STA 137 Homework 1

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2022-04-14

1.

Consider the time series:

$$x_t = \beta_1 + \beta_2 t + w_t$$

(a) Determine whether x_t is stationary:

To determine if x_t is stationary we need to check two conditions:

I. μ_x does not depend on t and is constant.
II. $\gamma_x(h)$ only depends on $h = |s - t|$

$$\mu_x = E(x_t) = E(\beta_1 + \beta_2 t + w_t) = \beta_1 + \beta_2 t$$
$$\text{since } E(w_t) = 0$$
$$\text{Here } \mu_x \text{ is not constant and depends on } t \text{ so } x_t \text{ is not stationary.}$$

(b) Show that the process $y_t = x_t - x_{t-1}$ is stationary.

To determine if y_t is stationary we need to check two conditions:

I. μ_y does not depend on t and is constant.
II. $\gamma_y(h)$ only depends on $h = |s - t|$

$$y_t = \beta_1 + \beta_2 t + w_t - (\beta_1 + \beta_2(t-1) + w_{t-1}) = \beta_2 + w_t - w_{t-1}$$
$$\mu_y = E(y_t) = E(\beta_2 + w_t - w_{t-1}) = \beta_2$$
$$\text{Since } E(w_t) = E(w_{t-1} = 0)$$
$$\text{The first condition is satisfied because } \beta_2 \text{ is a constant that does not depend on } t$$

$$\gamma_y(h) = Cov(y_{t+h}, y_t) =$$

$$Cov(\beta_2 + w_{t-h} - w_{t-h}, \beta_2 + w_t - w_{t-1}) =$$

$$E((\beta_2 + w_{t+h} - w_{t+h-1}) - \beta_2, (\beta_2 + w_t - w_{t-1}) - \beta_2)$$

$$\gamma_y(h) = E((w_{t+h} - w_{t+h-1})(w_t - w_{t-1})) =$$

$$E[(w_{t+h})(w_t)+(w_{t+h})(w_{t-1})-(w_{t+h-1})(w_t)+(w_{t+h-1})(w_{t-1})]$$

$$\gamma_y(h) = \begin{cases} 2\sigma_w^2 & h=0 \\ -\sigma_w^2 & h=|1| \\ 0 & h \geq |2| \end{cases}$$

Since the autocovariance function only depends on the lag $h = |s-t|$

$y_t = x_t - x_{t-1}$ is stationary

(c) Show that the mean of the moving average

$$v_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$$

is $\beta_1 + \beta_2 t$, and give a simplified expression for the autocovariance function.

$$\mu_v = E(v_t) = \frac{1}{2q+1} \sum_{j=-q}^{q} E(x_{t-j}) =$$

$$\frac{1}{2q+1} \sum_{j=-q}^{q} E(\beta_1 + \beta_2(t-j) + w_{t-j}) =$$

$$\frac{1}{2q+1} \left[\beta_1(2q+1) + \beta_2 t(2q+1) - \beta_2 \sum_{j=-q}^{q} j + \sum_{j=-q}^{q} E(w_{t-j}) \right] =$$

$$\frac{1}{2q+1} \left[\beta_1(2q+1) + \beta_2 t(2q+1) - 0 + 0 \right] =$$

$$\beta_1 + \beta_2 t$$

The autocovariance function is

$$\gamma_v(h) = Cov(v_{t+h}, v_t) = Cov(\frac{1}{2q+1} \sum_{j=-q}^{q} x_{t+h-j}, \frac{1}{2q+1} \sum_{k=-q}^{q} x_{t-k}) =$$

Let $j = h + k$ so we can have sums both in terms of k

$$\frac{1}{(2q+1)^2} Cov(\sum_{h+k=-q}^{q} x_{t-k}, \sum_{k=-q}^{q} x_{t-k}) =$$

$$\frac{1}{(2q+1)^2} Cov(\sum_{h+k=-q}^{q} \beta_1 + \beta_2(t-k) + w_{t-k}, \sum_{k=-q}^{q} \beta_1 + \beta_2(t-k) + w_{t-k}) =$$

$$\frac{1}{(2q+1)^2} Cov(\sum_{h+k=-q}^{q} w_{t-k}, \sum_{k=-q}^{q} w_{t-k}) =$$

$$\frac{(2q+1-|h|)\sigma_w^2}{(2q+1)^2}$$

$$\gamma_v(h) = \begin{cases} \frac{(2q+1-|h|)\sigma_w^2}{(2q+1)^2} & 0 \leq |h| \leq q \\ 0 & |h| > q \end{cases}$$

1.7 For a moving average process of the form:

$$x_t = w_{t-1} + 2w_t + w_{t+1}$$

(a) Autocovariance Function

$$\mu_x = 0$$

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov(w_{t+h-1} + 2w_{t+h} + w_{t+h+1}, w_{t-1} + 2w_t + w_{t+1}) =$$

$$E\left[(w_{t+h-1} + 2w_{t+h} + w_{t+h+1})(w_{t-1} + 2w_t + w_{t+1})\right] =$$

$$\begin{aligned} E[(w_{t+h+1})(w_{t-1}) + (w_{t+h+1})(2w_t) + (w_{t+h+1})(w_{t+1}) +& \\ (2w_{t+h})(w_{t-1}) + (2w_{t+h})(2w_t) + (2w_{t+h})(w_{t+1}) +& \\ (w_{t+h+1})(w_{t-1}) + (w_{t+h+1})(2w_t) + (w_{t+h+1})(w_{t+1})]& \end{aligned}$$

$$\gamma_x(h) = \begin{cases} 6\sigma_w^2 & h = 0 \\ 4\sigma_w^2 & |h| = 1 \\ \sigma_w^2 & |h| = 2 \\ 0 & |h| \geq 3 \end{cases}$$

(b) Autocorrelation Function

$$\rho_x(0) = \frac{\gamma_x(0)}{\gamma_x(0)} = \frac{6\sigma_w^2}{6\sigma_w^2} = 1$$

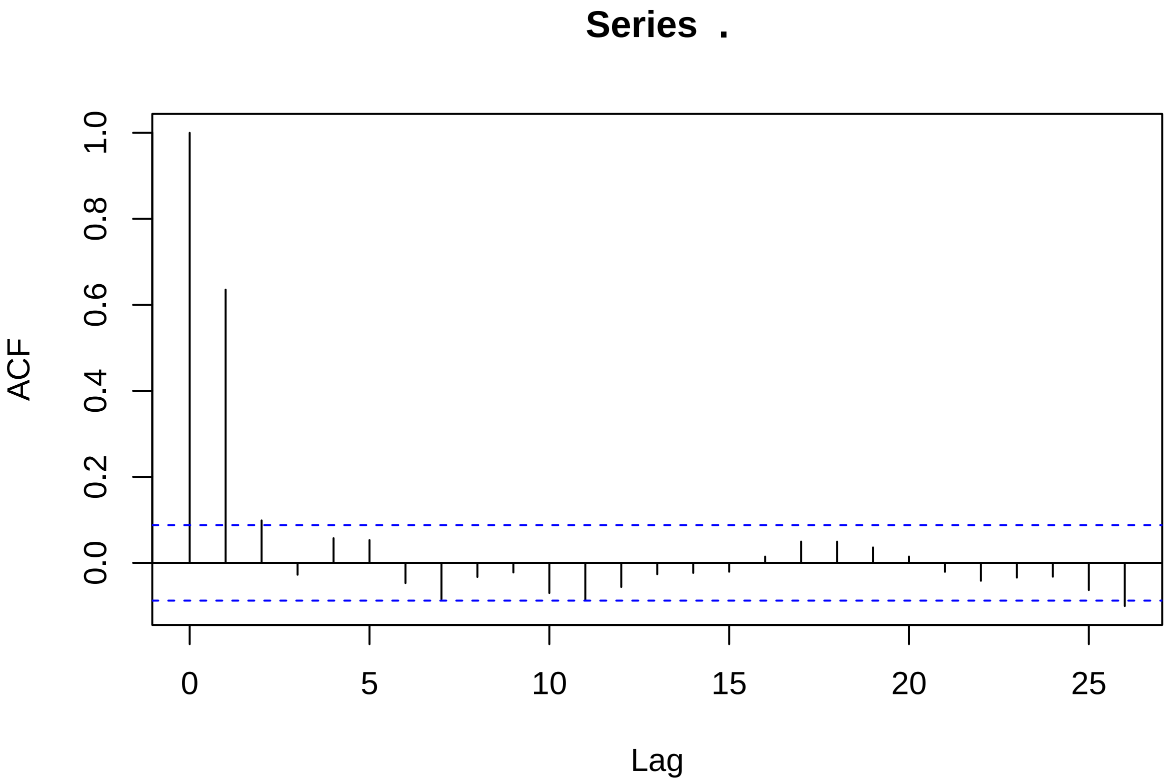
$$\rho_x(1) = \frac{\gamma_x(1)}{\gamma_x(0)} = \frac{4\sigma_w^2}{6\sigma_w^2} = \frac{2}{3}$$

$$\rho_x(2) = \frac{\gamma_x(2)}{\gamma_x(0)} = \frac{\sigma_w^2}{6\sigma_w^2} = \frac{1}{6}$$

$$\rho_x(3) = \frac{\gamma_x(3)}{\gamma_x(0)} = \frac{0}{6\sigma_w^2} = 0$$

$$\rho_x(h) = \begin{cases} 1 & h = 0 \\ \frac{2}{3} & |h| = 1 \\ \frac{1}{6} & |h| = 2 \\ 0 & |h| \geq 3 \end{cases}$$

```r
library(magrittr)
set.seed(42)
(rnorm(500) %>% stats::filter(filter = c(1,2,1),method = "convolution")) %>%  na.omit %>%  acf
```



1.13 Consider the two series

$$x_t = w_t$$

$$y_t = w_t - \theta w_{t-1} + u_t$$

(a) Express the ACF, $\rho_y(h)$,for h = 0,±1,±2,... of the series yt as a function of σ_w^2, σ_u^2 and θ.

$$\gamma_y(h) = Cov(y_{t+h}, y_t) = Cov(w_{t+h} - \theta w_{t+h-1} + u_{t+h}, w_t - \theta w_{t-1} + u_t) =$$

$$\begin{aligned} E[(w_{t+h})(w_t) - (w_{t+h})(\theta w_{t-1}) + (w_{t+h})(u_t) \\ -(\theta w_{t+h-1})(w_t) + (\theta w_{t+h-1})(\theta w_{t-1}) - (\theta w_{t+h-1})(u_t) + \\ (u_{t+h})(w_t) - (u_{t+h})(\theta w_{t-1}) + (u_{t+h})(u_t)] \end{aligned}$$

$$\gamma_y(h) = \begin{cases} (1+\theta^2)\sigma_w^2 + \sigma_u^2 & h = 0 \\ -\sigma_w^2 & |h| = 1 \\ 0 & |h| \geq 2 \end{cases}$$

$$\rho_y(0) = \frac{\gamma_y(0)}{\gamma_y(0)} = \frac{(1+\theta^2)\sigma_w^2 + \sigma_u^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2} = 1$$

$$\rho_y(\pm 1) = \frac{\gamma_y(1)}{\gamma_y(0)} = \frac{-\sigma_w^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2} = \frac{-\sigma_w^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2}$$

$$\rho_y(\pm 2) = \frac{\gamma_y(2)}{\gamma_y(0)} = \frac{0}{(1+\theta^2)\sigma_w^2 + \sigma_u^2} = 0$$

$$\rho_y(h) = \begin{cases} 1 & h = 0 \\ \frac{-\sigma_w^2}{(1+\theta^2)\sigma_w^2+\sigma_u^2} & |h| = 1 \\ 0 & |h| \geq 2 \end{cases}$$

(b) Determine the CCF, $\rho_{xy}(h)$ relating x_t and y_t

$$\gamma_x(h) = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

$$\gamma_{xy}(h) = Cov(x_{t+h}, y_t) = Cov(w_{t+h}, w_t - \theta w_t + u_t) =$$

$$E[(w_{t+h})(w_t) - (w_{t+h})(\theta w_{t-1}) + (w_{t+h})(u_t)]$$

$$\gamma_{xy}(h) = \begin{cases} \sigma_w^2 & h = 0 \\ -\theta\sigma_w^2 & h = -1 \\ 0 & otherwise \end{cases}$$

$$\rho_{xy}(0) = \frac{\gamma_{xy}(0)}{\sqrt{\gamma_x(0)\gamma_y(0)}} = \frac{\sigma_w^2}{\sqrt{\sigma_w^2[(1+\theta^2)\sigma_w^2 + \sigma_u^2]}} = \frac{\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}$$

$$\rho_{xy}(1) = \frac{\gamma_{xy}(1)}{\sqrt{\gamma_x(0)\gamma_y(0)}} = \frac{0}{\sqrt{\sigma_w^2[(1+\theta^2)\sigma_w^2 + \sigma_u^2]}} = 0$$

$$\rho_{xy}(-1) = \frac{\gamma_{xy}(-1)}{\sqrt{\gamma_x(0)\gamma_y(0)}} = \frac{-\theta\sigma_w^2}{\sqrt{\sigma_w^2[(1+\theta^2)\sigma_w^2 + \sigma_u^2]}} = \frac{-\theta\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}$$

$$\rho_{xy}(2) = \frac{\gamma_{xy}(2)}{\sqrt{\gamma_x(0)\gamma_y(0)}} = \frac{0}{\sqrt{\sigma_w^2[(1+\theta^2)\sigma_w^2 + \sigma_u^2]}} = 0$$

$$\rho_{xy}(-2) = \frac{\gamma_{xy}(-2)}{\sqrt{\gamma_x(0)\gamma_y(0)}} = \frac{0}{\sqrt{\sigma_w^2[(1+\theta^2)\sigma_w^2 + \sigma_u^2]}} = 0$$

$$\rho_{xy}(h) = \begin{cases} \frac{\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2+\sigma_u^2}} & h = 0 \\ \frac{-\theta\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2+\sigma_u^2}} & h = -1 \\ 0 & otherwise \end{cases}$$

(c) Show that x_t and y_t are jointly stationary:

We know from previous problems that x_t and y_t are stationary

$\gamma_{xy}(h)$ only depends on $h = |s - t|$

So x_t and y_t are jointly stationary