# Statistical Methods: Homework 4

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## 1 3.10.4

The complement of the event is P($Y_1$¿.6)∪P($Y_5$¡.6).
These are disjoint events.
Therefore P($Y_1$¿.6)∪P($Y_5$¡.6)= P($Y_1$¿.6) + P($Y_5$¡.6).
And P($Y_1$¿.6) = $[P(Y < .6)]^5$)

$$P(Y > .6) = \int_{.6}^{1} 2ydy = (.64)^5$$
$$P(Y > .6) = .107$$

Also

$$P(Y_5 < .6) = (\int_{0}^{.6} 2ydy)^5 = (.36)^5$$
$$P(Y_5 < .6) = .006$$

Therefore,

$$P(Y_1 < .6) \cup (Y_5 > .6) = 1 - .107 - .006 = .887$$

## 2 3.10.6

$f_Y(y) = e^\lambda$ And, $F_Y(y) = 1 - e^\lambda$

$$f(y_{min} < Y) = \int_0^t n \cdot f_Y(y)(1 - F_Y(y))^{n-1}dy = \int_0^t ne^{-ny}dy$$
$$f(y_{min} < Y) = -e^{-ny}\Big|_0^t = 1 - e^{-nt}$$

So using t=.2 and Probability = .9, we need to solve $.9 \leq 1 - e^{-.2n}$.
This happens with n¿11.5, so n should be greater than 12. $n \geq 12$

# 3 3.10.8

Find the pdf for $y_i$

$$f_{y_i}(y) = \frac{n!}{(i-1)!(n-i)!} F_y(y)^{i-1}(1-F_y(y))^{n-i} f_Y(y)dy$$

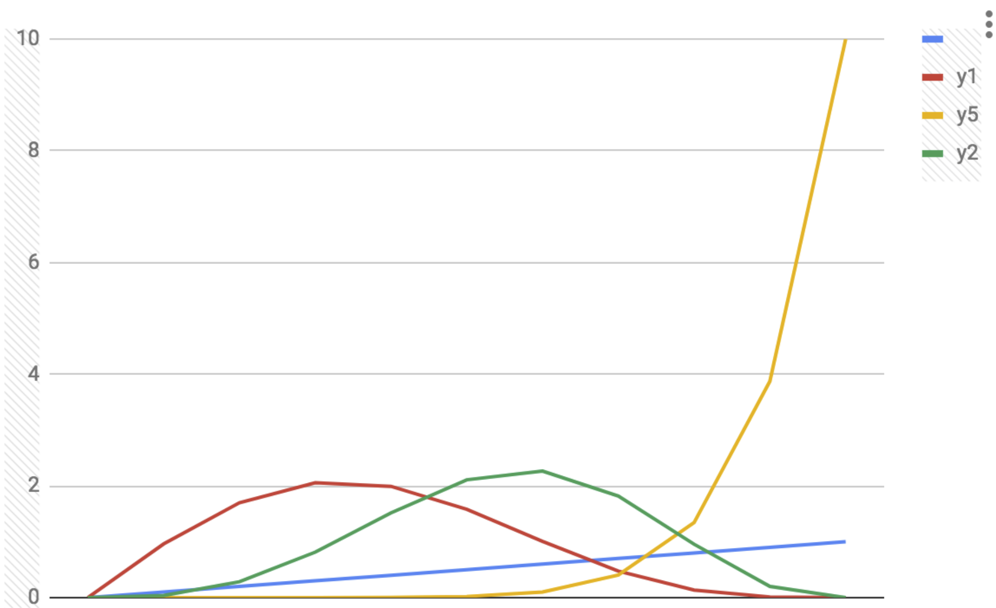
So, $f_Y(y) = 2y$ and $F_Y(y) = \frac{2y^2}{2} = y^2$

$$f_{y_i}(y) = \frac{n!}{(i-1)!(n-i)!}(y^2)^{i-1}(1-y^2)^{n-i}2ydy$$

letting i = 5

$$f_{y_5}(y) = \frac{5!}{(5-1)!(5-5)!}(y^2)^4(1-y^2)^5 2ydy$$

$$f_{y_5}(y) = 10y^9$$




# 4 3.10.10

So for the $N^{th}$ observation to be the smallest it must be smaller than each of the other observations in the sample. Being as they are identically distributed, the probability of any $Y_i$ being smaller or larger than another $Y_j$ is .5. That is $P(Y_i) > P(Y_j) = .5$ and $P(Y_i) < P(Y_j) = .5$ for all $i \neq j$.

So there are n-i of these comparisons that have to happen, and there is only 1 way to arrange the samples. Therefore

$$P(Smallest\ Value\ in\ n^{th} spot) = (\frac{1}{2})^{n-1}$$

# 5 3.10.12

This question is essentially asking what is the expected value of the first order statistic. We have a formula for this:

$$f_{y_1}(Y) = \int_0^\infty nF_y(Y)^0[1 - F_y(Y)]^{n-1} f_y(y)dy$$

Simplify this and take the expected value.

$$E[Y_1] = \int_0^\infty n[1 - e^{\lambda y}]^0[e^{-\lambda y}]^{n-1}\lambda e^{-\lambda y} \cdot ydy$$

$$E[Y_1] = \int_0^\infty n\frac{[e^{-\lambda y}]^n}{e^{-\lambda y}}\lambda e^{-\lambda y} \cdot ydy$$

$$E[Y_1] = \int_0^\infty n\lambda e^{-(\lambda \cdot n)y} \cdot ydy$$

This is just an exponential distribution with parameter $\lambda' = n \cdot \lambda$. Therefore we know the expected value of this distribution is $E[Y_1] = \frac{1}{n \cdot \lambda}$, as required by the question, and as we have shown in a previous assignment.

# 6 3.12.4

We have $p_x(k) = \frac{3}{4}^k \frac{1}{4}$.

Therefore

$$M_x(t) = \sum_\infty^0 e^{tk}\frac{3}{4}^k\frac{1}{4} = \frac{1}{4}\sum_0^\infty (\frac{3e^t}{4})^k$$

$$\frac{1}{4}\sum_0^\infty (\frac{3e^t}{4})^k = \frac{1}{4} \cdot \frac{1}{1 - \frac{3e^t}{4}}$$

So, $M_x(t) = \frac{1}{4} \cdot \frac{1}{1 - \frac{3e^t}{4}}$ and this converges when $e^t < \frac{4}{3}$.

# 7 3.12.7

To find the MGF we find $E[e^{tk}]$:

$$E[e^{tk}] = \sum_0^\infty e^{-\lambda}e^{tk}\frac{\lambda^k}{k!} = \frac{1}{e^\lambda}\sum_0^\infty \frac{(\lambda t)^k}{k!} = \frac{e^{t\lambda}}{e^\lambda} = e^{\lambda(t-1)}$$

## 8 3.12.10

Find $E[Y^4]$ of $y\ exp(\lambda), y > 0$.

We find the MGF of the exponential distribution

$$\int_0^\infty e^{ty}\lambda e^{-\lambda y}dy = \int_0^\infty \lambda e^{-(\lambda-t)y}dy$$

We substitute $(\lambda - t)y = u$ in the integral. So $\frac{du}{\lambda - t} = dy$. And so,

$$MGF = \frac{\lambda}{\lambda - t}(-e^{-u})\Big|_{u=0}^{u=\infty} = \frac{\lambda}{\lambda - t}(0 - (-1)) = \frac{\lambda}{\lambda - t}$$

No we go through the thrilling exercise of taking repeated partial derivatives 4 times of this MGF.

$$\frac{d}{dt}\frac{\lambda}{\lambda - t} = \lambda(\lambda - t)^{-2}$$

$$\frac{d^2 f}{dt^2} = \frac{d}{dt}\frac{df}{dt} = 2\lambda(\lambda - t)^-3$$

From this we can see the pattern that the nth moment is $n!\frac{\lambda}{(\lambda-t)^{n+1}}$

Therefore the forth moment is: $24\frac{\lambda}{(\lambda-t)^5}$

Evaluating this at t=0, $E[Y^4] = \frac{24}{\lambda^5}$

## 9 3.12.23

$X - poi(\lambda)$, and $f_x(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

Also,

$M_x(t) = e^{-\lambda + \lambda e^t}$

a) We consider $W = 3X$.

Then,

$$M_W(t) = M_X(3t) = e^{(\lambda(e^{(3t)-1)})}.$$

Based on this result we see W is not distributed as a poisson with a parameter because the MGF doesn't match what we would expect to see. So, in response to part a, No $W - Poi(\lambda^{'})$

b) We consider $W_2 = 3X + 1$.

We know $M_W(t) = e^{-3\lambda + 3\lambda e^t}$ from part a.

So $M_{W_2} = e^t \cdot e^{-3\lambda + 3\lambda e^t}$.

This is not a Poisson moment generating function. Therefore it is not distributed as a poisson random Variable.

## 10 R Programming assignment

**Part 1:**

```
event_1 <- 0
invcdf <- function(y) sqrt(y)
for (i in 1:1000){
  res<-sort(invcdf(runif(5)))
  if (res[1] < .6){
    if (res[5]>.6){
      event_1=event_1+1
    }
    else{
      warning("5th not greater than .6")
    }
  }
  else{
    warning("1st not less than .6")
  }
}


 j = event_1/1000
```

The value of J is .885. This is very close to our theoretical value of .887.

**Part 2:**

```
event_2 <- 0
invcdf_two <- function(y) -log((1 - y))

for (i in 1:1000){
  res_one <- sort(invcdf_two(runif(12)))[1]
  if (res_one < .2){
    event_2 = event_2 + 1
  }
}
```

The value of event_2 is 911. So, .911 is the calculated probability, which is close to our expected value of .9 which we found analytically.