Loewner landmark

Bridge between realization, approximation and identification

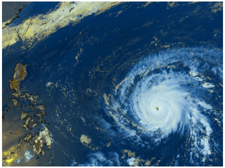
Charles Poussot-Vassal
October 11, 2024

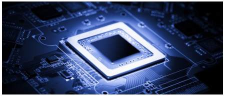
"Merge data and physics using computational sciences and engineering"

Forewords

Dynamical models, what for?

Dynamical models are centrals tools in engineering...





Digitalisation and computer-based modeling for

- simulation, optimisation, understanding
- control, estimation, analysis...

However

Finite machine precision, computational burden, memory management and actual solvers

- induces important time consumption
- generate inaccurate results
- limit the class of models to deal with
- limit the amount of data to treat

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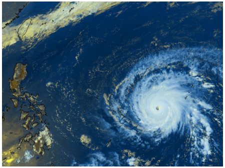
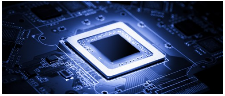

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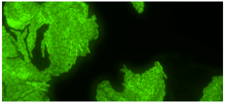
Develop robust and efficient tools to construct simplified dynamical models

… and in systems and control engineering

- **for verification and validation**
  (μ, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
- **for detection**
  (fault isolation, param. estim.)
- **for uncertainty propagation**
  (Multi Disc. Optim., robust optim.)
- **for feedback control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)



Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

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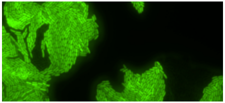

Complex models

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$\Longrightarrow$

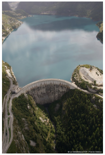
Model simplification

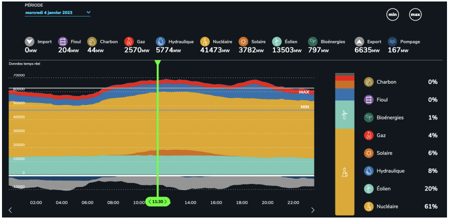
Simplified models

- reduced sim. time
- memory saving
- accurate results
- rational model

Hydraulic-driven electricity

- Dams & run-of-the-river ($\approx 10\%$)
- Run-of-the-river ($\approx 5\%$)
- Rely on open-channel hydraulic systems
- Need for analysis and control





January 4th, 2023

Forewords

Use-case: hydroelectric open-channel (model-driven)

Time-domain

$$
\begin{aligned}
\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/S)}{\partial x} + gS\frac{\partial H}{\partial x} &= gS(I - J),
\end{aligned}
$$

Frequency-domain

$$
\begin{aligned}
h(s,x) &= G_i(s,x)q_i(s) - G_o(s,x)q_o(s) \\
G_i(s,x) &= \frac{\lambda_1(s)e^{\lambda_2(s)L+\lambda_1(s)x} - \lambda_2(s)e^{\lambda_1(s)L+\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\
G_o(s,x) &= \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})}
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Both are x-position dependent, hard to simulate in practice...

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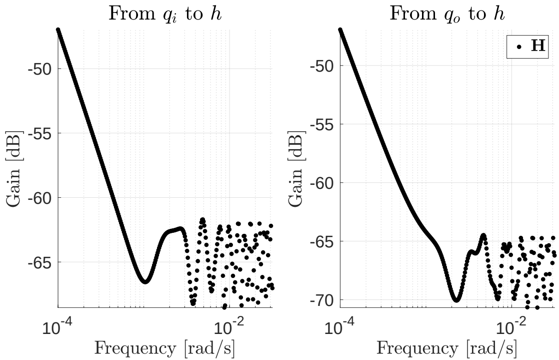

Model (integral and delay action)

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

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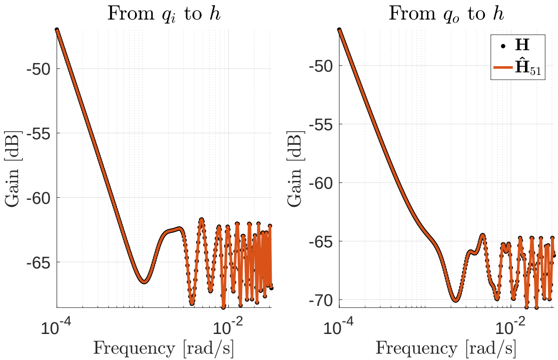

$E\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ (full interpolant, unstable)

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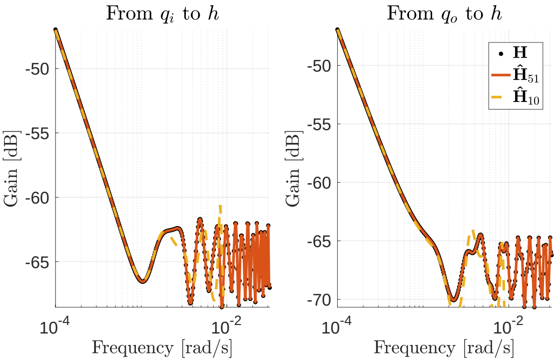

$\hat{E}\dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{B}\mathbf{u}$ (reduced, stable, integrator)

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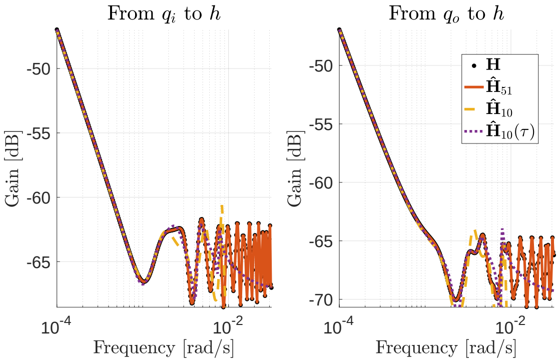

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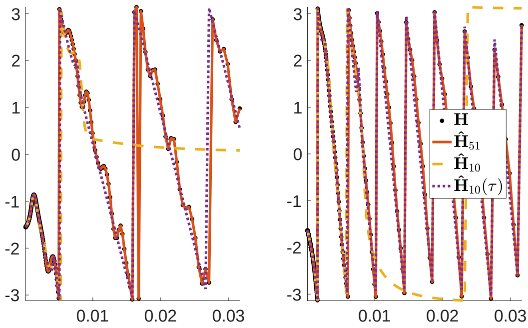

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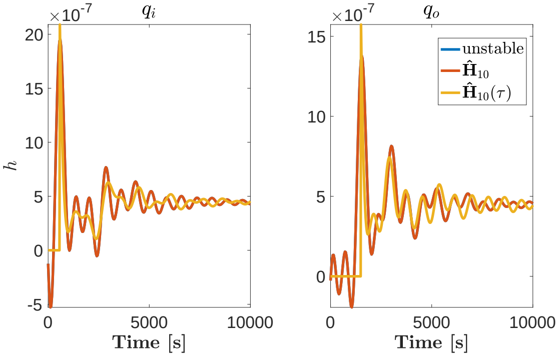

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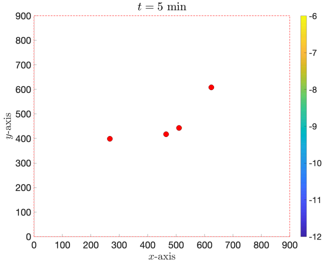
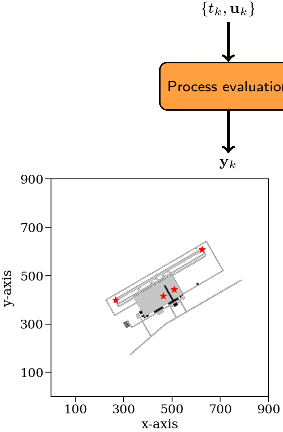
Use-case: pollutant modeling (data-driven)

Numerical data

- 3h LES complex simulation $\approx 7,500$h
- Time data over $\gg 1,000$ grid points





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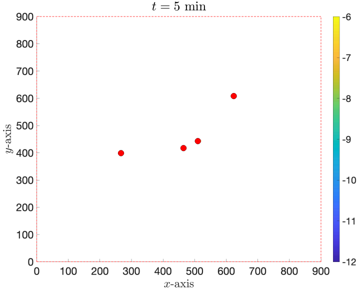
$\Longrightarrow$

Model Approximation & Inference

Accurate models

- $r = 30$
- **Q-ODE** model
- $\approx$ 1h (std. laptop)



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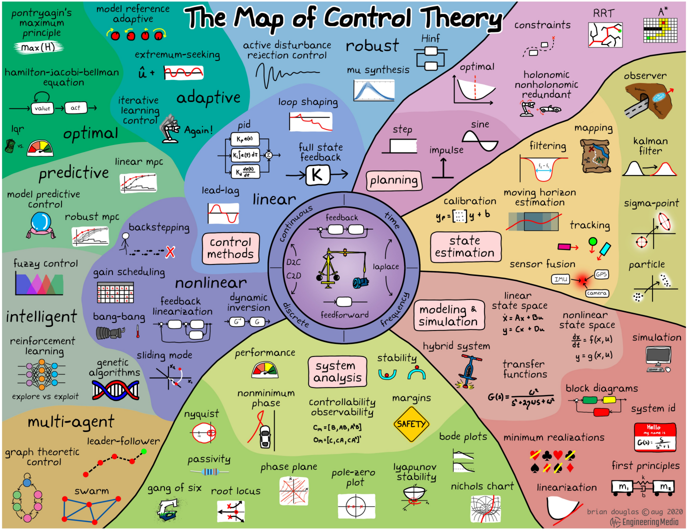

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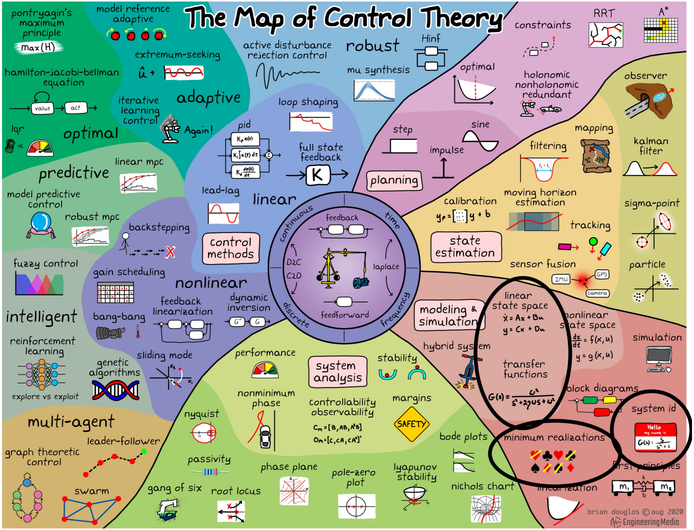

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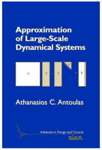
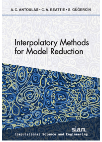


Antoulas/Beattie/Gugercin (2020)



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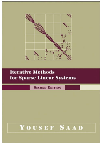

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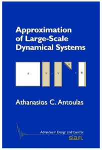

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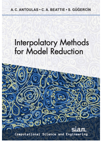

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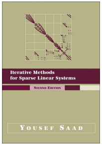

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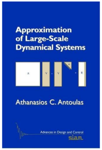

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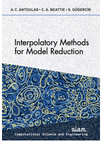

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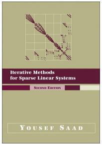

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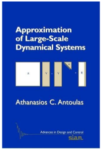

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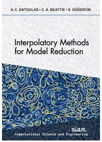

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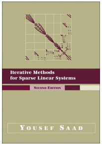

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Today's talk

Part 1 (reminder)

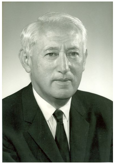
- Linear dynamical systems
- Realization and transfer functions

Part 2 (Loewner)

- Realization minimality
- Data-driven approximation
- Barycentric form

Part 3 (Loewner extended)

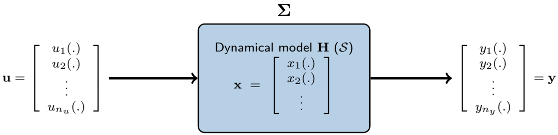
- Linear passive model (& pH)
- Linear parametric model
- Some non-linear models



Karel Löwner (Czech)
1893 - 1968
Ph.D. advisor: G.A. Pick

Content

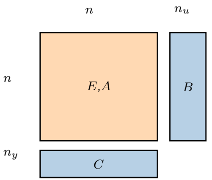
A dynamical model H (or $\mathcal{S}$) is a function mapping input u to output y signals of a system Σ



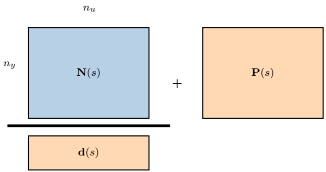
Let us stick (mainly) to linear systems only

Linear dynamical systems

Realization and transfer functions



$$\mathcal{S}:\begin{cases} E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$



$$\begin{aligned} \mathbf{H}(s) &= C\left(sE - A\right)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s) \end{aligned}$$

Realizations

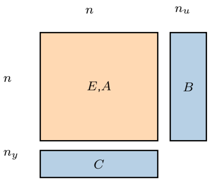
- E, A, B, C are (real) matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Realizations are infinite
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

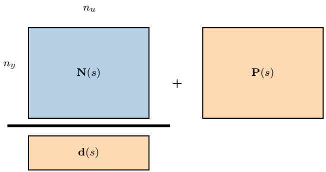
Transfer functions

- $\mathbf{H}$ is a (complex) function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Transfer functions are unique
- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$,
  $\mathbf{y}(s) \in \mathbb{C}^{n_y}$

Linear dynamical systems

Realization and transfer functions





$$\mathcal{S} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$

$$\begin{aligned} \mathbf{H}(s) &= C\left(sE - A\right)^{-1} B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s) \end{aligned}$$

Realizations

- E, A, B, C are (real) matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Realizations are infinite
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

Transfer functions

- $\mathbf{H}$ is a (complex) function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Transfer functions are unique
- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$,
  $\mathbf{y}(s) \in \mathbb{C}^{n_y}$

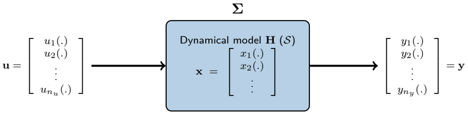
Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \to \mathbf{x} \to \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \to \mathbf{y}$



Linear dynamical systems

Linear finite dimensional models

Structures

- **L-ODE**
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1}$$

ODE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x \end{array}$$

Singularities of matrix pencil (A, E)

$$\Lambda(\mathcal{S}) = \Lambda(-1, 1) = \{-1\}$$

Structures
- L-ODE
- **L-ODE / DAE-1**
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Singularities of matrix pencil (A, E)

$$\Lambda(\mathcal{S}_1) = \Lambda(-1, 1) = \{-1\}$$

Structures

- L-ODE
- **L-ODE / DAE-1**
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + 2u \\ 0 & = & -x_2 + 2u = x_2 - 2u \\ y & = & x_1 + x_2 \end{array}$$

Singularities of matrix pencil (A, E)[a]

$$\Lambda(\mathcal{S}_2) = \Lambda\left(\left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]\right) = \{-1, \infty\}$$

[a] $B^T = \left[\begin{array}{cc} 2 & -2 \end{array}\right]$ and $C = \left[\begin{array}{cc} 1 & 1 \end{array}\right]$

Structures

- L-ODE
- **L-ODE / DAE-1**
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathcal{S}_2$ (canonical form)

$$\left(\left[\begin{array}{c|c} A_1 = -1 & \\ \hline & I_{n_2} = 1 \end{array} \right], \left[\begin{array}{c|c} I_{n_1} = 1 & \\ \hline & N = 0 \end{array} \right] \right)$$

Index is the k-nilpotent degree of N

- Finite dynamic modes
  $n_1 = 1$
- Infinite dynamic (impulsive) modes
  $\mathbf{rank}(E) - n_1 = \mathbf{rank}(N) = 1 - 1 = 0$
- Non dynamic modes
  $n - \mathbf{rank}(E) = 2 - 1 = 1$

Structures
- L-ODE
- L-ODE / DAE-1
- **L-DAE**
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil (A, E)

$$\Lambda(\mathcal{S}) = \{-1, \infty, \infty\}$$

- Finite dynamic modes $n_1 = 1$
- Impulsive modes $\mathbf{rank}(E) - n_1 = 2 - 1 = 1$
- Non dynamic modes $n - \mathbf{rank}(E) = 3 - 2 = 1$

Linear dynamical systems

Linear finite dimensional models

Structures

- L-ODE
- L-ODE / DAE-1
- **L-DAE**
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil (A, E)

$$\Lambda(\mathcal{S}) = \{-1, \infty, \infty\}$$

- Finite dynamic modes $n_1 = 1$
- Impulsive modes $\mathbf{rank}(E) - n_1 = 2 - 1 = 1$
- Non dynamic modes $n - \mathbf{rank}(E) = 3 - 2 = 1$

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- **L-DDE**
- L-PDE

Transfer function

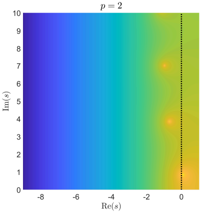
$$\mathbf{H}(s) = \frac{1}{s + e^{-ps}}$$

L-DDE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x(t-p) + u \\ y & = & x \end{array}$$

Singularities (periodic)

$$\Lambda(\mathcal{S}) = \{\omega \text{ s.t. } s + \cos(s) + \imath \sin(s) = 0\}$$



Structures

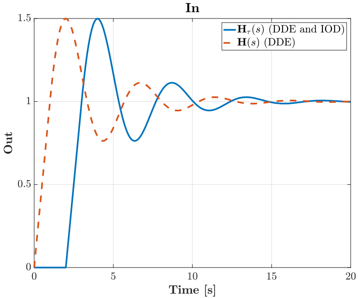
- L-ODE
- L-ODE / DAE-1
- L-DAE
- **L-DDE**
- L-PDE

Transfer function

$$\mathbf{H}_\tau(s) = \frac{1}{s + e^{-ps}} e^{-2s}$$

L-DDE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x(t-p) + u(t-2) \\ y & = & x \end{array}$$



Linear dynamical systems

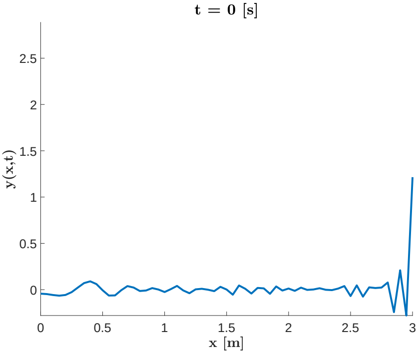
Linear infinite dimensional models

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$



Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$

L-PDE "realization" $\mathcal{S}$

$$\begin{array}{rcl} \dfrac{\partial \tilde{y}(x,t)}{\partial x} + 2x \dfrac{\partial \tilde{y}(x,t)}{\partial t} & = & 0 \\ \tilde{y}(x,0) & = & 0 \\ \tilde{y}(0,t) & = & \dfrac{1}{\sqrt{t}} \star \tilde{u}_f(0,t) \\ \dfrac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0,s) & = & u_f(0,s) \end{array}$$

Singularities

$$\Lambda(\mathcal{S}) = \{0, \lambda_1, \overline{\lambda_1}\}$$

Linear dynamical systems

Why all this? What is common?

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$

Linear dynamical systems

Why all this? What is common?

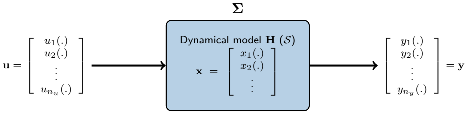
Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



Linear dynamical systems

Why all this? What is common?

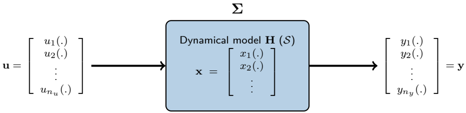
Structures
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Data
- Time-domain
- **Frequency-domain**

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



(Time-domain) $\{t_i, \mathbf{G}(t_i)\}_{i=1}^{N}$

(Frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^{N}$

Data

Content

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{v}_i^T\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^T & i = 1, \ldots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

MIMO tangential interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$
\begin{array}{ll}
\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\} & j = 1, \dots, k \\
\{\mu_i, \mathbf{l}_i^T, \mathbf{v}_i^T\} & i = 1, \dots, q
\end{array}
$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$
\begin{array}{rcll}
\mathbf{H}(\lambda_j)\mathbf{r}_j & = & \mathbf{w}_j & j = 1, \dots, k \\
\mathbf{l}_i^T \mathbf{H}(\mu_i) & = & \mathbf{v}_i^T & i = 1, \dots, q
\end{array}
$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

The right data can be expressed as:

$$
\begin{array}{rcl}
\Lambda & = & \mathbf{diag}\left[\lambda_1, \dots, \lambda_k\right] \in \mathbb{C}^{k \times k}, \\
\mathbf{R} & = & \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{n_u \times k} \\
\mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y \times k}
\end{array}
$$

and the left data can be expressed as:

$$
\begin{array}{rcl}
\mathbf{M} & = & \mathbf{diag}\left[\mu_1, \dots, \mu_q\right] \in \mathbb{C}^{q \times q} \\
\mathbf{L}^T & = & \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{n_y \times q} \\
\mathbf{V}^T & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{n_u \times q}
\end{array}
$$

The right data can be expressed as:

$$\begin{array}{rcl} \boldsymbol{\Lambda} & = & \mathbf{diag}\,[\lambda_1,\dots,\lambda_k] \in \mathbb{C}^{k\times k}, \\ \mathbf{R} & = & \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{n_u\times k} \\ \mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y\times k} \end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcl} \mathbf{M} & = & \mathbf{diag}\,[\mu_1,\dots,\mu_q] \in \mathbb{C}^{q\times q} \\ \mathbf{L}^T & = & \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{n_y\times q} \\ \mathbf{V}^T & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{n_u\times q} \end{array}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^T\mathbf{r}_1 - \mathbf{l}_1^T\mathbf{w}_1}{\mu_1-\lambda_1} & \dots & \frac{\mathbf{v}_1^T\mathbf{r}_k - \mathbf{l}_1^T\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^T\mathbf{r}_1 - \mathbf{l}_q^T\mathbf{w}_1}{\mu_q-\lambda_1} & \dots & \frac{\mathbf{v}_q^T\mathbf{r}_k - \mathbf{l}_q^T\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L} - \mathbb{L}\boldsymbol{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^T \mathbf{r}_1 - \mathbf{l}_1^T \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^T \mathbf{r}_k - \mathbf{l}_1^T \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^T \mathbf{r}_1 - \mathbf{l}_q^T \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^T \mathbf{r}_k - \mathbf{l}_q^T \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The shifted Loewner matrix is:

$$\mathbb{M} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^T \mathbf{r}_1 - \mathbf{l}_1^T \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^T \mathbf{r}_k - \mathbf{l}_1^T \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^T \mathbf{r}_1 - \mathbf{l}_q^T \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^T \mathbf{r}_k - \mathbf{l}_q^T \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{M} - \mathbb{M}\mathbf{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\mathbf{\Lambda}$$

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^T C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^T C(\mu_q E - A)^{-1} \end{bmatrix}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1} B\mathbf{r}_1, \ldots, (\lambda_k E - A)^{-1} B\mathbf{r}_k,\right]$$

of size $q \times n$ and $n \times k$ respectively, be the generalised tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^T \mathbf{r}_j - \mathbf{l}_i^T \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^T C(\mu_j E - A)^{-1} E(\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E[\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{M}]_{ij} &= \frac{\mu_i \mathbf{v}_i^T \mathbf{r}_j - \mathbf{l}_i^T \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^T C(\mu_j E - A)^{-1} A(\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A[\mathcal{R}_k]_j \end{aligned}$$

Loewner

Loewner pencil

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, let us define :

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of size $q \times n$ and $n \times k$ respectively, be the generalised tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^T \mathbf{r}_j - \mathbf{l}_i^T \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^T C(\mu_j E - A)^{-1} E(\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

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$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^T C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^T C(\mu_q E - A)^{-1} \end{bmatrix}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1} B\mathbf{r}_1, \dots, (\lambda_k E - A)^{-1} B\mathbf{r}_k,\right]$$

of size $q \times n$ and $n \times k$ respectively, be the generalised tangential observability and controllability matrices. Then,

$$\begin{array}{rcl} [\mathbb{L}]_{ij} & = & \dfrac{\mathbf{v}_i^T \mathbf{r}_j - \mathbf{l}_i^T \mathbf{w}_j}{\mu_i - \lambda_j} \\ & = & -\mathbf{l}_i^T C(\mu_j E - A)^{-1} E(\lambda_i E - A)^{-1} B\mathbf{r}_j \\ & = & -[\mathcal{O}_q]_i E[\mathcal{R}_k]_j \end{array}$$

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Assume that $k = q$, and let, be a regular pencil. Then

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is a descriptor realization of minimal interpolant of the data, *i.e.* , the rational function $\mathbf{H}(s) = \mathbf{W}(\mathbb{M} - s\mathbb{L})^{-1}\mathbf{V}$ interpolates the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}[\xi\mathbb{L} - \mathbb{M}] = \mathbf{rank}[\mathbb{L}, \; \mathbb{M}] = \mathbf{rank}\begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L}, \; \mathbb{M}] = Y\Sigma_l\tilde{X}^T, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = \tilde{Y}\Sigma_r X^*, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -Y^*\mathbb{L}X, \quad A = -Y^*\mathbb{M}X, B = -Y^*\mathbf{V}, C = \mathbf{W}X$$

Assume that $k = q$, and let, be a regular pencil. Then

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is a descriptor realization of minimal interpolant of the data, *i.e.* , the rational function $\mathbf{H}(s) = \mathbf{W}(\mathbb{M} - s\mathbb{L})^{-1}\mathbf{V}$ interpolates the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}[\xi\mathbb{L} - \mathbb{M}] = \mathbf{rank}[\mathbb{L}, \; \mathbb{M}] = \mathbf{rank}\begin{bmatrix} \mathbb{L} \\ \mathbb{M} \end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L}, \; \mathbb{M}] = Y\Sigma_l \tilde{X}^T, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{M} \end{bmatrix} = \tilde{Y}\Sigma_r X^*, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -Y^*\mathbb{L}X, \quad A = -Y^*\mathbb{M}X, B = -Y^*\mathbf{V}, C = \mathbf{W}X$$

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \dots, k$; $i = 1, \dots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

Facts

- underlying rational (r) order

$$\begin{aligned} r &= \mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) \\ &= \mathbf{rank}([\mathbb{L}, \mathbb{M}]) \\ &= \mathbf{rank}([\mathbb{L}^H, \mathbb{M}^H]^H) \end{aligned}$$

- and McMillan (ν) order

$$\nu = \mathbf{rank}(\mathbb{L})$$

- Both $\mathbb{L}$ and $\mathbb{M}$ are input-output independents.

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

Loewner

Loewner examples (simple case)

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 2$

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, $(\mathbb{M}, \mathbb{L})$ pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Loewner

Loewner examples (rectangular case)

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Sampled with

$$\begin{array}{c} \lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3 \\ \mu_1 = -1, \mu_2 = -2 \end{array}$$

Loewner

Loewner examples (rectangular case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization rectangular "$n = 2 \times 3$"

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{\dagger}\mathbf{V}$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{\dagger}\mathbf{V} \\ &= \frac{1}{s^2 - 4.650e - 16s + 1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

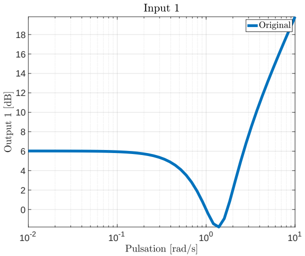
Loewner

Loewner examples (lot of data)

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Sampled with

$$\lambda_{1\dots 20} = [1, 2, \dots, 20]$$
$$\mu_{1\dots 20} = [1.5, 2.5, \dots, 20.5]$$



Loewner

Loewner examples (lot of data)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

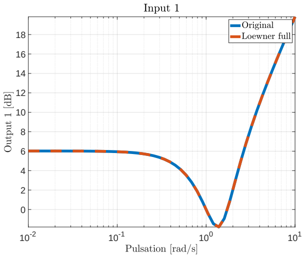
Realization $n = 20$

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Sampled with

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Loewner

Loewner examples (lot of data)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

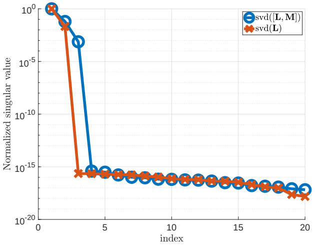
$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Sampled with

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Loewner

Loewner examples (lot of data)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

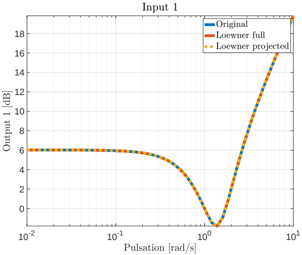
$r = 3$ and $\nu = 2$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}X(-sY^T\mathbb{L}X + Y^T\mathbb{M}X)^{-1}\dots \\ &\quad Y^T\mathbf{V} \\ &= \frac{s^2+s+2}{s+1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Sampled with

$$\lambda_{1\dots 20} = [1, 2, \dots, 20]$$
$$\mu_{1\dots 20} = [1.5, 2.5, \dots, 20.5]$$



Loewner

Loewner examples (complex dynamical tippe top case)

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Sampled with

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

Leads to

$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Loewner

Loewner examples (complex dynamical tippe top case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$$\mathbf{H}(s) = \frac{(1 + 2.22e - 16\imath)}{s^2 + (1+\imath)s + (1+\imath)}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Sampled with

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2 + \imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

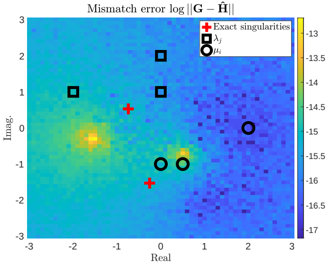
$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\hat{\mathbb{L}} = \begin{bmatrix} -0.207 + 0.9568\imath & -0.1276 - 0.0294\imath \\ -0.0438 - 0.0818\imath & 0.039 - 0.1089\imath \end{bmatrix}$$

$$\hat{\mathbb{M}} = \begin{bmatrix} 0.4738 - 0.0560\imath & 0.0089 - 0.3392\imath \\ 0.2253 - 0.1607\imath & -0.0342 + 0.1308\imath \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Sampled with

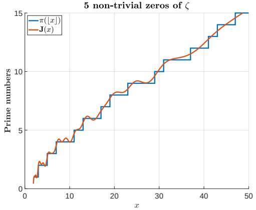
$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

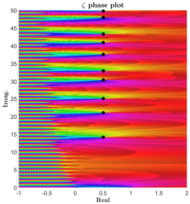
Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\hat{\mathbb{L}} = \begin{bmatrix} -0.207 + 0.9568\imath & -0.1276 - 0.0294\imath \\ -0.0438 - 0.0818\imath & 0.039 - 0.1089\imath \end{bmatrix}$$

$$\hat{\mathbb{M}} = \begin{bmatrix} 0.4738 - 0.0560\imath & 0.0089 - 0.3392\imath \\ 0.2253 - 0.1607\imath & -0.0342 + 0.1308\imath \end{bmatrix}$$

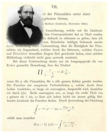




$$\pi(\lfloor x \rfloor) \approx \mathbf{Ri}(x) - \sum_{\rho_r} \mathbf{Ri}(x^{\rho_r}) - \sum_{\rho_i} \mathbf{Ri}(x^{\rho_i})$$

A continuation $(z \in \mathbb{C}_{\Re(z)\neq 1})$[a]

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

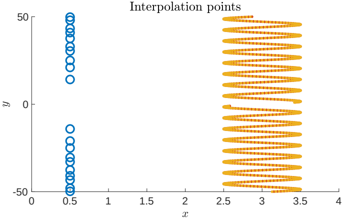


[a] Authors warmly thank **David Louapre** for the Python code allowing to compute the Riemann prime counting function.

C. P-V., I.V. Gosea, P. Vuillemin and A.C. Antoulas, *"Identifying the non-trivial zeros of the Riemann zeta function for prime counting function approximation in the Loewner framework"*, 10th International Conference on Curves and Surfaces, Arcachon, France, June, 2022.

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\pi(\lfloor x \rfloor) \approx \mathbf{Ri}(x) - \sum_{\rho_r} \mathbf{Ri}(x^{\rho_r}) - \sum_{\rho_i} \mathbf{Ri}(x^{\rho_i})$$

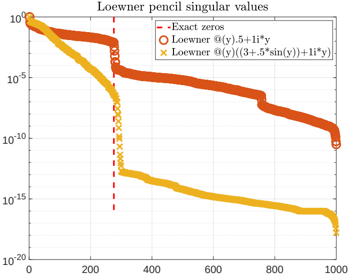
A continuation $(z \in \mathbb{C}_{\Re(z) \neq 1})$

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

C. P-V., I.V. Gosea, P. Vuillemin and A.C. Antoulas, *"Identifying the non-trivial zeros of the Riemann zeta function for prime counting function approximation in the Loewner framework"*, 10th International Conference on Curves and Surfaces, Arcachon, France, June, 2022.

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



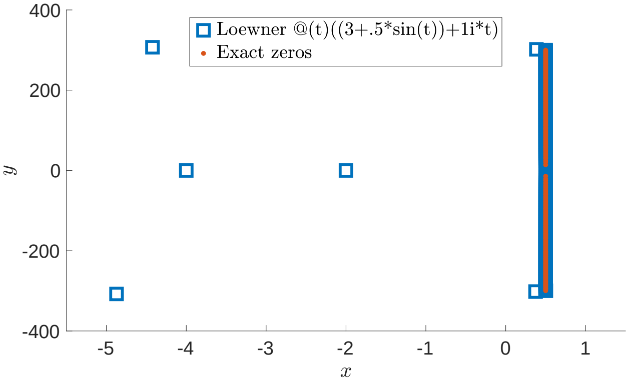
$$\pi(\lfloor x \rfloor) \approx \mathbf{Ri}(x) - \sum_{\rho_r} \mathbf{Ri}(x^{\rho_r}) - \sum_{\rho_i} \mathbf{Ri}(x^{\rho_i})$$

A continuation $(z \in \mathbb{C}_{\Re(z)\neq 1})$

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

C. P-V., I.V. Gosea, P. Vuillemin and A.C. Antoulas, *"Identifying the non-trivial zeros of the Riemann zeta function for prime counting function approximation in the Loewner framework"*, 10th International Conference on Curves and Surfaces, Arcachon, France, June, 2022.

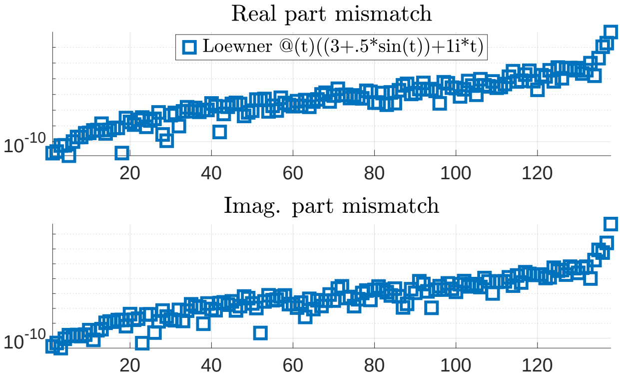


C. P-V., I.V. Gosea, P. Vuillemin and A.C. Antoulas, *"Identifying the non-trivial zeros of the Riemann zeta function for prime counting function approximation in the Loewner framework"*, 10th International Conference on Curves and Surfaces, Arcachon, France, June, 2022.

J-P. Berrut and L.N. Trefethen, *"Barycentric Lagrange interpolation"*, SIAM review, 2004.

Loewner

Loewner examples (Riemann)



C. P-V., I.V. Gosea, P. Vuillemin and A.C. Antoulas, *"Identifying the non-trivial zeros of the Riemann zeta function for prime counting function approximation in the Loewner framework"*, 10th International Conference on Curves and Surfaces, Arcachon, France, June, 2022.

J-P. Berrut and L.N. Trefethen, *"Barycentric Lagrange interpolation"*, SIAM review, 2004.

MIMO interpolation problem

Given the right the left data (λ_i and μ_j are distinct):

$$\begin{array}{ll} \{\lambda_i, \mathbf{w}_i\} & i = 1, \dots, k \\ \{\mu_j, \mathbf{v}_j^T\} & j = 1, \dots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_i) & = & \mathbf{w}_i & i = 1, \dots, k \\ \mathbf{H}(\mu_j) & = & \mathbf{v}_j^T & j = 1, \dots, q \end{array}$$

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{n_y q \times n_u k}$$ and let $\mathbb{L}\mathbf{c} = 0$

Consider system $\mathbf{G}$ in barycentric form

$$\mathbf{G}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{, where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \dots, \lambda_k\}, \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ and solving

$$\mathbb{L}\mathbf{c} = 0$$

leads to $\mathbf{H}$ in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}\mathbf{w}}_{C} \underbrace{\begin{bmatrix} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c} \end{bmatrix}^{-1}}_{\Phi(s)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$$\mathbf{L}_{t,x,n} = \begin{bmatrix} t - x_1 & x_2 - t & & \\ t - x_1 & & x_3 - t & \\ & & & \ddots \end{bmatrix} \in \mathbb{R}^{n \times (n+1)}$$

Consider system $\mathbf{G}$ in barycentric form

$$\mathbf{G}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{, where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \dots, \lambda_k\}, \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ and solving

$$\mathbb{L}\mathbf{c} = 0$$

leads to $\mathbf{H}$ in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{cw}}_{C} \underbrace{\begin{bmatrix} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c} \end{bmatrix}^{-1}}_{\mathbf{\Phi}(s)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$$\mathbf{L}_{t,x,n} = \begin{bmatrix} t - x_1 & x_2 - t & & \\ t - x_1 & & x_3 - t & \\ & & & \ddots \end{bmatrix} \in \mathbb{R}^{n \times (n+1)}$$

Loewner

Loewner barycentric examples (simple case)

$$\mathbf{G}(s) = \frac{2}{s+1}$$

Sampled with

$\lambda_1 = 1, \lambda_2 = 3$ and $\mu_1 = 2, \mu_2 = 4$

Leads to

$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2}$ and $\mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$

Loewner

Loewner barycentric examples (simple case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 2$

$$\mathbf{H}(s) = C\mathbf{\Phi}(s)^{-1}B = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{2}{s+1}$$

Sampled with

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\ker(\mathbb{L}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$C = \mathbf{cw} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \mathbf{L}_{s,\lambda,1} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} s-1 & 3-s \\ -1 & 2 \end{bmatrix}$$

Loewner

Loewner barycentric examples (complex dynamical tippe top case)

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Sampled with

$$\begin{array}{c} \lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath \\ \mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath \end{array}$$

Leads to

$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Loewner

Loewner barycentric examples (complex dynamical tippe top case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

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Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

Realization $n = 3$

$$C = \begin{bmatrix} \times & \times & \times \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\Phi}(s) = \begin{bmatrix} s-\imath & 2\imath - s & \\ s-\imath & & -2+\imath-s \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\ker(\mathbb{L}) = \begin{bmatrix} 0.76471 - 0.94118\imath \\ -2+2\imath \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Resulting in

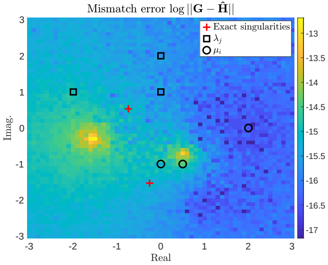
$$\frac{1 - 1.665e-16\imath}{s^2 + (1+1i)s + (1+1i)}$$

Loewner

Loewner barycentric examples (complex dynamical tippe top case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



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Sampled with

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$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

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Content

Loewner extensions

More structures and properties

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- L-pH
- pL-DAE
- B-DAE
- Q-DAE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$

- (Q, J, R, G, P, N, S) and $\mathbf{H}(s)$
- (E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
- (A, B, C, N) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$
- (A, B, C, Q) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$

Loewner extensions

More structures and properties

Structures

L-ODE

L-ODE / DAE-1

L-DAE

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pL-DAE

B-DAE

Q-DAE

Model

- (A, B, C) and $\mathbf{H}(s)$
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- (E, A, B, C) and $\mathbf{H}(s)$
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- (E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
- (A, B, C, N) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$
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Loewner extensions

Passive & pH

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Passivity

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u} \end{array}$$

$$\mathbf{\Phi}(s) = \mathbf{H}^T(-s) + \mathbf{H}(s)$$

Passive $= \mathbf{\Phi}(\imath\omega) > 0$ & stable & $D \succ 0$

Structures

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Passivity

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Passive $= \mathbf{\Phi}(\imath\omega) > 0$ & stable & $D \succ 0$

pH

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^T Q\mathbf{x} + (N+S)\mathbf{u} \end{array}$$

$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^T & N \end{bmatrix}$ and $\mathcal{W} = \begin{bmatrix} R & P \\ P^T & S \end{bmatrix}$

satisfy $\mathcal{V} = -\mathcal{V}^T$, $\mathcal{W} = \mathcal{W}^T \succeq 0$ and $Q = Q^T \succeq 0$

Loewner extensions

Structures

- L-ODE
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- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Loewner extensions

Passive & pH

Structures

- L-ODE
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Transfer function

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

L-pH realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{x} & = & (0-1)1x + (-2-1.4142)u \\ y & = & (-2+1.4142)x + (0+2)u \end{array}$$

where $\mathcal{V} = -\mathcal{V}^T$, $\mathcal{W} = \mathcal{W}^T \succeq 0$ and $Q = Q^T \succeq 0$

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} 1 & 1.4142 \\ 1.4142 & 2 \end{bmatrix}$$

Loewner extensions

Passive & pH

The right data can be expressed as:

$$\begin{array}{rcl} \Lambda & = & \mathbf{diag}\,[\lambda_1, \ldots, \lambda_k] \in \mathbb{C}^{k\times k}, \\ \mathbf{R} & = & \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{m\times k} \\ \mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{m\times k} \end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcccl} \mathbf{M} & = & -\Lambda^H & = & \mathbf{diag}\,[\mu_1, \ldots, \mu_q] \in \mathbb{C}^{q\times q} \\ \mathbf{L}^T & = & \mathbf{R}^H & = & \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{m\times q} \\ \mathbf{V}^T & = & -\mathbf{W}^H & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{m\times q} \end{array}$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions

Passive & pH

The right data can be expressed as:

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\Lambda &= \textbf{diag}\,[\lambda_1, \dots, \lambda_k] \in \mathbb{C}^{k\times k},\\
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\mathbf{V}^T &= -\mathbf{W}^H &&= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{m\times q}
\end{aligned}$$

Spectral zeros are $(\lambda_j, \mathbf{r}_j)$ from standard Loewner (n zeros in the open right half-plane)

$$\begin{bmatrix} 0 & A & B \\ A^T & 0 & C^T \\ B^T & C & D+D^T \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} 0 & E & 0 \\ E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions

Passive & pH

The right data can be expressed as:

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\Lambda & = & \mathbf{diag}\,[\lambda_1,\dots,\lambda_k] \in \mathbb{C}^{k\times k},\\
\mathbf{R} & = & \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k\end{bmatrix} \in \mathbb{C}^{m\times k}\\
\mathbf{W} & = & \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k\end{bmatrix} \in \mathbb{C}^{m\times k}
\end{array}
$$

and the left data can be expressed as:

$$
\begin{array}{rcccl}
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\mathbf{V}^T & = & -\mathbf{W}^H & = & \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q\end{bmatrix} \in \mathbb{C}^{m\times q}
\end{array}
$$

Construct an $m \times m$ real transfer function $\mathbf{H}(s)$ of McMillan degree n using self-conjugate interpolation conditions as follows:

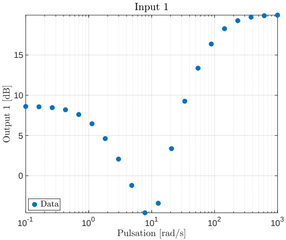
$$\mathbf{H}(\infty) = D \ , \ \mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{r}_j^H\mathbf{H}(-\overline{\lambda_j}) = -\mathbf{w}_j^H$$

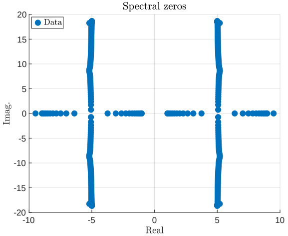
where $D + D^T \succ 0$ and $\mathbb{L} \succ 0$, then $\mathcal{S}$ of $\mathbf{H}(s)$ is a normalised pH form.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions

Passive & pH example (RLC circuit)

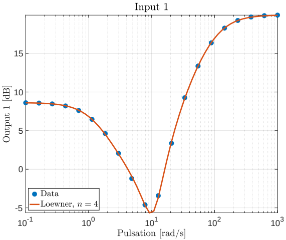


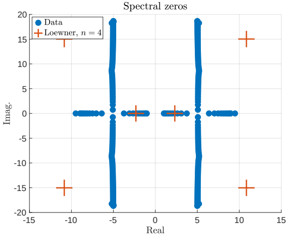


$$\mathbf{G}(s) = \text{RLC circuit } (n = 200)$$

Loewner extensions

Passive & pH example (RLC circuit)





$\mathbf{G}(s) =$ RLC circuit $(n = 200)$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

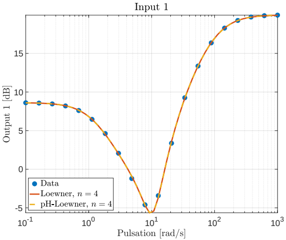
Loewner extensions

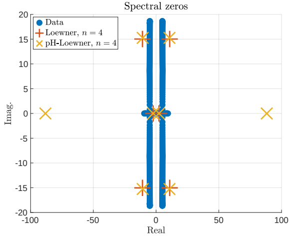
Passive & pH example (RLC circuit)

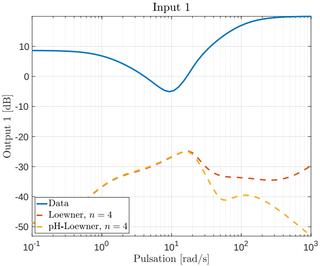
$$\mathbf{G}(s) = \text{RLC circuit } (n = 200)$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(-\overline{\lambda_j}) = -\mathbf{w}_j^H$$







Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

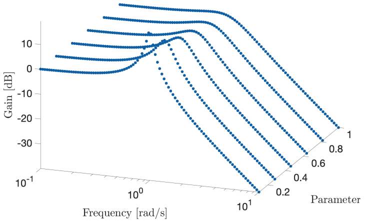
- L-pH
- **pL-DAE**
- B-DAE
- Q-DAE

Transfer function (degree $r = 2$ in s and $q = 1$ in p)

$$\mathbf{G}(s, p) = \frac{1}{s^2 + ps + 1}$$

Sampled as

$$\begin{array}{rcl} [s_1, \dots, s_{100}] & = & \imath[10^{-1}, \dots, 10] \\ [p_1, \dots, p_6] & = & [0.1, \dots, 1] \end{array}$$



Loewner extensions

Parametric models

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE
- L-pH
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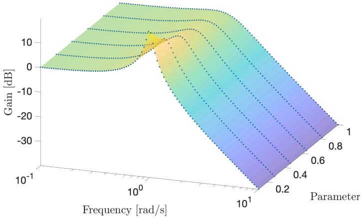

Loewner extensions

Parametric Loewner barycentric

SISO/MIMO parametric interpolation problem

Given the data (λ_i, μ_k, π_j and ν_l are distinct):

$$
\begin{array}{rcl}
[s_1,\ldots,s_N] & = & [\lambda_1,\ldots,\lambda_{\overline{n}}] \cup [\mu_1,\ldots,\mu_{\underline{n}}] \\
[p_1,\ldots,p_M] & = & [\pi_1,\ldots,\pi_{\overline{m}}] \cup [\nu_1,\ldots,\nu_{\underline{m}}] \\
\mathbf{\Phi} & = & \left[\begin{array}{c|c} \mathbf{w}_{ij} & \mathbf{\Phi}_{12} \\ \hline \mathbf{\Phi}_{21} & \mathbf{v}_{kl} \end{array}\right]
\end{array}
$$

we seek $\mathcal{S}(p) : (E, A(p), B(p), C(p))$, whose transfer function is $\mathbf{H}(s,p) = C(p)(sE - A(p))^{-1}B(p)$ s.t.

$$
\begin{array}{rcll}
\mathbf{H}(\lambda_i,\pi_j) & = & \mathbf{w}_{ij} & i = 1\ldots\overline{n}/j = 1,\ldots,\underline{n} \\
\mathbf{H}(\mu_k,\nu_l) & = & \mathbf{v}_{kl}^T & k = 1\ldots\overline{m}/l = 1,\ldots,\underline{m}
\end{array}
$$

A.C. Ionita and A.C. Antoulas, *"Data-Driven Parametrized Model Reduction in the Loewner Framework"*, SIAM on Scientific Computing, vol. 36(3).

T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, *"Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework"*, submitted to SIAM.

Loewner extensions

Rational parametric functions and barycentric form

$$
\begin{aligned}
[s_1, \dots, s_N] &= [\lambda_1, \dots, \lambda_{\overline{n}}] \cup [\mu_1, \dots, \mu_{\underline{n}}] \\
[p_1, \dots, p_N] &= [\pi_1, \dots, \pi_{\overline{m}}] \cup [\nu_1, \dots, \nu_{\underline{m}}] \\
\mathbf{\Phi} &= \left[\begin{array}{c|c} \mathbf{w}_{ij} & \mathbf{\Phi}_{12} \\ \hline \mathbf{\Phi}_{21} & \mathbf{v}_{kl} \end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{L}_2]_{i,j}^{k,l} &= \frac{\mathbf{v}_{kl} - \mathbf{w}_{ij}}{(\mu_k - \lambda_i)(\nu_l - \pi_j)} \\
[\mathbb{L}_{\lambda_i}] &= \text{one variable Loewner of } i\text{th row of } \mathbf{\Phi} \\
[\mathbb{L}_{\pi_j}] &= \text{one variable Loewner of } j\text{th column of } \mathbf{\Phi}
\end{aligned}
$$

$$
\widehat{\mathbb{L}}_2 \mathbf{c} = \begin{bmatrix} \mathbb{L}_2 \\ \mathbb{L}_\lambda \\ \mathbb{L}_\pi \end{bmatrix} \mathbf{c} = 0
$$

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Loewner extensions

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A.C. Ionita and A.C. Antoulas, *"Data-Driven Parametrized Model Reduction in the Loewner Framework"*, SIAM on Scientific Computing, vol. 36(3).

Two-variable rational interpolation

For any partitioning with $\overline{n} \geq n+1$ and $\overline{m} \geq m+1$, which follows the structure of the Lagrange basis, the rank of the two variable Loewner matrix satisfies

$$\mathbf{rank}(\mathbb{L}_2) = \overline{nm} - (\overline{n} - n)(\overline{m} - m).$$

Then, setting $(\overline{n}, \overline{m}) = (n+1, m+1)$ leads to a singular $\mathbb{L}_2$ with rank equal to $(n+1)(m+1) - 1$. Then, $\mathbf{H}(s,p)$ is recovered by the barycentric form with

$$\alpha_{ij} = c_{ij} \text{ and } \beta_{ij} = c_{ij}\mathbf{w}_{ij}$$

given by the vector $\mathbf{c} = [c_{1,1} \dots c_{1,m+1} | \dots | c_{n+1,1} \dots {}_{n+1,m+1}]^T$ computed from the null space of the Loewner matrix $\widehat{\mathbb{L}}_2$, *i.e.*

$$\widehat{\mathbb{L}}_2 \mathbf{c} = 0.$$

$$\mathbf{H}(s,p) = \frac{\sum_{i=1}^{r+1} \sum_{j=1}^{q+1} \frac{c_{ij}\mathbf{w}_{ij}}{(s-\lambda_i)(p-\pi_j)}}{\sum_{i=1}^{r+1} \sum_{j=1}^{q+1} \frac{c_{ij}}{(s-\lambda_i)(p-\pi_j)}}$$

Loewner extensions

Rational parametric functions and barycentric form ($p = n_u$)

$$E = \begin{bmatrix} I_p & -I_p & & \\ \vdots & & \ddots & \\ I_p & & & -I_p \\ 0 & 0 & \dots & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} \lambda_1 I_p & -\lambda_2 I_p & & \\ \vdots & & \ddots & \\ \lambda_1 I_p & & & -\lambda_{\overline{n}} I_p \\ \alpha_1(p) & \alpha_2(p) & \dots & \alpha_{\overline{n}}(p) \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_1(p) & \beta_2(p) & \dots & \beta_{\overline{n}}(p) \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & \dots & I_p \end{bmatrix}^T$$

where

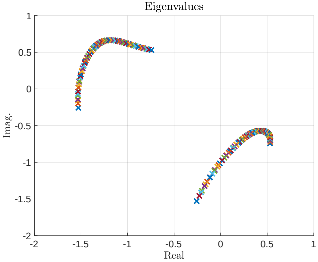
$$\alpha_i(p) = \sum_{j=1}^{\overline{m}} c_{ij} \mathbf{q}_j(p) \qquad \beta_i(p) = \sum_{j=1}^{\overline{m}} \mathbf{w}_{ij} c_{ij} \mathbf{q}_j(p) \qquad \mathbf{q}_j(p) = \prod_{j'=1, j' \neq i}^{n} (p - \pi_{j'})$$

T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, *"Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework"*, submitted to SIAM.

Loewner extensions

Examples (complex parametric dynamical tippe top case)

$$\mathbf{G}(s,p) = \frac{1}{s^2 + (1+p^2\imath)s + (p+\imath)}$$



Loewner extensions

Examples (complex parametric dynamical tippe top case)

Rational function satisfies

$\mathbf{H}(\lambda_i, \pi_j) = \mathbf{w}_{ij}$ and $\mathbf{H}(\mu_k, \nu_l) = \mathbf{v}_{kl}$

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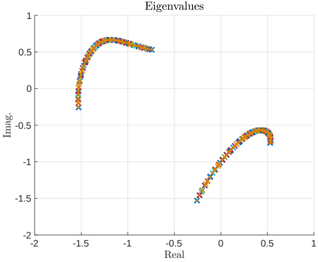
Function orders 2 in s and 2 in p.
Realization order is of order 3.

$$\begin{aligned} \mathbf{H}(s, p) &= (C + C_1 p + C_2 p^2) \ldots \\ & (sE - A - A_1 p - A_2 p^2)^{-1} B \end{aligned}$$



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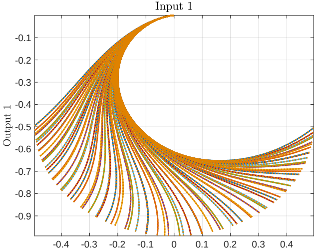
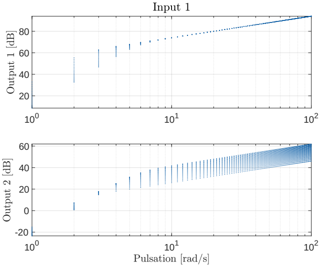
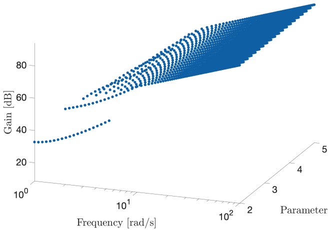


Loewner extensions

Examples (pL-DAE)



$$\mathbf{G}(s,p) = \frac{\begin{bmatrix} s^2+p+1000s^4 \\ p^2s^4 \end{bmatrix}}{2s^3+3p^3+0.1s^2p-1}$$



Loewner extensions

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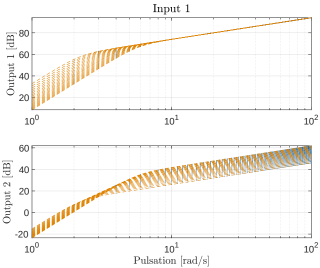

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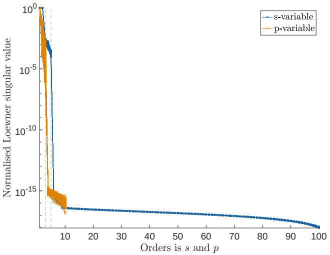

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Ranks drop at 5 and 3. Realization of order 7.

Loewner extensions

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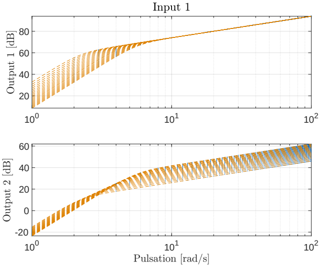

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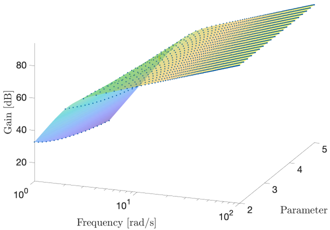

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Loewner extensions

Nonlinear model

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- L-pH
- pL-DAE
- **B-DAE**
- Q-DAE

Nonlinear **ODE** $\mathcal{S}$ (Langmuir kinetic)

$$\dot{x} = -\alpha x + \beta(x_0 - x)u \text{ and } y = x$$

B-ODE $\mathcal{S}$

$$\dot{x} = \underbrace{-\alpha}_{A} x + \underbrace{x_0\beta}_{B} u \underbrace{-\beta}_{N} xu \text{ and } y = \underbrace{1}_{C} x$$

Transfer function is a multivariate coupled infinite cascade of linear systems

$$\begin{array}{rcl} \mathbf{H}_1(s_1) & = & C\mathbf{\Phi}(s_1)B \\ \mathbf{H}_2(s_1,s_2) & = & C\mathbf{\Phi}(s_2)N\mathbf{\Phi}(s_1)B \\ \mathbf{H}_3(s_1,s_2,s_3) & = & C\mathbf{\Phi}(s_3)N\mathbf{\Phi}(s_2)N\mathbf{\Phi}(s_1)B \\ & \vdots & \end{array}$$

where $\mathbf{\Phi}(s) = (sE - A)^{-1} = (s+\alpha)^{-1}$

$$\mathbf{H}_1(s_1) = \frac{\beta x_0}{s_1+\alpha} \; , \; \mathbf{H}_2(s_1,s_2) = \frac{-\beta^2 x_0}{(s_2+\alpha)(s_1+\alpha)}$$

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$$\ddot{x}(t) + \delta\dot{x}(t) + \alpha x(t) + \beta x^3(t) = u(t),$$

Q-ODE $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_1 & = & -\delta x_1 - \alpha x_2 - \beta z_1 + u \\ \dot{x}_2 & = & x_1 \\ \dot{z}_1 & = & 3x_1 z_2 \\ \dot{z}_2 & = & 2x_1 x_2 \end{array}$$

or

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + Q(\mathbf{x} \otimes \mathbf{x}) \text{ and } \mathbf{y} = C\mathbf{x}$$

Transfer function is a multivariate coupled infinite cascade of linear systems

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Loewner extensions

Examples (Bilinear Langmuir kinetic)

$$\dot{x} = -0.5x + 4.5u - 0.5xu \ , \ y = x$$

$$\mathbf{G}_1(s_1) = \frac{4.5}{s_1 + 0.5}$$

$$\mathbf{G}_2(s_1, s_2) = \frac{-2.25}{(s_2 + 0.5)(s_1 + 0.5)}$$

A.C. Antoulas, I.V. Gosea and A.C. Ionita, *"Model reduction of bilinear systems in the Loewner framework"*, SIAM Journal on Scientific Computing 38 (5), 2016.

Loewner extensions

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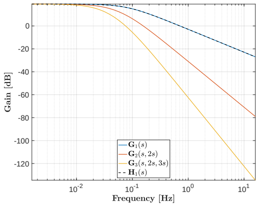

Rational function satisfies

$\mathbf{H}(\lambda_j) = \mathbf{G}_1(\lambda_j)$ and $\mathbf{H}(\mu_i) = \mathbf{G}_1(\mu_i)$

$\dot{x} = -0.5x + 2.121u$ and $y = 2.121x$

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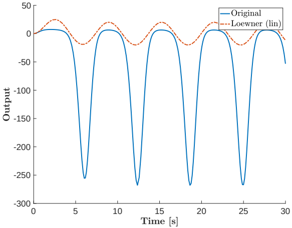

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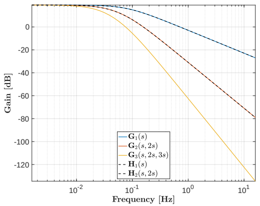

Right/left multi-tuples $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$

$$\begin{array}{rcl} \boldsymbol{\lambda} & = & \{\lambda_1\},\{\lambda_2,\lambda_1\}\dots \\ \boldsymbol{\mu} & = & \{\mu_1\},\{\mu_1,\mu_2\}\dots \end{array}$$

$$\mathbf{H}_1(\lambda_1) = \mathbf{G}_1(\lambda_1)\ ,\ \mathbf{H}_1(\mu_1) = \mathbf{G}_1(\mu_1)$$

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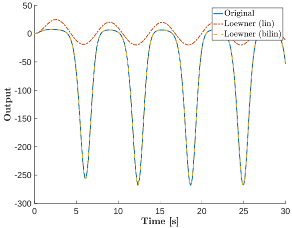

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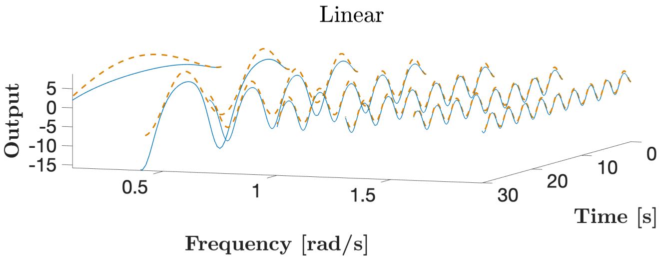
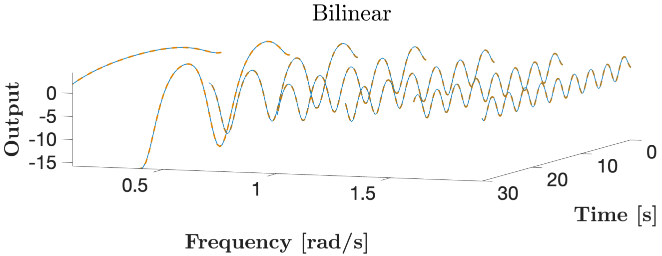




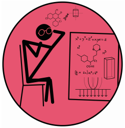
Content

Conclusions

Loewner... a versatile tool

- solves the LTI realization problem
- solves data-driven model reduction
- solves data-driven model approximation
- ... and pH, parametric, bilinear, quadratic...

  $\rightarrow$ direct impact in engineers life



... still so much to do

- Technical references and slides at https://sites.google.com/site/charlespoussotvassal/
- MOR Digital Systems numerical suite http://mordigitalsystems.fr/

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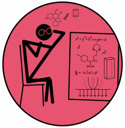

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mor Digital Systems

Loewner landmark

Bridge between realization, approximation and identification

Charles Poussot-Vassal
October 11, 2024

"Merge data and physics using computational sciences and engineering"

Conclusions

The map of mathematics (by Dominic Walliman)