More Math Review

Summations:

$$\sum_{i=1}^{n} i^d = \Theta(n^{d+1}) \quad \forall d \geq 0$$

$$\sum_{i=1}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \begin{cases} \Theta(c^n) & \forall c>1 \\ \Theta(1) & \forall c<1 \end{cases}$$

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + \Theta(1)$$

$$\sum_{i=1}^{n} \log i = n\log n - \Theta(n)$$

Ex 1.
```
for i = 1 to n do
    for j = 1 to n do          — Θ(n)              }
    for j = 1 to i do          }                   } Θ(i²+n)
        for k = 1 to i do      } Θ(i²)             }
```

$$\text{total} = \Theta\Big(\sum_{i=1}^{n}(i^2+n)\Big) = \Theta\Big(n^2 + \sum_{i=1}^{n} i^2\Big) = \Theta(n^2+n^3) = \Theta(n^3)$$

Ex 2.

"Build Heap"
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for i = n down to 1 do
    j = i
    while j ≤ n do      } Θ(log n/i)
        j = 2j
```

$$\text{total} = \Theta\Big(\sum_{i=1}^{n}\big(\log \tfrac{n}{i}\big)\Big) = \Theta\Big(\sum_{i=1}^{n}(\log n - \log i)\Big)$$

same factor

$$= \Theta\Big(n\log n - \sum_{i=1}^{n}\log i\Big) = \Theta\big(n\log n - (n\log n - \Theta(n))\big) = \Theta(n)$$

Recurrence:

Method 1: "Recurrence Tree"

- expand for k iterations, get tree of terms
- set k to reach base case
- sum across rows then levels

Ex 1. $T(n) = \begin{cases} 2T(n/2) + n^2 & n>1 \\ 3 & n=1 \end{cases}$

assume n power of 2

$$T(n) = \begin{matrix} & n^2 & \\ T(n/2) & & T(n/2) \end{matrix} = \begin{matrix} & & n^2 & & \\ & (\frac{n}{2})^2 & & (\frac{n}{2})^2 & \\ T(n/4) & T(n/4) & & \cdots & \cdots \end{matrix}$$

$= \cdots =$

$n^2 \longrightarrow n^2$

$(\frac{n}{2})^2 \quad (\frac{n}{2})^2 \longrightarrow 2(\frac{n}{2})^2 = \frac{n^2}{2}$

$(\frac{n}{4})^2 \quad (\frac{n}{4})^2 \quad (\frac{n}{4})^2 \quad (\frac{n}{4})^2 \longrightarrow 4(\frac{n}{4})^2 = \frac{n^2}{4}$

set $k = \log n$

$T(\frac{n}{2^k}) \quad T(\frac{n}{2^k}) \quad \cdots \longrightarrow 2^k T(1) = 2^k \cdot 3$

$$\text{total} = n^2\Big(\sum_{i=0}^{k-1}\frac{1}{2^i}\Big) + 2^k \cdot 3 = n^2\,\Theta(1) + 3n = \Theta(n^2)$$

Ex 2. $T(n) = \begin{cases} 5T(n/4) + \sqrt{n} & n>1 \\ 1 & n=1 \end{cases}$

$\sqrt{n} \longrightarrow \sqrt{n}$

$\sqrt{\frac{n}{4}} \cdots \cdots \longrightarrow 5\sqrt{\frac{n}{4}} = \frac{5}{2}\sqrt{n}$

$\longrightarrow 25\sqrt{\frac{n}{16}} = \frac{25}{4}\sqrt{n}$

$T(\frac{n}{4^k}) \longrightarrow 5^k T(1) = 5^k$

set $k = \log_4 n$

$$\text{total} = \sqrt{n}\sum_{i=0}^{k-1}\Big(\frac{5}{2}\Big)^i + 5^k$$

$$= \Theta\Big(\sqrt{n}\Big(\frac{5}{2}\Big)^{\log_4 n} + 5^{\log_4 n}\Big)$$

$$= \Theta\big(\sqrt{n}\, n^{\log_4 5 - \log_4 2} + n^{\log_4 5}\big)$$

$$= \Theta(n^{\log_4 5})$$

Method 2: The "Master Method" lookup the table

let $T(n) = \begin{cases} a \cdot T(n/b) + f(n) & n > n_0 \\ c & n \le n_0 \end{cases}$

set $d = \log_b a$ and ε be some small positive number (e.g. $\varepsilon = 0.001$)

case 1: $f(n) = O(n^{d-\varepsilon}) \Rightarrow T(n) = \Theta(n^d)$

case 2: $f(n) = \Theta(n^d) \Rightarrow T(n) = \Theta(n^d \log n)$

case 3: $\frac{f(n)}{n^{d+\varepsilon}}$ is increasing $\Rightarrow T(n) = \Theta(f(n))$

Ex 0. MergeSort $a=2$. $b=2$. $f(n)=n$. $d=1$

$\Rightarrow$ case 2. $T(n) = \Theta(n \log n)$

Ex 1 $T(n) = 2T(n/2) + n^2$

$a=2$. $b=2$. $f(n)=n^2$. $d=1$. pick $\varepsilon = 0.001$

$\frac{f(n)}{n^{d+\varepsilon}} = n^{0.999}$ increasing

$\Rightarrow T(n) = \Theta(n^2)$

Ex Bad. $T(n) = 2T(n/2) + \frac{n}{\log n}$

$a=2$. $b=2$ $f(n) = \frac{n}{\log n}$. $d=1$.. ~~case 2~~

case 1: $\frac{n}{\log n} = O(n^{d-\varepsilon}) = O(^{1-\varepsilon})$ no ε satisfy ✗

case 3: $\frac{\frac{n}{\log n}}{n^{1+\varepsilon}} = \frac{1}{\log n \cdot n^{\varepsilon}}$ decreasing ✗

Master Method doesn't work

No case matches

ans $= O(n \log \log n)$

Method 3 "Guess and Check"

1) guess form of solution $T(n) \le \ldots$ 2) induction 3) fill in consts

Ex 1. $T(n) = \begin{cases} 2T(n/2) + n^2 & n>1 \\ 3 & n=1 \end{cases}$ guess: $\Theta(n^2)$. $(O(n^2): T(n) \le cn^2)$

Base case: $T(1) = 3 \le c \Rightarrow c \ge 3$

Induction: assume $T(n/2) \le c(\frac{n}{2})^2$ c must be same. const must be same

$T(n) = 2T(n/2) + n^2 \le (\frac{c}{2}+1)n^2 \le cn^2 \Rightarrow c \ge 2$

$\therefore$ Pick $c=3$. $T(n) \le 3n^2 \in O(n^2)$

$T(n) = 2T(n/2) + n^2 \ge n^2 \in \Omega(n^2)$ $\Rightarrow \Theta(n^2)$

Ex 2. $T(n) = \begin{cases} 3T(\lfloor n/2 \rfloor) + 4T(\lfloor n/4 \rfloor) + 1 & n>4 \\ 1 & \text{else} \end{cases}$

guess: $T(n) \le cn^2$ (or cn^b to further get b)

Base case: $n = 1,2,3,4$ $T(n) = 1 \le c \Rightarrow c \ge 1$

Induction: assume $T(\lfloor n/4 \rfloor) \le c(\lfloor n/4 \rfloor)^2$ $T(\lfloor n/2 \rfloor) \le c(\lfloor n/2 \rfloor)^2$

$T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + 4T(\lfloor \frac{n}{4} \rfloor) + 1$

$\le 3c(\frac{n}{2})^2 + 4c(\frac{n}{4})^2 + 1 = cn^2 + 1$ ... can't go further

Change Hypothesis: $T(n) \le cn^2 - c'$

Base case: $T(n) = 1 \le c - c'$. $\Rightarrow c \ge 1 + c'$

Induction: $T(n) \le 3c(\frac{n}{2})^2 - 3c' + 4c(\frac{n}{4})^2 - 4c' + 1$

$= cn^2 + 1 - 7c' \le cn^2 - c'$

$\Rightarrow c' \ge \frac{1}{6}$

set $c' = \frac{1}{6}$, $c = \frac{7}{6}$. $T(n) \le \frac{7}{6}n^2 - \frac{1}{6} = O(n^2)$

If were to get $\Theta(n^2)$, need to use $T(n) \ge cn^2 - c'$

to have $T(n) = \Omega(n^2)$