

# Introduction to Machine Learning Problems: Logistic Regression

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1. Suggest possible response variables and predictors for the following classification problems. For each problem, indicate how many classes there are. There is no single correct answer.
  - (a) Given an audio sample, to detect the gender of the voice.
  - (b) A electronic writing pad records motion of a stylus and it is desired to determine which letter or number was written. Assume a segmentation algorithm is already run which indicates very reliably the beginning and end time of the writing of each character.

2. Suppose that a logistic regression model for a binary class label  $y = 0, 1$  is given by

$$P(y = 1|\mathbf{x}) = \frac{1}{1 + e^{-z}}, \quad z = \beta_0 + \beta_1 x_1 + \beta_2 x_2,$$

where  $\beta = [1, 2, 3]^\top$ . Describe the following sets:

- (a) The set of  $\mathbf{x}$  such that  $P(y = 1|\mathbf{x}) > P(y = 0|\mathbf{x})$ .
  - (b) The set of  $\mathbf{x}$  such that  $P(y = 1|\mathbf{x}) > 0.8$ .
  - (c) The set of  $x_1$  such that  $P(y = 1|\mathbf{x}) > 0.8$  and  $x_2 = 0.5$ .
3. A data scientist is hired by a political candidate to predict who will donate money. The data scientist decides to use two predictors for each possible donor:
    - $x_1$  = the income of the person (in thousands of dollars), and
    - $x_2$  = the number of websites with similar political views as the candidate the person follow on Facebook.

To train the model, the scientist tries to solicit donations from a randomly selected subset of people and records who donates or not. She obtains the following data:

Income (thousands \$), $x_{i1}$	30	50	70	80	100
Num websites, $x_{i2}$	0	1	1	2	1
Donate (1=yes or 0=no), $y_i$	0	1	0	1	1

- (a) Draw a scatter plot of the data labeling the two classes with different markers.

- (b) Find a linear classifier that makes at most one error on the training data. The classifier should be of the form,

$$\hat{y}_i = \begin{cases} 1 & \text{if } z_i > 0 \\ 0 & \text{if } z_i < 0, \end{cases} \quad z_i = \mathbf{w}^\top \mathbf{x}_i + b.$$

What is the weight vector  $\mathbf{w}$  and bias  $b$  in your classifier?

- (c) Now consider a logistic model of the form,

$$P(y_i = 1 | \mathbf{x}_i) = \frac{1}{1 + e^{-z_i}}, \quad z_i = \mathbf{w}^\top \mathbf{x}_i + b.$$

Using  $\mathbf{w}$  and  $b$  from the previous part, which sample  $i$  is the *least* likely (i.e.  $P(y_i | \mathbf{x}_i)$  is the smallest). If you do the calculations correctly, you should not need a calculator.

- (d) Now consider a new set of parameters

$$\mathbf{w}' = \alpha \mathbf{w}, \quad b' = \alpha b,$$

where  $\alpha > 0$  is a positive scalar. Would using the new parameters change the values  $\hat{y}$  in part (b)? Would they change the likelihoods  $P(y_i | \mathbf{x}_i)$  in part (c)? If they do not change, state why. If they do change, qualitatively describe the change as a function of  $\alpha$ .

4. Suppose we collect data for a group of students in a machine learning class with variables  $X_1$  = hours studied,  $X_2$  = undergrad GPA, and  $Y$  = receive an A. We fit a logistic regression and produce estimated coefficient,  $\beta_0 = -6$ ,  $\beta_1 = 0.05$ ,  $\beta_2 = 1$ .

- (a) Estimate the probability that a student who studies for 40 h and has an undergrad GPA of 3.5 gets an A in the class.
- (b) How many hours would the student in part (a) need to study to have a 50 % chance of getting an A in the class?

5. The loss function for logistic regression for binary classification is the binary cross entropy defined as

$$J(\boldsymbol{\beta}) = \sum_{i=1}^N \ln(1 + e^{z_i}) - y_i z_i$$

where  $z_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$  for two features  $x_{1,i}$  and  $x_{2,i}$ .

- (a) What are the partial derivatives of  $z_i$  with respect to  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ .
- (b) Compute the partial derivatives of  $J(\boldsymbol{\beta})$  with respect to  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ . You should use the chain rule of differentiation.
- (c) Can you find the close form expressions for the optimal parameters  $\hat{\beta}_0$ ,  $\hat{\beta}_1$ , and  $\hat{\beta}_2$  by putting the derivatives of  $J(\boldsymbol{\beta})$  to 0? What methods can be used to optimize the loss function  $J(\boldsymbol{\beta})$ ?