Statistical methods: Homework 5

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1 3.7.4

Find c if $f_{X,Y}(x,y) = cxy$ for X and Y defined over the triangle whose vertices are the points (0, 0), (0, 1), and (1, 1).

Answer:

$$c\int_0^1\int_0^x xydydx = 1$$

$$c\int_0^1 x\frac{y^2}{2}\Big|_0^x dx = c\int_0^1 x\frac{x^2}{2} = 1$$

$$c\int_0^1 \frac{x^3}{2}dx = c\frac{x^4}{8}\Big|_0^1 = \frac{c}{8} = 1$$

Thus $c = 8$.

2 3.7.20

For each of the following joint pdfs, find $f_X(x)$and $f_Y(y)$.

(a) $f_{X,Y}(x,y) = \frac{1}{2}, 0 \le x \le y \le 2$

$f_X(x) = \int_x^2 \frac{1}{2}dy = \frac{1}{2}y\Big|_x^2 = \frac{2-x}{2} = 1 - \frac{x}{2}$

$f_Y(y) = \int_0^y \frac{1}{2}dx = \frac{1}{2}x\Big|_0^y = \frac{y}{2}x$

(b) $f_{X,Y}(x,y) = \frac{1}{x}, 0 \le y \le x \le 1$

$f_X(x) = \int_0^x \frac{1}{x}dy = \frac{y}{x}\Big|_x^0 = 1$

$f_Y(y) = \int_y^1 \frac{1}{x}dx = \ln(x)\Big|_y^1 = -\ln(y)$

(c) $f_{X,Y}(x,y) = 6x, 0 \le x \le 1, 0 \le y \le 1-x$

$f_X(x) = \int_0^{1-x} 6xdy = 6xy\Big|_0^{1-x} = 6(x - x^2)$

$f_Y(y) = \int_1^0 6xdx = 3x^2 = 3$

3 3.7.28

For each of the following joint pdfs, find $F_X(x,y)$.

(a) $f_{X,Y}(x,y) = \frac{1}{2}, 0 \le x \le y \le 2$

$$\int_0^u \int_u^x \frac{1}{2} dydx = \int_0^u \frac{1}{2} y\Big|_x^v = \frac{1}{2}\int_0^u v - xdx = \frac{1}{2}(vx - \frac{x^2}{2}\Big|_0^u = \frac{1}{2}(vu - \frac{u^2}{2})$$

(b) $f_{X,Y}(x,y) = \frac{1}{x}, 0 \le y \le x \le 1$

$$\int_0^v \int_y^u \frac{1}{x} dxdy = \int_0^v \ln(x)\Big|_y^u = \int_0^v \ln(u) - \ln(y)dy = v\ln(u) - v\ln(v) + v$$

(c) $f_{X,Y}(x,y) = 6x, 0 \le x \le 1, 0 \le y \le 1 - x$

$$\int_0^u \int_0^v 6xdxdy = \int_0^v 3x^2\Big|_0^v dy = 3u^2 y\Big|_0^v = 3u^2 v$$

4 3.7.42

Are the random variables X and Y independent if $f_{X,Y}(x,y) = \frac{2}{3}(x+2y), 0 \le x \le 1, 0 \le y \le 1$?

Answer: $f_X(x) = \frac{2}{3}\int_0^1 (x+2y)dy = \frac{2}{3}\left[(xy+y^2)\right]\Big|_0^1 = \frac{2(x+1)}{3}$

$f_Y(y) = \frac{2}{3}\int_0^1 (x+2y)dx = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]\Big|_0^1 = \frac{2}{3}(\frac{1}{2} + 2y)$

If the variables are independent then $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$.

We show $f_X(x) \cdot f_Y(y)$

$$f_X(x)_Y(y) = \frac{2(x-1)}{3} \cdot \frac{2}{3}(\frac{1}{2} + 2y) \ne \frac{2}{3}(x+2y)$$

Therefore random variables x and y are not independent.

5 3.8.2

Let $f_Y(y) = \frac{3}{14}(1+y^2), 0 \le y \le 2$. Define the random variable W by $W = 3Y + 2$. Find $f_W(w)$. Be sure to specify the values of w for which $f_W(w) \ne 0$.

Answer:

$$W = 3Y + 2 \leftrightarrow Y = \frac{W-2}{3}$$

And if $0 \le y \le 2$ then w(0) = 2 and w(2)= 8 so $2 \le w \le 8$.

$$f_W(w) = \frac{1}{|3|}\frac{3}{14} + \left[\frac{w-2}{3}\right]^2 = \frac{1}{126}(9+w^2-4w+4) = \frac{1}{126}(9+w^2-4w+4) = \frac{1}{126}(w^2-4w+13), 2 \le w \le 8$$

6 3.8.10

Suppose the velocity of a gas molecule of mass m is a random variable with pdf $f_Y(y) = ay^2e^{-by^2}, y \geq 0$, where a and b are positive constants depending on the gas. Find the pdf of the kinetic energy, $W = (m/2)Y^2$, of such a molecule.

$P(W < w) = P(\frac{m}{2}Y^2 \leq w) = F_y(\sqrt{\frac{2w}{m}})$

So,

$$f_w(w) = f_y(\sqrt{\frac{2w}{m}}) \cdot \frac{1}{2\sqrt{\frac{2w}{m}}} = \sqrt{\frac{m}{2}}\frac{1}{2\sqrt{w}}f_Y(\sqrt{\frac{2w}{m}}) = a\sqrt{\frac{w}{2m}}e^{-2\cdot\frac{bw}{m}}$$

7 3.8.12

Let X and Y be two independent random variables. Given the marginal pdfs indicated below, find the cdf of Y/X. (Hint: Consider two cases, $0 \leq w \leq 1$ and $1 < w$.)

(a) $f_X(x) = 1, 0 \leq x \leq 1$, and $f_Y(y) = 1, 0 \leq y \leq 1$

Case 1 $w \leq 1$ and $\frac{1}{w} \geq 1$ so we can limit the integral at 1.

$$f_W(w) = \int_0^1 x(1)(1)dx = \frac{1}{2}$$

Case 2 $w \geq 1$ then we limit the integral at $\frac{1}{w}$

$$f_W(w) = \int_0^{\frac{1}{w}} x = \frac{1}{2w^2}$$

(b) $f_X(x) = 2x, 0 \leq x \leq 1$, and $f_Y(y) = 2y, 0 \leq y \leq 1$

Similarly if $w \leq 1$ the $\frac{1}{w} \geq 1$, therefore we limit the integral at 1.

$$f_W(w) = \int_0^1 x2xwxdx = 1$$

Case 2 $w \geq 1$ then we limit the integral at $\frac{1}{w}$

$$f_W(w) = \int_0^{\frac{1}{w}} x2x2wxdx = \frac{1}{w^3}$$

8 3.8.13

Suppose that X and Y are two independent random variables, where $f_X(x) = xe^{-x}, x \geq 0$, and $f_Y(y) = e^{-y}, y \geq 0$. Find the pdf of Y/X.

$$f_W(w) = \int_0^w |x| \, f_X(x) f_y(wx)$$

$$f_W(w) = \int_0^w xxe^{-x}e^{-wx}dx$$

$$f_W(w) = \int_0^w x^2 e^{-x(1-w)}dx$$

Try This, Multiply by $1 = \frac{1-w}{1-w}$,

$$f_W(w) = \frac{1}{1-w}\int_0^w x^2(1-w)e^{-x(1-w)}dx$$

This is basically the expected value of x^2 against an exponential distribution with parameter $\lambda = 1 - w$.

Then we also know for the exponential distribution the $Var(x) = E[x^2] - E[x]^2$, and for the exponential distribution is $Var(x) = \frac{1}{\lambda^2} = E[x^2] - \left[\frac{1}{\lambda}\right]^2$.

Therefore, $E[x^2] = \frac{2}{\lambda^2}$. Going back to our integral,

$$f_W(w) = \frac{1}{1-w}\int_0^w x^2(1-w)e^{-x(1-w)}dx = \frac{1}{1-w}E\Big[x^2\Big] = \frac{1}{1-w}\cdot\frac{2}{(1-w)^2} = \frac{2}{(1-w)^3}, w \geq 0$$

9 3.9.5

Suppose that X_i is a random variable for each $E(X_i) = \mu \neq 0, i = 1, 2, ..., n$. Under what conditions will the following be true?

$$E\left(\sum_{i=1}^n a_i X_i\right) = \mu$$

Answer:

$$E\left(\sum_{i=1}^n a_i X_i\right) = E\Big[a_1X_1 + a_2X_2 ... a_nX_n\Big] = a_1E\Big[X_1\Big] + a_2E\Big[X_2\Big]...a_nE\Big[X_n\Big] = a_1\mu + a_2\mu ... a_n\mu = \mu\sum_{i=1}^n a_i$$

Therefore, $E\left(\sum_{i=1}^n a_i X_i\right) = \mu \sum_{i=1}^n a_i = \mu$

When $\sum_{i=1}^n a_i = 1$.

10 3.9.8

Suppose two fair dice are tossed. Find the expected value of the product of the faces showing.

Answer:
So we know that the dice rolls are independent.
Therefore $E\Big(X_1 \cdot X_2\Big) = E\Big(X_1\Big) \cdot E\Big(X_2\Big)$
Now the expectation of a dice roll is $\sum_1^6 i \cdot \frac{1}{6} = \frac{7}{2}$
So, $E\Big(X_1 \cdot X_2\Big) = E\Big(X_1\Big) \cdot E\Big(X_2\Big) = \frac{7}{2} \cdot \frac{7}{2} = 12.25$

11 3.11.4

Five cards are dealt from a standard poker deck. Let X be the number of aces received, and Y the number of kings. Compute $P(X = 2|Y = 2)$.

Answer:
So this is essentially a ratio of hypergeometric probabilities.

$$\frac{\frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}}{\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}} = \frac{\binom{4}{2}\binom{44}{1}}{\binom{48}{3}} = .015$$

12 3.11.16

Suppose that X and Y are distributed according to the joint pdf

$$f_{X,Y}(x, y) = \frac{2}{5} \cdot (2x + 3y), 0 \leq x \leq 1, 0 \leq y \leq 1$$

(a) $f_X(x)$

$f_X(x) = \frac{2}{5} \int_0^1 2x + 3ydy = \frac{2}{5}(2xy + \frac{3y^2}{2})\Big|_0^1 = \frac{2}{5}(2x + \frac{3}{2})$

(b) $f_{Y|x}(y)$

$f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2x+3y}{2x+\frac{3}{2}} = \frac{4x+6y}{4x+3}$

(c) $P(\frac{1}{4} \leq Y \leq \frac{3}{4}|X = \frac{1}{2})$

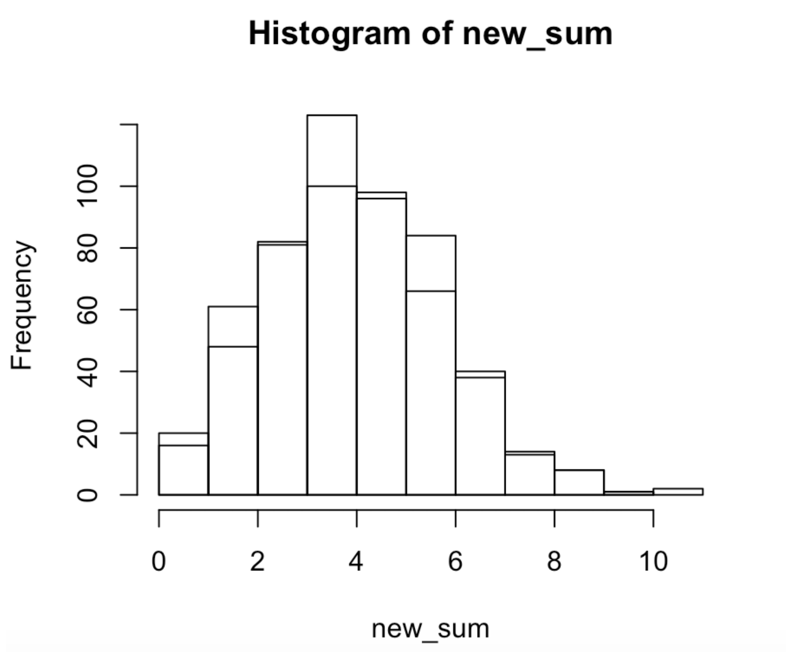
$$f_{Y|\frac{1}{2}} = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{4\frac{1}{2} + 6y}{4\frac{1}{2} + 3} dy = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{2 + 6y}{5} dy = \frac{1}{5}(2y3y^3)\Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{10}{20} = \frac{1}{2}$$

(d) $E(Y|x)$

$$E(Y|x) = \int_0^1 y \cdot \frac{4x + 6y}{4x + 3} = \frac{1}{4x + 3} \int_0^1 4xy + 6y^2 = \frac{2xy^2 + 2y^3}{4x + 3}\Big|_0^1 = \frac{2x + 2y}{4x + 3}$$

PROGRAMMING PROBLEM 1

```
 results = rbinom(500, 10, .3)

results2 = rbinom(500, 5, .3)

w_one = rbinom(500, 15, .3)

new_sum=results+results2

hist(new_sum)

hist(w_one, add=T)
```



PROGRAMMING PROBLEM 2

```
X=rexp(5000, rate = 1)
Y=rexp(5000, rate = 1)
W=Y/X

invcdf <- function(r) {1/(1-r)}-1
rands=runif(5000)
Wdev=invcdf(rands)

W = sort(W)
```

```
W = W[1:4750]

Wdev = sort(Wdev)
Wdev = Wdev[1:4750]

plot(W, Wdev)
```