Linearized $f(R)$ Gravity: Gravitational Radiation & Solar System Tests

Christopher P. L. Berry* and Jonathan R. Gair†
Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA, United Kingdom
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We investigate the linearized form of metric $f(R)$-gravity, assuming that $f(R)$ is analytic about $R = 0$ so it may be expanded as $f(R) = R + a_2 R^2/2 + \dots$. Gravitational radiation is modified, admitting an extra mode of oscillation, that of the Ricci scalar. We derive the energy-momentum pseudotensor for the radiation. We also present weak-field metrics for simple sources. These demonstrate that Kerr (or Schwarzschild) black holes do not exist in $f(R)$-gravity. We apply the metrics to tests that could constrain $f(R)$. We show that light deflection experiments cannot distinguish $f(R)$-gravity from general relativity as both have an effective post-Newtonian parameter $\gamma = 1$. We find that planetary precession rates are enhanced relative to general relativity; from the orbit of Mercury we derive the bound $|a_2| \lesssim 1.2 \times 10^{18}$ m^2. Eöt-Wash experiments provide a stricter bound $|a_2| \lesssim 2 \times 10^{-9}$ m^2. Although the former is weaker, it is still of interest in the case that the effective form of $f(R)$ is modified in different regions, perhaps through the chameleon mechanism. Assuming the laboratory bound is universal, we conclude that the propagating Ricci scalar mode cannot be excited by astrophysical sources.

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I. INTRODUCTION TO $f(R)$ THEORY

General relativity (GR) is a well tested theory of gravity [?]; however it is still exciting to explore alternative theories. This is motivated by the need to explain dark matter and dark energy in cosmology, trying to formulate a quantizable theory of gravity, or simple curiosity regarding the uniqueness of GR. One of the simplest extensions to standard GR is the class of $f(R)$ theories [? ?].

In this work we will look at metric $f(R)$-gravity. We focus on the modifications to gravitational radiation and possible solar system tests that can be used to constrain the theory. We begin with a review of the $f(R)$ field equations. In Sec. II we derive the linearized equations and in Sec. III we apply these to find wave solutions. These results are largely known in the literature, but are worked out here *ab initio*; they are included as a compendium of useful results within a consistent system of notation. We derive the energy-momentum pseudotensor for gravitational radiation in Sec. IV. In Sec. V we look at the effects of introducing a source term and derive the weak-field metrics for a point source, a slowly rotating point source and a uniform density sphere, recovering some results known for quadratic theories of gravity. These are used in Sec. VI derive constraints for $f(R)$ based upon solar system and laboratory tests.

Throughout this work we will use the time-like sign convention of Landau and Lifshitz [?]:

1. The metric has signature $(+, -, -, -)$.
2. The Riemann tensor is defined as $R^{\mu}{}_{\nu\sigma\rho} = \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} - \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\mu}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\sigma\nu}$.
3. The Ricci tensor is defined as the contraction $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$.

Greek indices are used to represent spacetime indices $\mu = \{0, 1, 2, 3\}$ and lowercase Latin indices are used for spatial indices $i = \{1, 2, 3\}$. Natural units with $c = 1$ will be used throughout, but factors of G will be retained.

A. The Action & Field Equations

General relativity may be derived from the Einstein-Hilbert action [? ?]

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G} \int R\sqrt{-g}\,\mathrm{d}^4x. \tag{1}$$

*cplb2@ast.cam.ac.uk
†jgair@ast.cam.ac.uk

In $f(R)$ theory we make a simple modification of the action to include an arbitrary function of the Ricci scalar R such that [?]

$$S[g] = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\,\mathrm{d}^4x. \tag{2}$$

Including the function $f(R)$ gives extra freedom in defining the behaviour of gravity; while this action may not encode the true theory of gravity it could contain sufficient information to act as an effective field theory, correctly describing phenomenological behaviour [?]. We will assume that $f(R)$ is analytic about $R=0$ so that it can be expressed as a power series [? ? ? ?]

$$f(R) = a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \dots \tag{3}$$

Since the dimensions of $f(R)$ must be the same as of R, $[a_n] = [R]^{(1-n)}$. To link to GR we will set $a_1 = 1$; any rescaling can be absorbed into the definition of G.

The field equations are obtained by a variational principle; there are several ways of achieving this. To derive the Einstein field equations from the Einstein-Hilbert action one may use the standard metric variation or the Palatini variation [?]. Both approaches can be used for $f(R)$, however they yield different results [? ?]. Following the metric formalism, one varies the action with respect to the metric $g^{\mu\nu}$. Following the Palatini formalism one varies the action with respect to both the metric $g^{\mu\nu}$ and the connection $\Gamma^{\rho}{}_{\mu\nu}$, which are treated as independent quantities: the connection is not the Levi-Civita metric connection.[1]

Finally, there is a third version of $f(R)$-gravity: metric-affine $f(R)$-gravity [? ?]. This goes beyond the Palatini formalism by supposing that the matter action is dependent on the variational independent connection. Parallel transport and the covariant derivative are divorced from the metric. This theory has its attractions: it allows for a natural introduction of torsion. However, it is not a metric theory of gravity and so cannot satisfy all the postulates of the Einstein equivalence principle [?]: a free particle does not necessarily follow a geodesic and so the effects of gravity might not be locally removed [?]. The implications of this have not been fully explored, but for this reason we will not consider the theory further.

We will restrict our attention to metric $f(R)$-gravity. This is preferred as the Palatini formalism has undesirable properties: static spherically symmetric objects described by a polytropic equation of state are subject to a curvature singularity [? ?]. Varying the action with respect to the metric $g^{\mu\nu}$ produces

$$\begin{aligned}\delta S \;=\; & \frac{1}{16\pi G}\int \Big\{ f'(R)\sqrt{-g}\left[R_{\mu\nu} - \nabla_\mu\nabla_\nu \; + g_{\mu\nu}\Box\right] \\ & - f(R)\frac{1}{2}\sqrt{-g}g_{\mu\nu}\Big\}\,\delta g^{\mu\nu}\,\mathrm{d}^4x,\end{aligned} \tag{4}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian and a prime denotes differentiation with respect to R. Proceeding from here requires certain assumptions regarding surface terms. In the case of the Einstein-Hilbert action these gather into a total derivative. It is possible to subtract this from the action to obtain a well-defined variational quantity [? ?]. This is not the case for general $f(R)$ [?]. However, since the action includes higher-order derivatives of the metric we are at liberty to fix more degrees of freedom at the boundary, in so doing eliminating the importance of the surface terms [? ?]. There is no well described prescription for this so we proceed directly to the field equations.

The vacuum field equations are

$$f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu} = 0. \tag{5}$$

Taking the trace of our field equation gives

$$f'R + 3\Box f' - 2f = 0. \tag{6}$$

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[1] Requiring that the metric and Palatini formalisms produce the same field equations, assuming an action that only depends on the metric and Riemann tensor, results in Lovelock gravity [?]. Lovelock gravities require the field equations to be divergence free and no more than second order; in four dimensions the only possible Lovelock gravity is GR with a potentially non-zero cosmological constant [? ? ?].

If we consider a uniform flat spacetime $R = 0$, this equation gives [?]

$$a_0 = 0. \tag{7}$$

In analogy to the Einstein tensor, we define

$$\mathcal{G}_{\mu\nu} = f' R_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' - \frac{f}{2} g_{\mu\nu}, \tag{8}$$

so that in a vacuum

$$\mathcal{G}_{\mu\nu} = 0. \tag{9}$$

B. Conservation Of Energy-Momentum

If we introduce matter with a stress-energy tensor $T_{\mu\nu}$, the field equations become

$$\mathcal{G}_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{10}$$

Acting upon this with the covariant derivative

$$\begin{aligned} 8\pi G \nabla^\mu T_{\mu\nu} &= \nabla^\mu \mathcal{G}_{\mu\nu} \\ &= R_{\mu\nu} \nabla^\mu f' + f' \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \\ &\quad - \left(\Box \nabla_\nu - \nabla_\nu \Box \right) f'. \end{aligned} \tag{11}$$

The second term contains the covariant derivative of the Einstein tensor and so is zero. The final term can be shown to be

$$\begin{aligned} \left(\Box \nabla_\nu - \nabla_\nu \Box \right) f' &= g^{\mu\sigma} \left(\nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\mu \nabla_\sigma \right) f' \\ &= R_{\tau\nu} \nabla^\tau f', \end{aligned} \tag{12}$$

which is a useful geometric identity [?]. Using this

$$\begin{aligned} 8\pi G \nabla^\mu T_{\mu\nu} &= R_{\mu\nu} \nabla^\mu f' - R_{\mu\nu} \nabla^\mu f' \\ &= 0. \end{aligned} \tag{13}$$

Consequently energy-momentum is a conserved quantity in the same way as in GR, as is expected from the symmetries of the action.

II. LINEARIZED THEORY

We start our investigation of $f(R)$ by looking at linearized theory. This is a weak-field approximation that assumes only small deviations from a flat background, greatly simplifying the field equations. Just as in GR, the linearized framework provides a natural way to study gravitational waves (GWs). We will see that the linearized field equations will reduce down to flat-space wave equations: GWs are as much a part of $f(R)$-gravity as of GR.

Consider a perturbation of the metric from flat Minkowski space such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \tag{14}$$

where, more formally, we mean that $h_{\mu\nu} = \varepsilon H_{\mu\nu}$ for a small parameter ε.[2] We will consider terms only to $\mathcal{O}(\varepsilon)$. Thus, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{15}$$

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[2] It is because we wish to perturb about flat spacetime that we have required $f(R)$ to be analytic about $R = 0$.

where we have used the Minkowski metric to raise the indices on the right, defining

$$h^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}. \tag{16}$$

Similarly, the trace h is given by

$$h = \eta^{\mu\nu}h_{\mu\nu}. \tag{17}$$

All quantities denoted by "h" are strictly $\mathcal{O}(\varepsilon)$.

The linearized connection is

$$\Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \tag{18}$$

To $\mathcal{O}(\varepsilon)$ the covariant derivative of any perturbed quantity will be the same as the partial derivative. The Riemann tensor is

$$R^{(1)\lambda}{}_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu\partial_\nu h^\lambda_\rho + \partial^\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\rho h^\lambda_\nu - \partial^\lambda\partial_\nu h_{\mu\rho}), \tag{19}$$

where we have raised the index on the differential operator with the background Minkowski metric. Contracting gives the Ricci tensor

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}(\partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}), \tag{20}$$

where the d'Alembertian operator is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Contracting this with $\eta^{\mu\nu}$ gives the first order Ricci scalar

$$R^{(1)} = \partial_\mu\partial_\rho h^{\rho\mu} - \Box h. \tag{21}$$

To $\mathcal{O}(\varepsilon)$ we can write $f(R)$ as a Maclaurin series

$$f(R) = a_0 + R^{(1)}; \tag{22}$$
$$f'(R) = 1 + a_2 R^{(1)}. \tag{23}$$

As we are perturbing from a Minkowski background where the Ricci scalar vanishes, we use (7) to set $a_0 = 0$. Inserting these into (8) and retaining terms to $\mathcal{O}(\varepsilon)$ yields

$$\mathcal{G}^{(1)}{}_{\mu\nu} = R^{(1)}{}_{\mu\nu} - \partial_\mu\partial_\nu(a_2 R^{(1)}) + \eta_{\mu\nu}\Box(a_2 R^{(1)}) - \frac{R^{(1)}}{2}\eta_{\mu\nu}. \tag{24}$$

Now consider the linearized trace equation, from (6)

$$\begin{aligned}\mathcal{G}^{(1)} &= R^{(1)} + 3\Box(a_2 R^{(1)}) - 2R^{(1)} \\ \mathcal{G}^{(1)} &= 3a_2\Box R^{(1)} - R^{(1)},\end{aligned} \tag{25}$$

where $\mathcal{G}^{(1)} = \eta^{\mu\nu}\mathcal{G}^{(1)}{}_{\mu\nu}$. This is the massive inhomogeneous Klein-Gordon equation. Setting $\mathcal{G} = 0$, as for a vacuum, we obtain the standard Klein-Gordon equation

$$\Box R^{(1)} + \Upsilon^2 R^{(1)} = 0, \tag{26}$$

defining the reciprocal length (squared)

$$\Upsilon^2 = -\frac{1}{3a_2}. \tag{27}$$

For a physically meaningful solution $\Upsilon^2 > 0$: we constrain $f(R)$ such that $a_2 < 0$ [? ? ? ?]. From Υ we define a reduced Compton wavelength

$$\lambda_R = \frac{1}{\Upsilon} \tag{28}$$

associated with this scalar mode.

The next step is to substitute in $h_{\mu\nu}$ to try to find wave solutions. We want a quantity $\overline{h}_{\mu\nu}$ that will satisfy a wave equation, related to $h_{\mu\nu}$ by

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + A_{\mu\nu}. \tag{29}$$

In GR we use the trace-reversed form where $A_{\mu\nu} = -(h/2)\eta_{\mu\nu}$. This will not suffice here, but let us look for a similar solution

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + B_{\mu\nu}. \tag{30}$$

The only rank two tensors in our theory are: $h_{\mu\nu}$, $\eta_{\mu\nu}$, $R^{(1)}{}_{\mu\nu}$, and $\partial_\mu\partial_\nu$; $h_{\mu\nu}$ has been used already, and we wish to eliminate $R^{(1)}{}_{\mu\nu}$, so we will try the simpler option based around $\eta_{\mu\nu}$. We want $B_{\mu\nu}$ to be $\mathcal{O}(\varepsilon)$; since we have already used h, we will try the other scalar quantity $R^{(1)}$. Therefore, we construct an ansatz

$$\overline{h}_{\mu\nu} = h_{\mu\nu} + \left(ba_2R^{(1)} - \frac{h}{2}\right)\eta_{\mu\nu}, \tag{31}$$

where a_2 has been included to ensure dimensional consistency and b is a dimensionless number. Contracting with the background metric yields

$$\overline{h} = 4ba_2R^{(1)} - h, \tag{32}$$

so we can eliminate h in our definition of $\overline{h}_{\mu\nu}$ to give

$$h_{\mu\nu} = \overline{h}_{\mu\nu} + \left(ba_2R^{(1)} - \frac{\overline{h}}{2}\right)\eta_{\mu\nu}. \tag{33}$$

Just as in GR, we have the freedom to perform a gauge transformation [**?** **?**]: the field equations are gauge invariant since we started with a function of the gauge invariant Ricci scalar. We will assume a Lorenz, or de Donder, gauge choice

$$\nabla^\mu\overline{h}_{\mu\nu} = 0; \tag{34}$$

to first order this is

$$\partial^\mu\overline{h}_{\mu\nu} = 0. \tag{35}$$

Subject to this, from (20), the Ricci tensor is

$$\begin{aligned} R^{(1)}{}_{\mu\nu} \;&=\; -\frac{1}{2}\left[2ba_2\partial_\mu\partial_\nu R^{(1)} + \Box\left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2}\eta_{\mu\nu}\right)\right. \\ &\quad \left. + \frac{b}{3}(R^{(1)} + \mathcal{G}^{(1)})\eta_{\mu\nu}\right]. \end{aligned} \tag{36}$$

Using this with (25) in (24) gives

$$\begin{aligned} \mathcal{G}^{(1)}{}_{\mu\nu} \;&=\; \frac{2-b}{6}\mathcal{G}^{(1)}\eta_{\mu\nu} - \frac{1}{2}\Box\left(\overline{h}_{\mu\nu} - \frac{\overline{h}}{2}\eta_{\mu\nu}\right) \\ &\quad -(b+1)\left(a_2\partial_\mu\partial_\nu R^{(1)} + \frac{1}{6}R^{(1)}\eta_{\mu\nu}\right). \end{aligned} \tag{37}$$

Picking $b = -1$ the final term vanishes, thus we set [**?** **?**]

$$\overline{h}_{\mu\nu} \;=\; h_{\mu\nu} - \left(a_2R^{(1)} + \frac{h}{2}\right)\eta_{\mu\nu} \tag{38a}$$

$$h_{\mu\nu} \;=\; \overline{h}_{\mu\nu} - \left(a_2R^{(1)} + \frac{\overline{h}}{2}\right)\eta_{\mu\nu}. \tag{38b}$$

From (21) the Ricci scalar is

$$\begin{aligned} R^{(1)} &= \Box\left(a_2 R^{(1)} - \frac{\overline{h}}{2}\right) - \Box(-4a_2 R^{(1)} - \overline{h}) \\ &= 3a_2 \Box R^{(1)} + \frac{1}{2}\Box\overline{h}. \end{aligned} \tag{39}$$

For consistency with (25), we require

$$-\frac{1}{2}\Box\overline{h} = \mathcal{G}^{(1)}. \tag{40}$$

Inserting this into (37), with $b = -1$, we see

$$-\frac{1}{2}\Box\overline{h}_{\mu\nu} = \mathcal{G}^{(1)}{}_{\mu\nu}; \tag{41}$$

we have our wave equation.

Should a_2 be sufficiently small that it can be regarded an $\mathcal{O}(\varepsilon)$ quantity, we recover the usual GR formulae to leading order within our analysis.

III. GRAVITATIONAL RADIATION

Having established two wave equations, (25) and (41), we now investigate their solutions. Consider waves in a vacuum, such that $\mathcal{G}_{\mu\nu} = 0$. Using a standard Fourier decomposition

$$\overline{h}_{\mu\nu} = \widehat{h}_{\mu\nu}(k_\rho)\exp\left(ik_\rho x^\rho\right), \tag{42}$$

$$R^{(1)} = \widehat{R}(q_\rho)\exp\left(iq_\rho x^\rho\right), \tag{43}$$

where k_μ and q_μ are 4-wavevectors. From (41) we know that k_μ is a null vector, so for a wave traveling along the z-axis

$$k^\mu = \omega(1, 0, 0, 1), \tag{44}$$

where ω is the angular frequency. Similarly, from (25)

$$q^\mu = \left(\Omega, 0, 0, \sqrt{\Omega^2 - \Upsilon^2}\right), \tag{45}$$

for frequency Ω. These waves do not travel at c, but have a group velocity

$$v(\Omega) = \frac{\sqrt{\Omega^2 - \Upsilon^2}}{\Omega}, \tag{46}$$

provided that $\Upsilon^2 > 0$, $v < 1 = c$. For $\Omega < \Upsilon$, we find an evanescently decaying wave. The traveling wave is dispersive. For waves made up of a range of frequency components there will be a time delay between the arrival of the high frequency and low frequency constituents. This may make it difficult to reconstruct a waveform, should the scalar mode be observed with a GW detector.

From the gauge condition (34) we find that k^μ is orthogonal to $\widehat{h}_{\mu\nu}$,

$$k^\mu \widehat{h}_{\mu\nu} = 0, \tag{47}$$

in this case

$$\widehat{h}_{0\nu} + \widehat{h}_{3\nu} = 0. \tag{48}$$

Let us consider the implications of (40) using equations (25) and (32),

$$\begin{aligned} \Box\left(4a_2 R^{(1)} + h\right) &= 0 \\ \Box h &= -\frac{4}{3}R^{(1)}. \end{aligned} \tag{49}$$

For non-zero $R^{(1)}$ (as required for the Ricci mode) there is no way to make a gauge choice such that the trace h will vanish [? ?]. This is distinct from in GR. It is possible, however, to make a gauge choice such that the trace $\overline{h}$ will vanish. Consider a gauge transformation generated by ξ_μ which satisfies $\Box\xi_\mu = 0$, and so has a Fourier decomposition

$$\xi_\mu = \widehat{\xi}_\mu \exp\left(ik_\rho x^\rho\right). \tag{50}$$

A transformation

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial^\rho\xi_\rho, \tag{51}$$

would ensure both conditions (34) and (41) are satisfied [?]. Under such a transformation

$$\widehat{h}_{\mu\nu} \to \widehat{h}_{\mu\nu} + i\left(k_\mu\widehat{\xi}_\nu + k_\nu\widehat{\xi}_\mu - \eta_{\mu\nu}k^\rho\widehat{\xi}_\rho\right). \tag{52}$$

We may therefore impose four further constraints (one for each $\widehat{\xi}_\mu$) upon $\widehat{h}_{\mu\nu}$. We take these to be

$$\widehat{h}_{0\nu} = 0, \qquad \widehat{h} = 0. \tag{53}$$

This might appear to be five constraints, however we have already imposed (48), and so setting $\widehat{h}_{00} = 0$ automatically implies $\widehat{h}_{03} = 0$. In this gauge we have

$$\begin{aligned} h_{\mu\nu} &= \overline{h}_{\mu\nu} - a_2 R^{(1)}\eta_{\mu\nu}, \\ h &= -4a_2 R^{(1)}. \end{aligned} \tag{54}$$

Thus $\overline{h}_{\mu\nu}$ behaves just as its GR counterpart, so we can define

$$\left[\widehat{h}_{\mu\nu}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

where h_+ and $h_\times$ are constants representing the amplitudes of the two transverse polarizations of gravitational radiation.

It is important that our solutions reduce to those of GR in the event that $f(R) = R$. In our linearized approach this corresponds to $a_2 \to 0$, $\Upsilon^2 \to \infty$. We see from (45) that in this limit it would take an infinite frequency to excite a propagating Ricci mode, and evanescent waves would decay away infinitely quickly. Therefore there would be no detectable Ricci modes and we would only observe the two polarizations found in GR. Additionally, $\overline{h}_{\mu\nu}$ would simplify to its usual trace-reversed form.

IV. ENERGY-MOMENTUM TENSOR

We expect gravitational radiation to carry energy-momentum. Unfortunately, it is difficult to define a proper energy-momentum tensor for a gravitational field: as a consequence of the equivalence principle it is possible to transform to a freely falling frame, eliminating the gravitational field and any associated energy density at a given point, although we can still define curvature in the neighbourhood of this point [? ?]. We will do nothing revolutionary here, but will follow the approach of Isaacson [? ?]. The full field equations (5) have no energy-momentum tensor for the gravitational field on the right-hand side. However, by expanding beyond the linear terms we can find a suitable energy-momentum pseudotensor for GWs. Expanding $\mathcal{G}_{\mu\nu}$ in orders of $h_{\mu\nu}$

$$\mathcal{G}_{\mu\nu} = \mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} + \mathcal{G}^{(1)}{}_{\mu\nu} + \mathcal{G}^{(2)}{}_{\mu\nu} + \dots \tag{56}$$

We use (B) for the background term instead of (0) to avoid potential confusion regarding its order in ε. So far we have assumed that our background is flat, however we can imagine that should the gravitational radiation carry energy-momentum then this would act as a source of curvature for the background. This is a second-order effect that may be encoded, to accuracy of $\mathcal{O}(\varepsilon^2)$, as

$$\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} = -\mathcal{G}^{(2)}{}_{\mu\nu}. \tag{57}$$

By shifting $\mathcal{G}^{(2)}{}_{\mu\nu}$ to the right-hand side we create an effective energy-momentum tensor. As in GR we will average over several wavelengths, assuming that the background curvature is on a larger scale [?],

$$\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu} = -\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle . \tag{58}$$

By averaging we probe the curvature in a macroscopic region about a given point in spacetime, yielding a gauge invariant measure of the gravitational field strength. The averaging can be thought of as smoothing out the rapidly varying ripples of the radiation, leaving only the coarse-grained component that acts as a source for the background curvature.[3] The energy-momentum pseudotensor for the radiation is

$$t_{\mu\nu} = -\frac{1}{8\pi G}\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle . \tag{59}$$

Having made this provisional identification, we must set about carefully evaluating the various terms in (56). We begin as in Sec. II by defining a total metric

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}, \tag{60}$$

where $\gamma_{\mu\nu}$ is the background metric. This changes our definition for $h_{\mu\nu}$: instead of being the total perturbation from flat Minkowski, it is the dynamical part of the metric with which we associate radiative effects. Since we know that $\mathcal{G}^{(\mathrm{B})}{}_{\mu\nu}$ is $\mathcal{O}(\varepsilon^2)$, we decompose our background metric as

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}, \tag{61}$$

where $j_{\mu\nu}$ is $\mathcal{O}(\varepsilon^2)$ to ensure that $R^{(\mathrm{B})\lambda}{}_{\mu\nu\rho}$ is also $\mathcal{O}(\varepsilon^2)$. Therefore its introduction will make no difference to the linearized theory.

We will consider terms only to $\mathcal{O}(\varepsilon^2)$. We identify $\Gamma^{(1)\rho}{}_{\mu\nu}$ from (18). There is one small subtlety: whether we use the background metric $\gamma^{\mu\nu}$ or $\eta^{\mu\nu}$ to raise indices of ∂_μ and $h_{\mu\nu}$. Fortunately, to the accuracy considered here, it does not make a difference; however, we will consider the indices to be changed with $\gamma^{\mu\nu}$. We will not distinguish between ∂_μ and $\nabla^{(\mathrm{B})}{}_\mu$, the covariant derivative for the background metric: to the order of accuracy required covariant derivatives would commute and $\nabla^{(\mathrm{B})}{}_\mu$ behaves just like ∂_μ. Thus

$$\begin{aligned} \Gamma^{(1)\rho}{}_{\mu\nu} &= \frac{1}{2}\gamma^{\rho\lambda}\left[\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2R^{(1)}\gamma_{\lambda\nu}\right) + \partial_\nu\left(\overline{h}_{\lambda\mu}\right.\right. \\ &\qquad \left.\left. - a_2R^{(1)}\gamma_{\lambda\mu}\right) - \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2R^{(1)}\gamma_{\mu\nu}\right)\right], \end{aligned} \tag{62}$$

and

$$\begin{aligned} \Gamma^{(2)\rho}{}_{\mu\nu} &= -\frac{1}{2}h^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\ &= -\frac{1}{2}\left(\overline{h}^{\rho\lambda} - a_2R^{(1)}\gamma^{\rho\lambda}\right)\left[\partial_\mu\left(\overline{h}_{\lambda\nu} - a_2R^{(1)}\gamma_{\lambda\nu}\right)\right. \\ &\qquad \left. + \partial_\nu\left(\overline{h}_{\lambda\mu} - a_2R^{(1)}\gamma_{\lambda\mu}\right) - \partial_\lambda\left(\overline{h}_{\mu\nu} - a_2R^{(1)}\gamma_{\mu\nu}\right)\right]. \end{aligned} \tag{63}$$

For the Ricci tensor we can use our linearized expression, (20), for the first order term,

$$R^{(1)}{}_{\mu\nu} = 2a_2\partial_\mu\partial_\nu R^{(1)} + \frac{1}{6}R^{(1)}\gamma_{\mu\nu}. \tag{64}$$

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[3] By averaging we do not try to localise the energy of a wave to within a wavelength; for the massive Ricci scalar mode we always consider scales greater than λ_R.

The next term is

$$
\begin{aligned}
R^{(2)}{}_{\mu\nu} &= \partial_\rho \Gamma^{(2)}{}^{\rho}{}_{\mu\nu} - \partial_\nu \Gamma^{(2)}{}^{\rho}{}_{\mu\rho} + \Gamma^{(1)}{}^{\rho}{}_{\mu\nu}\Gamma^{(1)}{}^{\sigma}{}_{\rho\sigma} \\
&\quad -\Gamma^{(1)}{}^{\rho}{}_{\mu\sigma}\Gamma^{(1)}{}^{\sigma}{}_{\rho\nu} \\
&= \frac{1}{2}\left\{ \frac{1}{2}\partial_\mu \overline{h}_{\sigma\rho}\partial_\nu \overline{h}^{\sigma\rho} + \overline{h}^{\sigma\rho}\left[\partial_\mu\partial_\nu \overline{h}_{\sigma\rho} \right.\right. \\
&\quad + \partial_\sigma\partial_\rho\left(\overline{h}_{\mu\nu} - a_2 R^{(1)}\gamma_{\mu\nu}\right) - \partial_\nu\partial_\rho\left(\overline{h}_{\sigma\mu}\right. \\
&\quad \left.\left. - a_2 R^{(1)}\gamma_{\sigma\mu}\right) - \partial_\mu\partial_\rho\left(\overline{h}_{\sigma\nu} - a_2 R^{(1)}\gamma_{\sigma\nu}\right)\right] \\
&\quad + \partial^\rho \overline{h}^{\sigma}_{\nu}\left(\partial_\rho \overline{h}_{\sigma\mu} - \partial_\sigma \overline{h}_{\rho\mu}\right) - a_2 \partial^\sigma R^{(1)}\partial_\sigma \overline{h}_{\mu\mu} \\
&\quad + a_2^2\left(2R^{(1)}\partial_\mu\partial_\nu R^{(1)} + 3\partial_\mu R^{(1)}\partial_\nu R^{(1)}\right. \\
&\quad \left.\left. + R^{(1)}\Box^{(B)}R^{(1)}\gamma_{\mu\nu}\right)\right\}. \qquad (65)
\end{aligned}
$$

The d'Alembertian is $\Box^{(B)} = \gamma^{\mu\nu}\partial_\mu\partial_\nu$.

To find the Ricci scalar we contract the Ricci tensor with the full metric. To $\mathcal{O}(\varepsilon^2)$,

$$
\begin{aligned}
R^{(B)} &= \gamma^{\mu\nu}R^{(B)}{}_{\mu\nu} \qquad (66) \\
R^{(1)} &= \gamma^{\mu\nu}R^{(1)}{}_{\mu\nu} \qquad (67) \\
R^{(2)} &= \gamma^{\mu\nu}R^{(2)}{}_{\mu\nu} - h^{\mu\nu}R^{(1)}{}_{\mu\nu} \\
&= \frac{3}{4}\partial_\mu \overline{h}_{\sigma\rho}\partial^\mu \overline{h}^{\sigma\rho} - \frac{1}{2}\partial^\rho \overline{h}^{\sigma\mu}\partial_\sigma \overline{h}_{\rho\mu} - 2a_2 \overline{h}^{\mu\nu}\partial_\mu\partial_\nu R^{(1)} \\
&\quad + a_2 {R^{(1)}}^2 + \frac{3a_2}{2}\partial_\mu R^{(1)}\partial^\mu R^{(1)}. \qquad (68)
\end{aligned}
$$

Using these

$$
\begin{aligned}
f^{(B)} &= R^{(B)} \qquad (69) \\
f^{(1)} &= R^{(1)} \qquad (70) \\
f^{(2)} &= R^{(2)} + \frac{a_2}{2}{R^{(1)}}^2, \qquad (71)
\end{aligned}
$$

and

$$
\begin{aligned}
f'^{(B)} &= a_2 R^{(B)} \qquad (72) \\
f'^{(0)} &= 1 \qquad (73) \\
f'^{(1)} &= a_2 R^{(1)} \qquad (74) \\
f'^{(2)} &= a_2 R^{(2)} + \frac{a_3}{2}{R^{(1)}}^2. \qquad (75)
\end{aligned}
$$

We list a zeroth order term for clarity.

Combining all of these

$$
\begin{aligned}
\mathcal{G}^{(2)}{}_{\mu\nu} &= R^{(2)}{}_{\mu\nu} + f'^{(1)}R^{(1)}{}_{\mu\nu} - \partial_\mu\partial_\nu f'^{(2)} + \Gamma^{(1)}{}^{\rho}{}_{\nu\mu}\partial_\rho f'^{(1)} + \gamma_{\mu\nu}\gamma^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(2)} - \gamma_{\mu\nu}\gamma^{\rho\sigma}\Gamma^{(1)}{}^{\lambda}{}_{\sigma\rho}\partial_\lambda f'^{(1)} \\
&\quad -\gamma_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(1)} + h_{\mu\nu}\gamma^{\rho\sigma}\partial_\rho\partial_\sigma f'^{(1)} - \frac{1}{2}f^{(2)}\gamma_{\mu\nu} - \frac{1}{2}f^{(1)}h_{\mu\nu} \\
&= R^{(2)}{}_{\mu\nu} + a_2\left(\gamma_{\mu\nu}\Box^{(B)} - \partial_\mu\partial_\nu\right)R^{(2)} - \frac{1}{2}R^{(2)}\gamma_{\mu\nu} + a_3\left(\gamma_{\mu\nu}\Box^{(B)} - \partial_\mu\partial_\nu\right){R^{(1)}}^2 - \frac{1}{6}\overline{h}_{\mu\nu}R^{(1)} \\
&\quad -a_2\gamma_{\mu\nu}\overline{h}^{\sigma\rho}\partial_\sigma\partial_\rho R^{(1)} + \frac{a_2}{2}\partial^\rho R^{(1)}\left(\partial_\mu \overline{h}_{\rho\nu} + \partial_\nu \overline{h}_{\rho\mu} - \partial_\rho \overline{h}_{\mu\nu}\right) + a_2\left(R^{(1)}R^{(1)}{}_{\mu\nu} + \frac{1}{4}{R^{(1)}}^2\gamma_{\mu\nu}\right) \\
&\quad -a_2^2\left(\partial_\mu R^{(1)}\partial_\nu R^{(1)} + \frac{1}{2}\gamma_{\mu\nu}\partial^\rho R^{(1)}\partial_\rho R^{(1)}\right). \qquad (76)
\end{aligned}
$$

It is simplest to split this up for the purposes of averaging. Since we average over all directions at each point, gradients average to zero [?]

$$\langle \partial_\mu V \rangle = 0. \tag{77}$$

As a corollary of this we have

$$\langle U \partial_\mu V \rangle = - \langle V \partial_\mu U \rangle . \tag{78}$$

Repeated application of this, together with our gauge condition, (34), and wave equations, (25) and (41), allows us to eliminate many terms. Those that do not average to zero are

$$\begin{aligned} \left\langle R^{(2)}{}_{\mu\nu} \right\rangle &= \left\langle -\frac{1}{4}\partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} + \frac{a_2^2}{2} \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right. \\ &\quad \left. + \frac{a_2}{6}\gamma_{\mu\nu} R^{(1)} \right\rangle ; \end{aligned} \tag{79}$$

$$\left\langle R^{(2)} \right\rangle = \left\langle \frac{3a_2}{2} {R^{(1)}}^2 \right\rangle ; \tag{80}$$

$$\left\langle R^{(1)} R^{(1)}{}_{\mu\nu} \right\rangle = \left\langle a_2 R^{(1)} \partial_\mu \partial_\nu R^{(1)} + \frac{1}{6}\gamma_{\mu\nu} {R^{(1)}}^2 \right\rangle . \tag{81}$$

Combining these gives

$$\left\langle \mathcal{G}^{(2)}{}_{\mu\nu} \right\rangle = \left\langle -\frac{1}{4}\partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} - \frac{3a_2^2}{2} \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle . \tag{82}$$

Thus we obtain the result

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \partial_\mu \overline{h}_{\sigma\rho} \partial_\nu \overline{h}^{\rho\sigma} + 6a_2^2 \partial_\mu R^{(1)} \partial_\nu R^{(1)} \right\rangle . \tag{83}$$

In the limit of $a_2 \to 0$ we obtain the familiar GR result as required. The GR result is also recovered if $R^{(1)} = 0$, as would be the case if the Ricci mode was not excited, for example if the frequency was below the cut off frequency Υ. Rewriting the pseudotensor in terms of metric perturbation $h_{\mu\nu}$, using (54),

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \partial_\mu h_{\sigma\rho} \partial_\nu h^{\rho\sigma} + \frac{1}{8}\partial_\mu h \partial_\nu h \right\rangle . \tag{84}$$

These results do not depend upon a_3 or higher order coefficients.

The pseudotensor could be used to constrain the parameter a_2 through observations of the energy and momentum carried away by GWs. Of particular interest would be a system with a frequency that evolved from $\omega < \Upsilon$ to $\omega > \Upsilon$ as then we would witness the switching on of the propagating Ricci mode. If we could accurately identify the cutoff frequency we could accurately measure a_2. However, see Sec. VI E for further discussion of why this is unlikely to happen.

V. $f(R)$ WITH A SOURCE

Having considered radiation in a vacuum, we now add a source term. We want a first order perturbation, so the linearized field equations are

$$\mathcal{G}^{(1)}{}_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{85}$$

We will again assume a Minkowski background, considering terms to $\mathcal{O}(\varepsilon)$ only. To solve the wave equations (25) and (41) with this source term we use a Green's function

$$\left(\Box + \Upsilon^2\right) \mathscr{G}_\Upsilon(x, x') = \delta(x - x'), \tag{86}$$

where $\Box$ acts on x. The Green's function is familiar as the Klein-Gordon propagator (up to a factor of $-i$) [?]

$$\mathscr{G}_\Upsilon(x, x') = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp\left[-ip \cdot (x - x')\right]}{\Upsilon^2 - p^2}. \tag{87}$$

This can be evaluated by a suitable contour integral to give

$$\mathscr{G}_\Upsilon(x, x') = \begin{cases} \displaystyle\int \frac{\mathrm{d}\omega}{2\pi} \exp\left[-i\omega(t-t')\right] \frac{1}{4\pi r} \exp\left[i\left(\omega^2 - \Upsilon^2\right)^{1/2} r\right] & \omega^2 > \Upsilon^2 \\ \displaystyle\int \frac{\mathrm{d}\omega}{2\pi} \exp\left[-i\omega(t-t')\right] \frac{1}{4\pi r} \exp\left[-\left(\Upsilon^2 - \omega^2\right)^{1/2} r\right] & \omega^2 < \Upsilon^2 \end{cases}, \tag{88}$$

where we have introduced $t = x^0$, $t' = x'^0$ and $r = |\boldsymbol{x} - \boldsymbol{x'}|$. For $\Upsilon = 0$

$$\mathscr{G}_0(x, x') = \frac{\delta(t - t' - r)}{4\pi r}, \tag{89}$$

the familiar retarded-time Green's function. We can use this to solve (41)

$$\begin{aligned} \overline{h}_{\mu\nu}(x) &= -16\pi G \int \mathrm{d}^4x'\, \mathscr{G}_0(x, x') T_{\mu\nu}(x') \\ &= -4G \int \mathrm{d}^3x' \frac{T_{\mu\nu}(t - r, \boldsymbol{x'})}{r}. \end{aligned} \tag{90}$$

This is exactly as in GR, so we can use standard results.

Solving for the scalar mode

$$R^{(1)}(x) = -8\pi G \Upsilon^2 \int \mathrm{d}^4x'\, \mathscr{G}_\Upsilon(x, x') T(x'). \tag{91}$$

To proceed further we must know the form of the trace $T(x')$. In general the form of $R^{(1)}(x)$ will be complicated.

A. The Newtonian Limit

Let us consider the limiting case of a Newtonian source, such that

$$T_{00} = \rho; \quad |T_{00}| \gg |T_{0i}|; \quad |T_{00}| \gg |T_{ij}|, \tag{92}$$

with a mass distribution of a stationary point source

$$\rho = M\delta(\boldsymbol{x'}). \tag{93}$$

This source does not produce any radiation. As in GR

$$\overline{h}_{00} = -\frac{4GM}{r}; \qquad \overline{h}_{0i} = \overline{h}_{ij} = 0. \tag{94}$$

Solving for the Ricci scalar

$$R^{(1)} = -2G\Upsilon^2 M \frac{\exp(-\Upsilon r)}{r}. \tag{95}$$

Combining these in (38b) yields a metric perturbation with non-zero elements

$$\begin{aligned} h_{00} &= -\frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]; \\ h_{ij} &= -\frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\delta_{ij}. \end{aligned} \tag{96}$$

Thus, to first order, the metric for a point mass in $f(R)$-gravity is [**? ? ?**]

$$\begin{aligned} \mathrm{d}s^2 &= \left\{1 - \frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}t^2 \\ &\quad - \left\{1 + \frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\right\}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \end{aligned} \tag{97}$$

This is not the linearized limit of the Schwarzschild metric (although it is recovered as $a_2 \to 0$, $\Upsilon \to \infty$). Therefore Schwarzschild black holes (BHs) do not exist in $f(R)$-gravity (with $a_2 \neq 0$) [**?**]. This metric has already been derived for the case of quadratic gravity, which includes terms like R^2 and $R_{\mu\nu}R^{\mu\nu}$ in the Lagrangian [**? ? ? ?**]. In linearized theory our $f(R)$ reduces to quadratic theory, as to first order $f(R) = R + a_2R^2/2$.

Extending this result to a slowly rotating source with angular momentum J; we then have the additional term [**?**]

$$\overline{h}^{0i} = -\frac{2G}{c^2r^3}\epsilon^{ijk}J_jx_k, \tag{98}$$

where ϵ^{ijk} is the Levi-Civita alternating tensor. The metric is

$$\begin{aligned} \mathrm{d}s^2 \;=\; & \left\{1 - \frac{2GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]\right\}\mathrm{d}t^2 \\ & + \frac{4GJ}{r^3}\left(x\mathrm{d}y - y\mathrm{d}x\right)\mathrm{d}t \\ & - \left\{1 + \frac{2GM}{r}\left[1 - \frac{\exp(-\Upsilon r)}{3}\right]\right\}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \end{aligned} \tag{99}$$

where z is the rotation axis. This is not the first order limit of the Kerr metric (aside from as $a_2 \to 0$, $\Upsilon \to \infty$).

It has been suggested that since $R = 0$ is a valid solution to the vacuum equations, the BH solutions of GR should also be solutions in $f(R)$ [**? ?**]. However we see that this is not the case: to have a BH you must have a source, and, because of (25), this forces R to be non-zero in the surrounding vacuum, although it will decay to zero at infinity [**?**]. BHs that exist in $f(R)$-gravity have a different structure than their GR counterparts. It should therefore be possible to distinguish between theories by observing the BHs that form.

Solving the full field equations to find the exact BH metric in $f(R)$ is difficult because of the higher-order derivatives that enter the equations. However, any solution must have the appropriate limiting form as given above.

In $f(R)$-gravity Birkhoff's theorem no longer applies: the metric about a spherically symmetric mass does not correspond to the equivalent of the Schwarzschild solution. The distribution of matter influences how the Ricci scalar decays, and consequently Gauss' theorem is not applicable. Repeating our analysis for a (non-rotating) sphere of uniform density and radius L

$$\overline{h}_{00} = -\frac{4GM}{r}; \qquad \overline{h}_{0i} = \overline{h}_{ij} = 0, \tag{100}$$

as in GR, and for the point mass, but

$$R^{(1)} = -6GM\frac{\exp(-\Upsilon r)}{r}\left[\frac{\Upsilon L\cosh(\Upsilon L) - \sinh(\Upsilon L)}{\Upsilon L^3}\right] \tag{101}$$

$$= -6GM\frac{\exp(-\Upsilon r)}{r}\Upsilon^2\Xi(\Upsilon L), \tag{102}$$

defining $\Xi(\Upsilon L)$ in the last line.[4] The metric perturbation thus has non-zero first order elements [**? ?**]

$$\begin{aligned} h_{00} &= -2GM\left[1 + \exp(-\Upsilon r)\Xi(\Upsilon L)\right]; \\ h_{ij} &= -2GM\left[1 - \exp(-\Upsilon r)\Xi(\Upsilon L)\right]\delta_{ij}. \end{aligned} \tag{103}$$

where we have assumed that $r > L$ at all stages.[5]

In the next section we will use these weak-field metrics with results from classic experimental tests of gravity to place constraints on $f(R)$.

VI. SOLAR SYSTEM & LABORATORY TESTS

A. Post-Newtonian Parameter γ

The parameterized post-Newtonian (PPN) formalism was created to quantify deviations from GR [**? ?**]. It is ideal for solar system tests. The only parameter we need to consider here is γ, which measures the space-curvature

---

[4] $\Xi(0) = 1/3$ is the minimum of $\Xi(\Upsilon L)$.
[5] Inside the source $R^{(1)} = -(6GM/L^3)[1 - (\Upsilon L + 1)\exp(-\Upsilon L) \times \sinh(\Upsilon r)/\Upsilon r]$.

produced by unit rest mass. The PPN metric has components

$$g_{00} = 1 + 2U; \qquad g_{ij} = -(1 + 2\gamma U)\delta_{ij}, \tag{104}$$

where for a point mass

$$U(r) = \frac{GM}{r}. \tag{105}$$

The $f(R)$ metric (97) is of a similar form, but there is not a direct correspondence because of the exponential. It has been suggested that this may be incorporated by changing the definition of the potential U [**?** **?**], then

$$\gamma = \frac{3 - \exp(-\Upsilon r)}{3 + \exp(-\Upsilon r)}. \tag{106}$$

As $\Upsilon \to \infty$, the GR value of $\gamma = 1$ is recovered. However, the experimental bounds for γ are derived assuming that it is a constant [**?**]. Since this is not the case, we will rederive the post-Newtonian, or $\mathcal{O}(\varepsilon)$, corrections to photon trajectories for a more general metric. We define

$$\mathrm{d}s^2 = P(r)\mathrm{d}t^2 - Q(r)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{107}$$

To post-Newtonian order, this has non-zero connection coefficients

$$\begin{aligned} {\Gamma^0}_{0i} &= \frac{P' x^i}{2r}; \qquad {\Gamma^i}_{00} = \frac{P' x^i}{2r}; \\ {\Gamma^i}_{jk} &= \frac{Q'(\delta_{ij}x^k + \delta_{ik}x^j - \delta_{jk}x^i)}{2r}. \end{aligned} \tag{108}$$

The photon trajectory is described by the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\sigma^2} + {\Gamma^\mu}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0, \tag{109}$$

for affine parameter σ. The time coordinate obeys

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\sigma^2} + {\Gamma^0}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0, \tag{110}$$

so we can rewrite the spatial components of (109) using t as an affine parameter [**?**]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \left({\Gamma^i}_{\nu\rho} - {\Gamma^0}_{\nu\rho}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = 0. \tag{111}$$

Since the geodesic is null we also have

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0. \tag{112}$$

To post-Newtonian accuracy these become

$$\begin{aligned} \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} &= -\left(\frac{P'}{2r} - \frac{Q'}{2r}\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|^2\right)x^i \\ &\quad + \frac{P' - Q'}{r}\boldsymbol{x}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}, \end{aligned} \tag{113}$$

$$0 = P - Q\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|^2. \tag{114}$$

The Newtonian, or zeroth order, solution of these is straight-line propagation at constant speed [**?**]

$$x^i_{\mathrm{N}} = n^i t; \qquad |\boldsymbol{n}| = 1. \tag{115}$$

The post-Newtonian trajectory can be written as

$$x^i = n^i t + x^i_{\mathrm{pN}} \tag{116}$$

where x^i_{pN} is the deviation from the straight-line. Substituting this into (113) and (114) gives

$$\begin{aligned} \frac{\mathrm{d}^2\boldsymbol{x}_{\mathrm{pN}}}{\mathrm{d}t^2} &= -\frac{1}{2}\boldsymbol{\nabla}(P-Q) + \boldsymbol{n}\cdot\boldsymbol{\nabla}(P-Q)\boldsymbol{n}, \quad (117)\\ \boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{x}_{\mathrm{pN}}}{\mathrm{d}t} &= \frac{P-Q}{2}. \quad (118) \end{aligned}$$

The post-Newtonian deviation only depends upon the difference $P-Q$. From (97)

$$\begin{aligned} P(r)-Q(r) &= -\frac{4GM}{r}\\ &= -4U(r). \end{aligned} \tag{119}$$

This is identical to in GR. The result holds not just for a point mass, we see, using (38b),

$$\begin{aligned} P(r)-Q(r) &= h_{00}+h_{ii} \qquad \text{(no sum)}\\ &= \overline{h}_{00}+\overline{h}_{ii}, \end{aligned} \tag{120}$$

and since $\overline{h}_{\mu\nu}$ obeys (41) exactly as in GR, there is no difference. We conclude that an appropriate definition for the post-Newtonian parameter is

$$\gamma = \frac{g_{00}+g_{ii}}{2U} - 1 \qquad \text{(no sum)}. \tag{121}$$

Using this, our $f(R)$ solutions have $\gamma = 1$. This agrees with the result found by Clifton [?].[6] Consequently, $f(R)$-gravity is indistinguishable form GR in this respect and is entirely consistent with the current observational value of $\gamma = 1 + (2.1 \pm 2.3)\times 10^{-5}$ [? ?].

B. The Weak-Field Metric

It is useful to transform the weak-field metric, (99), to the more familiar form

$$\mathrm{d}s^2 = A(\widetilde{r})\mathrm{d}t^2 + (4GJ/r)\sin^2\theta\mathrm{d}\phi\mathrm{d}t - B(\widetilde{r})\mathrm{d}\widetilde{r}^2 - \widetilde{r}^2\mathrm{d}\Omega^2. \tag{122}$$

The coordinate $\widetilde{r}$ is a circumferential measure, as in the Schwarzschild metric, as opposed to r, used in preceding sections, which is a radial distance (an isotropic coordinate) [? ?]. To simplify the algebra we introduce the Schwarzschild radius

$$r_{\mathrm{S}} = 2GM. \tag{123}$$

In the linearized regime, we require that the new radial coordinate satisfies

$$\begin{aligned} \widetilde{r}^2 &= \left\{1+\frac{r_{\mathrm{S}}}{r}\left[1-\frac{\exp(-\Upsilon r)}{3}\right]\right\}r^2 \quad (124)\\ \widetilde{r} &= r + \frac{r_{\mathrm{S}}}{2}\left[1-\frac{\exp(-\Upsilon r)}{3}\right]. \quad (125) \end{aligned}$$

This can be used as an implicit definition of r in terms of $\widetilde{r}$. To first order in r_{S}/r [?]

$$A(\widetilde{r}) = 1 - \frac{r_{\mathrm{S}}}{\widetilde{r}}\left[1+\frac{\exp(-\Upsilon r)}{3}\right]. \tag{126}$$

[6] Clifton [?] also gives PPN parameters $\beta = 1$, $\zeta_1 = 0$, $\zeta_3 = 0$ and $\zeta_4 = 0$, all identical to GR.

We see that the functional form of g_{00} is almost unchanged upon substituting $\widetilde{r}$ for r; however r is still in the exponential.

To find $B(\widetilde{r})$ we consider, using (125),

$$\begin{aligned}\frac{\mathrm{d}\widetilde{r}}{\widetilde{r}} &= \mathrm{d}\ln\widetilde{r} \\ &= \left\{\frac{1+\Upsilon r_{\mathrm{S}} r\exp(-\Upsilon r)/6\widetilde{r}}{1+(r_{\mathrm{S}}/2\widetilde{r})\left[1-\exp(-\Upsilon r)/3\right]}\right\}\frac{\mathrm{d}r}{\widetilde{r}}. \end{aligned} \tag{127}$$

Thus

$$\mathrm{d}\widetilde{r}^{\,2} = \frac{\widetilde{r}^{\,2}}{r^2}\left\{\frac{1+\Upsilon r_{\mathrm{S}} r\exp(-\Upsilon r)/6\widetilde{r}}{1+(r_{\mathrm{S}}/2\widetilde{r})\left[1-\exp(-\Upsilon r)/3\right]}\right\}^2 \mathrm{d}r^2. \tag{128}$$

The term in braces is $[B(\widetilde{r})]^{-1}$. We assume that in the weak-field

$$\varepsilon \sim \frac{r_{\mathrm{S}}}{r} \tag{129}$$

is small. Then the metric perturbations from Minkowski are small. Expanding to first order [**?**]

$$B(\widetilde{r}) = 1 + \frac{r_{\mathrm{S}}}{\widetilde{r}}\left[1+\frac{\exp(-\Upsilon r)}{3}\right] - \frac{\Upsilon r_{\mathrm{S}}\exp(-\Upsilon r)}{3}. \tag{130}$$

In the limit $\Upsilon \to \infty$, where we recover GR, $A(\widetilde{r})$ and $B(\widetilde{r})$ tend to their Schwarzschild forms.

C. Epicyclic Frequencies

One means of probing the nature of a spacetime is through observations of orbital motions [**?**]. We will consider the epicyclic motion produced by perturbing a circular orbit. There are two epicyclic frequencies associated with any circular-equatorial orbit, characterising perturbations in the radial and vertical directions respectively. We will start by deriving a general result for any metric of the form of (122), and then specialise to our $f(R)$ solution. We will work in the slow-rotation limit, keeping only linear terms in J.

An orbit in a spacetime described by (122) has as constants of motion: the orbiting particle's rest mass, the energy (per unit mass) of the orbit E and the z-component of the angular momentum (per unit mass) L_z. Using an over-dot to denote differentiation with respect to an affine parameter, which we identify as proper time τ,

$$E = A\dot{t} + (2GJ/r)\sin^2\theta\dot{\phi}; \tag{131}$$

$$L_z = \widetilde{r}^{\,2}\sin^2\theta\,\dot{\phi} - (2GJ/r)\sin^2\theta\dot{t}. \tag{132}$$

We will consider perturbations of circular-equatorial orbits, i.e., orbits such that $\dot{\tilde{r}} = \ddot{\tilde{r}} = \dot{\theta} = 0$ and $\theta = \pi/2$. The timelike geodesic equation can be written in the covariant form

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \frac{1}{2}\left(\partial_\mu g_{\rho\sigma}\right)u^\rho u^\sigma. \tag{133}$$

For a circular equatorial orbit, setting $\mu = \tilde{r}$ gives the frequency of the orbit $\omega_0 = \mathrm{d}\phi/\mathrm{d}t$ as

$$\omega_0 = -\frac{GJ}{\tilde{r}^3} \pm \frac{1}{2}\sqrt{\frac{2A'}{\tilde{r}} + \left(\frac{2GJ}{\tilde{r}^3}\right)^2} \tag{134}$$

in which a dash denotes $\mathrm{d}/\mathrm{d}\tilde{r}$ and the $+/-$ sign denotes prograde/retrograde orbits. The definition of proper time then gives

$$\dot{t} = \left(A + 4GJ\omega_0/\tilde{r} - \tilde{r}^2\omega_0\right)^{-1/2} \tag{135}$$

from which we get $\dot{\phi} = \omega_0\dot{t}$ and hence we have ω_0, E and L_z in terms of $\tilde{r}$.

From the Hamiltonian $\mathcal{H} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ we can obtain the general equation of motion for massive particles, using the substitutions

$$\dot{t} = \frac{Eg_{\phi\phi} + L_z g_{t\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} = \frac{E}{A} - \frac{2GJ}{A\tilde{r}^3}L_z, \qquad \dot{\phi} = \frac{Eg_{t\phi} + L_z g_{tt}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} = \frac{2GJE}{A\tilde{r}^3} + \frac{L_z}{\tilde{r}^2\sin^2\theta} \tag{136}$$

where the second equals sign in each case follows from linearising in J as appropriate for the slow-rotation limit. With these replacements, the general timelike geodesic equation takes the form

$$\dot{\tilde{r}}^2 + \frac{\tilde{r}^2}{B}\dot{\theta}^2 = \frac{E^2}{AB} - \frac{4GJEL_z}{AB\tilde{r}^3} - \frac{1}{B}\left(1 + \frac{L_z^2}{\tilde{r}^2\sin^2\theta}\right) = V(\tilde{r}, \theta, E, L_z). \tag{137}$$

To compute the epicyclic we imagine the orbit is perturbed by a small amount, while E and L_z are unchanged[7]. For radial perturbations

$$\tilde{r} = \overline{r} + \delta, \tag{138}$$

where $\overline{r}$ is the radius of the unperturbed orbit. If the perturbation is small the orbit undergoes small oscillations with frequency $\Omega_r^2 = -(1/2)(\partial^2 V/\partial\tilde{r}^2)|_{\overline{r},\theta=\pi/2}$. Small vertical perturbations $\theta = \pi/2 + \delta$ oscillate with frequency $\Omega_v^2 = -(1/2)(B(\overline{r})/\overline{r}^2)(\partial^2 V/\partial\tilde{\theta}^2)|_{\overline{r},\theta=\pi/2}$

We will denote $A(\overline{r}) = \overline{A}$, $B(\overline{r}) = \overline{B}$, $A'(\overline{r}) = \overline{A}'$ etc.. Substituting into (??) we find

$$\begin{aligned}
\dot{t}^2\Omega_r^2 &= -E^2\left(\frac{\overline{A}'^2}{\overline{A}^3\overline{B}} - \frac{\overline{A}''}{2\overline{A}^2\overline{B}} + \frac{\overline{A}'\overline{B}'}{\overline{A}^2\overline{B}^2} + \frac{\overline{B}'^2}{\overline{A}\,\overline{B}^3} - \frac{\overline{B}''}{2\overline{A}\,\overline{B}^2}\right) - \frac{\overline{B}''}{2\overline{B}^2} + \frac{\overline{B}'^2}{\overline{B}^3} - L_z^2\left(\frac{\overline{B}''}{2\overline{B}^2\overline{r}^2} - \frac{\overline{B}'^2}{\overline{B}^3\overline{r}^2} - \frac{2\overline{B}'}{\overline{B}^2\overline{r}^3} - \frac{3}{\overline{B}\overline{r}^4}\right) \\
&\quad + \frac{4GJEL_z}{r^3}\left(\frac{\overline{A}'^2}{\overline{A}^3\overline{B}} - \frac{\overline{A}''}{2\overline{A}^2\overline{B}} + \frac{\overline{A}'\overline{B}'}{\overline{A}^2\overline{B}^2} + \frac{\overline{B}'^2}{\overline{A}\,\overline{B}^3} - \frac{\overline{B}''}{2\overline{A}\,\overline{B}^2} + \frac{3}{\overline{r}}\left(\frac{\overline{A}'}{\overline{A}^2\overline{B}} + \frac{\overline{B}'}{\overline{A}\overline{B}^2}\right) + \frac{6}{\overline{A}\overline{B}\overline{r}^2}\right) \\
&= \frac{L_z^2}{\overline{r}^3\overline{B}}\left(\frac{\overline{A}''}{\overline{A}'} - \frac{2\overline{A}'}{\overline{A}} + \frac{3}{\overline{r}}\right) + \frac{6GJEL_z}{\overline{A}\overline{B}\overline{r}^4}\left(\frac{\overline{A}''}{\overline{A}'} + \frac{4}{\overline{r}}\right) \\
\dot{t}\Omega_v &= \frac{L_z}{\overline{r}^2}
\end{aligned} \tag{139}$$

This result holds for any metric of the general form (122), subject to the slow-rotation condition $J \ll 1$, which we have used to linearise in J at various stages.

Unless $\omega_0 = \Omega$ the elliptical motion will be asynchronous with the orbital motion: there will be precession of the periapsis. In one revolution the ellipse will precess about the focus by

$$\begin{aligned}
\varpi &= \omega_0\left(\frac{2\pi}{\Omega} - \frac{2\pi}{\omega_0}\right) \\
&= 2\pi\left(\frac{\omega_0}{\Omega} - 1\right)
\end{aligned} \tag{140}$$

where ω_0 is the frequency of the circular orbit, given in equation (134). The precession is cumulative, so a small deviation may be measurable over sufficient time. We are interested in whether or not the deviation arising from the $f(R)$ correction would be observable. In principle, the deviations will be observable if the orbit looks sufficiently different from orbits in the Kerr metric. To quantify the amount of difference, we need to identify orbits between the two spacetimes, and for circular equatorial orbits there is a natural way to do this, by identifying orbits with the same frequency, ω_0, since this is a gauge-invariant observable quantity. The quantity

$$\Delta(\omega_0, \Upsilon) = \Omega(\omega_0, \Upsilon) - \Omega(\omega_0, \Upsilon \to \infty) \tag{141}$$

characterises the rate of increase in the phase difference between the $f(R)$ trajectory and the Kerr trajectory with the same frequency and spin parameter[8]. A physical effect is in principle observable if it leads to a significant phase

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[7] It is not possible for the orbit to be perturbed without changing the energy or angular momentum. However, these corrections are quadratic in the amplitude of the perturbation, and so we can ignore them at linear order.

[8] NB By comparing the trajectory to the $\Upsilon \to \infty$ limit of the trajectory rather than the exact Kerr result ensures that we are taking the same slow rotation limit in both cases, and will not be dominated by $O(J^2)$ corrections.

Planet	Semimajor axis [?] $r/10^{11}$ m	Orbital period [?] $(2\pi/\omega_0)$/yr	Precession rate [?] $\Delta\varpi \pm \sigma_{\Delta\varpi}$/mas yr^{-1}	Eccentricity [?] e
Mercury	0.57909175	0.24084445	-0.040 ± 0.050	0.20563069
Venus	1.0820893	0.61518257	0.24 ± 0.33	0.00677323
Earth	1.4959789	0.99997862	0.06 ± 0.07	0.01671022
Mars	2.2793664	1.88071105	-0.07 ± 0.07	0.09341233
Jupiter	7.7841202	11.85652502	0.67 ± 0.93	0.04839266
Saturn	14.267254	29.42351935	-0.10 ± 0.15	0.05415060
Uranus	28.709722	83.74740682	-38.9 ± 39.0	0.04716771
Neptune	44.982529	163.7232045	-44.4 ± 54.0	0.00858587
Pluto	59.063762	248.0208	28.4 ± 45.1	0.24880766

TABLE I: Orbital properties of the eight major planets and Pluto. We take the semimajor orbital axis to be the flat-space distance r, not the coordinate $\widetilde{r}$. The eccentricity is not used in calculations, but is given to assess the accuracy of neglecting terms $\mathcal{O}(e^2)$.

shift in a gravitational waveform over the length of an observation. Thus, a simple criterion for the $f(R)$ theory to be distinguishable from GR would be that $\Delta T > 2\pi$. This is a significant oversimplification since we have assumed that only the orbital frequency has been matched to a Kerr value, while small changes in the other parameters such as the black hole mass and spin, the orbital eccentricity and inclination and so on could mimic the effects of an $f(R)$ deviation. On the other hand, we are also keeping the orbital frequency fixed whereas we will observe inspirals, and this tends to break the parameter degeneracies. Since we are interested in extreme-mass-ratio systems, for which the inspiral proceeds slowly, it is likely that we are being over-optimistic so these results can be considered upper bounds on what could be measureable. A fuller analysis accounting for parameter correlations and inspiral is beyond the scope of this paper.

The timescale of the systems we are considering is set by the black hole mass, and the quantities $M\omega_0$ and $M\Delta$ are mass-independent. A duration, $T_{\rm obs}$, of a typical EMRI observation with LISA will be of the order of a year, and so the criterion for detectability becomes

$$M\Delta = 9.8 \times 10^{-7} \left(\frac{M}{10^6 M_\odot}\right) \left(\frac{\rm yr}{T_{\rm obs}}\right). \tag{142}$$

In Figure 1 we shows the region of Υ/ω_0 parameter space in which $f(R)$ gravity could be distinguished from GR, as defined by this criterion. Each curve represents a particular choice for $M\Delta$, and the region below the curve is "detectable" in an observation characterised by that choice for $M\Delta$. Equation (142) indicates that the curve $M\Delta = 10^{-6}$ is what would be achieved in a 1 year observation for a $10^6 M_\odot$ mass black hole. The curves $M\Delta = 10^{-5}/10^{-7}$ are the corresponding results for a $10^7/10^5 M_\odot$ mass black hole, while the curve $M\Delta = 3 \times 10^{-7}$ represents what would be achieved in a 3 year observation and so on. We show results for two different choices of spin, $a = 0$ and $a = 0.5$, and it is clear that there is not too much difference between the two, although the vertical epicyclic frequency is only measurable for $a \neq 0$ since this coincides with the orbital frequency for $a = 0$ due to spherical symmetry. The results for the radial epicyclic frequency do not differ hugely between $a = 0$ and $a = 0.5$ in this weak field metric approximation. We note also that we show results only for prograde orbits. For $a \neq 0$, we can also compute results for retrograde orbits, and these differ from the prograde results but only by a small amount which is almost indistinguishable on the scale of these plots.

Our conclusion from Figure 1 is that, broadly speaking, we would be able to distinguish spacetimes with $\Upsilon \lesssim 1$. Somewhat larger values are accessible at higher frequencies, but this conclusion must be treated somewhat cautiously, as the inspiral would pass through that region fairly quickly, and those orbits correspond to relatively small values of the orbital radius at which the approximations that we made deriving the weak-field metric begin to break down. We note also that, for this criterion, the radial epicyclic frequency is always a more powerful probe than the vertical epicyclic frequency. This is to be expected, since the latter is generally smaller in magnitude than the former and so fewer cycles accumulate over a typical observation.

D. Planetary precessions

NB I have split this off into a different section since we will be reordering the paper but have not changed the text. Some changes will obviously have to be made at some point. We can apply this to the classic test of planetary precession in the solar system. Table I shows the orbital properties of the planets. We will use the deviation in perihelion precession rate from the GR prediction to constrain the value of ζ, and hence Υ and a_2. Since several of the deviations are negative, they cannot be explained by $f(R)$ corrections. This may be considered

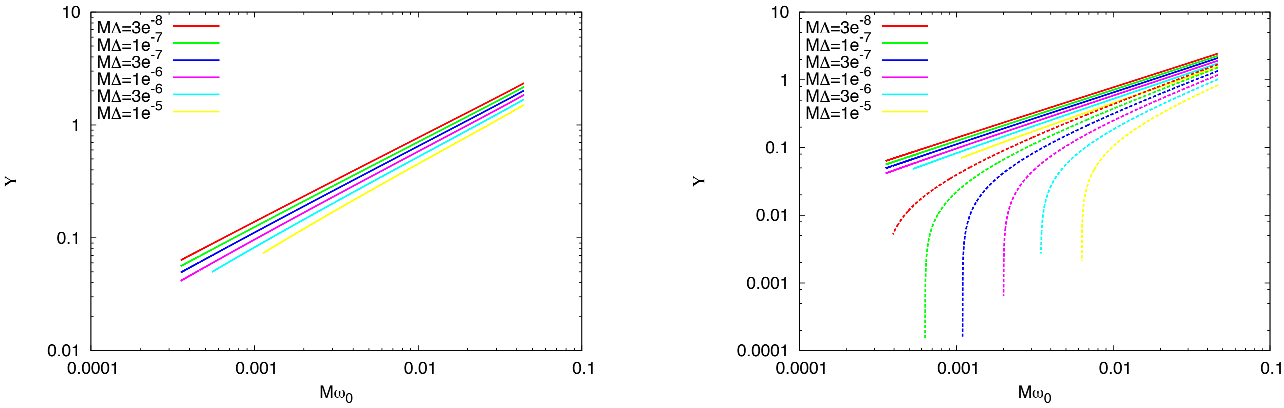

FIG. 1: Region of parameter space in which $f(R)$ theories can be distinguished from GR when the central black hole has spin $a = 0$ (left panel) or $a = 0.5$ (right panel). Each curve corresponds to a particular specification of the detectability criterion given in equation (142) in the text, as identified in the key. Dashed lines correspond to measurements of the vertical epicyclic frequency, while solid lines represent measurements of the radial epicyclic frequency. The region below the curve could be distinguishable in a LISA observation with that detectability value.

	Using $\sigma_{\Delta\varpi}$		Using $2\sigma_{\Delta\varpi}$	
Planet	$\Upsilon/10^{-11}$ m^{-1}	$\lvert a_2\rvert/10^{18}$ m^2	$\Upsilon/10^{-11}$ m^{-1}	$\lvert a_2\rvert/10^{18}$ m^2
Mercury	52.6	1.2	51.3	1.3
Venus	25.3	5.2	24.6	5.5
Earth	19.1	9.1	18.6	9.6
Mars	12.2	22	11.9	24
Jupiter	2.96	380	2.87	410
Saturn	1.69	1200	1.63	1200
Uranus	0.58	9800	0.56	11000
Neptune	0.35	28000	0.33	31000
Pluto	0.26	49000	0.25	55000

TABLE II: Bounds calculated using uncertainties in planetary perihelion precession rates. Υ must be greater than or equal to the tabulated value, $|a_2|$ must be less than or equal to the tabulated value.

as evidence against $f(R)$-gravity; however, all the precession rates are consistent with GR predictions ($\Delta\varpi = 0$). Thus we cannot conclusively rule out $f(R)$-gravity. Since the deviations are zero to within their uncertainties, we can use the size of these uncertainties to constrain the $f(R)$ correction. Table II shows the constraints for Υ and a_2 obtained by equating the uncertainty in the precession rate $\sigma_{\Delta\varpi}$ with the $f(R)$ correction, and similarly using twice the uncertainty $2\sigma_{\Delta\varpi}$.

The tightest constraint is obtained from the orbit of Mercury. Adopting a value of $\Upsilon \geq 5.2 \times 10^{-10}$ m^{-1}, the cutoff frequency for the Ricci mode is ≥ 0.16 s^{-1}. Therefore it could lie in the upper range of the LISA frequency band [**? ?**] or in the LIGO/Virgo frequency range [**? ? ?**]. However, as we will see in Sec. VI E, it is possible to place stronger constraints on Υ using laboratory experiments.

E. Fifth-Force Tests

From the metric (97) we see that a point mass has a Yukawa gravitational potential [**? ? ?**]

$$V(r) = \frac{GM}{r}\left[1 + \frac{\exp(-\Upsilon r)}{3}\right]. \tag{143}$$

Potentials of this form are well studied in fifth-force tests [? ? ?] which consider a potential defined by a coupling constant α and a length-scale λ such that

$$V(r) = \frac{GM}{r}\left[1 + \alpha \exp\left(-\frac{r}{\lambda}\right)\right]. \tag{144}$$

We are able to put strict constraints upon our length-scale λ_R, and hence a_2, since our coupling constant $\alpha_R = 1/3$ is relatively large. This can be larger for extended sources: comparison with (103) shows that for a uniform sphere $\alpha_R = \Xi(\Upsilon L) \geq 1/3$.

The best constraints at short distances come from the Eöt-Wash experiments, which use torsion balances [? ?]. These constrain $\lambda_R \lesssim 8 \times 10^{-5}$ m. Hence we determine $|a_2| \lesssim 2 \times 10^{-9}$ m^2. A similar result was obtained by Näf and Jetzer [?]. This would mean that the cutoff frequency for a propagating scalar mode would be $\gtrsim 4 \times 10^{12}$ s^{-1}. This is much higher than expected for astrophysical objects.

Fifth-force tests also permit λ_R to be large. This degeneracy can be broken using other tests, from Sec. VI C we know that the large range for λ_R is excluded by planetary precession rates.[9]

While the laboratory bound on λ_R may be strict compared to astronomical length-scales, it is still much greater than the expected characteristic gravitational scale, the Planck length ℓ_P. We might expect for a natural quantum theory, that $a_2 \sim \mathcal{O}(\ell_\mathrm{P}^2)$; however $\ell_\mathrm{P}^2 = 2.612 \times 10^{-70}$ m^2, thus the bound is still about 60 orders of magnitude greater than the natural value. The only other length-scale that we could introduce would be defined by the cosmological constant Λ. Using the concordance values [?] $\Lambda = 1.27 \times 10^{-52}$ m^{-2}; we see that $\Lambda^{-1} \gg |a_2|$. It is intriguing that if we combine these two length-scales we find $\ell_\mathrm{P}/\Lambda^{1/2} = 1.44 \times 10^{-9}$ m^2, which is on the order of the current bound. This is likely to be a coincidence, since there is nothing fundamental about the current level of precision. It would be interesting to see if the measurements could be improved to rule out a Yukawa interaction around this length-scale.

VII. SUMMARY & CONCLUSIONS

We have seen that gravitational radiation is modified in $f(R)$-gravity as the Ricci scalar is no longer constrained to be zero. In linearized theory we discover that there is an additional mode of oscillation, that of the Ricci scalar. This is only excited above a cutoff frequency; once a propagated mode is excited, it will carry additional energy-momentum away from the source. The two transverse GW modes are modified from their GR counterparts to include a contribution from the Ricci scalar, see (38a), allowing us to probe the curvature of the strong-field regions from which GWs originate. However, further study is needed in order to understand how GWs behave in a region with background curvature, in particular when R is non-zero.

Gravitational radiation is not the only way to test $f(R)$ theory. From linearized theory we have deduced the weak-field metrics for some simple mass distributions. These indicate that BH solutions are not the same as in GR. Additionally, Birkhoff's theorem no longer applies in $f(R)$-gravity.

Using these weak-field results it is possible to constrain the value of a_2. The strongest constraints come from fifth-force tests. Based upon the results of Eöt-Wash experiment, we find that $|a_2| \lesssim 2 \times 10^{-9}$ m^2. In this case we do not expect the propagating Ricci mode to be excited by astrophysical systems as the cutoff frequency is too high. Even in the absence of the Ricci mode, it may still be possible to use GWs to constrain $f(R)$-gravity through the dependence of the transverse polarization's dependence upon the Ricci scalar.

We have also derived the epicyclic frequency for near circular orbits. Some of the estimated deviations from GR precession rates are negative, which cannot be achieved with $f(R)$ corrections. Since all of the deviations are consistent with zero, we cannot use these as proof against $f(R)$. The bound obtain from precession rates is $|a_2| \lesssim 1.2 \times 10^{18}$ m^2. Although this bound is much larger than its fifth-force partner, it may still may be applicable: it is possible that $f(R)$-gravity is not universal, that it is different in different regions of space. For example, it is possible that $f(R)$-gravity is modified in the presence of matter via the chameleon mechanism [? ?]. In metric $f(R)$ this corresponds to a nonlinear effect arising from a large departure of the Ricci scalar from its background value [?]. The mass of the effective scalar degree of freedom then depends upon the density of its environment. In a region of high matter density, such as the Earth, the deviations from standard gravity would be exponentially suppressed due to a large effective Υ; while on cosmological scales, where the density is low, the scalar would have a small Υ, perhaps of the order H_0/c [? ?]. The chameleon mechanism allows $f(R)$ gravity to pass laboratory or solar system tests while remaining of interest for cosmology. In the context of gravitational radiation, this would mean that the Ricci scalar mode could freely propagate on cosmological scales [?]. Unfortunately, since the chameleon mechanism suppresses the effects of

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[9] This is supported by a result of Näf and Jetzer [?] obtained using the results of Gravity Probe B [?].

$f(R)$ in the presence of matter, this mode would have to be excited by something other than the movement of matter. Additionally since electromagnetic radiation and the standard transverse polarizations of gravitational radiation have a traceless energy-momentum tensor, they cannot excite the Ricci mode.[10] To be able to detect the Ricci mode we must observe it well away from any matter, which would cause it to become evanescent: a spaceborne detector such as LISA would be our only hope.

Alternatively, it may be that $f(R)$ is just an approximate effective theory, then the range of a particular parametrization's applicability could be limited to a specific domain. For example, we could imagine that the effective theory in the vicinity of a massive BH where the curvature is large is different from in the solar system where curvature is small, or $f(R)$ could evolve with cosmological epoch so that it varies with redshift. Since we have only discussed tests in the solar system we must be cautious with the range of applicability of our results.

An interesting extension to the work presented here is to consider the case when a_0 is non-zero. We could then consider an expansion about (anti-)de Sitter space. This is relevant because the current ΛCDM paradigm indicates that we live in a universe with a positive cosmological constant [**?**]. Such a study would naturally compliment an investigation into the effects of background curvature on propagation.

Acknowledgments

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[10] The contribution to the gravitational energy-momentum pseudotensor from a propagating Ricci mode does have a non-zero trace, see (83).