

Loewner Framework for data-driven reduced order modeling

... a bridge between realization, approximation and identification

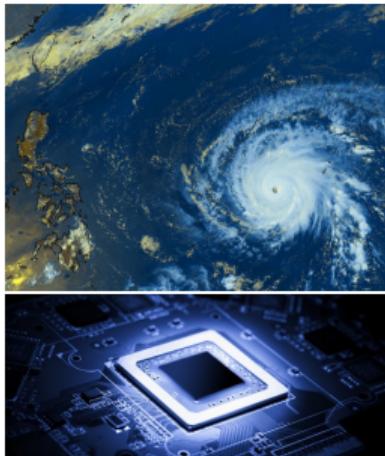
C. Poussot-Vassal
September 22, 2025



Forewords

Dynamical models, what for?

Dynamical models are centrals tools in engineering...



Digitalisation and computer-based modeling for

- ▶ simulation, optimisation, understanding
- ▶ control, estimation, analysis...

However

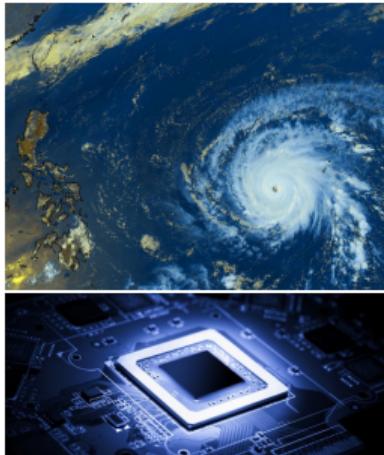
Finite machine precision, computational burden, memory management and actual solvers

- ▶ induces important time consumption
- ▶ generate inaccurate results
- ▶ limit the class of models to deal with
- ▶ limit the amount of data to treat

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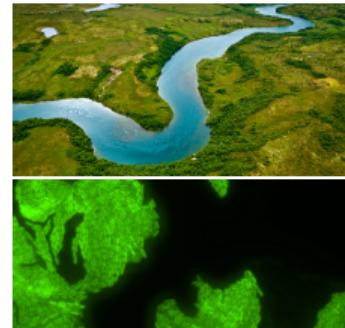
Develop robust and efficient tools to construct simplified dynamical models

Forewords

Dynamical models, what for?

... and in systems and control engineering

- ▶ **for verification and validation**
(μ , \mathcal{H}_∞ -norm, pseudo-spectra, Monte Carlo)
- ▶ **for detection**
(fault isolation, param. estim.)
- ▶ **for uncertainty propagation**
(Multi Disc. Optim., robust optim.)
- ▶ **for feedback control synthesis**
($\mathcal{H}_\infty/\mathcal{H}_2$ -norm, MPC, adaptive)



Complex models

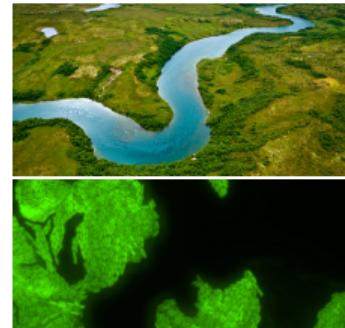
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Complex models

- ▶ important sim. time
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Model simplification

Simplified models

- ▶ reduced sim. time
- ▶ memory saving
- ▶ accurate results
- ▶ rational model

Forewords

Use-case: EDF hydroelectric open-channel (model-driven)

Hydraulic-driven electricity

- ▶ Dams & run-of-the-river ($\approx 10\%$)
- ▶ Run-of-the-river ($\approx 5\%$)
- ▶ Rely on [open-channel hydraulic systems](#)
- ▶ Need for analysis and control



January 4th, 2023

Forewords

Use-case: EDF hydroelectric open-channel (model-driven)

Time-domain

$$\begin{aligned}\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial(Q^2/S)}{\partial x} + gS \frac{\partial H}{\partial x} &= gS(I - J),\end{aligned}$$

Frequency-domain

$$\begin{aligned}h(s, x) &= G_i(s, x)q_i(s) - G_o(s, x)q_o(s) \\ G_i(s, x) &= \frac{\lambda_1(s)e^{\lambda_2(s)L + \lambda_1(s)x} - \lambda_2(s)e^{\lambda_1(s)L + \lambda_2(s)x}}{B_0s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\ G_o(s, x) &= \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{B_0s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})}\end{aligned}$$

Both are x -position dependent, hard to simulate in practice...



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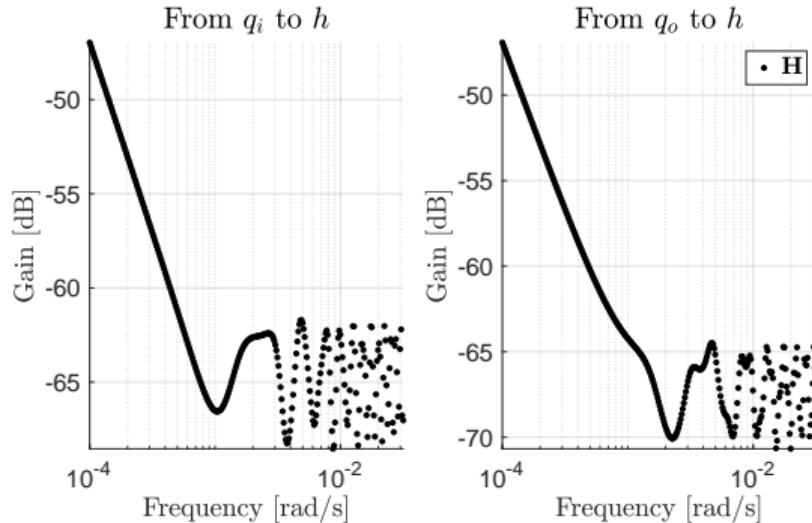
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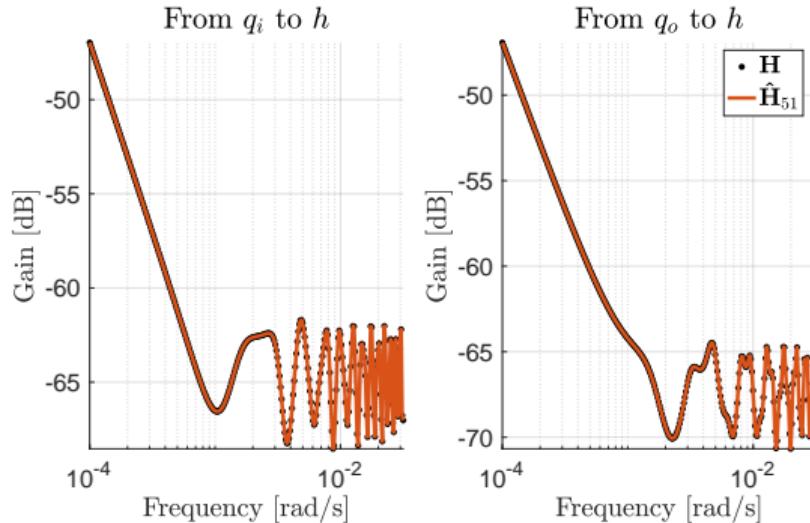
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(integral and delay action)
- ▶ $E\dot{\mathbf{x}} = A\mathbf{x} + Bu$
 H_{51}
(full interpolant, **unstable**)
- ▶ $\hat{E}\dot{\mathbf{x}} = \hat{A}\mathbf{x} + \hat{B}u$
 \hat{H}_{10}
(reduced, **stable, integrator**)
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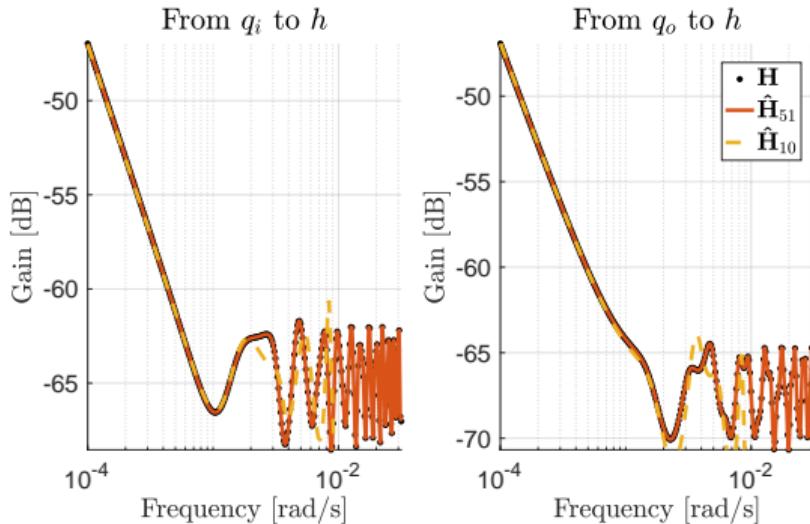
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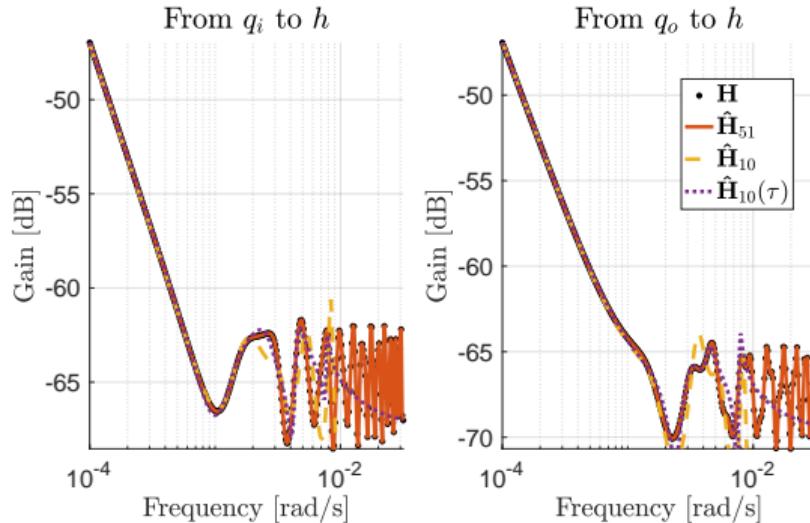
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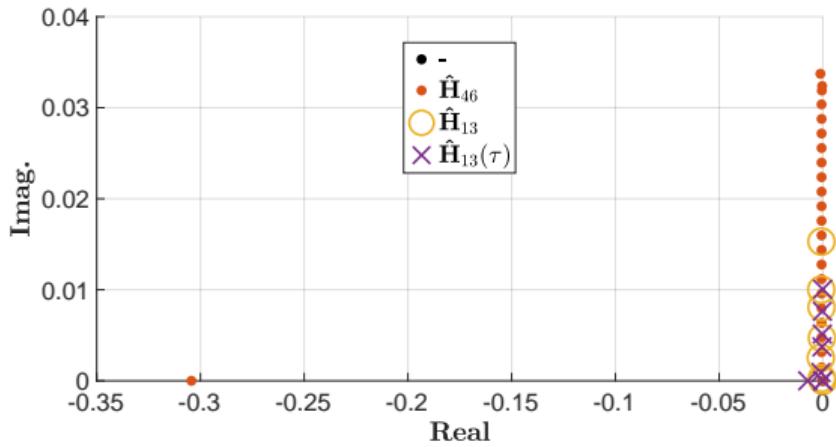
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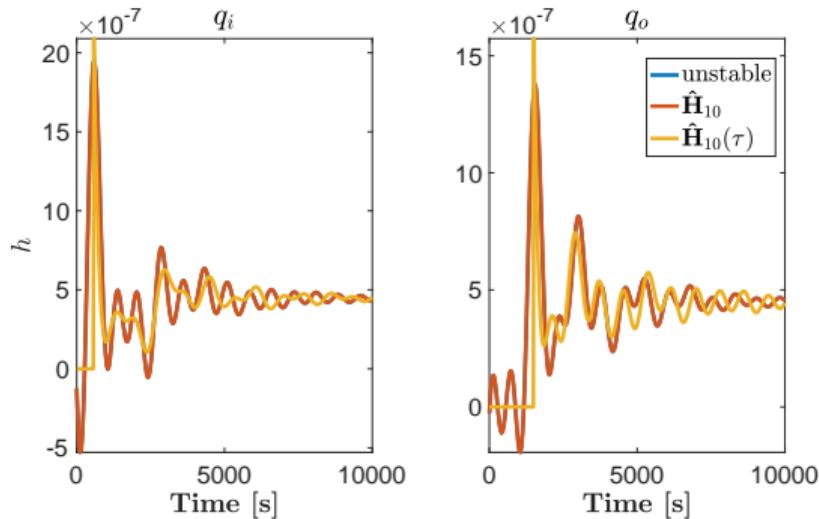
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ANTENNA RESEARCH

Antenna models

- ▶ to optimize parameters
- ▶ for polar computation

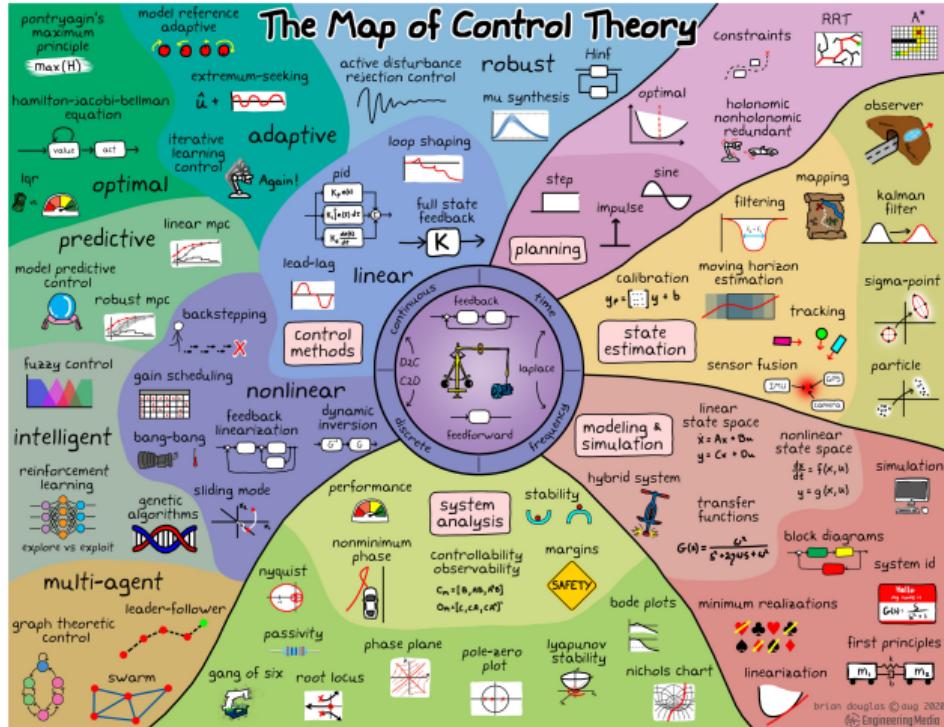
Blend physics from

- ▶ Maxwell equations
- ▶ Kirchoff equations

- ▶ Replace costly simulations by accurate simple model
- ▶ Preserve structure and properties (port-Hamiltonian)
- ▶ Allows for geometry optimization

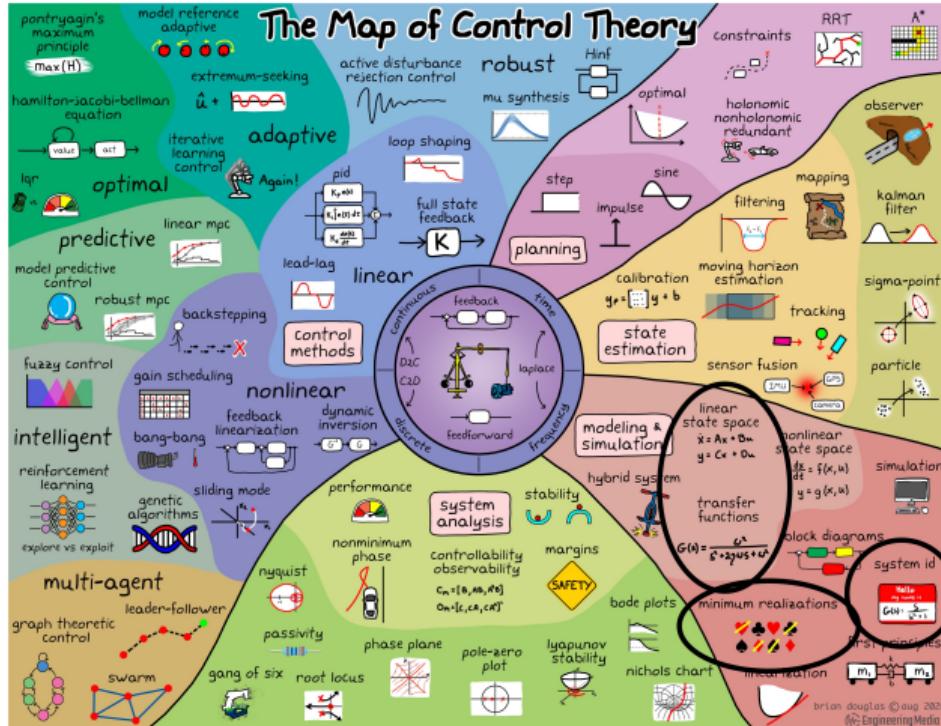
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The map of control theory (by Brian Douglas - <https://engineeringmedia.com/>)



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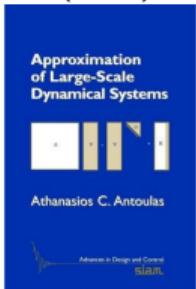
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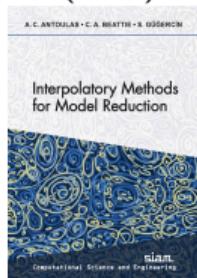
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Antoulas/Beattie/Gugercin
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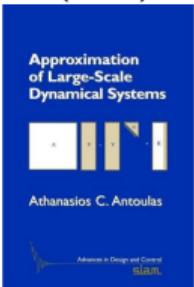


- ▶ Pencil & realization [Antoulas/Mayo/Trefethen/Embree/Ionita/...]
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 - ▶ B-LTI, Q-LTI [Antoulas/Benner/Gosea/Karachalios/Pontes/Willcox/P-V./...]
 - ▶ pHIS [Van-Dooren/Beattie/Gugercin/Benner/Schwerdtner/Matignon/...]
- side. Stability analysis [Vuillemin/P-V.]
side. Control [Kergos/Vuillemin/P-V.]
side. Discretization [Vuillemin/P-V.]
appli. Aircraft gust, vibration & flutter [Quero/Vuillemin/Reis/P-V.]
appli. Prime counting [Antoulas/Gosea/Vuillemin/P-V.]

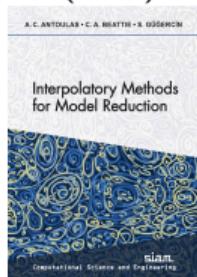
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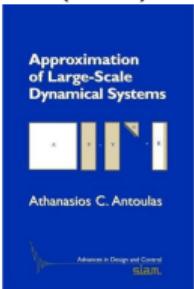
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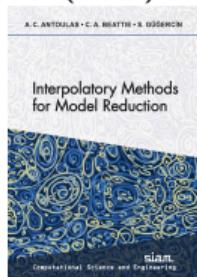
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Today's talk

Part 1 (reminder)

- ▶ Linear dynamical systems
- ▶ Realization and transfer functions

Part 2 (Loewner)

- ▶ Realization minimality
- ▶ Data-driven approximation
- ▶ Barycentric form

Part 3 (Loewner extended)

- ▶ Linear passive model (& pH)
- ▶ Linear parametric model
- ▶ Nonlinear model (quadratic & bilinear)



*Karel Löwner (Czech)
1893 - 1968
Ph.D. advisor: G.A. Pick*

Content

Forewords

Linear dynamical systems

Loewner

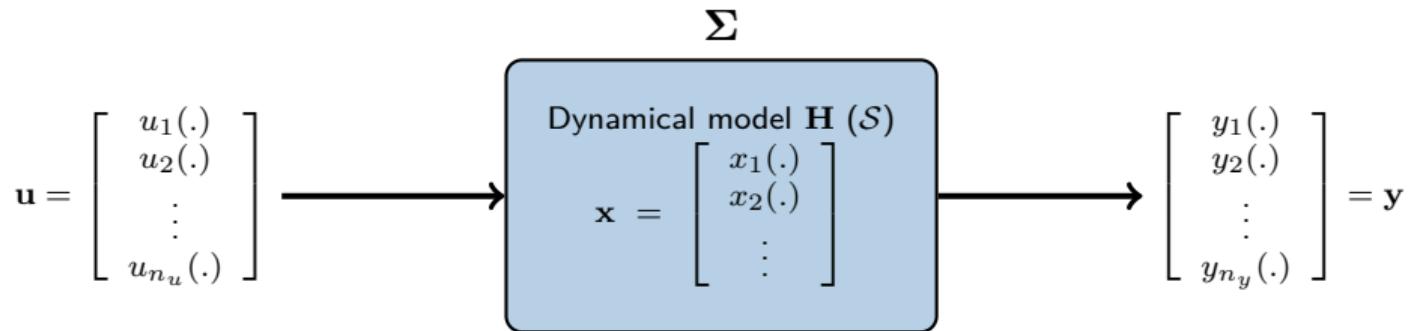
Loewner extensions

Conclusions

Linear dynamical systems

Simplifications

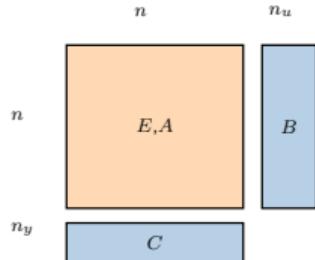
A dynamical model H (or \mathcal{S}) is a function mapping input u to output y signals of a system Σ



Let us stick (mainly) to linear systems only

Linear dynamical systems

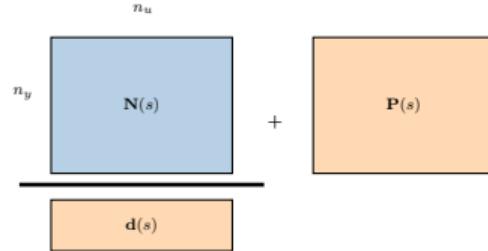
Realization and transfer functions



$$\mathcal{S} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$

Realizations

- E, A, B, C are (real) matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Realizations are infinite
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$
 $\mathbf{y}(t) \in \mathbb{R}^{n_y}$
 $\mathbf{x}(t) \in \mathbb{R}^n$



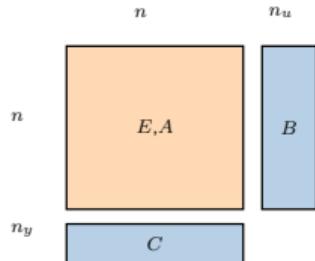
$$\begin{aligned} \mathbf{H}(s) &= C(sE - A)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s) \end{aligned}$$

Transfer functions

- \mathbf{H} is a (complex) function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Transfer functions are unique
- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$
 $\mathbf{y}(s) \in \mathbb{C}^{n_y}$

Linear dynamical systems

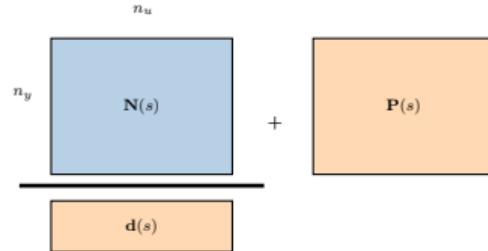
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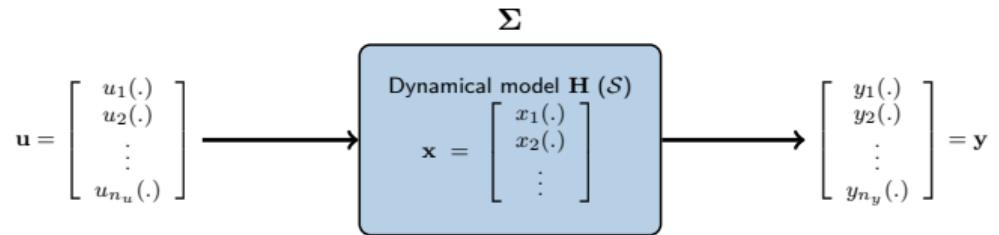
Model, data and structures

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

$$\begin{array}{ll} \text{(Time-domain)} & \mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y} \\ \text{(Frequency-domain)} & \mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y} \end{array}$$



Linear dynamical systems

Linear finite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

$$H(s) = \frac{2}{s+1}$$

ODE realization \mathcal{S}

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x\end{aligned}$$

Singularities λ and zeros z

$$\begin{aligned}\lambda_{\mathcal{S}} &= \text{eig}(A, E) \\ &= \Lambda(-1, 1) \\ &= \{-1\} \\ z_{\mathcal{S}} &= \text{eig}([A \ B; \ C \ D], \text{blkdiag}(E, \text{zeros}(ny, nu))) \\ &= \Lambda\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\ &= \{\infty, \infty\}\end{aligned}$$

Linear dynamical systems

Linear finite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization \mathcal{S}_1

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x + 2u\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}_1} = \Lambda(-1, 1) = \{-1\}$$

Linear dynamical systems

Linear finite dimensional models

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L-ODE / DAE-1

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L-PDE

Transfer function

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DAE index-1 realization \mathcal{S}_2

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2u \\ 0 &= -x_2 + 2u = x_2 - 2u \\ y &= x_1 + x_2\end{aligned}$$

Singularities of matrix pencil $(A, E)^a$

$$\lambda_{\mathcal{S}_2} = \Lambda \left(\left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & \\ \hline & 1 \end{array} \right] \right) = \{-1, \infty\}$$

$$^a B^\top = \left[\begin{array}{cc} 2 & -2 \end{array} \right] \text{ and } C = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$$

Linear dynamical systems

Linear finite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization \mathcal{S}_2 (canonical form)

$$\left(\left[\begin{array}{c|c} A_1 = -1 & \\ \hline I_{n_2} = 1 & \end{array} \right], \left[\begin{array}{c|c} I_{n_1} = 1 & \\ \hline N = 0 & \end{array} \right] \right)$$

Index is the k -nilpotent degree of N

- ▶ Finite dynamic modes
 $n_1 = 1$
- ▶ Infinite dynamic (impulsive) modes
 $\text{rank}(E) - n_1 = \text{rank}(N) = 1 - 1 = 0$
- ▶ Non dynamic modes
 $n - \text{rank}(E) = 2 - 1 = 1$

Linear dynamical systems

Linear finite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2 + s + 2}{s+1}$$

DAE index-2 realization \mathcal{S}

$$\begin{aligned}\dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ x_2 &= -x_3 + u = x_3 - u \\ y &= x_1 + x_2 + 2x_3\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}} = \{-1, \infty, \infty\}$$

- ▶ Finite dynamic modes $n_1 = 1$
- ▶ Impulsive modes $\text{rank } (E) - n_1 = 2 - 1 = 1$
- ▶ Non dynamic modes $n - \text{rank } (E) = 3 - 2 = 1$

Linear dynamical systems

Linear finite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

$$H(s) = \frac{2}{s+1} + s = \frac{s^2 + s + 2}{s+1}$$

DAE index-2 realization \mathcal{S}

$$\begin{aligned}\dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ x_2 &= -x_3 + u = x_3 - u \\ y &= x_1 + x_2 + 2x_3\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}} = \{-1, \infty, \infty\}$$

- ▶ Finite dynamic modes $n_1 = 1$
- ▶ Impulsive modes $\text{rank } (E) - n_1 = 2 - 1 = 1$
- ▶ Non dynamic modes $n - \text{rank } (E) = 3 - 2 = 1$

Linear dynamical systems

Linear finite dimensional models (about observability, controllability, minimality...)

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

- ▶ Reachability matrix

$$\mathcal{R}_n(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

- ▶ Observability matrix

$$\mathcal{O}_n(A, C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2 C^\top & \cdots & (A^\top)^n C^\top \end{bmatrix}^\top$$

- ▶ Minimality: both controllable and observable.

- ▶ Connection with Markov parameters

$$H_0 = D, H_k = CA^{k-1}B \ (k > 1)$$

$$\mathcal{O}_n \mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

Linear dynamical systems

Linear infinite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function

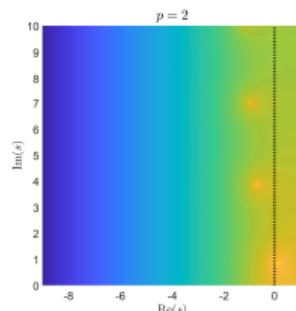
$$\mathbf{H}(s) = \frac{1}{s + e^{-ps}}$$

L-DDE realization \mathcal{S}

$$\begin{aligned}\dot{x} &= -x(t-p) + u \\ y &= x\end{aligned}$$

Singularities (periodic)

$$\lambda_{\mathcal{S}} = \{\omega \text{ s.t. } s + \cos(s) - i \sin(s) = 0\}$$



Linear dynamical systems

Linear infinite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

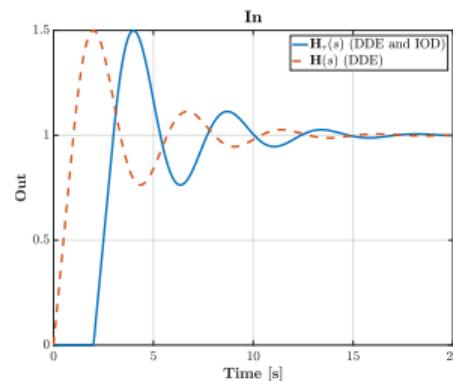
L-PDE

Transfer function

$$\mathbf{H}_\tau(s) = \frac{1}{s + e^{-ps}} e^{-2s}$$

L-DDE realization \mathcal{S}

$$\begin{aligned}\dot{x} &= -x(t-p) + u(t-2) \\ y &= x\end{aligned}$$



Linear dynamical systems

Linear infinite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$

Linear dynamical systems

Linear infinite dimensional models

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$

L-PDE "realization" \mathcal{S}

$$\begin{aligned}\frac{\partial \tilde{y}(x, t)}{\partial x} + 2x \frac{\partial \tilde{y}(x, t)}{\partial t} &= 0 \\ \tilde{y}(x, 0) &= 0 \\ \tilde{y}(0, t) &= \frac{1}{\sqrt{t}} \star \tilde{u}_f(0, t)\end{aligned}$$

$$\frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0, s) = u_f(0, s)$$

Singularities

$$\lambda_{\mathcal{S}} = \{0, \lambda_1, \bar{\lambda}_1\}$$

Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

- ▶ (A, B, C) and $\mathbf{H}(s)$
- ▶ (A, B, C, D) and $\mathbf{H}(s)$
- ▶ (E, A, B, C) and $\mathbf{H}(s)$
- ▶ $(A_i \dots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- ▶ $\mathbf{H}(s)$

Linear dynamical systems

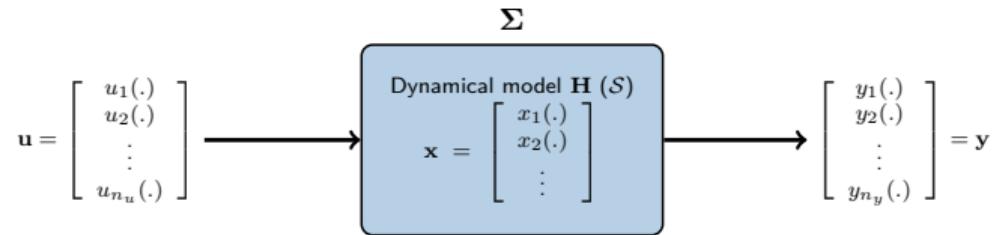
Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

$$\begin{array}{ll} \text{(Time-domain)} & \mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y} \\ \text{(Frequency-domain)} & \mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y} \end{array}$$



Linear dynamical systems

Why all this? What is common?

Structures

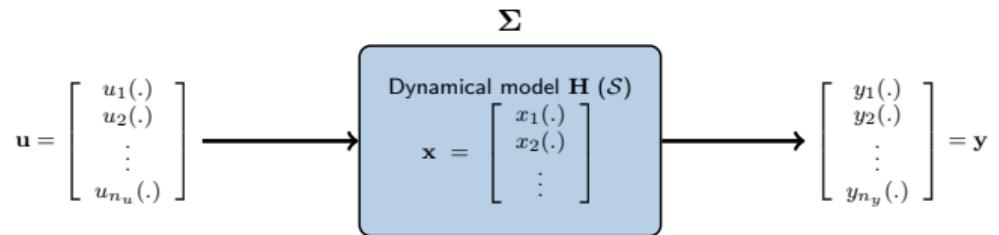
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Data

- Time-domain
- Frequency-domain

Model

$$\begin{array}{ll} \text{(Time-domain)} & \mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y} \\ \text{(Frequency-domain)} & \mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y} \end{array}$$



$$\begin{array}{ll} \text{(Time-domain)} & \{t_i, \mathbf{G}(t_i)\}_{i=1}^N \\ \text{(Frequency-domain)} & \{z_i, \mathbf{G}(z_i)\}_{i=1}^{\bar{N}} \end{array}$$

Data

Content

Forewords

Linear dynamical systems

Loewner

Loewner extensions

Conclusions

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{aligned}\{\lambda_j, \mathbf{w}_j\} \quad j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} \quad i = 1, \dots, q\end{aligned}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{aligned}\mathbf{H}(\lambda_j) &= \mathbf{w}_j \quad j = 1, \dots, k \\ \mathbf{H}(\mu_i) &= \mathbf{v}_i^\top \quad i = 1, \dots, q\end{aligned}$$

 A.J. Mayo and A.C. Antoulas, "*A framework for the solution of the generalized realization problem*", Linear Algebra and its Applications, vol. 425(2-3), 2007.

 I.V. Gosea, C. P-V. and A.C. Antoulas, "*Data-driven modeling and control of large-scale dynamical systems in the Loewner framework*", Handbook in Numerical Analysis, vol. 23, January 2022.

Loewner Matrix

Given a **row / left** array of pairs of complex numbers $\{\mu_i, \mathbf{v}_i\}$, $i = 1, \dots, q$, and a **column / right** array of pairs of complex numbers $\{\lambda_j, \mathbf{w}_j\}$, $j = 1, \dots, k$ with μ_i and λ_j distinct, the associated Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}.$$

If there is an underlying function \mathbf{H} then

$$\mathbf{w}_j = \mathbf{H}(\lambda_j) \text{ and } \mathbf{v}_i = \mathbf{H}(\mu_i).$$

We will present the Loewner framework in connection with

- (i) rational interpolation and consequently in connection with
- (ii) reduced-order modeling of linear dynamical systems given frequency domain data.

The Loewner matrix rank encodes the minimal complexity of the solutions of the interpolation problem.

$$P := \{(x_i, y_i) : i = 1, \dots, N, x_i \neq x_j, i \neq j\}$$

The idea behind the present approach to rational interpolation is to use a formula for rational interpolants which is similar to the one defining the Lagrange polynomial. First we partition the array P in two disjoint subarrays:

$$\begin{aligned} P_c &:= \{(\lambda_j, \mathbf{w}_j) \mid j = 1, \dots, k\} \\ P_r &:= \{(\mu_i, \mathbf{v}_i) \mid i = 1, \dots, q\} \end{aligned}$$

e.g. $N = k + q$

$$\begin{aligned} \lambda_j &= x_j & \mathbf{w}_j &= y_j & j &= 1, \dots, k \\ \mu_i &= x_{k+i} & \mathbf{v}_i &= y_{k+i} & i &= 1, \dots, q \end{aligned}$$

Using array P_c , for constants α_j and \mathbf{w}_j , consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^k \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0, \quad \alpha_j \neq 0$$

For constants α_j and \mathbf{w}_j , consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^k \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0, \quad \alpha_j \neq 0$$

solving for \mathbf{H} , we get

$$\mathbf{H}(s) = \frac{\sum_{j=1}^k \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^k \frac{\alpha_j}{s - \lambda_j}}$$

It follows that

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j$$

ensures P_c constraints by construction. This is the barycentric (rational) Lagrange interpolation formula.

The free parameters α_j can be determined so that the additional constraints contained in array P_r are satisfied:

$$\mathbf{H}(\mu_i) = \mathbf{v}_i$$

Then it follows $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^T - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^T - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^T - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^T - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k.$$

Lemma

Given the rational function \mathbf{H} and an array of points P , where $y_j = \mathbf{H}(x_j)$ and x_j is not a pole of \mathbf{H} , let \mathbb{L} be a $q \times k$ Loewner matrix, for some partitioning P_c, P_r of P . Then

$$q, k \geq \deg(\mathbf{H}) \Rightarrow \text{rank } \mathbb{L} = \deg(\mathbf{H})$$

Loewner

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

$$\begin{cases} P_c &:= \{(\lambda_j; \mathbf{w}_j), j = 1, \dots, k\} \\ P_r &:= \{(\mu_i; \mathbf{v}_i), i = 1, \dots, q\} \end{cases}$$

Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q \times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

$$M\mathbb{L} - \mathbb{L}\Lambda = V - W$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\sum_{j=1}^k \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^k \frac{\alpha_j}{s - \lambda_j}}$$

Null space

$$\text{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k$$

Data generated from $\mathbf{H}(s) = (s^2 + 4)/(s + 1)$ of complexity (2), with $k = 3, q = 4$

$$\left. \begin{array}{rcl} \lambda_j & = & [1, 3, 5] \\ \mu_i & = & [2, 4, 6, 8] \end{array} \right\} \xrightarrow{\mathbf{H}} \left\{ \begin{array}{rcl} \mathbf{w}_j & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_i & = & [8/3, 4, 40/7, 68/9] \end{array} \right.$$

Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix} \in \mathbb{C}^{4 \times 3}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Null space

$$\text{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Consider system \mathbf{H} in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)}, \text{ where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^n (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \dots, \lambda_k\}, \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ and solving

$$\mathbf{L}\mathbf{c} = 0$$

leads to \mathbf{H} in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}^\top \mathbf{w}}_C \underbrace{\left[\begin{array}{c} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c}^\top \end{array} \right]}_{\Phi(s)^{-1}}^{-1} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B$$

$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s - \lambda_1 & \lambda_2 - s & & \\ s - \lambda_1 & & \lambda_3 - s & \\ \vdots & & \ddots & \end{bmatrix} \in \mathbb{R}^{n \times (n+1)} \text{ and } \Phi(s) \in \mathbb{C}^{(n+1) \times (n+1)}$$

Consider system \mathbf{H} in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)}, \text{ where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^n (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \dots, \lambda_k\}, \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ and solving

$$\mathbf{L}\mathbf{c} = 0$$

leads to \mathbf{H} in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}^\top \mathbf{w}}_C \underbrace{\left[\begin{array}{c} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c}^\top \end{array} \right]}_{\Phi(s)^{-1}}^{-1} \underbrace{\left[\begin{array}{c} 0 \\ 1 \end{array} \right]}_B$$

$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s - \lambda_1 & \lambda_2 - s & & \\ s - \lambda_1 & & \lambda_3 - s & \\ \vdots & & \ddots & \end{bmatrix} \in \mathbb{R}^{n \times (n+1)} \text{ and } \Phi(s) \in \mathbb{C}^{(n+1) \times (n+1)}$$

Loewner

Loewner barycentric examples (simple case, CAS=1)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

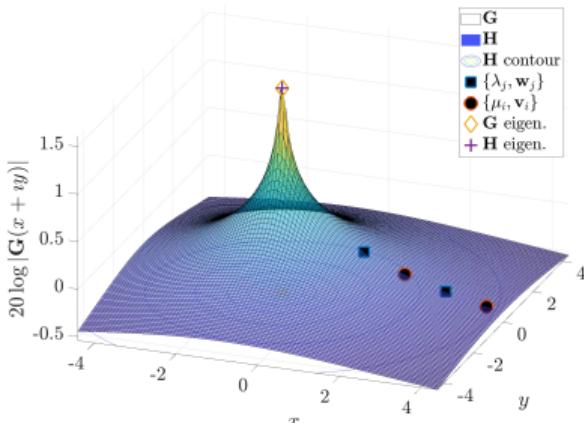
$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\mathbb{L} = \left[\begin{array}{cc} \frac{\frac{2}{3}-1}{\frac{2}{3}-1} & \frac{\frac{2}{3}-\frac{1}{2}}{\frac{2}{3}-3} \\ \frac{\frac{2}{3}-1}{\frac{2}{5}-1} & \frac{\frac{2}{5}-\frac{1}{2}}{\frac{2}{5}-3} \end{array} \right] = \left[\begin{array}{cc} -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 2$, $\mathbf{H}(s) = C\Phi(s)^{-1}B = \mathbf{G}(s)$



$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\ker(\mathbb{L}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{c}$$

$$C = \mathbf{c}^\top \mathbf{w} = \begin{bmatrix} -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \mathbf{L}_{s,\lambda,1} \\ \mathbf{c}^\top \end{bmatrix} = \begin{bmatrix} s-1 & 3-s \\ -1 & 2 \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1 + i)s + (1 + i)}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= i, \lambda_2 = 2i, \lambda_3 = -2 + i \\ \mu_1 &= -i, \mu_2 = 2, \mu_3 = 0.5 - i\end{aligned}$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\mathbb{L} = \begin{bmatrix} -\frac{1}{5} + \frac{3}{5}i & -\frac{1}{34} + \frac{13}{34}i & \frac{5}{17} + \frac{3}{17}i \\ \frac{17}{290} + \frac{59}{290}i & \frac{57}{986} + \frac{75}{986}i & \frac{31}{986} - \frac{63}{986}i \\ -\frac{46}{265} + \frac{108}{265}i & -\frac{37}{901} + \frac{209}{901}i & \frac{118}{901} + \frac{64}{901}i \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 3$

$$C = [c_1 \mathbf{w}_1 \quad c_2 \mathbf{w}_2 \quad c_3 \mathbf{w}_3], \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} s - i & 2i - s & -2 + i - s \\ s - i & c_2 & c_3 \\ c_1 & & \end{bmatrix}$$

Resulting in

$$\frac{-i}{-s^2i + s(1-i) + (1-i)} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+i)s + (1+i)}$$

Evaluated at

$$\begin{aligned} \lambda_1 &= i, \lambda_2 = 2i, \lambda_3 = -2 + i \\ \mu_1 &= -i, \mu_2 = 2, \mu_3 = 0.5 - i \end{aligned}$$

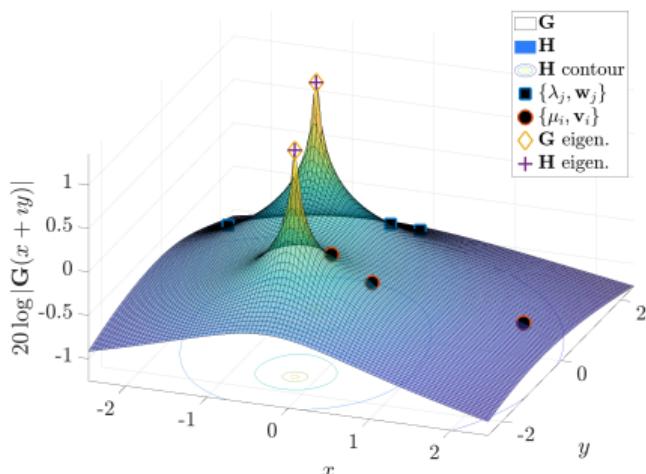
Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\ker(\mathbb{L}) = \begin{bmatrix} 13/17 - 16i/17 \\ -2 + 2i \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2 + (1 + i)s + (1 + i)}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= i, \lambda_2 = 2i, \lambda_3 = -2 + i \\ \mu_1 &= -i, \mu_2 = 2, \mu_3 = 0.5 - i\end{aligned}$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

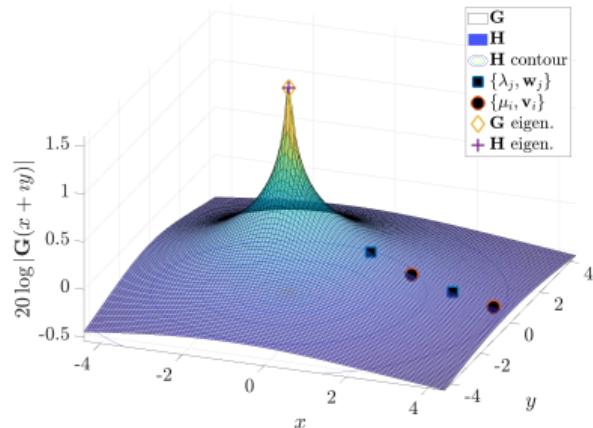
$$\ker(\mathbb{L}) = \begin{bmatrix} 13/17 - 16i/17 \\ -2 + 2i \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

```
% Model (CAS=1)
S = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP , responses
la = [1 3]; k = length(la);
mu = [2 4]; q = length(mu);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
LL = lf.loewnerMatrix(la,mu,W,V);
c = null(sym(LL));
% Realization
C = c.*W(:).';
B = zeros(k,1); B(end) = 1;
PHI = [s-la(1)*ones(k-1,1) diag(la(2:end))-s*eye(k-1); c.'];
Hr = simplify(C*(PHI\B));
```

```
% Model (CAS=1)
S = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP , responses
la = [1 3]; k = length(la);
mu = [2 4]; q = length(mu);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
LL = lf.loewnerMatrix(la,mu,W,V);
c = null(sym(LL));
% Realization
C = c.*W(:).';
B = zeros(k,1); B(end) = 1;
PHI = [s-la(1)*ones(k-1,1) diag(la(2:end))-s*eye(k-1); c.'];
Hr = simplify(C*(PHI\B));
```



SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{aligned} \{\lambda_j, \mathbf{w}_j\} \quad j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} \quad i = 1, \dots, q \end{aligned}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{aligned} \mathbf{H}(\lambda_j) &= \mathbf{w}_j \quad j = 1, \dots, k \\ \mathbf{H}(\mu_i) &= \mathbf{v}_i^\top \quad i = 1, \dots, q \end{aligned}$$

 A.J. Mayo and A.C. Antoulas, "*A framework for the solution of the generalized realization problem*", Linear Algebra and its Applications, vol. 425(2-3), 2007.

 I.V. Gosea, C. P-V. and A.C. Antoulas, "*Data-driven modeling and control of large-scale dynamical systems in the Loewner framework*", Handbook in Numerical Analysis, vol. 23, January 2022.

MIMO tangential interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{aligned} \{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\} \quad j = 1, \dots, k \\ \{\mu_i, \mathbf{l}_i^\top, \mathbf{v}_i^\top\} \quad i = 1, \dots, q \end{aligned}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{aligned} \mathbf{H}(\lambda_j)\mathbf{r}_j &= \mathbf{w}_j \quad j = 1, \dots, k \\ \mathbf{l}_i^\top \mathbf{H}(\mu_i) &= \mathbf{v}_i^\top \quad i = 1, \dots, q \end{aligned}$$

 A.J. Mayo and A.C. Antoulas, "*A framework for the solution of the generalized realization problem*", Linear Algebra and its Applications, vol. 425(2-3), 2007.

 I.V. Gosea, C. P-V. and A.C. Antoulas, "*Data-driven modeling and control of large-scale dynamical systems in the Loewner framework*", Handbook in Numerical Analysis, vol. 23, January 2022.

The **right data** can be expressed as:

$$\begin{aligned}\mathbf{\Lambda} &= \mathbf{diag} [\lambda_1, \dots, \lambda_k] \in \mathbb{C}^{k \times k}, \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{n_u \times k} \\ \mathbf{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y \times k}\end{aligned}$$

and the **left data** can be expressed as:

$$\begin{aligned}\mathbf{M} &= \mathbf{diag} [\mu_1, \dots, \mu_q] \in \mathbb{C}^{q \times q} \\ \mathbf{L}^\top &= \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{n_y \times q} \\ \mathbf{V}^\top &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{n_u \times q}\end{aligned}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

With this notation \mathbb{L} satisfy the Sylvester equation : $\mathbf{ML} - \mathbb{L}\mathbf{\Lambda} = \mathbf{VR} - \mathbf{LW}$.

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$$\begin{aligned}\mathbf{\Lambda} &= \mathbf{diag} [\lambda_1, \dots, \lambda_k] \in \mathbb{C}^{k \times k}, \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{n_u \times k} \\ \mathbf{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y \times k}\end{aligned}$$

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The **Loewner matrix** in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

With this notation \mathbb{L} satisfy the **Sylvester equation** : $\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

The Loewner matrix is:

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$$\mathbf{M}\mathbb{L} - \mathbb{L}\boldsymbol{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The shifted Loewner matrix is:

$$\mathbb{M} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \dots & \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{M} - \mathbb{M}\boldsymbol{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\boldsymbol{\Lambda}$$

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1} B \mathbf{r}_1, \dots, (\lambda_k E - A)^{-1} B \mathbf{r}_k \right] \in \mathbb{C}^{n \times k}$$

be the **generalized tangential observability** and **controllability matrices**. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E(\lambda_i E - A)^{-1} B \mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

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Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , \quad B = \mathbf{V}, \quad C = \mathbf{W},$$

is a **minimal descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\text{rank } [\xi\mathbb{L} - \mathbb{M}] = \text{rank } [\mathbb{L}, \mathbb{M}] = \text{rank } \begin{bmatrix} \mathbb{L} \\ \mathbb{M} \end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L}, \mathbb{M}] = \mathbf{Y}\Sigma_L \tilde{\mathbf{X}}^\top, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{M} \end{bmatrix} = \tilde{\mathbf{Y}}\Sigma_r \mathbf{X}^H, \quad \mathbf{Y}, \mathbf{X} \in \mathbb{C}^{N \times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -\mathbf{Y}^H \mathbb{L} \mathbf{X}, \quad A = -\mathbf{Y}^H \mathbb{M} \mathbf{X}, \quad B = -\mathbf{Y}^H \mathbf{V}, \quad C = \mathbf{W} \mathbf{X}$$

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Being known the full state realization \mathcal{S}

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

The Petrov-Galerkin reduced order model $\hat{\mathcal{S}}$ is obtained with projectors $W, V \in \mathbb{C}^{n \times r}$ as

$$\begin{aligned} W^\top EV\dot{\hat{\mathbf{x}}}(t) &= W^\top AV\hat{\mathbf{x}}(t) + W^\top B\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{aligned}$$

The above equations assume $W = \text{span}(\mathcal{W})$, $V = \text{span}(\mathcal{V})$ selected such that they meet the approximation objective:

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t), \quad , \quad \hat{\mathbf{x}}_0 = W^\top \mathbf{x}_0$$

Choosing two different bases V' and W' that respectively span the same subspaces \mathcal{V} and \mathcal{W} result in the same reconstructed solution $\mathbf{x}(t)$. Thus, subspaces are relevant, not basis.

Now recall,

$$\mathcal{O}E\mathcal{R} = \mathbb{L} \quad \text{and} \quad \mathcal{O}A\mathcal{R} = \mathbb{M},$$

where the generalized reachability and observability matrices are (complex conjugation trick for realness)

$$\mathcal{R} = \begin{bmatrix} \mathbf{r}_1^\top B^\top (\lambda_1 E - A)^{-\top} \\ \bar{\mathbf{r}}_1^\top B^\top (\bar{\lambda}_1 E - A)^{-\top} \\ \vdots \\ \mathbf{r}_{k/2}^\top B^\top (\lambda_{k/2} E - A)^{-\top} \\ \bar{\mathbf{r}}_{k/2}^\top B^\top (\bar{\lambda}_{k/2} E - A)^{-\top} \end{bmatrix}^\top \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{l}_1 C(\mu_1 E - A)^{-1} \\ \bar{\mathbf{l}}_1 C(\bar{\mu}_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_{q/2} C(\mu_{q/2} E - A)^{-1} \\ \bar{\mathbf{l}}_{q/2} C(\bar{\mu}_{q/2} E - A)^{-1} \end{bmatrix}.$$

$$(Y^\top J^H \mathcal{O})^\top E(\mathcal{R}JX) = \mathbb{L}, \quad \text{and} \quad (Y^\top J^H \mathcal{O})^\top A(\mathcal{R}JX) = \mathbb{M},$$

$$J_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \quad \text{and} \quad J = I_{k/2} \otimes (J_0 \otimes I_m),$$

Therefore,

$$\underbrace{(Y^\top J^H \mathcal{O})^\top}_{W^\top} E \underbrace{(\mathcal{R}JX)}_V = \mathbb{L}, \quad \text{and} \quad \underbrace{((Y^\top J^H \mathcal{O})^\top A}_{W^\top} \underbrace{(\mathcal{R}JX)}_V = \mathbb{M},$$

Therefore,

$$W = Y^\top J^H \mathcal{O} \quad \text{and} \quad V = \mathcal{R}JX.$$

and

$$\mathbf{x}(t) \approx V \hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R}JX.$$

can be recovered (approximated) solely from ROM simulation.



M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, "A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations", journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek \mathbf{H} s.t.

$$\mathbf{H}(\lambda_j) \mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$j = 1, \dots, k; i = 1, \dots, q.$$

Rational interpolation

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$



A.C. Antoulas, S. Lefteriu and A.C. Ionita, "[Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction](#)", Model Reduction and Approximation: Theory and Algorithms, 2016.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek \mathbf{H} s.t.

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Rational interpolation
 $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

- ▶ underlying rational (r) order

$$\begin{aligned} r &= \text{rank} (\xi\mathbb{L} - \mathbb{M}) \\ &= \text{rank} ([\mathbb{L}, \mathbb{M}]) \\ &= \text{rank} ([\mathbb{L}^H, \mathbb{M}^H]^H) \end{aligned}$$

- ▶ and McMillan (ν) order

$$\nu = \text{rank} (\mathbb{L})$$

- ▶ \mathbb{L} and \mathbb{M} are input-output independents.
- ▶ Minimal realization
- ▶ If \mathcal{S} is known, then internal state $\mathbf{x} \in \mathbb{R}^n$ is recovered by

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R}JX.$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

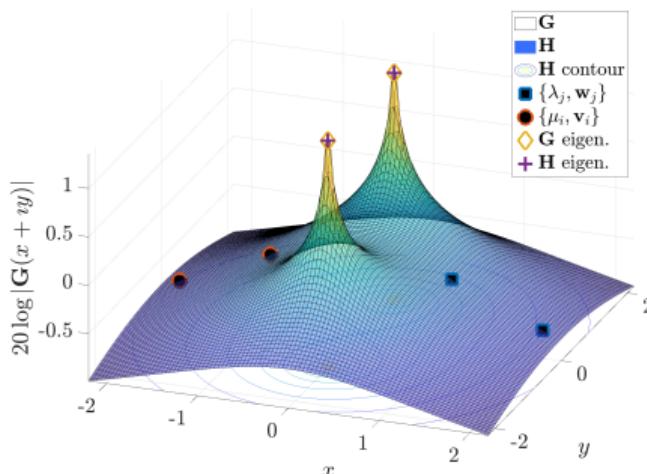
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Loewner

Loewner examples (simple case, CAS=4)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

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$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Loewner

Loewner examples (simple case, CAS=4)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\text{rank } (\xi \mathbb{L} - \mathbb{M}) = r$$

$$\text{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, (\mathbb{M}, \mathbb{L}) pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

$$\begin{aligned}\mathbf{w}_1 &= \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5} \\ \mathbf{v}_1 &= \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.\end{aligned}$$

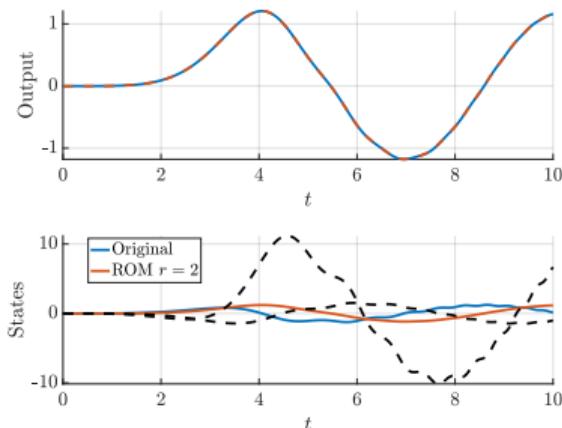
$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal variables



$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

$$\begin{aligned}\mathbf{w}_1 &= \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5} \\ \mathbf{v}_1 &= \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.\end{aligned}$$

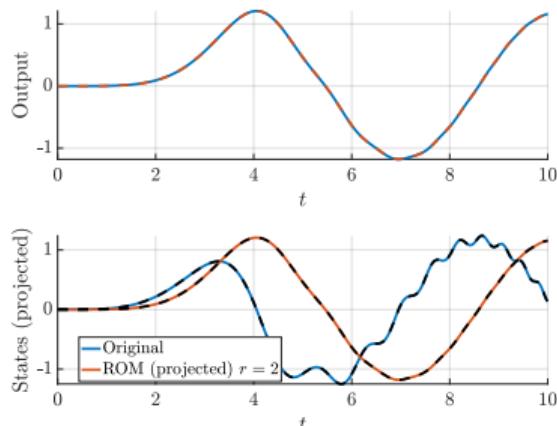
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$$\mathbf{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal **projected** variables



$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

$$\begin{aligned}\mathbf{w}_1 &= \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5} \\ \mathbf{v}_1 &= \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.\end{aligned}$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

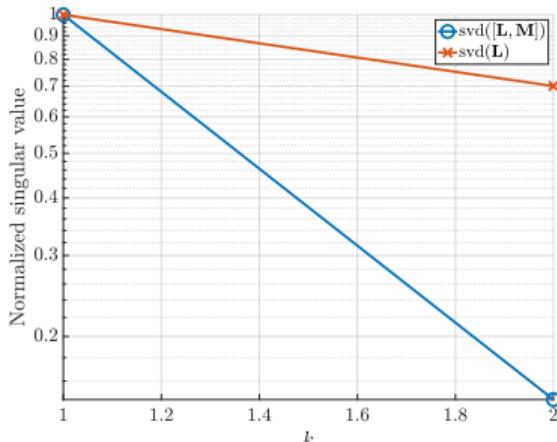
Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2, \lambda_3 = 3 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization rect.: $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^\dagger \mathbf{V}$



$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2, \lambda_3 = 3 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\text{rank } (\xi \mathbb{L} - \mathbb{M}) = r$$

$$\text{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}(-s\mathbb{L} + \mathbb{M})^\dagger \mathbf{V} \\ &= \frac{1}{s^2 - 4.650e - 16s + 1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= 1, \lambda_2 = 2, \lambda_3 = 3 \\ \mu_1 &= -1, \mu_2 = -2\end{aligned}$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

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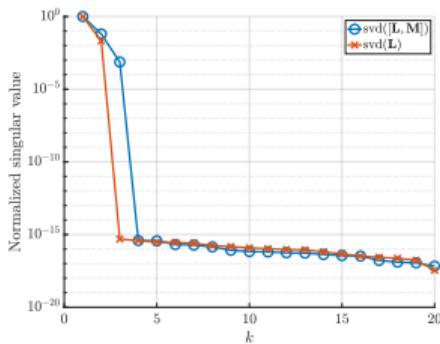
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

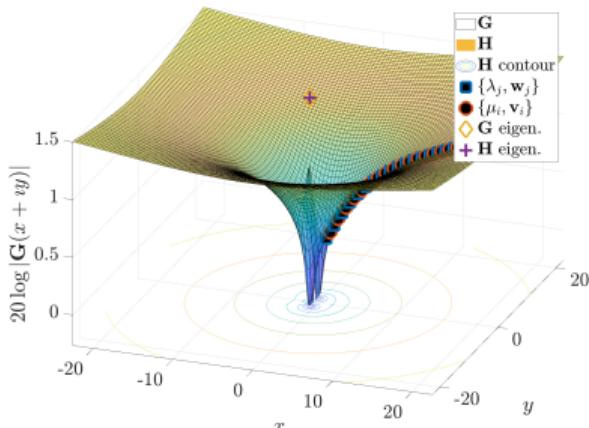
$$\begin{aligned}\lambda_{1\dots 20} &= [1, 2, \dots, 20] \\ \mu_{1\dots 20} &= [1.5, 2.5, \dots, 20.5]\end{aligned}$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

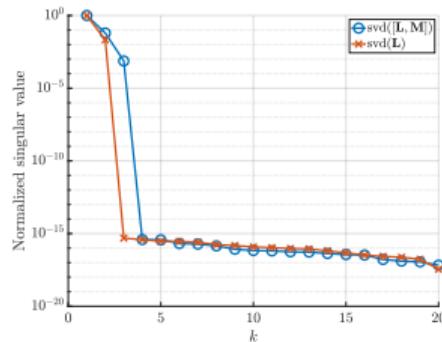
Realization $n = 20$: $\mathbf{H}(s) = \mathbf{W}(-s\mathbf{L} + \mathbf{M})^{-1}\mathbf{V}$



$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\begin{aligned}\lambda_{1\dots 20} &= [1, 2, \dots, 20] \\ \mu_{1\dots 20} &= [1.5, 2.5, \dots, 20.5]\end{aligned}$$



Rational function satisfies

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Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\text{rank } (\xi \mathbb{L} - \mathbb{M}) = r = 3$$

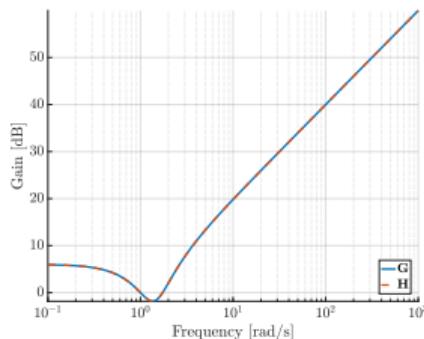
$$\text{rank } (\mathbb{L}) = \nu = 2$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\dots 20} = [1, 2, \dots, 20]$$

$$\mu_{1\dots 20} = [1.5, 2.5, \dots, 20.5]$$



Rational function satisfies

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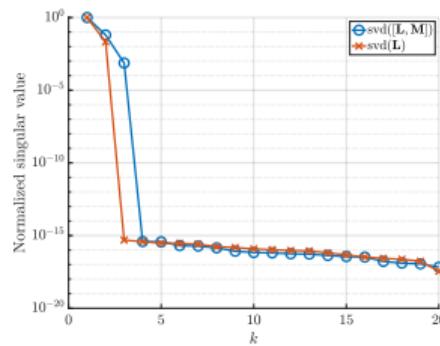
$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W} \mathbf{X} (-s \mathbf{Y}^T \mathbb{L} \mathbf{X} + \mathbf{Y}^T \mathbb{M} \mathbf{X})^{-1} \mathbf{Y}^T \mathbf{V} \\ &= \frac{s^2 + s + 2}{s + 1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\dots 20} = [1, 2, \dots, 20]$$

$$\mu_{1\dots 20} = [1.5, 2.5, \dots, 20.5]$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1 + i)s + (1 + i)}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= i, \lambda_2 = 2i, \lambda_3 = -2 + i \\ \mu_1 &= -i, \mu_2 = 2, \mu_3 = 0.5 - i\end{aligned}$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\text{rank } (\xi \mathbb{L} - \mathbb{M}) = r$$

$$\text{rank } (\mathbb{L}) = \nu$$

$$\mathbf{H}(s) = \frac{(1 + 2.22e - 16i)}{s^2 + (1 + i)s + (1 + i)}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1 + i)s + (1 + i)}$$

Evaluated at

$$\begin{aligned}\lambda_1 &= i, \lambda_2 = 2i, \lambda_3 = -2 + i \\ \mu_1 &= -i, \mu_2 = 2, \mu_3 = 0.5 - i\end{aligned}$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

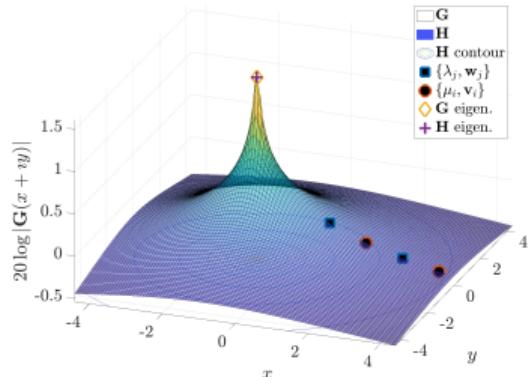
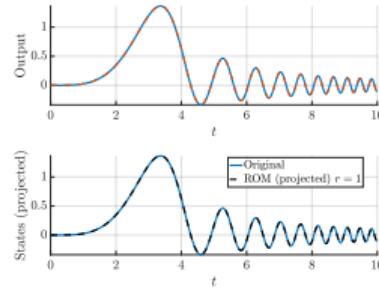
$$\begin{aligned}\hat{\mathbb{L}} &= \begin{bmatrix} -0.207 + 0.9568i & -0.1276 - 0.0294i \\ -0.0438 - 0.0818i & 0.039 - 0.1089i \end{bmatrix} \\ \hat{\mathbb{M}} &= \begin{bmatrix} 0.4738 - 0.0560i & 0.0089 - 0.3392i \\ 0.2253 - 0.1607i & -0.0342 + 0.1308i \end{bmatrix}\end{aligned}$$

```
% Model (CAS = 1)
S = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP , responses , tangent
la = [1 3];
mu = [2 4];
k = length(la);
q = length(mu);
R = ones(1,k);
L = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```

```
% Model (CAS = 1)
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[A,B,C,D,E] = dssdata(S);
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% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```



Content

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Linear dynamical systems

Loewner

Loewner extensions

Conclusions

Loewner extensions

More structures and properties

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- L-pH
- pL-DAE
- B-DAE
- Q-DAE

Model

- ▶ (A, B, C) and $\mathbf{H}(s)$
- ▶ (A, B, C, D) and $\mathbf{H}(s)$
- ▶ (E, A, B, C) and $\mathbf{H}(s)$
- ▶ $(A_i \dots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- ▶ $\mathbf{H}(s)$

- ▶ (Q, J, R, G, P, N, S) and $\mathbf{H}(s)$
- ▶ (E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
- ▶ (A, B, C, N) and $\mathbf{H}(s_1, s_2, \dots, s_k)$
- ▶ (A, B, C, Q) and $\mathbf{H}(s_1, s_2, \dots, s_k)$

Loewner extensions

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Loewner extensions

Passive & port-Hamiltonian

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L-pH

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Passivity

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}\end{aligned}$$

$$\Phi_{\mathbf{H}}(s) = \mathbf{H}(s) + \mathbf{H}^{\top}(-s)$$

satisfy $\Phi_{\mathbf{H}}(\omega) > 0$, $\text{Re}(\lambda_S) < 0$ and $D \succ 0$.

pH

$$\begin{aligned}\dot{\mathbf{x}} &= (J - R)Q\mathbf{x} + (G - P)\mathbf{u} \\ \mathbf{y} &= (G + P)^{\top}Q\mathbf{x} + (N + S)\mathbf{u}\end{aligned}$$

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^{\top} & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^{\top} & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^{\top}$, $\mathcal{W} = \mathcal{W}^{\top} \succeq 0$ and $Q = Q^{\top} \succeq 0$

Loewner extensions

Passive & port-Hamiltonian

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Transfer function

$$\mathbf{H}(s) = \frac{2s + 4}{s + 1}$$

ODE realization \mathcal{S}_1

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x + 2u\end{aligned}$$

Passivity

- ▶ $\text{Re}(\Lambda(A)) = -1 < 0$
- ▶ $D = 2 \succ 0$
- ▶ $\Phi_{\mathbf{H}}(\imath\omega) = \mathbf{H}(\imath\omega) + \mathbf{H}(-\imath\omega)^{\top} > 0$

$$\begin{aligned}\Phi_{\mathbf{H}}(s) &= \frac{4(s^2 - 2)}{s^2 - 1} \\ \Phi_{\mathbf{H}}(\imath\omega) &= \frac{4(\omega^2 + 2)}{\omega^2 + 1}\end{aligned}$$

Loewner extensions

Passive & port-Hamiltonian

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ODE realization \mathcal{S}_1

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x + 2u\end{aligned}$$

L-pH realization \mathcal{S}_2

$$\begin{aligned}\dot{\mathbf{x}} &= (J - R)Q\mathbf{x} + (G - P)\mathbf{u} = (0 - 1)1\mathbf{x} + (-2 - \sqrt{2})\mathbf{u} \\ \mathbf{y} &= (G + P)^\top Q\mathbf{x} + (N + S)\mathbf{u} = (-2 + \sqrt{2})\mathbf{x} + (0 + 2)\mathbf{u}\end{aligned}$$

where $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

Loewner extensions

Passive & port-Hamiltonian

The transfer function $\mathbf{H}(s)$ is

- ▶ strictly passive if it is **strictly positive-real** and **asymptotically stable**.
- ▶ (**non strictly**) passive if it is **positive-real** and **stable**.

The rational transfer function $\mathbf{H}(s)$ is **strictly positive-real** if $\Phi_{\mathbf{H}}(\imath\omega) > 0$ and **positive-real** if $\Phi_{\mathbf{H}}(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$, $\Phi_{\mathbf{H}}(\imath\omega) = \mathbf{H}(\imath\omega)^H + \mathbf{H}(\imath\omega)$

Real $X \succ 0$ is a **(strict)** passivity certificate for realization $\mathcal{S} : (A, B, C, D)$ iff.

$$W(X, \mathcal{S}) = \begin{bmatrix} -A^T X - XA & C^T - XB \\ (\star)^T & D + D^T \end{bmatrix} \succeq (\succ) 0$$

 P. Benner, P. Goyal and P. Van-Dooren, "*Identification of Port-Hamiltonian Systems from Frequency Response Data*", Systems & Control Letters, vol. 143, 2020.

LTI pH model

LTI pH system model of a proper transfer function \mathbf{H} , has the state-space form

$$\begin{aligned}\dot{\mathbf{x}} &= (J - R)Q\mathbf{x} + (G - P)\mathbf{u} \\ \mathbf{y} &= (G + P)^\top Q\mathbf{x} + (N + S)\mathbf{u}\end{aligned}$$

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satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

LTI strictly passive & LTI pH equivalence

If Q , \mathcal{W} are invertible, $X = Q$ is a strict passivity certificate (i.e. $W(X, \mathcal{S}) \succ 0$) of the normalized pH system, $\mathcal{S} : (A, B, C, D)$, it follows that \mathcal{S} , can always be transformed in the pH realization.

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Loewner extensions

Passive & port-Hamiltonian

The **right data** can be expressed as:

$$\begin{aligned}\mathbf{\Lambda} &= \mathbf{diag} [\lambda_1, \dots, \lambda_k] \in \mathbb{C}^{k \times k}, \\ \mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{m \times k} \\ \mathbf{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{m \times k}\end{aligned}$$

and the **left data** can be expressed as:

$$\begin{aligned}\mathbf{M} &= -\mathbf{\Lambda}^H = \mathbf{diag} [\mu_1, \dots, \mu_q] \in \mathbb{C}^{q \times q} \\ \mathbf{L}^\top &= \mathbf{R}^H = \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{m \times q} \\ \mathbf{V}^\top &= -\mathbf{W}^H = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{m \times q}\end{aligned}$$

Spectral zeros are $(\lambda_j, \mathbf{r}_j)$ from standard Loewner (n zeros in the open right half-plane)

$$\begin{bmatrix} 0 & A & B \\ A^\top & 0 & C^\top \\ B^\top & C & D + D^\top \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} 0 & E & 0 \\ E^\top & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$



A.C. Antoulas, "A new result on passivity preserving model reduction", Systems & Control Letters, vol. 54, 2005.



P. Benner, P. Goyal and P. Van-Dooren, "Identification of Port-Hamiltonian Systems from Frequency Response Data", Systems & Control Letters, vol. 143, 2020.

Loewner extensions

Passive & port-Hamiltonian

The **right data** can be expressed as:

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Passive & port-Hamiltonian

Then $\mathbf{H}(s) \in \mathbb{C}^{m \times m}$ of McMillan degree n and \mathcal{S} is a normalized pH form and satisfies

$$\mathbf{H}(\infty) = D, \mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j, \mathbf{r}_j^H \mathbf{H}(-\overline{\lambda_j}) = -\mathbf{w}_j^H, D + D^\top \succ 0 \text{ and } \mathbb{L} \succ 0$$

By construction, one obtains an Hermitian $\mathbb{L} \in \mathbb{C}^{r \times r}$ and a skew symmetric $\mathbb{M} \in \mathbb{C}^{r \times r}$ matrix. By setting, $\mathbf{H}(\infty) = D$, one recovers an $m \times m$ real transfer function $\hat{\mathbf{H}}$. As $\mathbb{L} \succ 0$, one may apply the Cholesky decomposition $\mathbb{L} = T^\top T$. Then the *normalized pH model* is obtained as $\Sigma_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, D)$,

$$\mathbf{S} := \begin{bmatrix} -T\hat{A}T^{-1} & -T\hat{B} \\ \hat{C}T^{-1} & D \end{bmatrix},$$

one obtains the equivalent pH-form by solving

$$\begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} := \frac{\mathbf{S} - \mathbf{S}^\top}{2} \text{ and } \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix} := \frac{\mathbf{S} + \mathbf{S}^\top}{2}.$$

 P. Benner, P. Goyal and P. Van-Dooren, "Identification of Port-Hamiltonian Systems from Frequency Response Data", Systems & Control Letters, vol. 143, 2020.

Loewner extensions

Passive & port-Hamiltonian

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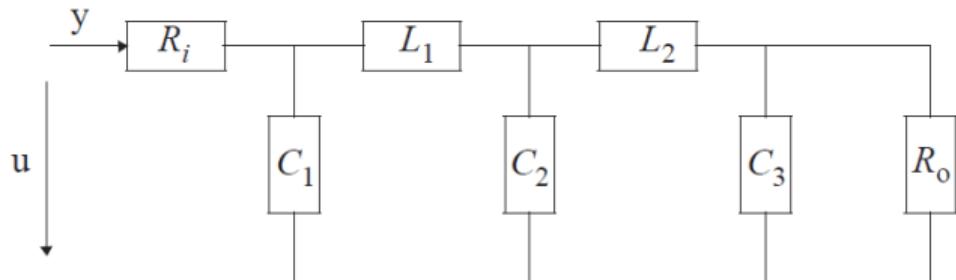
 P. Benner, P. Goyal and P. Van-Dooren, "Identification of Port-Hamiltonian Systems from Frequency Response Data", Systems & Control Letters, vol. 143, 2020.

Loewner extensions

RLC ladder network (by A.C. Antoulas, CAS='siso_passive_aca')

Variables & parameters

- ▶ x_1 : voltage across C_1
- ▶ x_2 : current across L_1
- ▶ x_3 : voltage across C_2
- ▶ x_4 : current across L_2
- ▶ x_5 : voltage across C_3
- ▶ u : voltage
- ▶ y : current
- ▶ $n = 5$ internal variables
- ▶ $C_i = \frac{1}{10} \text{ F}$
- ▶ $L_i = \frac{1}{10} \text{ H}$
- ▶ $R_i = \frac{1}{2} \Omega$
- ▶ $R_o = 5\Omega$



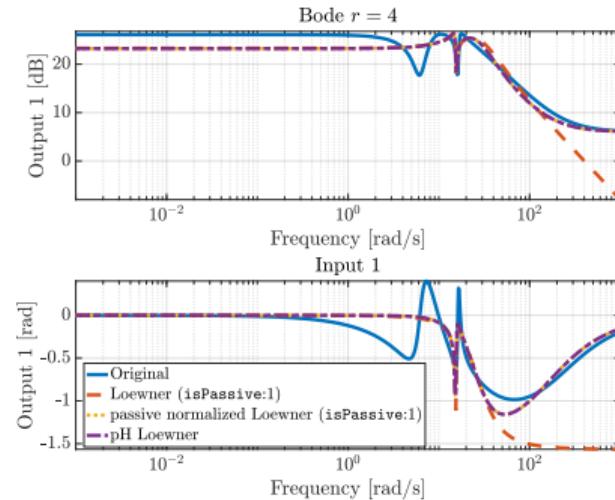
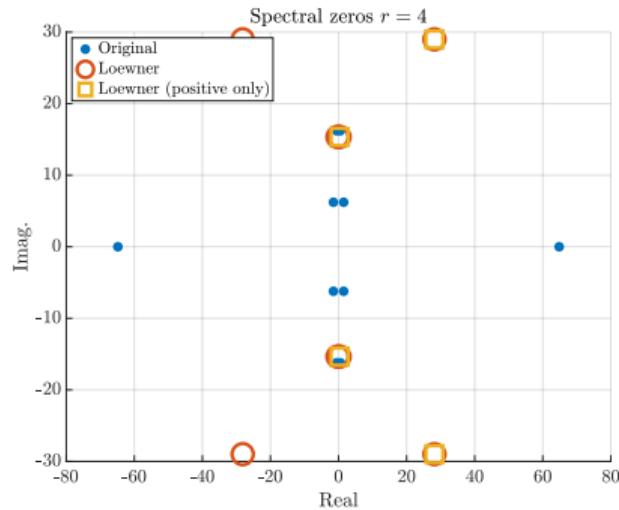
$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix}, B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \end{bmatrix}, D = 2$$

$$G(s) = \frac{2(s^5 + 222s^4 + 840s^3 + 66600s^2 + 118000s + 2220000)}{s^5 + 22s^4 + 440s^3 + 6600s^2 + 38000s + 220000}$$

Loewner extensions

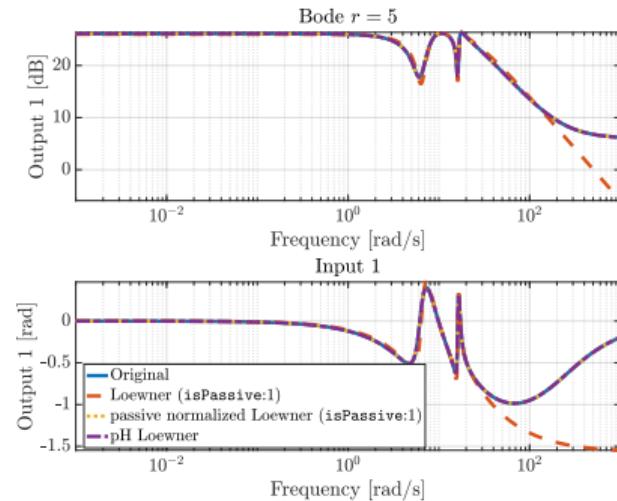
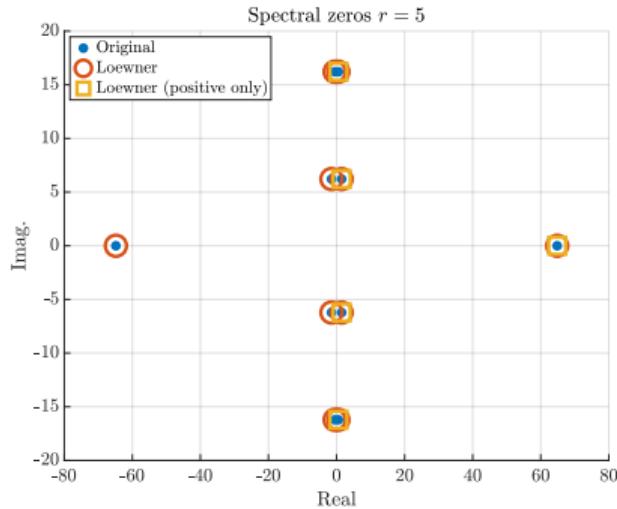
RLC ladder network (by A.C. Antoulas, CAS='siso_passive_aca')



Here **all models result stable and passive**; passive normalized allows pH reconstruction

Loewner extensions

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Loewner extensions

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Model

$$J = \begin{pmatrix} 0 & -54.7 & -39.1 & 15.0 & 66.2 \\ 54.7 & 0 & -16.3 & 0.0922 & 0.782 \\ 39.1 & 16.3 & 0 & 0.0412 & 0.45 \\ -15.0 & -0.0922 & -0.0412 & 0 & -6.32 \\ -66.2 & -0.782 & -0.45 & 6.32 & 0 \end{pmatrix} \quad G = \begin{pmatrix} -49.6 \\ -4.79 \\ -5.15 \\ 0.922 \\ 6.23 \end{pmatrix} \quad P = \begin{pmatrix} 45.0 \\ -5.27 \\ -5.65 \\ 1.21 \\ 8.16 \end{pmatrix}$$

$$R = \begin{pmatrix} 29.8 & 54.3 & 38.6 & -14.8 & -64.3 \\ 54.3 & -1.11 & -0.482 & 0.243 & 1.44 \\ 38.6 & -0.482 & -0.968 & 0.197 & 1.13 \\ -14.8 & 0.243 & 0.197 & -1.59 & -0.504 \\ -64.3 & 1.44 & 1.13 & -0.504 & -4.16 \end{pmatrix} \quad N = 0 \quad S = 2 \quad Q = I_5$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, CAS='siso_passive_aca')

- $\mathcal{W} = \mathcal{W}^\top \succeq 0$

$$\mathcal{W} = \begin{pmatrix} 29.8 & 54.3 & 38.6 & -14.8 & -64.3 & 45.0 \\ 54.3 & -1.11 & -0.482 & 0.243 & 1.44 & -5.27 \\ 38.6 & -0.482 & -0.968 & 0.197 & 1.13 & -5.65 \\ -14.8 & 0.243 & 0.197 & -1.59 & -0.504 & 1.21 \\ -64.3 & 1.44 & 1.13 & -0.504 & -4.16 & 8.16 \\ 45.0 & -5.27 & -5.65 & 1.21 & 8.16 & 2.0 \end{pmatrix}$$

- $\mathcal{V} = -\mathcal{V}^\top,$

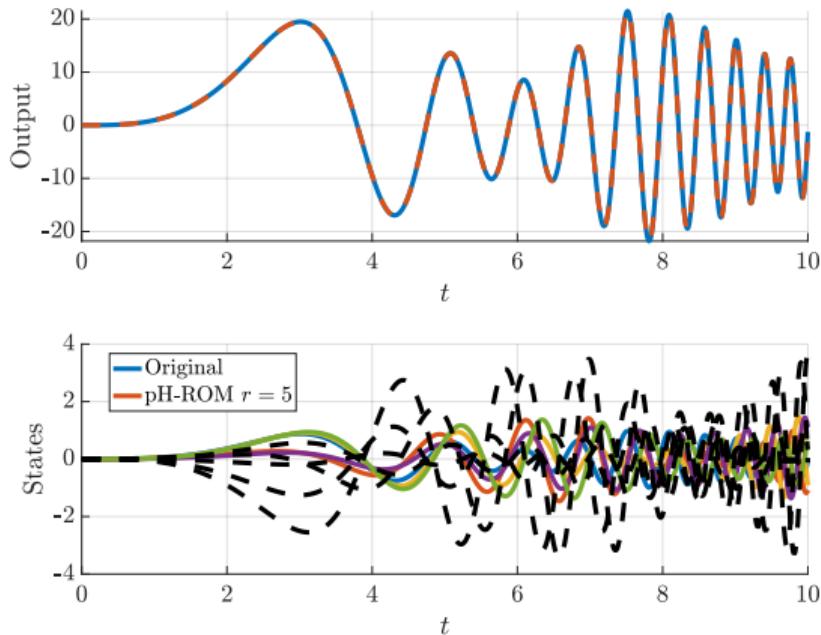
$$\mathcal{V} = \begin{pmatrix} 0 & 54.7 & 39.1 & -15.0 & -66.2 & 49.6 \\ -54.7 & 0 & 16.3 & -0.0922 & -0.782 & 4.79 \\ -39.1 & -16.3 & 0 & -0.0412 & -0.45 & 5.15 \\ 15.0 & 0.0922 & 0.0412 & 0 & 6.32 & -0.922 \\ 66.2 & 0.782 & 0.45 & -6.32 & 0 & -6.23 \\ -49.6 & -4.79 & -5.15 & 0.922 & 6.23 & 0 \end{pmatrix}$$

- $Q = Q^\top \succeq 0$

$$\Lambda(Q) = 1$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, CAS='siso_passive_aca')



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

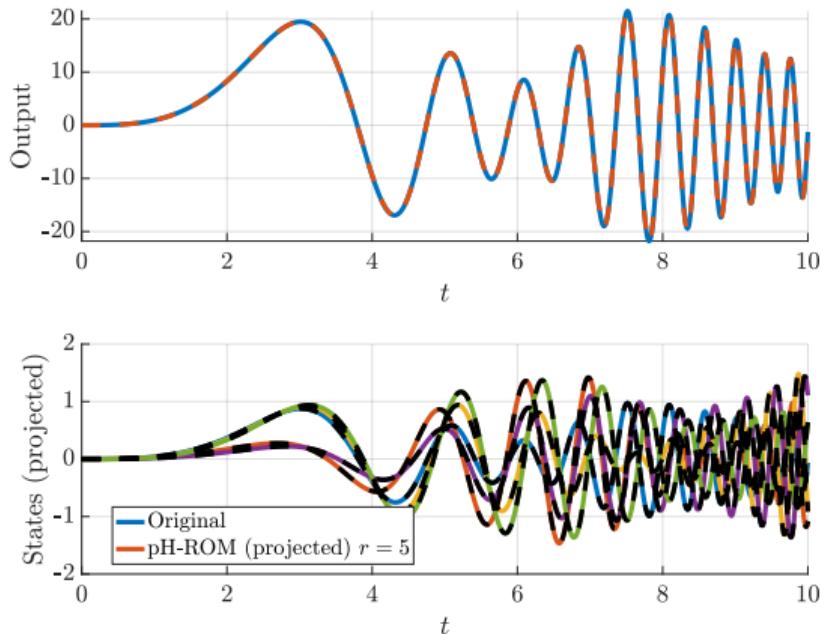
$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{aligned}\mathcal{H}(t) &= \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) \\ \mathcal{D}(t) &= \hat{\mathbf{x}}(t)^\top (J - R)\hat{\mathbf{x}}(t) \\ \mathcal{E}(t) &= \mathbf{u}(t)^\top \hat{\mathbf{y}}(t)\end{aligned}$$

Loewner extensions

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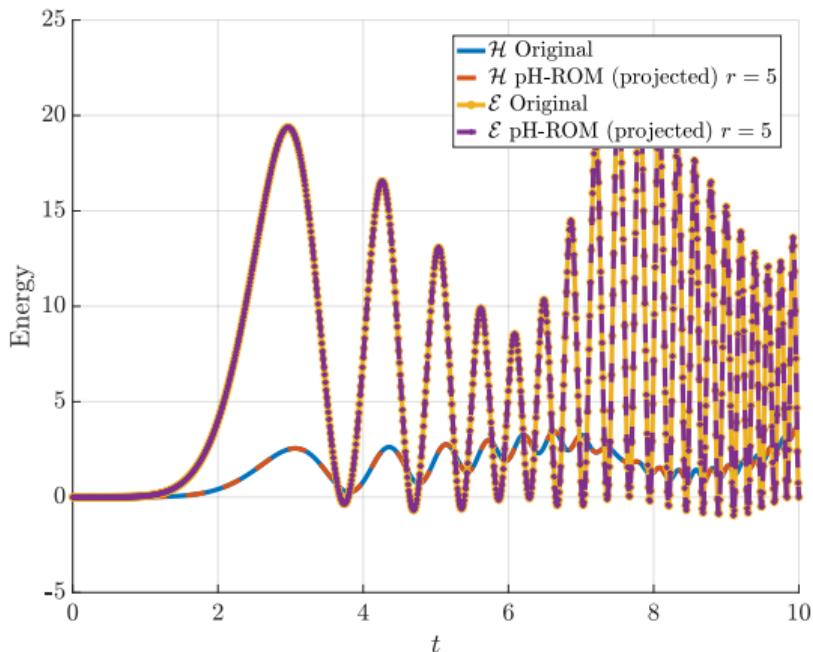
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Loewner extensions

Parametric models

Structures

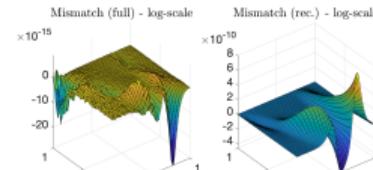
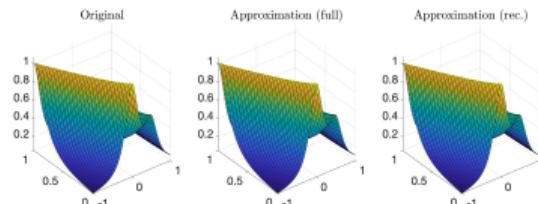
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- L-pH
- pL-DAE**
- B-DAE
- Q-DAE

$$\mathbf{G}({}^1s, {}^2s) = \mathbf{H}(s, p) = \frac{1}{1 + 25(s + p)^2} + \frac{0.5}{1 + 25(s - 0.5)^2} + \frac{0.1}{p + 25}$$

Evaluated as

$$\begin{matrix} {}^1s \\ [-1, 1] \end{matrix} \quad \times \quad \begin{matrix} {}^2s \\ [0, 1] \end{matrix} \quad \text{tab}_2 \in \mathbb{R}^{21 \times 21}$$



Loewner extensions

1-D case

$$\begin{cases} P_c^{(1)} &:= \left\{ \left({}^1\lambda_{j_1}; \mathbf{w}_{j_1} \right), j_1 = 1, \dots, k_1 \right\} \\ P_r^{(1)} &:= \left\{ \left({}^1\mu_{i_1}; \mathbf{b}_{i_1} \right), i_1 = 1, \dots, q_1 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{b}_{i_1} - \mathbf{w}_{j_1}}{{}^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

$$\mathbf{M}_1 \mathbb{L}_1 - \mathbb{L}_1 \boldsymbol{\Lambda}_1 = \mathbb{V}_1 \mathbf{R}_1 - \mathbf{L}_1 \mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}({}^1s) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{{}^1s - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1s - {}^1\lambda_{j_1}}}$$

Null space

$$\text{span } (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Loewner extensions

1-D case (example)

Data generated from $\mathbf{H}(^1s) = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$ of complexity (2)

$$\begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [2, 4, 6, 8] \end{array} \left. \right\} \xrightarrow{\mathbf{H}} \begin{array}{lcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{b}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Loewner extensions

2-D case

$$\begin{cases} P_c^{(2)} &:= \left\{ \left({}^1\lambda_{j_1}, {}^2\lambda_{j_2}; \mathbf{w}_{j_1, j_2} \right), j_1 = 1, \dots, k_1 \quad j_2 = 1, \dots, k_2 \right\} \\ P_r^{(2)} &:= \left\{ \left({}^1\mu_{i_1}, {}^2\mu_{i_2}; \mathbf{b}_{i_1, i_2} \right), i_1 = 1, \dots, q_1 \quad i_2 = 1, \dots, q_2 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1, j_2}^{i_1, i_2} = \frac{\mathbf{b}_{i_1, i_2} - \mathbf{w}_{j_1, j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

$$\begin{cases} \mathbf{M}_2 \mathbb{X} - \mathbb{X} \boldsymbol{\Lambda}_2 &= \mathbb{V}_2 \mathbf{R}_2 - \mathbf{L}_2 \mathbb{W}_2 \\ \mathbf{M}_1 \mathbb{L}_2 - \mathbb{L}_2 \boldsymbol{\Lambda}_1 &= \mathbb{X} \end{cases}$$

Lagrangian form

$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

Null space

$$\text{span } (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \begin{bmatrix} c_{1,1} \\ \vdots \\ \hline c_{1,k_2} \\ \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{bmatrix} \in \mathbb{C}^{k_1 k_2}$$

Loewner extensions

2-D case (example)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [0, 2, 4] \\ {}^2\lambda_{j_2} & = & [-1, -3] \\ {}^2\mu_{i_2} & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array} \right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array} \right]$$

Loewner extensions

2-D case (example)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [0, 2, 4] \\ {}^2\lambda_{j_2} & = & [-1, -3] \\ {}^2\mu_{i_2} & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Lagrangian form

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s, t)$$

Loewner extensions

n-D case

$$\begin{cases} P_c^{(n)} := \left\{ (^1\lambda_{j_1}, ^2\lambda_{j_2}, \dots, ^n\lambda_{j_n}; \mathbf{w}_{j_1, j_2, \dots, j_n}), \ j_l = 1, \dots, k_l, \ l = 1, \dots, n \right\} \\ P_r^{(n)} := \left\{ (^1\mu_{i_1}, ^2\mu_{i_2}, \dots, ^n\mu_{i_n}; \mathbf{b}_{i_1, i_2, \dots, i_n}), \ i_l = 1, \dots, q_l, \ l = 1, \dots, n \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n} = \frac{\mathbf{b}_{i_1, i_2, \dots, i_n} - \mathbf{w}_{j_1, j_2, \dots, j_n}}{(^1\mu_{i_1} - ^1\lambda_{j_1}) \cdots (^n\mu_{i_n} - ^n\lambda_{j_n})}$$

$$\begin{cases} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \boldsymbol{\Lambda}_n &= \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ \mathbf{M}_{n-1} \mathbb{X}_2 - \mathbb{X}_2 \boldsymbol{\Lambda}_{n-1} &= \mathbb{X}_1, \\ &\vdots \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \boldsymbol{\Lambda}_1 &= \mathbb{X}_{n-1}. \end{cases}$$

Lagrangian form

$$\mathbf{g}(^1s, \dots, ^ns) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n} \mathbf{w}_{j_1, \dots, j_n}}{(^1s - ^1\lambda_{j_1}) \cdots (^n s - ^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n}}{(^1s - ^1\lambda_{i_1}) \cdots (^n s - ^n\lambda_{i_n})}}$$

Null space

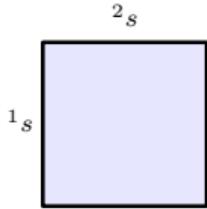
$$\text{span } (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \begin{bmatrix} c_{1, \dots, 1} \\ \vdots \\ \hline c_{1, \dots, k_n} \\ \vdots \\ \hline c_{k_1, \dots, 1} \\ \vdots \\ c_{k_1, \dots, k_n} \end{bmatrix} \in \mathbb{C}^{k_1 \cdots k_n}$$

Loewner extensions

n-D case - ReLU

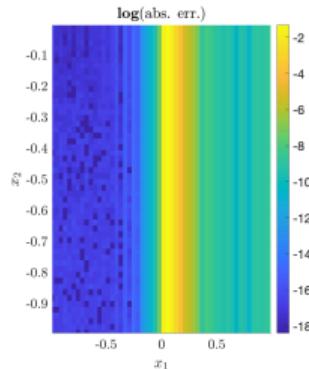
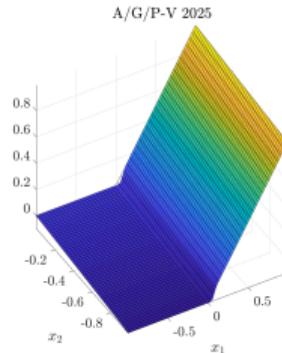
$$\mathbf{H}(^1s, ^2s) = \mathbf{ReLU}(^1s) + \frac{1}{100} ^2s$$



$$\begin{matrix} ^1s & \times & ^2s \\ [-1, 1] & \times & [0, 1] \end{matrix}$$

$$\mathbf{tab}_2 \in \mathbb{R}^{20 \times 20}$$

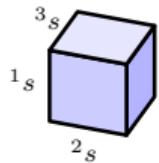
≈ 3.45 Ko ('real')



Loewner extensions

n-D case - Airbus flutter case

$$\Sigma(^1s, ^2s, ^3s) = \Sigma(s, \textcolor{red}{m}, \textcolor{orange}{v}) : s^2 M(\textcolor{red}{m})x(s) + sB(\textcolor{red}{m})x(s) + K(\textcolor{red}{m})x(s) - G(s, \textcolor{orange}{v}) = u(s), \mathbf{y}(s) = C\mathbf{x}(s)$$



$$\begin{matrix} ^1s \\ \iota[10, 35] \end{matrix} \quad \times \quad \begin{matrix} ^2s \\ [\underline{m}, \overline{m}] \end{matrix} \quad \times \quad \begin{matrix} ^3s \\ [\underline{v}, \overline{v}] \end{matrix}$$

$$\mathbf{tab}_3 \in \mathbb{C}^{300 \times 10 \times 10}$$

≈ 468.75 Ko ('complex')



A. dos Reis de Souza et al., "Aircraft flutter suppression: from a parametric model to robust control", ECC, 2023.

Loewner extensions

n-D case - Borehole function

$$\mathbf{H}(^1s, \dots, ^8s) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$

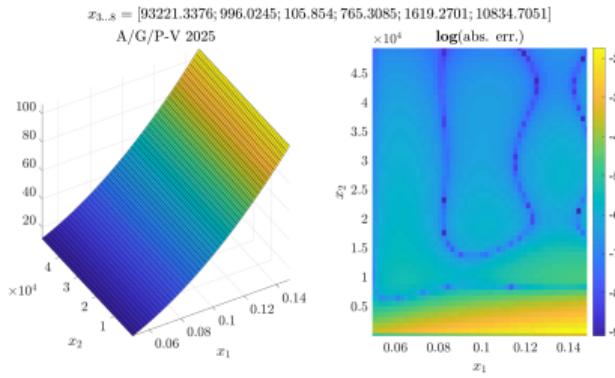


$${}^1s \quad \times \quad \cdots \quad \times \quad {}^8s \\ [r_w, \overline{r_w}] \quad \times \quad \cdots \quad \times \quad [\underline{K_w}, \overline{K_w}]$$

$$\text{tab}_8 \in \mathbb{C}^{8 \times 8 \times \cdots \times 8}$$

≈130 Mo ('real')

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	1.02e + 04	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	1.02e+04	39.1	0.00456	2.93e-09	0.0611



S. Surjanovic, "Borehole function", <https://www.sfu.ca/~ssurjano/borehole.html>.

Content

Forewords

Linear dynamical systems

Loewner

Loewner extensions

Conclusions

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Loewner... a versatile tool

- ▶ solves the LTI realization problem
 - ▶ solves data-driven model reduction
 - ▶ solves data-driven model approximation
 - ▶ ... and pH, parametric
 - ▶ ... and also, bilinear, quadratic...
- direct impact in engineers life



... still so much to do

- ▶ Technical references and slides at
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<https://arxiv.org/abs/2405.00495>
- ▶ MOR Digital Systems numerical suite
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mor Digital
Systems

Loewner Framework for data-driven reduced order modeling

... a bridge between realization, approximation and identification

C. Poussot-Vassal
September 22, 2025



Conclusions

The map of mathematics (by Dominic Walliman)

