# STA 137 Homework 4

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**1. Using the difference equation approach to compute the ACF for the ARMA(1,1) model** $x_t = \phi x_{t-1} + w_t + \theta w_{t-1}, |\phi| < 1$

$$\gamma(h) = cov(x_{t+h}, x_t) = cov(\sum_{j=1}^{p} \phi_j x_{t+h-j} + \sum_{j=0}^{q} \theta_j w_{t+h-j}, x_t) =$$

$$\sum_{j=1}^{p} \phi_j cov(x_{t+h-j}, x_t) + \sum_{j=0}^{q} \theta_j cov(w_{t+h-j}, x_t) =$$

$$\sum_{j=1}^{p} \phi_j \gamma(h-j) + \sigma^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$$

$$h \geq 0$$

$$cov(w_{t+h-j}, x_t) = cov(w_{t+h-j}, \sum_{k=0}^{\infty} \psi_k w_{t-k}) = \psi_{j-h}\sigma^2$$

Since $|\phi| < 1$ we can assume this is a causal ARMA process and can write its general homogeneous equation.

$$\gamma(h) - \phi\gamma(h-1) = 0$$

The general solution to this difference equation is:

$$\gamma(h) = c\phi^h$$

The initial conditions can be found using:

$$\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) = \sigma^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$$

$$0 \leq h < max(p, q+1)$$

$$\gamma(0) - \phi\gamma(-1) = \sigma^2 \sum_{j=0}^{1} \theta_j \psi_{j-h} =$$

$$\gamma(0) - \phi\gamma(1) = \sigma^2[\theta_0\psi_0 + \theta_1\psi_1]$$

$$\theta_0 = \psi_0 = 1$$

$$\psi_1 = \phi + \theta$$

$$\gamma(0) = \phi\gamma(1) + \sigma^2[1 + \theta\phi + \theta^2]$$

$$\gamma(1) = \phi\gamma(0) + \sigma^2\theta$$

Now we can solve for $\gamma_(0)$ and $\gamma(1)$

$$\phi\gamma(1) + \sigma^2[1 + \theta\phi + \theta^2] = \frac{\gamma(1) - \sigma^2\theta}{\phi}$$
$$\phi^2\gamma(1) + \phi\sigma^2 + \theta\phi^2\sigma^2 + \theta^2\phi\sigma^2 + \sigma^2\theta = \gamma(1)$$
$$\phi\sigma^2 + \theta\phi^2\sigma^2 + \theta^2\phi\sigma^2 + \sigma^2\theta = \gamma(1) - \phi^2\gamma(1)$$
$$\frac{\sigma^2(\phi + \theta\phi^2 + \theta^2\phi + \theta)}{1 - \phi^2} = \gamma(1)$$
$$\frac{\sigma^2(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2} = \gamma(1)$$

$$\phi\gamma(0) + \sigma^2\theta = \frac{\gamma(0) - \sigma^2[1 + \theta\phi + \theta^2]}{\phi}$$
$$\phi^2\gamma(0) + \phi\sigma^2\theta + \sigma^2[1 + \theta\phi + \theta^2] = \gamma(0)$$
$$\sigma^2[\phi\theta + 1 + \phi\theta + \theta^2] = \gamma(0) - \phi^2\gamma(0)$$
$$\frac{\sigma^2(1 + 2\theta\phi + \theta^2)}{1 - \phi^2} = \gamma(0)$$

The general solution to the difference equation is $\gamma(h) = \frac{\gamma(1)}{\phi}\phi^h$ since $c = \frac{\gamma(1)}{\phi}$

$$\gamma(h) = \sigma^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}\phi^{h-1}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\rho(h) = \frac{\sigma^2\frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\phi^{h-1}}{\frac{\sigma^2(1+2\theta\phi+\theta^2)}{1-\phi^2}}$$

$$\rho(h) = \frac{(1 + \theta\phi)(\phi + \theta)}{(1 + 2\theta\phi + \theta^2)}\phi^{h-1}$$

**2. When $\theta = 0$, we get the ARMA(1,0) model which is the AR(1) model. Check whether your solution matches that of equation (3.8) of the textbook**

When $\theta = 0$ the resulting equation is:

$$\rho(h) = \frac{(1 + \theta\phi)(\phi + \theta)}{(1 + 2\theta\phi + \theta^2)}\phi^{h-1}$$

$$\rho(h) = \phi\phi^{h-1} = \phi^h$$

This result is the same as in equation 3.8

**3. When $\phi = 0$, we get the ARMA(0,1) model which is the MA(1) model. Check whether your solution matches that of Example 3.5 of the textbook**

When $\phi = 0$ the resulting equation is:

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{(1+2\theta\phi+\theta^2)}\phi^{h-1}$$

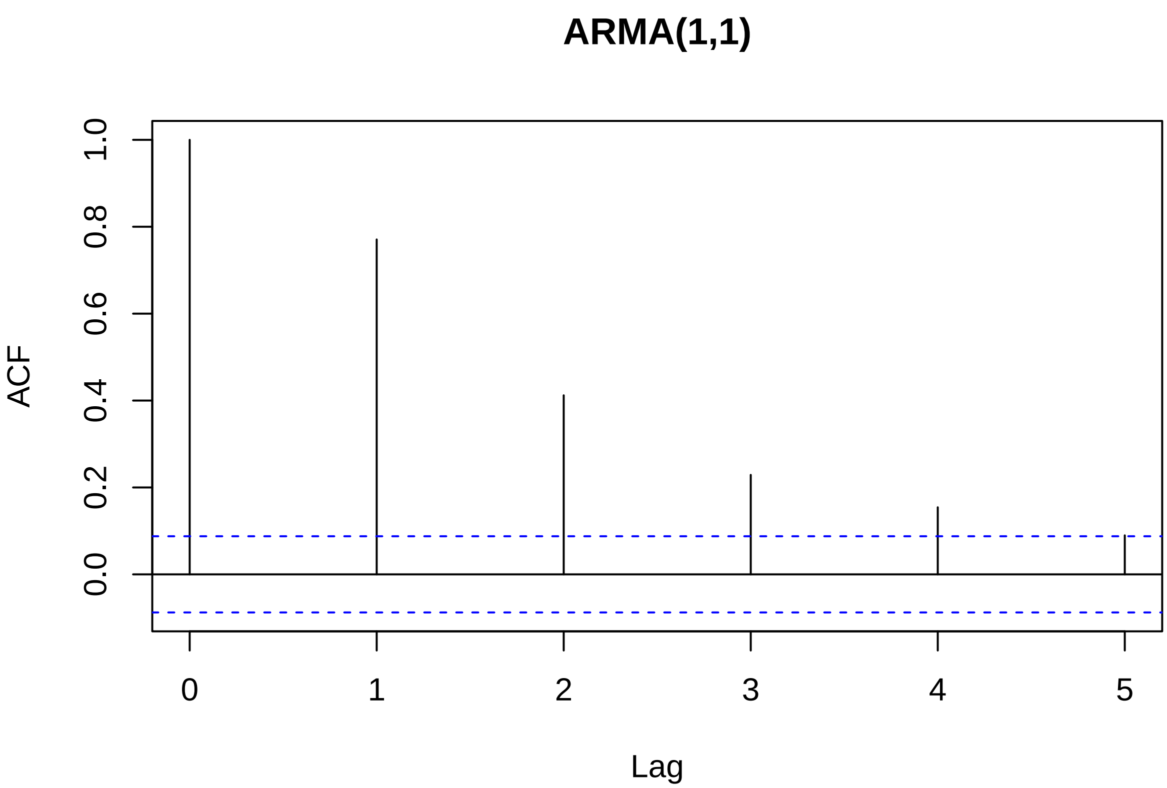
$$\rho(h) = \frac{\theta}{1+\theta^2}$$

This result is the same as in example 3.5

**4. Plot the sample ACFs when $\theta = 0.9$ and $\phi = 0.6$ using R for a few lags h = 0,1,2,3,4,5. Check whether they are close to your theoretical derivations.**

```r
library(magrittr)
library(kableExtra)
set.seed(42)

arima.sim(list(ar = c(0.6),ma = c(0.9)),500) %>% acf(lag.max = 5,main = "ARMA(1,1)")
```




```r
ARMAacf(ar = c(0.6), ma = c(0.9), lag.max = 5) %>% t %>% kbl
```

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.799308 | 0.4795848 | 0.2877509 | 0.1726505 | 0.1035903 |

Here we can see that for a sample from 500 random normally distributed variables the sample ACF values for each lag h are very close to the theoretical ACF values