CS-UY 4563: Lecture 24
Reinforcement Learning

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SOME MATERIAL WE DIDN'T HAVE TIME FOR

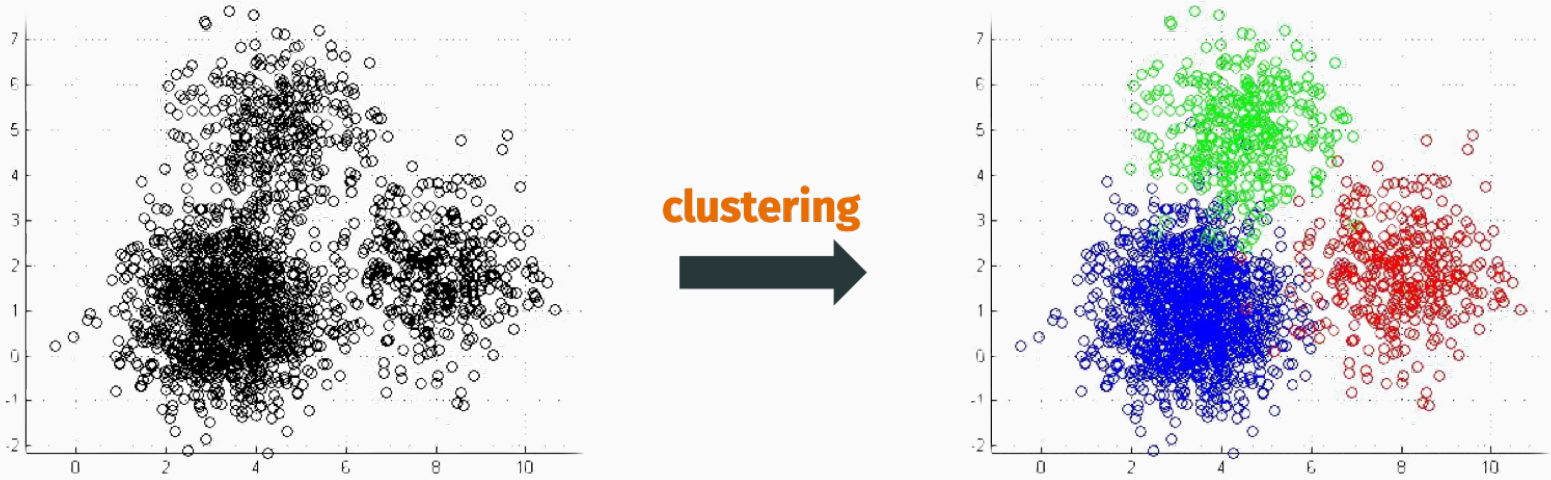
Supervised learning:

- **Decision trees.** Very effective model for problems with few features. Difficult to train, but heuristics work well in practice.
- **Boosting.** Approach for combining several "weak" models to obtain better overall accuracy than any one model alone.

Unsupervised learning:

- **Adversarial models.** Modern alternative to auto-encoders that performs very well for lots of interesting problems, especially in generative ML.
- Clustering. Hugely important for data exploration and visualization.

Important unsupervised learning task:

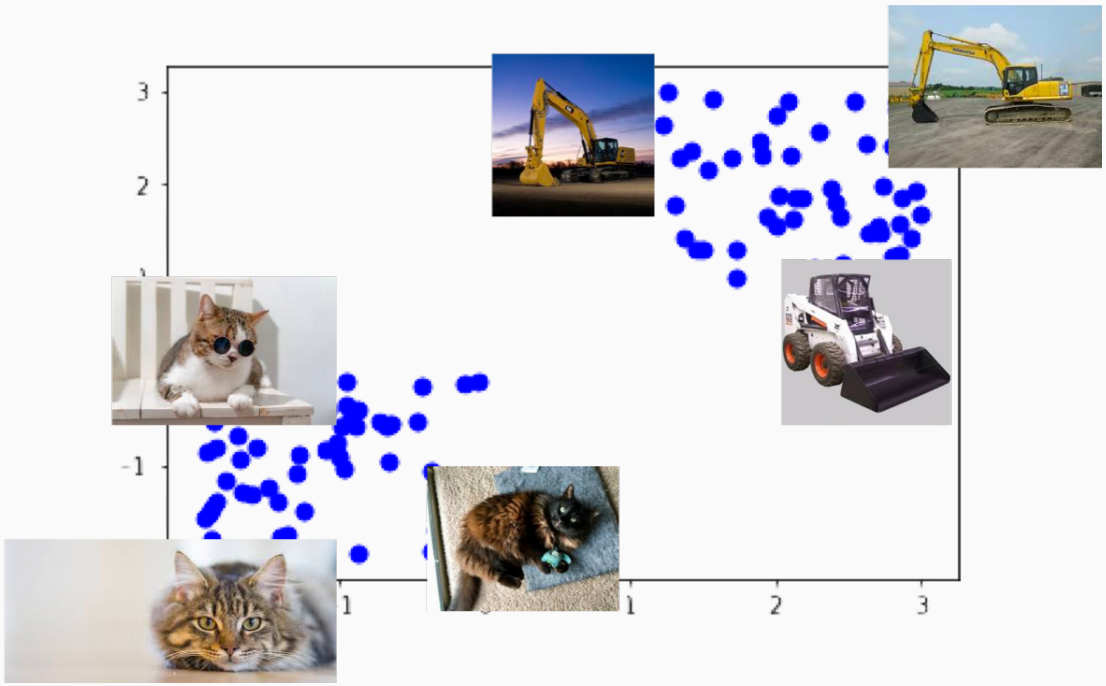


Separate unlabeled data into natural clusters.

- Exploratory data analysis.
- Categorizing and grouping data.
- Visualizing data.

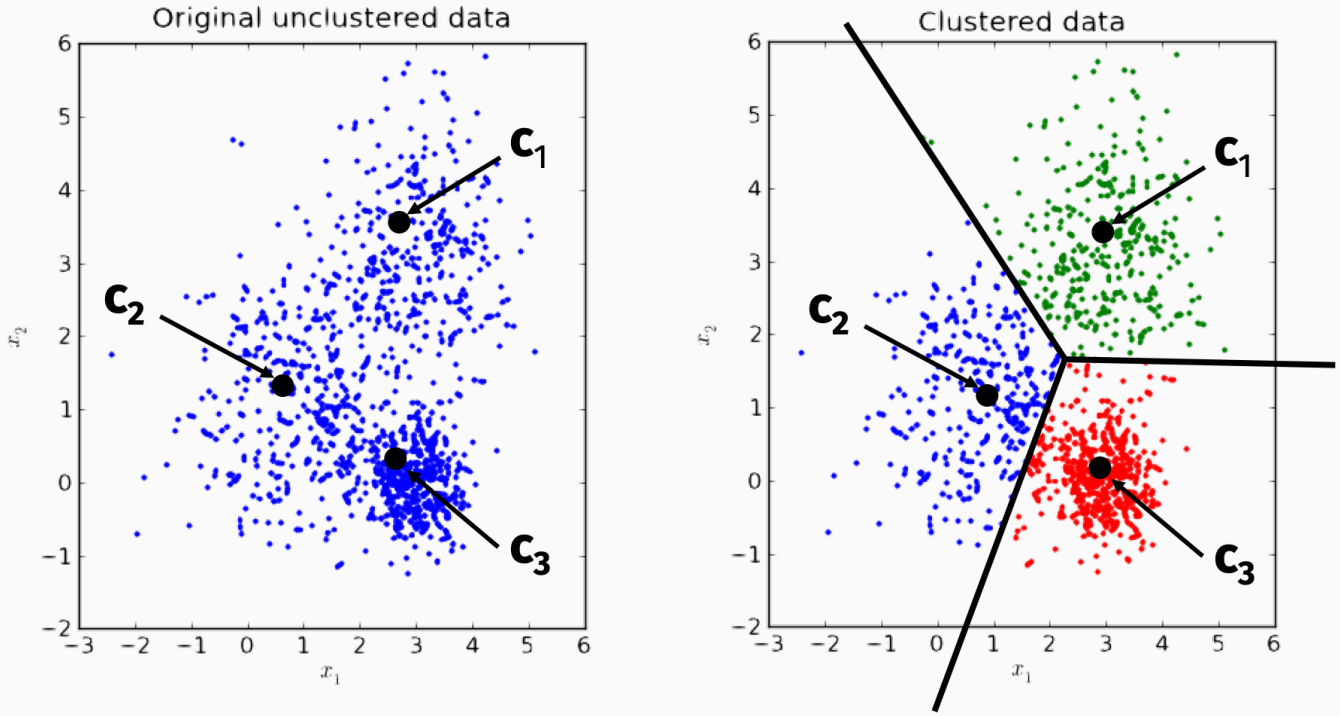
Example application:

Images of Cats.



Find sub-classes in your data which you did not know about.
Helps you decide how to adjust features or improve data set for a supervised application.

DATA CLUSTERING



k-center clustering:

- Choose centers $\vec{c}_1, \ldots, \vec{c}_k \in \mathbb{R}^d$.
- Assign data point $\vec{x}$ to cluster i if $\vec{c}_i$ is the "nearest" center.
- Can use any distance metric.

K-CENTER CLUSTERING

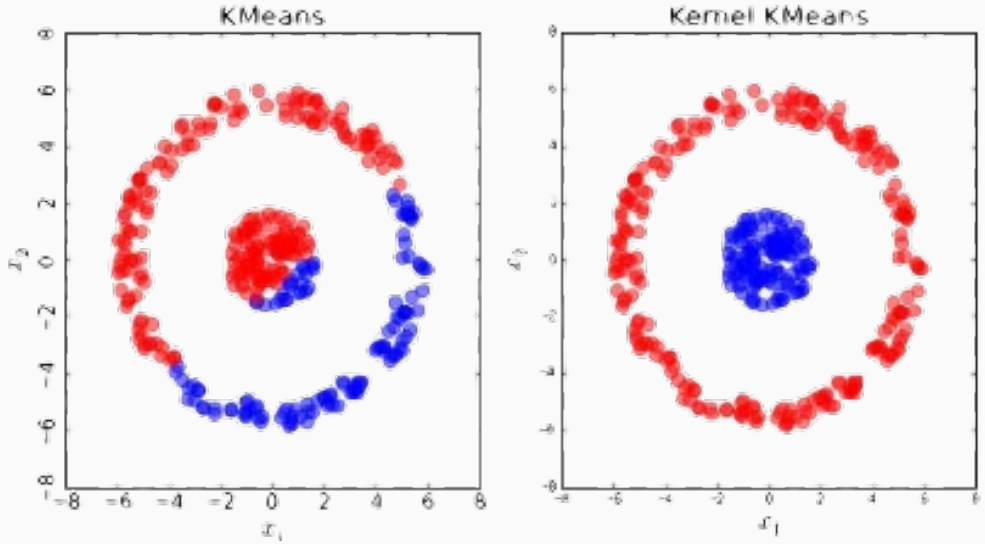
Given data points $\vec{x}_1, \ldots, \vec{x}_n$ and distance metric $\Delta(\vec{x}, \vec{c}) \to \mathbb{R}$, choose $\vec{c}_1, \ldots, \vec{c}_k$ to minimize:

$$Cost(\vec{c}_1, \ldots, \vec{c}_k) = \sum_{i=1}^{n} \min_j \Delta(\vec{x}_i, \vec{c}_j).$$

In general this could be a hard optimization problem.

Common choice: Use Euclidean distance. I.e. set $\Delta(\vec{x}, \vec{c}) = \|\vec{x} - \vec{c}\|_2^2$.

- If $k = 1$, optimal choice for c_1 is the centroid $\frac{1}{n}\sum_{i=1}^{n} \vec{x}_n$. For large k the problem is NP-hard.
- Can be solved efficiently in practice using optimization techniques known as **alternating minimization**. Called "Llyod's algorithm" when applied to k-means clustering.
- Euclidean k-means can only identify linearly separable clusters.



Today: Give flavor of the area and insight into one algorithm (Q-learning) which has been successful in recent years.

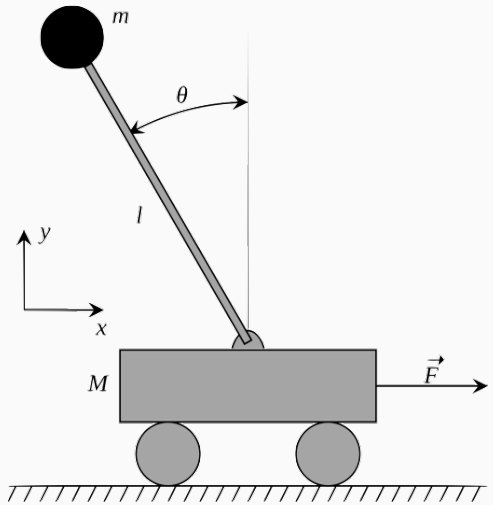
Basic setup:[1]

- **Agent** interacts with **environment** over time $1, \ldots, t$.
- Takes repeated sequence of **actions**, $a_1, \ldots, a_t$ which effect the environment.
- **State** of the environment over time denoted $s_1, \ldots, s_t$.
- Earn **rewards** $r_1, \ldots, r_t$ depending on actions taken and states reached.
- Goal is to maximize reward over time.

[1] Slide content adapted from: `http://cs231n.stanford.edu/`

REINFORCEMENT LEARNING EXAMPLES

Classic inverted pendulum problem:



- **Agent:** Cart/software controlling cart.
- **State:** Position of the car, pendulum head, etc.
- **Actions:** Move cart left or move right.
- **Reward:** 1 for every time step that $|\theta| < 90°$ (pendulum is upright). 0 when $|\theta| = 90°$

REINFORCEMENT LEARNING EXAMPLES

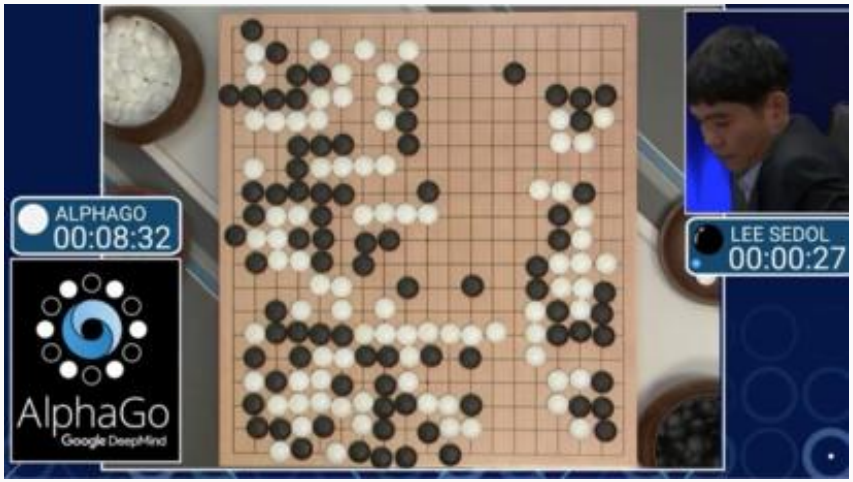
This problem has a long history in **Control Theory.** Other applications of classical control:

- Semi-autonomous vehicles (airplanes, helicopters, rockets, etc.)
- Industrial processes (e.g. controlling large chemical reactions)
- Robotics

control theory : reinforcement learning :: stats : machine learning

REINFORCEMENT LEARNING EXAMPLES

Strategy games, like Go:

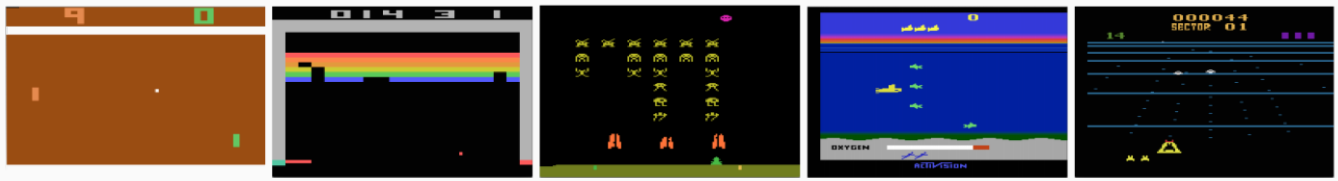


- **State:** Position of all pieces on board.
- **Actions:** Place new piece.
- **Reward:** 1 if in winning position at time t. 0 otherwise.

This is a sparse reward problem. Payoff only comes after many times steps, which makes the problem very challenging.

REINFORCEMENT LEARNING EXAMPLES

Video games, like classic Atari games:



- **State:** Raw pixels on the screen (sometimes there is also hidden state which can't be observed by the player).
- **Actions:** Actuate controller (up,down,left,right,click).
- **Reward:** 1 if point scored at time t.

Model problem as a **Markov Decision Process (MDP)**:

- $\mathcal{S}$: Set of all possible states. $|\mathcal{S}| = n$.
- $\mathcal{A}$: Set of all possible actions. $|\mathcal{A}| = k$.
- **Reward function**
  $R(s, a) : \mathcal{S} \times \mathcal{A} \to$ probability distribution over $\mathbb{R}$. $r_t \sim R(s_t, a_t)$.
- **State transition function**
  $P(s, a) : \mathcal{S} \times \mathcal{A} \to$ probability distribution over $\mathcal{S}$. $s_{t+1} \sim P(s_t, a_t)$.

Why is this called a Markov decision process? What does the term Markov refer to?

Goal: Learn a **policy** $\Pi : \mathcal{S} \to \mathcal{A}$ from states to actions which maximized expected cumulative reward.

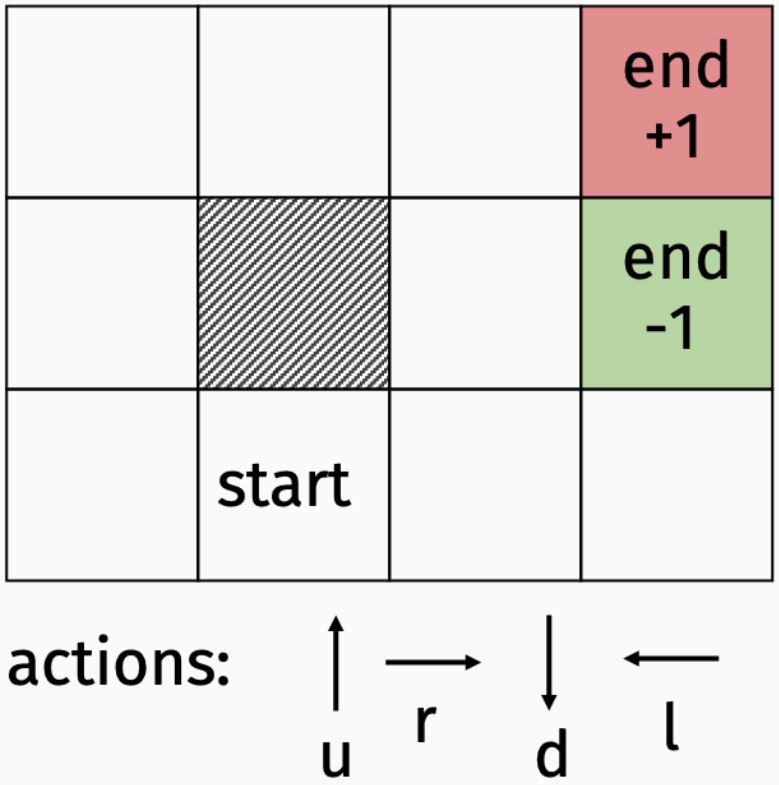
- Start is state s_0.
- For $t = 0 \ldots, T$
  - $r_t \sim R(s_t, \Pi(s_t))$.
  - $s_{t+1} \sim P(s_t, \Pi(s_t))$.

The **time horizon** T could be finite (game with fixed number of steps) or infinite (stock investing). Goal is to maximize:

$$reward(\Pi) = \mathbb{E} \sum_{t=0}^{T} r_t$$

$[s_0, a_0, r_0], [s_1, a_1, r_1], \ldots, [s_t, a_t, r_t]$ is called a **trajectory** of the MDP under policy Π.

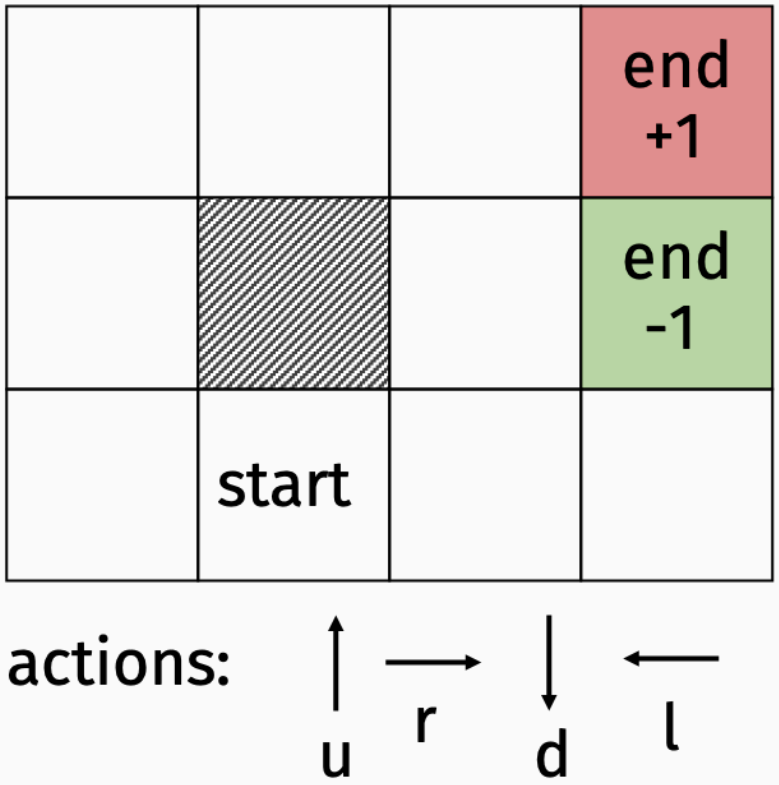
SIMPLE EXAMPLE: GRIDWORLD



- $r_t = -.01$ if not at an end position. ± 1 if at end position.
- $P(s_t, a)$: 50% of the time move in the direction indicated by a. 50% of the time move in a random direction.

What is the optimal policy Π?

SIMPLE EXAMPLE: GRIDWORLD



- $r_t = -.5$ if not at an end position. ± 1 if at end position.
- $P(s_t, a)$: 50% of the time move in the direction indicated by a. 50% of the time move in a random direction.

What is the optimal policy Π?

DISCOUNT FACTOR

For infinite or very long times horizon games (large T), we often introduce a **discount factor** γ and seek instead to find a policy Π which minimizes:

$$\mathbb{E} \sum_{t=0}^{T} \gamma^t r_t$$

where $r_t \sim R(s_t, \Pi(s_t))$ and $s_{t+1} \sim P(s_t, \Pi(s_t))$ as before.

From now on assume $T = \infty$. We can do this without loss of generality by adding a time parameter to state and moving into an "end state" with no additional rewards once the time hits T.

VALUE FUNCTION AND Q FUNCTION

Two important definitions.

- **Value function:** $V^{\Pi}(s) = \mathbb{E}_{\Pi, s_0=s} \sum_{t \geq 0} \gamma^t r_t$. Measures the expected return if we start in state s and follow policy Π.
- **Q-function:** $Q^{\Pi}(s, a) = \mathbb{E}_{\Pi, s_0=s, a_0=a} \sum_{t \geq 0} \gamma^t r_t$. Measures the expected return if we start in state s, play action a, and then follow policy Π.

$$Q^*(s, a) = \max_{\Pi} \mathbb{E}_{\Pi, s_0=s, a_0=a} \sum_{t \geq 0} \gamma^t r_t.$$

If we knew the function Q^*, we would immediately know an optimal policy. Whenever we're in state s, we should always play action $a^* = \arg\max_Q^*(s, a)$.

BELLMAN EQUATION

Q^* satisfies what's known as a Bellman equation:

$$Q^*(s,a) = \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q^*(s',a') \right].$$

Value Iteration: Used fixed point iteration to find Q^*:

- Initialize Q^0 (e.g. randomly).
- For $i = 1, \ldots, z$:
  - $Q^i = \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q^{i-1}(s',a') \right]$

Possible to prove that $Q^i \to Q^*$ as $i \to \infty$.

Note that many details are involved in this computation.

Need to handle the expectations on the right hand side by randomly sampling trajectories from the MDP.

CENTRAL ISSUE IN REINFORCEMENT LEARNING

Bigger issue: Even writing down Q^* is intractable... This is a function over $|\mathcal{S}|^{|\mathcal{A}|}$ possible inputs. Even for relatively simple games, $|\mathcal{S}|$ is gigantic...

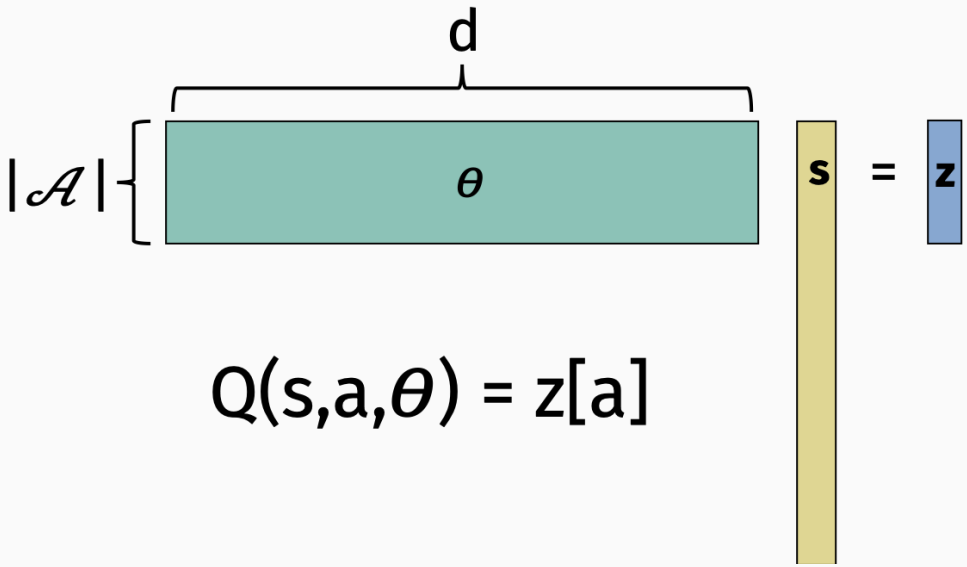
Back of the envelope calculations:

- **Tic-tac-toe:** $3^{(3\times 3)} \approx 20,000$
- **Chess:** $\approx 10^{43}$ (due to Claude Shannon).
- **Go:** $3^{(19*\times 19)} \approx 10^{171}$.
- **Atari:** $128^{(210\times 160)} \approx 10^{71,000}$.

Number of atoms in the universe: $\approx 10^{82}$.

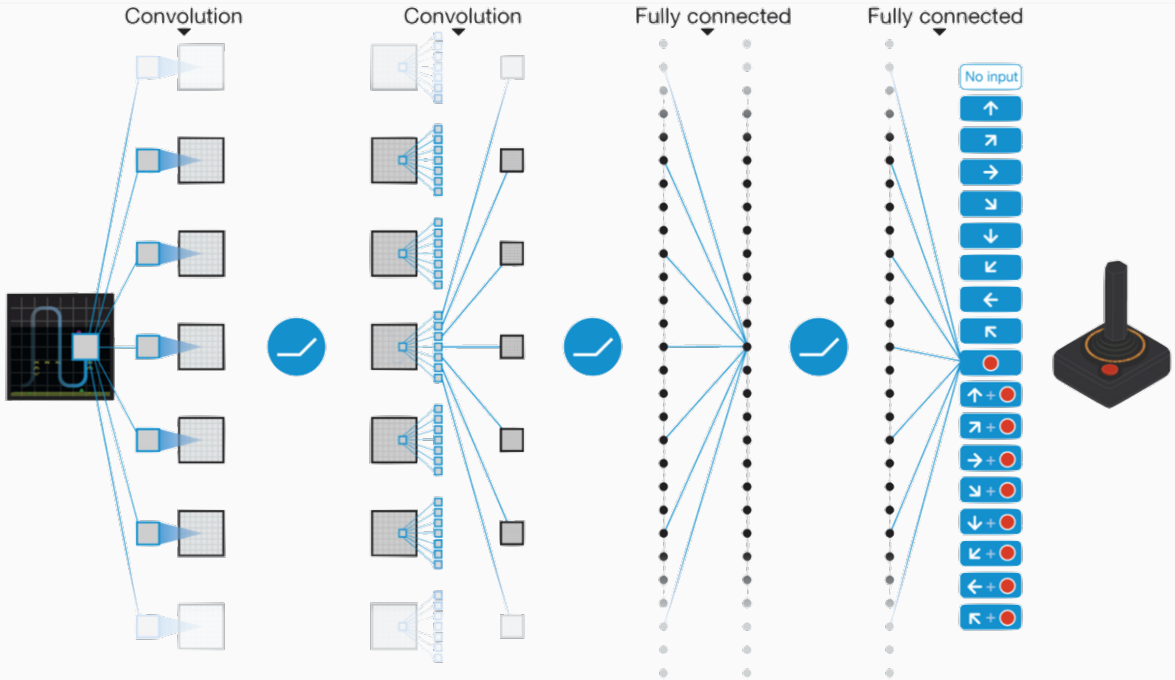
Learn a **simpler** function $Q(s, a, \theta) \approx Q^*(s, a)$ parameterized by a small number of parameters θ.

Example: Suppose our state can be represented by a vector in $\mathbb{R}^d$ and our action a by an integer in $1, \ldots, |\mathcal{A}|$. We could use a linear function where θ is a small matrix:



Learn a **simpler** function $Q(s, a, \theta) \approx Q^*(s, a)$ parameterized by a small number of parameters θ.

Example: Could also use a (deep) neural network.



DeepMind: "Human-level control through deep reinforcement learning", Nature 2015.

If $Q(s, a, \theta)$ is a good approximation to $Q^*(s, a)$ then we have an approximately optimal policy: $\tilde{\Pi}^*(s) = \arg\max_a Q(s, a, \theta)$.

- Start in state s_0.
- For $t = 1, 2, \ldots$
  - $a^* = \arg\max_a Q(s, a, \theta)$
  - $s_t \sim P(s_{t-1}, a^*)$

How do we find an optimal θ? If we knew $Q^*(s, a)$ could use supervised learning, but the true Q function is infeasible to compute.

Q-LEARNING W/ FUNCTION APPROXIMATION

Find θ which satisfies the Bellman equation:

$$Q^*(s,a) = \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q^*(s',a') \right]$$

$$Q(s,a,\theta) \approx \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q(s,a,\theta) \right].$$

Should be true for all a, s. Should also be true for $a, s \sim \mathcal{D}$ for any distribution $\mathcal{D}$:

$$\mathbb{E}_{s,a \sim \mathcal{D}} Q(s,a,\theta) \approx \mathbb{E}_{s,a \sim \mathcal{D}} \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q(s,a,\theta) \right].$$

Loss function:

$$L(\theta) = \mathbb{E}_{s,a \sim \mathcal{D}} \left(y - Q(s,a,\theta) \right)^2$$

where $y = \mathbb{E}_{s' \sim P(s,a)} \left[R(s,a) + \gamma \max_{a'} Q(s',a',\theta) \right]$.

Q-learning w/ function approximation

Minimize loss with **gradient descent**:

$$\nabla L(\theta) = -2 \cdot \mathbb{E}_{s,a \sim \mathcal{D}} \nabla Q(s, a, \theta) \cdot \left[R(s, a) + \gamma \max_{a'} Q(s', a', \theta) - Q(s, a, \theta) \right]$$

In practice use stochastic gradient:

$$\nabla L(\theta, s, a) = -2 \cdot \nabla Q(s, a, \theta) \cdot \left[R(s, a) + \gamma \max_{a'} Q(s', a', \theta) - Q(s, a, \theta) \right]$$

- Initialize θ_0
- For $i = 0, 1, 2, \ldots$
  - Choose random $s, a \sim \mathcal{D}$.
  - Set $\theta_{i+1} = \theta_i - \eta \cdot \nabla L(\theta_i, s, a)$

where η is a learning rate parameter.

Q-learning w/ function approximation

- Initialize θ_0
- For $i = 0, 1, 2, \ldots$
  - Choose random $s, a \sim \mathcal{D}$.
  - Set $\theta_{i+1} = \theta_i - \nabla L(\theta_i, s, a)$.

What is the distribution $\mathcal{D}$?

- **Random play**: Choose uniformly over reachable states + actions.

Wasteful: Seeks to approximate Q^* well in parts of the state-action space that don't actually matter for optimal play. Would require a ton of samples.

Q-learning w/ function approximation

More common approach: Play according to current guess for optimal policy, with some random "off-policy" exploration. The $\mathcal{D}$ is the distribution over states/actions results form this play. Note that $\mathcal{D}$ changes over time...

ϵ-greedy approach:

- Initialize s_0.
- For $t = 0, 1, 2, \ldots,$
  - $a_i = \begin{cases} \arg\max_a Q(s_t, a, \theta_{curr}) & \text{with probabilty } (1-\epsilon) \\ \text{random action} & \text{with probabilty } \epsilon \end{cases}$

Exploration-exploitation tradeoff. Increasing ϵ = more **exploration**.

REFERENCES

Lots of other details we don't have time for! References:

- Original DeepMind Atari paper: `https://www.cs.toronto.edu/~vmnih/docs/dqn.pdf`, which is very readable.
- Stanford lecture video: `https://www.youtube.com/watch?v=lvoHnicueoE` and slides: `http://cs231n.stanford.edu/slides/2017/cs231n_2017_lecture14.pdf`

Important concept we did not cover: **experience replay**.

ATARI DEMO

https://www.youtube.com/watch?v=V1eYniJ0Rnk