COMPSCI 514: Midterm Review

1 Concepts to Study

Foundational Probability Concepts + Concentration Bounds

- Linearity of expectation and variance.
- Markov's inequality, Chebyshev's inequality (should know from memory).
- Union bound (should know from memory).
- General idea of higher moment inequalities.
- Chernoff and Bernstein bounds (don't need to memorize the exact bounds, but should be able to apply if given).
- General idea of law of large numbers and central limit theorem.
- Technique of breaking random variables into sums of indicator random variables.
- Averaging to reduce error.
- Median trick.

Random Hashing and Related Algorithms

- Random hash functions.
- Definitions of 2-universal and pairwise independent hash functions (should have memorized).
- Application of random hashing to load balancing.
- Hashing for Distinct Elements. Understand the 'idealized' algorithm where we hash to real numbers. Don't need to understand details of HyperLogLog
- Bloom Filters. Don't need to have formulas memorized.
- MinHash for Jaccard similarity.
- Idea of locality sensitive hashing. How it is used for similarity search (with hash signatures and repeated tables). Idea of s-curve tuning (don't need to memorize formula).

Other

- Frequent elements problem definition and setup.
- High level idea of Boyer-Moore and Misra-Gries, but don't need to know in detail.
- Count-min sketch and analysis.
- The Johnson-Lindenstrauss Lemma. Don't need to memorize, but should understand and be able to apply if given.
- Do not need to be able to recreate the JL proof, but should understand the ideas behind it.

2 Practice Questions

Work in progress. Check back to see if more questions have been added.

Probability, Expectation, Variance:

1. Exercises 2.1, 2.3, 2.4, 2.28, 2.41 of *Foundations of Data Science* (https://www.cs.cornell.edu/jeh/book.pdf)

2. Show that for any $\mathbf{X}$, $\mathbb{E}[\mathbf{X}^2] \geq \mathbb{E}[\mathbf{X}]^2$.

3. Show that for independent $\mathbf{X}$ and $\mathbf{Y}$, $\mathbb{E}[\mathbf{X} \cdot \mathbf{Y}] = \mathbb{E}[\mathbf{X}] \cdot \mathbb{E}[\mathbf{Y}]$.

4. Show that for independent $\mathbf{X}$ and $\mathbf{Y}$ with $\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{Y}] = 0$, $Var[\mathbf{X} \cdot \mathbf{Y}] = Var[\mathbf{X}] \cdot Var[\mathbf{Y}]$. **Hint:** use part (3).

5. For the statements below, indicate if they are **always true**, **sometimes true**, or **never true**. Give a sentence explaining why.

   (a) $\Pr[\mathbf{X} = s \cap \mathbf{Y} = t] > \Pr[\mathbf{X} = s]$. ALWAYS SOMETIMES NEVER

   (b) $\Pr[\mathbf{X} = s \cup \mathbf{Y} = t] \leq \Pr[\mathbf{X} = s] + \Pr[\mathbf{Y} = t]$. ALWAYS SOMETIMES NEVER

   (c) $\Pr[\mathbf{X} = s \cap \mathbf{Y} = t] = \Pr[\mathbf{X} = s] \cdot \Pr[\mathbf{Y} = t]$. ALWAYS SOMETIMES 4 NEVER

Concentration Inequalities:

1. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be the number of visitors to a website on n consecutive days. These are independent and identically distributed random variables. We have $\mathbb{E}[\mathbf{X}_i] = 20,000$ and $Var[\mathbf{X}_i] = 100,000,000$.

   (a) Give an upper bound on the probability that on day i, more than $40,000$ visitors hit the website.

   (b) Let $\bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$ be the average number of visitors over n days. What are $\mathbb{E}[\bar{\mathbf{X}}]$ and $Var[\bar{\mathbf{X}}]$?

   (c) Give an upper bound on the probability that $\bar{\mathbf{X}} \geq 25,000$, for $n = 100$.

2. Assume there are 1000 registered users on your site $u_1, \ldots, u_{1000}$, and in a given day, each user visits the site with some probability p_i. The event that any user visits the site is independent of what the other users do. Assume that $\sum_{i=1}^{1000} p_i = 500$.

   (a) Let $\mathbf{X}$ be the number of users that visit the site on the given day. What is $\mathbb{E}[\mathbf{X}]$.

   (b) Apply a Chernoff bound to show that $\Pr[\mathbf{X} \geq 600] \leq .01$.

Random Hashing Algorithms:

1. Exercises 6.1, 6.2, 6.6, 6.7, 6.10, 6.19, 6.22, 6.23 of *Foundations of Data Science*

2. Consider a hash function mapping m-bit strings to a single bit – $\mathbf{h} : \{0,1\}^m \to \{0,1\}$. We generate $\mathbf{h}$ by selecting a random position i from $1, \ldots, m$. Then let $\mathbf{h}(x) = x(i)$, the value of x at position i. Note that after i is chosen, it remains fixed, when we apply $\mathbf{h}$ to different inputs.

   (a) Given $x, y \in \{0,1\}^m$ with hamming distance $\|x - y\|_0$ (i.e., x and y have different bit values in $\|x - y\|_0$ positions), what is $\Pr[\mathbf{h}(x) = \mathbf{h}(y)]$.

(b) Is $\mathbf{h}$ a locality sensitive hash function?

(c) Let m be the number of all possible 5-singles in a document (i.e., all possible strings of 5 English words). If x and y are indicator vectors of the 5-shingles in two different documents, why do we expect them to be very sparse (i.e., each only have a few bits set to 1)?

(d) Why might might MinHash and Jaccard similarity be more useful in the situation of (c) than the hash function $\mathbf{h}$ and Hamming distance.

3. Use a Chernoff bound to show that if we hash n items into a table with n buckets, with probability $\geq 1 - \delta$, the maximum number of items in a single bucket is upper bounded by $O(\log n/\delta)$.