

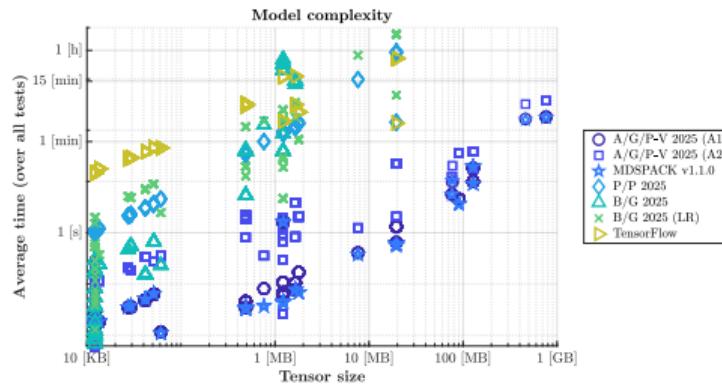
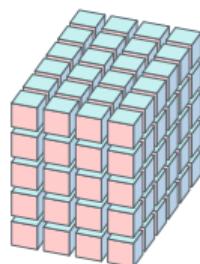
# The Loewner framework... in the eye of the tensor

The Kolmogorov superposition theorem, the curse of dimensionality & benchmark

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Presentation at ONERA UDSG

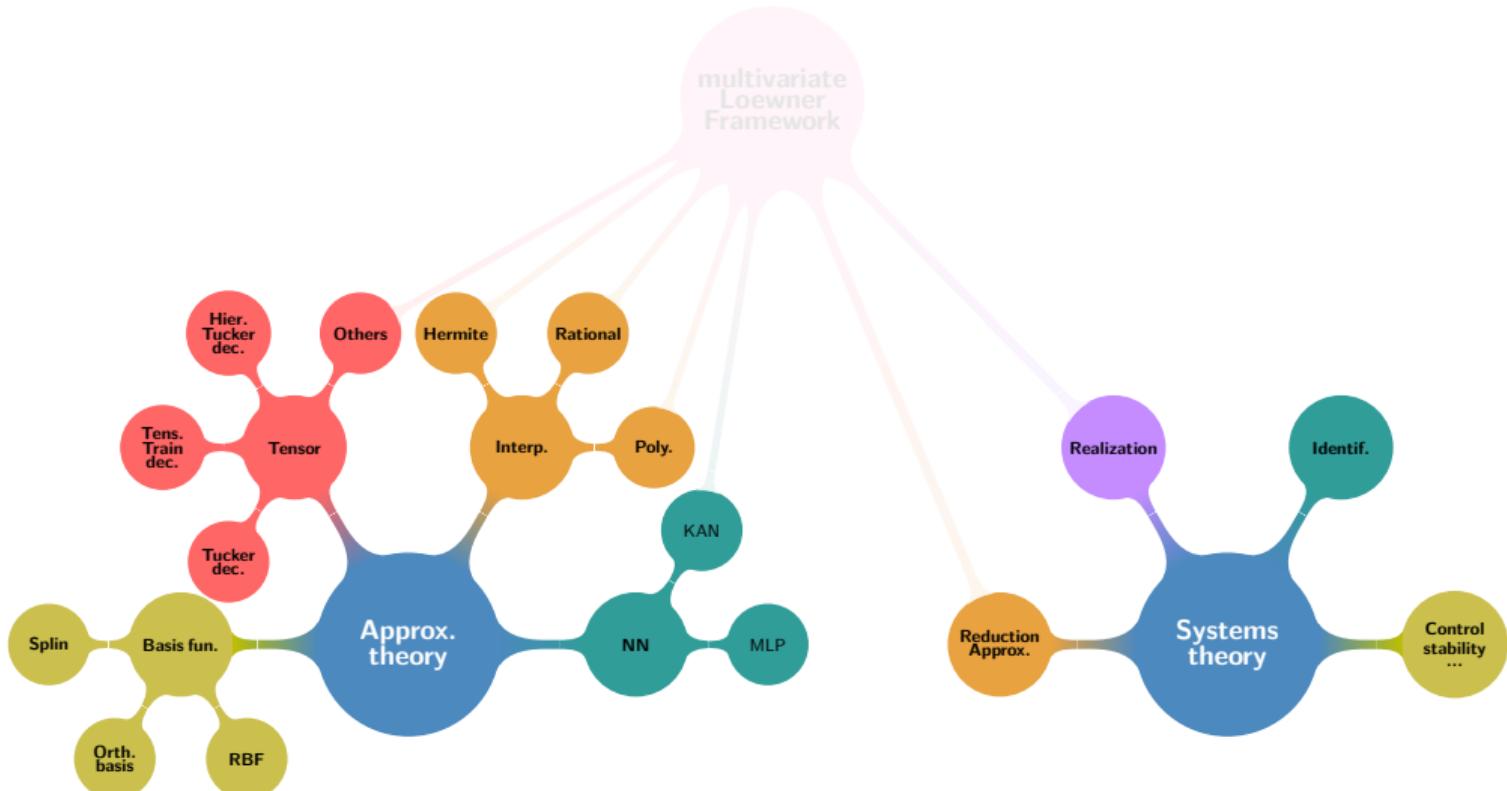
<https://arxiv.org/abs/2405.00495>  
<https://arxiv.org/abs/2506.04791>  
<https://github.com/cpoussot/mLF>

[in SIAM Review - Research Spotlight]  
[extensive benchmark]  
[research code]



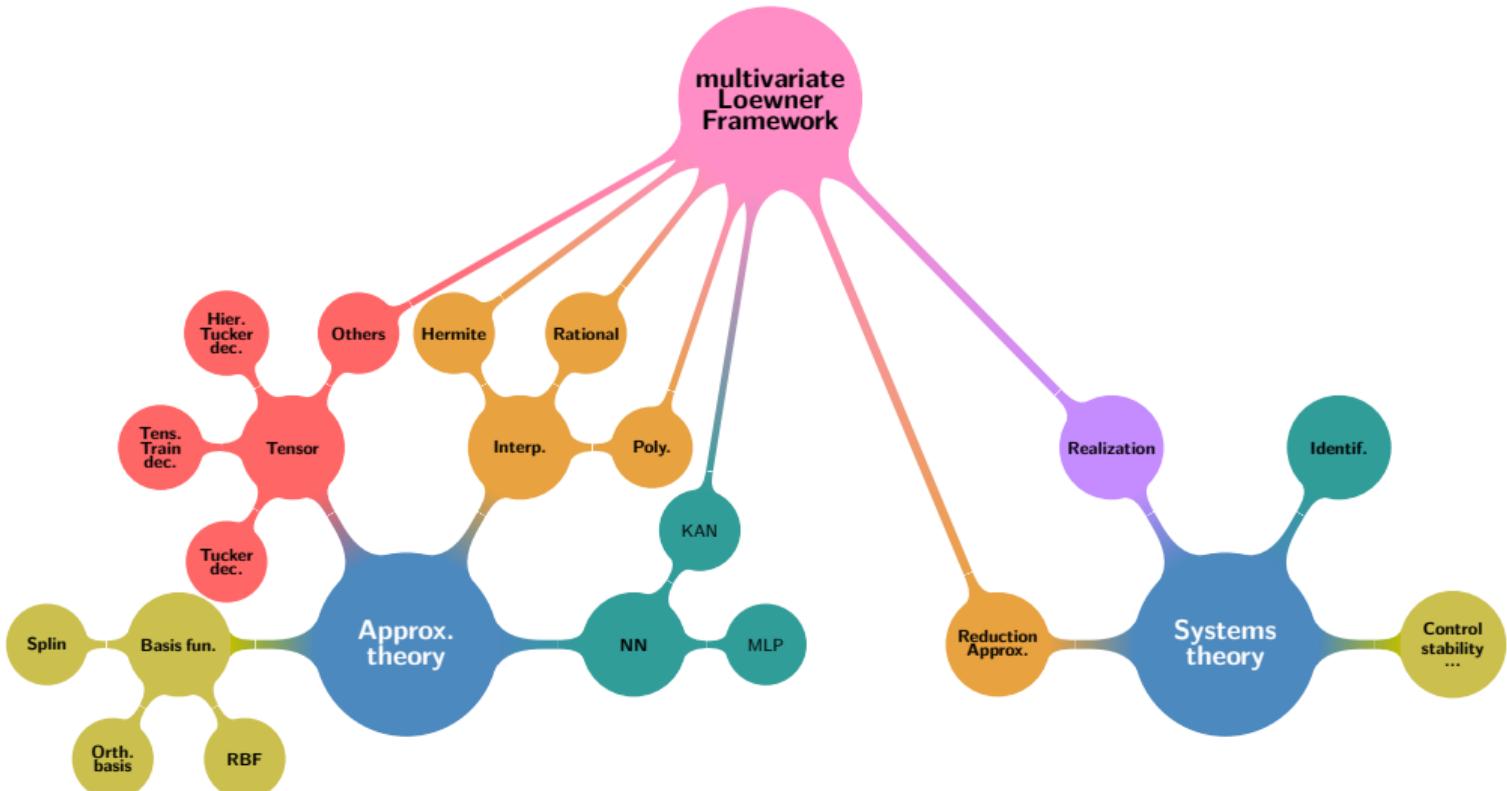
# Forewords

Approximation & systems theory... where we stand



# Forewords

Approximation & systems theory... where we stand



# Forewords

## Starting (motivating) examples - Borehole function

$$\mathbf{H}({}^1x, \dots, {}^8x) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$$\begin{matrix} {}^1x \\ [r_w, \overline{r_w}] \end{matrix} \quad \times \quad \cdots \quad \times \quad \begin{matrix} {}^8x \\ [K_w, \overline{K_w}] \end{matrix}$$

$\text{tab}_8 \in \mathbb{C}^{8 \times 8 \times \dots \times 8}$

$\approx 130 \text{ Mo ('real')}$

$r_w \in [0.05, 0.15]$	radius of borehole (m)
$r \in [100, 50\,000]$	radius of influence (m)
$T_u \in [63\,070, 115\,600]$	transmissivity of upper aquifer ( $\text{m}^2/\text{yr}$ )
$H_u \in [990, 1110]$	potentiometric head of upper aquifer (m)
$T_l \in [63.1, 116]$	transmissivity of lower aquifer ( $\text{m}^2/\text{yr}$ )
$H_l \in [700, 820]$	potentiometric head of lower aquifer (m)
$L \in [1120, 1680]$	length of borehole (m)
$K_w \in [9855, 12\,045]$	hydraulic conductivity of borehole (m/yr)



# Forewords

## Starting (motivating) examples - Borehole function

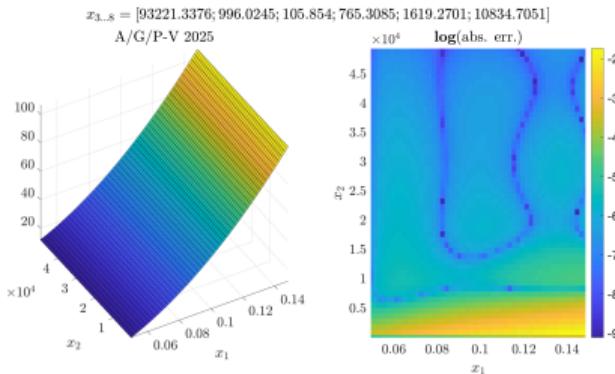
$$\mathbf{H}({}^1x, \dots, {}^8x) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$$\begin{matrix} {}^1x \\ [r_w, \overline{r_w}] \end{matrix} \quad \times \quad \cdots \quad \times \quad \begin{matrix} {}^8x \\ [K_w, \overline{K_w}] \end{matrix}$$

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# Forewords

## Starting (motivating) examples - Borehole function

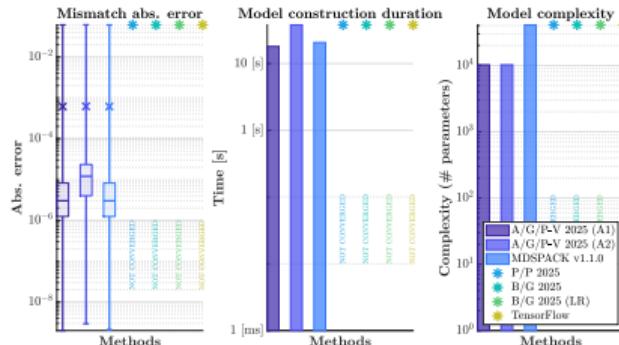
$$\mathbf{H}({}^1x, \dots, {}^8x) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$${}^1x \quad \times \quad \cdots \quad \times \quad {}^8x \\ [\underline{r_w}, \overline{r_w}] \quad \times \quad \cdots \quad \times \quad [\underline{K_w}, \overline{K_w}]$$

$\text{tab}_8 \in \mathbb{C}^{8 \times 8 \times \dots \times 8}$

$\approx 130 \text{ Mo ('real')}$



# Forewords

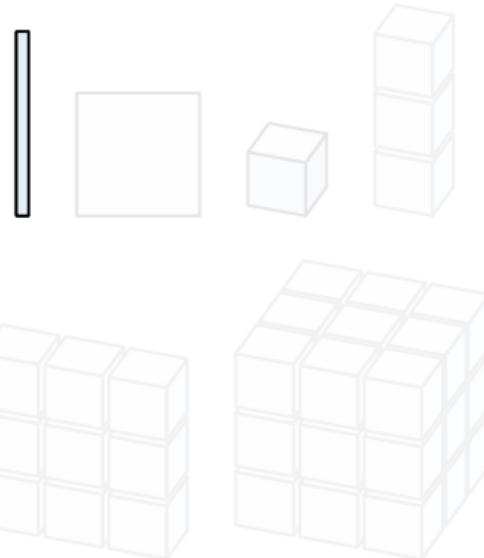
## Data (and tensors)

Column / Row data

$${}^1\mathbf{x} = \left. {}^1\lambda_{j_1}, {}^1\mu_{i_1} \right\} \xrightarrow{\mathbf{H}({}^1x)} \left\{ \mathbf{w}_{j_1}, \mathbf{v}_{i_1} \right.$$

${}^1x$	
${}^1\lambda_{1,\dots,k_1}$	$\mathbf{W}_{k_1}$
${}^1\mu_{1,\dots,q_1}$	$\mathbf{V}_{q_1}$

Tensors (1-D) tab<sub>1</sub>



# Forewords

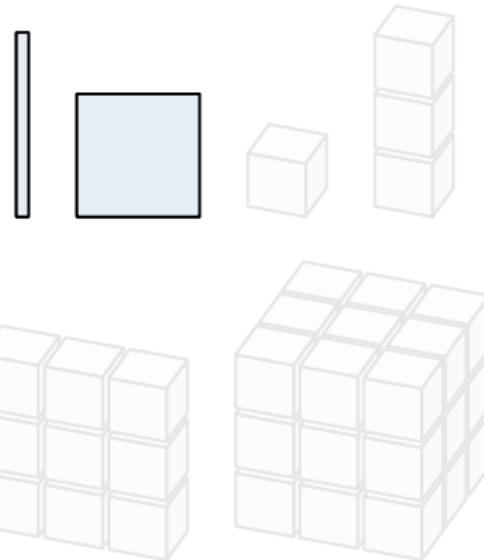
## Data (and tensors)

Column / Row data

$$\begin{matrix} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{matrix} \left. \right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2}, \mathbf{v}_{i_1, i_2} \end{array} \right.$$

${}^1x$	${}^2x$	
${}^1\lambda_{1, \dots, k_1}$	$\mathbf{W}_{k_1, k_2}$	$\phi_{cr}$
${}^1\mu_{1, \dots, q_1}$	$\phi_{rc}$	$\mathbf{V}_{q_1, q_2}$

Tensors (2-D) tab<sub>2</sub>



# Forewords

## Data (and tensors)

### Column / Row data

$$\left. \begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\mathbf{x} = {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x, {}^3x)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2, j_3}, \mathbf{v}_{i_1, i_2, i_3} \end{array} \right.$$

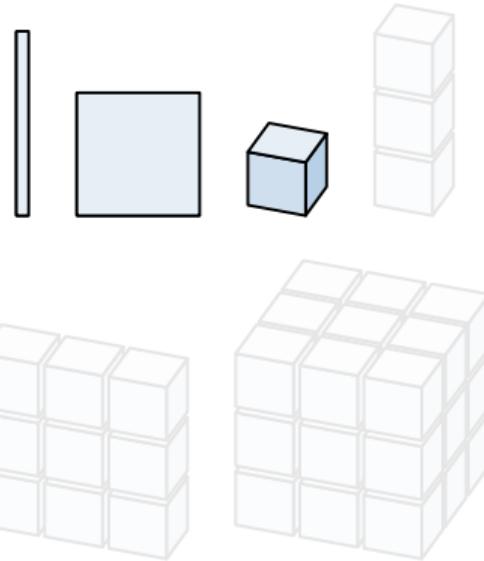
$${}^3x = {}^3\lambda_{1, \dots, k_3}$$

${}^1x$	${}^2x$	${}^2\lambda_{1, \dots, k_2}$	${}^2\mu_{1, \dots, q_2}$
${}^1\lambda_{1, \dots, k_1}$	$\mathbf{W}_{k_1, k_2, k_3}$	$\phi_{crc}$	
${}^1\mu_{1, \dots, q_1}$	$\phi_{rcc}$	$\phi_{rrc}$	

$${}^3x = {}^3\mu_{1, \dots, q_3}$$

${}^1x$	${}^2x$	${}^2\lambda_{1, \dots, k_2}$	${}^2\mu_{1, \dots, q_2}$
${}^1\lambda_{1, \dots, k_1}$	$\phi_{crr}$	$\phi_{crr}$	
${}^1\mu_{1, \dots, q_1}$	$\phi_{rcr}$	$\mathbf{V}_{q_1, q_2, q_3}$	

### Tensors (3-D) $\text{tab}_3$



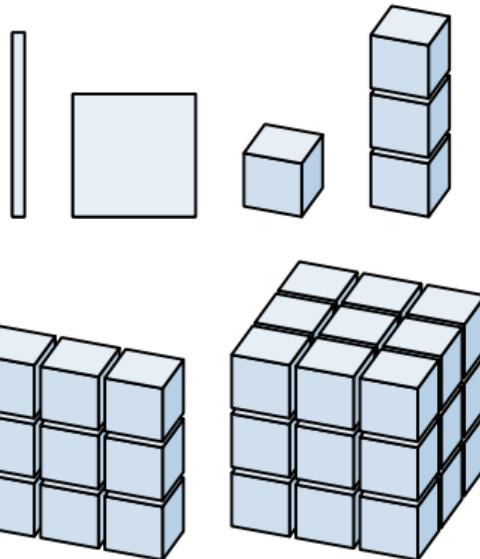
# Forewords

## Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\mathbf{x} = {}^3\lambda_{j_3}, {}^3\mu_{i_3} \\ \vdots \\ {}^n\mathbf{x} = {}^n\lambda_{j_n}, {}^n\mu_{i_n} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1x, \dots, {}^n x)} \left\{ \mathbf{w}_{j_1, \dots, j_n}, \mathbf{v}_{i_1, \dots, i_n} \right\}$$

Tensors ( $n$ -D)  $\text{tab}_n$



# Forewords

Contributions claim & trajectory of the presentation

## List of contributions

- ▶  $n$ -D tensor data to  $n$ -D Loewner matrix  $\mathbb{L}_n$
- ▶  $n$ -variable transfer functions
- ▶ Taming the curse of dimensionality
  - » in computation effort (flop)
  - » in storage needs (Bytes)
  - » in accuracy
- ▶  $n$ -variable **decoupling**
  - » **KST** formulation for rational functions
  - » connection with **KAN**
- ▶ Comparison with **MLP**, **KAN**, **AAA**



- 
- 📚 A.C. Antoulas, I-V. Gosea and C. P-V., "*On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality*", SIAM Review, November, 2025 (<https://arxiv.org/abs/2405.00495>).
  - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "*Tensor-based multivariate function approximation: methods benchmarking and comparison*", June, 2025 <https://arxiv.org/abs/2506.04791>.
  - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "*mLF package*", <https://github.com/cpoussot/mLF>.

# Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} &:= \left\{ \left( {}^1\lambda_{j_1}; \mathbf{w}_{j_1} \right), j_1 = 1, \dots, k_1 \right\} \\ P_r^{(1)} &:= \left\{ \left( {}^1\mu_{i_1}; \mathbf{v}_{i_1} \right), i_1 = 1, \dots, q_1 \right\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{}^1\mu_{i_1} - {}^1\lambda_{j_1}$$

## Lagrangian form

$$\mathbf{g}({}^1x) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}$$

## Null space

$$\text{span } (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

# Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from  $\mathbf{H}^{(1)}x = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$  of complexity (2)

$$\left. \begin{array}{rcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [2, 4, 6, 8] \end{array} \right\} \xrightarrow{\mathbf{H}} \left\{ \begin{array}{rcl} {}^{\textcolor{orange}{w}}_{\textcolor{blue}{j}_1} & = & [5/2, 13/4, 29/6] \\ {}^{\textcolor{red}{v}}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array} \right.$$

## Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

## Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

## Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

# Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} &:= \left\{ (\textcolor{brown}{1}\lambda_{j_1}, \textcolor{brown}{2}\lambda_{j_2}; \mathbf{w}_{j_1, j_2}), \ j_1 = 1, \dots, k_1 \quad j_2 = 1, \dots, k_2 \right\} \\ P_r^{(2)} &:= \left\{ (\textcolor{violet}{1}\mu_{i_1}, \textcolor{violet}{2}\mu_{i_2}; \mathbf{v}_{i_1, i_2}), \ i_1 = 1, \dots, q_1 \quad i_2 = 1, \dots, q_2 \right\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1, j_2}^{i_1, i_2} = \frac{\mathbf{v}_{i_1, i_2} - \mathbf{w}_{j_1, j_2}}{(\textcolor{violet}{1}\mu_{i_1} - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{violet}{2}\mu_{i_2} - \textcolor{brown}{2}\lambda_{j_2})}$$

## Lagrangian form

$$\mathbf{g}(\textcolor{violet}{1}x, \textcolor{brown}{2}x) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{(\textcolor{violet}{1}x - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{brown}{2}x - \textcolor{brown}{2}\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2}}{(\textcolor{violet}{1}x - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{brown}{2}x - \textcolor{brown}{2}\lambda_{j_2})}}$$

## Null space

$$\text{span } (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \begin{bmatrix} c_{1,1} \\ \vdots \\ \hline c_{1,k_2} \\ \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{bmatrix} \in \mathbb{C}^{k_1 k_2}$$

# Multi-variate data, function & Loewner matrix

## 2-D case (example)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\left. \begin{array}{l} {}^1\lambda_{j_1} = [1, 3, 5] \\ {}^1\mu_{i_1} = [0, 2, 4] \\ {}^2\lambda_{j_2} = [-1, -3] \\ {}^2\mu_{i_2} = [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[ \begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

**Loewner matrix**

$$\mathbb{L}_2 = \left[ \begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \hline \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \hline \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array} \right]$$

**Null space**

$$\mathbf{c}_2 = \left[ \begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array} \right]$$

# Multi-variate data, function & Loewner matrix

## 2-D case (example)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\left. \begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [0, 2, 4] \\ {}^2\lambda_{j_2} & = & [-1, -3] \\ {}^2\mu_{i_2} & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[ \begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

### Lagrangian form

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s, t)$$

# Multi-variate data, function & Loewner matrix

*n-D case*

$$\begin{cases} P_c^{(n)} := \left\{ (^1\lambda_{j_1}, ^2\lambda_{j_2}, \dots, ^n\lambda_{j_n}; \mathbf{w}_{j_1, j_2, \dots, j_n}), \quad j_l = 1, \dots, k_l, \quad l = 1, \dots, n \right\} \\ P_r^{(n)} := \left\{ (^1\mu_{i_1}, ^2\mu_{i_2}, \dots, ^n\mu_{i_n}; \mathbf{v}_{i_1, i_2, \dots, i_n}), \quad i_l = 1, \dots, q_l, \quad l = 1, \dots, n \right\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n} = \frac{\mathbf{v}_{i_1, i_2, \dots, i_n} - \mathbf{w}_{j_1, j_2, \dots, j_n}}{(^1\mu_{i_1} - ^1\lambda_{j_1}) \cdots (^n\mu_{i_n} - ^n\lambda_{j_n})}$$

## Lagrangian form

$$\mathbf{g}(^1x, \dots, ^nx) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n} \mathbf{w}_{j_1, \dots, j_n}}{(^1x - ^1\lambda_{j_1}) \cdots (^n x - ^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n}}{(^1x - ^1\lambda_{j_1}) \cdots (^n x - ^n\lambda_{j_n})}}$$

## Null space

$$\text{span } (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

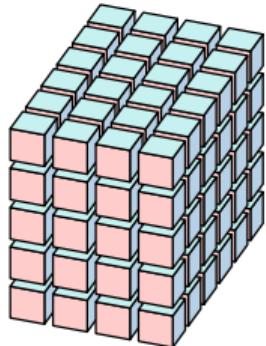
$$\mathbf{c}_n = \begin{bmatrix} c_{1, \dots, 1} \\ \vdots \\ \hline c_{1, \dots, k_n} \\ \vdots \\ \hline c_{k_1, \dots, 1} \\ \vdots \\ c_{k_1, \dots, k_n} \end{bmatrix} \in \mathbb{C}^{k_1 \cdots k_n}$$

# Taming the curse of dimensionality

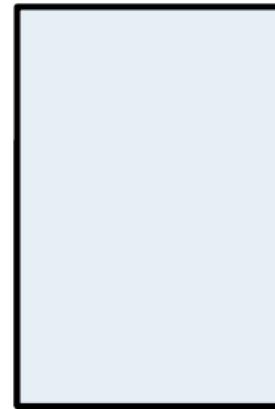
*n*-variable Loewner matrix operator

$$\begin{aligned} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \dots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1) \times \dots \times (k_n+q_n)} &\longrightarrow \mathbb{C}^{\textcolor{magenta}{Q} \times \textcolor{orange}{K}} \\ (\textcolor{brown}{1}\lambda_{j_1}, \textcolor{brown}{1}\mu_{i_1}, \dots, \textcolor{brown}{n}\lambda_{j_n}, \textcolor{brown}{n}\mu_{i_n}, \mathbf{tab}_n) &\longmapsto \mathbb{L}_n \end{aligned}$$

*n*-D tensor  $\mathbf{tab}_n$

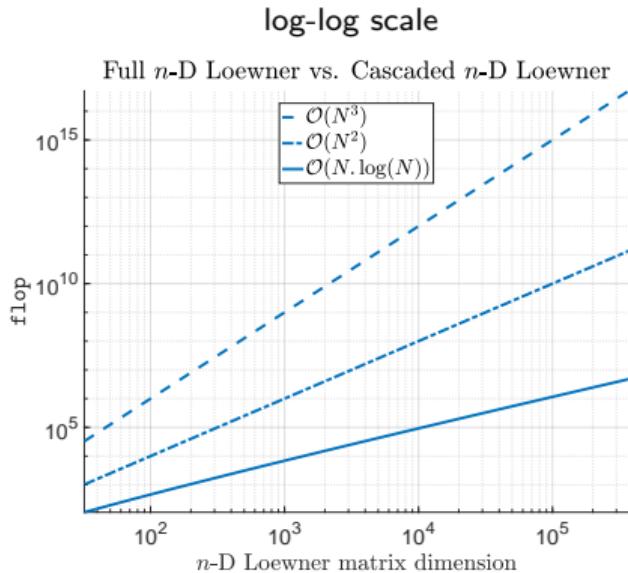


matrix  $\mathbb{L}_n$



# Taming the curse of dimensionality

Null space flop and memory issues



(rows)  $Q = q_1 q_2 \dots q_n$  and  
(columns)  $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

## Computational issue

Note that  $Q \times K$  matrix SVD flop estimation is

- ▶  $QK^2$  (if  $Q > K$ )
- ▶  $N^3$  (if  $Q = K = N$ )

## Storage issue

Note that  $Q \times K$  matrix storage estimation is

- ▶ in real double  $QK \frac{8}{2^{20}}$  MB
- ▶ in complex double  $2QK \frac{8}{2^{20}}$  MB

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

${}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1x$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_1 = 1$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_2 = 3$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\lambda_3 = 5$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_1 = 0$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_2 = 2$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$
${}^1\mu_3 = 4$				

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  ${}^1x$ , for  
 ${}^2x = {}^2\lambda_2 = -3$
- 3  $\mathbb{L}_1$  along  ${}^2x$  for  
 ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space  $\mathbf{c}_2^\top =$   

$$[ {}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3 ]^\top$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

${}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1x$				
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  ${}^1x$ , for

$${}^2x = {}^2\lambda_2 = -3$$

- 3  $\mathbb{L}_1$  along  ${}^2x$  for

$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$\left[ {}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

${}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1x$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_1 = 1$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_2 = 3$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\lambda_3 = 5$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_1 = 0$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_2 = 2$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$
${}^1\mu_3 = 4$				

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  ${}^1x$ , for

$${}^2x = {}^2\lambda_2 = -3$$

- 3  $\mathbb{L}_1$  along  ${}^2x$  for

$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$\left[ {}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

${}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1x$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_1 = 1$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_2 = 3$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\lambda_3 = 5$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_1 = 0$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_2 = 2$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$
${}^1\mu_3 = 4$				

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  ${}^1x$ , for

$${}^2x = {}^2\lambda_2 = -3$$

- 3  $\mathbb{L}_1$  along  ${}^2x$  for

$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$[ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 ]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

## 2-D case

### Theorem: 2-D to 1-D

Being given the tableau  $\text{tab}_2$  tensor in response of the 2-variables  $\mathbf{H}({}^1x, {}^2x)$  function, the null space of the corresponding 2-D Loewner matrix  $\mathbb{L}_2$ , is spanned by

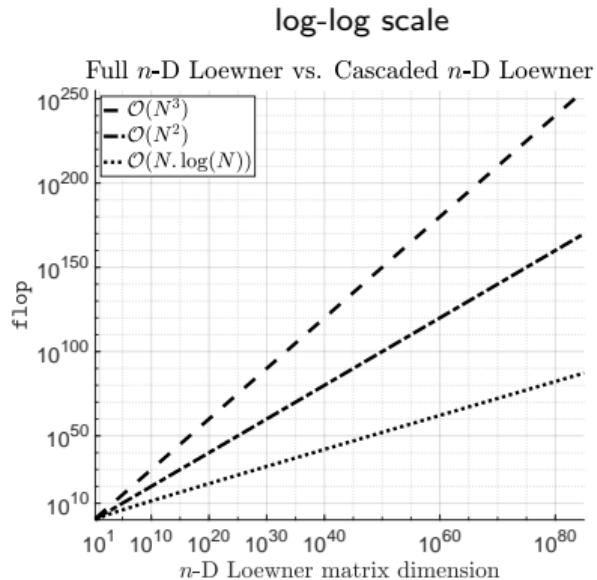
$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[ \mathbf{c}_1^{{}^2\lambda_1} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \vdots \\ {}^1\lambda_{k_1} \end{bmatrix}_1, \dots, \mathbf{c}_1^{{}^2\lambda_{k_2}} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \vdots \\ {}^1\lambda_{k_1} \end{bmatrix}_{k_2} \right],$$

where

- ▶  $\mathbf{c}_1^{{}^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{{}^1\lambda_{k_1}})$ ,  
i.e. the null space of the **1-D Loewner matrix** for frozen  ${}^1x = {}^1\lambda_{k_1}$ , and
- ▶  $\mathbf{c}_1^{{}^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{{}^2\lambda_{j_2}})$ ,  
i.e. the  $j_2$ -th null space of the **1-D Loewner matrices** for frozen  ${}^2x = \{{}^2\lambda_1, \dots, {}^2\lambda_{k_2}\}$ .

# Taming the curse of dimensionality

Null space - flop complexity



(rows)  $Q = q_1 q_2 \dots q_n$  and  
(columns)  $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

## Computational issue

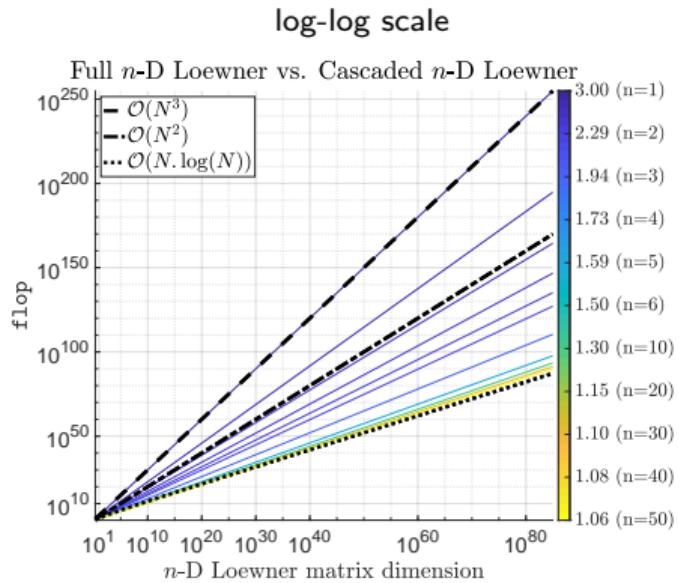
Note that  $Q \times K$  matrix SVD flop estimation is

- ▶  $QK^2$  (if  $Q > K$ )
- ▶  $N^3$  (if  $Q = K = N$ )

⇒ The CURSE of dimensionality

# Taming the curse of dimensionality

Null space - flop complexity



## Theorem: Worst case complexity

$k$  interpolation points per variables.

$$\overline{\text{flop}_1} = k^3 \frac{1 - k^n}{1 - k} = k^3 \frac{1 - N}{1 - k},$$

which is a ( $n$  finite) geometric series of ratio  $k$ .

⇒ The CURSE of dimensionality is TAMED

$$\begin{aligned}\mathcal{O}(N^3) &\rightarrow \mathcal{O}(N^{2.29}) && \text{for } n = 2 \\ &\rightarrow \mathcal{O}(N^{1.94}) && \text{for } n = 3 \\ &\vdots \\ &\rightarrow \mathcal{O}(N^{1.06}) && \text{for } n = 50\end{aligned}$$

# Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB})$$

## Full $n$ -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where  $N = k_1 k_2 \cdots k_n$ , needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example:  $N = 46,080$

Memory: 31.64 GB

flop:  $9.78 \cdot 10^{13}$

# Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB)}$$

## Full $n$ -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where  $N = k_1 k_2 \cdots k_n$ , needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example:  $N = 46,080$

Memory: 31.64 GB

flop:  $9.78 \cdot 10^{13}$

## Cascaded $n$ -D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\bar{k} \times \bar{k}}$$

where  $\bar{k} = \max_j k_j$ , needs

$$\frac{8}{2^{20}} \bar{k}^2 \text{ MB}$$

Example:  $\bar{k} = 20$

Memory: 6.25 KB

flop:  $8.13 \cdot 10^5$

# Variables decoupling, KST and KANs

Kolmogorov Superposition Theorem and Hilbert's 13th problem

## Kolmogorov, Arnold, Kahane, Lorentz and Sprecher

For every continuous function  $f : \mathbb{I}^n \mapsto \mathbb{R}$  and any  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist

- ▶ real numbers  $\lambda_1, \dots, \lambda_n$ ;
- ▶ continuous functions  $\Phi_k : \mathbb{I} \mapsto \mathbb{R}$ ,  $k = 1, \dots, 2n + 1$ ;
- ▶ a continuous function  $g : \mathbb{R} \mapsto \mathbb{R}$ ;

such that:

$$\forall (^1x, \dots, {}^nx) \in \mathbb{I}^n, \quad f(^1x, \dots, {}^nx) = \sum_{k=1}^{2n+1} g(\lambda_1 \Phi_k(^1x) + \dots + \lambda_n \Phi_k({}^nx))$$

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to its 13th problem) is wrong."



A.G. Vitushkin, "On Hilbert's thirteenth problem and related questions", Russian Math. Surveys 59:1, pp. 11-25.

# Variables decoupling, KST and KANs

Loewner and KST

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[ \mathbf{c}_1^{\frac{2}{\lambda_1}} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_1, \dots, \mathbf{c}_1^{\frac{2}{\lambda_{k_2}}} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_{k_2} \right],$$

## Variable decoupling

Given data  $\text{tab}_n$ , the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{nx}}_{\text{Bary}(nx)} \odot \underbrace{(\mathbf{c}^{n-1x} \otimes \mathbf{1}_{k_n})}_{\text{Bary}(n-1x)} \odot \underbrace{(\mathbf{c}^{n-2x} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\text{Bary}(n-2x)} \odot \dots \odot \underbrace{(\mathbf{c}^1x \otimes \mathbf{1}_{k_n \dots k_2})}_{\text{Bary}(1x)}.$$

where  $\mathbf{c}^l x$  denotes the vectorized barycentric coefficients related to the  $l$ -th variable.

This is decoupling !

# Variables decoupling, KST and KANs

Loewner and KST

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[ \mathbf{c}_1^{2\lambda_1} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_1, \dots, \mathbf{c}_1^{2\lambda_{k_2}} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_{k_2} \right],$$

## Variable decoupling

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where  $\mathbf{c}^l x$  denotes the vectorized barycentric coefficients related to the  $l$ -th variable.

This is decoupling !

# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $H = {}^1x \cdot {}^2x$ )

$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\left( \begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\mathbf{c}^{{}^2x} = \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix}$$

$$\mathbf{c}^{{}^1x} = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $\mathbf{H} = {}^1x \cdot {}^2x$ )

$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\left( \begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \mathbf{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\mathbf{c}^{{}^2x} = \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix}$$

$$\mathbf{c}^{{}^1x} = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

$$\mathbf{D} = \begin{pmatrix} \overbrace{\mathbf{c}^{{}^1x} \cdot \mathbf{Lag}^{{}^1x}}^{\mathbf{Bary}^{{}^1x}} & \overbrace{\mathbf{c}^{{}^2x} \cdot \mathbf{Lag}^{{}^2x}}^{\mathbf{Bary}^{{}^2x}} \\ -\frac{1.0}{{}^1x+1.0} & -\frac{1.0}{{}^2x+1.0} \\ -\frac{1.0}{{}^1x+1.0} & \frac{1}{{}^2x-1.0} \\ \frac{1}{{}^1x-1.0} & -\frac{1.0}{{}^2x+1.0} \\ \frac{1}{{}^1x-1.0} & \frac{1}{{}^2x-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\sum_{i-\text{th row}} \prod_{j-\text{th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i-\text{th row}} \mathbf{w} \cdot \prod_{j-\text{th col}} [\mathbf{D}]_{i,j}$$

# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $H = {}^1x \cdot {}^2x$ )

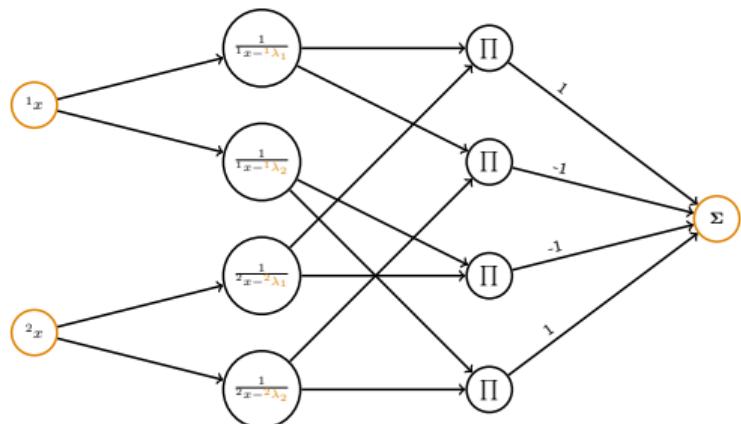
$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\left( \begin{array}{cccc} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\begin{aligned} \mathbf{c}^{{}^2x} &= \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\ \mathbf{c}^{{}^1x} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix} \end{aligned}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

Denominator Network view



# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $\mathbf{H} = {}^1x \cdot {}^2x$ )

## KST via Loewner

$$\begin{aligned} &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{{}^c j_1, j_2 \mathbf{w}_{j_1, j_2}}{\binom{{}^1 x - 1}{\lambda_{j_1}} \binom{{}^2 x - 2}{\lambda_{j_2}}}} \\ &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{{}^c j_1, j_2}{\binom{{}^1 x - 1}{\lambda_{j_1}} \binom{{}^2 x - 2}{\lambda_{j_2}}}} \\ &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp \left( \log(\mathbf{w}_{j_1, j_2}) + \log(\mathbf{Bary}_{j_1}^{{}^1 x}) + \log(\mathbf{Bary}_{j_2}^{{}^2 x}) \right)}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp \left( \log(\mathbf{Bary}_{j_1}^{{}^1 x}) + \log(\mathbf{Bary}_{j_2}^{{}^2 x}) \right)} \end{aligned}$$

## Decoupled barycentric weights

$$\begin{array}{ll} \overbrace{\mathbf{c}^{{}^1 x} \cdot \mathbf{Lag}({}^1 x)}^{\mathbf{Bary}({}^1 x)} & \overbrace{\mathbf{c}^{{}^2 x} \cdot \mathbf{Lag}({}^2 x)}^{\mathbf{Bary}({}^2 x)} \\ -\frac{1.0}{1x+1.0} & -\frac{1.0}{2x+1.0} \\ -\frac{1.0}{1x+1.0} & \frac{1}{2x-1.0} \\ \frac{1}{1x-1.0} & -\frac{1.0}{2x+1.0} \\ \frac{1}{1x-1.0} & \frac{1}{2x-1.0} \end{array}$$

This is the solution of KST for rational forms !

# Comparisons

## Some competitors

Rat. app [B/G 2025]

- ▶ Lagrangian interpolation theorem
- ▶ p-AAA

KAN [P/P 2025]

- ▶ Kolmogorov Arnold theorem
- ▶ Kolmogorov Arnold Network

MLP [TensorFlow by Google - Keras 2025]

- ▶ Universal approximation theorem
- ▶ Multi Layer Perceptron
- ▶ Dense connected / ReLU / ADAM / 1000 it. / rand. init.

### TensorFlow interface (Python code)

```
1 import numpy as np
2 import math
3 import matplotlib.pyplot as plt
4 import cmath
5 from keras.models import Sequential
6 from matplotlib import cm
7
8 # Data
9 def f(x):
10     y = pow(x[1], 0)*x[0]
11     y += 1/2*x[1]*x[0] + np.abs(x[0]) + 3/10*x[1]**3
12     y = x[1]*x[0]*x[1] ##
13     y = np.exp(x[0]*x[1]**3)/(log(x[0]*x[1]**2-1.44)*(pow(x[1], 2)-1.44))
14     y = np.exp(x[0]*x[1]**3)/(log(x[0]*x[1]**2-1.44)*(pow(x[1], 2)-1.44))
15     y = pow(np.abs(x[0]*x[1]**3), 1/3)
16     y = pow(x[1], 2) + pow(x[1], 1, 2) + x[1]*x[0] + x[1], 1 + (pow(x[1], 0), 3) + pow(x[1], 1, 2) + 4) ##
17     return np.transpose(np.array([y]))
18
19 # n = 2
20 N1 = 40
21 N2 = 40
22 x1 = np.linspace(-1, 1, N1)
23 x2 = np.linspace(0, 1, N2)
24
25 # IP
26 N = 11440
27 lab = np.zeros(N, 1)
```



L. Balicki and S. Gugercin, "*Multivariate Rational Approximation via Low-Rank Tensors and the p-AAA Algorithm*", SISC, 2025.



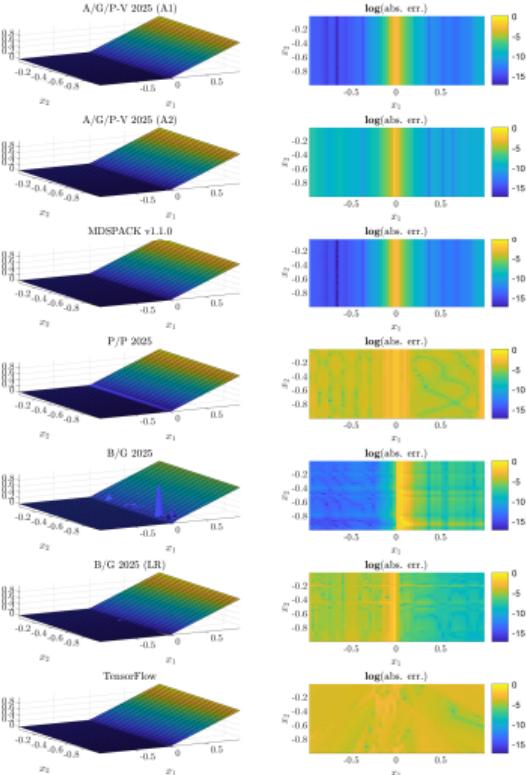
M. Poluektov and A. Polar, "*Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method*",  
<https://arxiv.org/abs/2305.08194>.



M. Abadi et al., "*TensorFlow: Large-scale machine learning on heterogeneous systems, 2015*", Software available from tensorflow.org.

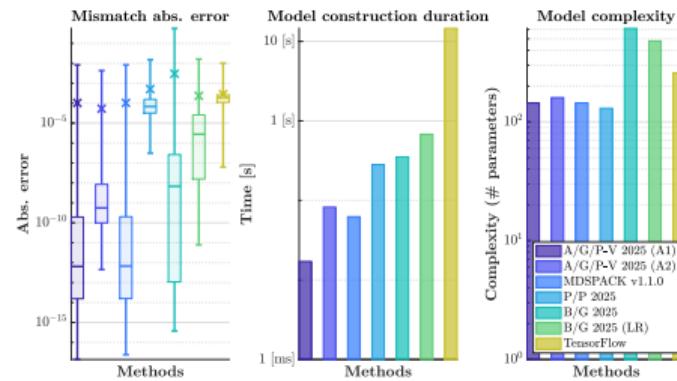
# Comparisons

## Irrational functions (example #1)



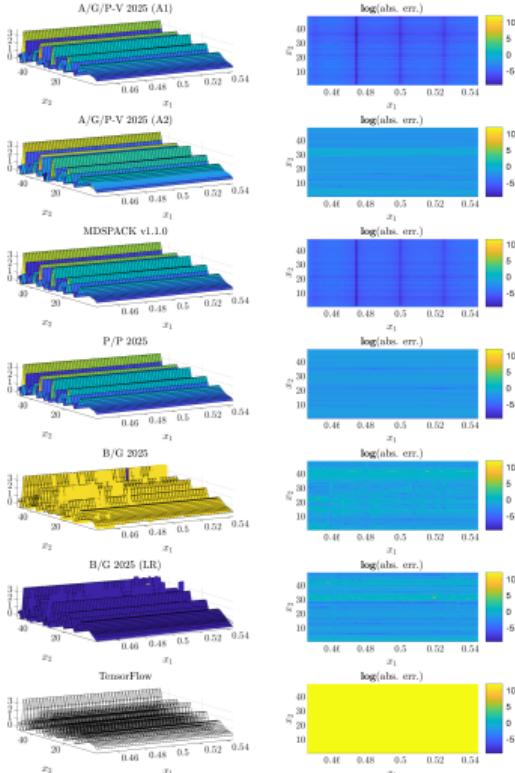
$$\text{ReLU}(\begin{pmatrix} 1 \\ x \end{pmatrix}) + \frac{1}{100} \begin{pmatrix} 2 \\ x \end{pmatrix}$$

- ▶ Reference: Personal communication, [none]
- ▶ Domain:  $\mathbb{R}$
- ▶ Tensor size: 12.5 KB ( $40^2$  points)
- ▶ Bounds:  $(-1, 1) \times (-1, -10^{-10})$



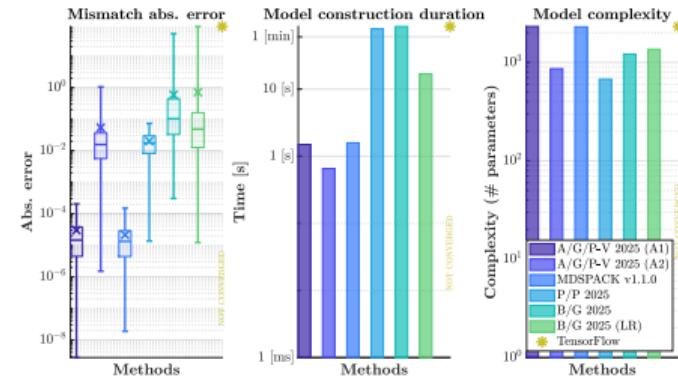
# Comparisons

## Irrational functions (example #34)



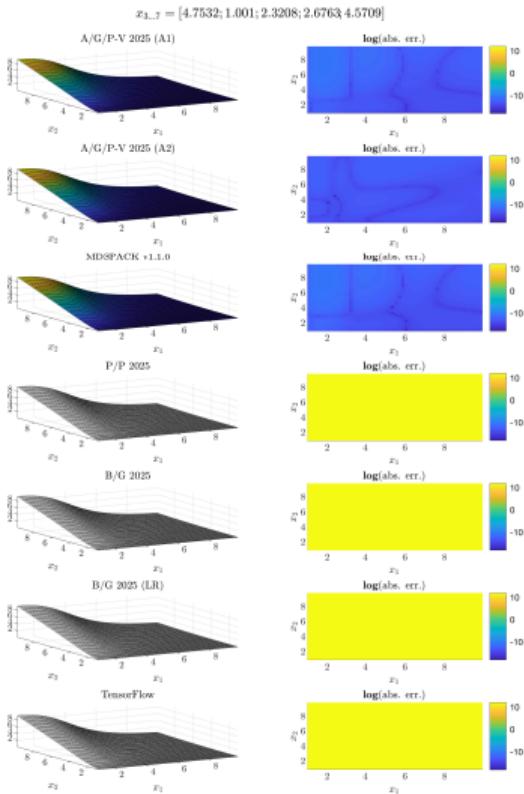
$$\operatorname{Re}(\zeta(1^1 x + i^2 x))$$

- ▶ Riemann  $\zeta$  function (real part), [none]
- ▶ Domain:  $\mathbb{R}$
- ▶ Tensor size: 1.22 MB ( $400^2$  points)
- ▶ Bounds:  $\left( \frac{9}{20}, \frac{11}{20} \right) \times (1, 50)$



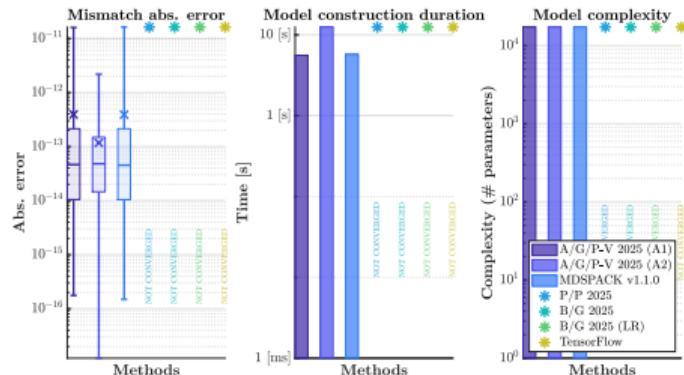
# Comparisons

## Irrational functions (example #43)



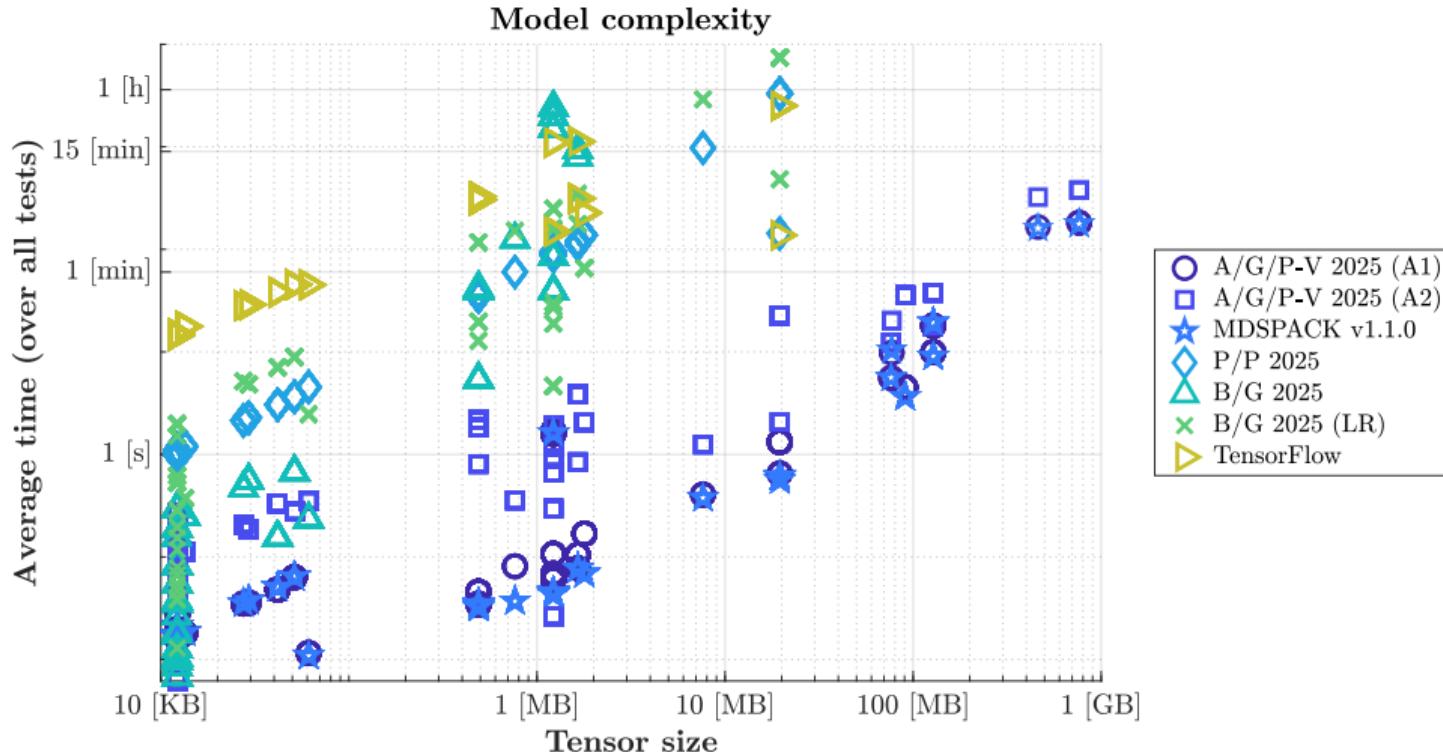
$$\frac{3x^2x^3 + 1}{1x^4 + 2x^2 \cdot 3x + 4x^2 + 5x + 6x^3 + 7x}$$

- ▶ Reference: Personal communication, [Riemann]
- ▶ Domain:  $\mathbb{R}$
- ▶ Tensor size: 76.3 MB ( $10^7$  points)
- ▶ Bounds:  $(1 \quad 10)^7$



# Comparisons

## Irrational functions (time, scalability)



# Conclusion

## Take home message

### Main contributions

From any  $n$ -th order multi-variate transfer function / data tensor

- ▶ Construct a transfer function in barycentric form
- ▶ Construct a realization with controlled complexity
- ▶ Tame the computational complexity
- ▶ Two algorithms (direct & iterative)
- ▶ Connection with Kolmogorov theorem
- ▶ Connection with Kolmogorov networks

### Side effects

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with

A.C. Antoulas [Rice Univ.]

I.V. Goșea [MPI]

P. Vuillemin [ONERA]

<https://arxiv.org/abs/2405.00495>

<https://arxiv.org/abs/2506.04791>

<https://github.com/cpoussot/mLF>

<https://cpoussot.github.io>



# In parting... if enough time

Numerical examples, 20-D example

$$\mathbf{H}(^1x, ^2x, \dots, ^{20}x) =$$

$$\frac{3 \cdot {}^1x^3 + 4 \cdot {}^8x + {}^{12}x + {}^{13}x \cdot {}^{14}x + {}^{15}x}{{}^1x + {}^2x^2 \cdot {}^3x + {}^4x + {}^5x + {}^6x + {}^7x \cdot {}^8x + {}^9x \cdot {}^{10}x \cdot {}^{11}x + {}^{13}x + {}^{13}x^3 \cdot \pi + {}^{17}x + {}^{18}x \cdot {}^{19}x - {}^{20}x}$$

## Statistics

- ▶ 20-D tensor of dimension ( $\geq 48$  TB in real double precision)
- ▶ Complexity: (3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)
- ▶  $n$ -D Loewner matrix  
 $6,291,456^2 \rightarrow 288$  TB of storage in real double precision
- ▶ Full SVD:  $2.49 \cdot 10^{20}$  flop and 288 TB  
Recursive SVD:  $5.43 \cdot 10^7$  flop and 0.12 KB
- ▶ error  $\approx 10^{-11}$

# In parting... if enough time

Numerical examples, 20-D example

$$\mathbf{H}(^1x, ^2x, \dots, ^{20}x) =$$

$$\frac{3 \cdot {}^1x^3 + 4 \cdot {}^8x + {}^{12}x + {}^{13}x \cdot {}^{14}x + {}^{15}x}{{}^1x + {}^2x^2 \cdot {}^3x + {}^4x + {}^5x + {}^6x + {}^7x \cdot {}^8x + {}^9x \cdot {}^{10}x \cdot {}^{11}x + {}^{13}x + {}^{13}x^3 \cdot \pi + {}^{17}x + {}^{18}x \cdot {}^{19}x - {}^{20}x}$$

## Statistics

- ▶ 20-D tensor of dimension ( $\geq 48$  TB in real double precision)
- ▶ Complexity: (3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
- ▶  $n$ -D Loewner matrix  
 $6,291,456^2 \rightarrow 288$  TB of storage in real double precision
- ▶ Full SVD:  $2.49 \cdot 10^{20}$  flop and 288 TB  
Recursive SVD:  $5.03 \cdot 10^7$  flop and 0.12 KB
- ▶ error  $\approx 10^{-11}$