Exploring Gravity
With Gravitational Waves & Strong-Field Tests

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References

Appendix A

The signal inner product

We wish to derive an inner product over the space of signals. We shall denote the product of signals g and h as $(g|h)$.

A.1 The Fourier transform

A.1.1 Basic properties

We will begin with some basic properties of Fourier transform. We shall define our transforamtions as

$$x(t) = \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \, \mathrm{d}f \tag{A.1a}$$

$$\widetilde{x}(f) = \int_{-\infty}^{\infty} x(t) \exp(-2\pi i f t) \, \mathrm{d}t. \tag{A.1b}$$

The Dirac delta-function arises as

$$\delta(f) = \int_{-\infty}^{\infty} \exp(-2\pi i f t) \, \mathrm{d}t \tag{A.2}$$

We shall use Plancherel's theorem which proves the unitarity of the Fourier transfromation

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \widetilde{x}(f) \exp(2\pi i f t) \, \mathrm{d}f \int_{-\infty}^{\infty} \widetilde{x}^*(f') \exp(-2\pi i f' t) \, \mathrm{d}f' \\
&= \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \widetilde{x}(f) \widetilde{x}^*(f') \delta(f' - f) \\
&= \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \, \mathrm{d}f.
\end{aligned} \tag{A.3}$$

A.1.2 Wiener-Knichin theorem

We begin by deriving the Wiener-Khinchin theorem. For a real signal we must have $\widetilde{x}(f) = \widetilde{x}^*(f)$, and since $\widetilde{x}(f) = \widetilde{x}^*(-f)$

$$|\widetilde{x}(f)|^2 = |\widetilde{x}(-f)|^2 . \tag{A.4}$$

We shall use $\langle \ldots \rangle$ to denote time averaging, then

$$\left\langle x^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [x(t)]^2 \,\mathrm{d}t. \tag{A.5}$$

Applying Plancherel's theorem for our real signal

$$\begin{aligned}\left\langle x^2 \right\rangle &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \,\mathrm{d}f \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\infty} |\widetilde{x}(f)|^2 \,\mathrm{d}f. \end{aligned} \tag{A.6}$$

The power spectrum $G(f)$ is defined as

$$G(f) = \lim_{T\to\infty} \frac{1}{T} \overline{|\widetilde{x}(f)|^2}, \tag{A.7}$$

where an overline represents an ensemble average. Therefore

$$\overline{\langle x^2 \rangle} = \int_0^\infty G(f)\,\mathrm{d}f. \tag{A.8}$$

If $x(t)$ is a randomly varying signal we can use the ergodic principle to equate a time average with an ensemble over multiple realisations: the two are equivalent. Hence $\overline{\langle x^2 \rangle} = \langle x^2 \rangle$ and we may drop the overline.

The correlation function for a random process is

$$\begin{aligned} C(\tau) &= \langle x(t)x(t+\tau) \rangle \qquad (A.9) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\,\mathrm{d}t \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t \int_{-\infty}^{\infty} \widetilde{x}(f)\exp(2\pi i f t)\,\mathrm{d}f \int_{-\infty}^{\infty} \widetilde{x}(f')\exp\left[2\pi i f'(t+\tau)\right]\mathrm{d}f' \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f'\, \widetilde{x}(f)\widetilde{x}(f')\exp(2\pi i f'\tau)\delta(f+f') \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\widetilde{x}(f)|^2 \exp(2\pi i f\tau)\,\mathrm{d}f. \qquad (A.10) \end{aligned}$$

We can rewrite this in terms of the power spectrum

$$C(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} G(f)\exp(2\pi i f\tau)\,\mathrm{d}f \tag{A.11}$$

$$= \int_0^\infty G(f)\cos(2\pi f\tau)\,\mathrm{d}f. \tag{A.12}$$

Inverting these

$$G(f) = 2\int_{-\infty}^{\infty} C(\tau)\exp(-2\pi i f\tau)\,\mathrm{d}\tau \tag{A.13}$$

$$= 4\int_0^\infty C(\tau)\cos(2\pi f\tau)\,\mathrm{d}\tau. \tag{A.14}$$

The power spectrum and correlation function are related to each other by the Fourier transform. This is the Wiener-Kinchin theorem.

A.2 Defining the inner product

A.2.1 Gaussian noise

We will consider a normally distributed noise signal $n(t)$ with zero mean and standard deviation σ_n. The variance is

$$\left\langle n^2 \right\rangle = C_n(0) = \sigma_n^2, \tag{A.15}$$

introducing the correlation function $C_n(\tau)$. If we have a measured signal $s(t)$ and a true signal $h(t)$, then the probability $p(s|h)$ is just the probability of the realisation of noise such that

$$s = h + n. \tag{A.16}$$

Let us consider a discrete signal $n_i \equiv n(t_i)$, with $t_i - t_j = (i-j)\Delta t$ $\{i, j = -N, \ldots, N\}$ and $\Delta T = 2T/(2N+1)$. For a single point

$$p(s_i|h_i) = \frac{1}{\sqrt{2\pi C_n(0)}} \exp\left[-\frac{1}{2}\frac{n_i^2}{C_n(0)}\right]. \tag{A.17}$$

Expanding this to the entire signal

$$p(s|h) = \frac{1}{\sqrt{(2\pi)^{2N+1} \det C_{n,\,ij}}} \exp\left[-\frac{1}{2}\sum_{k,l} C^{-1}_{kl} n_k n_l\right], \tag{A.18}$$

introducing short hand $C_{n,\,ij} \equiv C_n(t_i - t_j)$ and defining the inverse matrix C^{-1}_{kl} such that

$$\delta_{jl} = \sum_l C_{n,\,jk} C^{-1}_{kl}. \tag{A.19}$$

To transform to the continuum (and infinite duration) limit we identify

$$\lim_{T\to\infty;\Delta t\to 0} \sum_j \Delta t \to \lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t_j. \tag{A.20}$$

To change between Kronecker and Dirac deltas

$$\sum_j \delta_{jk} = \int_{-T}^{T} \delta(t_j - t_k)\,\mathrm{d}t_j, \tag{A.21}$$

hence

$$\delta(t_j - t_k) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\delta_{jk}. \tag{A.22}$$

Using the inverse matrix definition

$$\begin{aligned}
\exp(-2\pi i f t_k) &= \sum_j \exp(-2\pi i f t_j)\delta_{jk} \\
&= \sum_j \exp(-2\pi i f t_j) \sum_l C_{n,\,jl} C^{-1}_{lk} \\
&= \frac{1}{(\Delta t)^2} \sum_j \Delta t \exp(-2\pi i f t_j) \sum_l \Delta t\, C_{n,\,jl} C^{-1}_{lk}.
\end{aligned} \tag{A.23}$$

Taking the limit

$$\begin{aligned}\exp(-2\pi i f t_k) &= \lim_{T\to\infty;\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-T}^{T} \exp(-2\pi i f t_j)\,\mathrm{d}t_j \int_{-T}^{T} C_n(t_j - t_l) C^{-1}(t_l, t_k)\,\mathrm{d}t_l \\ &= \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{\infty} C_n(\tau) \exp(-2\pi i f \tau)\,\mathrm{d}\tau \int_{-\infty}^{\infty} C^{-1}(t_l, t_k) \exp(-2\pi i f t_l)\,\mathrm{d}t_l,\end{aligned} \tag{A.24}$$

where $\tau = t_j - t_l$. Defining the transformation

$$\widetilde{C^{-1}}(f, t_k) = \int_{-\infty}^{\infty} C^{-1}(t, t_k) \exp(-2\pi i f t)\,\mathrm{d}t, \tag{A.25}$$

and using the Wiener-Khinchin theorem to define power spectrum

$$S_n(f) = \lim_{T\to\infty} \frac{1}{T} \overline{|\widetilde{n}(f)|^2} \tag{A.26}$$

$$= 2\int_{-\infty}^{\infty} C_n(\tau) \exp(-2\pi i f \tau)\,\mathrm{d}\tau, \tag{A.27}$$

we have

$$\exp(-2\pi i f t_k) = \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \frac{S_n(f)}{2} \widetilde{C^{-1}}(f, t_k). \tag{A.28}$$

This can be rearranged to define $\widetilde{C^{-1}}(f, t_k)$.

The term in the exponential in equation (A.18) has the limit

$$\begin{aligned}\mathcal{H} &= \frac{1}{2} \lim_{T\to\infty;\Delta t\to 0} \sum_{j,k} C^{-1}_{jk} n_j n_k \\ &= \frac{1}{2} \lim_{T\to\infty;\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-T}^{T} \mathrm{d}t_j \int_{-T}^{T} \mathrm{d}t_k\, C^{-1}(t_j, t_k) n(t_j) n(t_k) \\ &= \frac{1}{2} \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{\infty} \mathrm{d}t_j \int_{-\infty}^{\infty} \mathrm{d}t_k \left\{ \int_{-\infty}^{\infty} \widetilde{C^{-1}}(f, t_k) \exp(2\pi i f t_j)\,\mathrm{d}f \right. \\ &\quad \left. \times \int_{-\infty}^{\infty} \widetilde{n}(f') \exp(2\pi i f' t_j)\,\mathrm{d}f' n(t_k) \right\} \\ &= \frac{1}{2} \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{\infty} \mathrm{d}t_k \int_{-\infty}^{\infty} \mathrm{d}f\, \widetilde{C^{-1}}(f, t_k) \widetilde{n}(-f) n(t_k) \\ &= \int_{-\infty}^{\infty} \mathrm{d}t_k \int_{-\infty}^{\infty} \mathrm{d}f\, \frac{\exp(-2\pi i f t_k)}{S_n(f)} \widetilde{n}^*(f) n(t_k) \\ &= \int_{-\infty}^{\infty} \frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)}\,\mathrm{d}f \\ &= \frac{1}{2}\left(n|n\right),\end{aligned} \tag{A.29}$$

defining the inner product

$$(g|h) = 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f)}{S_n(f)}\,\mathrm{d}f \tag{A.30}$$

$$= 2\int_{-\infty}^{\infty} \frac{\widetilde{g}^*(f)\widetilde{h}(f) + \widetilde{g}(f)\widetilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f. \tag{A.31}$$

This is a noise-weighted inner product over the space of real signals. The probability of the signal is

$$p(s|h) \propto \exp\left[-\frac{1}{2}(n|n)\right]. \tag{A.32}$$

A.2.2 Properties of the inner product

We will consider an ensemble average over multiple noise realisations, which is the same as a time average assuming stationarity of the noise spectrum

$$\begin{aligned}
\langle (n|n)\,(n|q)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(n(t+\tau)|p(t))\,(n(t+\tau)|q(t))\,\mathrm{d}\tau \\
&= \lim_{T\to\infty}\frac{2}{T}\int_{-T}^{T}\mathrm{d}\tau\left\{\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f)\exp(-2\pi i f\tau)\widetilde{p}(f)}{S_n(f)}\,\mathrm{d}f\right. \\
&\quad\left.\times\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f')\exp(-2\pi i f'\tau)\widetilde{q}(f')}{S_n(f')}\,\mathrm{d}f'\right\} \\
&= \lim_{T\to\infty}\frac{2}{T}\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f)\widetilde{n}(f)}{S_n(f)}\frac{\widetilde{p}(f)\widetilde{q}^*(f)}{S_n(f)}\,\mathrm{d}f\int_{-\infty}^{\infty}\frac{\widetilde{n}^*(f')}{S_n(f')}\,\mathrm{d}f' \\
&= (p|q)\,,
\end{aligned} \tag{A.33}$$

using the definition of the noise spectrum to obtain the final line.

Appendix B

The loss cone

When considering the orbits of stars about a massive black hole (MBH), the loss cone describes a region of velocity space that is depopulated because of tidal disruption.

A main sequence star may be disrupted by tidal forces before it is swallowed by a MBH, we will define the tidal disruption radius as r_{T}. We would expect any orbit that passes inside r_{T} to be depopulated unless stars may successfully escape to another orbit before being disrupted. Stars' velocities change because of gravitational interaction with other stars. Deflections can be modelled as a series of two-body encounters, the cumulative effect of which is a random walk in velocity space. Changes scale with the square-root of time, with the relaxation time-scale τ_{R} setting the scale.

Consider a typical star at a distance r from the MBH. We can decompose its motion into radial and tangential components as

$$v_{\mathrm{r}} = v\cos\theta \tag{B.1a}$$
$$v_{\perp} = v\sin\theta. \tag{B.1b}$$

Over a dynamical time-scale t_{dyn}, we expect that stars would change velocity by a typical amount

$$\theta_{\mathrm{D}} \approx \left(\frac{t_{\mathrm{dyn}}}{\tau_{\mathrm{R}}}\right)^{1/2}, \tag{B.2}$$

assuming this change is small. We define the loss cone angle θ_{LC} to describe the range of trajectories that will proceed to pass within a distance r_{T} of the MBH. By comparing the diffusion and loss cone angles we can deduce if we would expect orbits to be depleted: if $\theta_{\mathrm{D}} > \theta_{\mathrm{LC}}$ a star can safely diffuse out of the loss cone before it is destroyed, whereas if $\theta_{\mathrm{D}} < \theta_{\mathrm{LC}}$ a star will be disrupted before it can change its velocity sufficiently, leading to the depopulation of the orbit.

Frank and Rees first introduced the loss cone. They considered stars on nearly radial orbits. The orbital energy and angular momentum (per unit mass) of an object with eccentricity e and periapse radius r_{p} are

$$\mathcal{E} = -\frac{GM_{\bullet}(1-e)}{2r_{\mathrm{p}}} \tag{B.3}$$
$$\mathcal{J}^2 = GM_{\bullet}(1+e)r_{\mathrm{p}}, \tag{B.4}$$

where $M_\bullet$ is the MBH's mass. The angular momentum can also be defined as

$$\begin{aligned}\mathcal{J}^2 &= v_\perp^2 r^2 \\ &\simeq \theta^2 v^2 r^2,\end{aligned} \tag{B.5}$$

using the small angle approximation. Frank and Rees took the limit $e \to 1$, then setting $r_\mathrm{p} = r_\mathrm{T}$ to demarcate the limit of the loss cone, we can rearrange to find

$$\theta_\mathrm{LC} \simeq \frac{2GM_\bullet r_\mathrm{T}}{v^2 r^2}. \tag{B.6}$$

We need to find the speed at r. Frank and Rees use a typical value

$$v^2 \simeq 3\sigma^2, \tag{B.7}$$

where σ is the 1D velocity dispersion. They assume that the velocity dispersion is Keplerian within the core region where dynamics are dominated by the MBH, and is a constant outside of this

$$\sigma^2 \simeq \begin{cases} \frac{GM_\bullet}{r} & r < r_\mathrm{c} \\ \frac{GM_\bullet}{r_\mathrm{c}} & r < r_\mathrm{c} \end{cases}. \tag{B.8}$$

Here the core radius r_c is chosen such that

$$r_\mathrm{c} = \frac{GM_\bullet}{\sigma_0^2}, \tag{B.9}$$

where σ_0 is the 1D velocity dispersion far from the MBH. Substituting for v^2 in equation (B.6) gives

$$\theta_\mathrm{LC}^2 \simeq \begin{cases} \frac{2r_\mathrm{T}}{3r} & r < r_\mathrm{c} \\ \frac{2r_\mathrm{T}r_\mathrm{c}}{3r^2} & r < r_\mathrm{c} \end{cases}. \tag{B.10}$$

Frank and Rees make one final modification, introducing a gravitational focusing factor f such that

$$\theta_\mathrm{LC} \simeq f \begin{cases} \left(\frac{2r_\mathrm{T}}{3r}\right)^{1/2} & r < r_\mathrm{c} \\ \left(\frac{2r_\mathrm{T}r_\mathrm{c}}{3r^2}\right)^{1/2} & r < r_\mathrm{c} \end{cases}. \tag{B.11}$$

The focusing factor could be imagined as being the correction from assuming that stars travel along straight lines, such that $\tan\theta_\mathrm{LC} = r_\mathrm{T}/r$, to accounting for a Keplerian trajectory about the MBH.

It is unappealing to include an arbitrary, albeit order unitary, factor in the expression. Additionally, there are various approximations in the derivation which are restrictive. Considering the orbital energy for $v^2 = 3\sigma^2$ inside the core

$$\frac{3\sigma^2}{2} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet}{2r_\mathrm{T}} \tag{B.12}$$

$$\frac{3GM_\bullet}{2r} - \frac{GM_\bullet}{r} = -\frac{GM_\bullet(1-e)}{2r_\mathrm{T}} \tag{B.13}$$

$$\frac{r_\mathrm{T}}{r} = e - 1. \tag{B.14}$$

Since the radii must be positive, this enforces that $e \geq 1$: the orbits could be marginally bound at best. As we have taken the limit $e \to 1$, assuming that $r \gg r_{\mathrm{T}}$ this is still self-consistent. However, it is desirable to relax these conditions.

Let us consider an orbit with $r_{\mathrm{p}} = r_{\mathrm{T}}$, which gives the edge of the loss cone. The angular momentum (squared) is

$$\sin^2\theta_{\mathrm{LC}} v^2 r^2 = GM_{\bullet}(1+e)r_{\mathrm{T}}. \tag{B.15}$$

The energy is

$$\frac{v}{2} - \frac{GM_{\bullet}}{r} = -\frac{GM_{\bullet}}{2r_{\mathrm{T}}}. \tag{B.16}$$

Combining these to eliminate the velocity gives

$$\sin^2\theta_{\mathrm{LC}} = \frac{(1+e)r_{\mathrm{T}}^2}{2rr_{\mathrm{T}} - (1-e)r^2}. \tag{B.17}$$

This has been obtained without making any assumptions about the velocity dispersion or the position of the star. Since we have considered the Keplerian orbit, there should be no need to introduce a focusing factor.

This is similar in form to the classic result. Consider an orbit with eccentricity $e = 1 - \epsilon$, where ϵ is small. Let us choose the star to be at a characteristic distance set by its semimajor axis $a = r_{\mathrm{p}}/(1-e)$, such that

$$r = \frac{r_{\mathrm{T}}}{\epsilon}. \tag{B.18}$$

This ensures that $r \gg r_{\mathrm{T}}$. Therefore we have matched the assumptions of Frank and Rees. Substituting into our loss cone formula

$$\begin{aligned}\sin^2\theta_{\mathrm{LC}} &= \frac{(2-\epsilon)r_{\mathrm{T}}^2}{2rr_{\mathrm{T}} + \epsilon r^2} \\ &\simeq \frac{2r_{\mathrm{T}}}{3r},\end{aligned} \tag{B.19}$$

retaining terms to first order in ϵ. Since this is small, we can use the small angle approximation to recover the result of equation (B.10).