem to define power spectrum (Cutler & Flanagan 1994)

$$S_n(f) = \lim_{T\to\infty} \frac{1}{T} \overline{|\widetilde{n}(f)|^2} \tag{A.24}$$

$$= 2 \int_{-\infty}^{\infty} C_n(\tau) \exp(-2\pi i f \tau)\,\mathrm{d}\tau, \tag{A.25}$$

we have

$$\exp(-2\pi i f t_k) = \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \frac{S_n(f)}{2} \widetilde{C^{-1}}(f, t_k). \tag{A.26}$$

This can be rearranged to define $\widetilde{C^{-1}}(f, t_k)$ (Finn 1992).

The term in the exponential in equation (A.16) has the limit

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2} \lim_{T\to\infty;\ \Delta t\to 0} \sum_{j,k} C^{-1}_{jk} n_j n_k \\
&= \frac{1}{2} \lim_{T\to\infty;\ \Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-T}^{T} \mathrm{d}t_j \int_{-T}^{T} \mathrm{d}t_k\, C^{-1}(t_j, t_k) n(t_j) n(t_k) \\
&= \frac{1}{2} \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{\infty} \mathrm{d}t_k \int_{-\infty}^{\infty} \mathrm{d}f\, \widetilde{C^{-1}}(f, t_k) \widetilde{n}(-f) n(t_k) \\
&= \int_{-\infty}^{\infty} \frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)}\, \mathrm{d}f \\
&= \frac{1}{2} \left(n|n\right),
\end{aligned} \tag{A.27}$$

defining the inner product

$$(g|h) = 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f)}{S_n(f)}\, \mathrm{d}f = 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f) + \widetilde{g}(f)\widetilde{h}^*(f)}{S_n(f)}\, \mathrm{d}f. \tag{A.28}$$

This is a noise-weighted inner product over the space of real signals. The probability of the signal is

$$p(s|h) \propto \exp\left[-\frac{1}{2}\left(n|n\right)\right]. \tag{A.29}$$

A.2.2 Properties of the inner product

Consider an ensemble average over multiple noise realisations, which is the same as a time average assuming stationarity of the noise spectrum:

$$\begin{aligned}
\langle (n|n)\,(n|q)\rangle &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(n(t+\tau)|p(t)\right) \left(n(t+\tau)|q(t)\right) \mathrm{d}\tau \\
&= \lim_{T\to\infty} \frac{2}{T} \int_{-\infty}^{\infty} \frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)} \frac{\widetilde{p}(f)\widetilde{q}^*(f)}{S_n(f)}\, \mathrm{d}f \int_{-\infty}^{\infty} \frac{\widetilde{n}^*(f')}{S_n(f')}\, \mathrm{d}f' \\
&= (p|q)\,,
\end{aligned} \tag{A.30}$$

using the definition of the noise spectrum to obtain the final line.

Appendix B

The loss cone

When considering the orbits of stars about a massive black hole (MBH), the loss cone describes a region of velocity space that is depopulated because of tidal disruption (Frank & Rees 1976; Lightman & Shapiro 1977).

A main sequence star may be disrupted by tidal forces before it is swallowed by a MBH; we define the tidal disruption radius as r_{T}. We expect any orbit that passes inside r_{T} is depopulated unless stars can successfully escape to another orbit before being disrupted. Stars' velocities change because of gravitational interaction with other stars. Deflections can be modelled as a series of two-body encounters, the cumulative effect of which is a random walk in velocity space (Chandrasekhar 1960, chapter 2). Changes scale with the square-root of time, with the relaxation time-scale τ_{R} setting the scale.

Consider a typical star at a distance r from the MBH. We decompose its motion into radial and tangential components as

$$v_{\mathrm{r}} = v\cos\theta; \quad v_{\perp} = v\sin\theta. \tag{B.1}$$

Over a dynamical time-scale t_{dyn}, we expect that stars change velocity by a typical amount

$$\theta_{\mathrm{D}} \approx \left(\frac{t_{\mathrm{dyn}}}{\tau_{\mathrm{R}}}\right)^{1/2}, \tag{B.2}$$

assuming this change is small. We introduce the loss cone angle θ_{LC} to describe the range of trajectories that shall proceed to pass within a distance r_{T} of the MBH. By comparing the diffusion and loss cone angles we can deduce if we would expect orbits to be depleted: if $\theta_{\mathrm{D}} > \theta_{\mathrm{LC}}$ a star can safely diffuse out of the loss cone before it is destroyed, whereas if $\theta_{\mathrm{D}} < \theta_{\mathrm{LC}}$ a star is disrupted before it can change its velocity sufficiently, leading to the depopulation of the orbit.

Frank & Rees (1976) first introduced the loss cone. They considered stars on nearly radial orbits. The orbital energy and angular momentum (per unit mass) of an object with eccentricity e and periapse radius r_{p} are

$$\mathcal{E} = -\frac{GM_{\bullet}(1-e)}{2r_{\mathrm{p}}}; \tag{B.3}$$

$$\mathcal{J}^2 = GM_{\bullet}(1+e)r_{\mathrm{p}}, \tag{B.4}$$

where $M_{\bullet}$ is the MBH's mass. The angular momentum can also be defined as

$$\begin{aligned} \mathcal{J}^2 &= v_{\perp}^2 r^2 \\ &\simeq \theta^2 v^2 r^2, \end{aligned} \tag{B.5}$$

using the small angle approximation. Frank & Rees (1976) took the limit $e \to 1$ and then set $r_{\mathrm{p}} = r_{\mathrm{T}}$ to demarcate the limit of the loss cone; we rearrange to find

$$\theta_{\mathrm{LC}} \simeq \frac{2GM_{\bullet}r_{\mathrm{T}}}{v^2 r^2}. \tag{B.6}$$

We need to find the speed at r. Frank & Rees (1976) used a typical value

$$v^2 \simeq 3\sigma^2, \tag{B.7}$$

where σ is the 1D velocity dispersion. They assumed the velocity dispersion is Keplerian within the core region, where dynamics are dominated by the MBH, and is a constant outside of this

$$\sigma^2 \simeq \begin{cases} \dfrac{GM_\bullet}{r} & r < r_\mathrm{c} \\ \dfrac{GM_\bullet}{r_\mathrm{c}} & r < r_\mathrm{c} \end{cases}. \tag{B.8}$$

The core radius r_c is

$$r_\mathrm{c} = \frac{GM_\bullet}{\sigma_0^2}, \tag{B.9}$$

where σ_0 is the 1D velocity dispersion far from the MBH. Substituting for v^2 in equation (B.6) gives

$$\theta_\mathrm{LC}^2 \simeq \begin{cases} \dfrac{2r_\mathrm{T}}{3r} & r < r_\mathrm{c} \\ \dfrac{2r_\mathrm{T}r_\mathrm{c}}{3r^2} & r < r_\mathrm{c} \end{cases}. \tag{B.10}$$

Frank & Rees (1976) made one final modification, introducing a gravitational focusing factor f such that

$$\theta_\mathrm{LC} \simeq f \begin{cases} \left(\dfrac{2r_\mathrm{T}}{3r}\right)^{1/2} & r < r_\mathrm{c} \\ \left(\dfrac{2r_\mathrm{T}r_\mathrm{c}}{3r^2}\right)^{1/2} & r < r_\mathrm{c} \end{cases}. \tag{B.11}$$

The focusing factor could be imagined as the correction from assuming that stars travel along straight lines, such that $\tan\theta_\mathrm{LC} = r_\mathrm{T}/r$, to accounting for a Keplerian trajectory about the MBH.

It is unappealing to include an arbitrary, albeit order unitary, factor. Additionally, there are various restrictive approximations in the derivation. Considering the orbital energy for $v^2 = 3\sigma^2$ inside the core

$$\frac{3GM_\bullet}{2r} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{T}} \tag{B.12}$$

$$\implies \frac{r_\mathrm{T}}{r} = e - 1. \tag{B.13}$$

Since the radii must be positive, this enforces that $e \geq 1$: the orbits could be marginally bound at best. As we have taken the limit $e \to 1$, assuming that $r \gg r_\mathrm{T}$ this is still self-consistent. However, it is desirable to relax these conditions.

Let us consider an orbit with $r_\mathrm{p} = r_\mathrm{T}$, which gives the edge of the loss cone. The angular momentum squared is

$$\sin^2\theta_\mathrm{LC} v^2 r^2 = GM_\bullet(1+e)r_\mathrm{T}. \tag{B.14}$$

The energy is

$$\frac{v^2}{2} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{T}}. \tag{B.15}$$

Combining these to eliminate the velocity gives

$$\sin^2\theta_\mathrm{LC} = \frac{(1+e)r_\mathrm{T}^2}{2rr_\mathrm{T} - (1-e)r^2}. \tag{B.16}$$

This has been obtained without making any assumptions about the velocity dispersion or the position of the star. Since we have considered the Keplerian orbit, there should be no need to introduce a focusing factor.

This is similar in form to the classic result. Consider an orbit with eccentricity $e = 1 - \epsilon$, where ϵ is small. Let us choose the star to be at a characteristic distance set by its semimajor axis $a = r_\mathrm{p}/(1-e)$, such that

$$r = \frac{r_\mathrm{T}}{\epsilon}. \tag{B.17}$$

This ensures that $r \gg r_\mathrm{T}$. Therefore, we have matched the assumptions of Frank & Rees (1976). Substituting into our loss cone formula

$$\begin{aligned} \sin^2\theta_\mathrm{LC} &= \frac{(2-\epsilon)r_\mathrm{T}^2}{2rr_\mathrm{T} + \epsilon r^2} \\ &\simeq \frac{2r_\mathrm{T}}{3r}, \end{aligned} \tag{B.18}$$

retaining terms to first order in ϵ. Since this is small, we can use the small angle approximation to recover the result of equation (B.10).

Appendix C

Semirelativistic fluxes

The semirelativistic approximation for extreme-mass-ratio waveforms uses an exact geodesic of the background for the trajectory of the orbiting body, but only uses the flat-space radiation generation formula (Ruffini & Sasaki 1981). This is at the heart of the numerical kludge approximation. Gair, Kennefick & Larson (2005) derived analytic formulae for the fluxes of energy and angular momentum using the semirelativistic approximation for Schwarzschild geometry. These are useful for checking the accuracy of the numerical kludge waveforms.

The published expressions contain a number of (minor) errors; we rederive the correct forms. We consider an object of mass m orbiting about another of mass M, with a trajectory specified by eccentricity e and periapsis r_{p}. For this section we use geometric units with $G = c = 1$.

The geodesic equations in Schwarzschild are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2m}{r}\right)^{-1} E, \tag{C.1a}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \left(E^2 - 1\right) + \frac{2M}{r}\left(1 + \frac{L_z^2}{r^2}\right) - \frac{L_z^2}{r^2}, \tag{C.1b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{L_z}{r^2}, \tag{C.1c}$$

where t, r and ϕ are the usual Schwarzschild coordinates, τ is the proper time, and we have introduced specific energy E and azimuthal angular momentum L_z. Spherical symmetry has been exploited to set $\theta = \pi/2$ without loss of generality. For bound orbits, the radial equation has three roots, and can be written as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -\left(E^2 - 1\right)\frac{(r_{\mathrm{a}} - r)(r - r_{\mathrm{p}})(r - r_3)}{r^3}. \tag{C.2}$$

The turning points are the apoapsis, the periapsis and a third root; the orbit becomes unstable when $r_{\mathrm{p}} = r_3$. An eccentricity can be defined, in analogy to Keplerian orbits, such that

$$r_{\mathrm{a}} = \frac{1+e}{1-e} r_{\mathrm{p}}. \tag{C.3}$$

The third root is then

$$r_3 = \frac{2(1+e)M}{(1+e)r_{\mathrm{p}} - 4M} r_{\mathrm{p}}. \tag{C.4}$$

The last stable orbit with a given eccentricity, has periapse radius

$$r_{\mathrm{p,\,LSO}} = \frac{2(3+e)M}{1+e}. \tag{C.5}$$

Orbits that approach closer than this will plunge into the black hole.

The parameters $\{r_\mathrm{p}, e\}$ can be used to characterise orbits in place of $\{E, L_z\}$. The two are related by

$$E^2 = 1 - \frac{(1-e)\left[(1+e)r_\mathrm{p} - 4M\right]M}{\left[(1+e)r_\mathrm{p} - (3+e^2)\,M\right]r_\mathrm{p}}; \tag{C.6}$$

$$L_z^2 = \frac{(1+e)^2 M r_\mathrm{p}^2}{(1+e)r_\mathrm{p} - (3+e^2)\,M}. \tag{C.7}$$

Following the semirelativistic approximation, the fluxes of energy and angular momentum are derived by inserting the Schwarzschild geodesic into the flat-space radiation formulae, identifying the coordinate t with the flat-space time (Misner *et al.* 1973, chapter 36)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{5}\left\langle \frac{\mathrm{d}^3 I\!\!\!-_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 I\!\!\!-^{ij}}{\mathrm{d}t^3}\right\rangle, \tag{C.8}$$

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = -\frac{2}{5}\left\langle \frac{\mathrm{d}^2 I\!\!\!-_{xi}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 I\!\!\!-^{iy}}{\mathrm{d}t^3} - \frac{\mathrm{d}^2 I\!\!\!-_{yi}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 I\!\!\!-^{iz}}{\mathrm{d}t^3}\right\rangle, \tag{C.9}$$

where $I\!\!\!-_{ij} = I_{ij} - (1/3)I\delta_{ij}$ is the reduced mass quadrupole tensor and $\langle\ldots\rangle$ indicates averaging over several wavelengths (or periods). For a point particle, the mass quadrupole is

$$I^{jk} = \mu x^j x^k, \tag{C.10}$$

for trajectory $x^i(t)$. This is determined from the geodesic equations, and written as a function of r_p, e and r. To calculate the total change over one orbit we integrate r from r_p to r_a and back again. For this purpose it is easier to consider derivatives with respect to r. The integrands are rational functions of r and the square root of a cubic in r; the integrals can thus be written as a combination of elliptic integrals.

The integrals are of a general form

$$\mathscr{I}_n = \int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{M^{n+1}}{r^n\sqrt{(r_\mathrm{a}-r)(r-r_\mathrm{p})(r-r_3)r}}\,\mathrm{d}r. \tag{C.11}$$

By considering the derivative of $r^{-n}\sqrt{(r_\mathrm{a}-r)(r-r_\mathrm{p})(r-r_3)r}$ we may derive a recurrence relationship using integration by parts. After some rearrangement

$$\mathscr{I}_n = \frac{n-1}{2n-1}\mathscr{I}_{n-1} - \frac{2n-3}{2n-1}\frac{(r_\mathrm{a}+r_\mathrm{p}+r_3)M^2}{r_\mathrm{a}r_\mathrm{p}r_3}\mathscr{I}_{n-2} + \frac{2(n-1)}{2n-1}\frac{M^3}{r_\mathrm{a}r_\mathrm{p}r_3}\mathscr{I}_{n-3}. \tag{C.12}$$

Setting $n = 2$, the third term vanishes, hence the integrals $\mathscr{I}_0$ and $\mathscr{I}_1$ are sufficient to specify the series.[1] The zeroth integral can be evaluated using Gradshteyn & Ryzhik (2000, 3.147.6) as

$$\mathscr{I}_0 = \frac{2M}{r_\mathrm{p}}\sqrt{\frac{r_\mathrm{p}}{r_\mathrm{a}-r_3}}K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right], \tag{C.13}$$

where $K(k)$ is the complete elliptic integral of the first kind. The next integral can be evaluated using Gradshteyn & Ryzhik (2000, 3.149.6) as

$$\mathscr{I}_1 = \frac{2M^2}{r_\mathrm{p}r_3\sqrt{r_\mathrm{p}(r_\mathrm{a}-r_3)}}\left\{r_\mathrm{p}K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right] - (r_\mathrm{p}-r_3)\Pi\left[\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}, \sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right]\right\}, \tag{C.14}$$

where $\Pi(n, k)$ is the complete elliptic integral of the third kind. In this instance we may simplify using Olver *et al.* (2010, 19.6.2)

$$\Pi(k^2, k) = \frac{E(k)}{1-k^2} \tag{C.15}$$

[1] The integral $\mathscr{I}_{-1}$ could be calculated using Gradshteyn & Ryzhik (2000, 3.148.6).

to rewrite in terms of the complete elliptic integral of the second kind. Hence

$$\mathscr{I}_1 = \frac{2M^2}{r_3\sqrt{r_\mathrm{p}(r_\mathrm{a}-r_3)}}\left\{K\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right] - \frac{r_\mathrm{a}-r_3}{r_\mathrm{a}}E\left[\sqrt{\frac{(r_\mathrm{a}-r_\mathrm{p})r_3}{(r_\mathrm{a}-r_3)r_\mathrm{p}}}\right]\right\}. \tag{C.16}$$

Substituting in for the integrals, we find that the energy lost in one orbit is

$$\begin{aligned}\frac{M}{m}\Delta E = &-\frac{16M^{11}}{1673196525r_\mathrm{p}^6(1+e)^{19/2}\left\{(r_\mathrm{p}-2M)\left[(1+e)r_\mathrm{p}-2(1-e)M\right]\right\}^{5/2}} \\ &\times\left\{\sqrt{(1+e)\frac{r_\mathrm{p}}{M}-2(3-e)}E\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]f_1\left(\frac{r_\mathrm{p}}{M},e\right)\right. \\ &+\left.\frac{1+e}{\sqrt{(1+e)\left(r_\mathrm{p}/M\right)-2(3-e)}}K\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]f_2\left(\frac{r_\mathrm{p}}{M},e\right)\right\},\end{aligned} \tag{C.17}$$

where we have introduced functions

$$\begin{aligned}f_1(y,e) = {}& 4608(1-e)(1+e)^2\left(3+e^2\right)^2\left(2428691599+313957879e^2+1279504693e^4\right. \\ &\left.+63843717e^6\right)-192(1+e)^2\left(908960573673-155717471796e^2\right. \\ &-88736969547e^4-293676299040e^6-195313674237e^8-26635698156e^{10} \\ &\left.-346799201e^{12}\right)y+384(1+e)^3\left(336063804453-53956775638e^2-33318942522e^4\right. \\ &\left.-92857670352e^6-41764459155e^8-2765710514e^{10}\right)y^2 \\ &-16(1+e)^4\left(3418907055555-580720618635e^2-168432860626e^4\right. \\ &\left.-606890963686e^6-176495184865e^8-3768291999e^{10}\right)y^3 \\ &+32(1+e)^5\left(510454645597-92175635794e^2+26432814256e^4-28250211070e^6\right. \\ &\left.-5713846269e^8\right)y^4-4(1+e)^6\left(1107402703901-174239346926e^2\right. \\ &\left.+100957560852e^4+3707280110e^6-899162673e^8\right)y^5 \\ &+8(1+e)^7\left(143625217397-16032820010e^2+4238287541e^4+275190560e^6\right)y^6 \\ &-(1+e)^8\left(220627324753-14884378223e^2-1210713997e^4+14138955e^6\right)y^7 \\ &+8(1+e)^9\left(2922108518-46504603e^2-2407656e^4\right)y^8 \\ &-3(1+e)^{10}\left(241579935+6314675e^2-149426e^4\right)y^9 \\ &-4(1+e)^{11}\left(8608805-48992e^2\right)y^{10}-2(1+e)^{12}\left(1242083-16320e^2\right)y^{11} \\ &-184320(1+e)^{13}y^{12}-5120(1+e)^{14}y^{13}\end{aligned} \tag{C.18}$$

and

$$
\begin{aligned}
f_2(y,e) = {} & 3072(3-e)(3+e)\left(3+e^2\right)\left(7286074797-3299041125e^2+792940362e^4\right.\\
&\left.-1366777698e^6-369698151e^8-5932745e^{10}\right)-384(1+e)\left(2989180413711\right.\\
&-583867932642e^2-131661872359e^4-419423580924e^6-194293515951e^8\\
&\left.-3390301442e^{10}+1353430119e^{12}\right)y+64(1+e)^2\left(14825178681327\right.\\
&-2675442646782e^2-728511901515e^4-1837874368340e^6-591999524567e^8\\
&\left.-1856757710e^{10}+841581651e^{12}\right)y^2-32(1+e)^3\left(14292163934541\right.\\
&-2666166422089e^2-522582885086e^4-1347373382962e^6-307066297439e^8\\
&\left.-1675056789e^{10}\right)y^3+16(1+e)^4\left(9557748374919-1917809903861e^2\right.\\
&\left.-24258045506e^4-511875047746e^6-86779453317e^8-462078345e^{10}\right)y^4\\
&-8(1+e)^5\left(5390797838491-990602472036e^2+161182699002e^4\right.\\
&\left.-89978894004e^6-11363685245e^8\right)y^5+4(1+e)^6\left(2857676457065\right.\\
&\left.-351292910556e^2+79840371470e^4-2670080940e^6-463345647e^8\right)y^6\\
&-2(1+e)^7\left(1249768416047-79903103833e^2+12179840133e^4\right.\\
&\left.+482157413e^6\right)y^7+(1+e)^8\left(363565648057-10040939153e^2-318841465e^4\right.\\
&\left.+14611473e^6\right)y^8-2(1+e)^9\left(13862653487-100645509e^2-11015842e^4\right)y^9\\
&+(1+e)^{10}\left(518128485+16345427e^2-421398e^4\right)y^{10}\\
&+16(1+e)^{11}\left(1220639-13448e^2\right)y^{11}+2(1+e)^{12}\left(689123-18880e^2\right)y^{12}\\
&+153600(1+e)^{13}y^{13}+5120(1+e)^{14}y^{14}. \qquad \text{(C.19)}
\end{aligned}
$$

The angular momentum lost is

$$
\begin{aligned}
\frac{\Delta L_z}{m} = {} & -\frac{16M^{15/2}}{24249225(1+e)^{13/2}r_\mathrm{p}^{7/2}(r_\mathrm{p}-2M)^2\left[(1+e)r_\mathrm{p}-2(1-e)M\right]^2}\\
&\times\left\{\sqrt{(1+e)\frac{r_\mathrm{p}}{M}-2(3-e)}\,E\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]g_1\left(\frac{r_\mathrm{p}}{M},e\right)\right.\\
&\left.+\frac{(1+e)}{\sqrt{(1+e)\left(r_\mathrm{p}/M\right)-2(3-e)}}K\left[\sqrt{\frac{4eM}{(1+e)r_\mathrm{p}-2(3-e)M}}\right]g_2\left(\frac{r_p}{M},e\right)\right\} \qquad \text{(C.20)}
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(y,e) = {} & 169728(1-e)(1+e)^2\left(279297+219897e^2+106299e^4+9611e^6\right)\\
&-384(1+e)^2\left(192524061-13847615e^2-36165965e^4-20710173e^6-588532e^8\right)y\\
&+192(1+e)^3\left(235976417+13109547e^2-3369705e^4-3292707e^6\right)y^2\\
&-16(1+e)^4\left(813592799+112906199e^2+53843933e^4+602061e^6\right)y^3\\
&+16(1+e)^5\left(87491089+7247482e^2+4608349e^4\right)y^4+8(1+e)^6\left(9580616\right.\\
&\left.+6179243e^2-92047e^4\right)y^5-4(1+e)^7\left(3760123+272087e^2\right)y^6\\
&-(1+e)^8\left(1168355-35347e^2\right)y^7-71792(1+e)^9y^8-4120(1+e)^{10}y^9 \qquad \text{(C.21)}
\end{aligned}
$$

and

$$\begin{aligned} g_2(y,e) = {} & 339456(3-e)(3+e)\left(93099-10213e^2-18155e^4-10551e^6-420e^8\right) \\ & -1536(1+e)\left(319648410-35712133e^2-33099777e^4-11272311e^6+457187e^8\right)y \\ & +128(1+e)^2\left(2706209781-45415294e^2-103634296e^4-34056010e^6-130293e^8\right)y^2 \\ & -32(1+e)^3\left(3895435659+212168215e^2+4641265e^4-15197651e^6\right)y^3 \\ & +16(1+e)^4\left(1396737473+123722895e^2+27602127e^4-465119e^6\right)y^4 \\ & -16(1+e)^5\left(78148621+3035912e^2+3130827e^4\right)y^5 \\ & -16(1+e)^6\left(8005570+1485159e^2-47943e^4\right)y^6+2(1+e)^7\left(4015181+601959e^2\right)y^7 \\ & +(1+e)^8\left(737603-39467e^2\right)y^8+47072(1+e)^9y^9+4120(1+e)^{10}y^{10}. \end{aligned} \tag{C.22}$$

Taking limit $r_\mathrm{p} \to \infty$ should recover weak field results. Using series expansions of the elliptic integrals for small arguments

$$\begin{aligned} \frac{M}{m}\Delta E \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^{7/2} \\ & -\frac{192\pi}{5}\frac{1}{(1+e)^{9/2}}\left(1+\frac{31}{8}e^2+\frac{65}{32}e^4+\frac{1}{6}e^6\right)\left(\frac{M}{r_\mathrm{p}}\right)^{9/2}+\mathcal{O}\left(\frac{M^{11/2}}{r_\mathrm{p}^{11/2}}\right) \end{aligned} \tag{C.23}$$

$$\begin{aligned} \frac{\Delta L_z}{m} \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^2}\left(1+\frac{7}{8}e^2\right)\left(\frac{M}{r_\mathrm{p}}\right)^2 \\ & -\frac{192\pi}{5}\frac{1}{(1+e)^3}\left(1+\frac{35}{24}e^2+\frac{1}{4}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^3+\mathcal{O}\left(\frac{M^4}{r_\mathrm{p}^4}\right). \end{aligned} \tag{C.24}$$

The leading order terms correspond to the Keplerian results of Peters (1964).

For a parabolic orbit with $e = 1$, the energy loss reduces to

$$\frac{M}{m}\Delta E = -\frac{2^{7/2}M^{21/2}}{1673196525\left(r_\mathrm{p}-2M\right)^2 r_\mathrm{p}^{17/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)f_1\left(\frac{r_\mathrm{p}}{M}\right)+K\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)f_2\left(\frac{r_\mathrm{p}}{M}\right)\right] \tag{C.25}$$

where

$$\begin{aligned} f_1(y) = {} & -2y\left(27850061568-83550184704y+117662445984y^2-102686941680y^3\right. \\ & +64808064704y^4-33026468872y^5+12784148218y^6-2873196259y^7 \\ & \left.+185808888y^8+17119626y^9+2451526y^{10}+368640y^{11}+20480y^{12}\right) \end{aligned} \tag{C.26}$$

and

$$\begin{aligned} f_2(y) = {} & -72901570560+274404834816y-424693524096y^2 \\ & +378109481088y^3-249480499840y^4+154011967968y^5 \\ & -84437171728y^6+31689370996y^7-6231594434y^8+321950817y^9 \\ & +27462280y^{10}+4073612y^{11}+696320y^{12}+40960y^{13}. \end{aligned} \tag{C.27}$$

The angular momentum lost is

$$\frac{\Delta L_z}{m} = \frac{64M^7}{24249225{r_\mathrm{p}}^{11/2}\left(r_\mathrm{p}-2M\right)^{3/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)g_1\left(\frac{r_\mathrm{p}}{M}\right)+K\left(\sqrt{\frac{2M}{r_\mathrm{p}-2M}}\right)g_2\left(\frac{r_\mathrm{p}}{M}\right)\right], \tag{C.28}$$

where

$$\begin{aligned} g_1(y) = {} & 181817664y-363635328y^2-245236248y^3-49673460y^4 \\ & -7833906y^5+2016105y^6+283252y^7+35896y^8+4120y^9 \end{aligned} \tag{C.29}$$

and

$$
\begin{aligned}
g_2(y) = {} & 71285760 - 324389184y + 468548880y^2 - 277856496y^3 + 54521424y^4 \\
& + 6181872y^5 - 1630457y^6 - 238086y^7 - 31776y^8 - 4120y^9.
\end{aligned}
\tag{C.30}
$$