

# CS-UY 4563: Lecture 9

## Linear Classifiers, Logistic Regression

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NYU Tandon School of Engineering, Prof. Christopher Musco

# QUICK COMMENT ON PYTHON

Two ways to multiply every entry in a matrix by 2.

```
In [1]: 1 import numpy as np
        2 import time
```

```
In [2]: 1 n = 10000
        2 d = 10000
        3 X = np.random.randn(n,d)
        4 Y = np.zeros(X.shape)
```

```
In [3]: 1 t = time.time()
        2 for i in range(1,n):
        3     for j in range(1,d):
        4         Y[i,j] = 2*X[i,j]
        5 elapsed = time.time() - t
        6 print(str(elapsed) + " seconds")
```

75.23580312728882 seconds

```
In [4]: 1 t = time.time()
        2 Y = 2*X
        3 elapsed = time.time() - t
        4 print(str(elapsed) + " seconds")
```

0.20938587188720703 seconds

Second one is way faster...

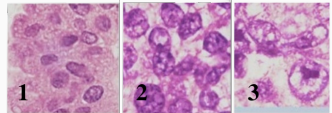
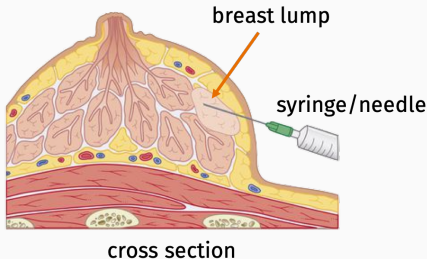
- Try to use matrix operations as much as possible!
- Slice indexing instead of loops. Broadcasting instead of entrywise multiplication.
- Review `demo2_more_numpy.ipynb`.
- This will make a huge difference in labs, and in your projects. Knowing how to work with matrices in an efficient way is one of the most important coding skills for machine learning and data science.

# CLASSIFICATION

## MOTIVATING PROBLEM

**Breast Cancer Biopsy:** Determine if a breast lump in a patient is malignant (cancerous) or benign (safe).

- Collect cells from lump using fine needle biopsy.
- Stain and examine cells under microscope.
- Based on certain characteristics (shape, size, cohesion) determine if likely malignant or not).



# MOTIVATING PROBLEM

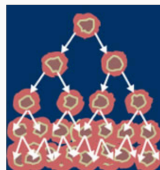
Demo: `demo_breast_cancer.ipynb`

Data: UCI machine learning repository

## Breast Cancer Wisconsin (Original) Data Set

Download: [Data Folder](#), [Data Set Description](#)

**Abstract:** Original Wisconsin Breast Cancer Database



Data Set Characteristics:	Multivariate	Number of Instances:	699	Area:	Life
Attribute Characteristics:	Integer	Number of Attributes:	10	Date Donated	1992-07-15
Associated Tasks:	Classification	Missing Values?	Yes	Number of Web Hits:	564320

[https://archive.ics.uci.edu/ml/datasets/breast+cancer+wisconsin+\(original\)](https://archive.ics.uci.edu/ml/datasets/breast+cancer+wisconsin+(original))

**Features:** 10 numerical scores about cell characteristics (Clump Thickness, Uniformity, Marginal Adhesion, etc.)

**Data:**  $(\vec{x}_1, y_1), \dots, (\vec{x}_n, y_n)$ .

$\vec{x}_i = [1, 5, 4 \dots, 2]$  contains score values.

Label  $y_i \in \{0, 1\}$  is 0 if benign cells, 1 if malignant cells.

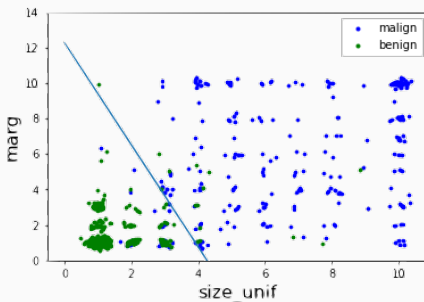
**Goal:** Based on scores (which would be collected manually, or even learned on their own using an ML algorithm) predict if a sample of cells is malignant or benign.

# LINEAR CLASSIFIER

Given vector of predictors  $\vec{x}_i \in \mathbb{R}^d$  (here  $d = 2$ ) find a parameter vector  $\vec{\beta} \in \mathbb{R}^d$  and threshold  $\lambda$ .

- Predict  $y_i = 0$  if  $\langle \vec{x}_i, \vec{\beta} \rangle \leq \lambda$ .
- Predict  $y_i = 1$  if  $\langle \vec{x}_i, \vec{\beta} \rangle > \lambda$

Line has equation  $\langle \vec{x}, \vec{\beta} \rangle = \lambda$ .



Hyperplane

Half-space



**Question:** How do we find a good linear classifier automatically?

**Loss minimization approach (first attempt):**

- **Model<sup>1</sup>:**

$$f_{\vec{\beta}}(\vec{x}) = \mathbb{1} \left[ \langle \vec{x}, \vec{\beta} \rangle > 0 \right]$$

- **Loss function: “0 – 1 Loss”**

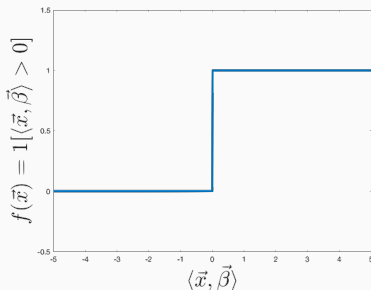
$$L(\vec{\beta}) = \sum_{i=1}^n |f_{\vec{\beta}}(\vec{x}_i) - y_i|$$

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<sup>1</sup> $\mathbb{1}[\text{event}]$  is the indicator function: it evaluates to 1 if the argument inside is true, 0 if false.

## 0 – 1 LOSS

Problem with 0 – 1 loss:



- The loss function  $L(\vec{\beta})$  is not differentiable because  $f_{\vec{\beta}}(\vec{x})$  is discontinuous.
- Impossible to take the gradient, very hard to minimize loss to find optimal  $\vec{\beta}$ .

## LINEAR CLASSIFIER VIA SQUARE LOSS

**Question:** How do we find a good linear classifier automatically?

**Loss minimization approach (second attempt):**

- **Model:**

$$f_{\vec{\beta}}(\vec{x}) = \mathbb{1} \left[ \langle \vec{x}, \vec{\beta} \rangle > 1/2 \right]$$

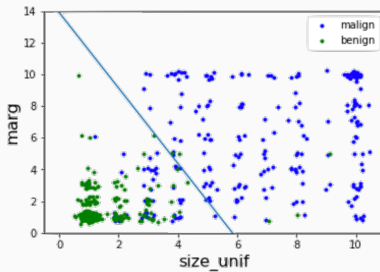
- **Loss function:** “Square Loss”

$$L(\vec{\beta}) = \sum_{i=1}^n (\langle \vec{x}_i, \vec{\beta} \rangle - y_i)^2$$

Intuitively tries to make  $\langle \vec{x}, \vec{\beta} \rangle$  close to 0 for examples in class 0, close too 1 for examples in class 1.

## LINEAR CLASSIFIER VIA SQUARE LOSS

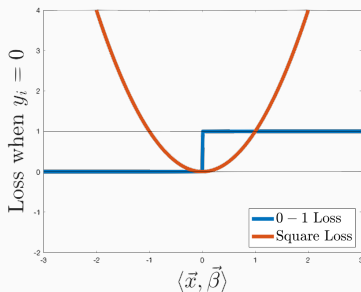
We can solve for  $\vec{\beta}$  by just solving a least squares multiple linear regression problem.



Do you see any issues here?

## LINEAR CLASSIFIER VIA SQUARE LOSS

Problem with square loss:



- Loss increases if  $\langle \vec{x}, \vec{\beta} \rangle < 0$  even if correct label is 0. Or if  $\langle \vec{x}, \vec{\beta} \rangle > 1$  even if correct label is 1. Or
- Intuitively we don't want to "punish" these cases.

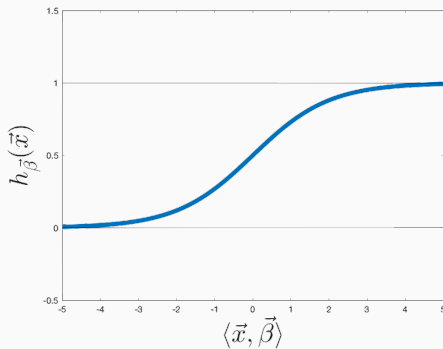
Let  $h_{\vec{\beta}}(\vec{x})$  be the **logistic function**:

$$h_{\vec{\beta}}(\vec{x}) = \frac{1}{1 + e^{-\langle \vec{\beta}, \vec{x} \rangle}}$$

# LOGISTIC REGRESSION

Let  $h_{\vec{\beta}}(\vec{x})$  be the **logistic function**:

$$h_{\vec{\beta}}(\vec{x}) = \frac{1}{1 + e^{-\langle \vec{\beta}, \vec{x} \rangle}}$$



We can think of  $h_{\vec{\beta}}(\vec{x})$  as mapping  $\langle \vec{\beta}, \vec{x} \rangle$  to a probability.

Loss minimization approach (what works!):

- **Model:** Let  $h_{\vec{\beta}}(\vec{x}) = \frac{1}{1+e^{-\langle \vec{\beta}, \vec{x} \rangle}}$

$$f_{\vec{\beta}}(\vec{x}) = \mathbb{1} \left[ h_{\vec{\beta}}(\vec{x}) > 1/2 \right]$$

- **Loss function:** “Logistic loss” aka “Cross-entropy loss”

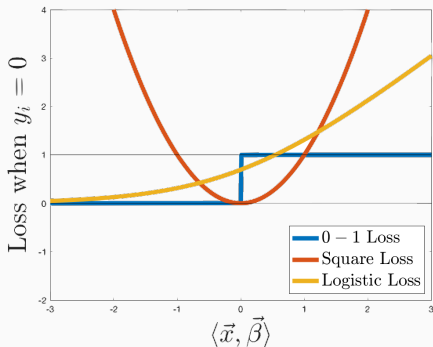
$$L(\vec{\beta}) = - \sum_{i=1}^n y_i \log(h_{\vec{\beta}}(\vec{x}_i)) + (1 - y_i) \log(1 - h_{\vec{\beta}}(\vec{x}_i))$$



# LOGISTIC LOSS

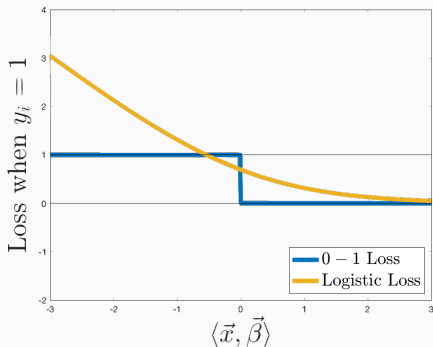
Logistic Loss:

$$L(\vec{\beta}) = -\sum_{i=1}^n y_i \log(h_{\vec{\beta}}(\vec{x})) + (1 - y_i) \log(1 - h_{\vec{\beta}}(\vec{x}))$$



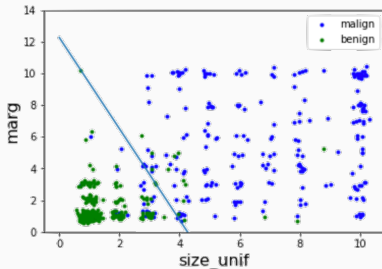
Logistic Loss:

$$L(\vec{\beta}) = -\sum_{i=1}^n y_i \log(h_{\vec{\beta}}(\vec{x})) + (1 - y_i) \log(1 - h_{\vec{\beta}}(\vec{x}))$$



## LOGISTIC LOSS

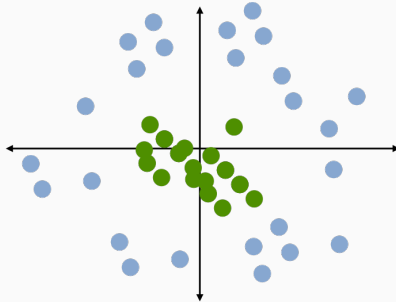
- Convex function, can be minimized using gradient descent (next lecture).
- Works well in practice.
- Good Bayesian motivation: see posted lecture notes if you are interested.



Fit using logistic regression/log loss.

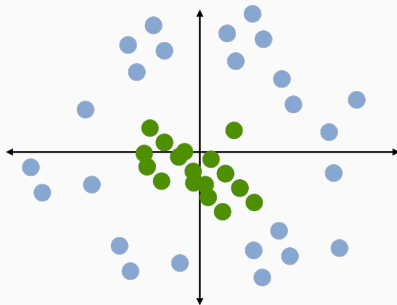
# NON-LINEAR TRANSFORMATIONS

How would we learn a classifier for this data:



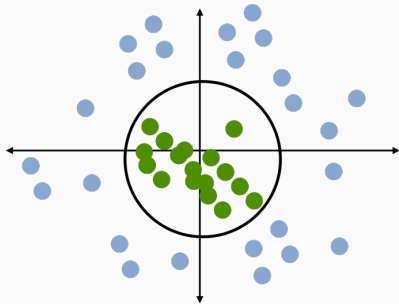
## NON-LINEAR TRANSFORMATIONS

How would we learn a classifier for this data using logistic regression?



## NON-LINEAR TRANSFORMATIONS

Transform each  $\vec{x} = [x_1, x_2]$  to  $\vec{x} = [1, x_1, x_2, x_1^2, x_2^2, x_1x_2]$



- Predict class 1 if  $x_1^2 + x_2^2 < \lambda$ .
- Predict class 0 if  $x_1^2 + x_2^2 \geq \lambda$ .

This is a linear classifier on our transformed data set. Logistic regression would learn  $\vec{\beta} = [0, 0, 0, 1, 1, 0]$ .

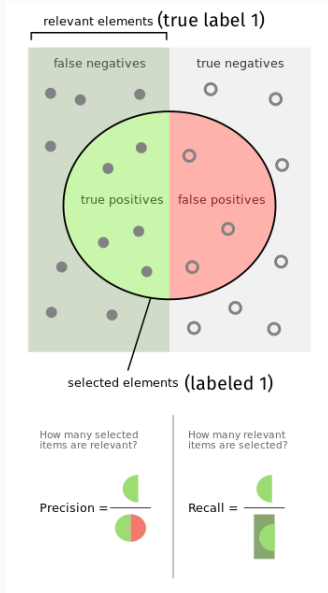
Once we have a classification algorithm, how do we judge its performance?

- **Simplest answer:** Error rate = fraction of data examples misclassified in test set.
- What are some issues with this approach?

# ERROR IN CLASSIFICATION

- **Precision:** Fraction of positively labeled examples (label 1) which are correct.
- **Recall:** Fraction of true positives that we labeled correctly with label 1.

**Question:** Which should we optimize for medical diagnosis?





### Possible logistic regression workflow:

- Learn  $\vec{\beta}$  and compute  $h_{\vec{\beta}}(\vec{x}_i) = \frac{1}{1+e^{-\langle \vec{x}_i, \vec{\beta} \rangle}}$  for all  $\vec{x}_i$ .
- Predict  $y_i = 0$  if  $h_{\vec{\beta}}(\vec{x}_i) \leq \lambda$ ,  $y_i = 1$  if  $h_{\vec{\beta}}(\vec{x}_i) > \lambda$ .
- Default value of  $\lambda$  is 1/2. Increasing  $\lambda$  improves precision. Decreasing  $\lambda$  improves recall.

This is very heuristic. There are other methods for handling “class imbalance” which can often lead to good overall error, but poor precision or recall. Techniques include weighting the loss function to care more about false negatives, or subsampling the larger class.

What about when  $y \in \{1, \dots, q\}$  instead of  $y \in \{0, 1\}$

Two options for multiclass data:

- One-vs.-all (most common, also called one-vs.-rest)
- One-vs.-one (slower, but can be more effective)

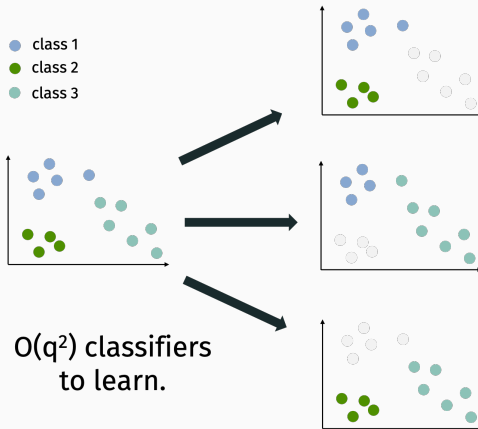
In both cases, we convert to multiple binary classification problem.

# ONE VS. REST



- For  $q$  classes train  $q$  classifiers. Obtain parameters  $\vec{\beta}_1, \dots, \vec{\beta}_q$ .
- Assign  $y$  to class  $i$  with maximum  $\langle \vec{\beta}_i, \vec{x} \rangle$ .

# ONE VS. REST



- For  $q$  classes train  $\frac{q(q-1)}{2}$  classifiers.
- Assign  $y$  to class which  $i$  which wins in the most number of head-to-head comparisons.

Hard case for one-vs.-all.



- One-vs.-one would be a better choice here.
- Also tends to work better when there is class in balance.

# ERROR IN (MULTICLASS) CLASSIFICATION

Confusion matrix for  $k$  classes:

Pred-->	1	2	...	K
Real↓				
1				
2				
...				
K				

- Entry  $i, j$  is the fraction of class  $i$  items classified as class  $j$ .
- Overall accuracy is the average of the diagonals.
- Useful to see whole matrix to visualize where errors occur.