Assignment #4

Lower limb kinematics and dynamics

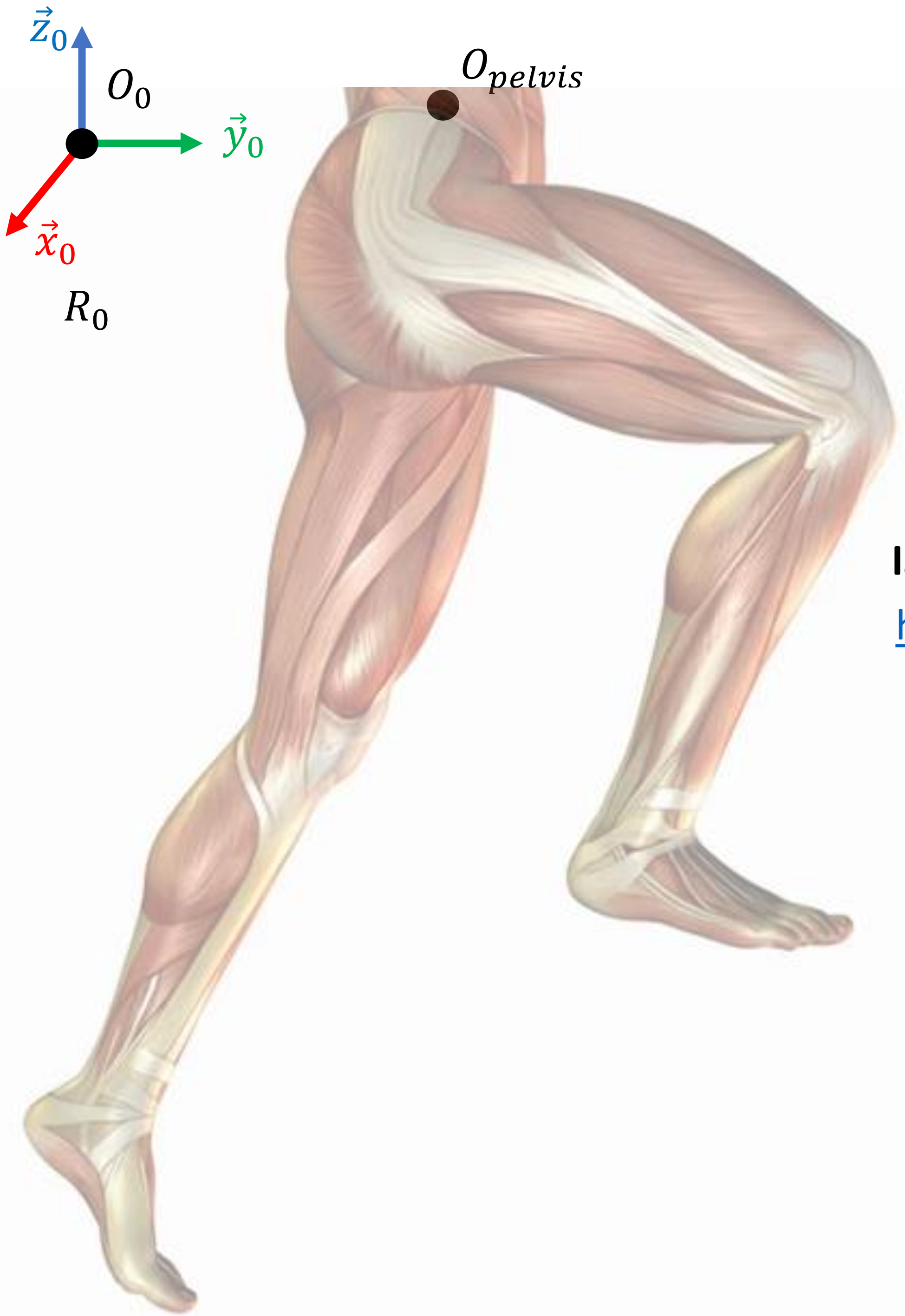
Kinematics and dynamics, lower limb

The assignment is accompanied by a python code. Experimental data (gait, run, jump) is available to be analyzed with the model developed here

Assignment goals

- Modeling the lower limb (2D model)
- Anatomical position of the joint center in the reference frame (forward kinematics)
- Joint angles (inverse kinematics)
- Joint torques (inverse dynamics)

Biomechanical model



We suppose know the orientation of the pelvis

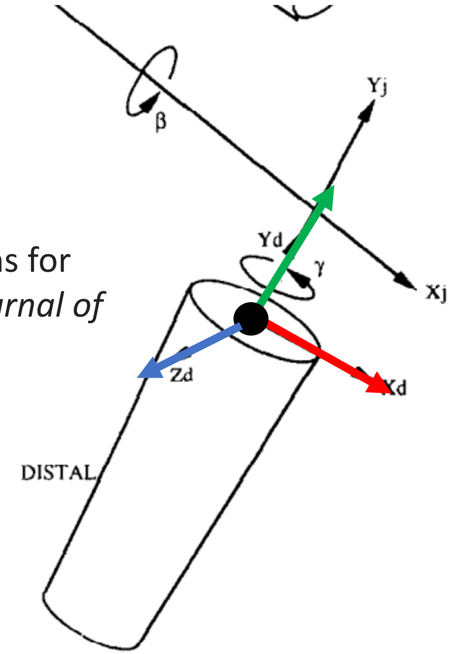
1. Define the **degrees of freedom of the system**
2. Locate the joint centres **and the coordinates system**
3. Express the rotation matrices **from one coordinate system to one other**

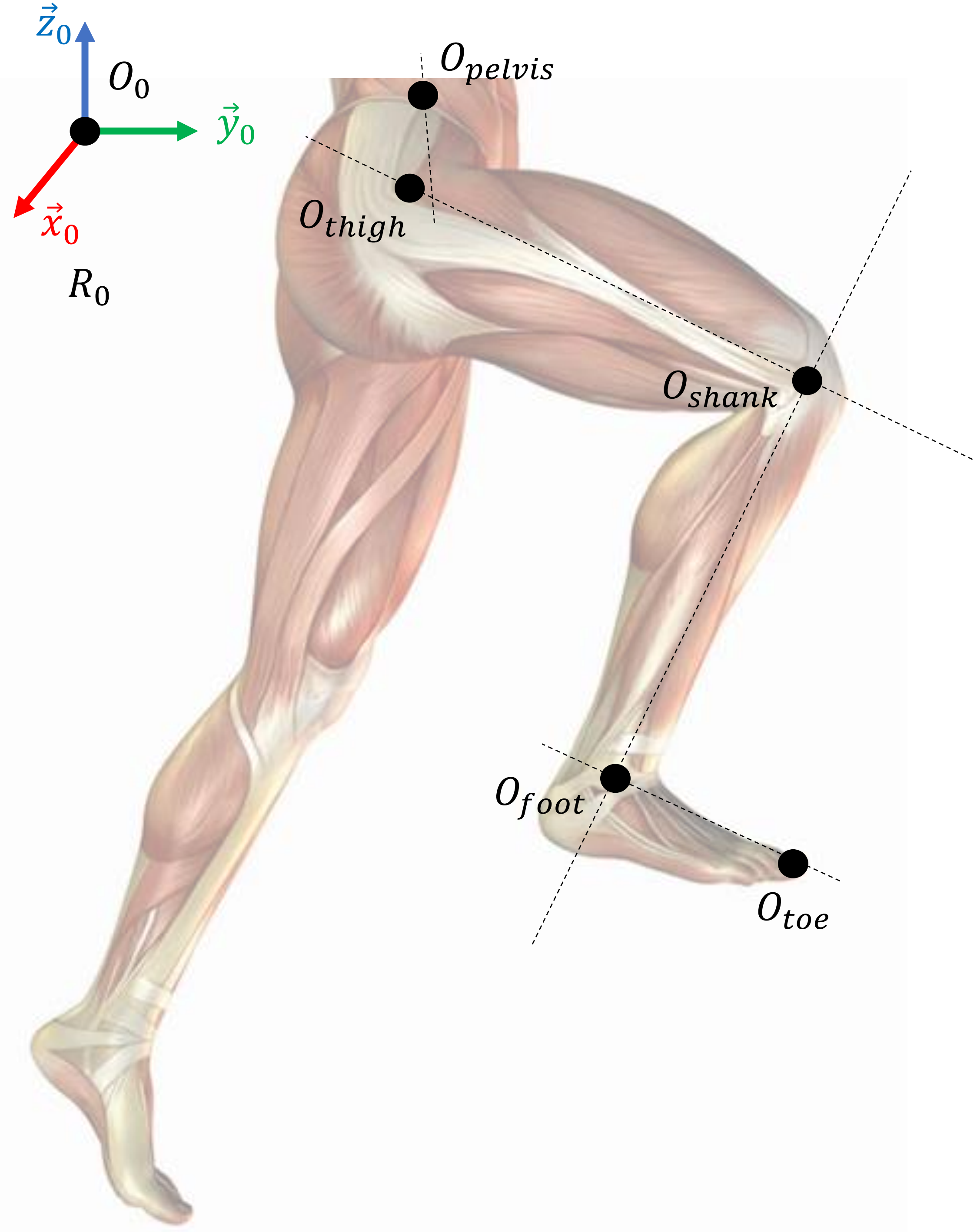
ISB recommendations (planar model)

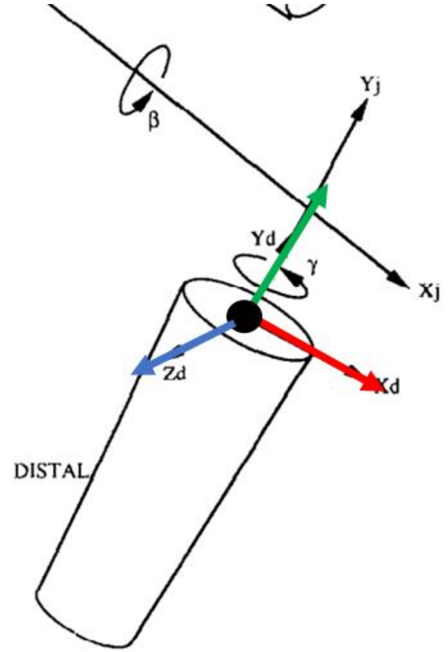
https://isbweb.org/activities/standards

Note the same as the 3D convention !

Wu, G., & Cavanagh, P. R. (1995). ISB recommendations for standardization in the reporting of kinematic data. *Journal of biomechanics*, *28*(10), 1257-1262.

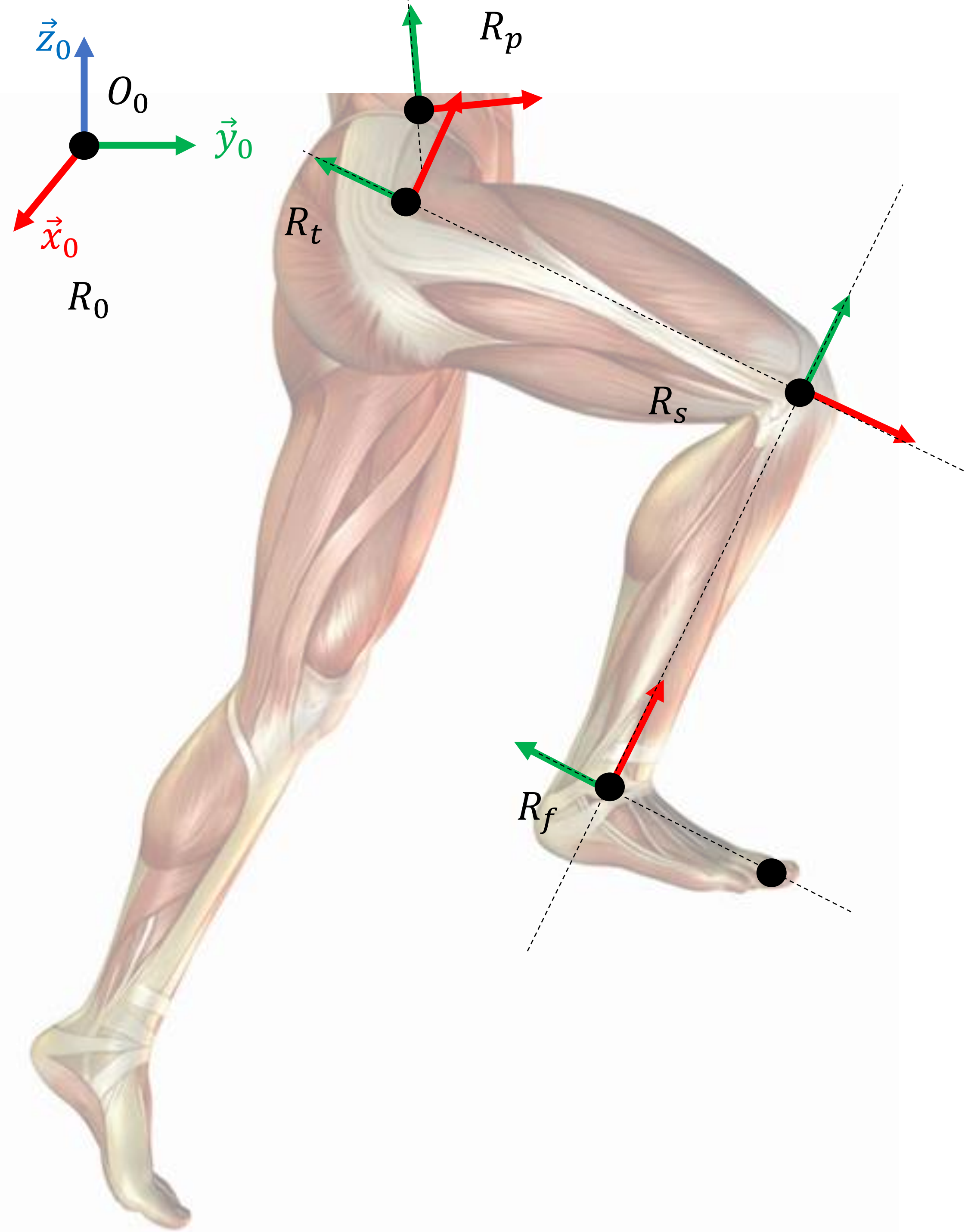


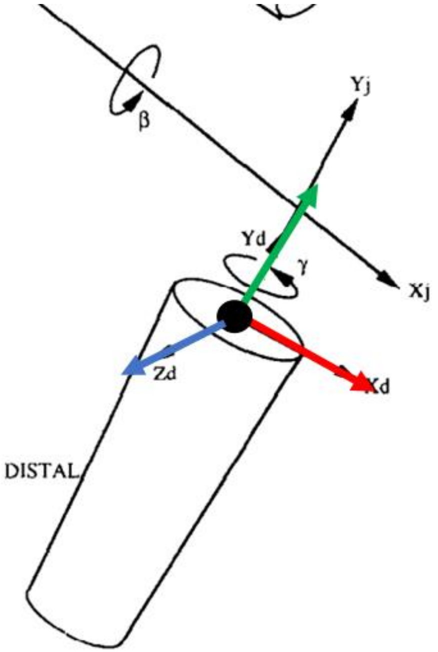




Joint centers

Longitudinal axes = y direction for all segments


$\vec{z}_0$
O_0
$\vec{y}_0$
$\vec{x}_0$
R_0
R_p
R_t
R_s
R_f


β
Yj
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γ
Xj
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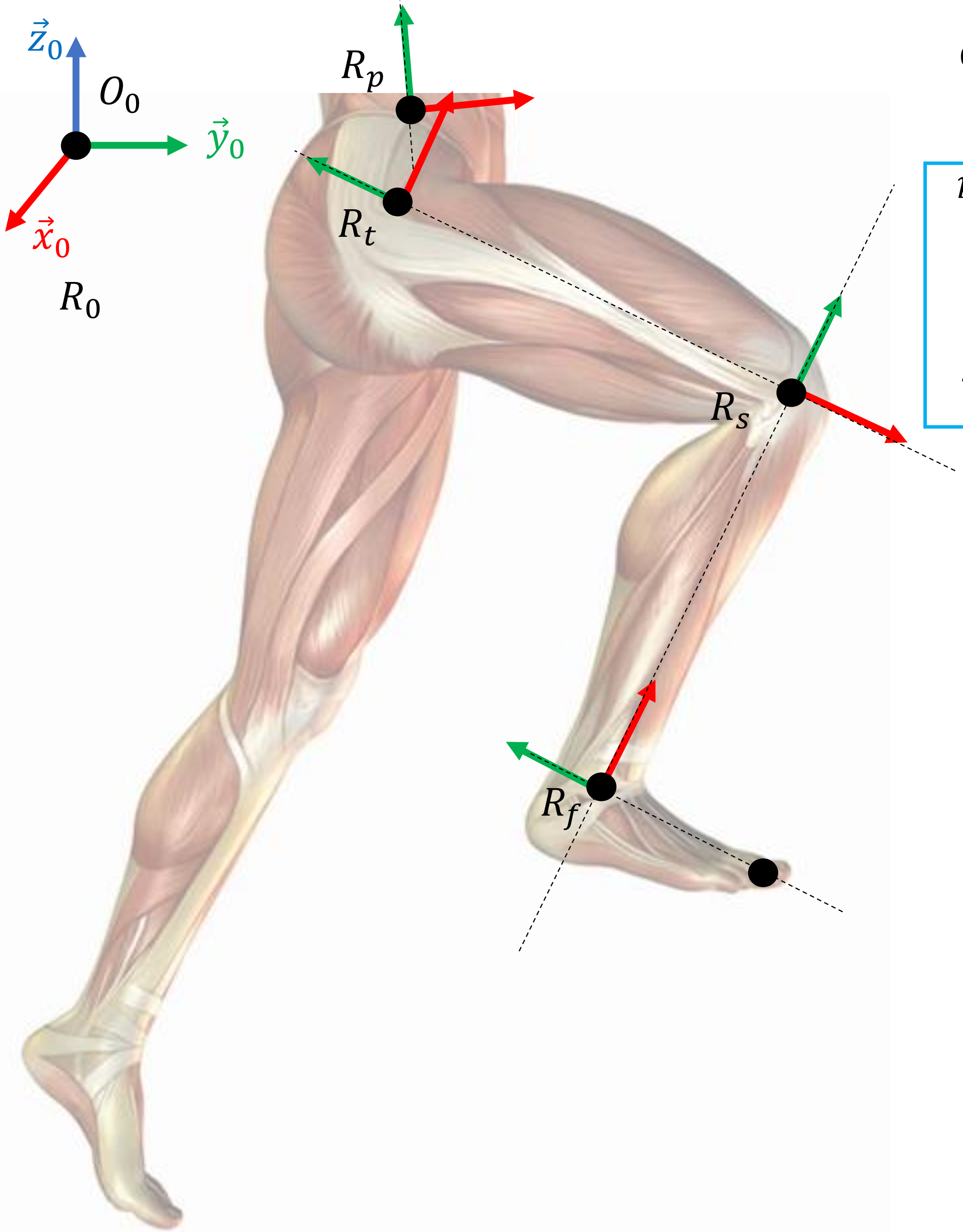
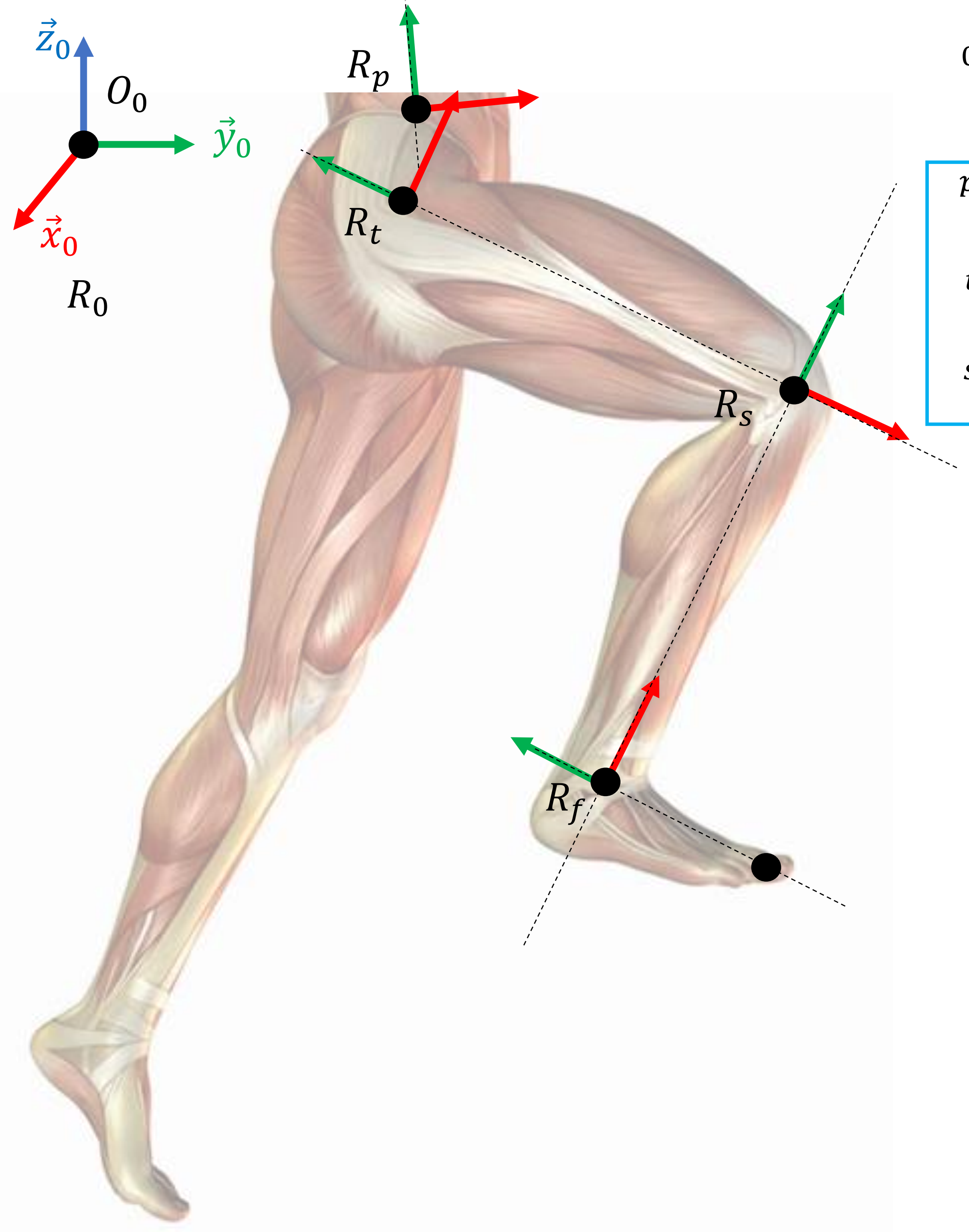


${}^{0}R_p$	Rotation matrix from pelvis to world (assumed to be known)
${}^{p}R_t$	Rotation matrix from thigh to pelvis
${}^{t}R_s$	Rotation matrix from shank to thigh
${}^{s}R_f$	Rotation matrix from foot to shank

How to compute them ?



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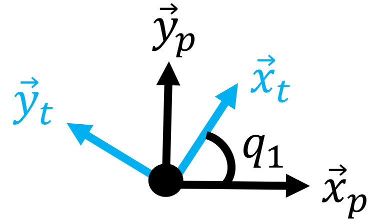
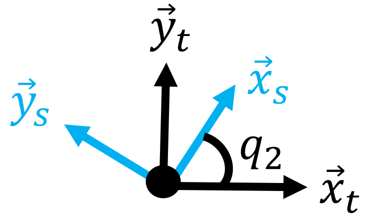
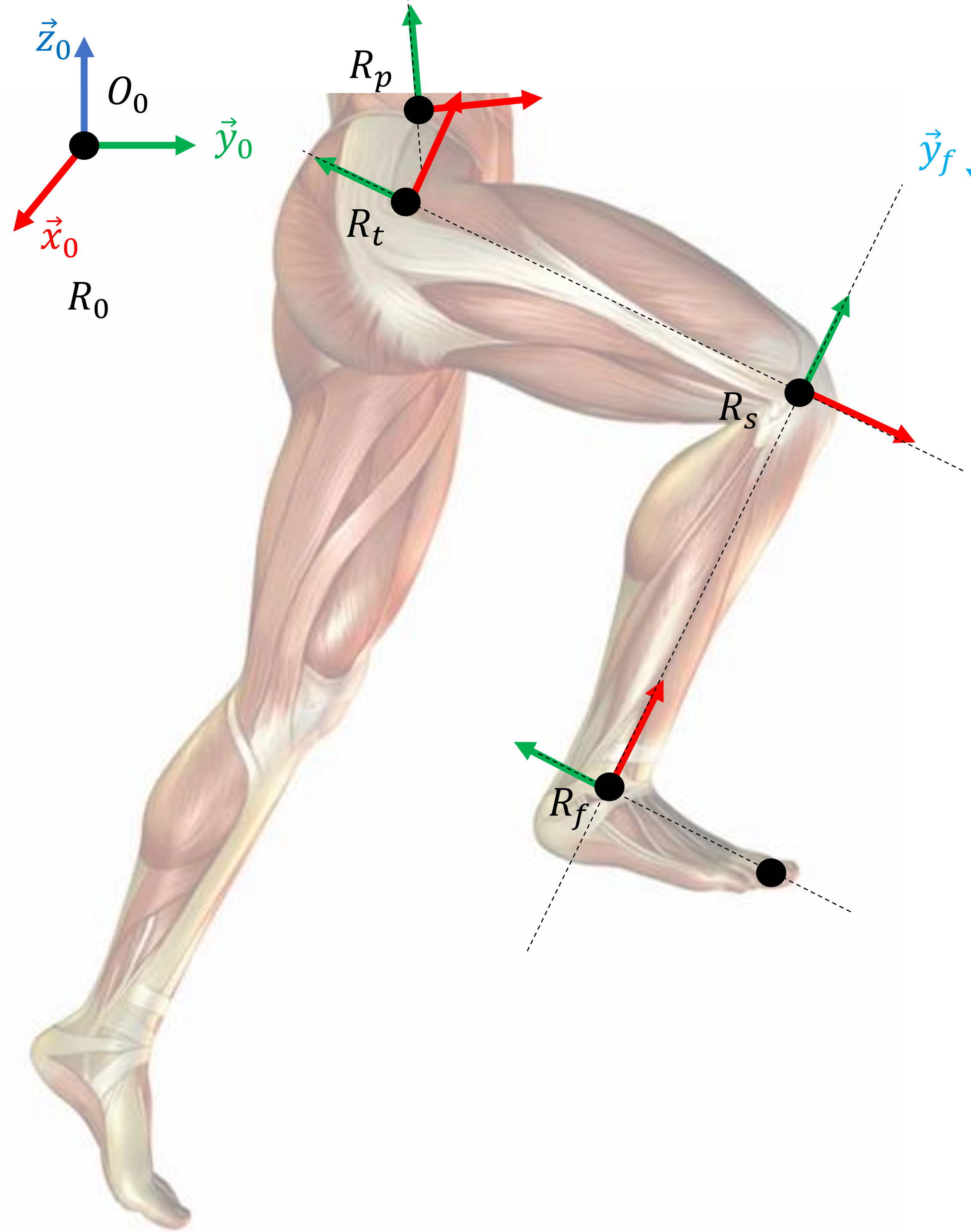


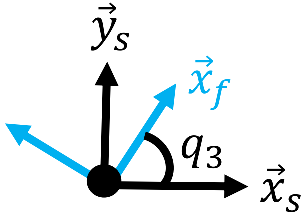
$$ {}^{p}R_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ {}^{t}R_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

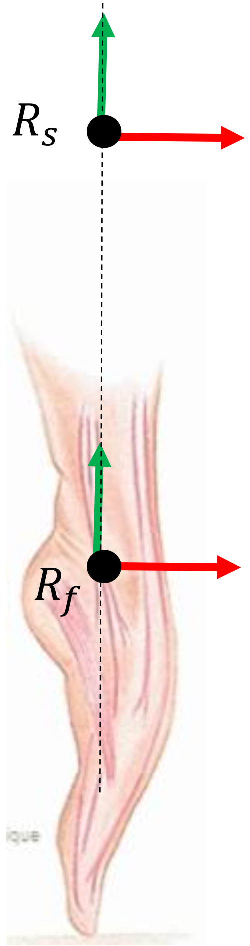
But for the foot…





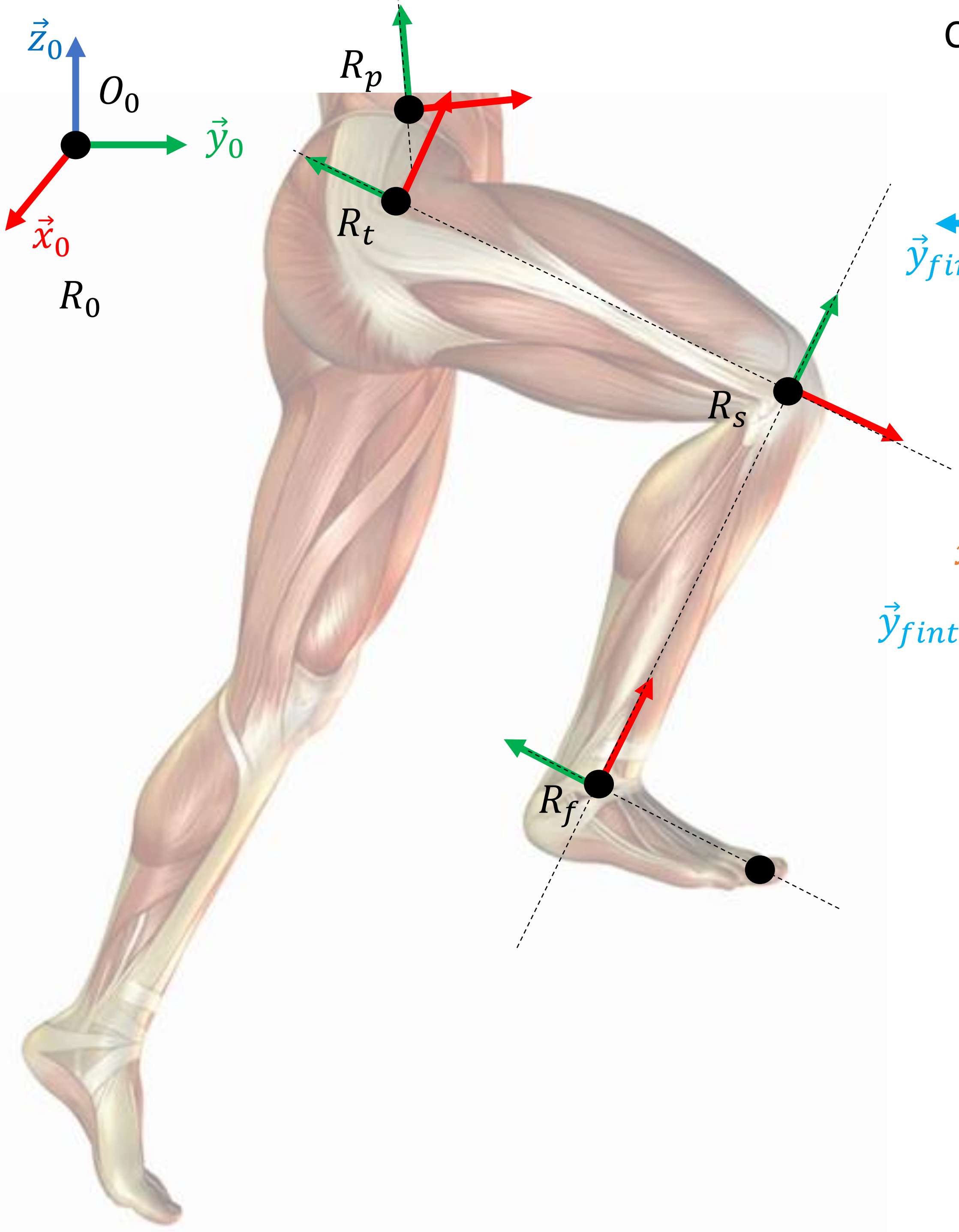
$$^{s}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

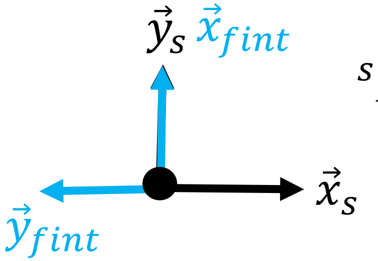
If q_3 is 0 …



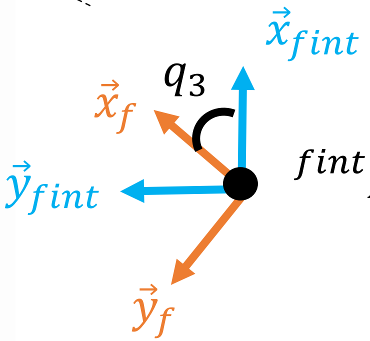
The anatomical (reference) position is not ok

Creating an intermediate coordinate system R_{fint}



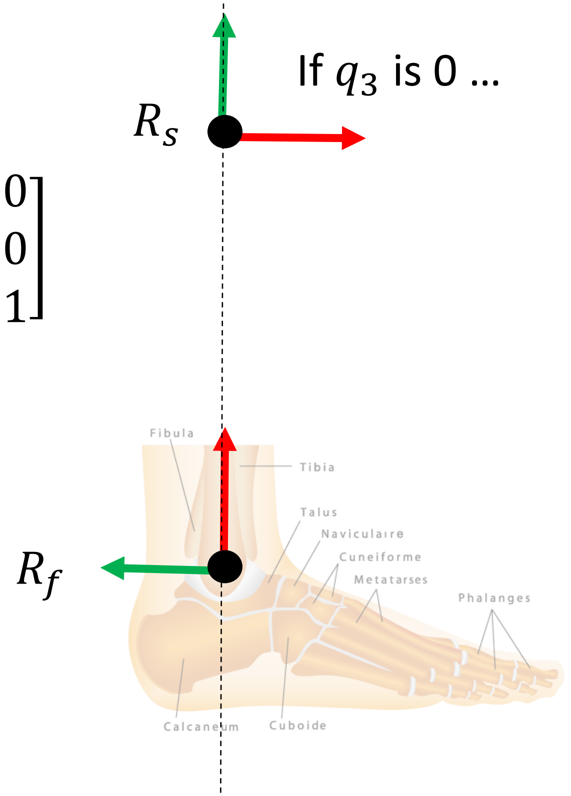


$$^{s}R_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

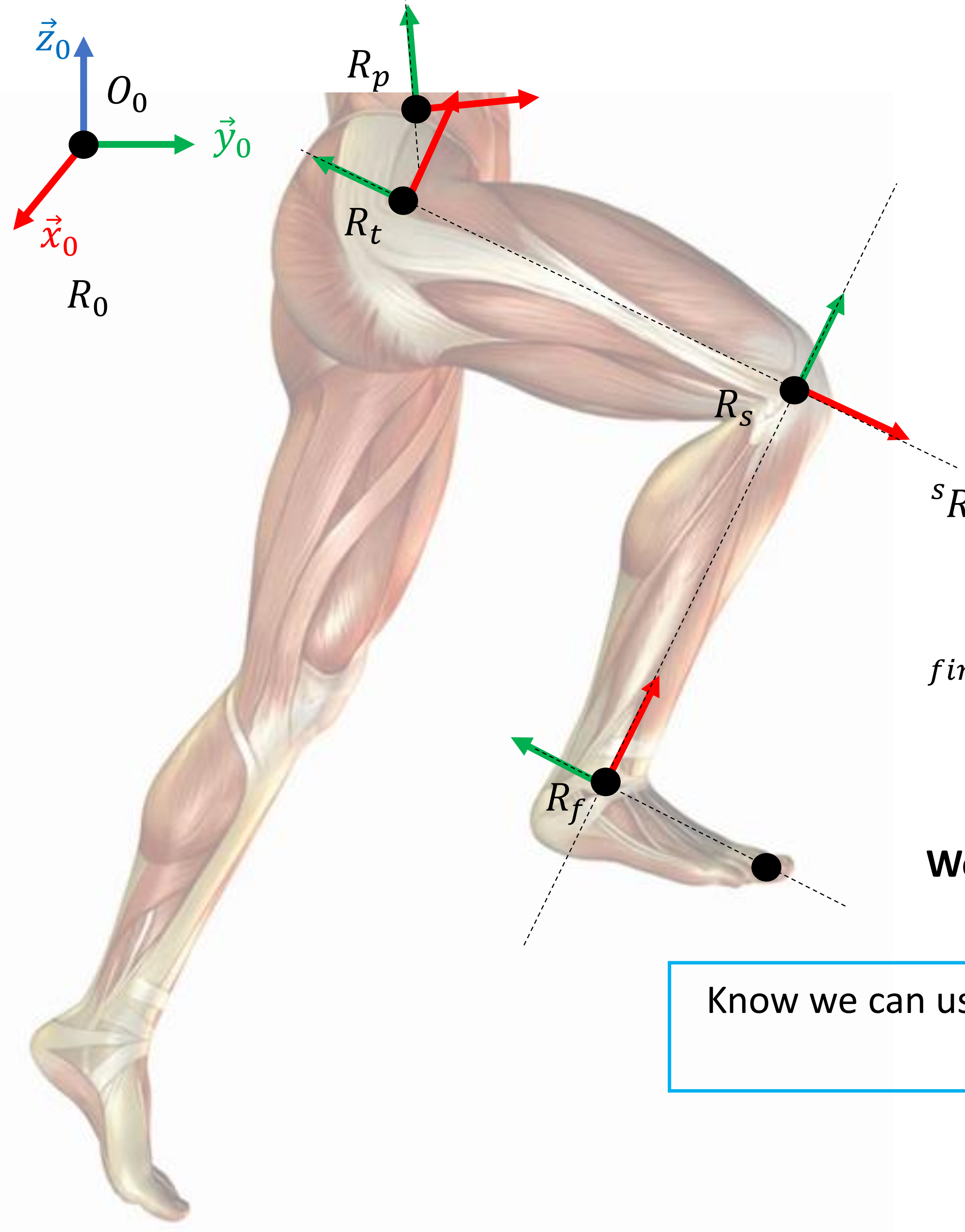


$$^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If q_3 is 0 ...



The anatomical position is ok



$$ {}^{p}R_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{t}R_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

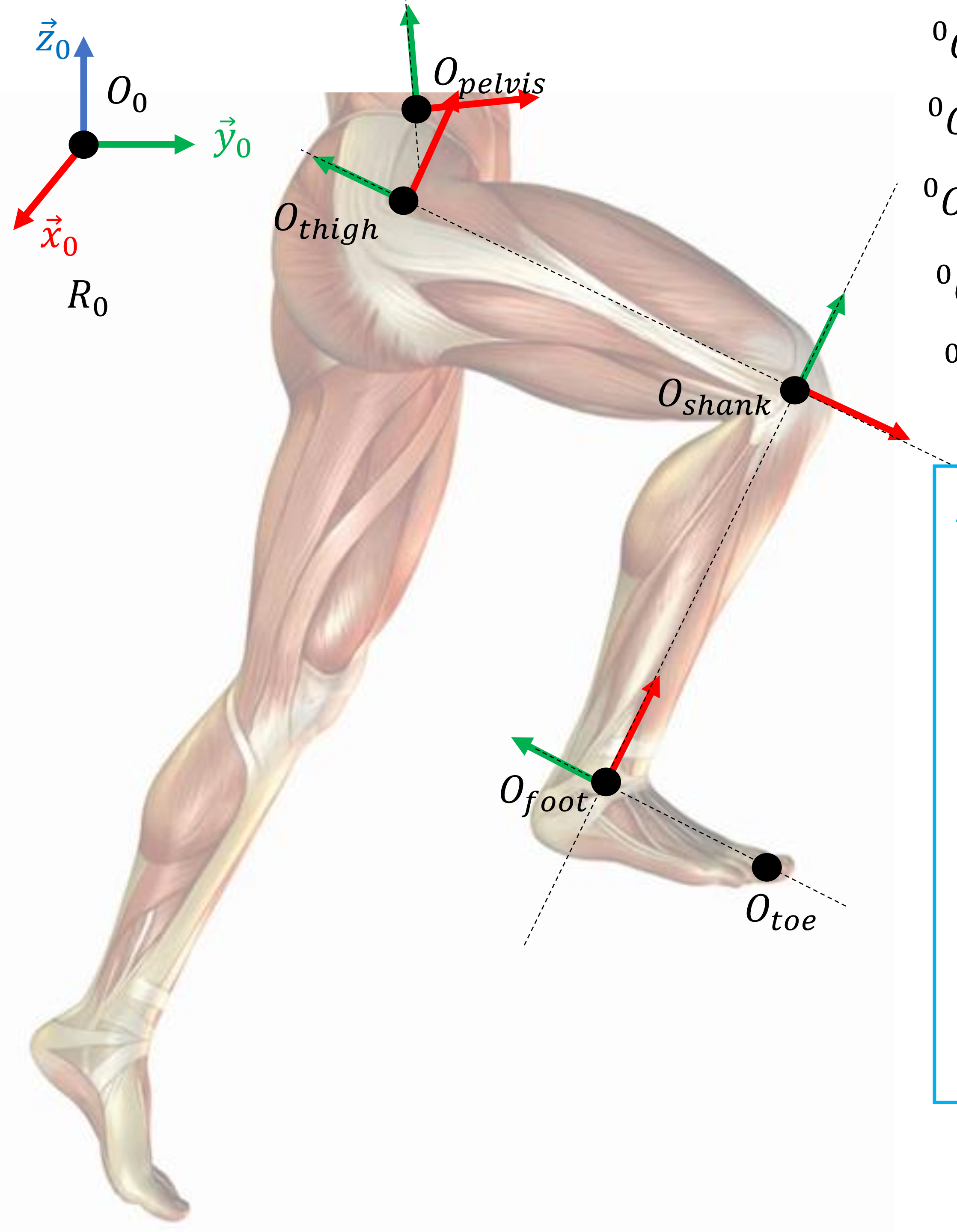
$$ {}^{s}R_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Well…ok

Know we can use that to compute forward and inverse kinematics

Forward Kinematics



$^0O_{pelvis}$ Origin of the pelvis in R_0 (assumed to be known)

$^0O_{thigh}$ Origin of the thigh in R_0 (expressed from ^0R_p)

$^0O_{shank}$ Origin of the shank in R_0 (expressed from $^0R_p\ , q_1$)

$^0O_{foot}$ Origin of the foot in R_0 (expressed from $^0R_p, q_1, q_2$)

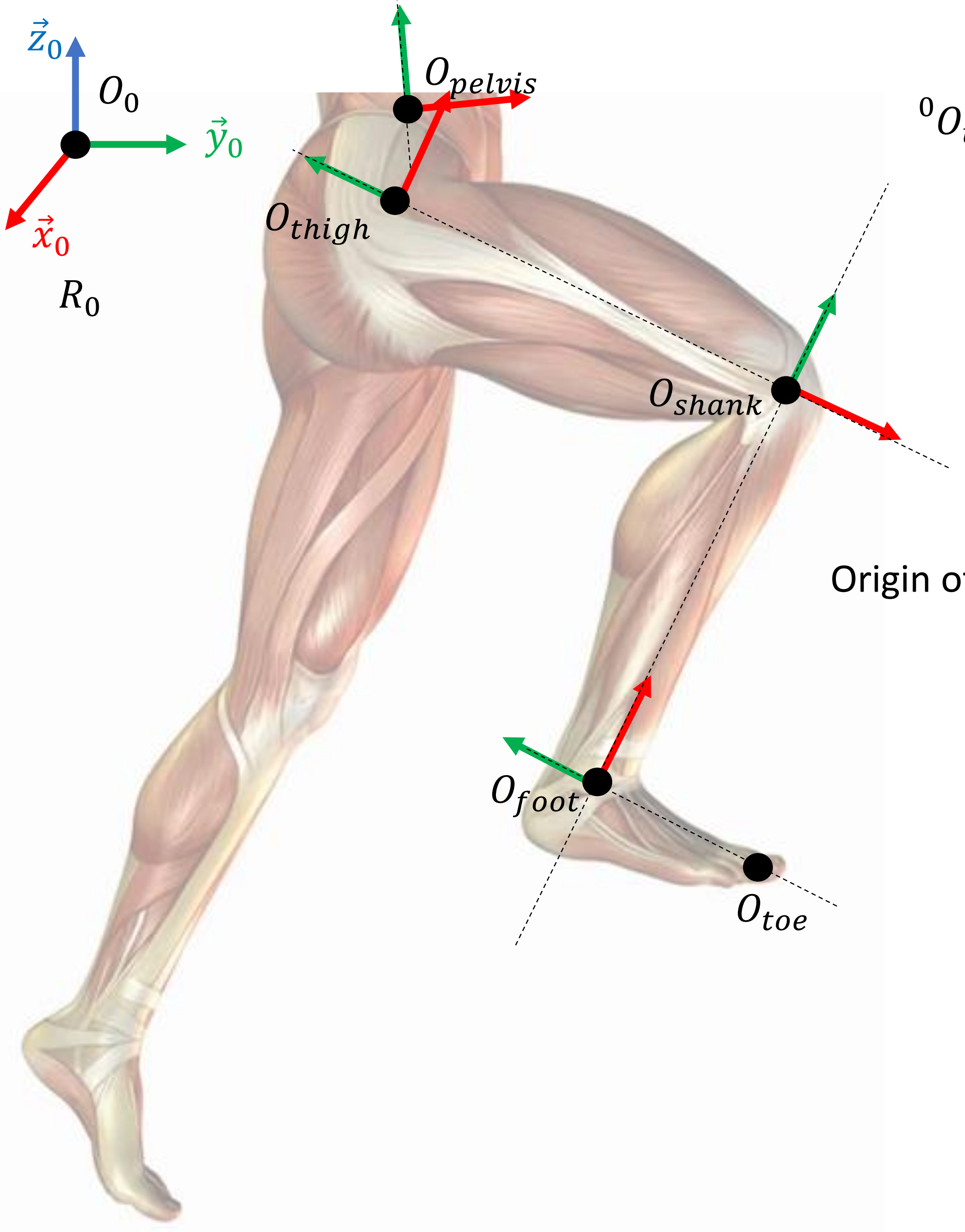
$^0O_{toe}$ Toe position in R_0 (expressed from $^0R_p, q_1, q_2, q_3$)

Assumptions

$$^0O_{pelvis} = \begin{bmatrix} x_{Op} \\ y_{Op} \\ 0 \end{bmatrix}$$ Assumed to be known

$$^0R_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Assumed to be known (for sake of simplicity, we assumed that this is the identity matrix in the following calculus)

We also assume to be known the position of the different joint centers on the segments



${}^0O_{thigh}$ Origin of the thigh in R_0 (expressed from ${}^0R_p$)

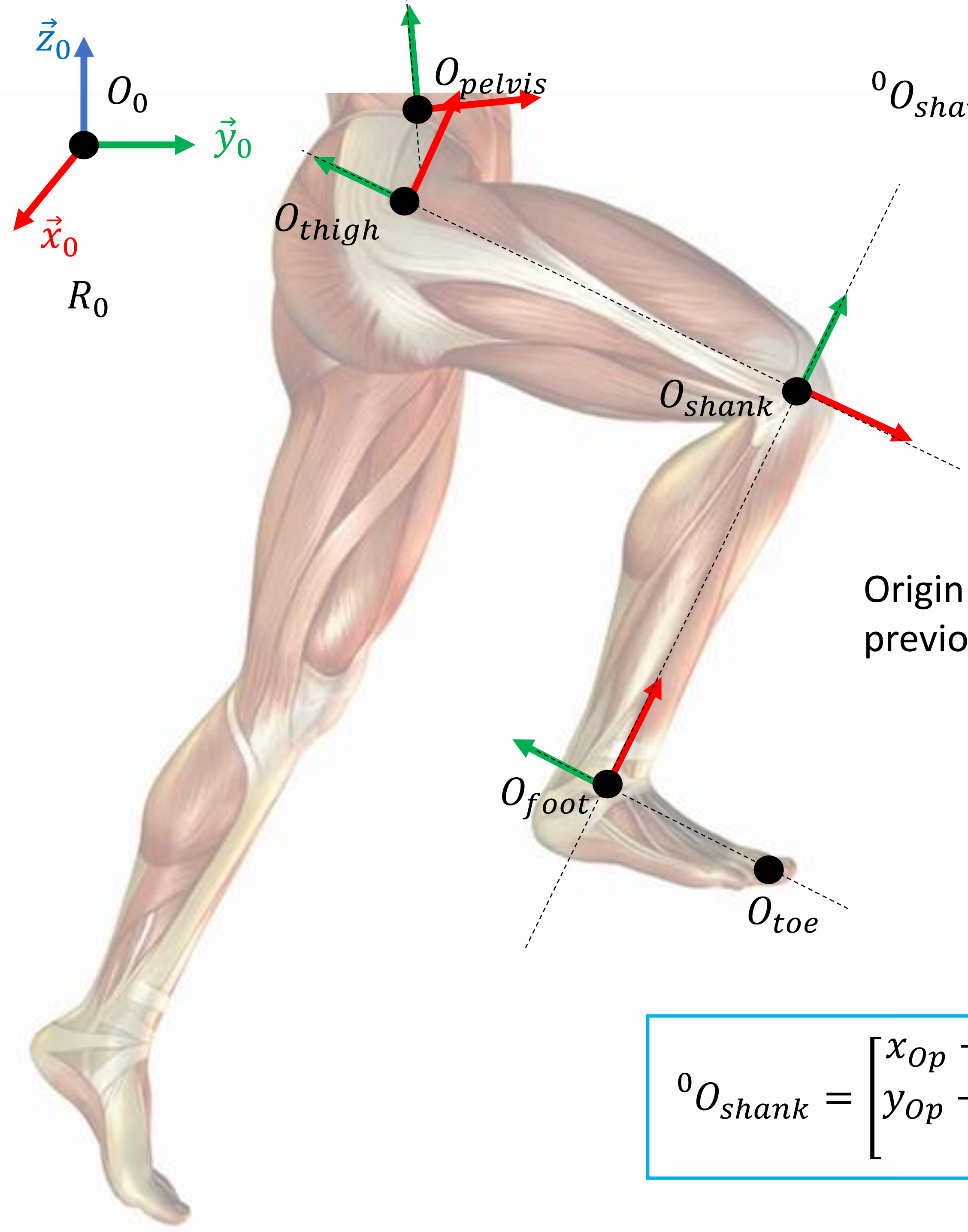
If we have ${}^pO_{thigh} = \begin{bmatrix} x_{Ot} \\ y_{Ot} \\ 0 \end{bmatrix}$ Origin of the thigh in R_p

then $${}^0O_{thigh} = {}^0O_{pelvis} + {}^0R_p\,{}^pO_{thigh}$$

Origin of the pelvis in R_0

Projecting the origin of the thigh in R_p in the basis B_0

$$\boxed{{}^0O_{thigh} = \begin{bmatrix} x_{Op} \\ y_{Op} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{Ot} \\ y_{Ot} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{Op} + x_{Ot} \\ y_{Op} + y_{Ot} \\ 0 \end{bmatrix}}$$



${}^0O_{shank}$ Origin of the shank in R_0 (expressed from ${}^0R_p$, q_1)

Let's consider ${}^tO_{shank} = \begin{bmatrix} 0 \\ y_{Os} \\ 0 \end{bmatrix}$ Origin of the shank in R_t

Alors $${}^0O_{shank} = {}^0O_{thigh} + {}^0R_t \, {}^tO_{shank}$$

Origin of the thigh in R_0 (see previous slide)

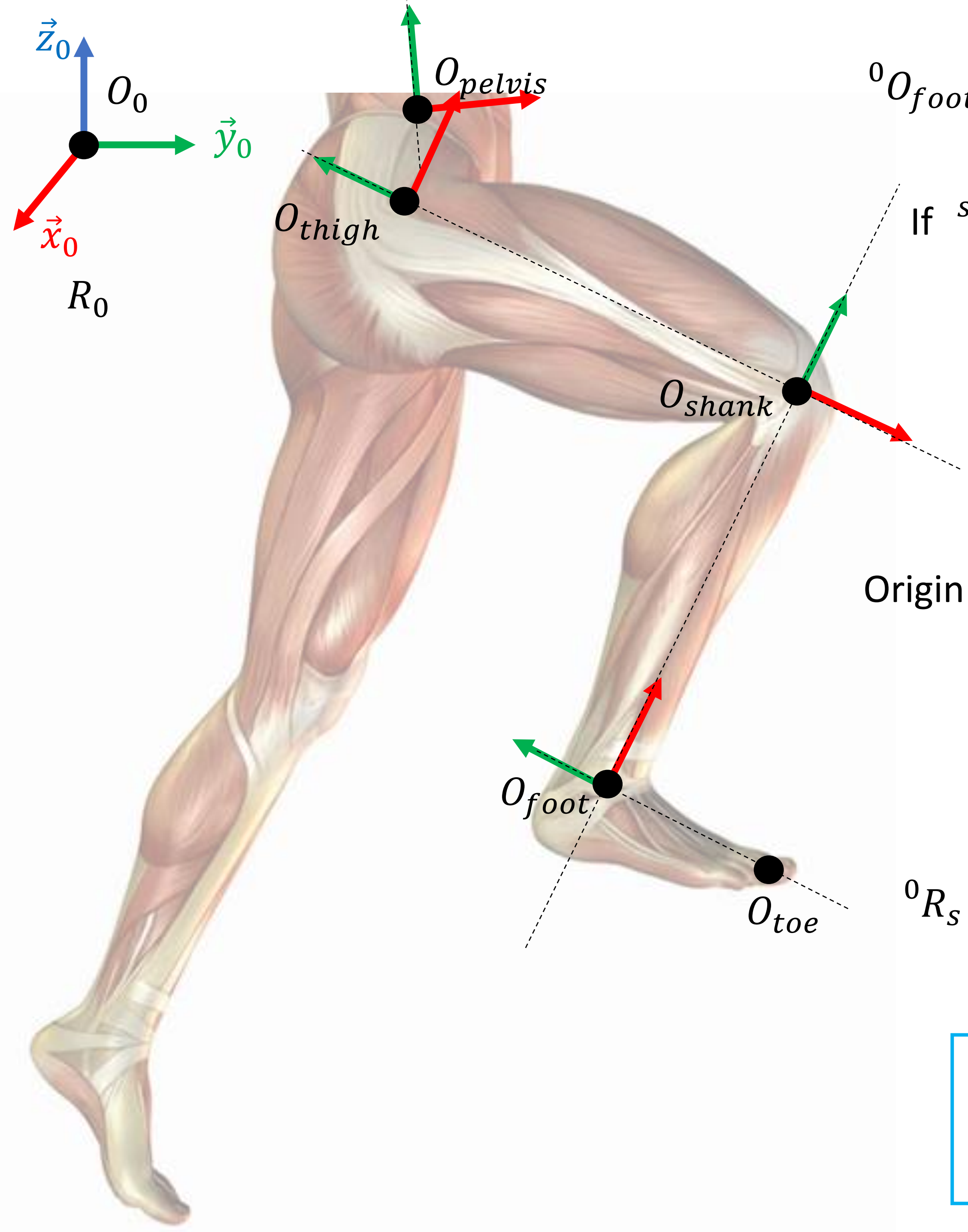
Projecting the origin of the shank in R_t in the basis B_0

$${}^0O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} \\ y_{Op} + y_{Ot} \\ 0 \end{bmatrix} + {}^0R_t \begin{bmatrix} 0 \\ y_{Os} \\ 0 \end{bmatrix}$$

$${}^0R_t = {}^0R_p \, {}^pR_t$$

$${}^0R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{{}^0O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1 \, y_{Os} \\ y_{Op} + y_{Ot} + \cos q_1 \, y_{Os} \\ 0 \end{bmatrix}}$$



${}^{0}O_{foot}$ Origin of the foot in R_0 (expressed from ${}^{0}R_p, q_1, q_2$)

If ${}^{s}O_{foot} = \begin{bmatrix} 0 \\ y_{Of} \\ 0 \end{bmatrix}$ Origin of the foot in R_s

Then $${}^{0}O_{foot} = {}^{0}O_{shank} + {}^{0}R_s \, {}^{s}O_{foot}$$

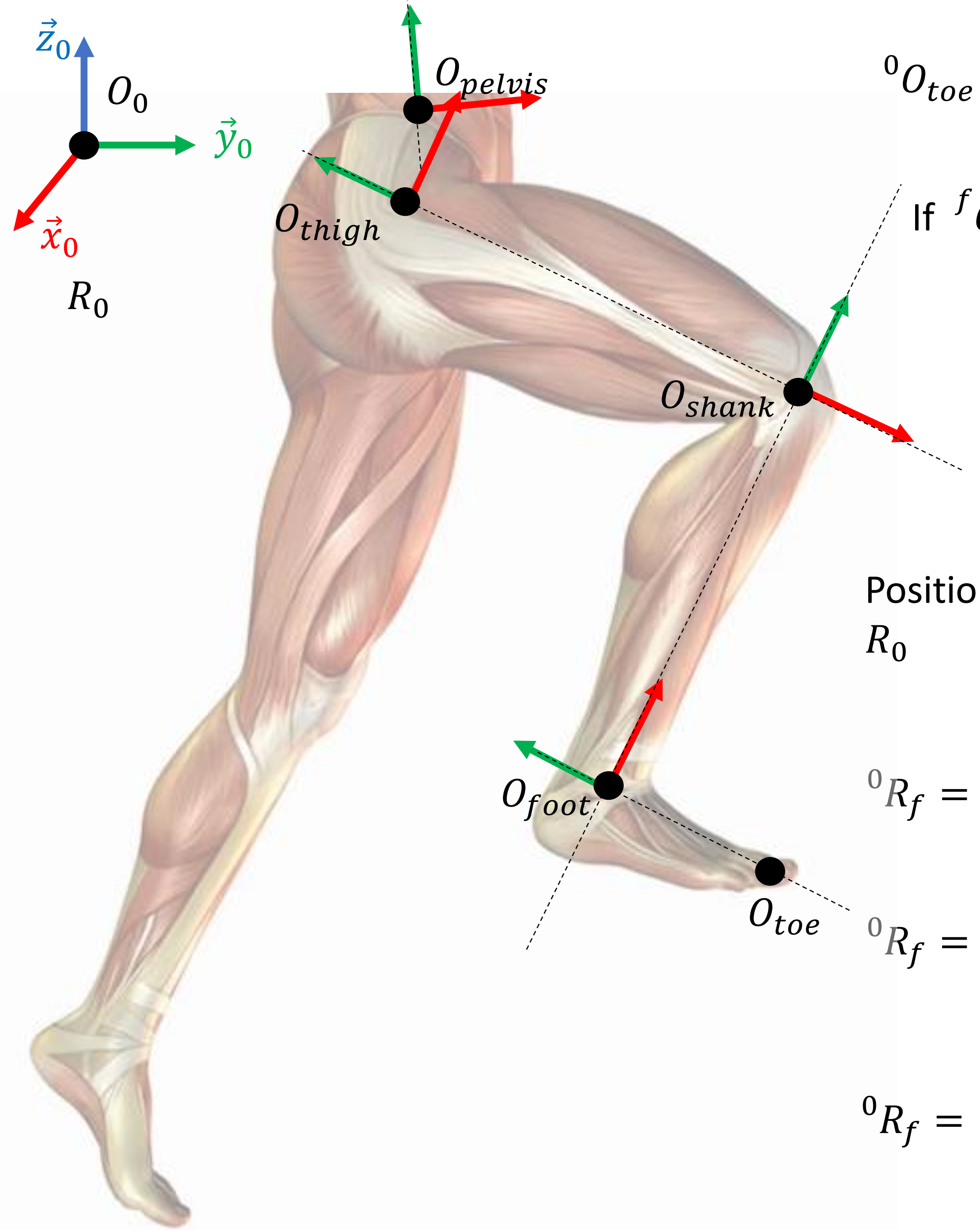
Origin of the shank in R_0

Projecting the position of the origin of the foot in R_s in the R_0 basis

$${}^{0}R_s = {}^{0}R_p \, {}^{p}R_t \, {}^{t}R_s$$

$${}^{0}R_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{{}^{0}O_{foot} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1 \, y_{Os} - \sin(q_1 + q_2) y_{Of} \\ y_{Op} + y_{Ot} + \cos q_1 \, y_{Os} - \cos(q_1 + q_2) y_{Of} \\ 0 \end{bmatrix}}$$



${}^0O_{toe}$ Position of the toes in R_0 (expressed from ${}^0R_p, q_1, q_2, q_3$)

If ${}^fO_{toe} = \begin{bmatrix} x_{Otoe} \\ y_{Otoe} \\ 0 \end{bmatrix}$ Position of the toes in R_f

then $${}^0O_{toe} = {}^0O_{foot} + {}^0R_f\,{}^fO_{toe}$$

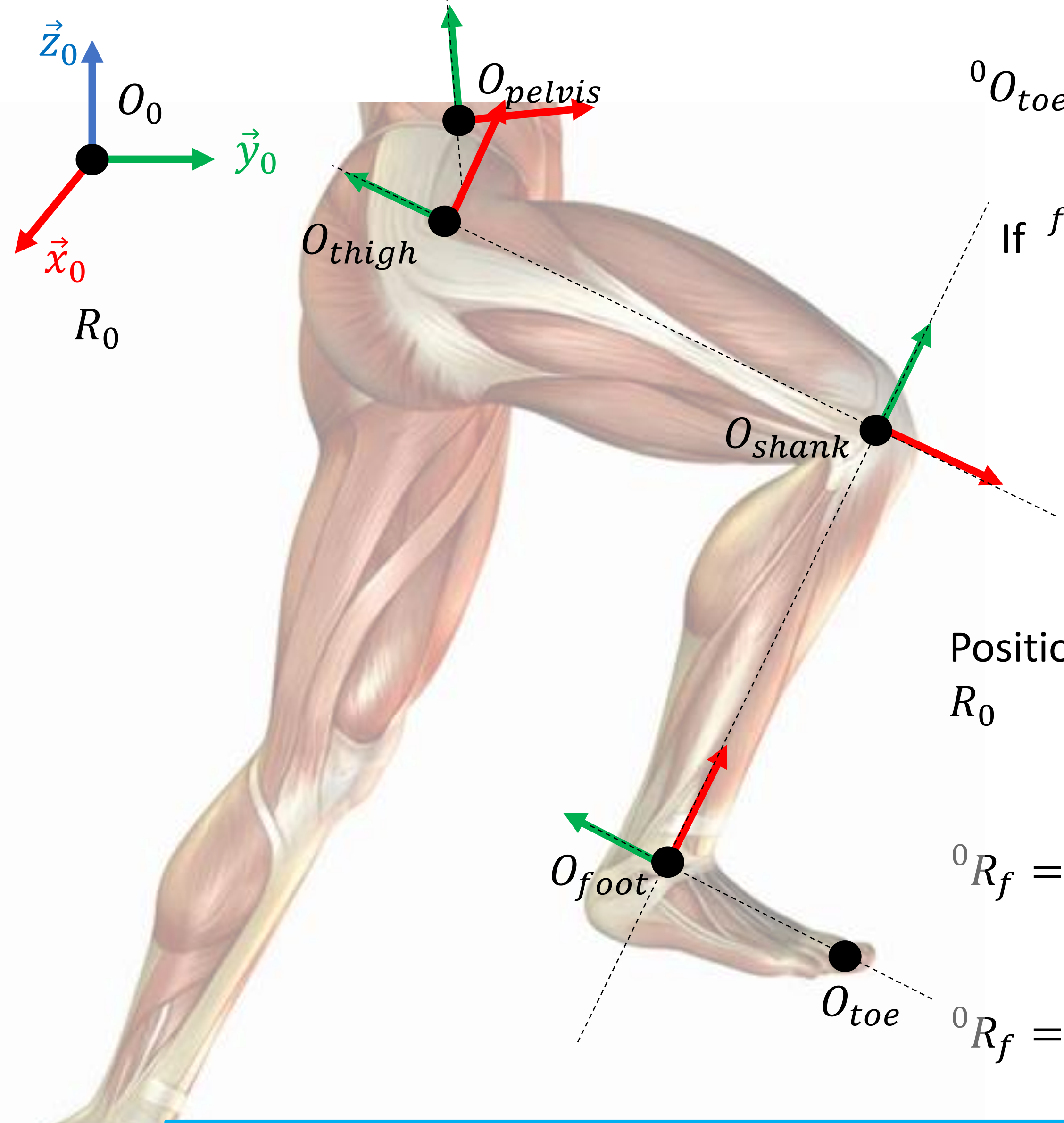
Position of the origin of the foot in R_0

Projecting the position of the toes in R_f in the basis B_0

$${}^0R_f = {}^0R_p\,{}^pR_t\,{}^tR_s\,{}^sR_{fint}\,{}^{fint}R_f$$

$${}^0R_f = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^0R_f = \begin{bmatrix} -\sin(q_1+q_2+q_3) & -\cos(q_1+q_2+q_3) & 0 \\ \cos(q_1+q_2+q_3) & -\sin(q_1+q_2+q_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



${}^0O_{toe}$ Position of the toes in R_0 (expressed from ${}^0R_p, q_1, q_2, q_3$)

If ${}^fO_{toe} = \begin{bmatrix} x_{Otoe} \\ y_{Otoe} \\ 0 \end{bmatrix}$ Position of the toes in R_f

then $${}^0O_{toe} = {}^0O_{foot} + {}^0R_f \, {}^fO_{toe}$$

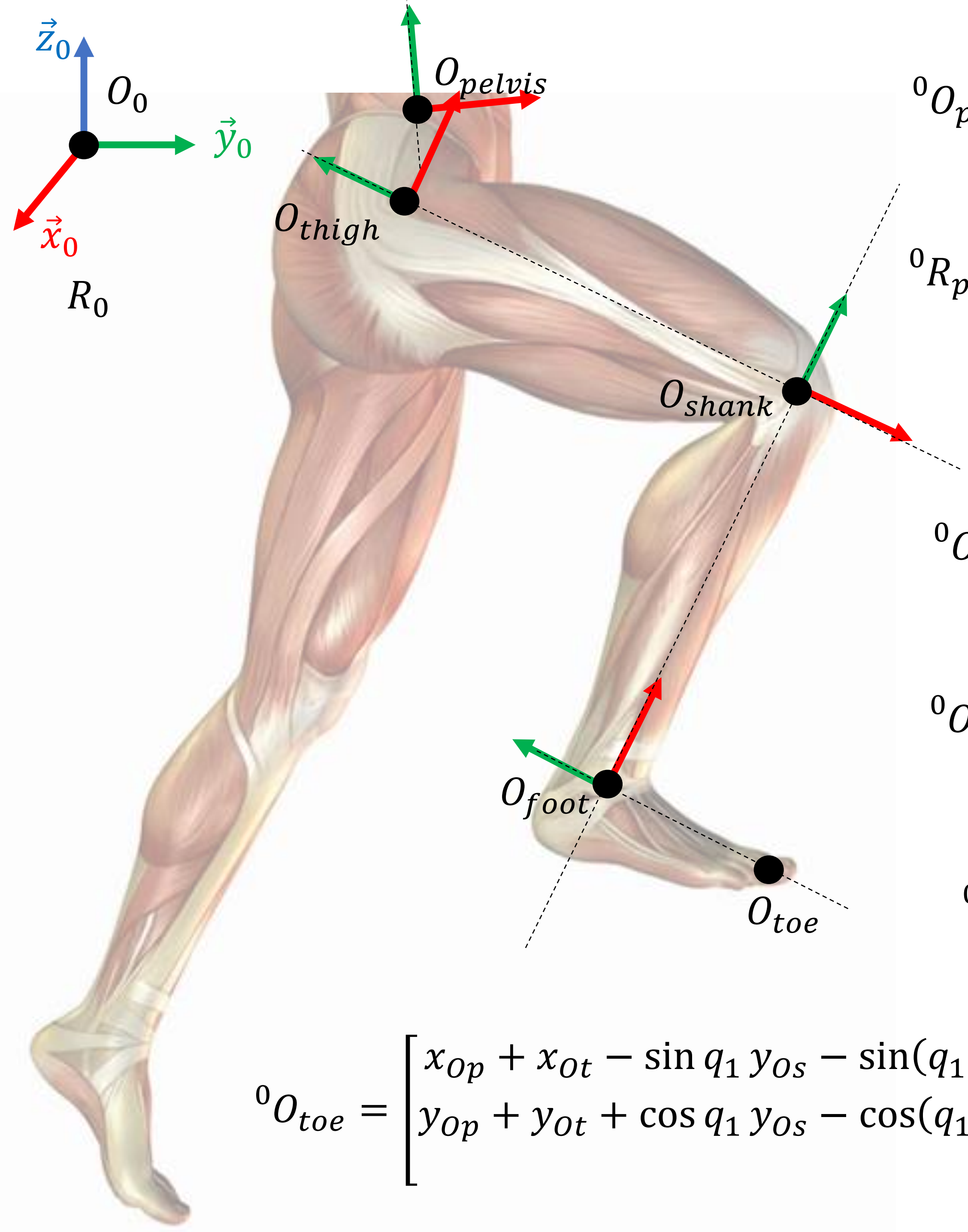
Position of the origin of the foot in R_0

Projecting the position of the toes in R_f in the basis B_0

$${}^0R_f = {}^0R_p \, {}^pR_t \, {}^tR_s \, {}^sR_{fint} \, {}^{fint}R_f$$

$${}^0R_f = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{{}^0O_{toe} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1 \, y_{Os} - \sin(q_1+q_2) y_{Of} - y_{Otoe} \cos(q_1+q_2+q_3) - x_{Otoe} \sin(q_1+q_2+q_3) \\ y_{Op} + y_{Ot} + \cos q_1 \, y_{Os} - \cos(q_1+q_2) y_{Of} + x_{Otoe} \cos(q_1+q_2+q_3) - y_{Otoe} \sin(q_1+q_2+q_3) \\ 0 \end{bmatrix}}$$



$$ {}^0O_{pelvis} = \begin{bmatrix} x_{Op} \\ y_{Op} \\ 0 \end{bmatrix} $$

$$ {}^0R_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0O_{thigh} = \begin{bmatrix} x_{Op} + x_{Ot} \\ y_{Op} + y_{Ot} \\ 0 \end{bmatrix} $$

$$ {}^0O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} \\ 0 \end{bmatrix} $$

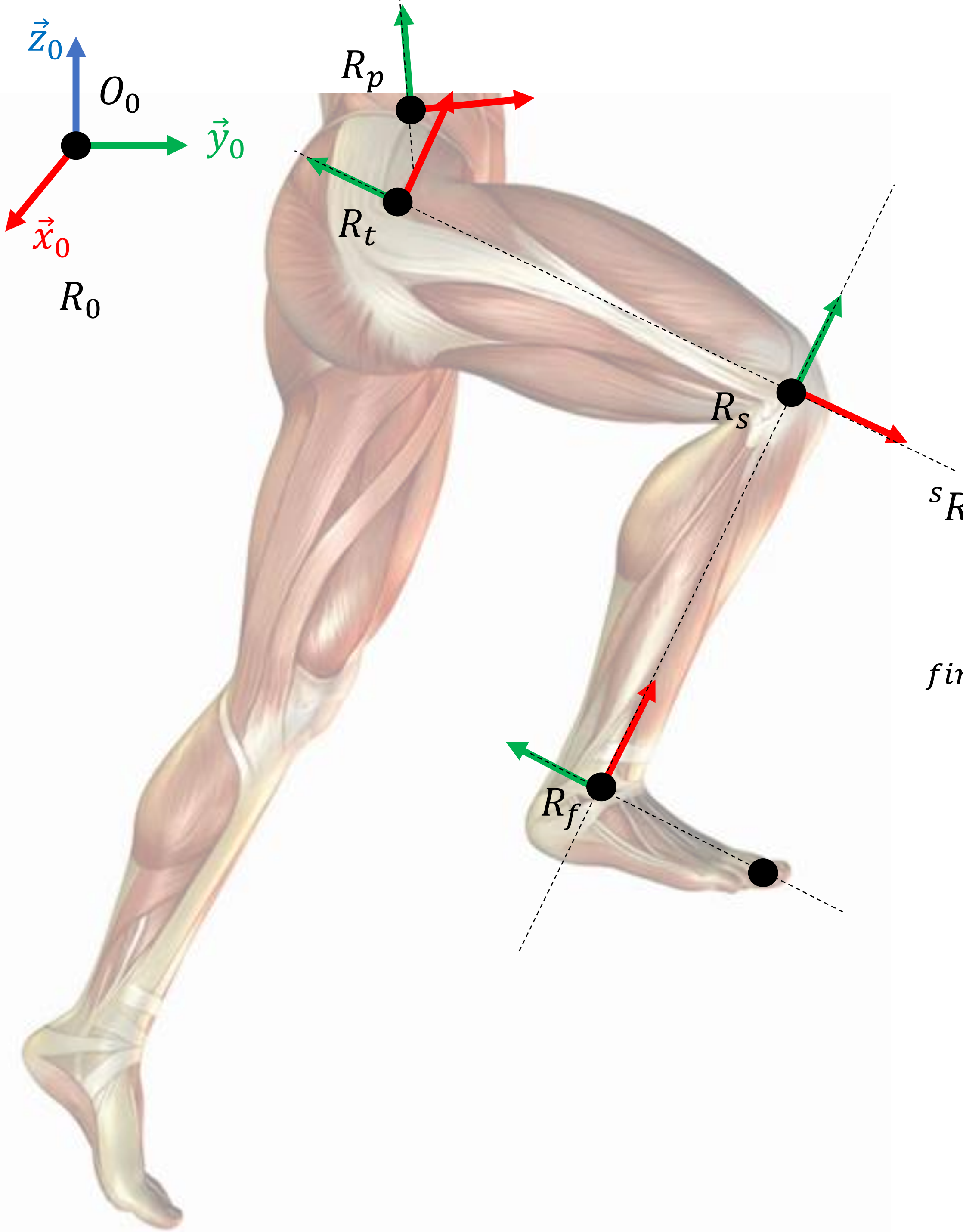
$$ {}^0O_{foot} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} - \sin(q_1 + q_2) y_{Of} \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} - \cos(q_1 + q_2) y_{Of} \\ 0 \end{bmatrix} $$

$$ {}^0O_{toe} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} - \sin(q_1 + q_2) y_{Of} - y_{Otoe} \cos(q_1 + q_2 + q_3) - x_{Otoe} \sin(q_1 + q_2 + q_3) \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} - \cos(q_1 + q_2) y_{Of} + x_{Otoe} \cos(q_1 + q_2 + q_3) - y_{Otoe} \sin(q_1 + q_2 + q_3) \\ 0 \end{bmatrix} $$

Next

- Implementing the model and kinematics into python
- Load and manipulate the data into pytho,
- Developing an inverse kinematics method to get hip (q_1), knee (q_2) et ankle (q_3) flexion angles against time
- See Lower_limb_IK.ipnyb

Inverse kinematics



$$^{p}R_t = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

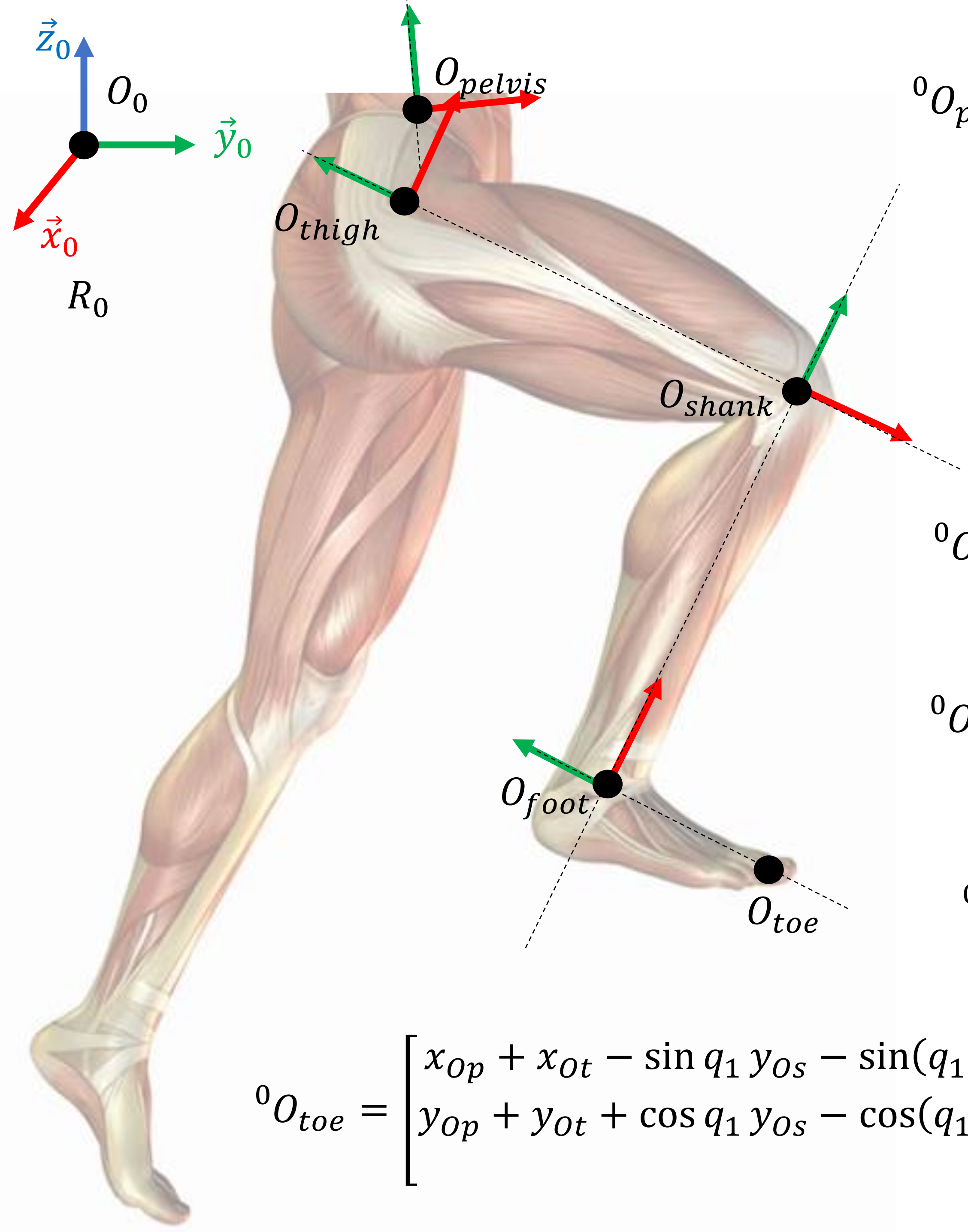
$$^{t}R_s = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{s}R_{fint} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{fint}R_f = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{0}R_p = \left[\ ^{0}\vec{x}_p \quad ^{0}\vec{y}_p \quad ^{0}\vec{z}_p \right] \longrightarrow$$

From the pelvis markers



$$^{0}O_{pelvis} = \begin{bmatrix} x_{Op} \\ y_{Op} \\ 0 \end{bmatrix}$$

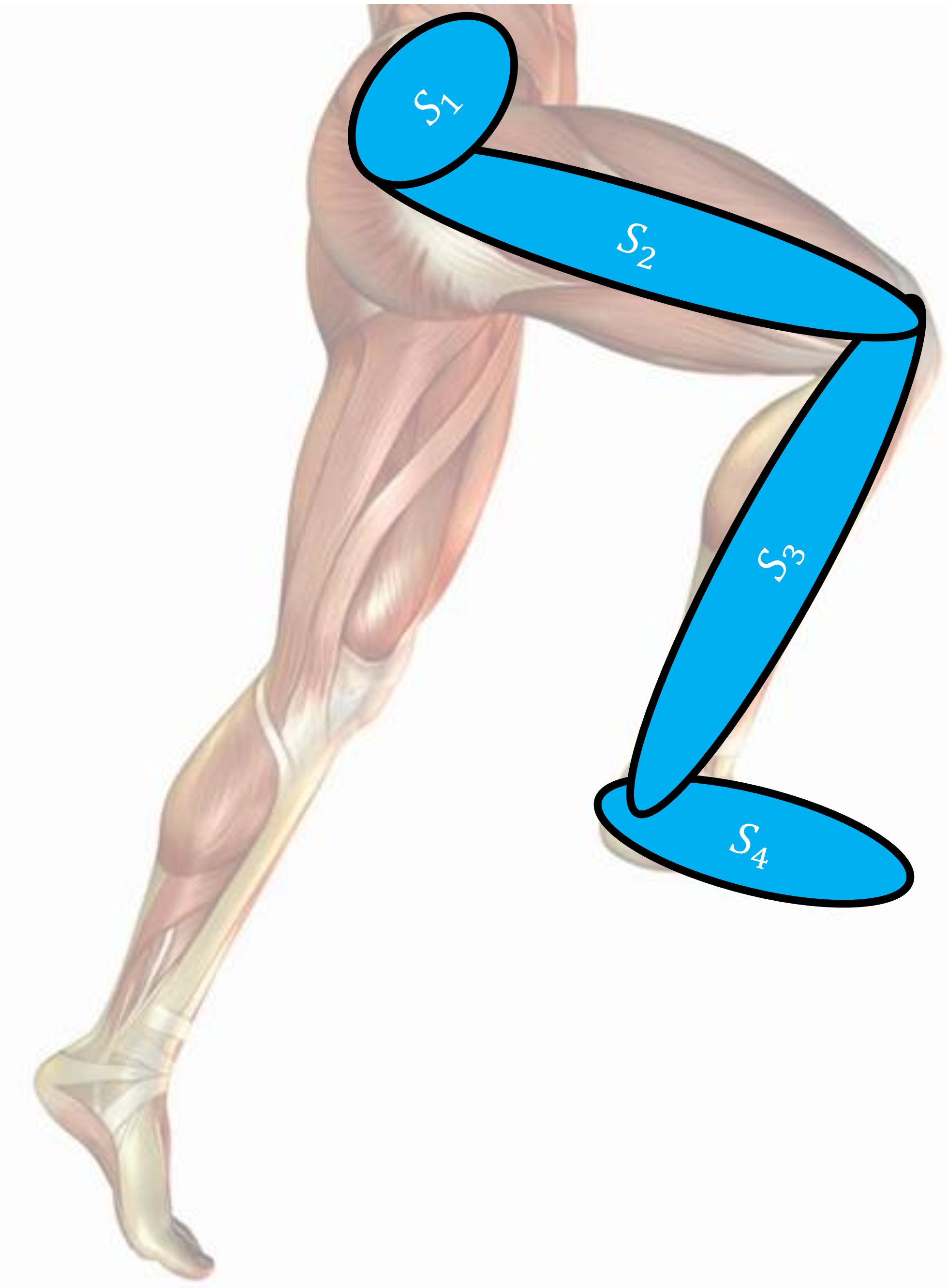
$$^{0}O_{thigh} = \begin{bmatrix} x_{Op} + x_{Ot} \\ y_{Op} + y_{Ot} \\ 0 \end{bmatrix}$$

$$^{0}O_{shank} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} \\ 0 \end{bmatrix}$$

$$^{0}O_{foot} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} - \sin(q_1 + q_2) y_{Of} \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} - \cos(q_1 + q_2) y_{Of} \\ 0 \end{bmatrix}$$

$$^{0}O_{toe} = \begin{bmatrix} x_{Op} + x_{Ot} - \sin q_1\, y_{Os} - \sin(q_1 + q_2) y_{Of} - y_{Otoe} \cos(q_1 + q_2 + q_3) - x_{Otoe} \sin(q_1 + q_2 + q_3) \\ y_{Op} + y_{Ot} + \cos q_1\, y_{Os} - \cos(q_1 + q_2) y_{Of} + x_{Otoe} \cos(q_1 + q_2 + q_3) - y_{Otoe} \sin(q_1 + q_2 + q_3) \\ 0 \end{bmatrix}$$

Inverse dynamics



Goal: get the flexion torques at hip, knee and ankle joints

Methods: isolate the segments one by one from bottom (foot) to the top (hip)

Assumption: we suppose know the kinematics !

Fundamental principle of dynamics

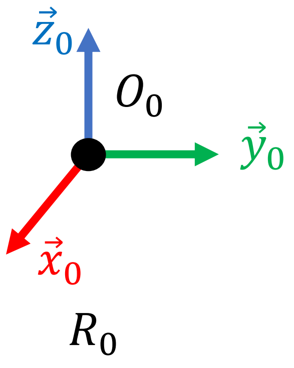
For a given solid S isolated, the dynamic wrench is equal to the external forces wrench

$$\boxed{\{D(S/R_0)\} = \{F(ext \rightarrow S)\}}$$

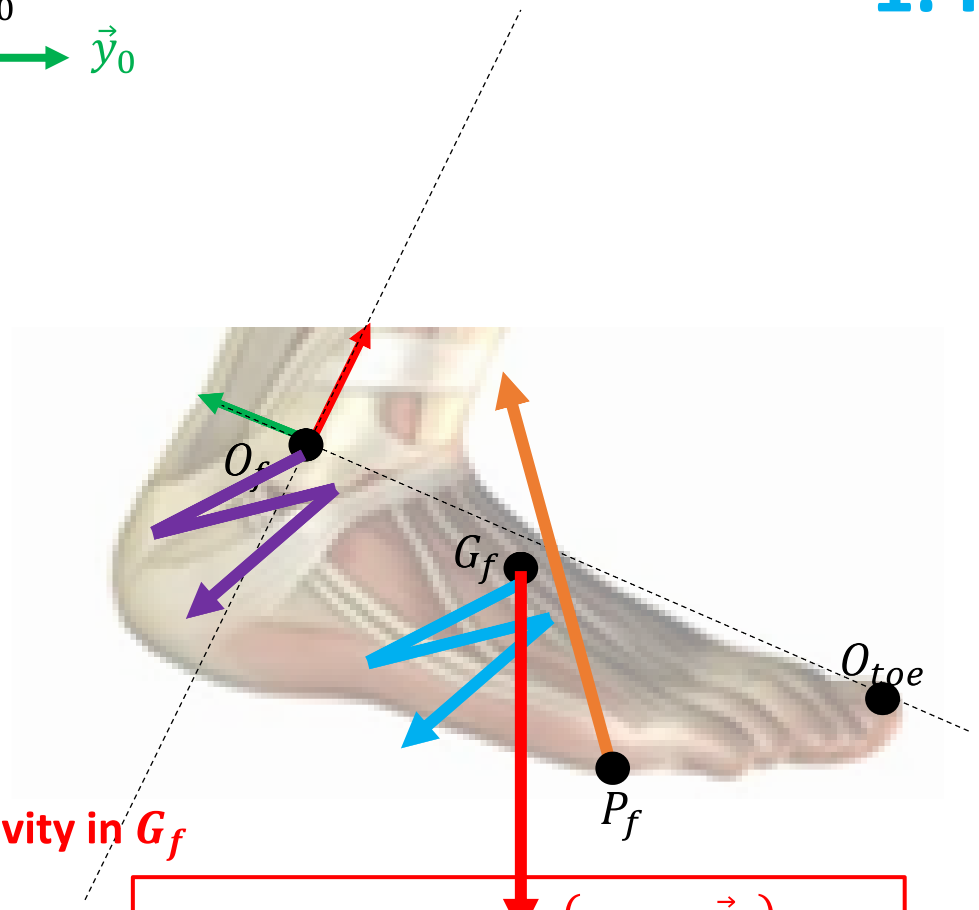
$$\boxed{\{D(S/R_0)\} = \begin{Bmatrix} m\vec{\Gamma}(G \in S/R_0) \\ \vec{\delta}(P \in S/R_0) \end{Bmatrix}_A}$$

$$\boxed{\{F(ext \rightarrow S)\} = \begin{Bmatrix} \sum_S \vec{F}(ext \rightarrow S/R_0) \\ \sum_S \vec{M}(P, F \rightarrow S/R_0) \end{Bmatrix}_P}$$

- Planar problem : 3 equations of motion(2 translations xy, 1 rotation z)
- $\vec{\Omega}(S_i/R_0) = \dot{\theta}_i \vec{z}_0$
- We should express all the quantities at a single point
- We should express all the quantities in the same basis



1. Foot S_4



Gravity in G_f

$$\{\tau(\bar{S}_4 \to S_4)_g\} = \begin{Bmatrix} -m_f g \vec{y}_0 \\ \vec{0} \end{Bmatrix}_{G_f}$$

External forces

Ground reaction forces measured by the force platforms at P_f

$$\{\tau(\bar{S}_4 \to S_4)\} = \begin{Bmatrix} F_x \vec{x}_0 + F_y \vec{y}_0 \\ \vec{0} \end{Bmatrix}_{P_f}$$

With $\overrightarrow{O_0 P_f} = x_{p_f} \vec{x}_0 + y_{p_f} \vec{y}_0$

(Known)

Shank action on the foot O_f

$$\{\tau(S_3 \to S_4)\} = \begin{Bmatrix} R_{f_x} \vec{x}_f + R_{f_y} \vec{y}_f \\ \Gamma_f \vec{z}_0 \end{Bmatrix}_{O_f}$$

From muscle action ! (unknown)

With $\overrightarrow{O_0 O_f} = x_{o_f} \vec{x}_0 + y_{o_f} \vec{y}_0$

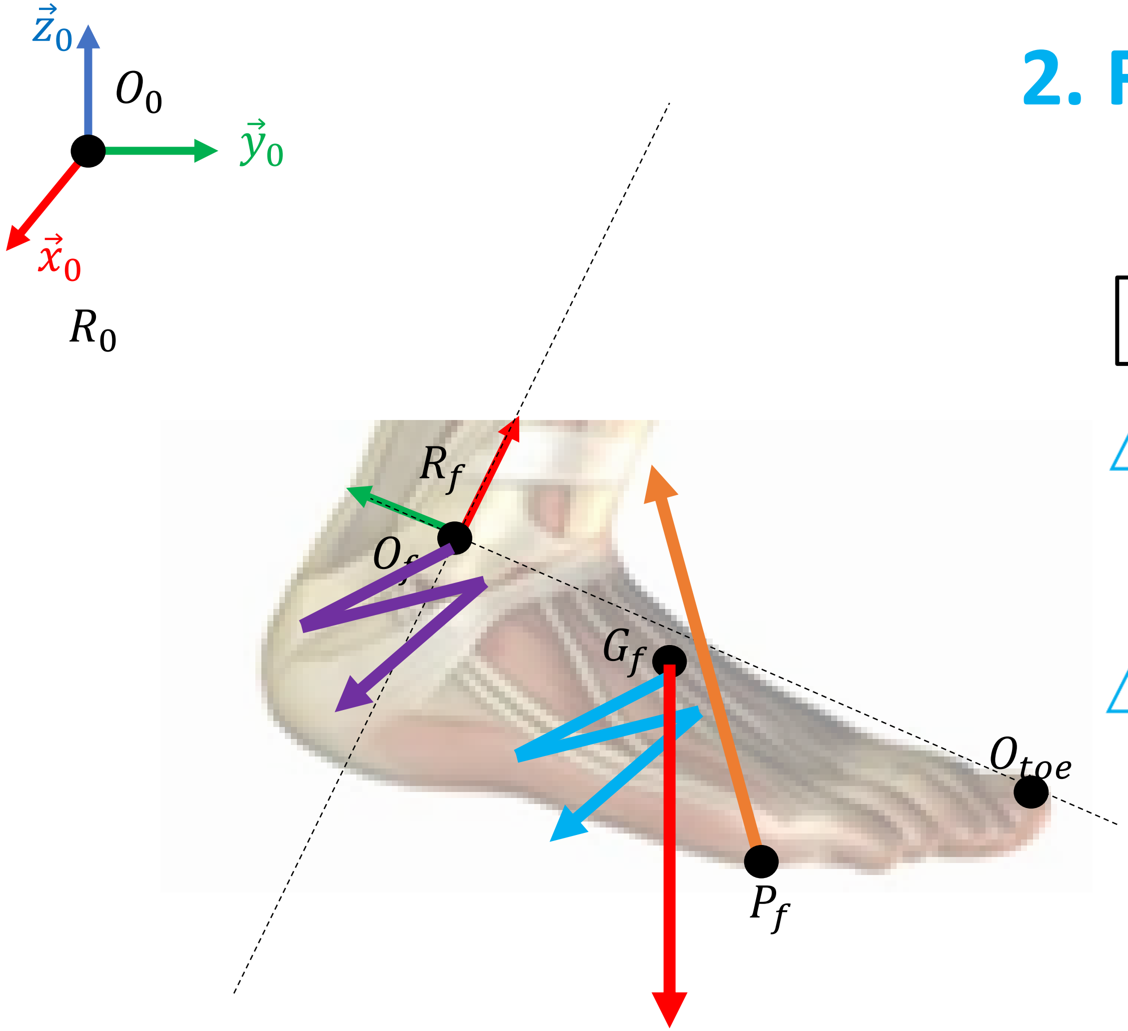
Acceleration quantities (dynamic wrench)

$$\{D(S_4/R_0)\} = \begin{Bmatrix} m_f \vec{\Gamma}(G_f \in S_4/R_0) \\ I_f \ddot{q}_f \vec{z}_0 \end{Bmatrix}_{G_f}$$

The motion is supposed to be known

$\vec{\Gamma}(G_f \in S_4/R_0) = \dfrac{d^2 \overrightarrow{O_0 G_f}}{dt^2}$ (known)

$\ddot{q}_f$ (knwon)



2. Foot equilibrium S_4

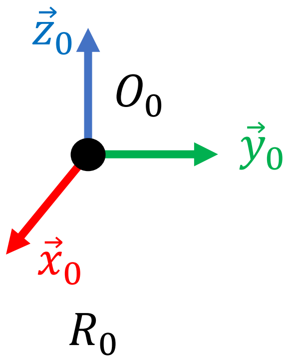
$$\boxed{\{\tau(\bar{S}_4 \rightarrow S_4)_g\} + \{\tau(\bar{S}_4 \rightarrow S_4)\} + \{\tau(S_3 \rightarrow S_4)\} = \{D(S_4/\,R_0)\}}$$

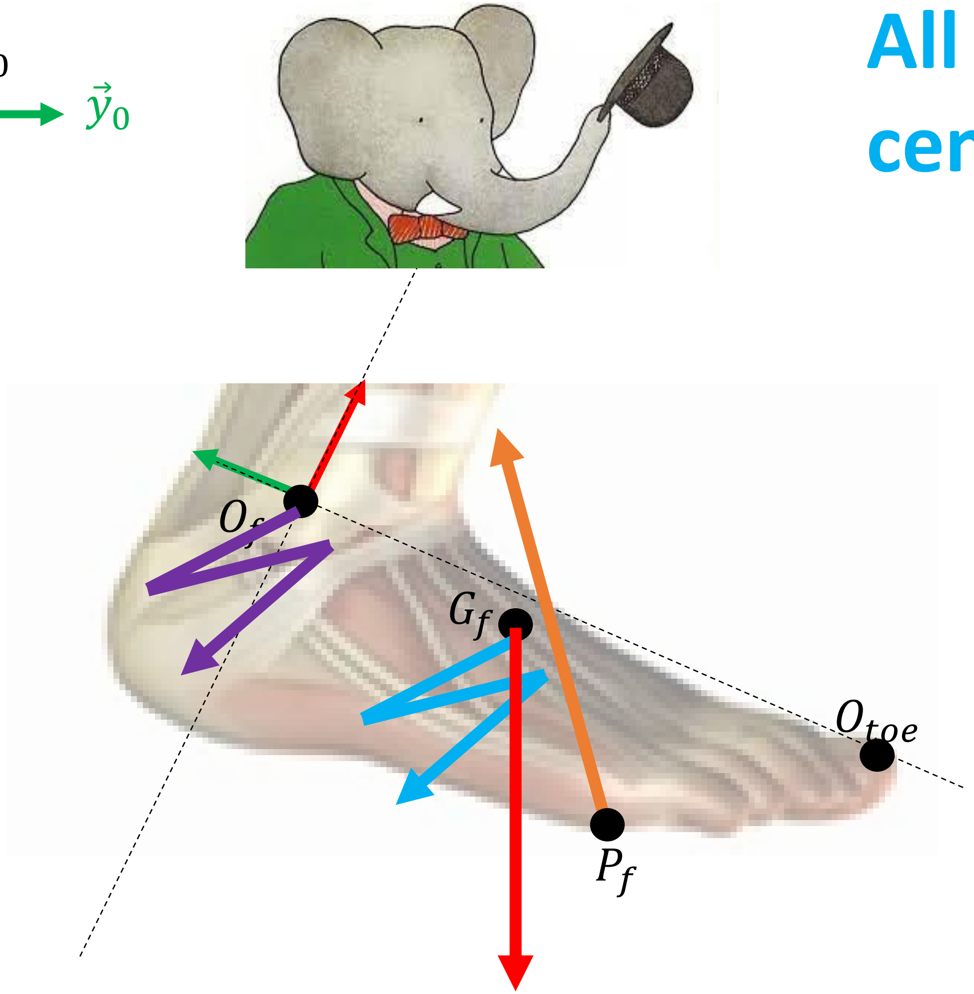
The point of application should be the same

All moments expressed at the ankle joint center O_f

The quantities should be expressed in the same basis

All in B_f (using ${}^{0}R_f$)





All moments expressed at the ankle joint center O_f

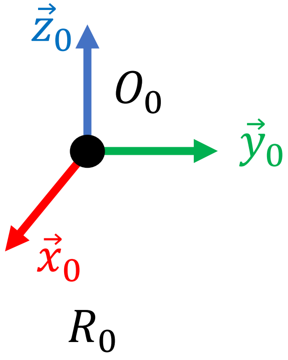
$$\{\tau(\bar{S}_4 \to S_4)\} = \begin{Bmatrix} F_x\vec{x}_0 + F_y\vec{y}_0 \\ \vec{0} \end{Bmatrix}_{P_f} = \begin{Bmatrix} F_x\vec{x}_0 + F_y\vec{y}_0 \\ \overrightarrow{O_fP_f} \wedge (F_x\vec{x}_0 + F_y\vec{y}_0) \end{Bmatrix}_{O_f}$$

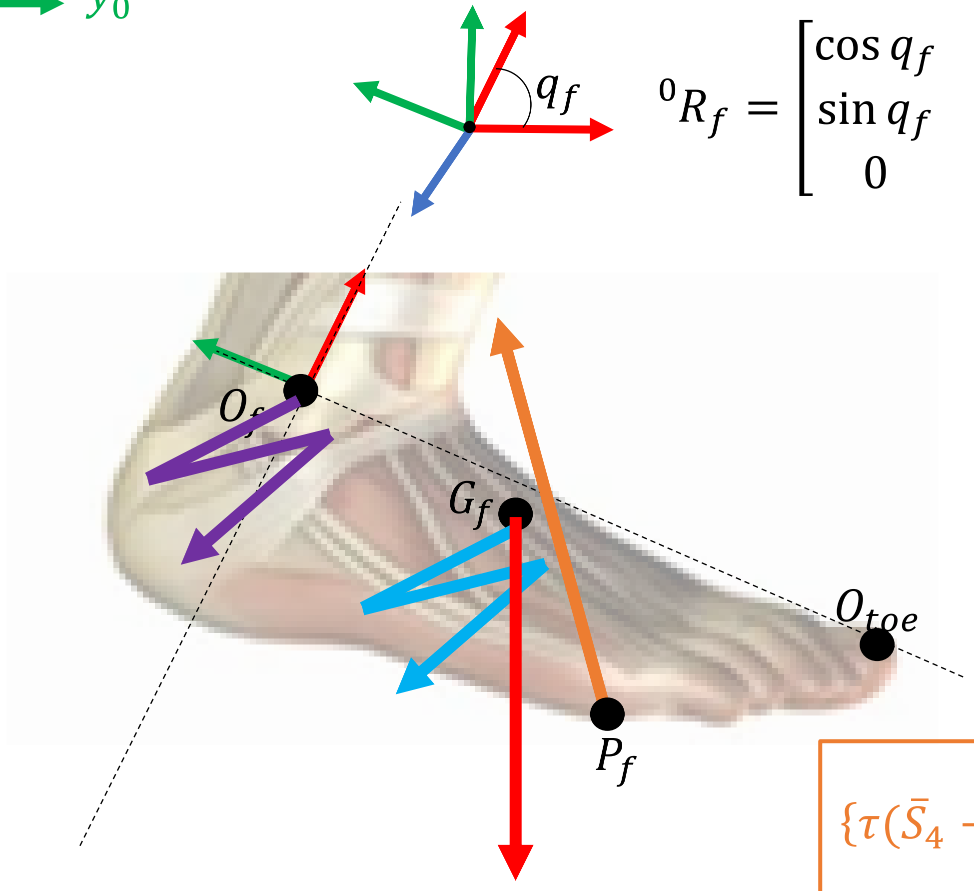
$$\{\tau(S_3 \to S_4)\} = \begin{Bmatrix} R_{f_x}\vec{x}_f + R_{f_y}\vec{y}_f \\ \Gamma_f\vec{z}_0 \end{Bmatrix}_{O_f}$$

$$\{\tau(\bar{S}_4 \to S_4)_g\} = \begin{Bmatrix} -m_f g\vec{y}_0 \\ \vec{0} \end{Bmatrix}_{G_f} = \begin{Bmatrix} -m_f g\vec{y}_0 \\ \overrightarrow{O_fG_f} \wedge (-m_f g\vec{y}_0) \end{Bmatrix}_{O_f}$$

$$\{D(S_4/R_0)\} = \begin{Bmatrix} m_f\vec{\Gamma}(G_f \in S_4/R_0) \\ I_f\ddot{q}_f\vec{z}_0 \end{Bmatrix}_{G_f} = \begin{Bmatrix} m_f\vec{\Gamma}(G_f \in S_4/R_0) \\ I_f\ddot{q}_f\vec{z}_0 + \overrightarrow{O_fG_f} \wedge m_f\vec{\Gamma}(G_f \in S_4/R_0) \end{Bmatrix}_{O_f}$$

3. Projecting in B_f





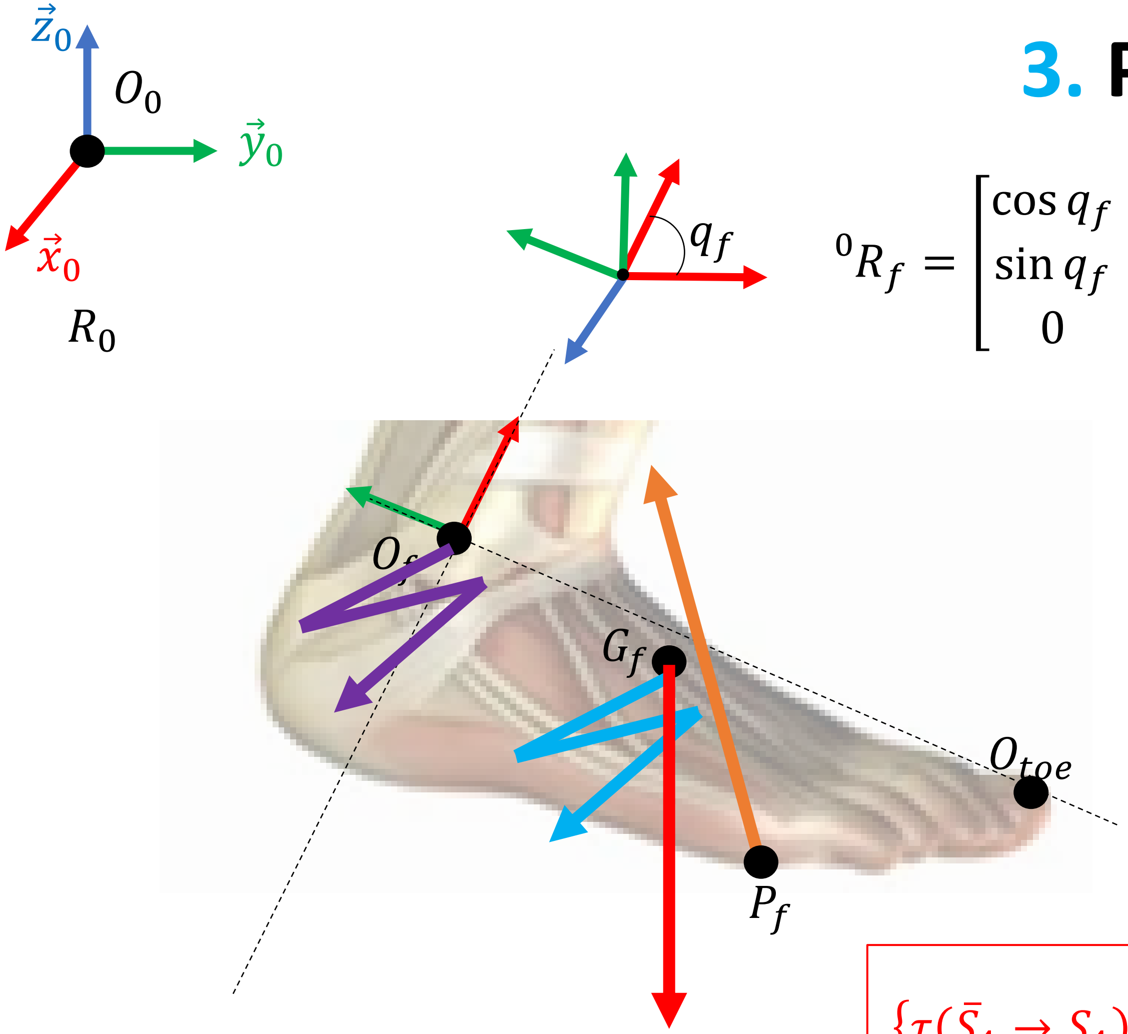
$${}^{0}R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{f}R_0 = {}^{0}R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\tau(\bar{S}_4 \rightarrow S_4)\} = \begin{Bmatrix} F_x\vec{x}_0 + F_y\vec{y}_0 \\ \overrightarrow{O_fP_f} \wedge (F_x\vec{x}_0 + F_y\vec{y}_0) \end{Bmatrix}_{O_f}$$

$${}^{0}F = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} \Rightarrow {}^{f}F = {}^{f}R_0\,{}^{0}F = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

$$\{\tau(\bar{S}_4 \rightarrow S_4)\} = \begin{Bmatrix} (F_x \cos q_f + F_y \sin q_f)\vec{x}_f + (-F_x \sin q_f + F_y \cos q_f)\vec{y}_f \\ \overrightarrow{O_fP_f} \wedge (F_x\vec{x}_0 + F_y\vec{y}_0) \end{Bmatrix}_{O_f}$$

The moment is already expressed around $\vec{z}_0$

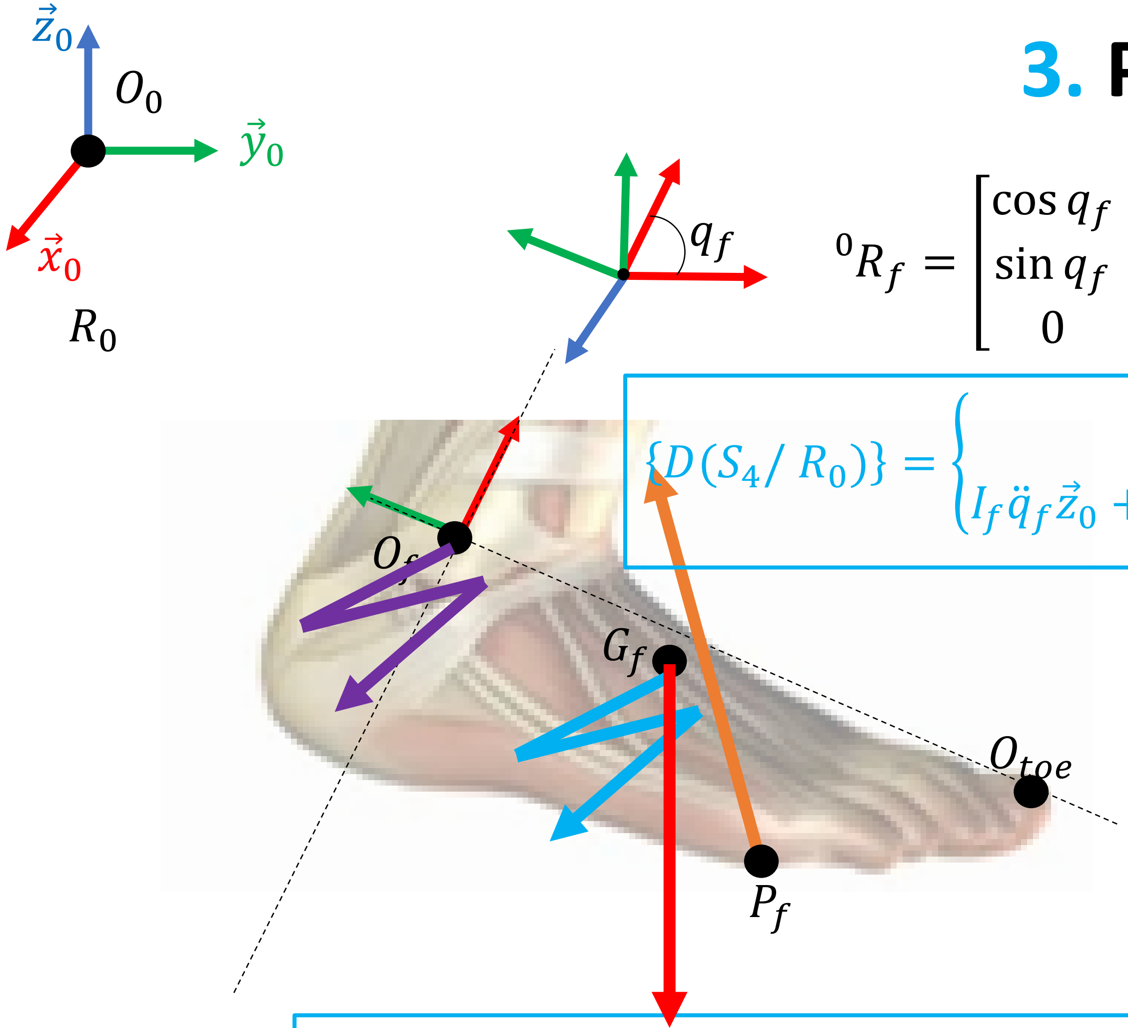


3. Projecting in B_f

$$^{0}R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad {}^{f}R_0 = {}^{0}R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\{\tau(\bar{S}_4 \to S_4)_g\} = \begin{Bmatrix} -m_f g \vec{y}_0 \\ \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0) \end{Bmatrix}_{O_f} = \begin{Bmatrix} -m_f g(\sin q_f \, \vec{x}_f + \cos q_f \, \vec{y}_f) \\ \overrightarrow{O_f G_f} \wedge (-m_f g \vec{y}_0) \end{Bmatrix}_{O_f}}$$

3. Projecting in B_f

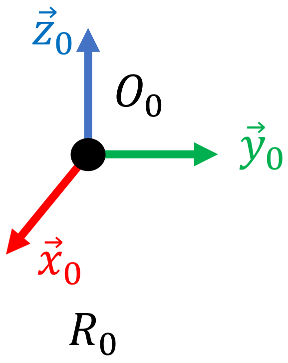


$$
{}^0R_f = \begin{bmatrix} \cos q_f & -\sin q_f & 0 \\ \sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^fR_0 = {}^0R_f^t = \begin{bmatrix} \cos q_f & \sin q_f & 0 \\ -\sin q_f & \cos q_f & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

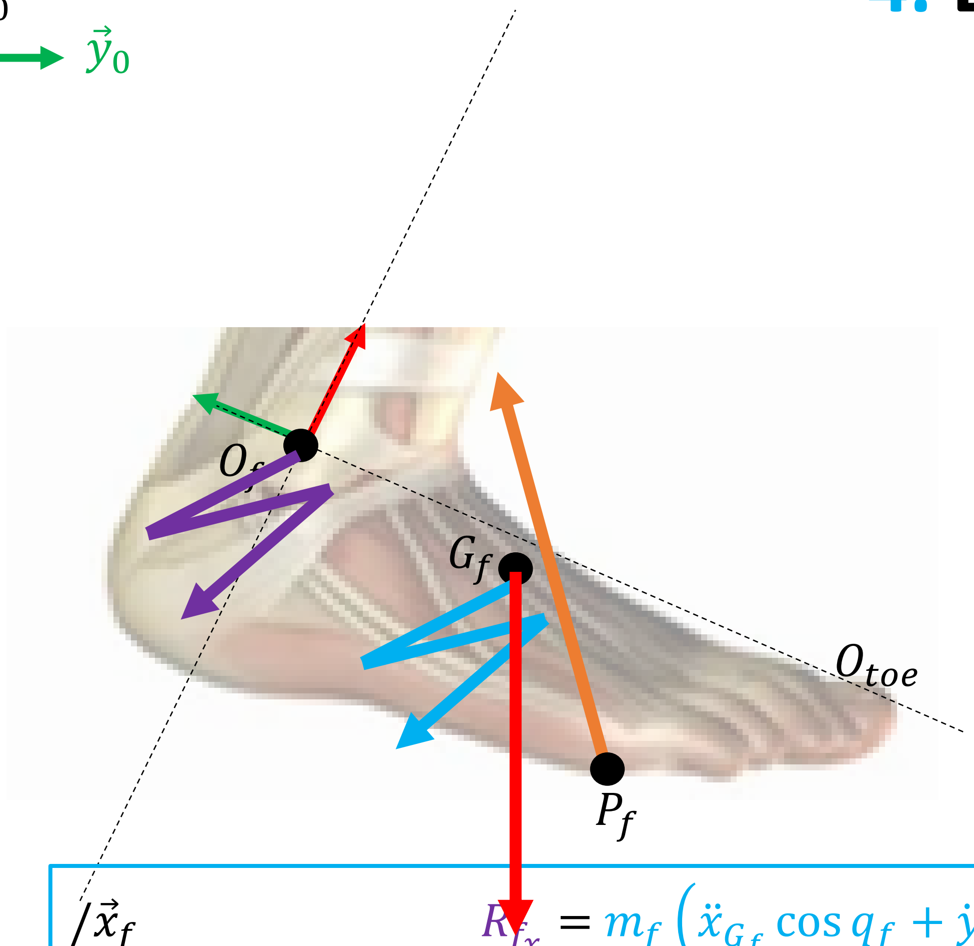
$$
\{D(S_4/\,R_0)\} = \begin{Bmatrix} m_f\vec{\Gamma}(G_f \in S_4/R_0) \\ I_f\ddot{q}_f\vec{z}_0 + \overrightarrow{O_fG_f}\wedge m_f\vec{\Gamma}(G_f \in S_4/R_0) \end{Bmatrix}_{O_f} = \begin{Bmatrix} m_f(\ddot{x}_{G_f}\vec{x}_0 + \ddot{y}_{G_f}\vec{y}_0) \\ I_f\ddot{q}_f\vec{z}_0 + \overrightarrow{O_fG_f}\wedge m_f\vec{\Gamma}(G_f \in S_4/R_0) \end{Bmatrix}_{O_f}
$$

Same principle

$$
\{D(S_4/\,R_0)\} = \begin{Bmatrix} m_f\left(\left(\ddot{x}_{G_f}\cos q_f + \ddot{y}_{G_f}\sin q_f\right)\vec{x}_f + \left(-\ddot{x}_{G_f}\sin q_f + \ddot{y}_{G_f}\cos q_f\right)\vec{y}_f\right) \\ I_f\ddot{q}_f\vec{z}_0 + \overrightarrow{O_fG_f}\wedge m_f\vec{\Gamma}(G_f \in S_4/R_0) \end{Bmatrix}_{O_f}
$$



4. Equilibrium



$$\boxed{\{\tau(\bar{S}_4 \rightarrow S_4)_g\} + \{\tau(\bar{S}_4 \rightarrow S_4)\} + \{\tau(S_3 \rightarrow S_4)\} = \{D(S_4/\,R_0)\}}$$

$$\boxed{\{\tau(S_3 \rightarrow S_4)\} = \{D(S_4/\,R_0)\} - \{\tau(\bar{S}_4 \rightarrow S_4)\} - \{\tau(\bar{S}_4 \rightarrow S_4)_g\}}$$

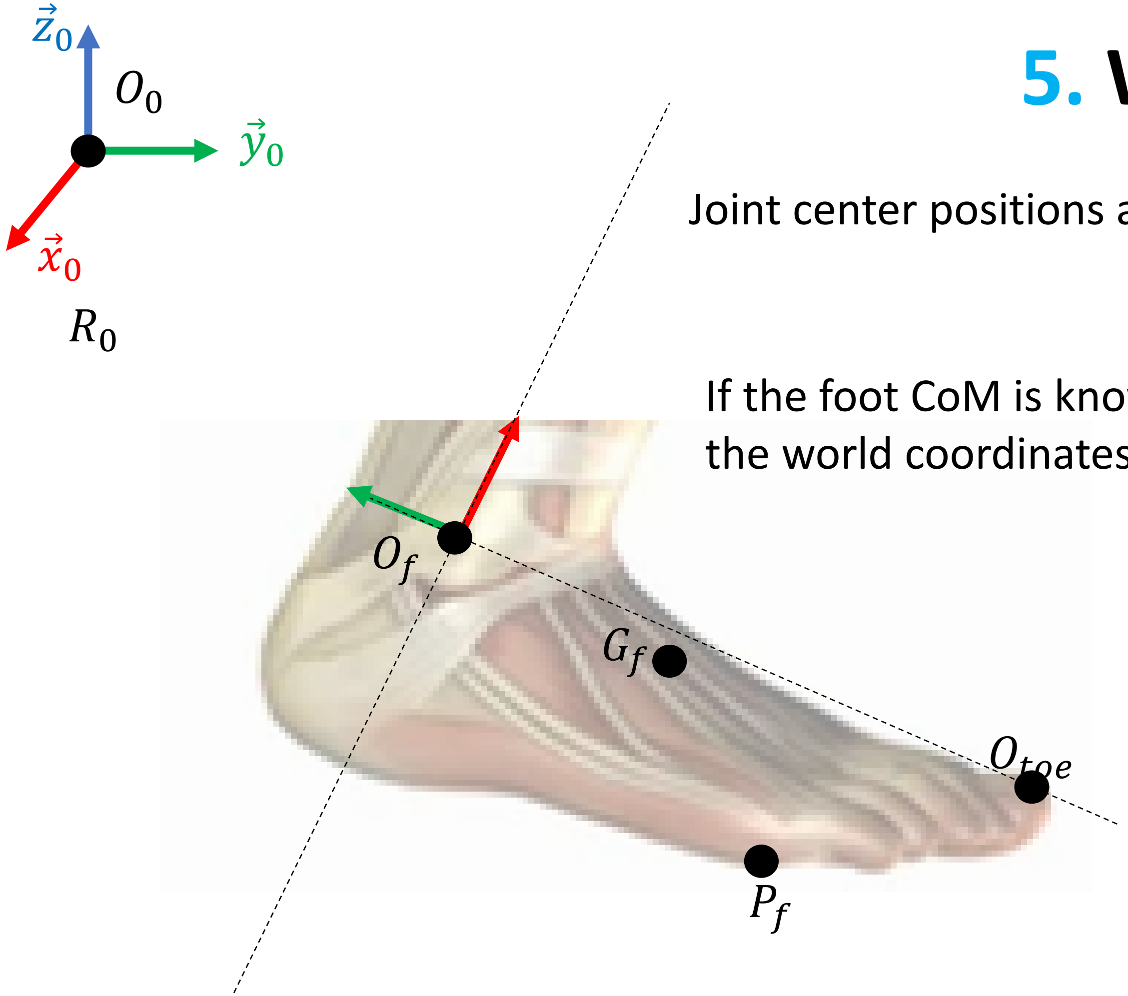
$/\vec{x}_f \qquad R_{f_x} = m_f\left(\ddot{x}_{G_f}\cos q_f + \ddot{y}_{G_f}\sin q_f\right) - \left(F_x \cos q_f + F_y \sin q_f\right) + m_f g \sin q_f$

$/\vec{y}_f \qquad R_{f_y} = m_f\left(-\ddot{x}_{G_f}\sin q_f + \ddot{y}_{G_f}\cos q_f\right) - \left(-F_x \sin q_f + F_y \cos q_f\right) + m_f g \cos q_f$

$/\vec{z}_f \qquad \Gamma_f = \left(I_f \ddot{q}_f \vec{z}_0 + \overrightarrow{O_f G_f} \wedge m_f \vec{\Gamma}(G_f \in S_4/R_0) - \overrightarrow{O_f P_f} \wedge \left(F_x \vec{x}_0 + F_y \vec{y}_0\right) - \overrightarrow{O_f G_f} \wedge \left(-m_f g \vec{y}_0\right)\right) . \vec{z}_0$

If the motion is known, then we get the torque at the ankle !

5. What about kinematics ?



Joint center positions are known

$${}^{0}O_f \text{ known}$$

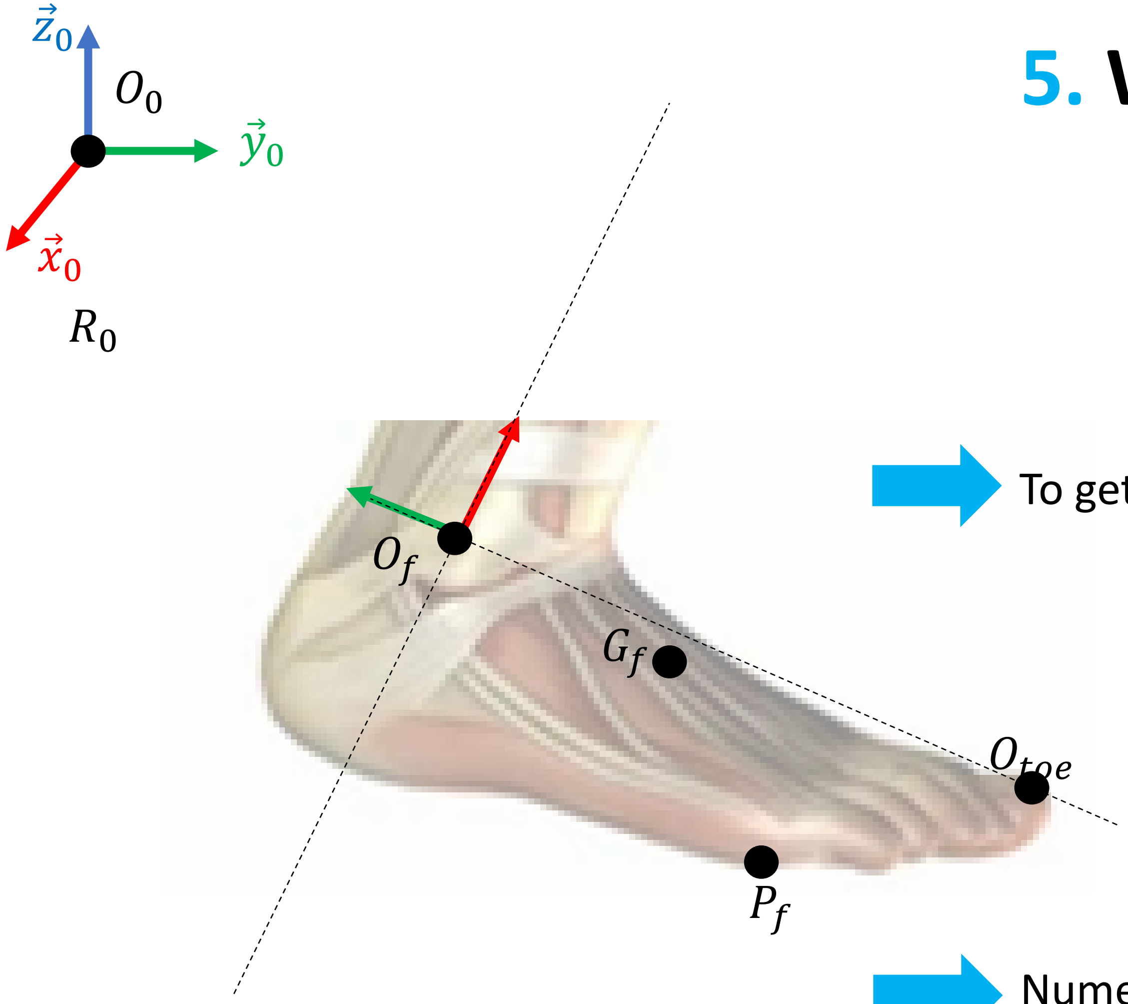
If the foot CoM is known in the foot coordinates system, then we can get its position in the world coordinates system

$${}^{f}G_f \text{ known}$$

$$\Downarrow$$

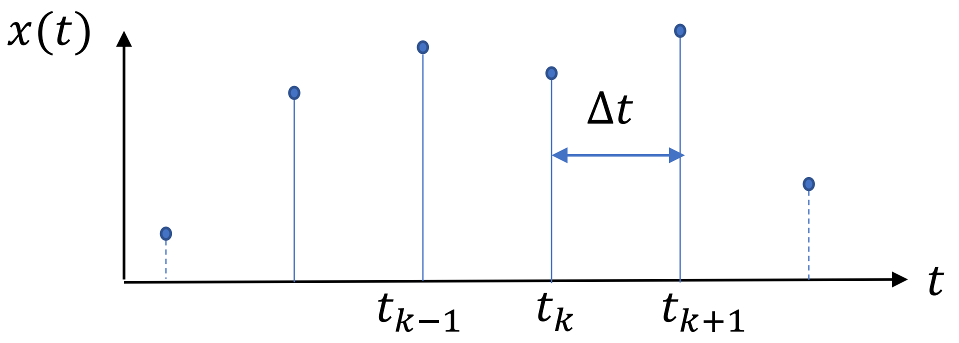
$${}^{0}G_f = {}^{0}O_f + {}^{0}R_f \, {}^{f}G_f$$

5. What about kinematics ?



${}^0G_f$ known

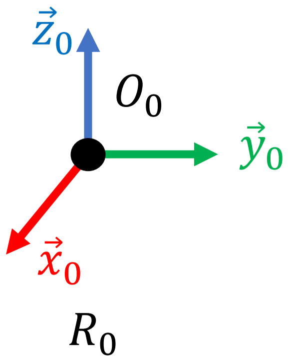
To get $\vec{\Gamma}(G_f \in S_4/R_0)$, we differentiate 2 times ${}^0G_f$



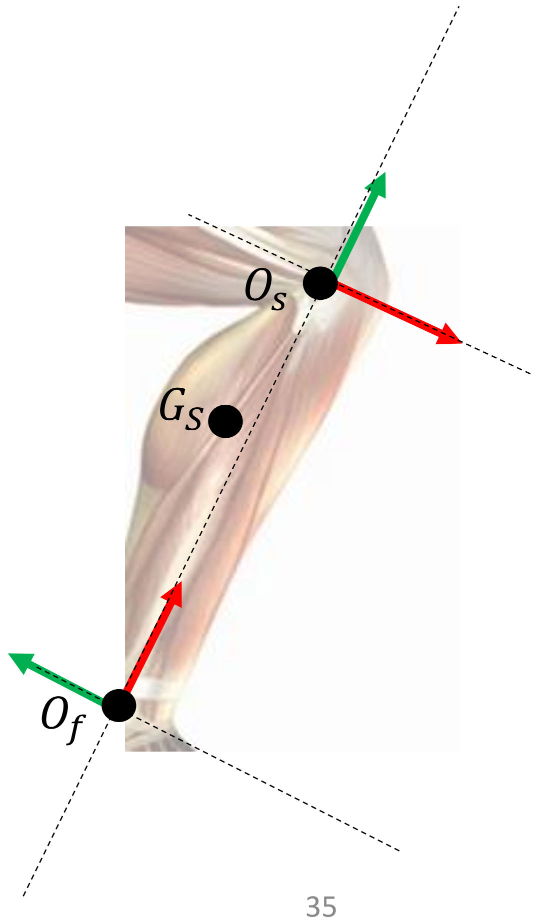
Numerical differentiation for a signal $x(t)$

$$\dot{x}_{t_k} = \frac{x(t_{k+1}) - x(t_{k-1})}{2\Delta t}$$ (2nd order centered finite difference)

Numerical differentiation amplifies noise…we should filter the data



Shank S_3



⚠ **The motion is known**

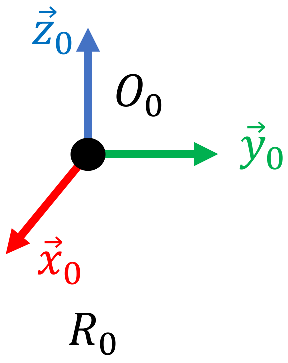
$\vec{\Gamma}(G_S \in S_3/R_0) = \frac{d^2\overrightarrow{O_0G_S}}{dt^2}$ (known)

$\ddot{q}_s$ (known)

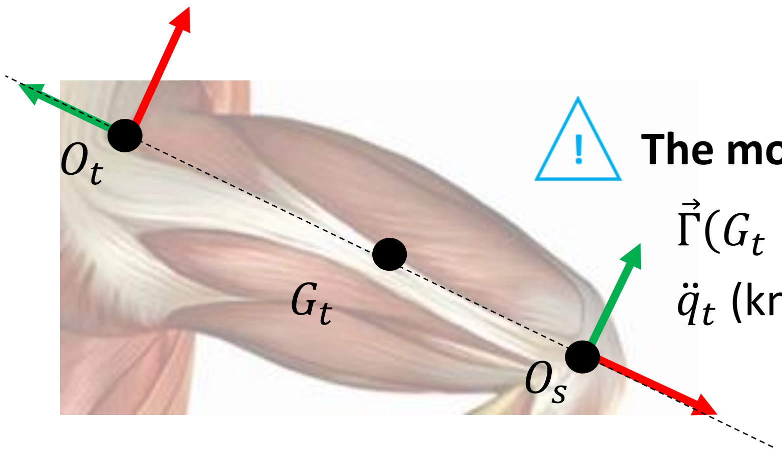
External forces ?

Acceleration quantities in G_S ?

Action of the thigh on the shank in O_S ?



Thigh S_2



The motion is supposed known

$\vec{\Gamma}(G_t \in S_2/R_0) = \frac{\mathrm{d}^2\overrightarrow{O_0G_t}}{dt^2}$ (known)

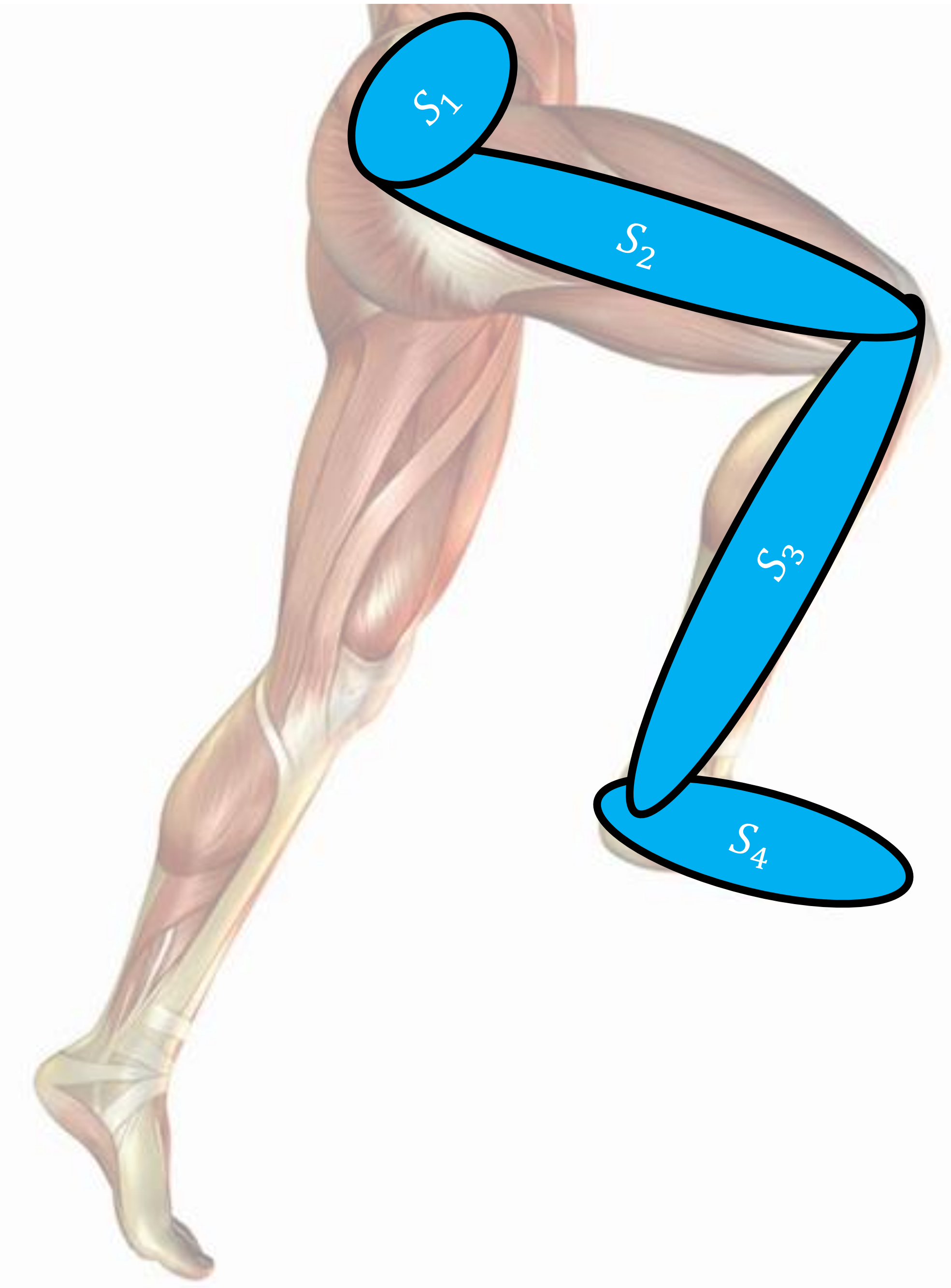
$\ddot{q}_t$ (known)

External forces ?

Acceleration quantities in G_t ?

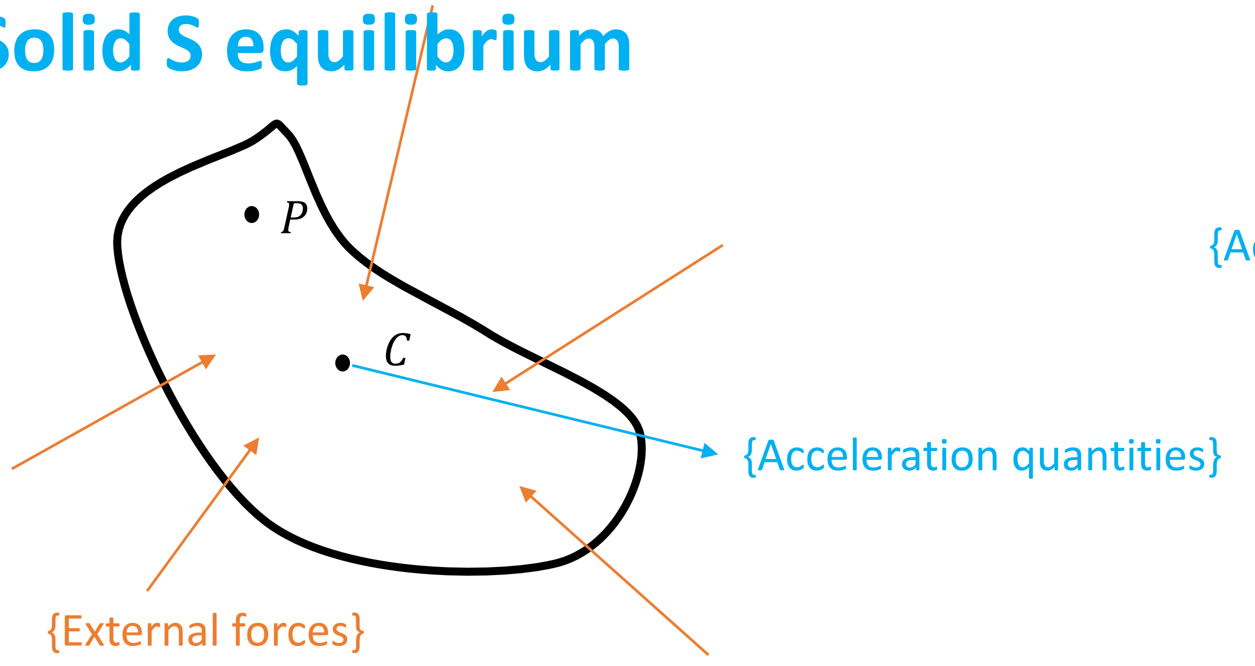
Action of the pelvis on the thigh in O_t ?

Generalisation



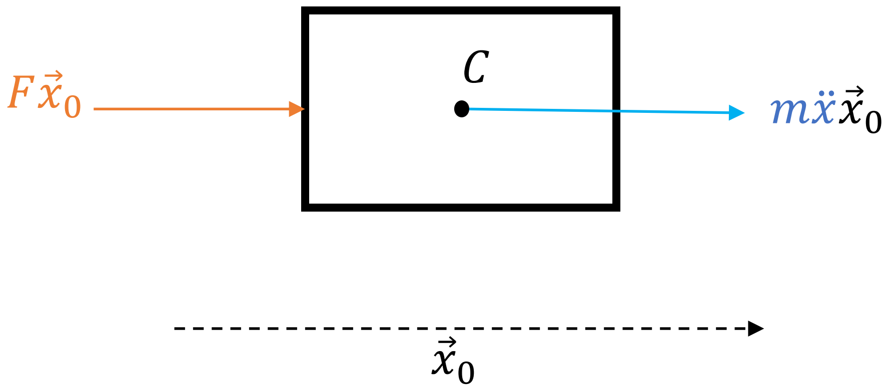
Recursive Newton-Euler Algorithm (RNEA)

Solid S equilibrium



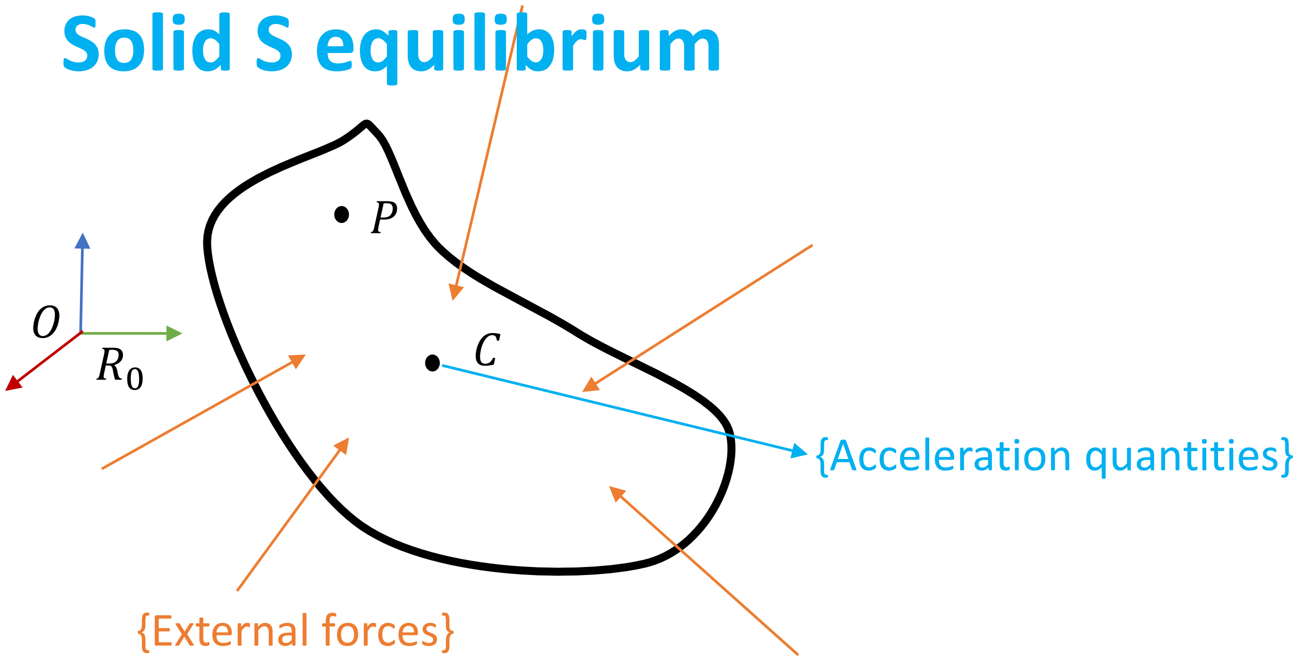
{Acceleration quantities}={External forces}

"For a solid in simple translation…"



$m\ddot{x} = F$

Solid S equilibrium

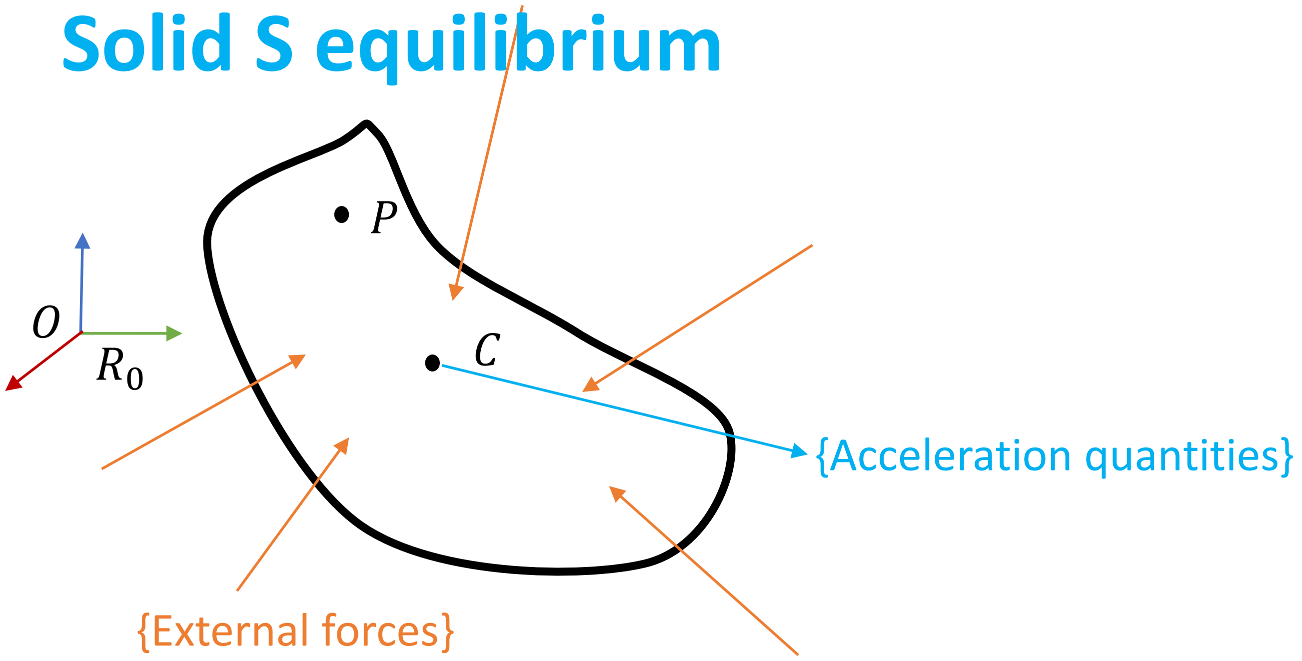


At the CoM

$$\begin{cases} \boldsymbol{f} = m\ddot{\boldsymbol{c}} & (1) \\ \boldsymbol{\tau}^{(c)} = \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} & (2) \end{cases}$$

$\boldsymbol{f}$	external forces
m	solid mass
$\boldsymbol{c}$	CoM position in R_0 (fixed)
$\boldsymbol{\omega}$	solid instantaneous velocity / R_0 and expressed in R_0
$\boldsymbol{I}$	inertia matrix expressed at $\boldsymbol{c}$ in R_0
$\boldsymbol{\tau}^{(c)}$	torque associated to the external forces expressed at $\boldsymbol{c}$ in R_0

Solid S equilibrium

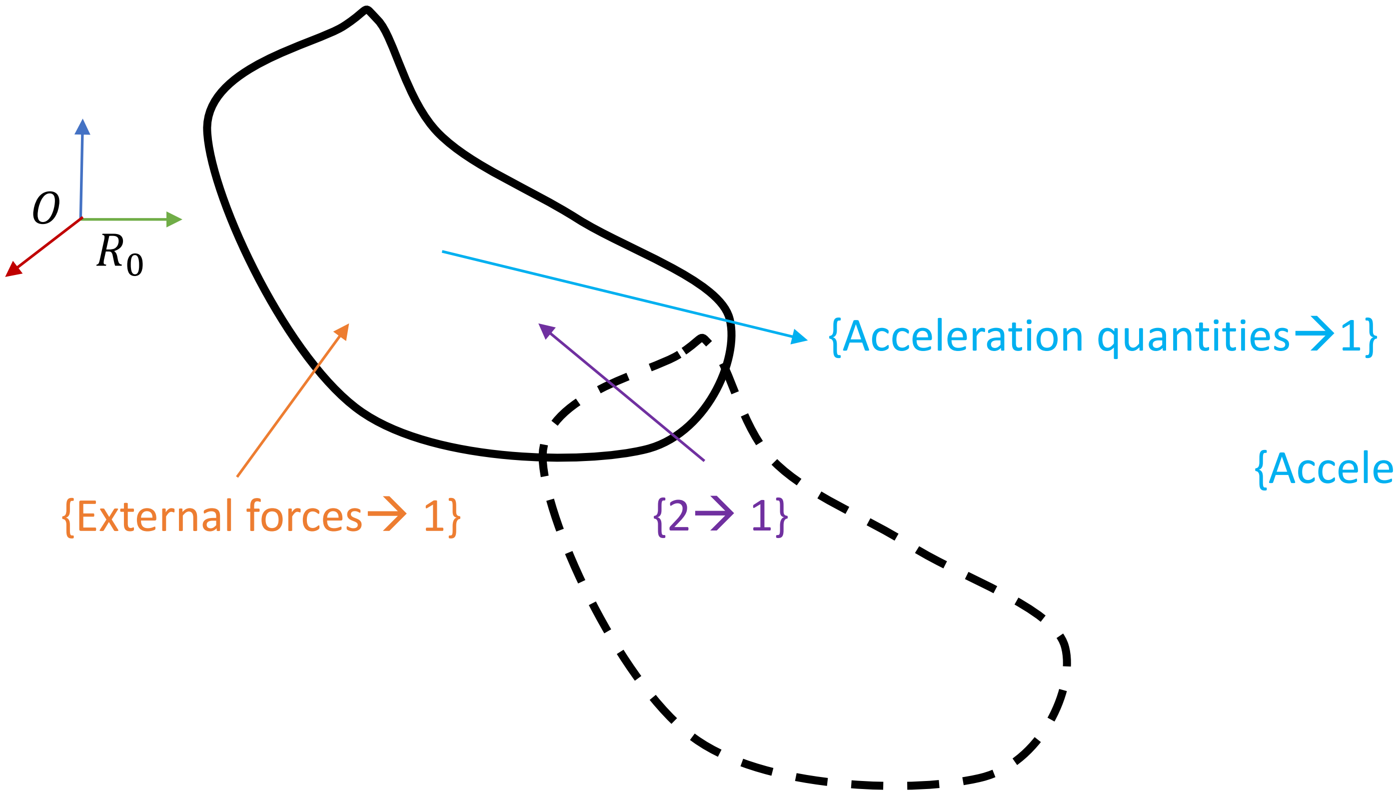


At the CoM

$$\begin{cases} \boldsymbol{f} = m\ddot{\boldsymbol{c}} & (1) \\ \boldsymbol{\tau}^{(c)} = \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} & (2) \end{cases}$$

$\boldsymbol{f}$	external forces → known (from measure/modeling)
m	solid mass → known (from modeling)
$\boldsymbol{c}$	CoM position in R_0 (fixed) → known (from measure)
$\boldsymbol{\omega}$	solid instantaneous velocity / R_0 and expressed in R_0 → known (from measure)
$\boldsymbol{I}$	inertia matrix expressed at $\boldsymbol{c}$ in R_0 → known (from modeling)
$\boldsymbol{\tau}^{(c)}$	torque associated to the external forces expressed at $\boldsymbol{c}$ in R_0 → known (from measure/modeling)

Equilibrium of an arborescent system



$$\overset{\text{Known}}{\{\text{Acceleration quantities}\rightarrow 1\}} = \overset{\text{Known}}{\{\text{External forces}\rightarrow 1\}} + \overset{\text{Computed !}}{\{2\rightarrow 1\}}$$

Equilibrium of an arborescent system

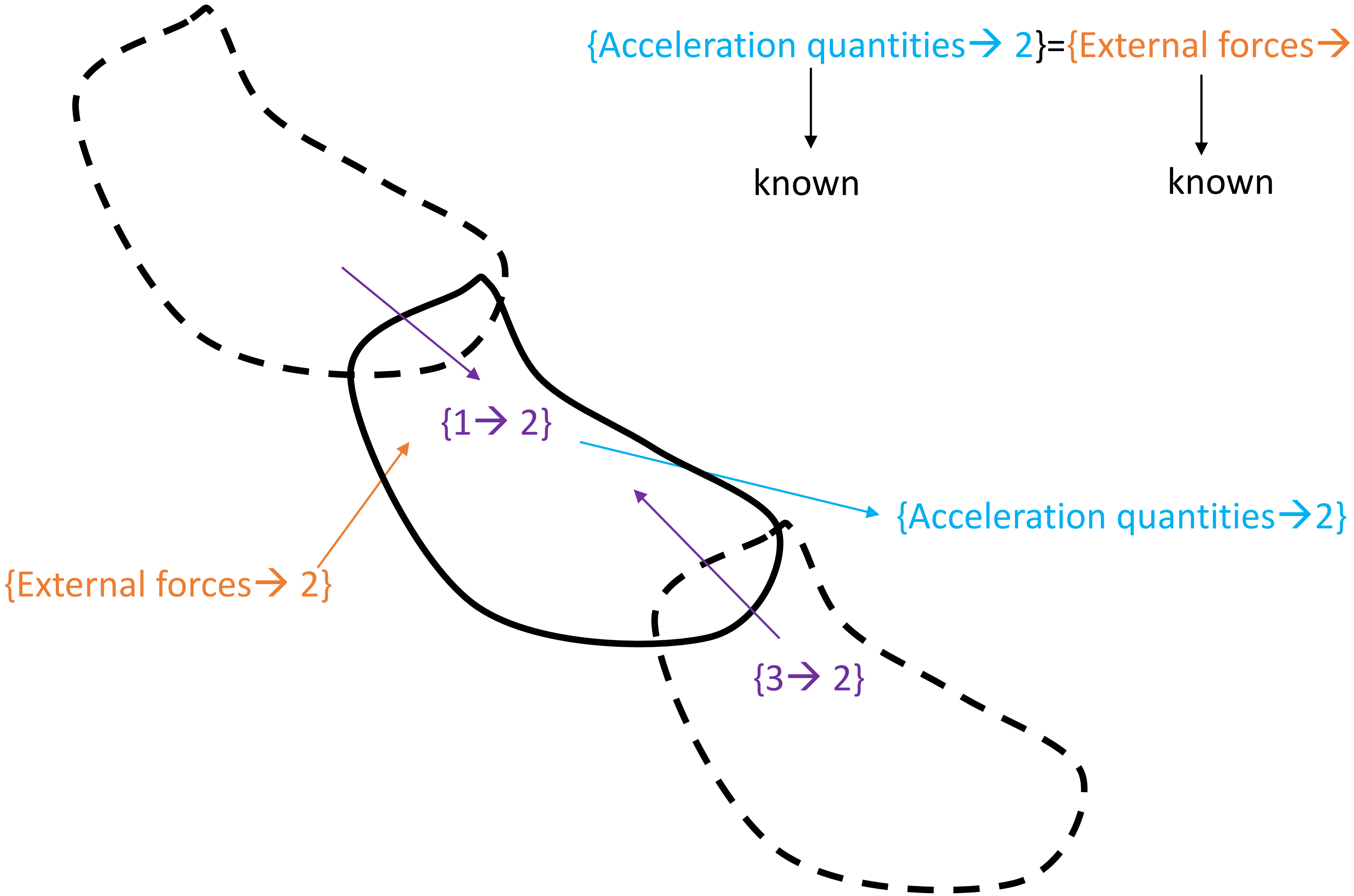
Computed

{Acceleration quantities→ 2}={External forces→2} +{1→2}+{3→2}

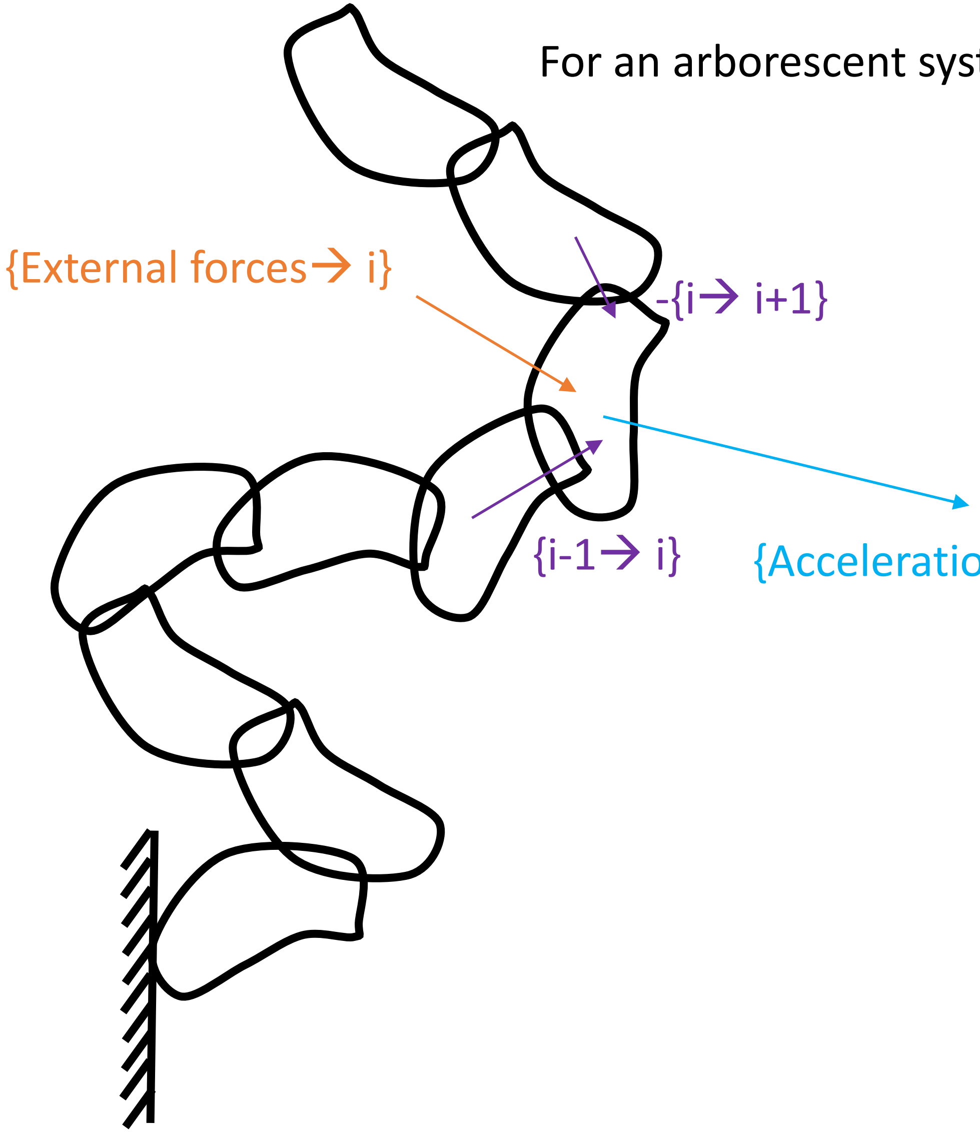
known

known

-{2→1}



RNEA



For an arborescent system of n_B solids

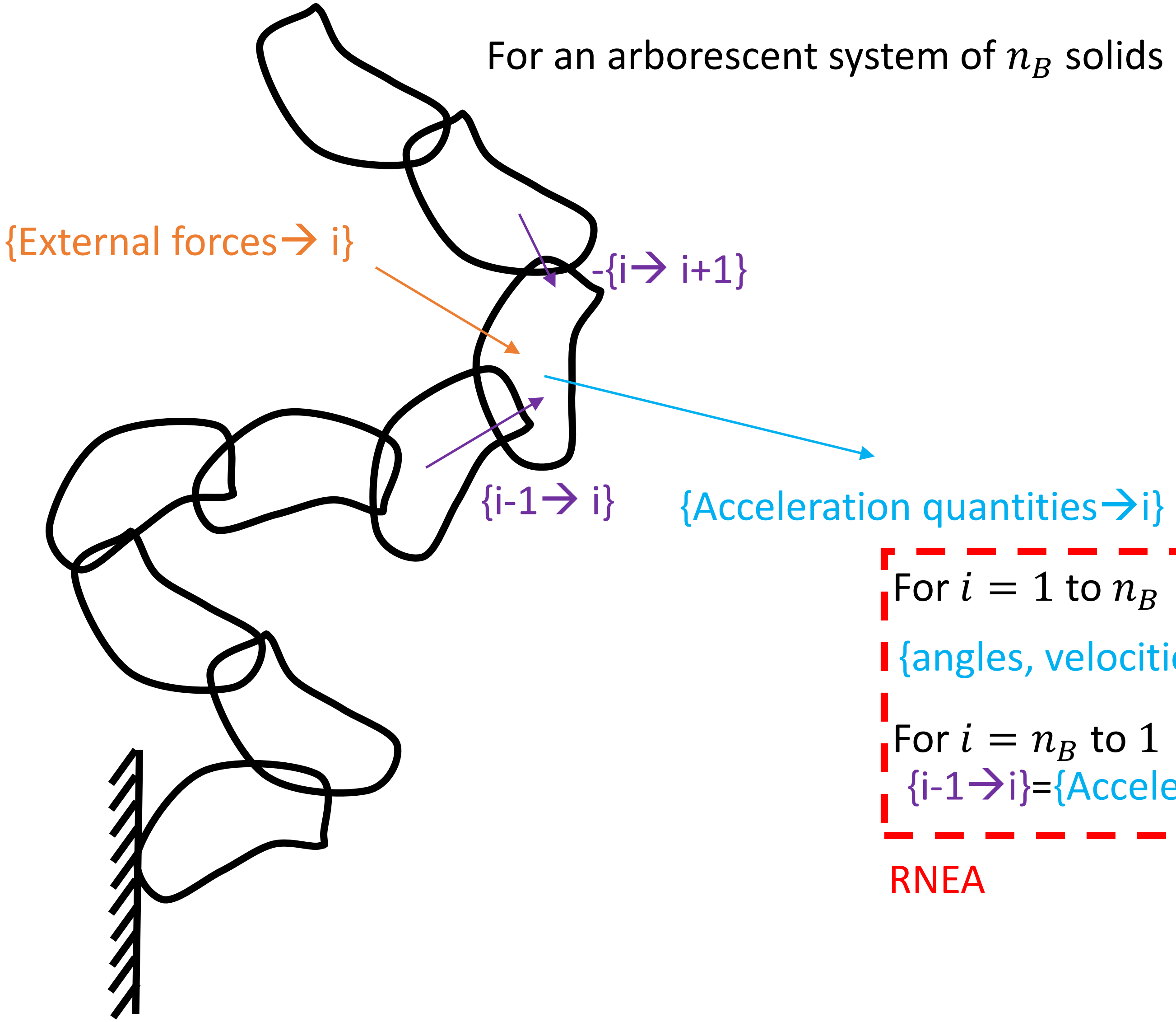
Pour $i = n_B$ à 1

{i-1→i}={Acceleration quantities→ i}-{External forces→i}+{i→ i+1}

Need to compute the acceleration quantities before !

RNEA

For an arborescent system of n_B solids



For $i = 1$ to n_B

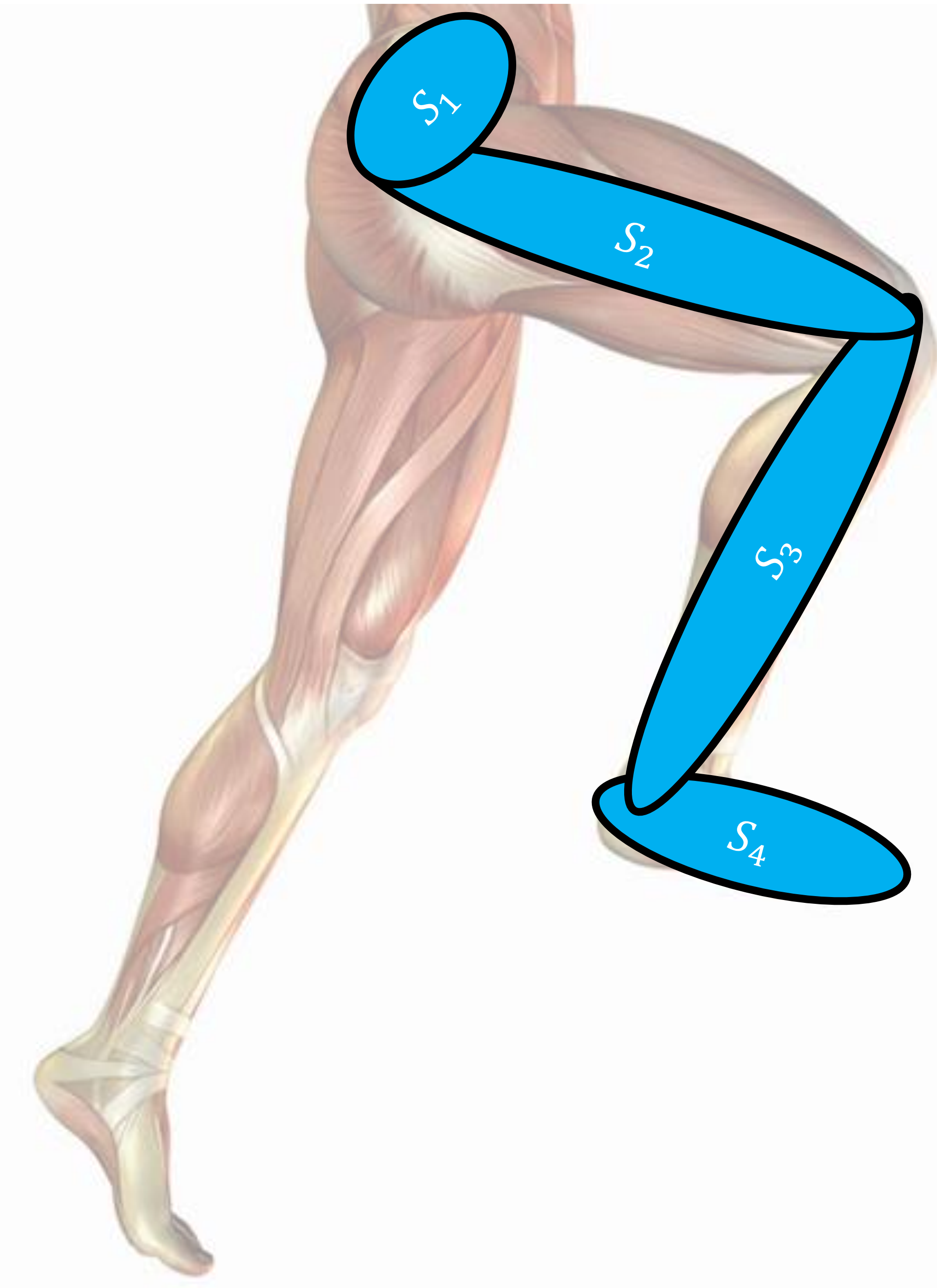
{angles, velocities, accelerations}_i =f(**q**,**dq**,**ddq**)

For $i = n_B$ to 1

{i-1→i}={Acceleration quantities→ i}-{External forces→i}+{i→ i+1}

RNEA

Application



Objectif: RNEA on a lower limb planar model

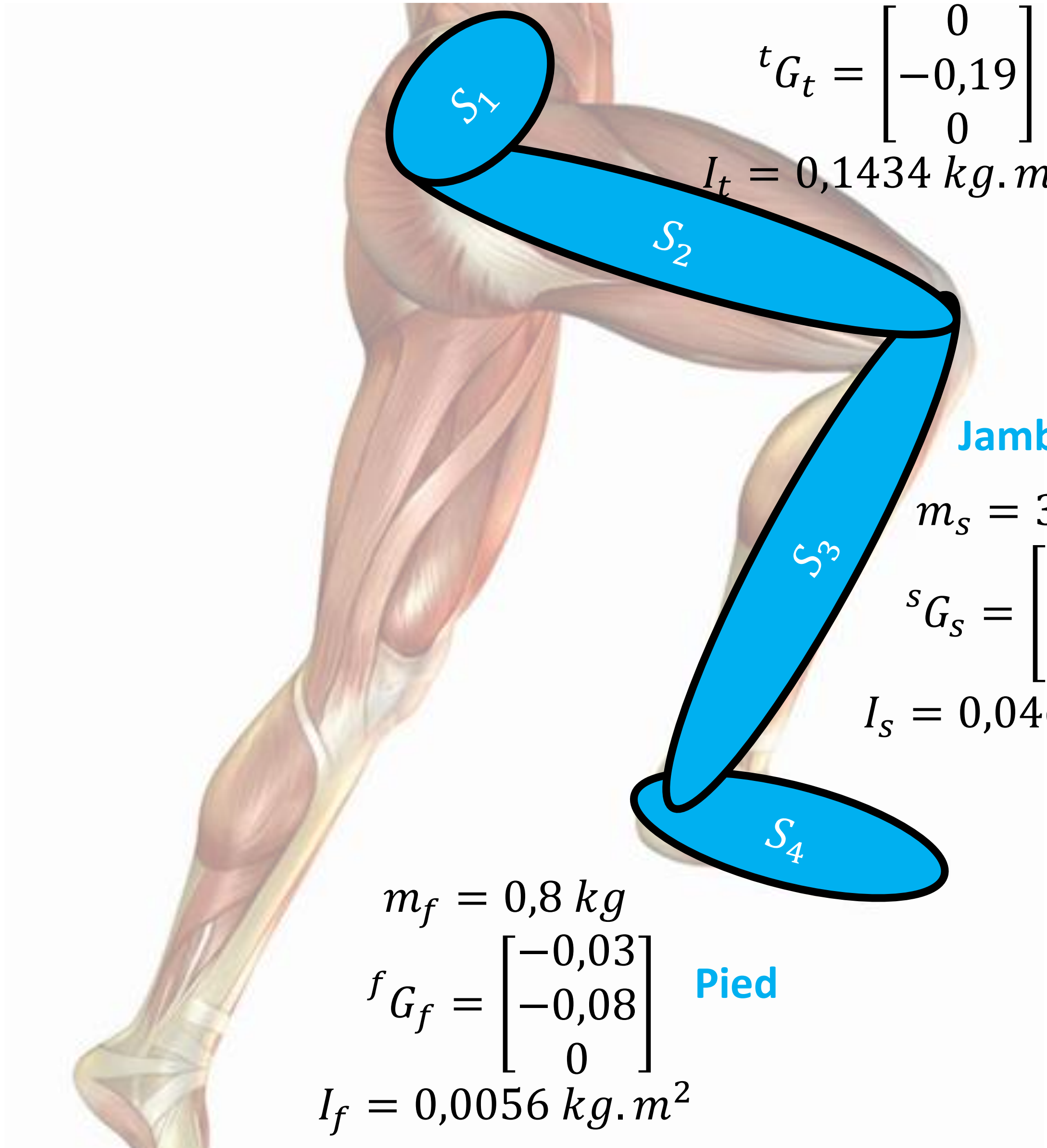
Supplementary data: inertial properties, external forces (from forces platforms)

Methods:

- Construct CoM trajectories
- Filter the data (low pass)
- Compute CoM acceleration and joint accelerations (**RNEA #1**)
- compute acceleration quantities
- Isolate from bottom to top the solid to compute the forces (**RNEA #2**)

➡ See Lower_limb_ID.ipnyb

Application



Cuisse

$$m_t = 8{,}25\ kg$$

$$^{t}G_t = \begin{bmatrix} 0 \\ -0{,}19 \\ 0 \end{bmatrix}$$

$$I_t = 0{,}1434\ kg.m^2$$

Jambe

$$m_s = 3{,}21\ kg$$

$$^{s}G_s = \begin{bmatrix} 0 \\ -0{,}186 \\ 0 \end{bmatrix}$$

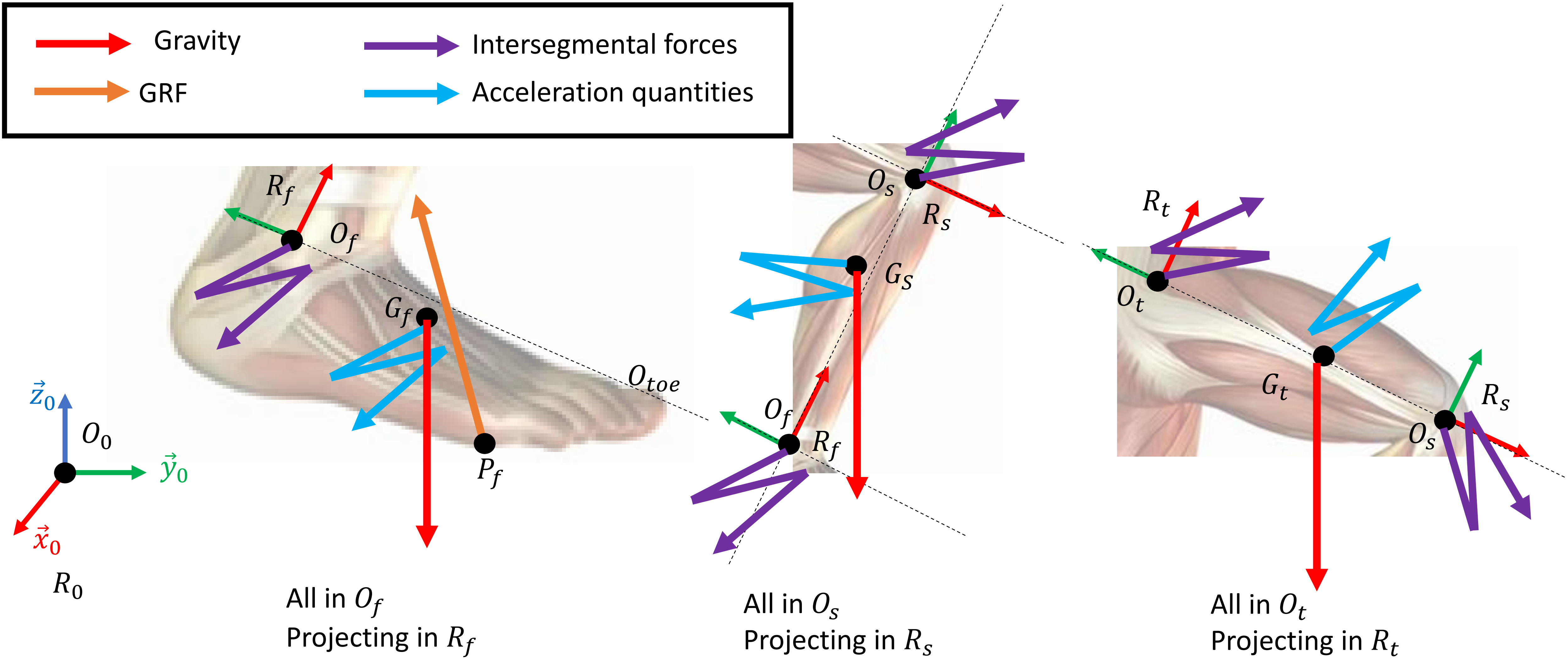
$$I_s = 0{,}0463\ kg.m^2$$

Pied

$$m_f = 0{,}8\ kg$$

$$^{f}G_f = \begin{bmatrix} -0{,}03 \\ -0{,}08 \\ 0 \end{bmatrix}$$

$$I_f = 0{,}0056\ kg.m^2$$



Gravity
Intersegmental forces
GRF
Acceleration quantities
R_f
O_f
G_f
O_{toe}
P_f
$\vec{z}_0$
O_0
$\vec{y}_0$
$\vec{x}_0$
R_0
All in O_f
Projecting in R_f
O_s
R_s
G_s
O_f
R_f
All in O_s
Projecting in R_s
R_t
O_t
G_t
R_s
O_s
All in O_t
Projecting in R_t