1 3.8.13

Suppose that X and Y are two independent random variables, where $f_X(x) = xe^{-x}, x \geq 0$, and $f_Y(y) = e^{-y}, y \geq 0$. Find the pdf of Y/X.

$$f_W(w) = \int_0^w |x|\, f_X(x) f_y(wx)$$

$$f_W(w) = \int_0^w xxe^{-x}e^{-wx}dx$$

$$f_W(w) = \int_0^w x^2 e^{-x(1-w)}dx$$

Try This, Multiply by $1 = \frac{1-w}{1-w}$,

$$f_W(w) = \frac{1}{1-w}\int_0^w x^2(1-w)e^{-x(1-w)}dx$$

This is basically the expected value of x^2 against an exponential distribution with parameter $\lambda = 1 - w$.

Then we also know for the exponential distribution the $Var(x) = E[x^2] - E[x]^2$, and for the exponential distribution is $Var(x) = \frac{1}{\lambda^2} = E[x^2] - \left[\frac{1}{\lambda}\right]^2$.

Therefore, $E[x^2] = \frac{2}{\lambda^2}$. Going back to our integral,

$$f_W(w) = \frac{1}{1-w}\int_0^w x^2(1-w)e^{-x(1-w)}dx = \frac{1}{1-w}E\left[x^2\right] = \frac{1}{1-w}\cdot\frac{2}{(1-w)^2} = \frac{2}{(1-w)^3}, w \geq 0$$