Exploring Gravity
With Gravitational Waves & Strong-Field Tests

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Thesis

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References

[1] Ruffini, R. & Sasaki, M.; *Progress of Theoretical Physics*; **66**(5):1627–1638; 1981.

[2] Peters, P. C.; *Physical Review*; **136**(4B):B1224–B1232; 1964.

Appendix A

Semirelativistic fluxes

The semirelativistic approximation for extreme-mass-ratio waveforms uses an exact geodesic of the background for the trajectory of the orbiting body, but only uses the flat-space radiation generation formula[1]. This is at the heart of the numerical kludge approximation. Gair, Kennefick and Larson derived analytic formulae for the fluxes of energy and angular momentum using the semirelativistic approximation for Schwarzschild geometry. These are useful for checking the accuracy of the numerical kludge waveforms.

The published expressions contain a number of (minor) errors, we reproduce the correct forms here. We assume an object of mass m orbiting about another of mass M with a trajectory specified by eccentricity e and periapsis r_{p}. For section we shall use geometric units with $G = c = 1$.

The energy lost in one orbit is

$$\begin{aligned}\frac{M}{m}\mathit{\Delta} E = &-\frac{16M^{11}}{1673196525 r_{\mathrm{p}}^{6}(1+e)^{19/2}\left\{(r_{\mathrm{p}}-2M)\left[(1+e)r_{\mathrm{p}}-2(1-e)M\right]\right\}^{5/2}} \\ &\times\left\{\sqrt{(1+e)\frac{r_{\mathrm{p}}}{M}-2(3-e)}\,E\left[\sqrt{\frac{4eM}{(1+e)r_{\mathrm{p}}-2(3-e)M}}\right] f_1\left(\frac{r_{\mathrm{p}}}{M}, e\right)\right. \\ &\left.+\frac{1+e}{\sqrt{(1+e)\left(r_{\mathrm{p}}/M\right)-2(3-e)}} K\left[\sqrt{\frac{4eM}{(1+e)r_{\mathrm{p}}-2(3-e)M}}\right] f_2\left(\frac{r_{\mathrm{p}}}{M}, e\right)\right\},\end{aligned} \qquad \text{(A.1)}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds,

$$\begin{aligned}
f_1(y,e) = {} & 4608(1-e)(1+e)^2\left(3+e^2\right)^2\left(2428691599+313957879e^2+1279504693e^4\right.\\
& \left.+\,63843717e^6\right)-192(1+e)^2\left(908960573673-155717471796e^2\right.\\
& -88736969547e^4-293676299040e^6-195313674237e^8-26635698156e^{10}\\
& \left.-\,346799201e^{12}\right)y+384(1+e)^3\left(336063804453-53956775638e^2-33318942522e^4\right.\\
& \left.-\,92857670352e^6-41764459155e^8-2765710514e^{10}\right)y^2\\
& -16(1+e)^4\left(3418907055555-580720618635e^2-168432860626e^4\right.\\
& \left.-\,606890963686e^6-176495184865e^8-3768291999e^{10}\right)y^3\\
& +32(1+e)^5\left(510454645597-92175635794e^2+26432814256e^4-28250211070e^6\right.\\
& \left.-\,5713846269e^8\right)y^4-4(1+e)^6\left(1107402703901-174239346926e^2\right.\\
& \left.+\,100957560852e^4+3707280110e^6-899162673e^8\right)y^5\\
& +8(1+e)^7\left(143625217397-16032820010e^2+4238287541e^4+275190560e^6\right)y^6\\
& -(1+e)^8\left(220627324753-14884378223e^2-1210713997e^4+14138955e^6\right)y^7\\
& +8(1+e)^9\left(2922108518-46504603e^2-2407656e^4\right)y^8\\
& -3(1+e)^{10}\left(241579935+6314675e^2-149426e^4\right)y^9\\
& -4(1+e)^{11}\left(8608805-48992e^2\right)y^{10}-2(1+e)^{12}\left(1242083-16320e^2\right)y^{11}\\
& -184320(1+e)^{13}y^{12}-5120(1+e)^{14}y^{13}
\end{aligned} \tag{A.2}$$

and

$$
\begin{aligned}
f_2(y,e) = {}& 3072(3-e)(3+e)\left(3+e^2\right)\left(7286074797-3299041125e^2+792940362e^4\right.\\
&-\left.1366777698e^6-369698151e^8-5932745e^{10}\right)-384(1+e)\left(2989180413711\right.\\
&-583867932642e^2-131661872359e^4-419423580924e^6-194293515951e^8\\
&-\left.3390301442e^{10}+1353430119e^{12}\right)y+64(1+e)^2\left(14825178681327\right.\\
&-2675442646782e^2-728511901515e^4-1837874368340e^6-591999524567e^8\\
&-\left.1856757710e^{10}+841581651e^{12}\right)y^2-32(1+e)^3\left(14292163934541\right.\\
&-2666166422089e^2-522582885086e^4-1347373382962e^6-307066297439e^8\\
&-\left.1675056789e^{10}\right)y^3+16(1+e)^4\left(9557748374919-1917809903861e^2\right.\\
&-\left.24258045506e^4-511875047746e^6-86779453317e^8-462078345e^{10}\right)y^4\\
&-8(1+e)^5\left(5390797838491-990602472036e^2+161182699002e^4\right.\\
&-\left.89978894004e^6-11363685245e^8\right)y^5+4(1+e)^6\left(2857676457065\right.\\
&-\left.351292910556e^2+79840371470e^4-2670080940e^6-463345647e^8\right)y^6\\
&-2(1+e)^7\left(1249768416047-79903103833e^2+12179840133e^4\right.\\
&+\left.482157413e^6\right)y^7+(1+e)^8\left(363565648057-10040939153e^2-318841465e^4\right.\\
&+\left.14611473e^6\right)y^8-2(1+e)^9\left(13862653487-100645509e^2-11015842e^4\right)y^9\\
&+(1+e)^{10}\left(518128485+16345427e^2-421398e^4\right)y^{10}\\
&+16(1+e)^{11}\left(1220639-13448e^2\right)y^{11}+2(1+e)^{12}\left(689123-18880e^2\right)y^{12}\\
&+153600(1+e)^{13}y^{13}+5120(1+e)^{14}y^{14}. \qquad\qquad \text{(A.3)}
\end{aligned}
$$

The angular momentum lost is

$$
\begin{aligned}
\frac{\mathit{\Delta} L_z}{m} = {}& -\frac{16M^{15/2}}{24249225(1+e)^{13/2}r_{\mathrm{p}}^{7/2}(r_{\mathrm{p}}-2M)^2\left[(1+e)r_{\mathrm{p}}-2(1-e)M\right]^2}\\
&\times\left\{\sqrt{(1+e)\frac{r_{\mathrm{p}}}{M}-2(3-e)}\,E\left[\sqrt{\frac{4eM}{(1+e)r_{\mathrm{p}}-2(3-e)M}}\right]g_1\left(\frac{r_{\mathrm{p}}}{M},e\right)\right.\\
&+\left.\frac{(1+e)}{\sqrt{(1+e)\left(r_{\mathrm{p}}/M\right)-2(3-e)}}K\left[\sqrt{\frac{4eM}{(1+e)r_{\mathrm{p}}-2(3-e)M}}\right]g_2\left(\frac{r_p}{M},e\right)\right\} \qquad \text{(A.4)}
\end{aligned}
$$

where

$$\begin{aligned} g_1(y,e) = {} & 169728(1-e)(1+e)^2\left(279297 + 219897e^2 + 106299e^4 + 9611e^6\right) \\ & - 384(1+e)^2\left(192524061 - 13847615e^2 - 36165965e^4 - 20710173e^6 - 588532e^8\right)y \\ & + 192(1+e)^3\left(235976417 + 13109547e^2 - 3369705e^4 - 3292707e^6\right)y^2 \\ & - 16(1+e)^4\left(813592799 + 112906199e^2 + 53843933e^4 + 602061e^6\right)y^3 \\ & + 16(1+e)^5\left(87491089 + 7247482e^2 + 4608349e^4\right)y^4 + 8(1+e)^6\left(9580616\right. \\ & \left. + 6179243e^2 - 92047e^4\right)y^5 - 4(1+e)^7\left(3760123 + 272087e^2\right)y^6 \\ & - (1+e)^8\left(1168355 - 35347e^2\right)y^7 - 71792(1+e)^9y^8 - 4120(1+e)^{10}y^9 \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} g_2(y,e) = {} & 339456(3-e)(3+e)\left(93099 - 10213e^2 - 18155e^4 - 10551e^6 - 420e^8\right) \\ & - 1536(1+e)\left(319648410 - 35712133e^2 - 33099777e^4 - 11272311e^6 + 457187e^8\right)y \\ & + 128(1+e)^2\left(2706209781 - 45415294e^2 - 103634296e^4 - 34056010e^6 - 130293e^8\right)y^2 \\ & - 32(1+e)^3\left(3895435659 + 212168215e^2 + 4641265e^4 - 15197651e^6\right)y^3 \\ & + 16(1+e)^4\left(1396737473 + 123722895e^2 + 27602127e^4 - 465119e^6\right)y^4 \\ & - 16(1+e)^5\left(78148621 + 3035912e^2 + 3130827e^4\right)y^5 \\ & - 16(1+e)^6\left(8005570 + 1485159e^2 - 47943e^4\right)y^6 + 2(1+e)^7\left(4015181 + 601959e^2\right)y^7 \\ & + (1+e)^8\left(737603 - 39467e^2\right)y^8 + 47072(1+e)^9y^9 + 4120(1+e)^{10}y^{10}. \end{aligned} \tag{A.6}$$

Taking limit $r_\mathrm{p} \to \infty$ should recover weak field results. Using series expansions of the elliptic integrals for small arguments

$$\begin{aligned} \frac{M}{m}\mathit{\Delta} E \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^{7/2} \\ & - \frac{192\pi}{5}\frac{1}{(1+e)^{9/2}}\left(1 + \frac{31}{8}e^2 + \frac{65}{32}e^4 + \frac{1}{6}e^6\right)\left(\frac{M}{r_\mathrm{p}}\right)^{9/2} + \mathcal{O}\left(\frac{M^{11/2}}{r_\mathrm{p}^{11/2}}\right) \end{aligned} \tag{A.7}$$

$$\begin{aligned} \frac{\mathit{\Delta} L_z}{m} \simeq {} & -\frac{64\pi}{5}\frac{1}{(1+e)^2}\left(1 + \frac{7}{8}e^2\right)\left(\frac{M}{r_\mathrm{p}}\right)^2 \\ & - \frac{192\pi}{5}\frac{1}{(1+e)^3}\left(1 + \frac{35}{24}e^2 + \frac{1}{4}e^4\right)\left(\frac{M}{r_\mathrm{p}}\right)^3 + \mathcal{O}\left(\frac{M^4}{r_\mathrm{p}^4}\right). \end{aligned} \tag{A.8}$$

The leading order terms correspond to the Keplerian results of[2].

For a parabolic orbit with $e = 1$, the energy loss reduces to

$$\frac{M}{m}\mathit{\Delta} E = -\frac{2^{7/2}M^{21/2}}{1673196525\left(r_\mathrm{p} - 2M\right)^2 r_\mathrm{p}^{17/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p} - 2M}}\right)f_1\left(\frac{r_\mathrm{p}}{M}\right) + K\left(\sqrt{\frac{2M}{r_\mathrm{p} - 2M}}\right)f_2\left(\frac{r_\mathrm{p}}{M}\right)\right] \tag{A.9}$$

where

$$f_1(y) = -2y\big(27850061568 - 83550184704y + 117662445984y^2 - 102686941680y^3 + 64808064704y^4 - 33026468872y^5 + 12784148218y^6 - 2873196259y^7 + 185808888y^8 + 17119626y^9 + 2451526y^{10} + 368640y^{11} + 20480y^{12}\big) \tag{A.10}$$

and

$$f_2(y) = -72901570560 + 274404834816y - 424693524096y^2 + 378109481088y^3 - 249480499840y^4 + 154011967968y^5 - 84437171728y^6 + 31689370996y^7 - 6231594434y^8 + 321950817y^9 + 27462280y^{10} + 4073612y^{11} + 696320y^{12} + 40960y^{13}. \tag{A.11}$$

The angular momentum lost is

$$\frac{\Delta L_z}{m} = \frac{64M^7}{24249225{r_\mathrm{p}}^{11/2}\left(r_\mathrm{p} - 2M\right)^{3/2}}\left[E\left(\sqrt{\frac{2M}{r_\mathrm{p} - 2M}}\right)g_1\left(\frac{r_\mathrm{p}}{M}\right) + K\left(\sqrt{\frac{2M}{r_\mathrm{p} - 2M}}\right)g_2\left(\frac{r_\mathrm{p}}{M}\right)\right], \tag{A.12}$$

where

$$g_1(y) = 181817664y - 363635328y^2 - 245236248y^3 - 49673460y^4 - 7833906y^5 + 2016105y^6 + 283252y^7 + 35896y^8 + 4120y^9 \tag{A.13}$$

and

$$g_2(y) = 71285760 - 324389184y + 468548880y^2 - 277856496y^3 + 54521424y^4 + 6181872y^5 - 1630457y^6 - 238086y^7 - 31776y^8 - 4120y^9. \tag{A.14}$$