Statistical methods: Homework 6

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(a) Let $0 < a < b$. Which number is larger?

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \; or \; \int_{-b}^{-a} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Answer: The two quantities are equal. We rely on the symmetry about 0 of the normal distribution to show this.
Consider only $f(z) = e^{-\frac{z^2}{2}}$. For any number a we can show that $f(-a) = f(a)$.

$$f(-a) = e^{-\frac{(-a)^2}{2}} = e^{-\frac{(-1)^2 a^2}{2}} = e^{-\frac{(1 \cdot a^2)}{2}} = f(a)$$

Therefore,

$$\begin{aligned}
\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} &= P(a < Z < b) = 1 - (P(Z < a) + P(Z > b)) \\
&= 1 - (P(Z > -a) + P(Z < -b))(by\ symmetry) \\
&= P(-b < Z < -a) \\
&= \int_{-b}^{-a} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}
\end{aligned}$$

(b) Let $a > 0$. Which number is larger?

$$\int_a^{a+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \; or \; \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Answer:
$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ is the larger quantity.

Consider only $f(z) = e^{-\frac{z^2}{2}}$

Then, $f'(z) = -ze^{-\frac{z^2}{2}}$, Clearly $f'(z) < 0 \; \forall \; z > 0$.

Now($a+1-a) = 1$ and also $(a+\frac{1}{2}-(a-\frac{1}{2})) = 1$ Therefore the integrals are over the same length domain.

But, $f(a+\epsilon) < f(a)$ because $f'(a) < 0$. Therefore the normal function is always less over the interval $[a, a+1]$, compared to $[a-\frac{1}{2}, a+\frac{1}{2}]$.

So we can then conclude

$$\int_a^{a+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$