Asymptotic Nontation:

Θ tight bound

O upper bound

Ω lower bound

o loose upper bound

ω loose lower bound

Def: let $f, g : \mathbb{N} \to \mathbb{R}$

1. $f(n) \in O(g(n)) \iff \exists\ c > 0\ \exists\ n_0,\ s.t.\ \forall\ n \geq n_0,\ f(n) \leq cg(n)$
2. $f(n) \in \Omega(g(n)) \iff \exists\ c > 0\ \exists\ n_0,\ s.t.\ \forall\ n \geq n_0,\ f(n) \leq cg(n)$
3. $f(n) \in \Theta(g(n)) \iff f(n) \in \Omega(g(n)),\ f(n) \in O(g(n))$
4. $f(n) \in o(g(n)) \iff \forall\ c > 0\ \exists\ n_0,\ s.t.\ \forall\ n \geq n_0,\ f(n) \leq cg(n)$
5. $f(n) \in \omega(g(n)) \iff \forall\ c > 0\ \exists\ n_0,\ s.t.\ \forall\ n \geq n_0,\ f(n) \geq cg(n)$

e.g.

$f(n) = 2n^2 + 8n - 10 \leq 2n^2 + 8n \leq 10n^2, \forall\ n \geq 1$

$f(n) = 2n^2 + 8n - 10 \geq 2n^2 - 10 \geq n^2, \forall\ n \geq \sqrt{10}$

Thus, $2n^2 + 8n - 10 \in \Theta(n^2)$

$f(n) = 3n + 2 \leq 4n, \forall\ n \geq 2, 4n \leq cn^2, \forall\ n \geq \frac{4}{c}$

Thus, $3n + 2 \in o(n^2),\ n_0 = max\{2, \frac{4}{c}\}$

Facts (connection with limits)

let $h = \lim_{x \to \infty} \frac{f(n)}{g(n)}$, then:

1. $f(n) \in o(g(n)) \iff h = 0$
2. $f(n) \in \omega(g(n)) \iff h = \infty$
3. $f(n) \in O(g(n)) \iff h < \infty$
4. $f(n) \in \Omega(g(n)) \iff h > 0$

e.g. Taking limits using L'Hopital's Rule

$$2^n \in \omega(n^2) \leftarrow \lim_{n \to \infty} \frac{2^n}{n^2} = \lim_{n \to \infty} \frac{2^n \ln 2}{2n} = \lim_{n \to \infty} \frac{2^n \ln 2 \ln 2}{2} = \infty$$

Ex. $f(n)$ vs $g(n)$

1.01^n vs n^{100} (exponential vs polynomial): $1.01^n = \omega(n^{100})$

$(\log n)^{100}$ vs $n^{0.01}$ (polylog vs polynomial): $(\log n)^{100} = o(n^{0.01})$

$n^3 + \frac{2^n}{10}$ vs $n^2 + \frac{3^n}{10}$ (drop lower order terms and constant factors): $n^3 + \frac{2^n}{10} = o(n^2 + \frac{3^n}{10})$

$n!$ vs $(n+1)!$ (factorial growth rate changes when $n \to n+1$): $n! = o((n+1)!)$

$2^{\log_5 n}$ vs $2^{\log_6 n}$ ($a^{\log_b n} = a^{\frac{\log_a n}{\log_a b}} = n^{\log_b a}$)

n vs $n^{1+\sin n}$ NONE! since $n^{1+\sin n}$ oscillates

Commonly seen complexity: Good $\to$ Bad

$...n...n\log n...n^2...n^3...1.01^n...2^n...100^n...n!...$

Slower: $...100^n...n!...n^n...2^{n^2}...2^{n!}...2^{2^{2^{\cdots}}}...$

Faster: $1...\log\log\log n...\log\log n...\log n...(\log n)^{100}...n^{0.01}...\sqrt{n}...n...$

$\log n! \in \Omega(n \log n)$

$2^{\sqrt{\log n}} \in o(n^{0.01})$ since $\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{n^{0.01}} = \lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{2^{0.01\log n}} = \lim_{n\to\infty} 2^{\sqrt{\log n} - 0.01\log n} = 2^{-\infty} = 0$

Remark: asymptotic notations can be used in equations and interpreted from left to right.

e.g. $n^2 = O(n^3) = O(n^4)$, $2n^3 + 3n + 4 = 2n^3 + o(n^2) = O(n^3)$

Summations:

$$\sum_{i=1}^{n} i^d = \Theta(n^{d+1}) \ \forall\ d \geq 0$$

$$\sum_{i=1}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \Theta(c^n) \ \forall\ c \geq 1$$

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + \Theta(1) \text{ (take integral)}$$

$$\sum_{i=1}^{n} \log i = \log n! = n\log n - \Theta(n) \text{ (take integral)}$$