STA 137 Homework 3

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3.1 For an MA(1), $x_t = w_t + \theta w_t - 1$, show that $|\rho_x(1)| \leq \frac{1}{2}$ for any number θ. For which values of θ does $\rho_x(1)$ attain its maximum and minimum?

First we need to find $\gamma_x(h)$

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov(w_{t+h} + \theta w_{t+h-1}, w_t + \theta w_t - 1) =$$
$$E[(w_{t+h})(w_t) + (w_{t+h})(\theta w_{t-1}) + (\theta w_{t+h-1})(w_t) + (\theta w_{t+h-1})(\theta w_{t-1})]$$
$$\gamma_x(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \theta\sigma^2 & |h| = 1 \\ 0 & otherwise \end{cases}$$

Now we need to define $|\rho_x(1)|$ and the maximum/minimum values for θ

$$|\rho_x(1)| = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{(1+\theta^2)}$$

$$\frac{d\theta}{d(1+\theta^2)} = \frac{1+\theta^2-\theta(2\theta)}{(1+\theta^2)^2} = \frac{1-\theta^2}{(1+\theta^2)^2}$$
$$\frac{1-\theta^2}{(1+\theta^2)^2} = 0$$
$$1 - \theta^2 = 0$$
$$1 = \theta^2$$
$$\theta = \pm 1$$

$\rho_x(1)$ achieves its maximum and minimun when $\theta = \pm 1$

$$|\rho_x(1)|, \theta = \pm 1$$
$$\frac{\theta}{(1+\theta^2)} = |\frac{1}{2}|$$

So $|\rho_x(1)| \leq |\frac{1}{2}|$ for all values of θ

3.4 Identify the following models as ARMA(p,q) models (watch out for parameter redundancy), and determine whether they are causal and/or invertible:

(a) $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$

We can write this expresson as:

$$x_t - 0.8x_{t-1} - 0.15x_{t-2} = w_t - 0.3w_{t-1} =$$

$$(1 - 0.8B - 0.15B^2)x_t = (1 - 0.3B)w_t =$$

$$(1 - 0.5B)(1 - 0.3B)x_t = (1 - 0.3B)w_t =$$

$$(1 - 0.5B)x_t = w_t$$

Now let's check if this ARMA(1,0) process is causual:

$$\phi(z) = 1 - 0.5z$$

$$1 - 0.5z = 0$$

$$1 = 0.5z$$

$$|z| = |2|$$

$$|z| > 1$$

So this ARMA(1,0) process is causal.

$$\phi(z)\psi(z) = \theta(z)$$

$$(1 - 0.5z)(1 + \psi_1 z + ...\psi_p z^p) = 1$$

To satisfy this equation $\psi = (0.5^1, 0.5^2, ...0.5^p)$ Now we can rewrite this process to a one-sided causal process:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

$$x_t = (1 - 0.5B)^{-1}w_t = (\sum_{j=0}^{\infty}(0.5B)^j)w_t = \sum_{j=0}^{\infty}(0.5)^j(B^j w_t) =$$

$$x_t = \sum_{j=0}^{\infty}(0.5)^j w_{t-j}$$

Since the expression in (a) is an AR(1) process it is inherently invertible with representation:

$$w_t = x_t - 0.5x_{t-1}$$

$$\pi = (1, -0.5, 0...)$$

(b)

$$x_t = x_{t-1} - 0.5x_{t-2} + w_t + w_{t-1} =$$

$$x_t - x_{t-1} + 0.5x_{t-2} = w_t - w_{t-1} =$$

$$(1 - B + 0.5B^2)x_t = (1 - B)w_t$$

Let's check if this ARMA(2,1) process is causal:

$$\phi(z) = 1 - z + 0.5z^2$$

$$1 - z + 0.5z^2 = 0$$

Using the quadratic formula:

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2} > 1$$

This ARMA(2,1) process is causal.

$$\phi(z)\psi(z) = \theta(z)$$

$$(1 - z + 0.5z^2)(1 + \psi_1 z + ...\psi_p z^p) = 1 - z$$

$$x_t = w_t + \sum_{j=2}^{\infty}(\psi_j)(w_{t-j})$$

$$\psi_j = (-0.5, -0.5, -0.25, ..., \psi_j - \psi_{j-1} + 0.5\psi_{j-2})$$
$$j = (2, 3, ..., \infty)$$
$$\psi_0 = 1, \psi_1 = 0$$

$$\theta(z) = 1 + z$$
$$1 - z = 0$$
$$z = 1$$

Since the root lies on the unit circle this ARMA(2,1) process is also not invertible.