

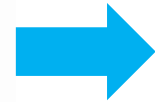
# Assignment #4

Lower limb kinematics and dynamics

# Kinematics and dynamics, lower limb

The assignment is accompanied by a python code. Experimental data (gait, run, jump) is available to be analyzed with the model developed here

## Assignment goals



Modeling the lower limb (2D model)



Anatomical position of the joint center in the reference frame (forward kinematics)



Joint angles (inverse kinematics)



Joint torques (inverse dynamics)



# Biomechanical model





We suppose know the orientation of the pelvis

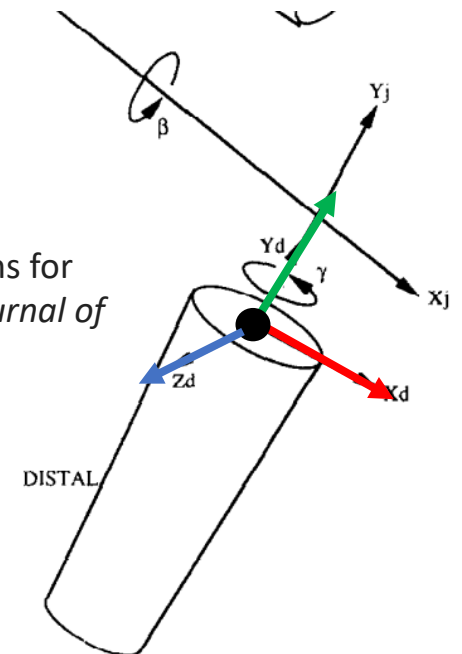
1. Define the **degrees of freedom of the system**
2. Locate the joint centres **and the coordinates system**
3. Express the rotation matrices **from one coordinate system to one other**

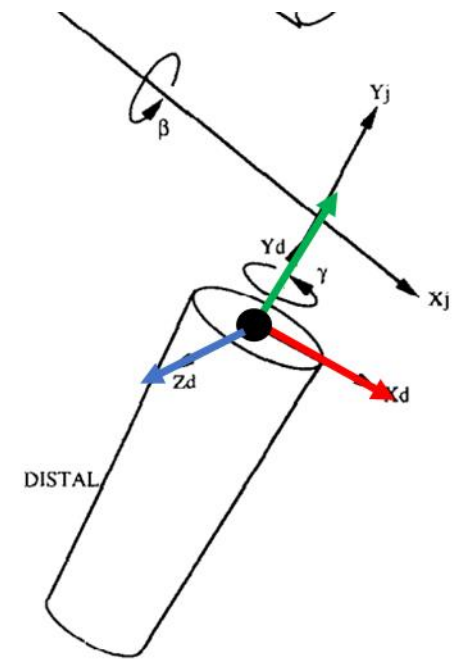
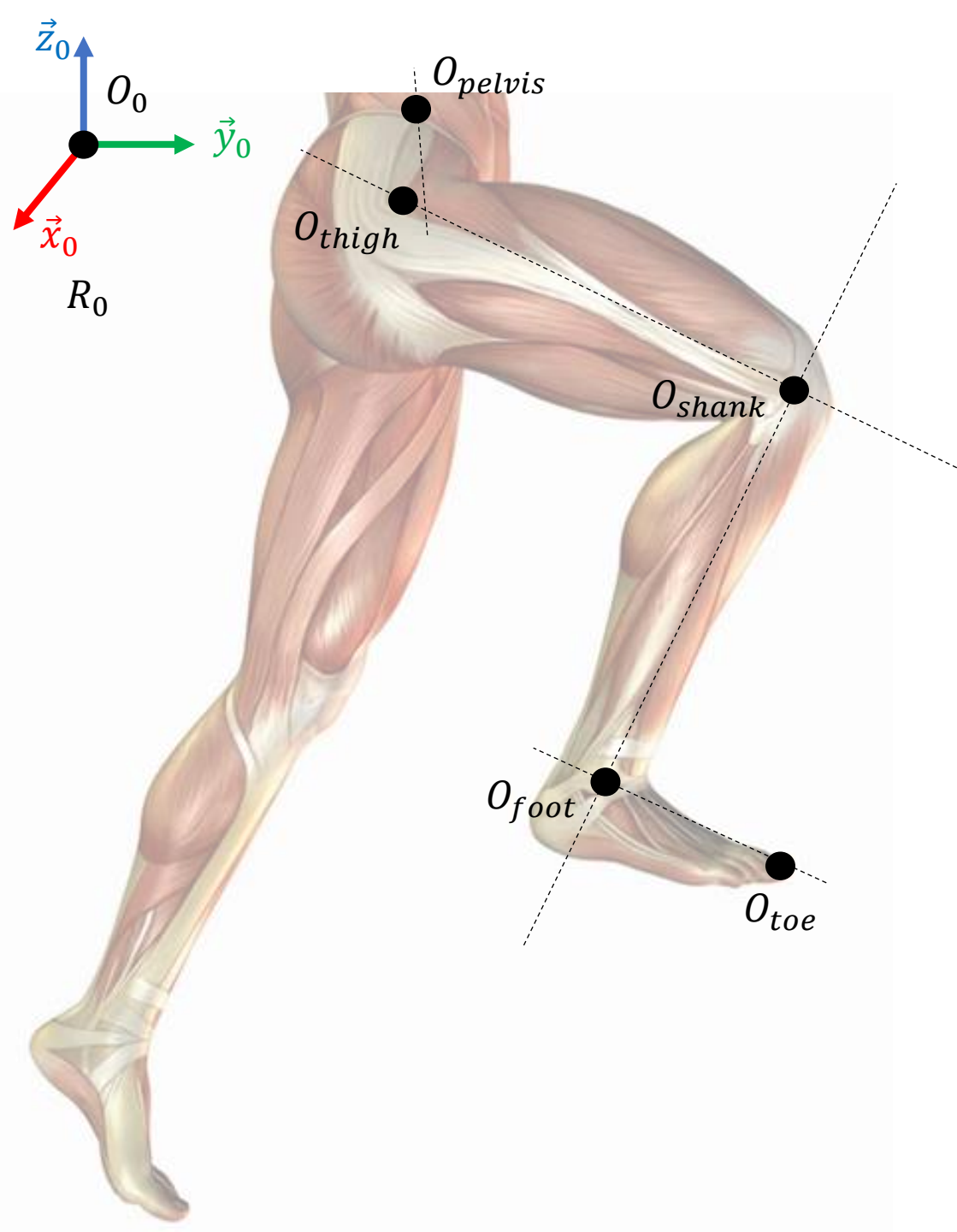
**ISB recommendations (planar model)**

<https://isbweb.org/activities/standards>

Wu, G., & Cavanagh, P. R. (1995). ISB recommendations for standardization in the reporting of kinematic data. *Journal of biomechanics*, 28(10), 1257-1262.

Note the same as the 3D convention !

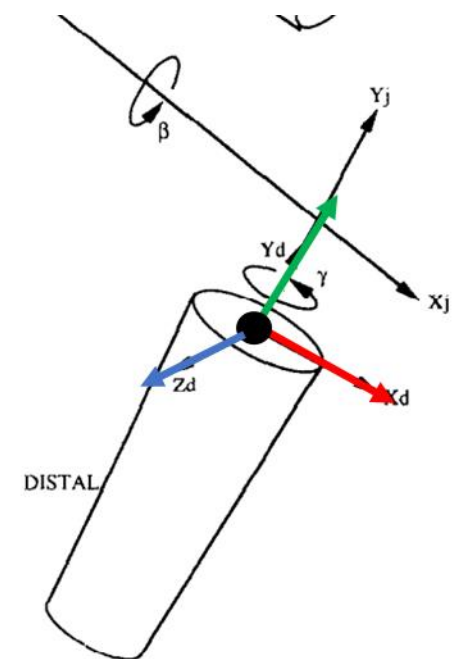
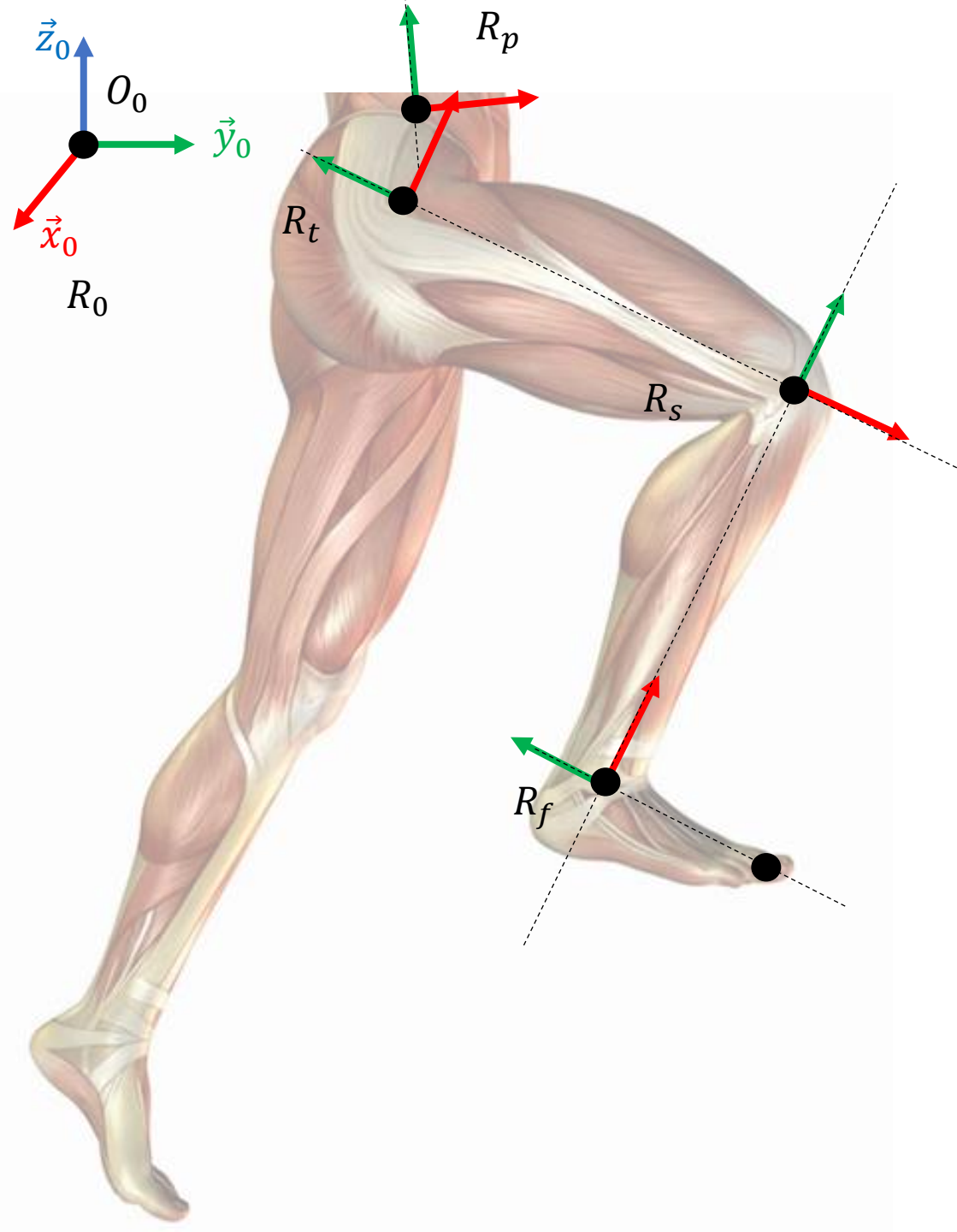


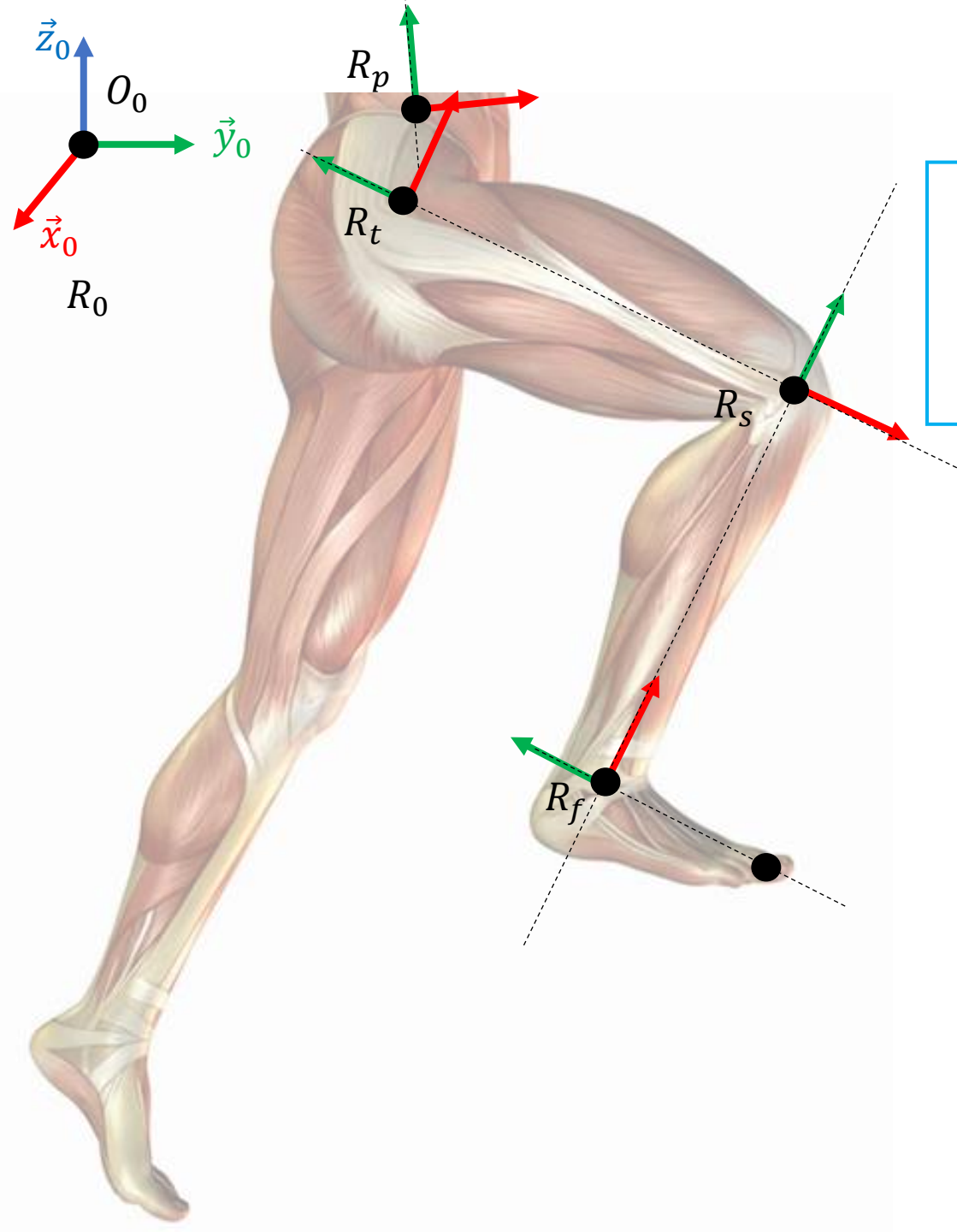


Joint centers



Longitudinal axes =  $y$  direction  
for all segments





${}^0R_p$  Rotation matrix from pelvis to world (assumed to be known)

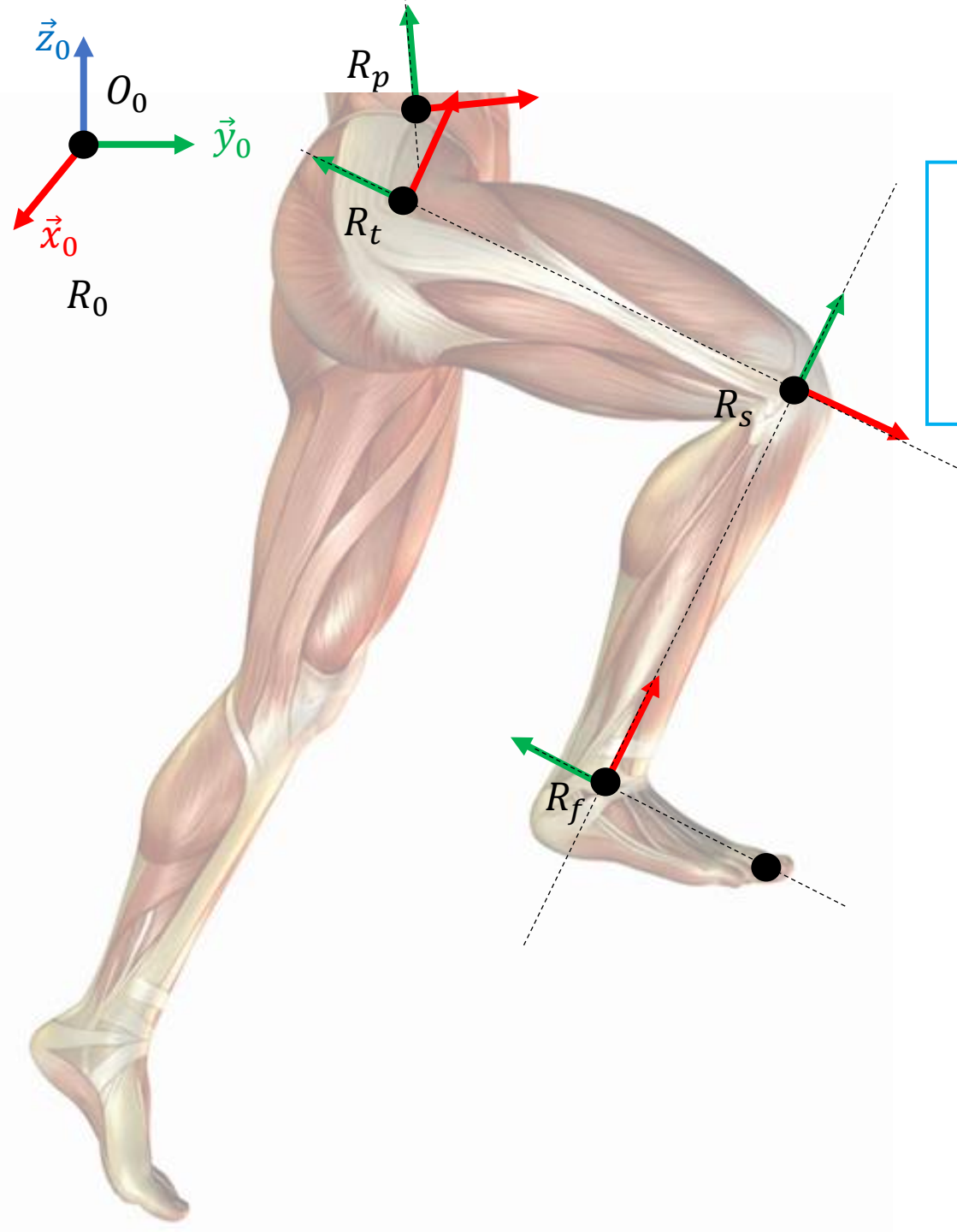
${}^pR_t$  Rotation matrix from thigh to pelvis

${}^tR_s$  Rotation matrix from shank to thigh

${}^sR_f$  Rotation matrix from foot to shank

How to compute them ?



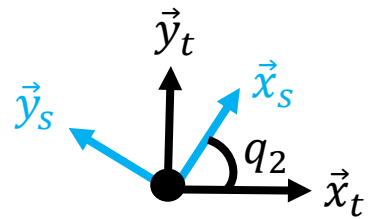
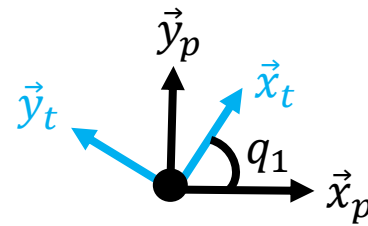


${}^0R_p$  Rotation matrix from pelvis to world (assumed to be known)

${}^pR_t$  Rotation matrix from thigh to pelvis

${}^tR_s$  Rotation matrix from shank to thigh

${}^sR_f$  Rotation matrix from foot to shank

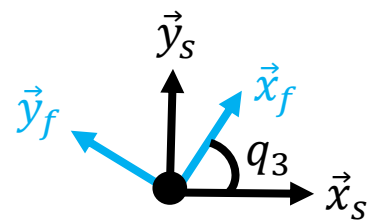
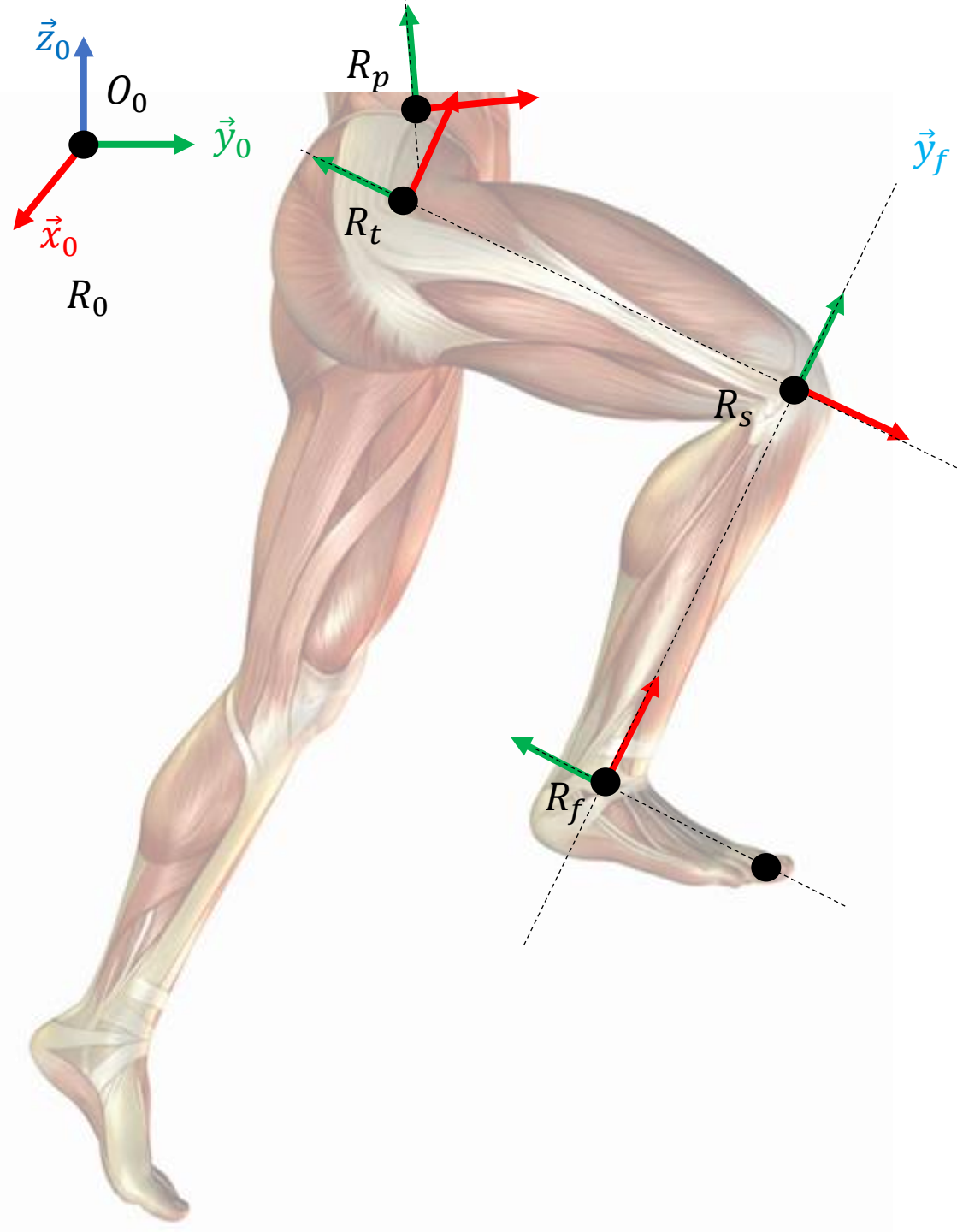


$${}^pR_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^tR_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

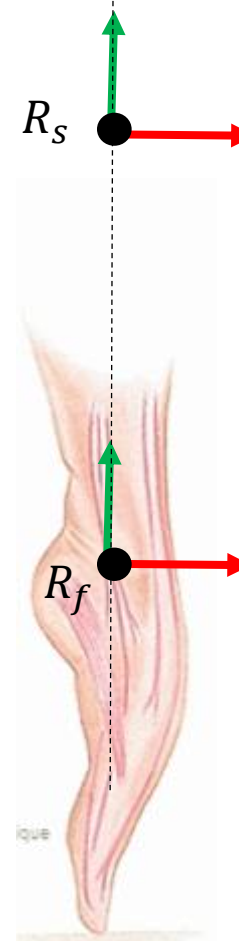
But for the foot...



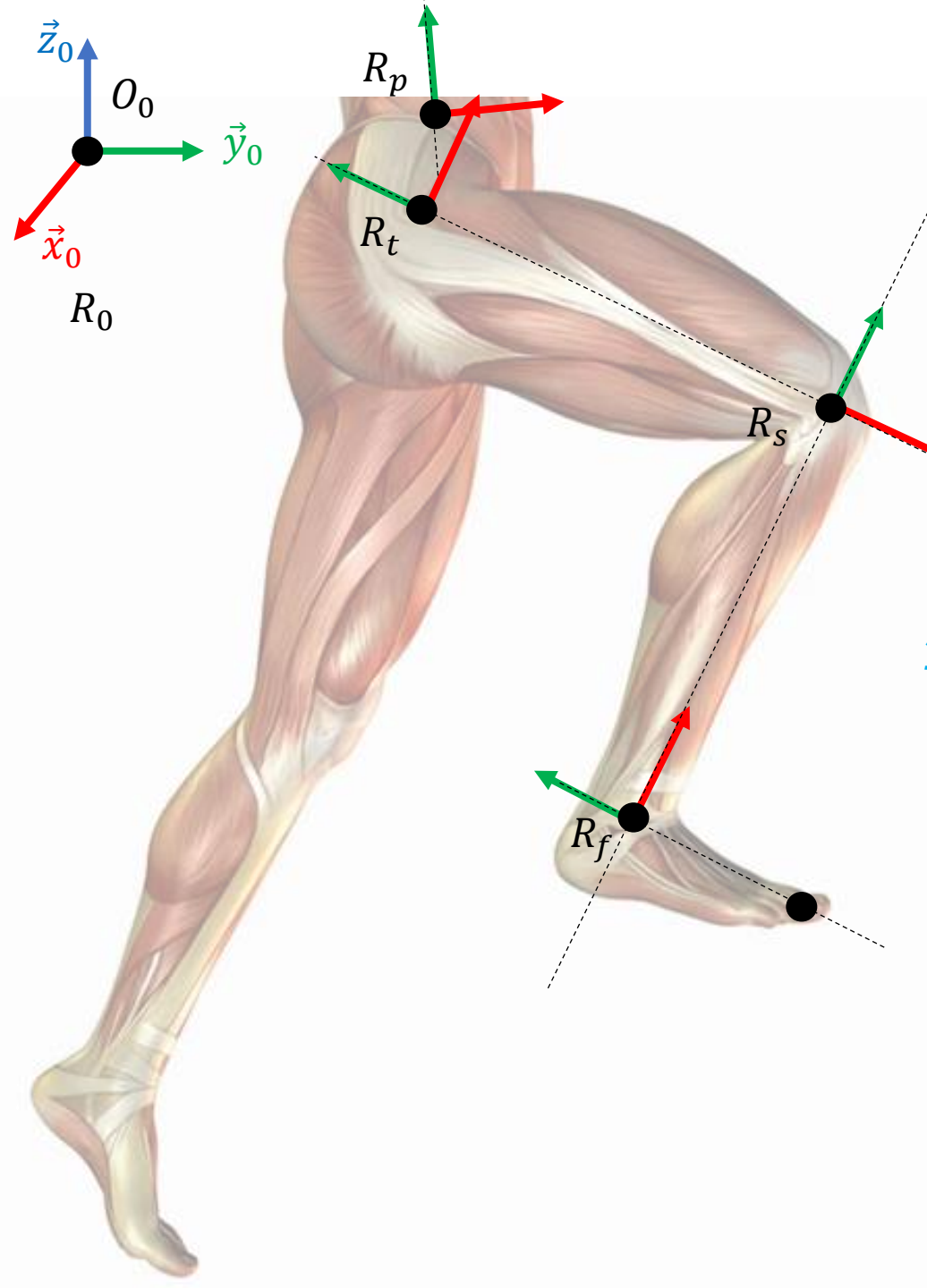


$${}^sR_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

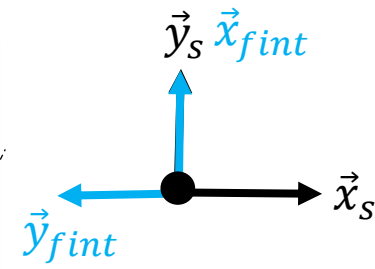
If  $q_3$  is 0 ...



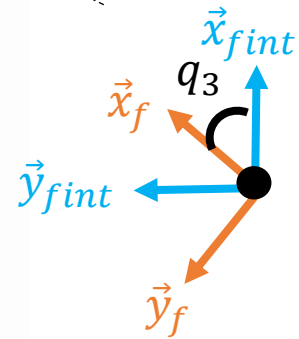
The anatomical (reference) position is not ok



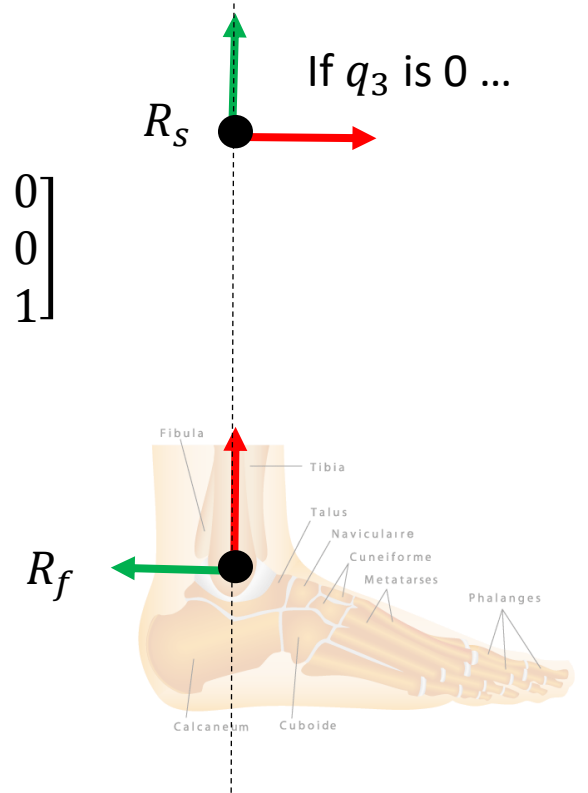
Creating an intermediate coordinate system  $R_{fint}$



$${}^sR_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

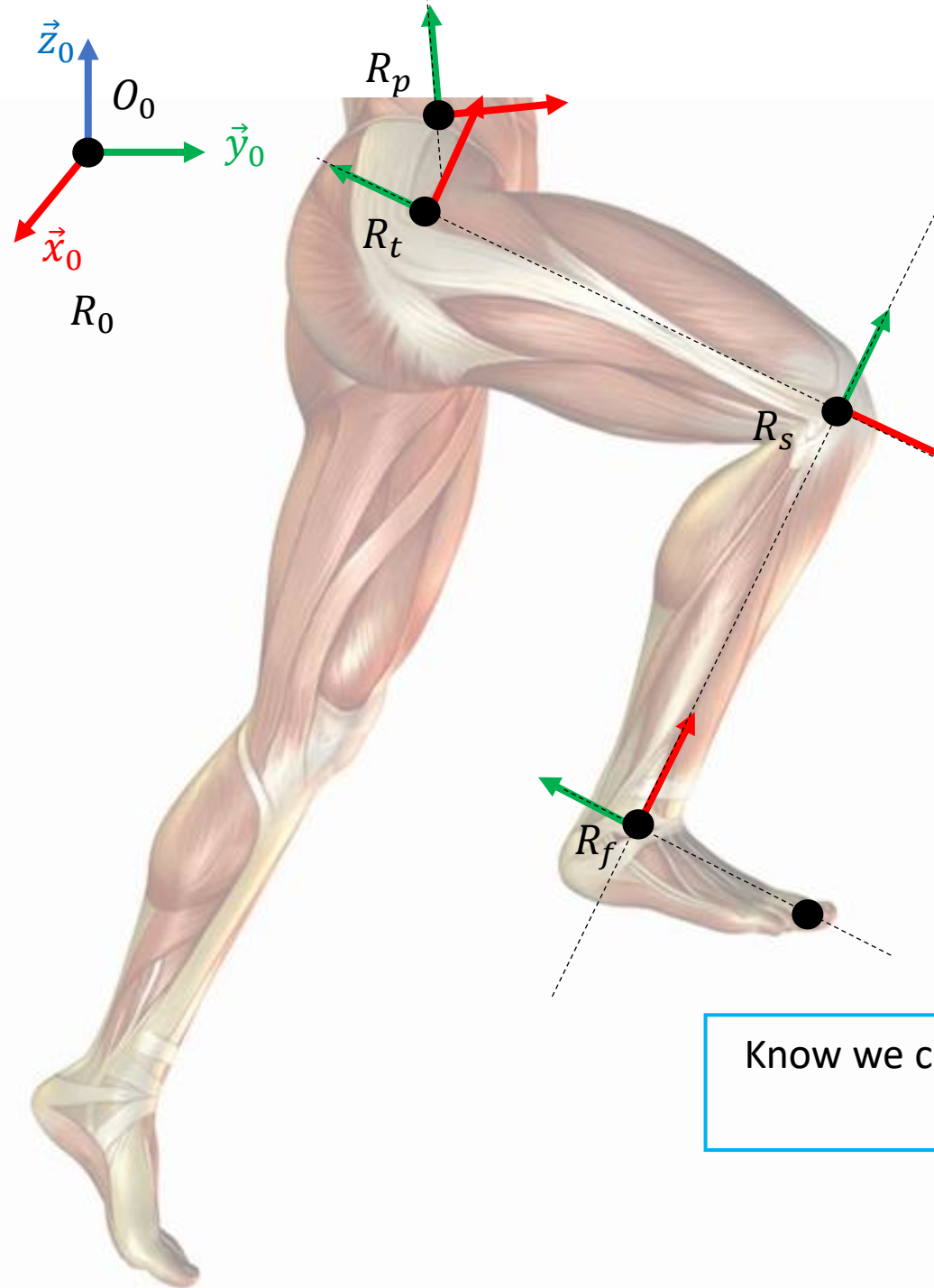


$${}^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



If  $q_3$  is 0 ...

The anatomical position is ok



$${}^pR_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^tR_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^sR_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

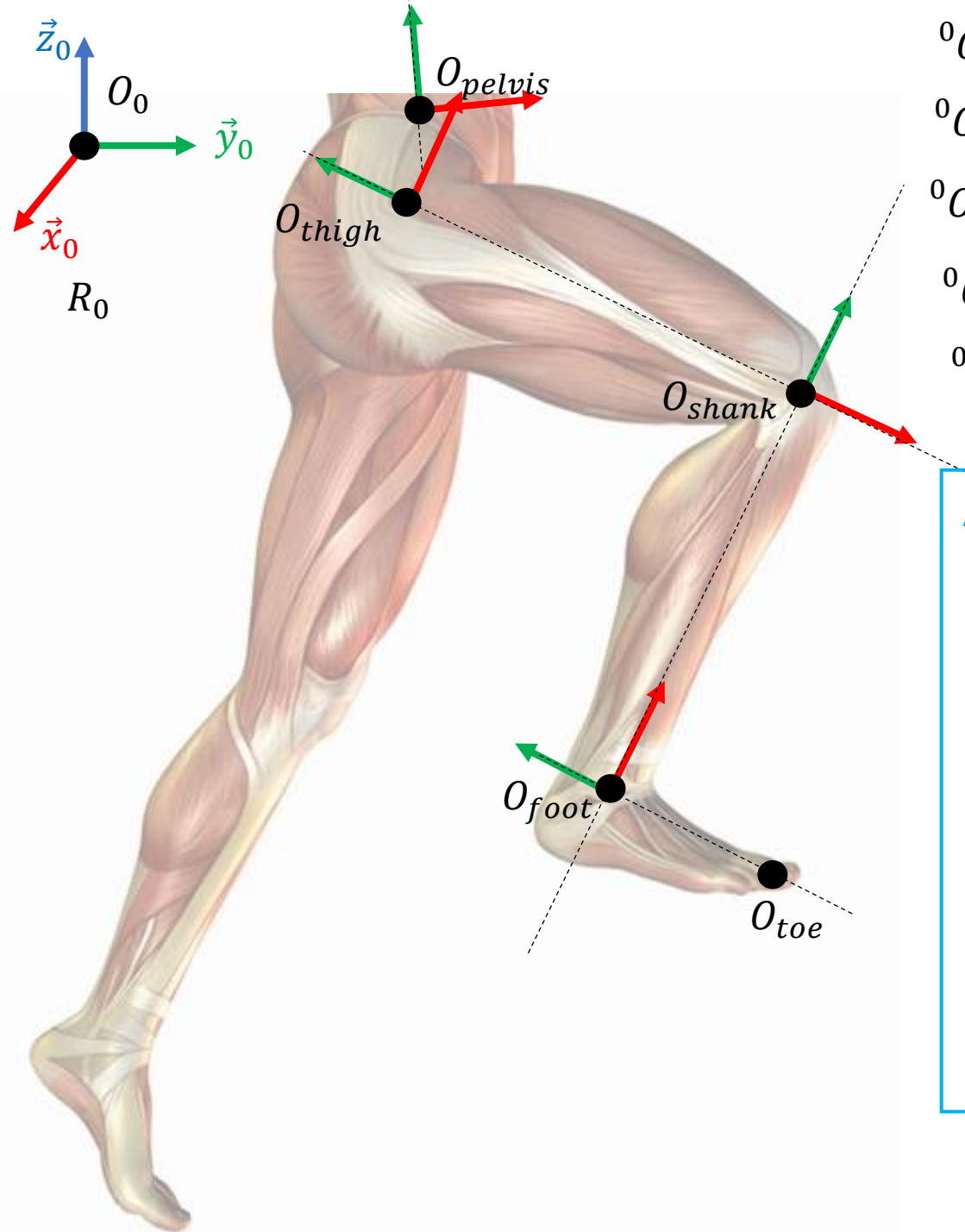
$${}^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Well...ok

Know we can use that to compute forward and inverse kinematics

# Forward Kinematics





${}^0O_{pelvis}$  Origin of the pelvis in  $R_0$  (assumed to be known)

${}^0O_{thigh}$  Origin of the thigh in  $R_0$  (expressed from  ${}^0R_p$ )

${}^0O_{shank}$  Origin of the shank in  $R_0$  (expressed from  ${}^0R_p, q_1$ )

${}^0O_{foot}$  Origin of the foot in  $R_0$  (expressed from  ${}^0R_p, q_1, q_2$ )

${}^0O_{toe}$  Toe position in  $R_0$  (expressed from  ${}^0R_p, q_1, q_2, q_3$ )

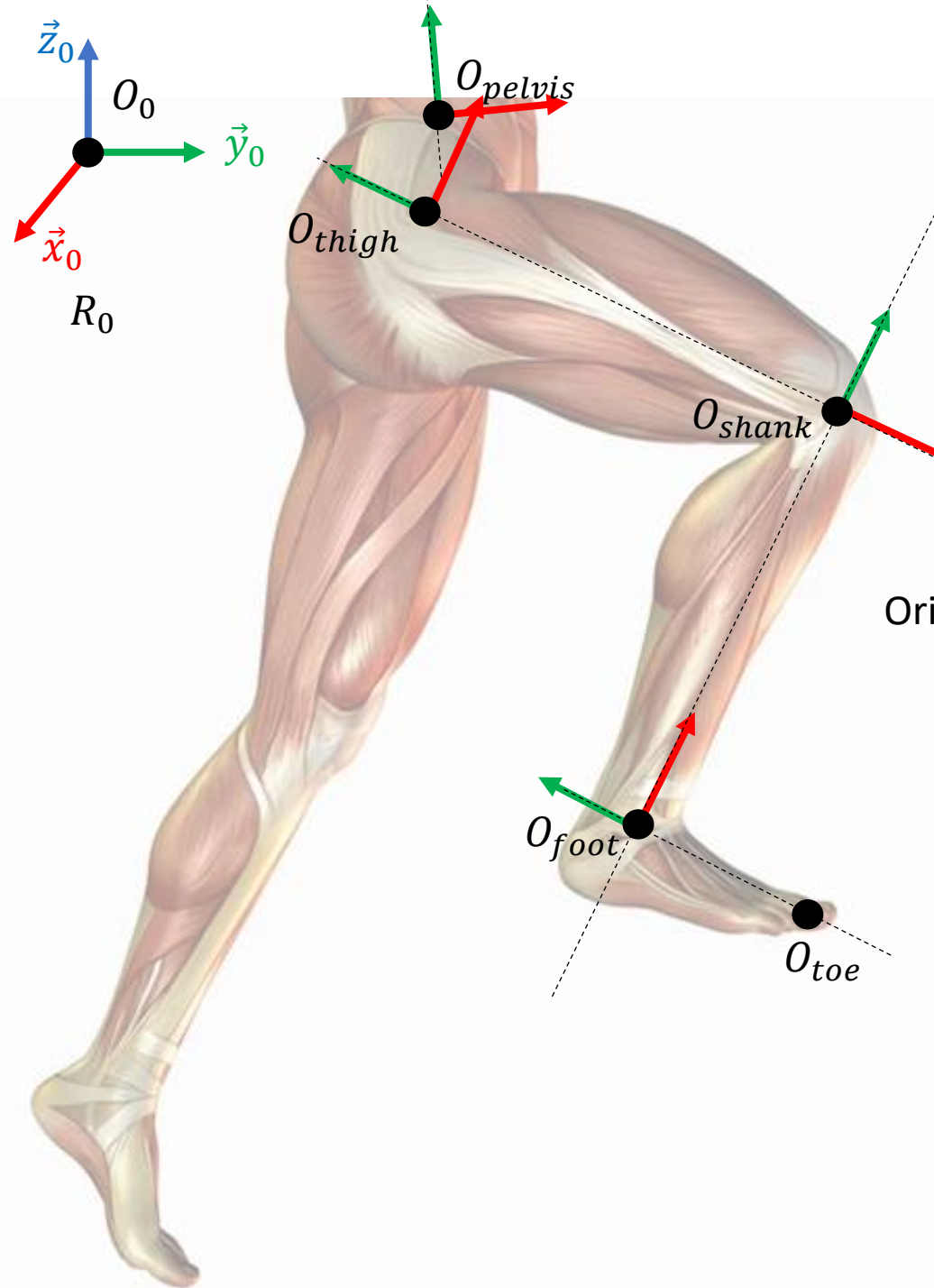
### Assumptions

$${}^0O_{pelvis} = \begin{bmatrix} x_{op} \\ y_{op} \\ 0 \end{bmatrix} \quad \text{Assumed to be known}$$

$${}^0R_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Assumed to be known (for sake of simplicity, we assumed that this is the identity matrix in the following calculus)}$$

We also assume to be known the position of the different joint centers on the segments





${}^0O_{thigh}$  Origin of the thigh in  $R_0$  (expressed from  ${}^0R_p$ )

If we have  ${}^pO_{thigh} = \begin{bmatrix} x_{ot} \\ y_{ot} \\ 0 \end{bmatrix}$  Origin of the thigh in  $R_p$

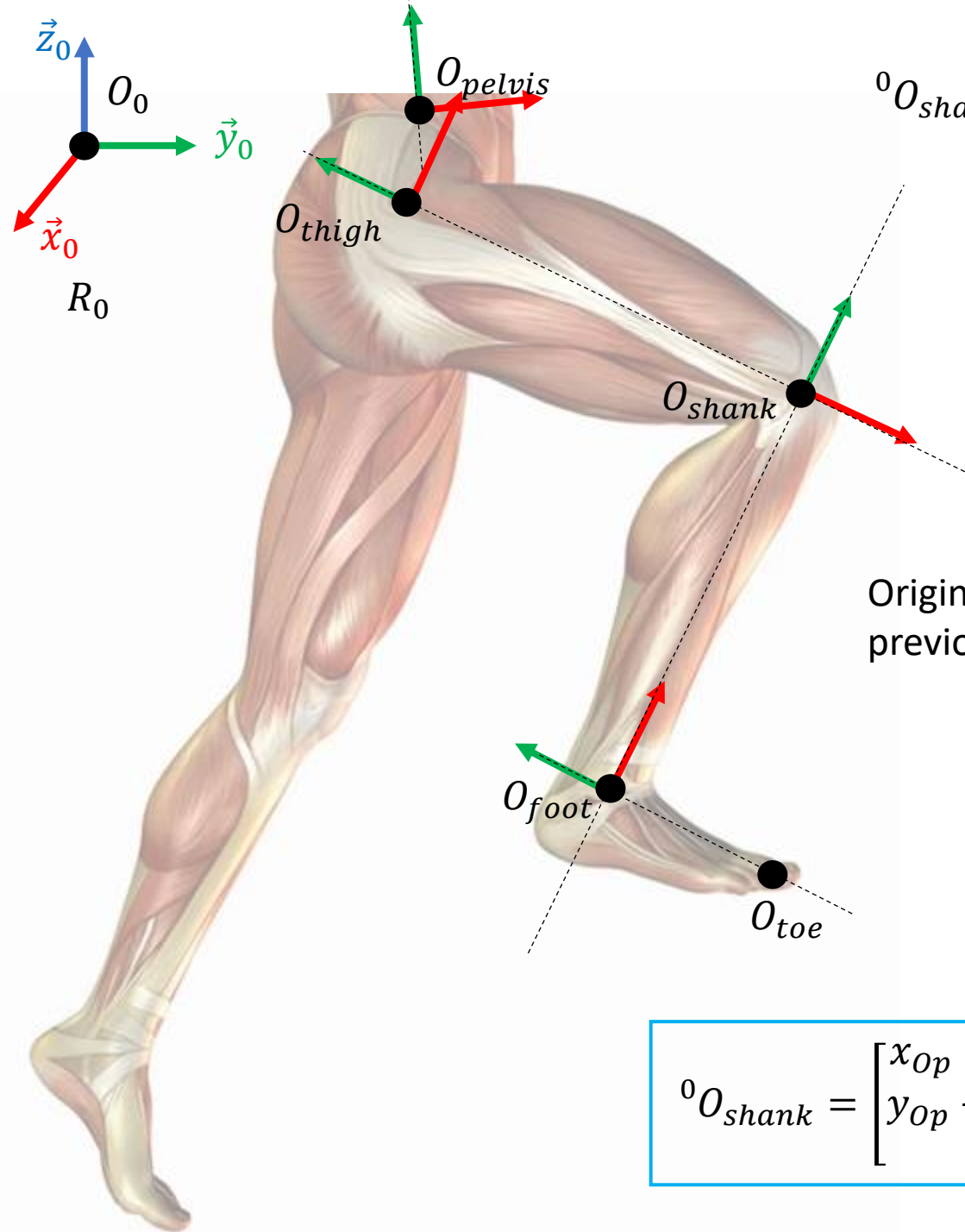
then

$${}^0O_{thigh} = {}^0O_{pelvis} + {}^0R_p {}^pO_{thigh}$$

Origin of the pelvis in  $R_0$

Projecting the origin of the thigh in  $R_p$  in the basis  $B_0$

$${}^0O_{thigh} = \begin{bmatrix} x_{op} \\ y_{op} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{ot} \\ y_{ot} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{op} + x_{ot} \\ y_{op} + y_{ot} \\ 0 \end{bmatrix}$$



${}^0O_{shank}$  Origin of the shank in  $R_0$  (expressed from  ${}^0R_p, q_1$ )

Let's consider  ${}^tO_{shank} = \begin{bmatrix} 0 \\ y_{Os} \\ 0 \end{bmatrix}$  Origin of the shank in  $R_t$

Alors  ${}^0O_{shank} = {}^0O_{thigh} + {}^0R_t {}^tO_{shank}$

Origin of the thigh in  $R_0$  (see previous slide)

Projecting the origin of the shank in  $R_t$  in the basis  $B_0$

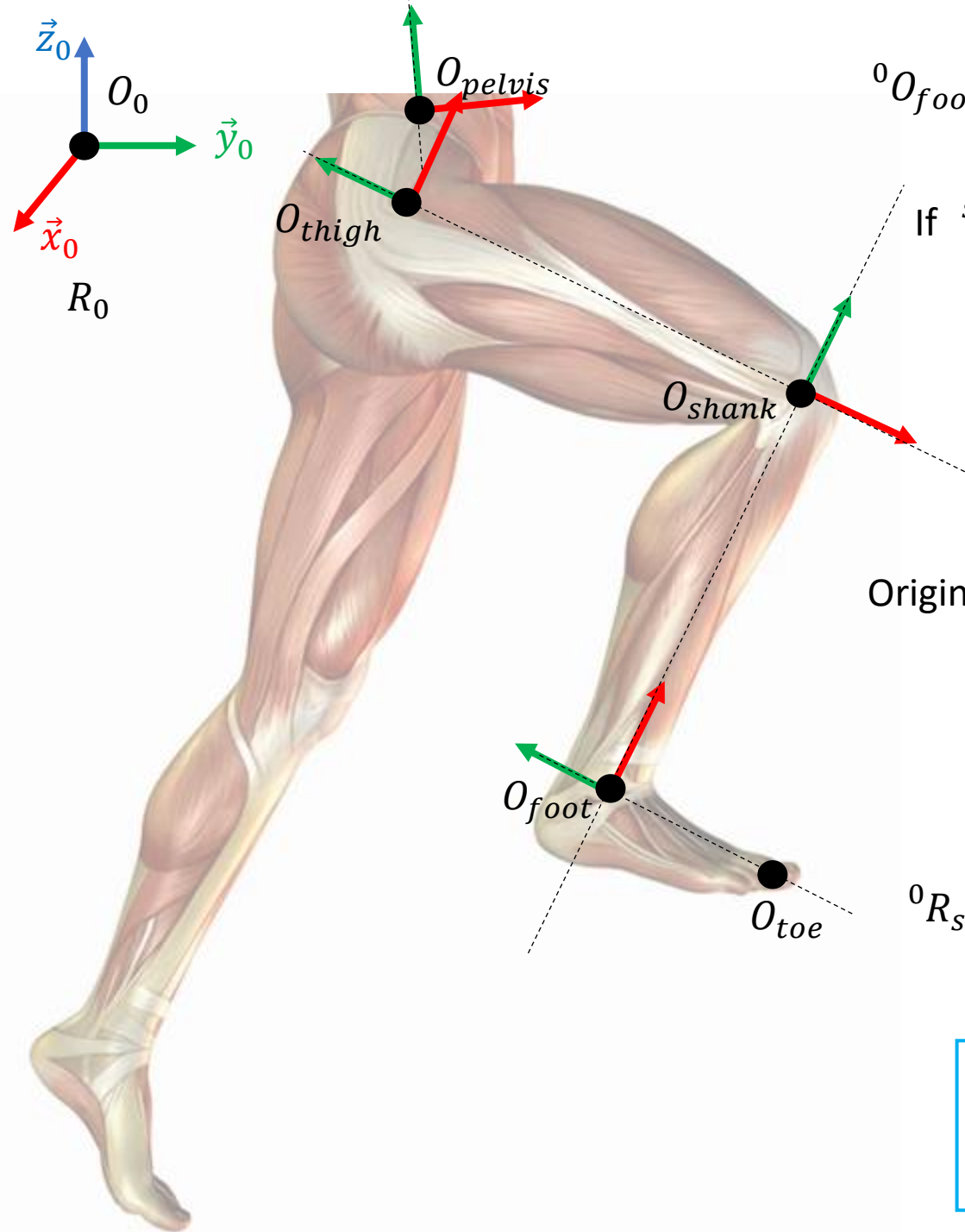
$${}^0O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} \\ y_{Op} + y_{Ot} \\ 0 \end{bmatrix} + {}^0R_t \begin{bmatrix} 0 \\ y_{Os} \\ 0 \end{bmatrix}$$

$${}^0R_t = {}^0R_p {}^pR_t$$

$${}^0O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1 y_{Os} \\ y_{Op} + y_{Ot} + \cos q_1 y_{Os} \\ 0 \end{bmatrix}$$

$${}^0R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





${}^0O_{foot}$  Origin of the foot in  $R_0$  (expressed from  ${}^0R_p, q_1, q_2$ )

$$\text{If } {}^sO_{foot} = \begin{bmatrix} 0 \\ y_{of} \\ 0 \end{bmatrix}$$

Origin of the foot in  $R_s$

Then 
$${}^0O_{foot} = {}^0O_{shank} + {}^0R_s {}^sO_{foot}$$

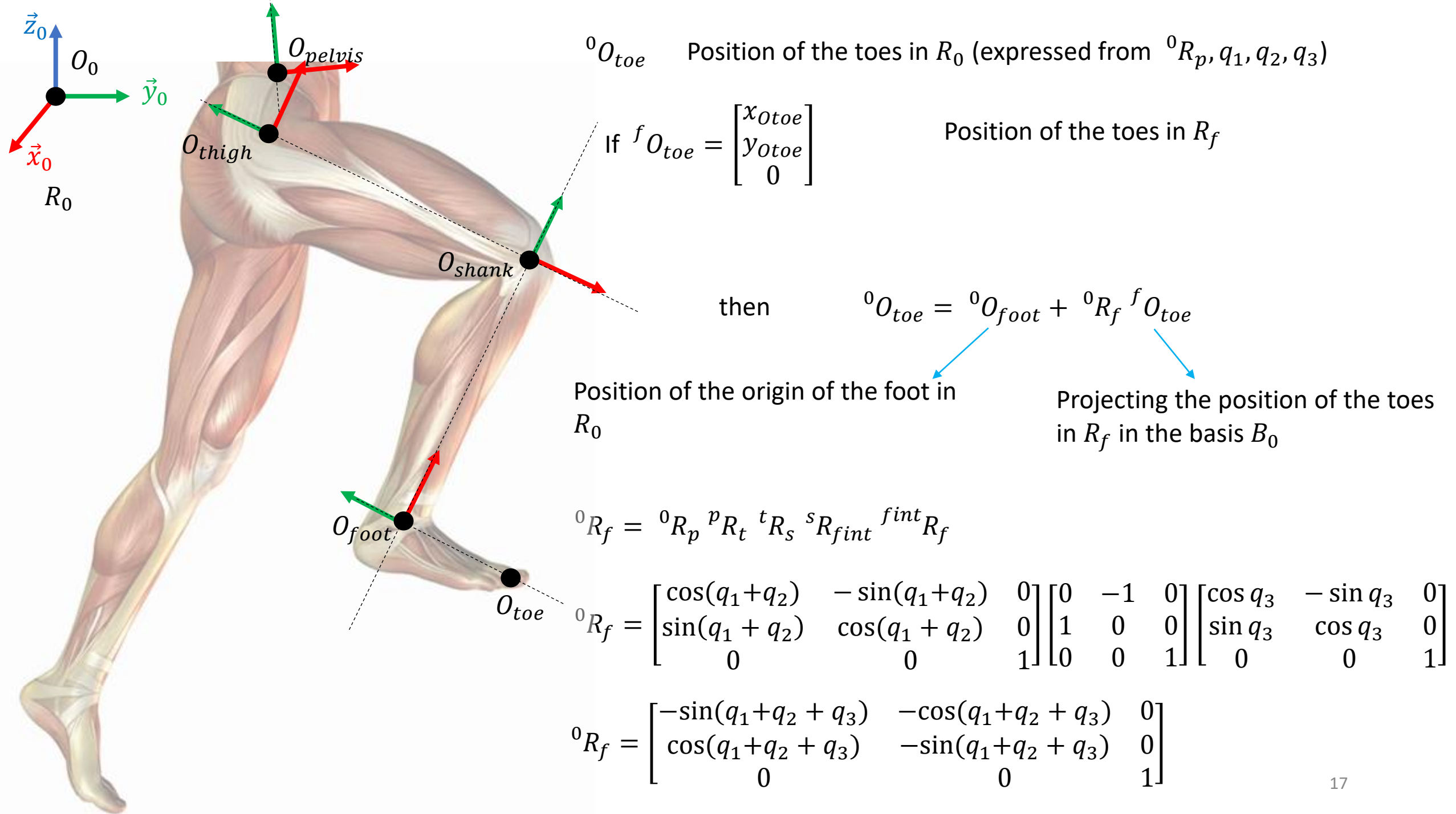
Origin of the shank in  $R_0$

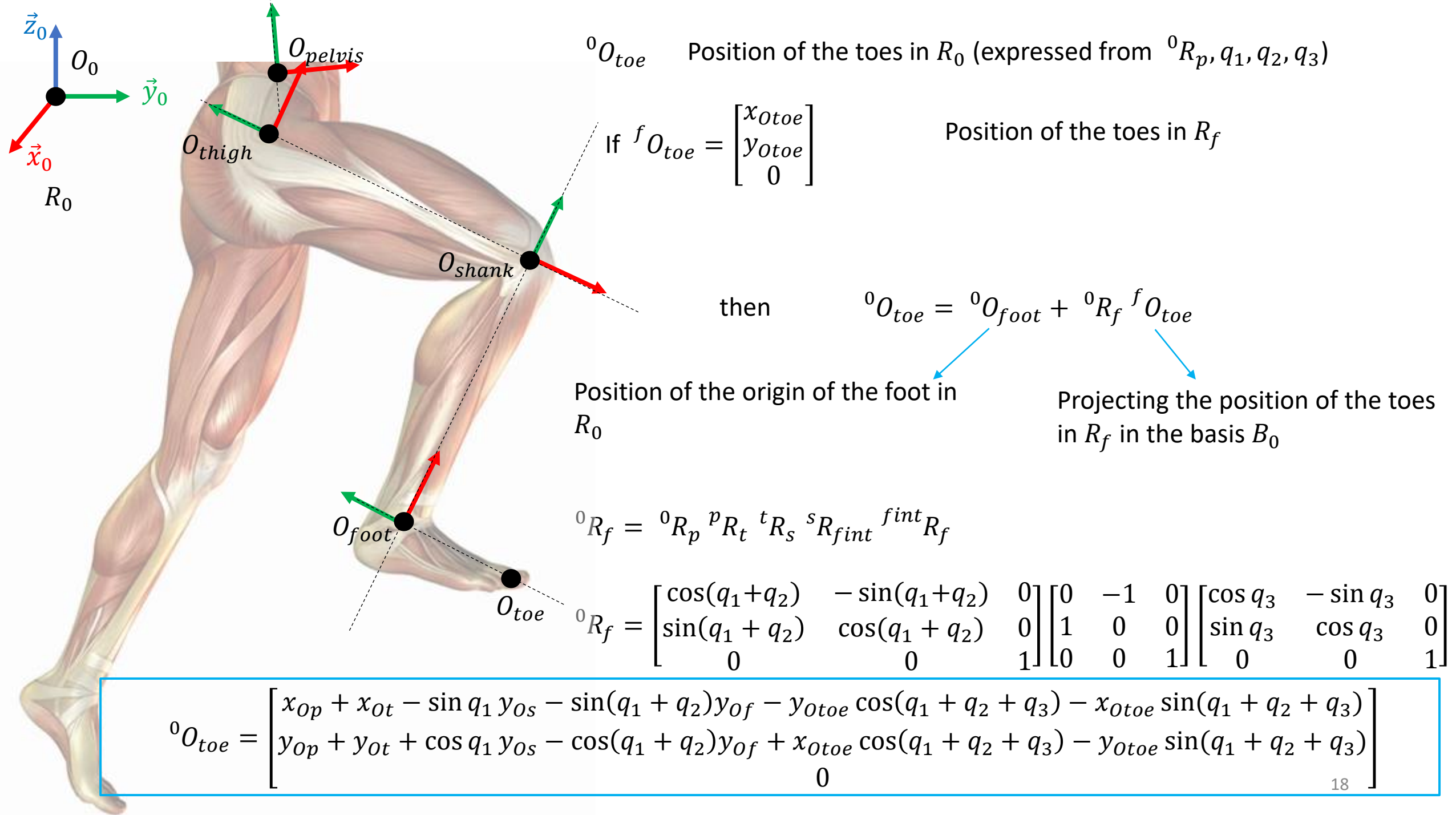
Projecting the position of the origin of the foot in  $R_s$  in the  $R_0$  basis

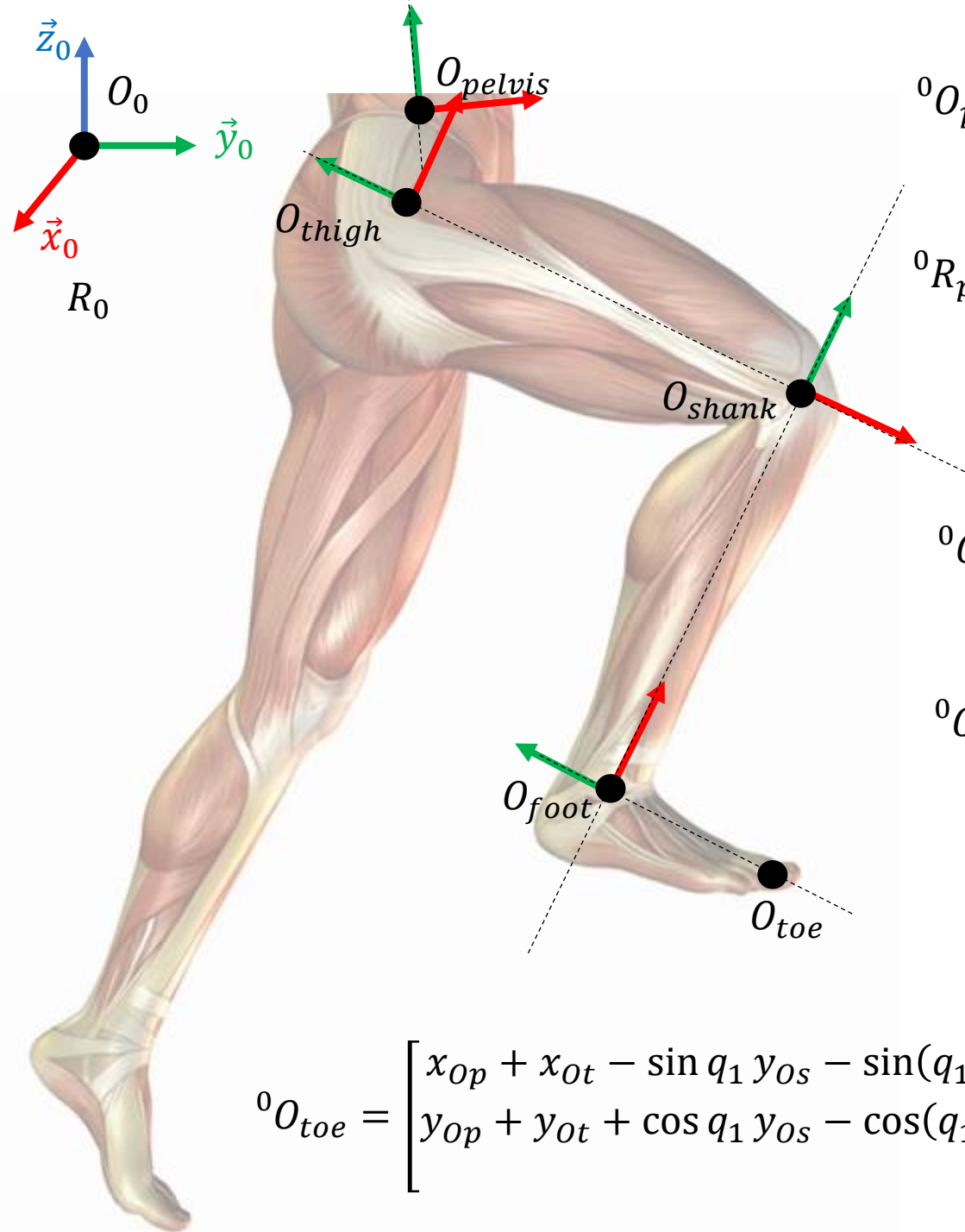
$${}^0R_s = {}^0R_p {}^pR_t {}^tR_s$$

$${}^0R_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^0O_{foot} = \begin{bmatrix} x_{op} + x_{ot} - \sin q_1 y_{os} - \sin(q_1 + q_2) y_{of} \\ y_{op} + y_{ot} + \cos q_1 y_{os} - \cos(q_1 + q_2) y_{of} \\ 0 \end{bmatrix}$$







$${}^0O_{pelvis} = \begin{bmatrix} x_{op} \\ y_{op} \\ 0 \end{bmatrix}$$

$${}^0R_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

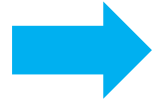
$${}^0O_{thigh} = \begin{bmatrix} x_{op} + x_{ot} \\ y_{op} + y_{ot} \\ 0 \end{bmatrix}$$

$${}^0O_{shank} = \begin{bmatrix} x_{op} + x_{ot} - \sin q_1 y_{os} \\ y_{op} + y_{ot} + \cos q_1 y_{os} \\ 0 \end{bmatrix}$$

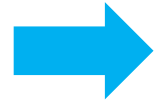
$${}^0O_{foot} = \begin{bmatrix} x_{op} + x_{ot} - \sin q_1 y_{os} - \sin(q_1 + q_2)y_{of} \\ y_{op} + y_{ot} + \cos q_1 y_{os} - \cos(q_1 + q_2)y_{of} \\ 0 \end{bmatrix}$$

$${}^0O_{toe} = \begin{bmatrix} x_{op} + x_{ot} - \sin q_1 y_{os} - \sin(q_1 + q_2)y_{of} - y_{otoe} \cos(q_1 + q_2 + q_3) - x_{otoe} \sin(q_1 + q_2 + q_3) \\ y_{op} + y_{ot} + \cos q_1 y_{os} - \cos(q_1 + q_2)y_{of} + x_{otoe} \cos(q_1 + q_2 + q_3) - y_{otoe} \sin(q_1 + q_2 + q_3) \\ 0 \end{bmatrix}$$

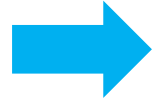
# Next



Implementing the model and kinematics into python



Load and manipulate the data into pytho,



Developing an inverse kinematics method to get hip ( $q_1$ ), knee ( $q_2$ ) et ankle ( $q_3$ ) flexion angles against time



See Lower\_limb\_IK.ipnyb

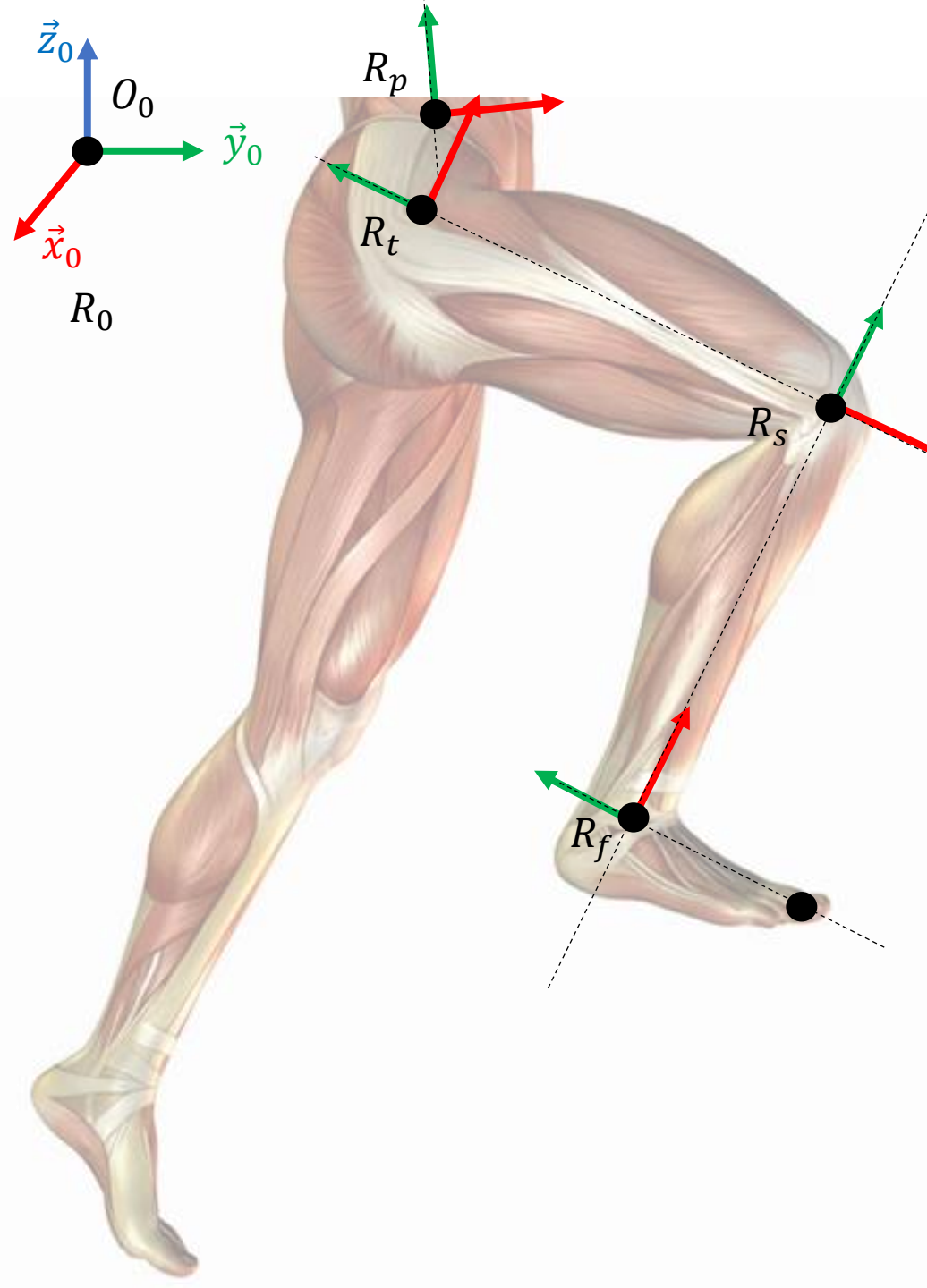
# Inverse kinematics



Get  $q_1, q_2, q_3$  from the experimental data







$${}^pR_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^tR_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^sR_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

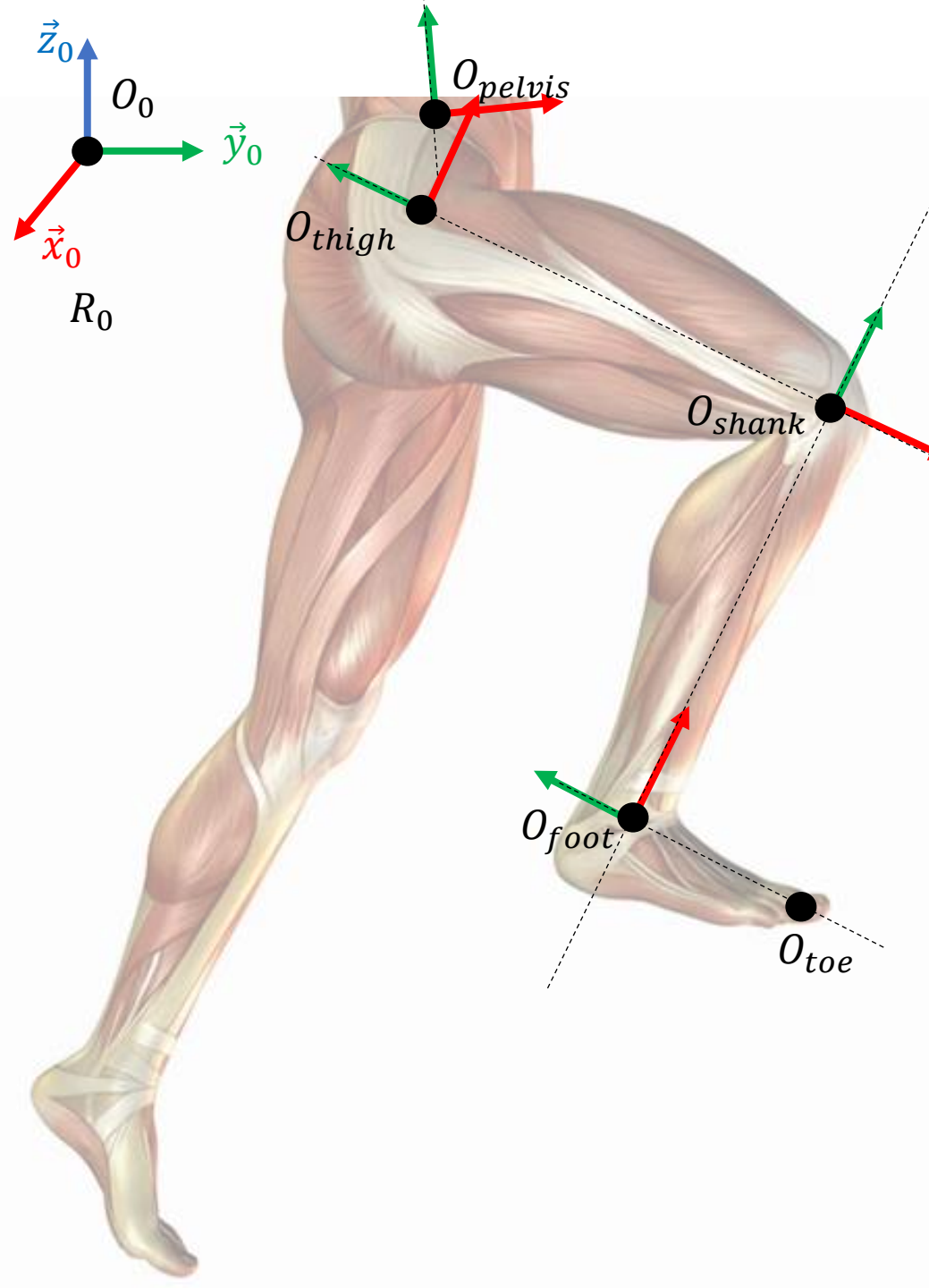
$${}^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^0R_p = \begin{bmatrix} {}^0\vec{x}_p & {}^0\vec{y}_p & {}^0\vec{z}_p \end{bmatrix}$$

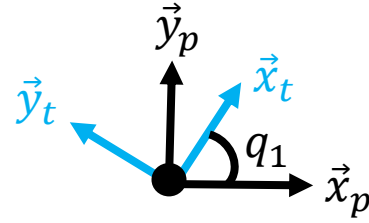


From the pelvis  
markers

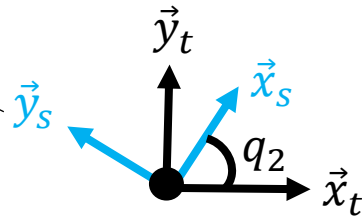




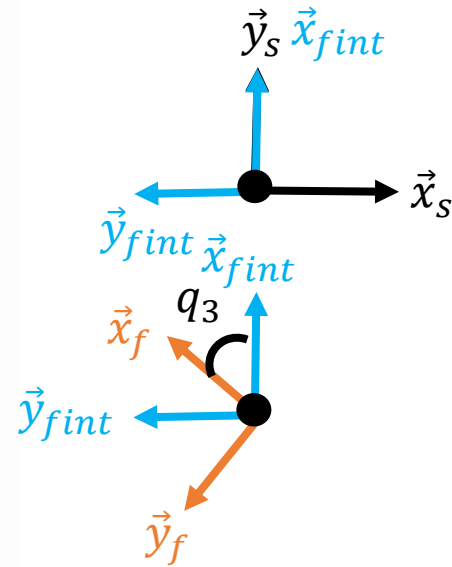
A simple way to do it



$$\vec{y}_t \cdot \vec{y}_p = \cos q_1$$

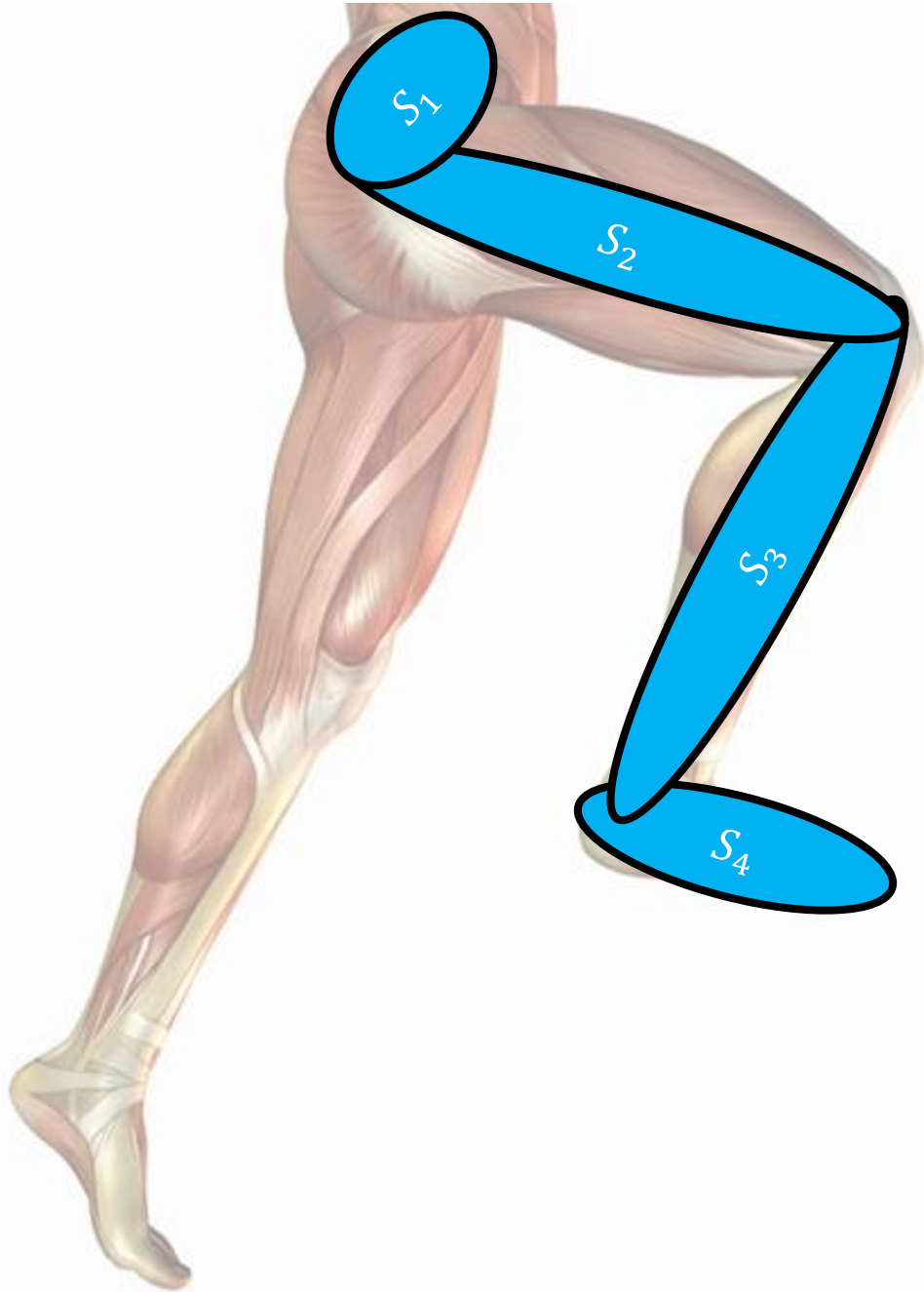


$$\vec{y}_s \cdot \vec{y}_t = \cos q_2$$



$$\vec{y}_f \cdot \vec{y}_s = \cos(q_1 + \frac{\pi}{2})$$

# Inverse dynamics



**Goal:** get the flexion torques at hip, knee and ankle joints

**Methods:** isolate the segments one by one from bottom (foot) to the top (hip)

**Assumption:** we suppose know the kinematics !

# Fundamental principle of dynamics

➡ For a given solid  $S$  **isolated**, the dynamic wrench is equal to the external forces wrench

$$\{D(S/R_0)\} = \{F(ext \rightarrow S)\}$$

$$\{D(S/R_0)\} = \left\{ \begin{array}{c} m\vec{\Gamma}(G \in S/R_0) \\ \vec{\delta}(P \in S/R_0) \end{array} \right\}_A$$

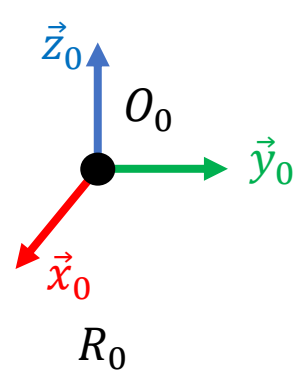
$$\{F(ext \rightarrow S)\} = \left\{ \begin{array}{c} \sum_S \vec{F}(ext \rightarrow S/R_0) \\ \sum_S \vec{M}(P, F \rightarrow S/R_0) \end{array} \right\}_P$$

⚠ Planar problem : 3 equations of motion(2 translations xy, 1 rotation z)

$$\vec{\Omega}(S_i/R_0) = \dot{\theta}_i \vec{z}_0$$

⚠ We should express all the quantities at a single point

⚠ We should express all the quantities in the same basis



# 1. Foot $S_4$

## External forces

**Ground reaction forces** measured by the force platforms at  $P_f$

$$\{\tau(\bar{S}_4 \rightarrow S_4)\} = \begin{Bmatrix} F_x \vec{x}_0 + F_y \vec{y}_0 \\ \vec{0} \end{Bmatrix}_{P_f}$$

With  $\overrightarrow{O_0 P_f} = x_{p_f} \vec{x}_0 + y_{p_f} \vec{y}_0$   
(Known)

## Shank action on the foot $O_f$

$$\{\tau(S_3 \rightarrow S_4)\} = \begin{Bmatrix} R_{fx} \vec{x}_f + R_{fy} \vec{y}_f \\ \underbrace{(\Gamma_f \vec{r}_0)}_{\text{From muscle action ! (unknown)}} \end{Bmatrix}_{O_f}$$

From muscle action !  
(unknown)  
With  $\overrightarrow{O_0 O_f} = x_{o_f} \vec{x}_0 + y_{o_f} \vec{y}_0$

**Gravity in  $G_f$**

$$\{\tau(\bar{S}_4 \rightarrow S_4)_g\} = \begin{Bmatrix} -m_f g \vec{y}_0 \\ \vec{0} \end{Bmatrix}_{G_f}$$

**! The motion is supposed to be known**

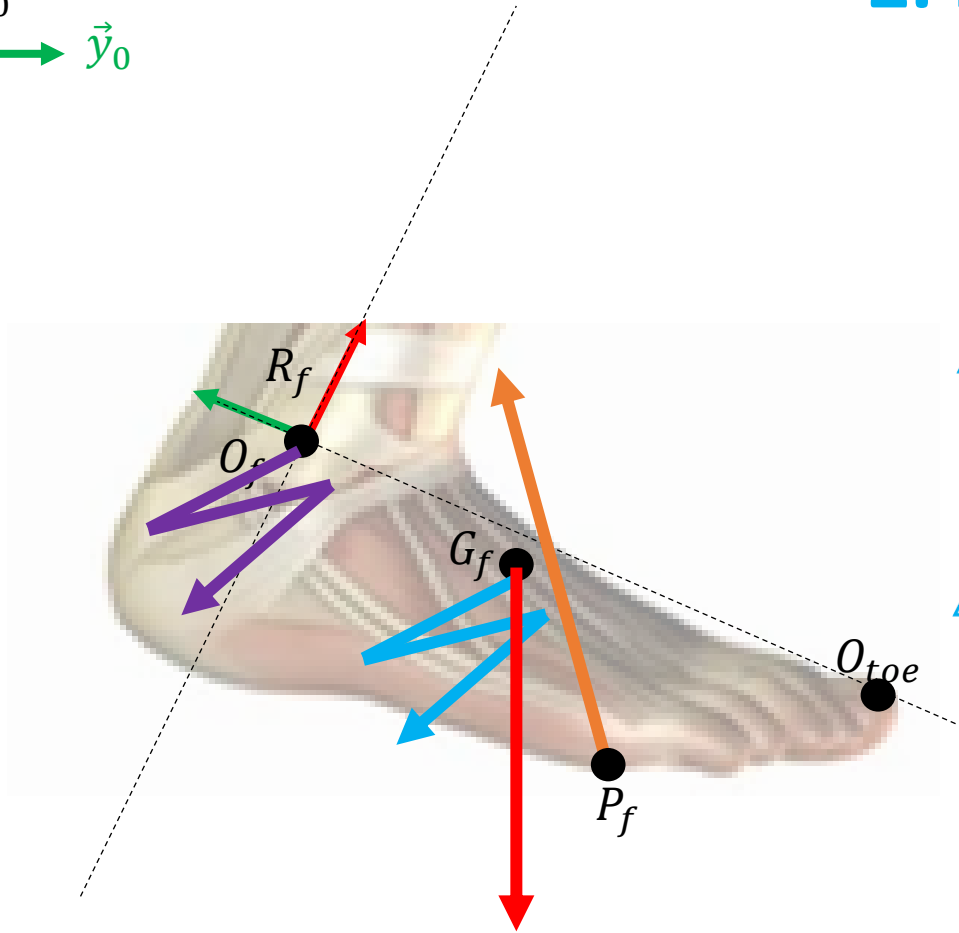
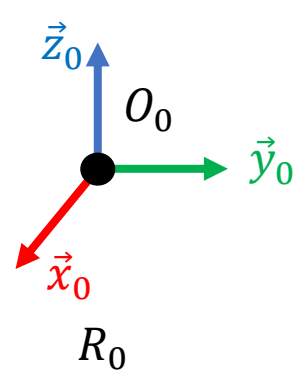
$$\vec{\Gamma}(G_f \in S_4/R_0) = \frac{d^2 \overrightarrow{O_0 G_f}}{dt^2} \text{ (known)}$$

$$\ddot{q}_f \text{ (known)}$$

## Acceleration quantities (dynamic wrench)

$$\{D(S_4/R_0)\} = \begin{Bmatrix} m_f \vec{\Gamma}(G_f \in S_4/R_0) \\ I_f \ddot{q}_f \vec{z}_0 \end{Bmatrix}_{G_f}$$

## 2. Foot equilibrium $S_4$



$$\{\tau(\bar{S}_4 \rightarrow S_4)_g\} + \{\tau(\bar{S}_4 \rightarrow S_4)\} + \{\tau(S_3 \rightarrow S_4)\} = \{D(S_4 / R_0)\}$$



The point of application should be the same



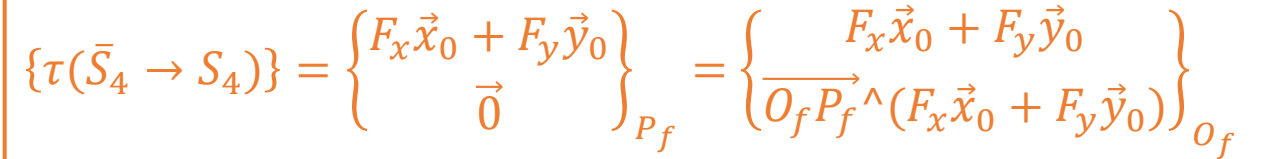
All moments expressed at the ankle joint center  $O_f$



The quantities should be expressed in the same basis



All in  $B_f$  (using  ${}^0R_f$ )

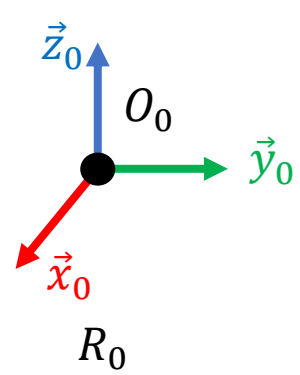


$$\{\tau(S_3 \rightarrow S_4)\} = \left\{ \begin{array}{c} R_{fx}\vec{x}_f + R_{fy}\vec{y}_f \\ \Gamma_f\vec{Z}_0 \end{array} \right\}_{O_f}$$

$$\{\tau(\bar{S}_4 \rightarrow S_4)_g\} = \left\{ \begin{matrix} -m_f g \vec{y}_0 \\ \vec{0} \end{matrix} \right\}_{G_f} = \left\{ \begin{matrix} -m_f g \vec{y}_0 \\ \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0) \end{matrix} \right\}_{O_f}$$

$$\{D(S_4/R_0)\} = \left\{ \begin{matrix} m_f \vec{\Gamma}(G_f \in S_4/R_0) \\ I_f \ddot{q}_f \vec{Z}_0 \end{matrix} \right\}_{G_f} = \left\{ \begin{matrix} m_f \vec{\Gamma}(G_f \in S_4/R_0) \\ I_f \ddot{q}_f \vec{Z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4/R_0) \end{matrix} \right\}_{O_f}^{28}$$

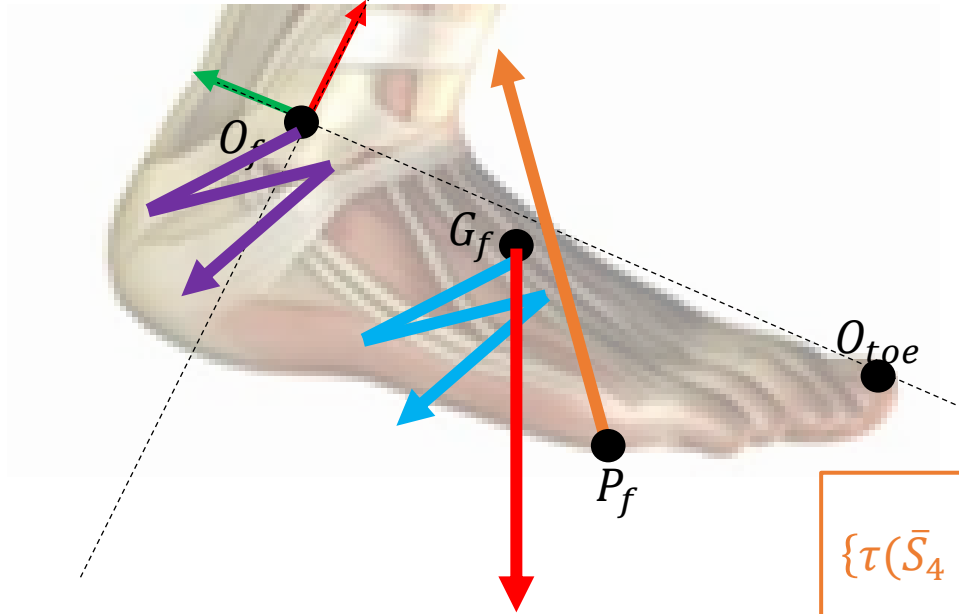
### 3. Projecting in $B_f$



$${}^0R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$${}^fR_0 = {}^0R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\{\tau(\bar{S}_4 \rightarrow S_4)\} = \left\{ \begin{array}{c} F_x \vec{x}_0 + F_y \vec{y}_0 \\ \overrightarrow{O_f P_f} \wedge (F_x \vec{x}_0 + F_y \vec{y}_0) \end{array} \right\}_{O_f}$$

$${}^0F = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} \Rightarrow {}^fF = {}^fR_0 {}^0F = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

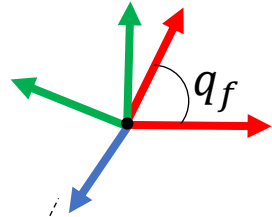
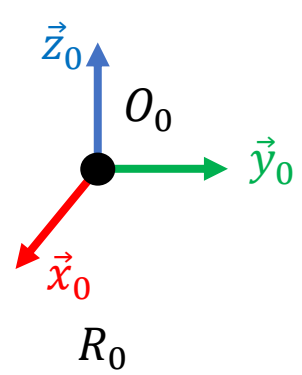
$$\{\tau(\bar{S}_4 \rightarrow S_4)\} = \left\{ \begin{array}{c} (F_x \cos q_f + F_y \sin q_f) \vec{x}_f + (-F_x \sin q_f + F_y \cos q_f) \vec{y}_f \\ \overrightarrow{O_f P_f} \wedge (F_x \vec{x}_0 + F_y \vec{y}_0) \end{array} \right\}_{O_f}$$



The moment is already expressed around  $\vec{z}_0$



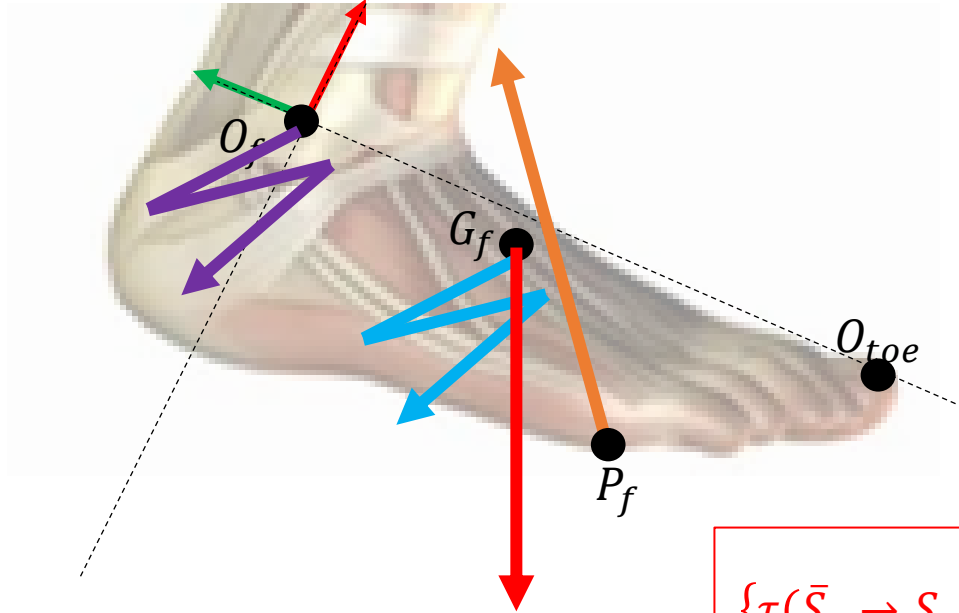
### 3. Projecting in $B_f$



$${}^0R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

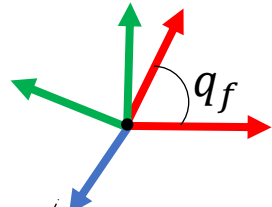
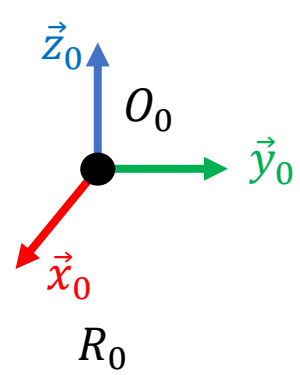


$${}^fR_0 = {}^0R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\{\tau(\bar{S}_4 \rightarrow S_4)_g\} = \left\{ \begin{matrix} -m_f g \vec{y}_0 \\ \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0) \end{matrix} \right\}_{O_f} = \left\{ \begin{matrix} -m_f g (\sin q_f \vec{x}_f + \cos q_f \vec{y}_f) \\ \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0) \end{matrix} \right\}_{O_f}$$

### 3. Projecting in $B_f$



$${}^0R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



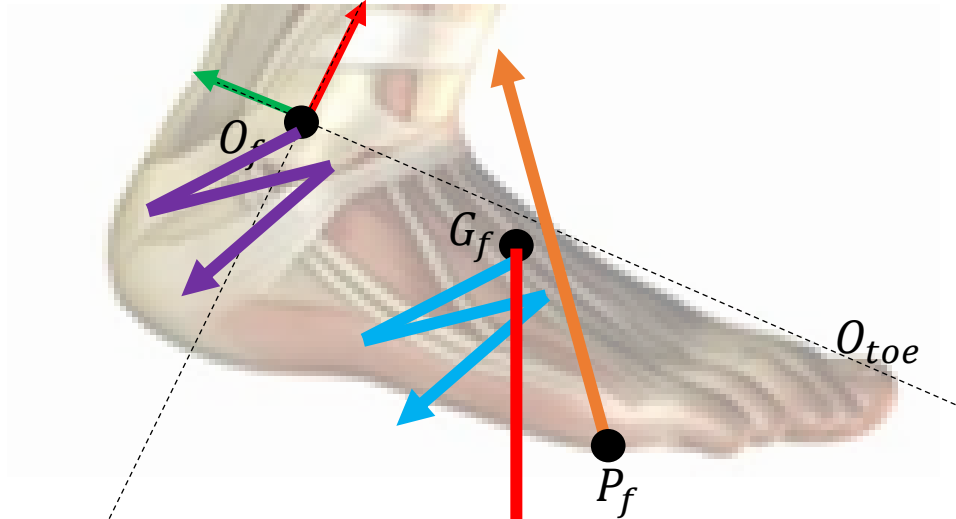
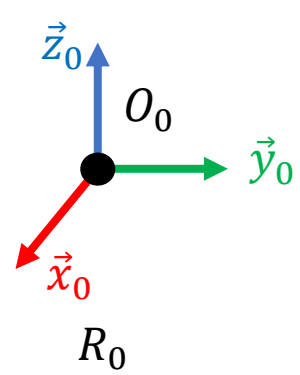
$${}^fR_0 = {}^0R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{D(S_4/R_0)\} = \left\{ \begin{array}{c} m_f \vec{\Gamma}(G_f \in S_4/R_0) \\ I_f \ddot{q}_f \vec{z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4/R_0) \end{array} \right\}_{O_f} = \left\{ \begin{array}{c} m_f (\ddot{x}_{G_f} \vec{x}_0 + \ddot{y}_{G_f} \vec{y}_0) \\ I_f \ddot{q}_f \vec{z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4/R_0) \end{array} \right\}_{O_f}$$

Same principle

$$\{D(S_4/R_0)\} = \left\{ \begin{array}{c} m_f ((\ddot{x}_{G_f} \cos q_f + \ddot{y}_{G_f} \sin q_f) \vec{x}_f + (-\ddot{x}_{G_f} \sin q_f + \ddot{y}_{G_f} \cos q_f) \vec{y}_f) \\ I_f \ddot{q}_f \vec{z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4/R_0) \end{array} \right\}_{O_f}$$

## 4. Equilibrium



$$\{\tau(\bar{S}_4 \rightarrow S_4)_g\} + \{\tau(\bar{S}_4 \rightarrow S_4)\} + \{\tau(S_3 \rightarrow S_4)\} = \{D(S_4 / R_0)\}$$



$$\{\tau(S_3 \rightarrow S_4)\} = \{D(S_4 / R_0)\} - \{\tau(\bar{S}_4 \rightarrow S_4)\} - \{\tau(\bar{S}_4 \rightarrow S_4)_g\}$$



/  $\vec{x}_f$

$$R_{fx} = m_f (\ddot{x}_{G_f} \cos q_f + \ddot{y}_{G_f} \sin q_f) - (F_x \cos q_f + F_y \sin q_f) + m_f g \sin q_f$$

/  $\vec{y}_f$

$$R_{fy} = m_f (-\ddot{x}_{G_f} \sin q_f + \ddot{y}_{G_f} \cos q_f) - (-F_x \sin q_f + F_y \cos q_f) + m_f g \cos q_f$$

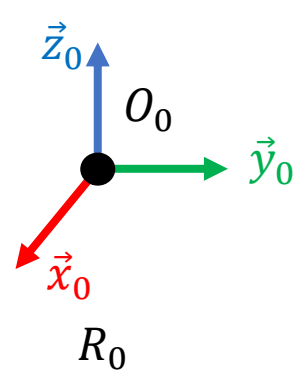
/  $\vec{z}_f$

$$\Gamma_f = (I_f \ddot{q}_f \vec{z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4 / R_0) - \overrightarrow{O_f P_f} \wedge (F_x \vec{x}_0 + F_y \vec{y}_0) - \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0)) \cdot \vec{z}_0$$



If the motion is known, then we get the torque at the ankle !

## 5. What about kinematics ?



Joint center positions are known

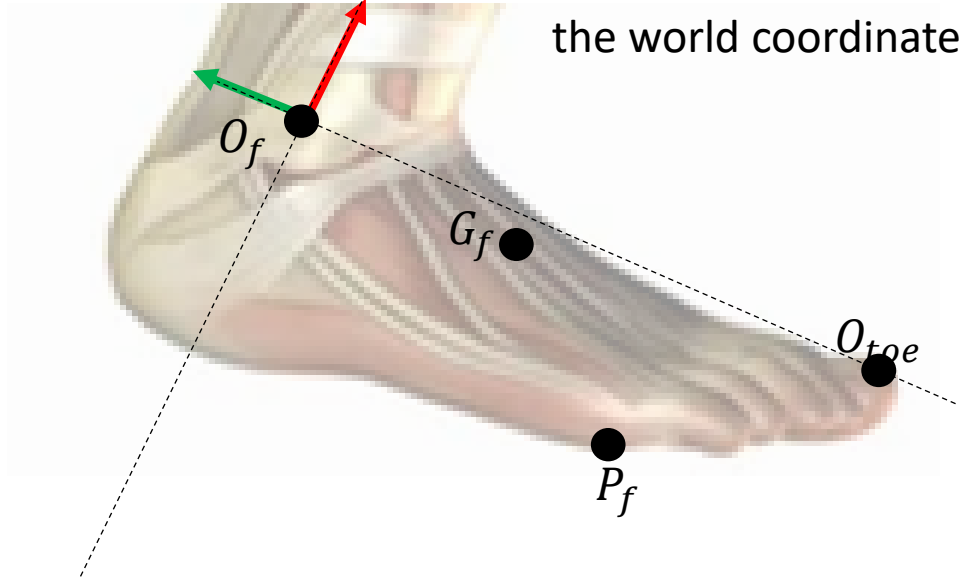
${}^0O_f$  known

If the foot CoM is known in the foot coordinates system, then we can get its position in the world coordinates system

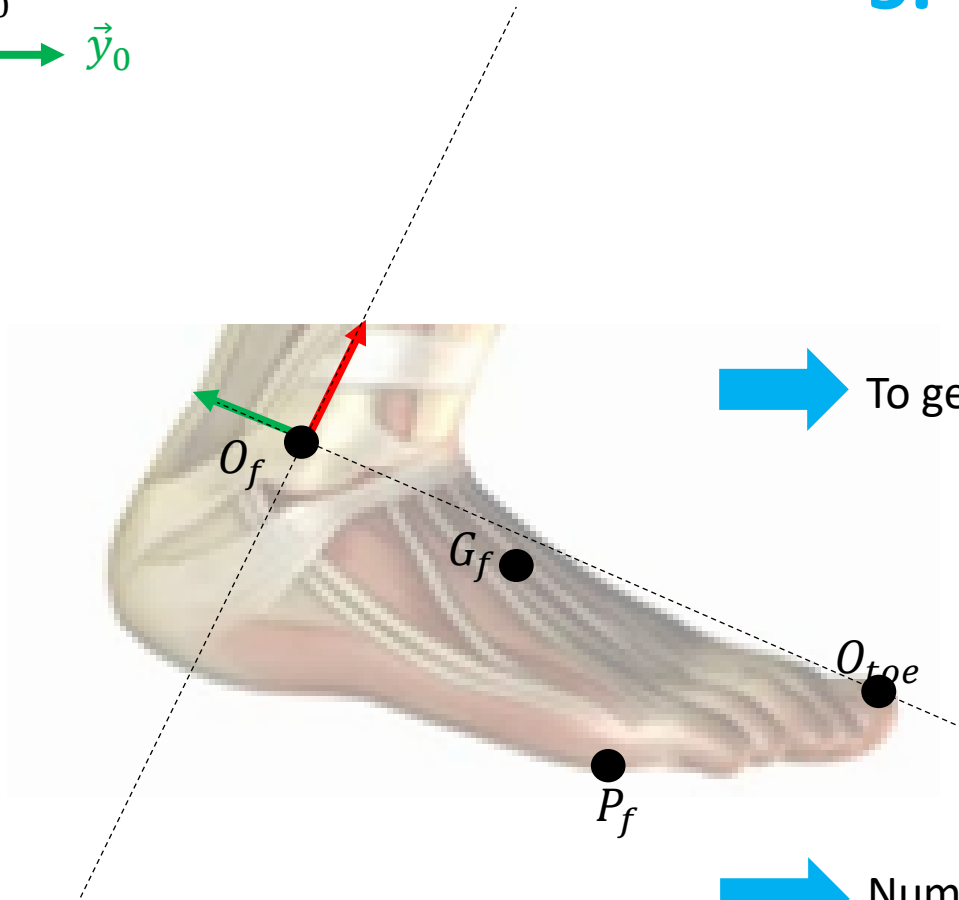
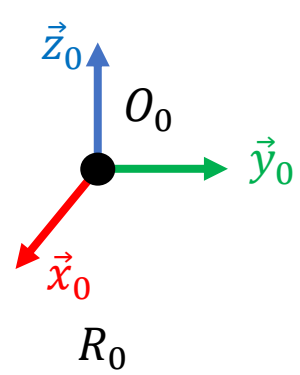
${}^fG_f$  known



$${}^0G_f = {}^0O_f + {}^0R_f {}^fG_f$$

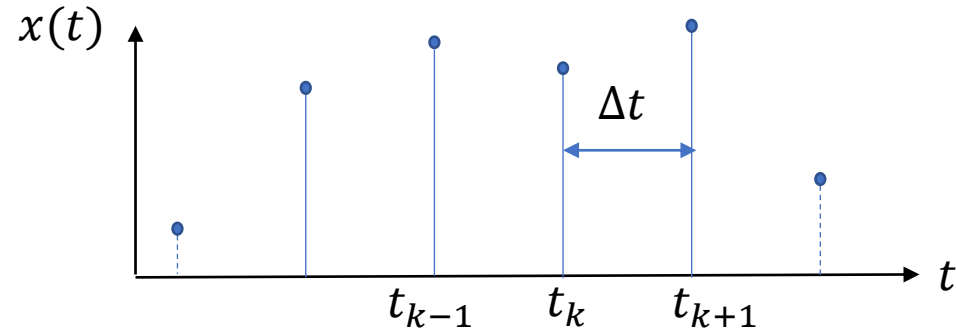


## 5. What about kinematics ?



${}^0G_f$  known

➡ To get  $\vec{\Gamma}(G_f \in S_4/R_0)$ , we differentiate 2 times  ${}^0G_f$



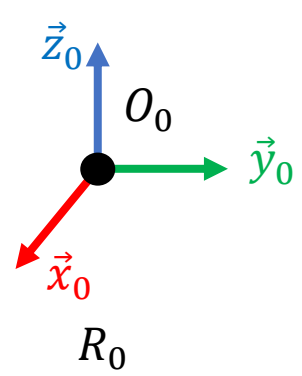
➡ Numerical differentiation for a signal  $x(t)$

$$\dot{x}_{t_k} = \frac{x(t_{k+1}) - x(t_{k-1}))}{2\Delta t} \quad (2^{\text{nd}} \text{ order centered finite difference})$$



Numerical differentiation amplifies noise...we should filter the data

# Shank $S_3$



The motion is known

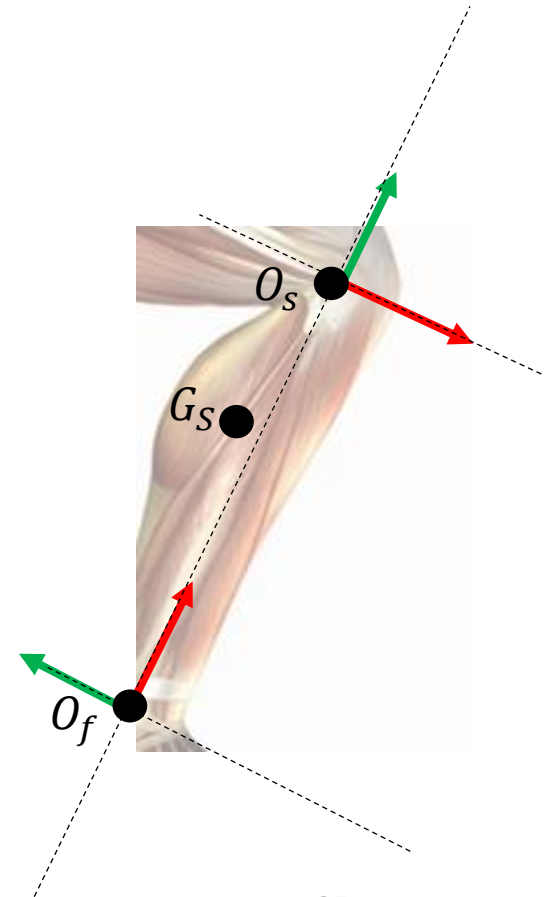
$$\vec{\Gamma}(G_S \in S_3/R_0) = \frac{d^2 \overrightarrow{O_0 G_S}}{dt^2} \text{ (known)}$$

$\ddot{q}_s$  (known)

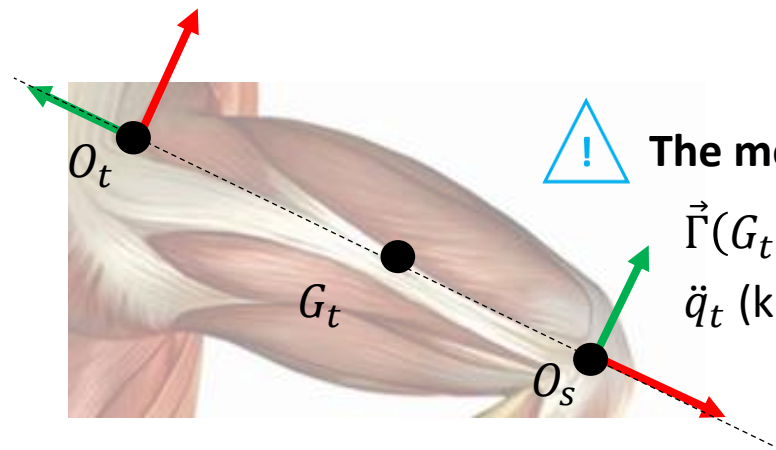
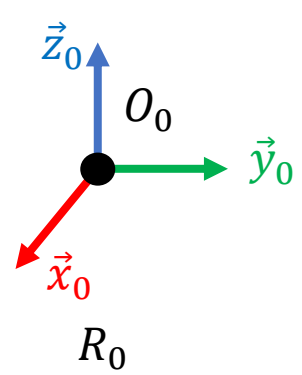
External forces ?

Acceleration quantities in  $G_S$  ?

Action of the thigh on the shank in  $O_S$  ?



# Thigh $S_2$



The motion is supposed known

$$\vec{\Gamma}(G_t \in S_2/R_0) = \frac{d^2 \overrightarrow{O_0 G_t}}{dt^2} \text{ (known)}$$

$$\ddot{q}_t \text{ (known)}$$

External forces ?

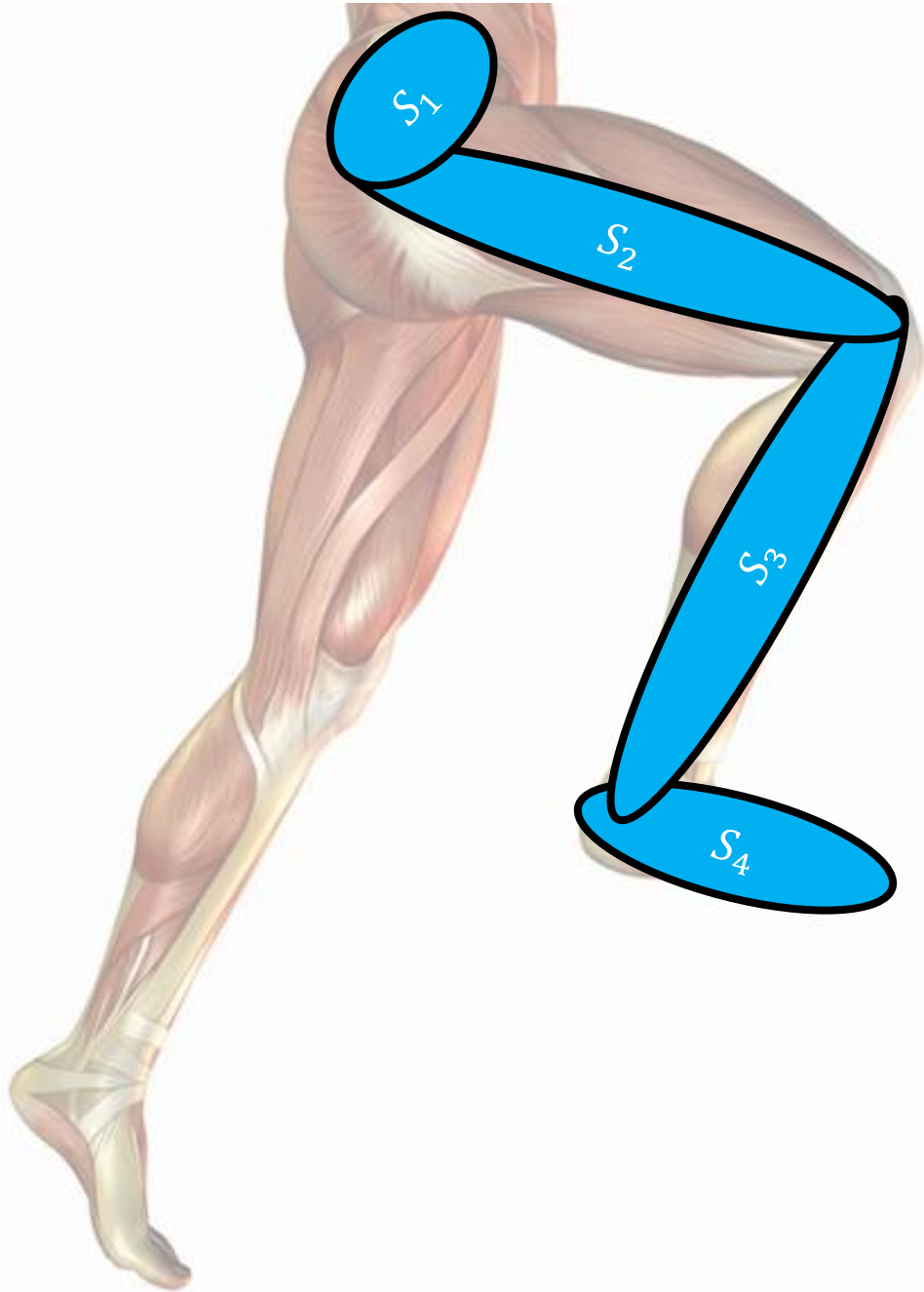
Acceleration quantities in  $G_t$  ?

Action of the pelvis on the thigh in  $O_t$  ?

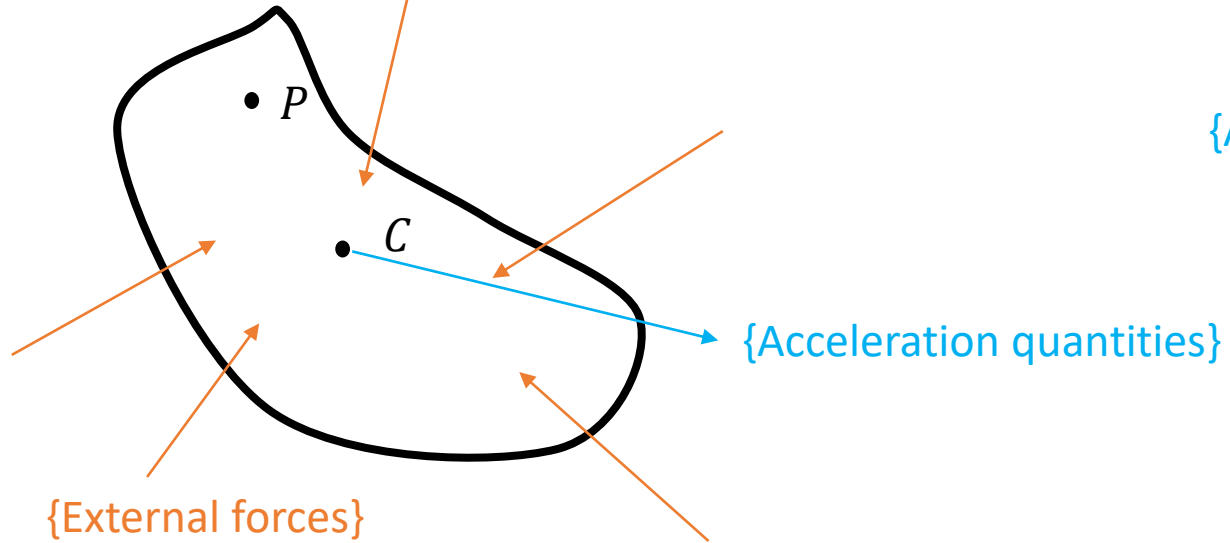


# Generalisation

**Recursive Newton-Euler Algorithm (RNEA)**

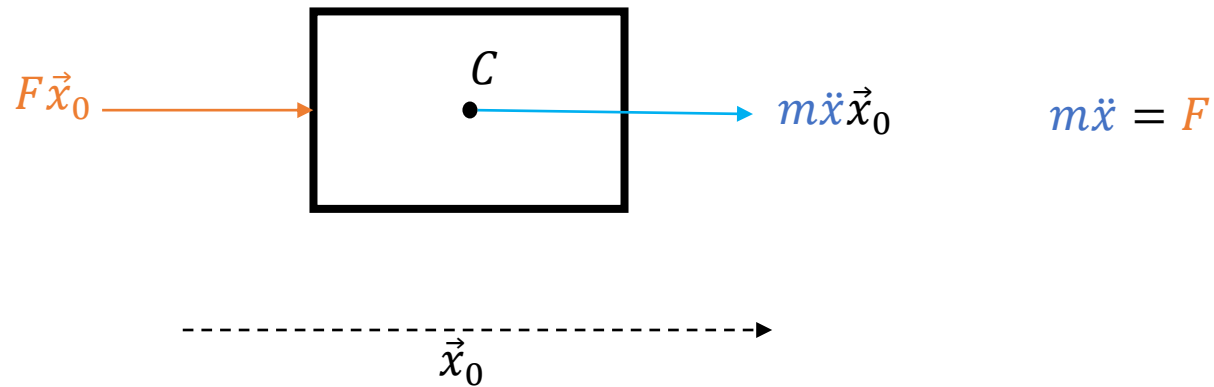


# Solid S equilibrium

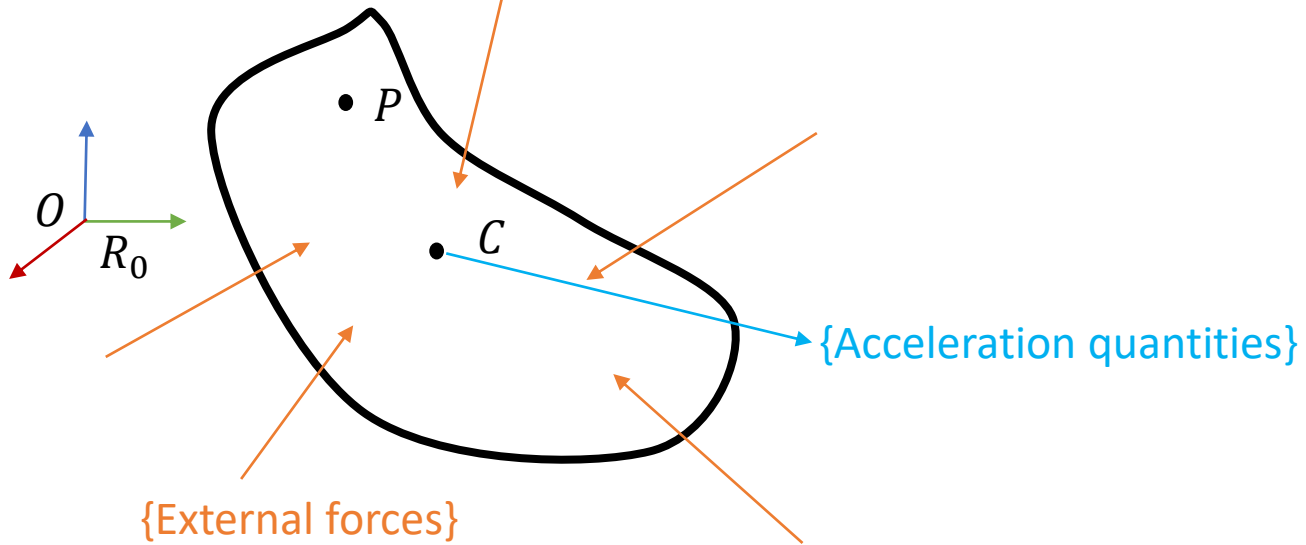


$$\{\text{Acceleration quantities}\} = \{\text{External forces}\}$$

“For a solid in simple translation...”



# Solid S equilibrium

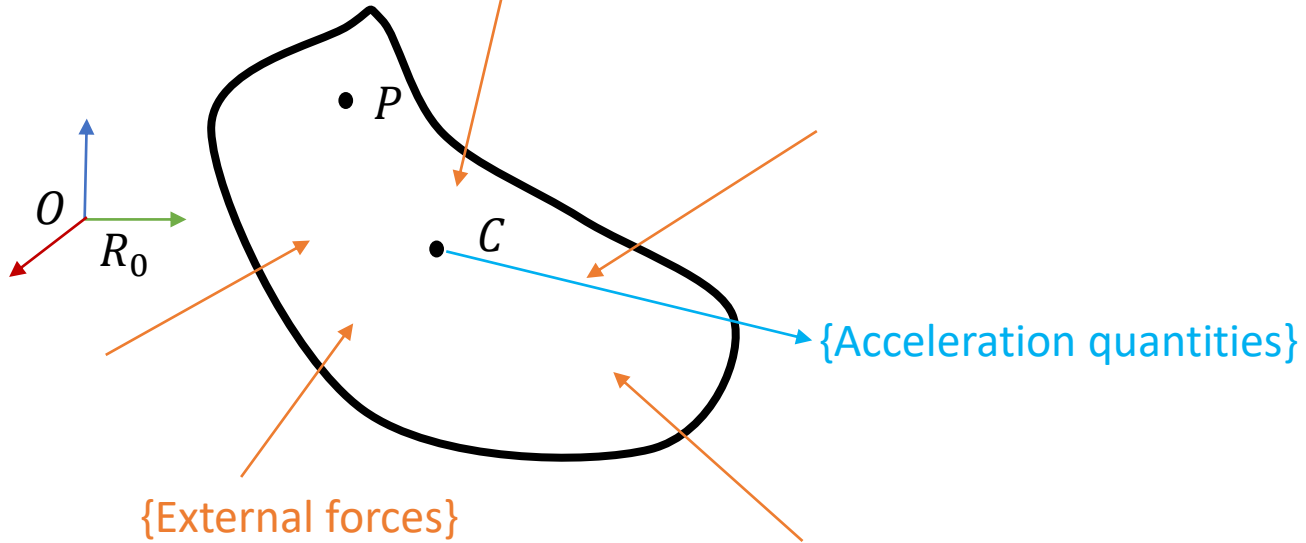


At the CoM

$$\begin{cases} \mathbf{f} = m\ddot{\mathbf{c}} & (1) \\ \boldsymbol{\tau}^{(c)} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} & (2) \end{cases}$$

- $\mathbf{f}$  external forces
- $m$  solid mass
- $\mathbf{c}$  CoM position in  $R_0$  (fixed)
- $\boldsymbol{\omega}$  solid instantaneous velocity /  $R_0$  and expressed in  $R_0$
- $\mathbf{I}$  inertia matrix expressed at  $\mathbf{c}$  in  $R_0$
- $\boldsymbol{\tau}^{(c)}$  torque associated to the external forces expressed at  $\mathbf{c}$  in  $R_0$

# Solid S equilibrium

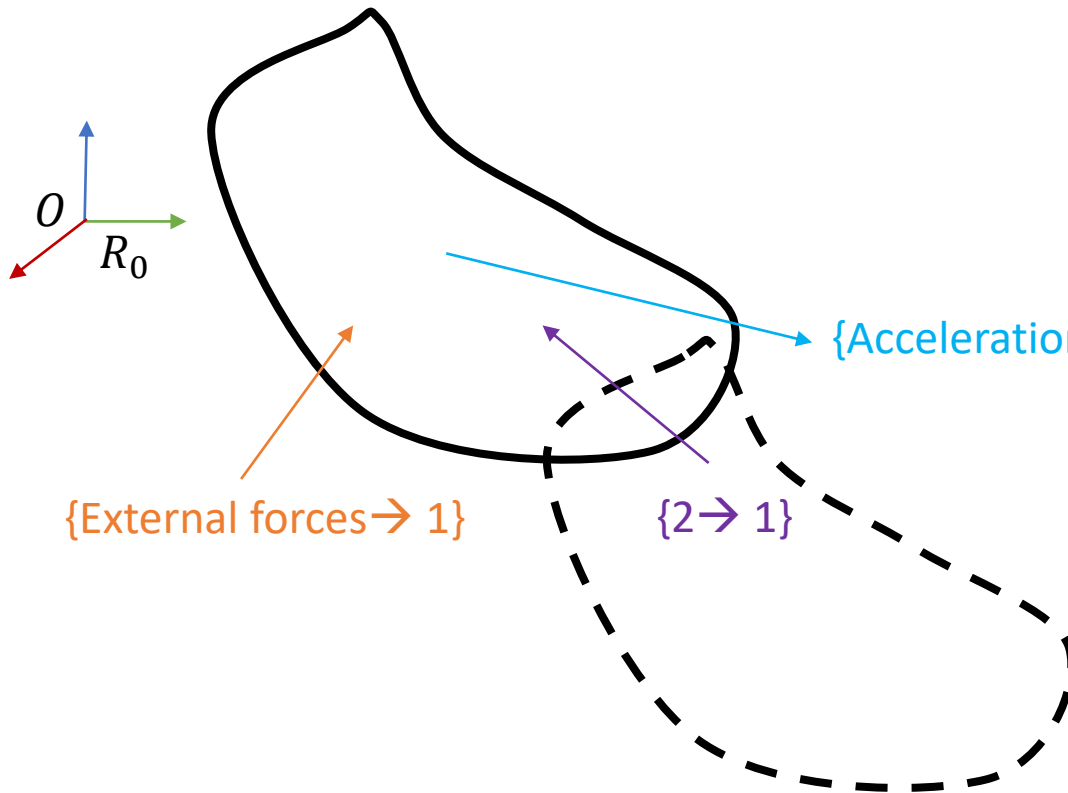


At the CoM

$$\begin{cases} \mathbf{f} = m\ddot{\mathbf{c}} & (1) \\ \boldsymbol{\tau}^{(c)} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} & (2) \end{cases}$$

- $\mathbf{f}$  external forces  $\rightarrow$  known (from measure/modeling)
- $m$  solid mass  $\rightarrow$  known (from modeling)
- $\mathbf{c}$  CoM position in  $R_0$  (fixed)  $\rightarrow$  known (from measure)
- $\boldsymbol{\omega}$  solid instantaneous velocity /  $R_0$  and expressed in  $R_0 \rightarrow$  known (from measure)
- $\mathbf{I}$  inertia matrix expressed at  $\mathbf{c}$  in  $R_0 \rightarrow$  known (from modeling)
- $\boldsymbol{\tau}^{(c)}$  torque associated to the external forces expressed at  $\mathbf{c}$  in  $R_0 \rightarrow$  known (from measure/modeling)

# Equilibrium of an arborescent system

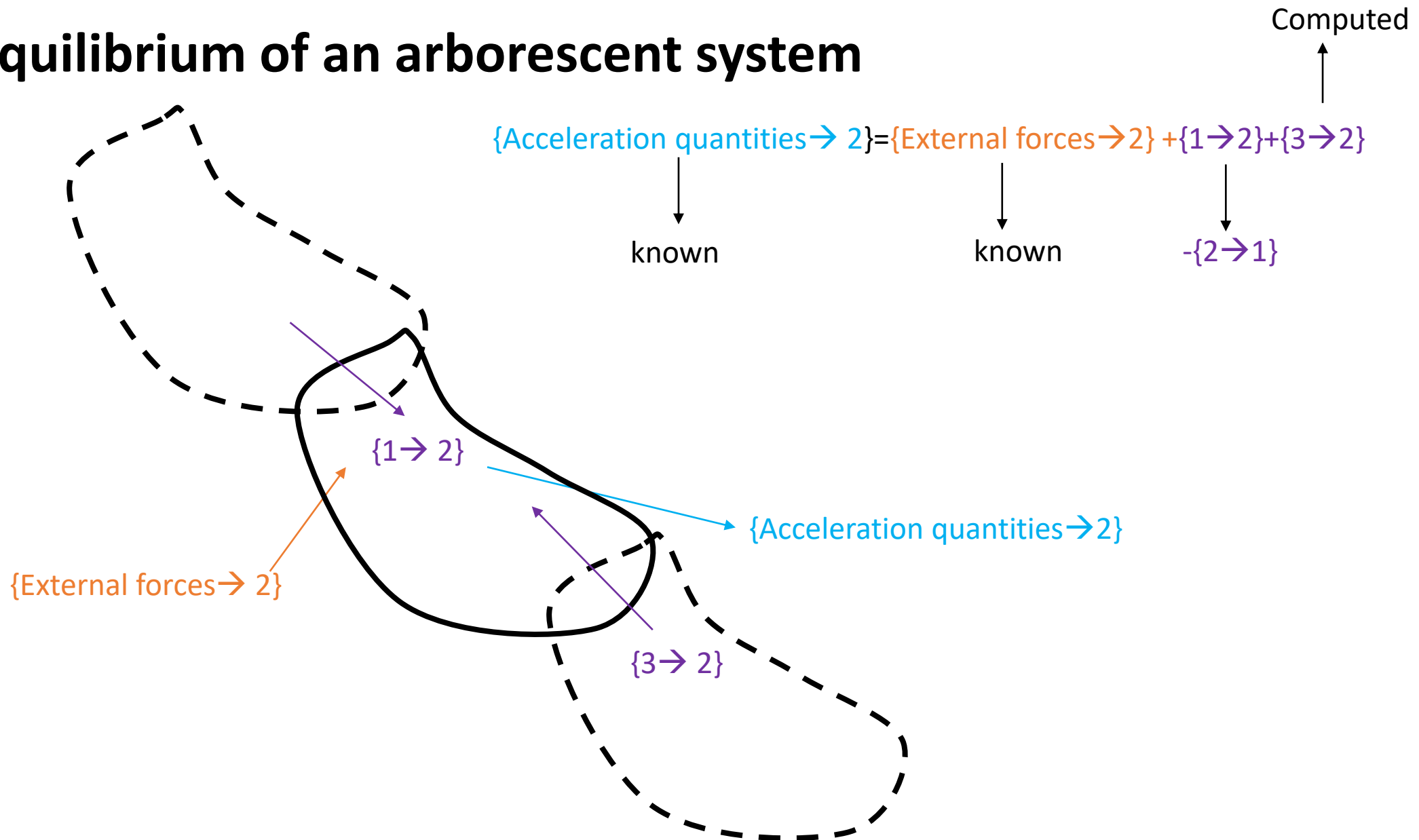


Known                      Known                      Computed !

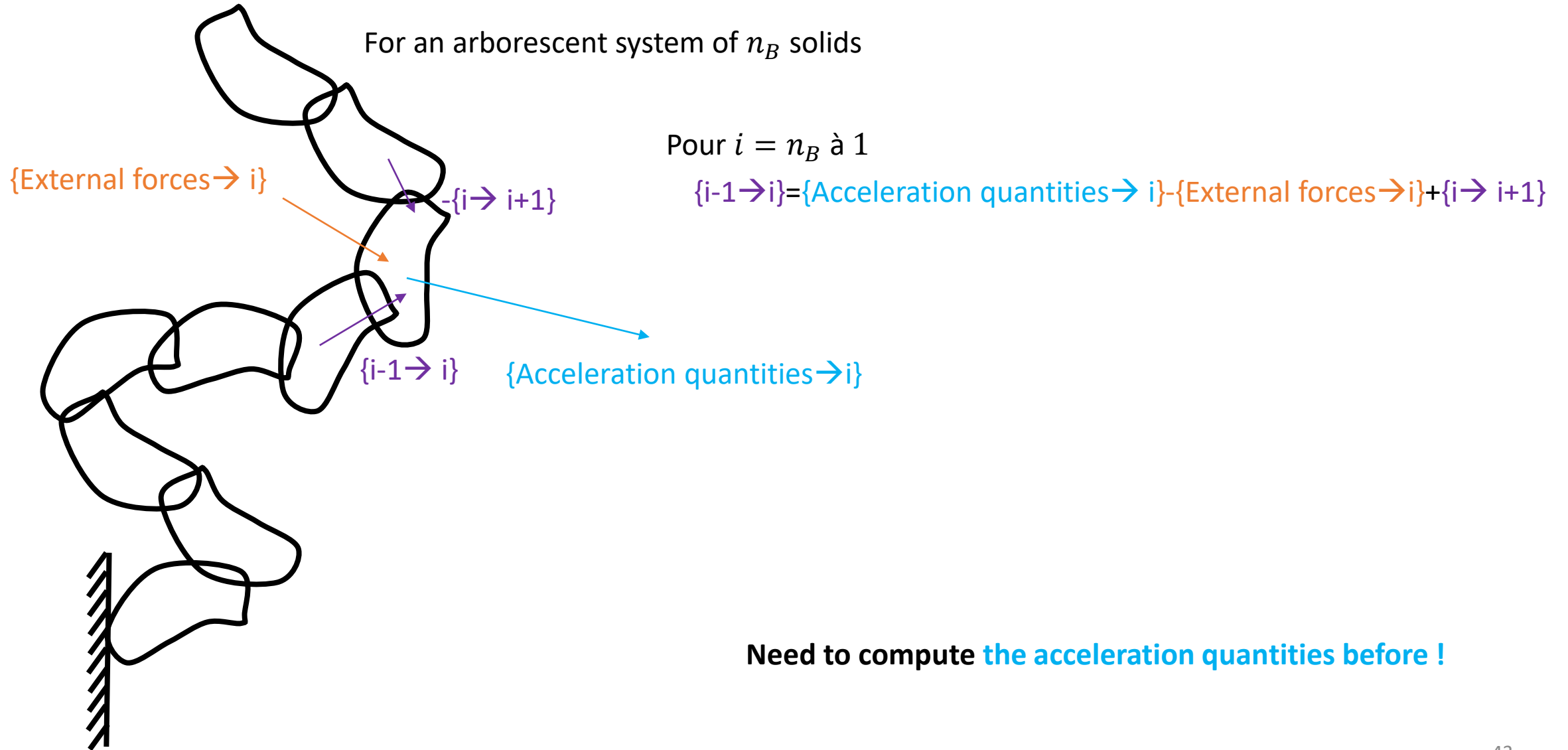
↑                                      ↑                                      ↑

$\{\text{Acceleration quantities} \rightarrow 1\} = \{\text{External forces} \rightarrow 1\} + \{2 \rightarrow 1\}$

# Equilibrium of an arborescent system



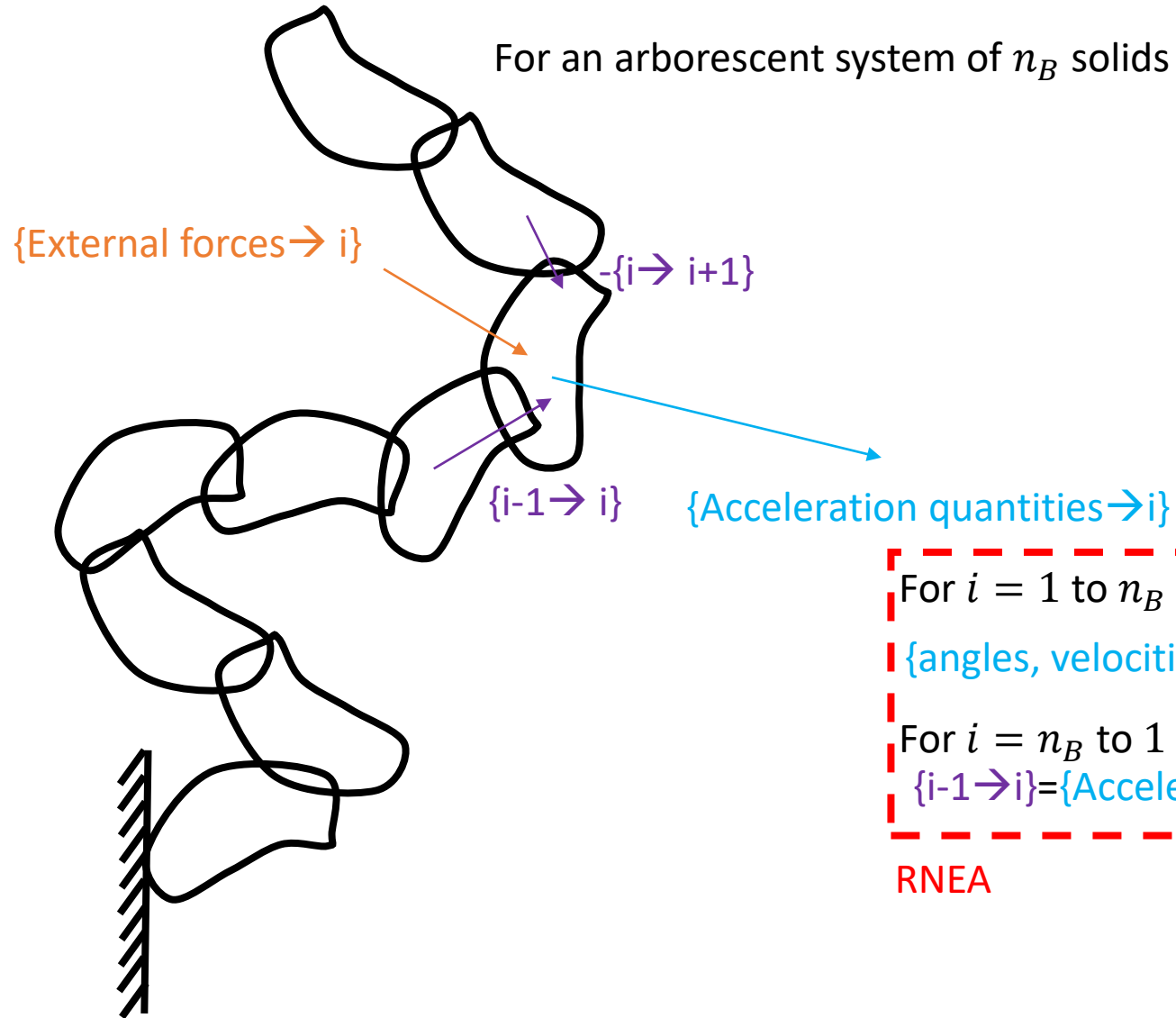
# RNEA



Need to compute **the acceleration quantities before !**



# RNEA



For  $i = 1$  to  $n_B$

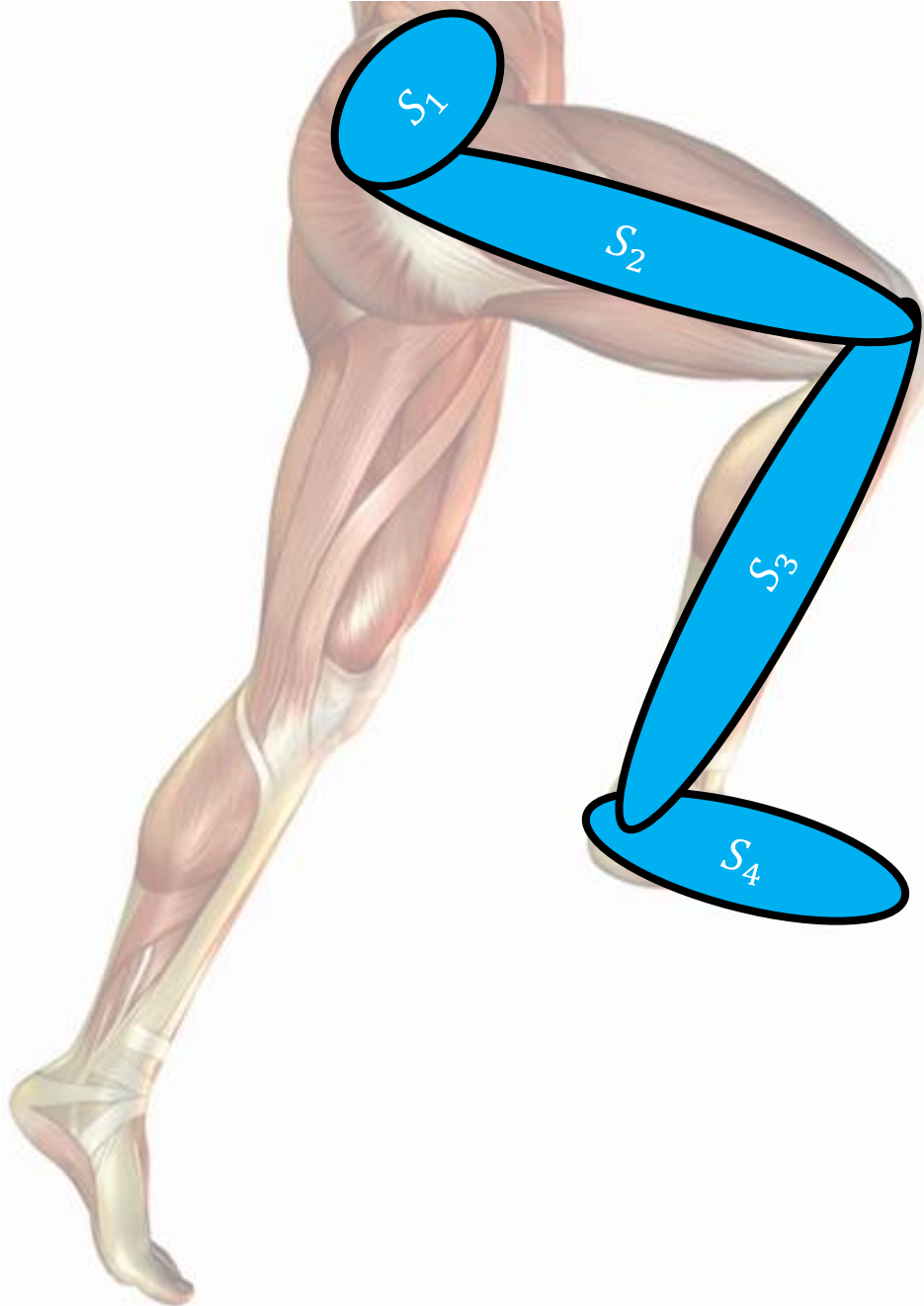
$\{\text{angles, velocities, accelerations}\}_i = f(\mathbf{q}, \mathbf{dq}, \mathbf{ddq})$

For  $i = n_B$  to 1

$\{i-1 \rightarrow i\} = \{\text{Acceleration quantities} \rightarrow i\} - \{\text{External forces} \rightarrow i\} + \{i \rightarrow i+1\}$

RNEA

# Application



**Objectif:** RNEA on a lower limb planar model

**Supplementary data:** inertial properties, external forces (from forces platforms)

## Methods:

- Construct CoM trajectories
- Filter the data (low pass)
- Compute CoM acceleration and joint accelerations (**RNEA #1**)
- compute acceleration quantities
- Isolate from bottom to top the solid to compute the forces (**RNEA #2**)



See Lower\_limb\_ID.ipnyb

# Application

