The multivariate Loewner framework...

The Kolmogorov superposition theorem, the curse of dimensionality & benchmark

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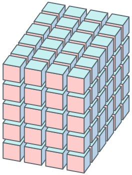
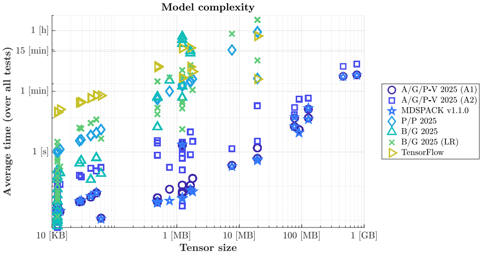
October 12, 2025

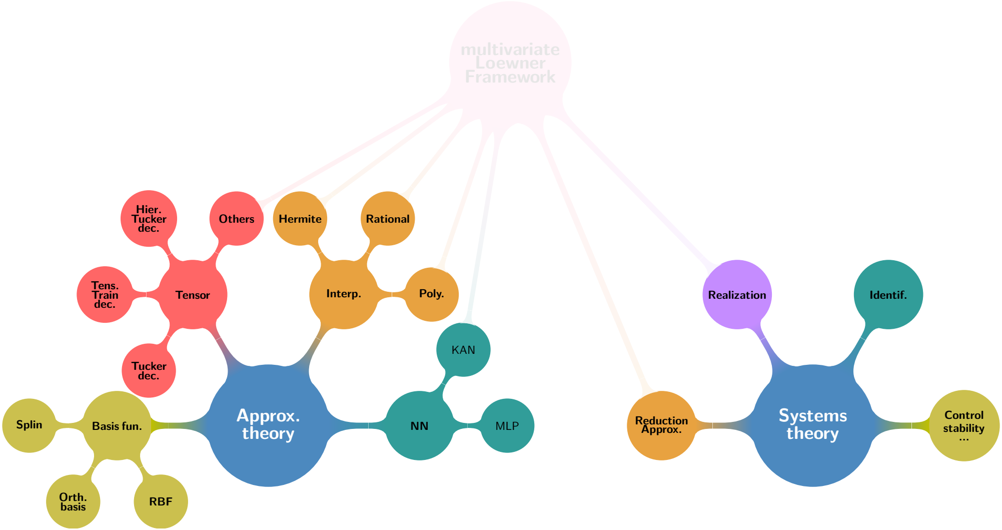
https://arxiv.org/abs/2405.00495 [in SIAM Review - Research Spotlight]
https://arxiv.org/abs/2506.04791 [extensive benchmark]
https://github.com/cpoussot/mLF [research code]





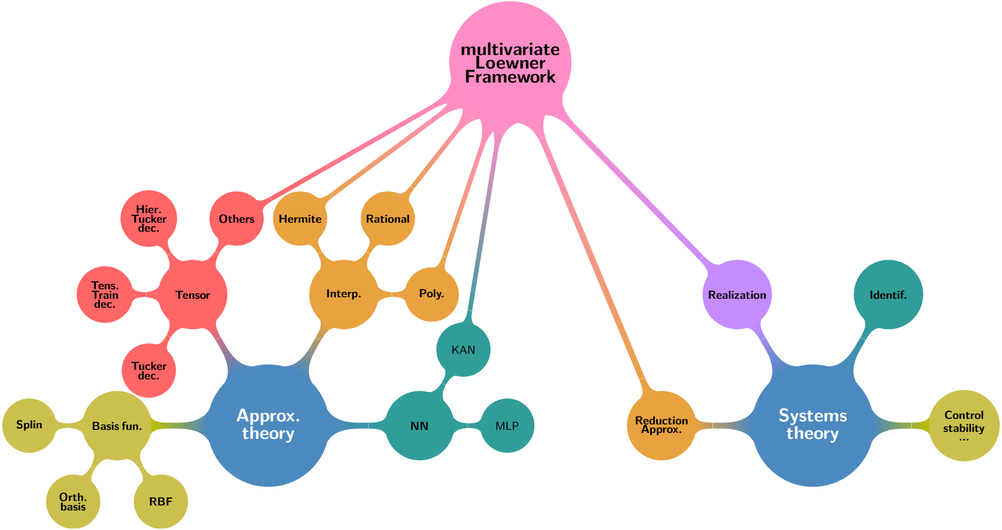
Forewords

Approximation & systems theory... where we stand



Forewords

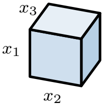
Approximation & systems theory... where we stand



Forewords

Starting (motivating) examples - Airbus flutter case

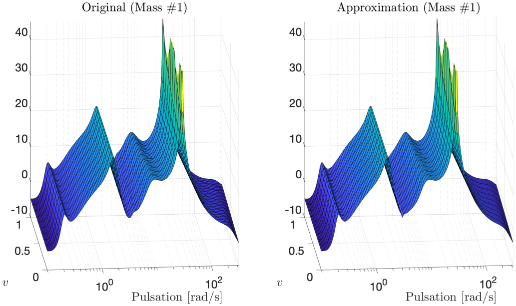
$$\mathbf{\Sigma}(x_1, x_2, x_3) = \mathbf{\Sigma}(s, m, v) : s^2 M(m)x(s) + sB(m)x(s) + K(m)x(s) - G(s, v) = u(s),\ \mathbf{y}(s) = C\mathbf{x}(s)$$



$$\begin{array}{ccccc} x_1 & \times & x_2 & \times & x_3 \\ \imath[10, 35] & \times & [\underline{m}, \overline{m}] & \times & [\underline{v}, \overline{v}] \end{array}$$

$$\textbf{tab}_3 \in \mathbb{C}^{300\times 10\times 10}$$

$\approx$`468.75 Ko ('complex')`

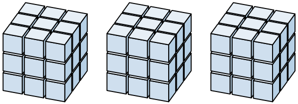


A. dos Reis de Souza et al., *"Aircraft flutter suppression: from a parametric model to robust control"*, ECC, 2023.

Forewords

Starting (motivating) examples - Borehole function

$$\mathbf{H}(x_1,\cdots,x_8) = \mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w) = \frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$



$$\begin{array}{ccccc} x_1 & \times & \cdots & \times & x_8 \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

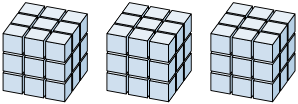
tab$_8 \in \mathbb{C}^{8\times 8\times\cdots\times 8}$

$\approx$`130 Mo ('real')`

$r_w \in [0.05, 0.15]$	radius of borehole (m)
$r \in [100, 50\,000]$	radius of influence (m)
$T_u \in [63\,070, 115\,600]$	transmissivity of upper aquifer (m^2/yr)
$H_u \in [990, 1110]$	potentiometric head of upper aquifer (m)
$T_l \in [63.1, 116]$	transmissivity of lower aquifer (m^2/yr)
$H_l \in [700, 820]$	potentiometric head of lower aquifer (m)
$L \in [1120, 1680]$	length of borehole (m)
$K_w \in [9855, 12\,045]$	hydraulic conductivity of borehole (m/yr)

S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

$$\mathbf{H}(x_1,\cdots,x_8)=\mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w)=\frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$



$$\begin{array}{ccccc} x_1 & \times & \cdots & \times & x_8 \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

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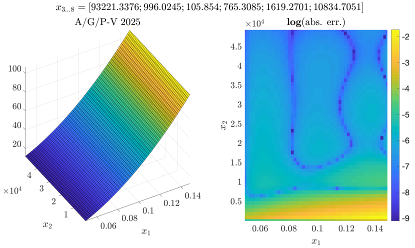
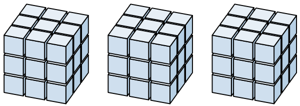


S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

$$\mathbf{H}(x_1,\cdots,x_8)=\mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w)=\frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$



$$\begin{array}{ccccc} x_1 & \times & \cdots & \times & x_8 \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

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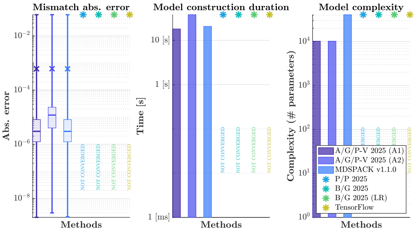
≈130 Mo ('real')



S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

Forewords

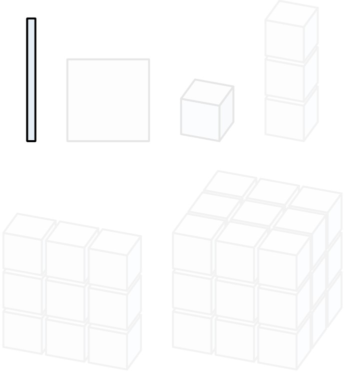
Data (and tensors)

Column / Row data

$$\mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \; \Big\} \xrightarrow{\mathbf{H}(x_1)} \Big\{ \; \mathbf{w}_{j_1}, \mathbf{v}_{i_1}$$

x_1	
$\lambda_1(1, \cdots, k_1)$	$\mathbf{W}_{k_1}$
$\mu_1(1, \cdots, q_1)$	$\mathbf{V}_{q_1}$

Tensors (1-D) tab_1

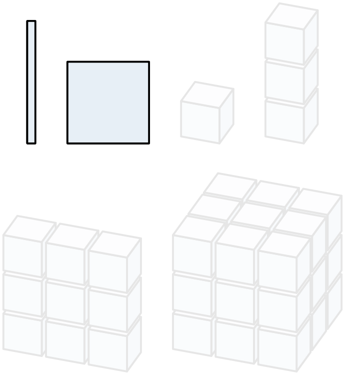


Column / Row data

$$\left.\begin{aligned}\mathbf{x}_1 &= \lambda_1(j_1), \mu_1(i_1)\\ \mathbf{x}_2 &= \lambda_2(j_2), \mu_2(i_2)\end{aligned}\right\} \xrightarrow{\mathbf{H}(x_1,x_2)} \left\{ \mathbf{w}_{j_1,j_2}, \mathbf{v}_{i_1,i_2} \right.$$

$x_1 \backslash x_2$	$\lambda_2(1,\cdots,k_2)$	$\mu_2(1,\cdots,q_2)$
$\lambda_1(1,\cdots,k_1)$	$\mathbf{W}_{k_1,k_2}$	ϕ_{cr}
$\mu_1(1,\cdots,q_1)$	ϕ_{rc}	$\mathbf{V}_{q_1,q_2}$

Tensors (2-D) tab_2



Column / Row data

$$\left.\begin{array}{l}\mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1)\\ \mathbf{x}_2 = \lambda_2(j_2), \mu_2(i_2)\\ \mathbf{x}_3 = \lambda_3(j_3), \mu_3(i_3)\end{array}\right\} \xrightarrow{\mathbf{H}(x_1,x_2,x_3)} \left\{ \mathbf{w}_{j_1,j_2,j_3}, \mathbf{v}_{i_1,i_2,i_3}\right.$$

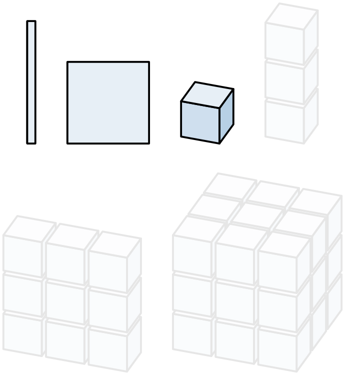
$x_3 = \lambda_3(1, \cdots, k_3)$

$x_1 \backslash x_2$	$\lambda_2(1, \cdots, k_2)$	$\mu_2(1, \cdots, q_2)$
$\lambda_1(1, \cdots, k_1)$	$\mathbf{W}_{k_1,k_2,k_3}$	ϕ_{crc}
$\mu_1(1, \cdots, q_1)$	ϕ_{rcc}	ϕ_{rrc}

$x_3 = \mu_3(1, \cdots, q_3)$

$x_1 \backslash x_2$	$\lambda_2(1, \cdots, k_2)$	$\mu_2(1, \cdots, q_2)$
$\lambda_1(1, \cdots, k_1)$	ϕ_{crr}	ϕ_{crr}
$\mu_1(1, \cdots, q_1)$	ϕ_{rcr}	$\mathbf{V}_{q_1,q_2,q_3}$

Tensors (3-D) tab_3



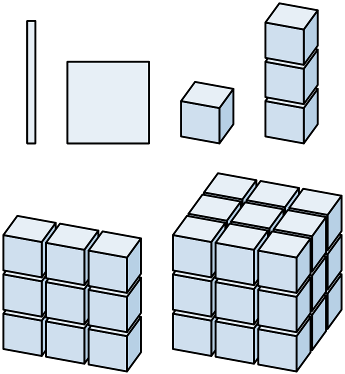
Forewords

Data (and tensors)

Column / Row data

$$
\left.\begin{array}{l}
\mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \\
\mathbf{x}_2 = \lambda_2(j_2), \mu_2(i_2) \\
\mathbf{x}_3 = \lambda_3(j_3), \mu_3(i_3) \\
\vdots \\
\mathbf{x}_n = \lambda_n(j_n), \mu_n(i_n)
\end{array}\right\} \xrightarrow{\mathbf{H}(x_1,\cdots,x_n)} \left\{ \mathbf{w}_{j_1,\cdots,j_n}, \mathbf{v}_{i_1} \right.
$$

Tensors (n-D) tab$_n$

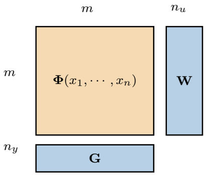


Data-driven model approximation

Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$

$$\hat{\mathbf{H}}(x_1, x_2, \ldots, x_n) = \mathbf{G}\mathbf{\Phi}(x_1, x_2, \ldots, x_n)^{-1}\mathbf{W} \in \mathbb{C}$$

that interpolates the data.



Connection to standard dynamical system realization

A linear-in-state dynamical system parameterized in terms of parameters included in $\mathcal{S} = [x_2, \ldots, x_n]^\top \subset \mathbb{C}^{n-1}$

$$\begin{aligned} \mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t;\mathcal{S}) &= \mathbf{A}(\mathcal{S})\mathbf{x}(t;\mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t;\mathcal{S}) &= \mathbf{C}\mathbf{x}(t;\mathcal{S}) \end{aligned}$$

equivalently

$$\hat{\mathbf{H}}(x_1, x_2, \ldots, x_n) = \mathbf{C}(\mathcal{S})\left[x_1\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1}\mathbf{B} \in \mathbb{C}.$$

Data-driven model approximation

Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$

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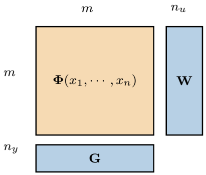

Connection to standard dynamical system realization

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equivalently

$$\hat{\mathbf{H}}(x_1, x_2, \ldots, x_n) = \mathbf{C}(\mathcal{S})\left[x_1\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1}\mathbf{B} \in \mathbb{C}.$$

Forewords

Where we stand (some references)

1-D Two-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization minimality
$\Rightarrow$ direct algorithm

1-D One-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ direct algorithm

1-D AAA (Adaptive Anderson Antoulas - one-sided)
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

2-D Parametric one-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization (non-minimal)
$\Rightarrow$ direct algorithm

3-D Parametric AAA
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

$>$3-D **few results**

J-P. Berrut and N. Trefethen, *"Barycentric Lagrange Interpolation"*, SIAM Review, 46(3), 2004.

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realisation problem"*, LAA, 425(2-3), 2007.

A.C. Ionita and A.C. Antoulas, *"Data-Driven Parametrized Model Reduction in the Loewner Framework"*, SIAM Journal on Scientific Computing, 36(3), 2014.

T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, *"Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework"*, SIAM Journal on Applied Dynamical Systems, 22(4), 2023.

A.C. Rodriguez, L. Balicki and S. Gugercin, *"The p-AAA algorithm for data driven modeling of parametric dynamical systems"*, SIAM Journal on Scientific Computing, 45(3), 2023.

Forewords

Contributions claim & trajectory of the presentation

- n-D tensor data to n-D Loewner matrix $\mathbb{L}_n$
- n-variable transfer functions
- n-variable generalized realization
- Taming the curse of dimensionality
  - » in computation effort (`flop`)
  - » in storage needs (`Bytes`)
  - » in accuracy
- n-variable **decoupling**
  - » **KST** formulation for rational functions
  - » connection with **KAN**
- Comparison with **MLP, KAN, AAA**

A.C. Antoulas, I-V. Gosea and C. P-V., *"On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality"*, SIAM Review, November, 2025 (`https://arxiv.org/abs/2405.00495`).

A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, *"Tensor-based multivariate function approximation: methods benchmarking and comparison"*, arXiv, June, 2025 (`https://arxiv.org/abs/2506.04791`).

A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, *"mLF package"*, GitHub (`https://github.com/cpoussot/mLF`).

Content

Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} := \{(\lambda_1(j_1); \mathbf{w}_{j_1}),\ j_1 = 1, \dots, k_1\} \\ P_r^{(1)} := \{(\mu_1(i_1); \mathbf{v}_{i_1}),\ i_1 = 1, \dots, q_1\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1,j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{\mu_1(i_1) - \lambda_1(j_1)}$$

$$\mathbf{M}_1\mathbb{L}_1 - \mathbb{L}_1\mathbf{\Lambda}_1 = \mathbb{V}_1\mathbf{R}_1 - \mathbf{L}_1\mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}(x_1) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1}\mathbf{w}_{j_1}}{x_1 - \lambda_1(j_1)}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{x_1 - \lambda_1(j_1)}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}(x_1) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\left.\begin{array}{rcl} \lambda_1(j_1) & = & [1,3,5] \\ \mu_1(i_1) & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} & := & \{(\lambda_1(j_1), \lambda_2(j_2); \mathbf{w}_{j_1,j_2}),\ j_1 = 1,\dots,k_1 \quad j_2 = 1,\dots,k_2\} \\ P_r^{(2)} & := & \{(\mu_1(i_1), \mu_2(i_2); \mathbf{v}_{i_1,i_2}),\ i_1 = 1,\dots,q_1 \quad i_2 = 1,\dots,q_2\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{(\mu_1(i_1) - \lambda_1(j_1))(\mu_2(i_2) - \lambda_2(j_2))}$$

$$\begin{cases} \mathbf{M}_2\mathbb{X} - \mathbb{X}\mathbf{\Lambda}_2 & = & \mathbb{V}_2\mathbf{R}_2 - \mathbf{L}_2\mathbb{W}_2 \\ \mathbf{M}_1\mathbb{L}_2 - \mathbb{L}_2\mathbf{\Lambda}_1 & = & \mathbb{X} \end{cases}$$

Lagrangian form

$$\mathbf{g}(x_1, x_2) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \begin{bmatrix} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{bmatrix} \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} \lambda_1(j_1) & = & [1,3,5] \\ \mu_1(i_1) & = & [0,2,4] \\ \lambda_2(j_2) & = & [-1,-3] \\ \mu_2(i_2) & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Loewner matrix

$$\mathbb{L}_2=\left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array}\right]$$

Null space

$$\mathbf{c}_2=\left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} \lambda_1(j_1) & = & [1,3,5] \\ \mu_1(i_1) & = & [0,2,4] \\ \lambda_2(j_2) & = & [-1,-3] \\ \mu_2(i_2) & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Lagrangian form

$$\mathbf{g}(s,t)=-\frac{\frac{1}{9(s-1)(t+1)}-\frac{1}{3(s-1)(t+3)}-\frac{2}{(s-3)(t+1)}+\frac{6}{(s-3)(t+3)}+\frac{25}{9(s-5)(t+1)}-\frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)}-\frac{5}{9(s-1)(t+3)}-\frac{10}{9(s-3)(t+1)}+\frac{14}{9(s-3)(t+3)}+\frac{7}{9(s-5)(t+1)}-\frac{1}{(s-5)(t+3)}}=\mathbf{H}(s,t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \{(\lambda_1(j_1), \lambda_2(j_2), \cdots, \lambda_n(j_n); \mathbf{w}_{j_1,j_2,\cdots,j_n}),\ j_l = 1, \ldots, k_l, \quad l = 1, \cdots, n\} \\ P_r^{(n)} := \{(\mu_1(i_1), \mu_2(i_2), \cdots, \mu_n(i_n); \mathbf{v}_{i_1,i_2,\cdots,i_n}),\ i_l = 1, \ldots, q_l, \quad l = 1, \cdots, n\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \frac{\mathbf{v}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{(\mu_1(i_1) - \lambda_1(j_1)) \cdots (\mu_n(i_n) - \lambda_n(j_n))}$$

$$\begin{cases} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \mathbf{\Lambda}_n & = & \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ & \vdots & \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \mathbf{\Lambda}_1 & = & \mathbb{X}_{n-1}. \end{cases}$$

Lagrangian form

$$\mathbf{g}(x_1, \cdots, x_n) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \left[\begin{array}{c} c_{1,\cdots,1} \\ \vdots \\ c_{1,\cdots,k_n} \\ \hline \vdots \\ \hline c_{k_1,\cdots,1} \\ \vdots \\ c_{k_1,\cdots,k_n} \end{array}\right] \in \mathbb{C}^{k_1 \cdots k_n}$$

Content

Multi-variate realization

1-D case (example cont'd)

Data generated from $\mathbf{H}(x_1) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}}$$

Lagrangian realization $\hat{\mathbf{H}}(s) = \mathbf{W}\mathbf{\Phi}(s)^{-1}\mathbf{G}$

$$\mathbf{\Phi}(s) = \left[\begin{array}{ccc} s-1 & 3-s & 0 \\ s-1 & 0 & 5-s \\ \hline -\frac{1}{3} & \frac{4}{3} & -1 \end{array}\right]$$

$$\mathbf{W} = \left[\begin{array}{ccc} \frac{5}{6} & -\frac{13}{3} & \frac{29}{6} \end{array}\right]$$

$$\mathbf{G}^\top = \left[\begin{array}{cc|c} 0 & 0 & -1 \end{array}\right]$$

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t)=-\frac{\frac{1}{9(s-1)(t+1)}-\frac{1}{3(s-1)(t+3)}-\frac{2}{(s-3)(t+1)}+\frac{6}{(s-3)(t+3)}+\frac{25}{9(s-5)(t+1)}-\frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)}-\frac{5}{9(s-1)(t+3)}-\frac{10}{9(s-3)(t+1)}+\frac{14}{9(s-3)(t+3)}+\frac{7}{9(s-5)(t+1)}-\frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}(s,t)=\mathbf{W}\mathbf{\Phi}(s,t)^{-1}\mathbf{G}$

$$\mathbf{\Phi}(s,t)=\left[\begin{array}{ccc|c|cc} s-1 & 3-s & 0 & 0 & 0 & 0\\ s-1 & 0 & 5-s & 0 & 0 & 0\\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1 & 0 & 0\\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 & 0 & 0\\ \hline \frac{1}{9} & -2 & \frac{25}{9} & 0 & t+1 & \frac{1}{2}\\ -\frac{1}{3} & 6 & -\frac{25}{3} & 0 & -t-3 & -\frac{1}{2} \end{array}\right]$$

$$\mathbf{W}=\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\mathbf{G}^\top=\left[\begin{array}{cc|cc|cc} 0 & 0 & 1/2 & -1/2 & 0 & 0 \end{array}\right]$$

$\rightarrow$ (3,3) block is unimodular !

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t)=-\frac{\frac{1}{9(s-1)(t+1)}-\frac{1}{3(s-1)(t+3)}-\frac{2}{(s-3)(t+1)}+\frac{6}{(s-3)(t+3)}+\frac{25}{9(s-5)(t+1)}-\frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)}-\frac{5}{9(s-1)(t+3)}-\frac{10}{9(s-3)(t+1)}+\frac{14}{9(s-3)(t+3)}+\frac{7}{9(s-5)(t+1)}-\frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}_{\mathsf{c}}(s,t)=\mathbf{W}_{\mathsf{c}}(t)\mathbf{\Phi}_{\mathsf{c}}(s,t)^{-1}\mathbf{G}_{\mathsf{c}}$

$$\mathbf{\Phi}_{\mathsf{c}}(s,t)=\left[\begin{array}{ccc|c} s-1 & 3-s & 0 & 0\\ s-1 & 0 & 5-s & 0\\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1\\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 \end{array}\right]$$

$$\mathbf{W}_{\mathsf{c}}(t)=\left[\begin{array}{ccc|c} -\frac{2t}{9} & 4t & -\frac{50t}{9} & 0 \end{array}\right]$$

$$\mathbf{G}_{\mathsf{c}}^{\top}=\left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\mathbf{g}(x_1,x_2,\cdots,x_n)=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}\mathbf{w}_{j_1,j_2,\cdots,j_n}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))\cdots(x_n-\lambda_n(j_n))}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))\cdots(x_n-\lambda_n(j_n))}},$$

Theorem: n-D Lagrangian realization

A $2\ell+\kappa-1=m$-th order realization $(\mathbf{G},\mathbf{\Phi},\mathbf{W})$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\hat{\mathbf{H}}(x_1,\cdots,x_n)=\mathbf{W}\mathbf{\Phi}(x_1,x_2,\cdots,x_n)^{-1}\mathbf{G}$, is given by,

$$\begin{aligned}
\mathbf{\Phi}(x_1,\cdots,x_n) &= \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m} \\
\mathbf{G} &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hline \mathbf{\Delta}(\ell,:)^\top \\ \hline \mathbf{0}_{\ell,1} \end{array}\right] \in \mathbb{C}^{m\times 1} \\
\mathbf{W} &= \left[\begin{array}{c|c|c} \mathbf{0}_{1,\kappa} & \mathbf{0}_{1,\ell-1} & -\mathbf{e}_\ell^\top \end{array}\right] \in \mathbb{C}^{1\times m}
\end{aligned}$$

where $\mathbb{A}^{\text{Lag}},\mathbb{B}^{\text{Lag}}\in\mathbb{C}^{\ell\times\kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

Multi-variate realization

Generalized n-D Lagrangian realization (focus on left / right variable sets)

$$\mathbf{\Phi}(x_1,\cdots,x_n) = \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned}
\mathbf{\Gamma} &= \mathbb{X}^{\text{Lag}}{}_1 \otimes \mathbb{X}^{\text{Lag}}{}_2 \otimes \cdots \otimes \mathbb{X}^{\text{Lag}}{}_k \in \mathbb{C}^{\kappa\times\kappa}[x_1,\cdots,x_k] \\
\mathbf{\Delta} &= \mathbb{X}^{\text{Lag}}{}_{k+1} \otimes \mathbb{X}^{\text{Lag}}{}_{k+2} \otimes \cdots \otimes \mathbb{X}^{\text{Lag}}{}_n \in \mathbb{C}^{\ell\times\ell}[x_{k+1},\cdots,x_n] \\
\mathbb{X}^{\text{Lag}}{}_j &= \left[\begin{array}{ccccc} \mathbf{x}_{1\,j} & -\mathbf{x}_{2\,j} & 0 & \cdots & 0 \\ \mathbf{x}_{1\,j} & 0 & -\mathbf{x}_{3\,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{x}_{1\,j} & 0 & \cdots & 0 & -\mathbf{x}_{n_j\,j} \\ \hline q_{1\,j} & q_{2\,j} & \cdots & q_{n_j-1\,j} & q_{n_j\,j} \end{array}\right] \in \mathbb{C}^{n_j\times n_j} \\
\mathbf{x}_{i\,j} &= x_j - \lambda_j(i)
\end{aligned}$$

Facts

- Left / right variables splitting

  $$\Gamma \text{ and } \Delta$$

- $\mathbb{X}^{\text{Lag}}{}_j$ is unimodular, i.e.

  $$\det(\mathbb{X}^{\text{Lag}}{}_j) = 1$$

- ...so are Γ and Δ

Multi-variate realization

Generalized n-D Lagrangian realization (focus on barycentric weights $\mathbb{A}^{\text{Lag}}$ and $\mathbb{B}^{\text{Lag}}$)

$$\mathbf{\Phi}(x_1,\cdots,x_n) = \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned} \mathbb{A}^{\text{Lag}} &= \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m+1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,m+1} \end{bmatrix} \\ \mathbf{c}_n &= \text{vec}\left(\mathbb{A}^{\text{Lag}}\right) \end{aligned}$$

Facts

- $\mathbb{A}^{\text{Lag}}$ is simply some rearrangement of $\mathcal{N}(\mathbb{L}_n) = \mathbf{c}_n$
- $\mathbb{B}^{\text{Lag}}$ follows

Multi-variate realization

Generalized n-D Lagrangian realization (control the complexity)

$$\mathbf{\Phi}(x_1,\cdots,x_n) = \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned} m &= 2\ell+\kappa-1 \\ \kappa &= \textstyle\prod_{j=1}^{k} n_j \\ \ell &= \textstyle\prod_{j=k+1}^{n} n_j \end{aligned}$$

Facts

- Γ gathers the first group of parameters $x_1,\cdots,x_k$
- Δ gathers the second group of parameters $x_{k+1},\cdots,x_n$

Complexity

Re-ordering allows complexity control, e.g. according to the order of each variable x_j

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}(x_1,x_2,x_3)=\mathbf{H}(s,t,p)=(s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$\begin{array}{rcl}\mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\ \mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]\end{array}$$

Arrangement #1

$(s)-(t,p)$, one obtains a realization of dimension $m=13$:

$$\kappa = 2 \text{ and } \ell = 2\times 3$$

$$\mathbf{\Delta}(s)=\mathbb{X}^{\mathrm{Lag}}{}_1(s)$$

$$\mathbf{\Gamma}(t,p)=\mathbb{X}^{\mathrm{Lag}}{}_2(t)\otimes\mathbb{X}^{\mathrm{Lag}}{}_3(p)$$

Arrangement #2

$(s,t)-(p)$, one obtains a realization of dimension $m=9$:

$$\kappa = 2\times 2 \text{ and } \ell = 3$$

$$\mathbf{\Delta}(s,t)=\mathbb{X}^{\mathrm{Lag}}{}_1(s)\otimes\mathbb{X}^{\mathrm{Lag}}{}_2(t)$$

$$\mathbf{\Gamma}(p)=\mathbb{X}^{\mathrm{Lag}}{}_3(p)$$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}(x_1,x_2,x_3)=\mathbf{H}(s,t,p)=(s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{array}{rcl}
\mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\
\mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{array}
$$

$$
\mathbf{\Phi}=\left[\begin{array}{cccc|cc|ccc}
(s-2)\ (t-1) & -(s-2)\ (t-3) & -(t-1)\ (s-4) & (s-4)\ (t-3) & 0 & 0 & 0 & 0 & 0 \\
1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 & 0 & 0 & 0 \\
-\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 & 0 & 0 & 0 \\
\frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p & 0 & 0 & 0 \\
\hline
\frac{1}{8} & -\frac{17}{56} & -\frac{9}{56} & \frac{19}{56} & 0 & 0 & p-5 & p-5 & \frac{1}{2} \\
-\frac{2}{7} & \frac{5}{7} & \frac{5}{14} & -\frac{11}{14} & 0 & 0 & 6-p & 0 & -1 \\
\frac{9}{56} & -\frac{23}{56} & -\frac{11}{56} & \frac{25}{56} & 0 & 0 & 0 & 7-p & \frac{1}{2}
\end{array}\right]
$$

$$
\mathbf{W}=-\mathbf{e}_9^\top \text{ and } \mathbf{G}^\top=\left[\begin{array}{c|ccc|c} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 & \mathbf{0}_{1,3}\end{array}\right]
$$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}(x_1,x_2,x_3)=\mathbf{H}(s,t,p)=(s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{array}{rcl}
\mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\
\mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{array}
$$

$$
\mathbf{\Phi}_c = \left[\begin{array}{cccc:cc}
(s-2)\ (t-1) & -(s-2)\ (t-3) & -(t-1)\ (s-4) & (s-4)\ (t-3) & 0 & 0 \\
1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0 \\
\frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0 \\
\hdashline
\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 \\
-\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 \\
\frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p
\end{array}\right]
$$

$$
\mathbf{W}_c(p) = \left[\begin{array}{cccc:cc} \frac{p}{28}+\frac{1}{14} & -\frac{3\,p}{28}-\frac{1}{14} & -\frac{p}{28}-\frac{1}{7} & \frac{3\,p}{28}+\frac{1}{7} & 0 & 0 \end{array}\right] \text{ and } \mathbf{G}_c^\top = \left[\begin{array}{c:ccc} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 \end{array}\right]
$$

Multi-variate realization

(Compressed) generalized n-D Lagrangian realization

$$\mathbf{g}(x_1,x_2,\cdots,x_n)=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}\mathbf{w}_{j_1,j_2,\cdots,j_n}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))\cdots(x_n-\lambda_n(j_n))}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}}{(x_1-\lambda_1(j_1))(x_2-\lambda_2(j_2))\cdots(x_n-\lambda_n(j_n))}},$$

Theorem: n-D Lagrangian compressed realization

A $\ell+\kappa-1=m$-th order realization $(\hat{\mathbf{G}}_c,\hat{\mathbf{\Phi}}_c,\hat{\mathbf{W}}_c)$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\mathbf{H}(x_1,\cdots,x_n)=\hat{\mathbf{W}}_c\hat{\mathbf{\Phi}}_c(x_1,x_2,\cdots,x_n)^{-1}\hat{\mathbf{G}}_c(x_{k+1},\cdots,x_n)$, is given by,

$$\begin{aligned}
\hat{\mathbf{\Phi}}_c(x_1,\cdots,x_n) &= \left[\begin{array}{c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} \\ \hline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top \end{array}\right] \in \mathbb{C}^{m\times m} \\
\hat{\mathbf{G}}_c(x_{k+1},\cdots,x_n) &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hline \mathbf{\Delta}(\ell,:)^\top \end{array}\right] \in \mathbb{C}^{m\times 1} \\
\hat{\mathbf{W}}_c &= \mathbf{e}_\ell^\top \mathbf{\Delta}^{-\top} \left[\begin{array}{c|c} \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} \end{array}\right] \in \mathbb{C}^{1\times m}
\end{aligned}$$

where $\mathbb{A}^{\mathrm{Lag}},\mathbb{B}^{\mathrm{Lag}}\in\mathbb{C}^{\ell\times\kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

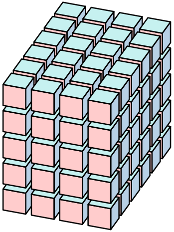
Content

Taming the curse of dimensionality

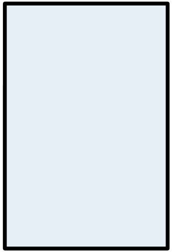
n-variable Loewner matrix operator[1]

$$
\begin{array}{ccc}
\mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q \times K} \\
(\lambda_1(j_1), \mu_1(i_1), \ldots, \lambda_n(j_n), \mu_n(i_n), \textbf{tab}_n) & \longmapsto & \mathbb{L}_n
\end{array}
$$

n-D tensor $\textbf{tab}_n$



matrix $\mathbb{L}_n$

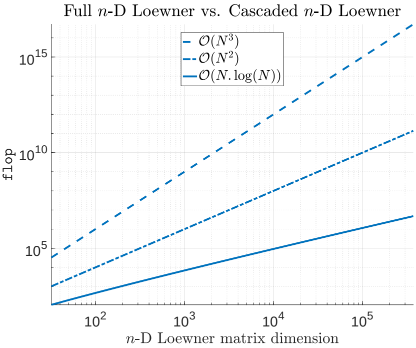


[1] where $Q = q_1 q_2 \cdots q_n$ and $K = k_1 k_2 \cdots k_n$.

Taming the curse of dimensionality

Null space `flop` and memory issues

log-log scale



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD `flop` estimation is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- in real double $QK\frac{8}{2^{20}}$ MB
- in complex double $2QK\frac{8}{2^{20}}$ MB

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$x_1 \backslash x_2$	$\lambda_2(1)=-1$	$\lambda_2(2)=-3$	$\mu_2(1)=-2$	$\mu_2(2)=-4$
$\lambda_1(1)=1$	$h_{1,1}=-\frac{1}{3}$	$h_{1,2}=-\frac{3}{5}$	$h_{1,3}=-\frac{1}{2}$	$h_{1,4}=-\frac{2}{3}$
$\lambda_1(2)=3$	$h_{2,1}=-\frac{9}{5}$	$h_{2,2}=-\frac{27}{7}$	$h_{2,3}=-3$	$h_{2,4}=-\frac{9}{2}$
$\lambda_1(3)=5$	$h_{3,1}=-\frac{25}{7}$	$h_{3,2}=-\frac{25}{3}$	$h_{3,3}=-\frac{25}{4}$	$h_{3,4}=-10$
$\mu_1(1)=0$	$h_{4,1}=0$	$h_{4,2}=0$	$h_{4,3}=0$	$h_{4,4}=0$
$\mu_1(2)=2$	$h_{5,1}=-1$	$h_{5,2}=-2$	$h_{5,3}=-\frac{8}{5}$	$h_{5,4}=-\frac{16}{7}$
$\mu_1(3)=4$	$h_{6,1}=-\frac{8}{3}$	$h_{6,2}=-6$	$h_{6,3}=-\frac{32}{7}$	$h_{6,4}=-\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along x_1, for $x_2=\lambda_2(2)=-3$
- 3 $\mathbb{L}_1$ along x_2 for $x_1=\{\lambda_1(1),\lambda_1(2),\lambda_1(3)\}$
- Scaled null space $\mathbf{c}_2^\top =$ $\left[\begin{array}{ccc} \mathbf{c}_1^{\lambda_1(1)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_1 & \mathbf{c}_1^{\lambda_1(2)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_2 & \mathbf{c}_1^{\lambda_1(3)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_3 \end{array}\right]^\top$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(x_1, x_2) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

x_1 \ x_2	$\lambda_2(1) = -1$	$= -3$	$\mu_2(1) = -2$	$\mu_2(2) = -4$
$\lambda_1(1) = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
$\lambda_1(2) = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
$\lambda_1(3) = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
$\mu_1(1) = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
$\mu_1(2) = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
$\mu_1(3) = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along x_1, for $x_2 = \lambda_2(2) = -3$

$$\mathbf{c}_1^{\lambda_2(2)} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

- 3 $\mathbb{L}_1$ along x_2 for $x_1 = \{\lambda_1(1), \lambda_1(2), \lambda_1(3)\}$
- Scaled null space $\mathbf{c}_2^\top =$
  $\left[\; \mathbf{c}_1^{\lambda_1(1)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_3 \;\right]^\top$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$x_1 \backslash x_2$	$\lambda_2(1)=-1$	$\lambda_2(2)=-3$	$\mu_2(1)=-2$	$\mu_2(2)=-4$
$\lambda_1(1)=1$	$h_{1,1}=-\frac{1}{3}$	$h_{1,2}=-\frac{3}{5}$	$h_{1,3}=-\frac{1}{2}$	$h_{1,4}=-\frac{2}{3}$
$\lambda_1(2)=3$	$h_{2,1}=-\frac{9}{5}$	$h_{2,2}=-\frac{27}{7}$	$h_{2,3}=-3$	$h_{2,4}=-\frac{9}{2}$
$\lambda_1(3)=5$	$h_{3,1}=-\frac{25}{7}$	$h_{3,2}=-\frac{25}{3}$	$h_{3,3}=-\frac{25}{4}$	$h_{3,4}=-10$
$\mu_1(1)=0$	$h_{4,1}=0$	$h_{4,2}=0$	$h_{4,3}=0$	$h_{4,4}=0$
$\mu_1(2)=2$	$h_{5,1}=-1$	$h_{5,2}=-2$	$h_{5,3}=-\frac{8}{5}$	$h_{5,4}=-\frac{16}{7}$
$\mu_1(3)=4$	$h_{6,1}=-\frac{8}{3}$	$h_{6,2}=-6$	$h_{6,3}=-\frac{32}{7}$	$h_{6,4}=-\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2=\begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along x_1, for $x_2=\lambda_2(2)=-3$
- 3 $\mathbb{L}_1$ along x_2 for $x_1=\{\lambda_1(1),\lambda_1(2),\lambda_1(3)\}$

$$\mathbf{c}_1^{\lambda_2(2)}=\begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(1)}=\begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(2)}=\begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(3)}=\begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

- Scaled null space $\mathbf{c}_2^\top=$

$$\left[\ \mathbf{c}_1^{\lambda_1(1)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_3\ \right]^\top$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(x_1,x_2)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$x_1 \backslash x_2$	$\lambda_2(1)=-1$	$\lambda_2(2)=-3$	$\mu_2(1)=-2$	$\mu_2(2)=-4$
$\lambda_1(1)=1$	$h_{1,1}=-\frac{1}{3}$	$h_{1,2}=-\frac{3}{5}$	$h_{1,3}=-\frac{1}{2}$	$h_{1,4}=-\frac{2}{3}$
$\lambda_1(2)=3$	$h_{2,1}=-\frac{9}{5}$	$h_{2,2}=-\frac{27}{7}$	$h_{2,3}=-3$	$h_{2,4}=-\frac{9}{2}$
$\lambda_1(3)=5$	$h_{3,1}=-\frac{25}{7}$	$h_{3,2}=-\frac{25}{3}$	$h_{3,3}=-\frac{25}{4}$	$h_{3,4}=-10$
$\mu_1(1)=0$	$h_{4,1}=0$	$h_{4,2}=0$	$h_{4,3}=0$	$h_{4,4}=0$
$\mu_1(2)=2$	$h_{5,1}=-1$	$h_{5,2}=-2$	$h_{5,3}=-\frac{8}{5}$	$h_{5,4}=-\frac{16}{7}$
$\mu_1(3)=4$	$h_{6,1}=-\frac{8}{3}$	$h_{6,2}=-6$	$h_{6,3}=-\frac{32}{7}$	$h_{6,4}=-\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2=\left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along x_1, for $x_2=\lambda_2(2)=-3$
- 3 $\mathbb{L}_1$ along x_2 for $x_1=\{\lambda_1(1),\lambda_1(2),\lambda_1(3)\}$

$$\mathbf{c}_1^{\lambda_2(2)}=\begin{bmatrix}\frac{5}{9}\\-\frac{14}{9}\\1\end{bmatrix},\ \mathbf{c}_1^{\lambda_1(1)}=\begin{bmatrix}-\frac{3}{5}\\1\end{bmatrix},\ \mathbf{c}_1^{\lambda_1(2)}=\begin{bmatrix}-\frac{5}{7}\\1\end{bmatrix},\ \mathbf{c}_1^{\lambda_1(3)}=\begin{bmatrix}-\frac{7}{9}\\1\end{bmatrix}$$

- Scaled null space $\mathbf{c}_2^\top=$

$$\begin{bmatrix}\mathbf{c}_1^{\lambda_1(1)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_1 & \mathbf{c}_1^{\lambda_1(2)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_2 & \mathbf{c}_1^{\lambda_1(3)\top}\cdot[\mathbf{c}_1^{\lambda_2(2)}]_3\end{bmatrix}^\top$$

Theorem: 2-D to 1-D

Being given the tableau $\mathbf{tab}_2$ tensor in response of the 2-variables $\mathbf{H}(x_1, x_2)$ function, the null space of the corresponding 2-D Loenwer matrix $\mathbb{L}_2$, is spanned by

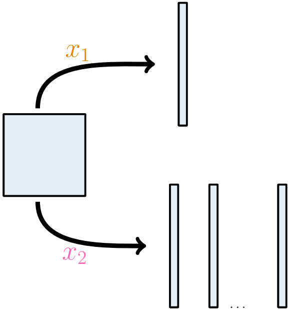
$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{\lambda_2(1)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_1, \cdots, \mathbf{c}_1^{\lambda_2(k_2)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_{k_2}\right],$$

where

- $\mathbf{c}_1^{\lambda_1(k_1)} = \mathcal{N}(\mathbb{L}_1^{\lambda_1(k_1)})$,
  i.e. the null space of the **1-D Loewner matrix** for frozen $x_1 = \lambda_1(k_1)$, and
- $\mathbf{c}_1^{\lambda_2(j_2)} = \mathcal{N}(\mathbb{L}_1^{\lambda_2(j_2)})$,
  i.e. the j_2-th null space of the **1-D Loewner matrices** for frozen $x_2 = \{\lambda_2(1), \cdots, \lambda_2(k_2)\}$.

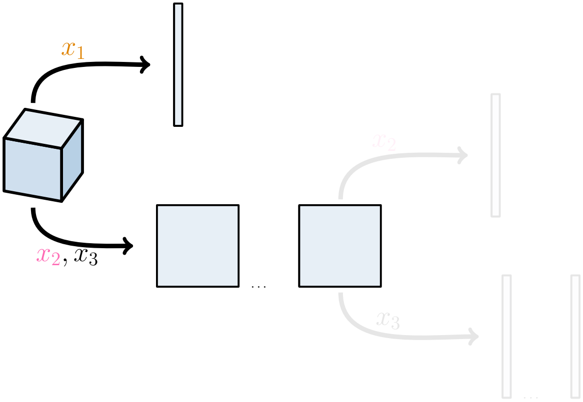
Taming the curse of dimensionality

2-D case



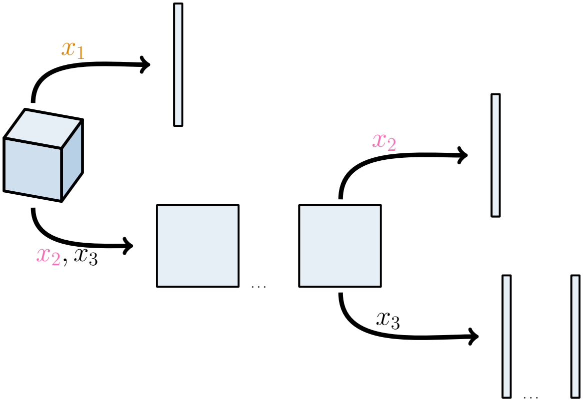
Taming the curse of dimensionality

3-D case



Taming the curse of dimensionality

3-D case



Theorem: n-D to $(n-1)$-D

Being given the tableau $\textbf{tab}_n$ tensor in response of the n-variables $\mathbf{H}(x_1, \cdots, x_n)$ function, the null space of the corresponding n-D Loenwer matrix $\mathbb{L}_n$, is spanned by

$$\mathcal{N}(\mathbb{L}_n) = \mathbf{vec}\left[c_{n-1}^{\lambda_1(1)} \cdot \left[\mathbf{c}_1^{(\lambda_2(k_2),\lambda_3(k_3),\cdots,\lambda_n(k_n))}\right]_1, \cdots, \mathbf{c}_{n-1}^{\lambda_1(k_1)} \cdot \left[\mathbf{c}_1^{(\lambda_2(k_2),\lambda_3(k_3),\cdots,\lambda_n(k_n))}\right]_{k_1} \right],$$

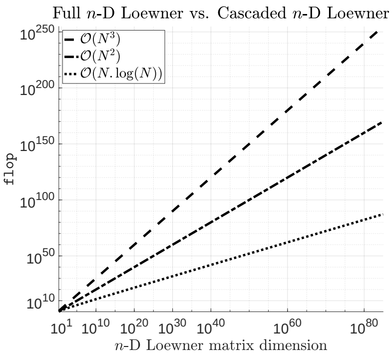
where

- $\mathbf{c}_1^{(\lambda_2(k_2),\lambda_3(k_3),\cdots,\lambda_n(k_n))}$ spans $\mathcal{N}(\mathbb{L}_1^{(\lambda_2(k_2),\lambda_3(k_3),\cdots,\lambda_n(k_n))})$, i.e. the null space of the **1-D Loewner matrix** for frozen $\{\lambda_{k_2\,2}, \lambda_{k_3\,3}, \cdots, \lambda_{k_n\,n}\}$, and
- $c_{n-1}^{\lambda_1(j_1)}$ spans $\mathcal{N}(\mathbb{L}_{n-1}^{\lambda_1(j_1)})$, i.e. the j_1-th null space of the $(n-1)$-**D Loewner matrix** for frozen $x_1 = \{\lambda_1(1), \cdots, \lambda_1(k_1)\}$.

Taming the curse of dimensionality

Null space - flop complexity

log-log scale



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD flop estimation is

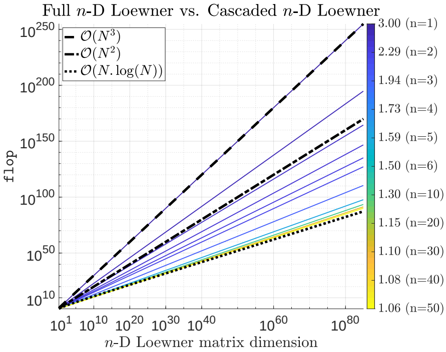
- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

$\Rightarrow$ **The CURSE of dimensionality**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Theorem: Recursive complexity

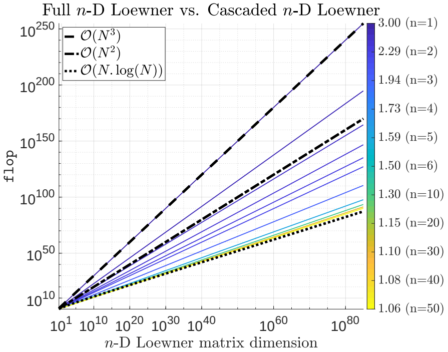
$$\texttt{flop}_1(n) = \sum_{j=1}^{n} \left(k_j^3 \prod_{l=1}^{j} k_{l-1} \right) \text{ where } k_0 = 1.$$

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Theorem: Worst case complexity

k interpolation points per variables.

$$\overline{\texttt{flop}_1} = k^3 \frac{1-k^n}{1-k} = k^3 \frac{1-N}{1-k},$$

which is a (n finite) geometric series of ratio k.

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

$$\begin{array}{llll} \mathcal{O}(N^3) & \rightarrow & \mathcal{O}(N^{2.29}) & \text{for } n=2 \\ & \rightarrow & \mathcal{O}(N^{1.94}) & \text{for } n=3 \\ & \vdots & & \\ & \rightarrow & \mathcal{O}(N^{1.06}) & \text{for } n=50 \end{array}$$

Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}}\prod_{l}^{n} q_l + k_l \ \ \texttt{MB} \text{ (example tableau } 2\cdot[20,6,4,6,8,2] = 2\cdot[k_1,k_2,k_3,k_4,k_5,k_6] \text{ needs } 45 \ \texttt{MB})$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N\times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}}N^2 \ \ \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
flop: $9.78\cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_{l}^{n} q_l + k_l \ \ \texttt{MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs } 45 \ \texttt{MB})$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \ \ \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
flop: $9.78 \cdot 10^{13}$

Cascaded n-D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\overline{k} \times \overline{k}}$$

where $\overline{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}} \overline{k}^2 \ \ \texttt{MB}$$

Example: $\overline{k} = 20$
Memory: 6.25 KB
flop: $8.13 \cdot 10^5$

Content

Kolmogorov, Arnold, Kahane, Lorentz and Sprecher

For every continuous function $\mathbf{f} : \mathbb{I}^n \mapsto \mathbb{R}$ and any $n \in \mathbb{N}$, $n \geq 2$, there exist

- real numbers $\lambda_1, \ldots, \lambda_n$;
- continuous functions $\Phi_k : \mathbb{I} \mapsto \mathbb{R}$, $k = 1, \ldots, 2n+1$;
- a continuous function $\mathbf{g} : \mathbb{R} \mapsto \mathbb{R}$;

such that:

$$\forall (x_1, \cdots, x_n) \in \mathbb{I}^n, \quad \mathbf{f}(x_1, \cdots, x_n) = \sum_{k=1}^{2n+1} \mathbf{g}(\lambda_1 \Phi_k(x_1) + \cdots + \lambda_n \Phi_k(x_n))$$

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to it's 13th problem) is wrong."

A.G. Vitushkin, *"On Hilbert's thirteenth problem and related questions"*, Russian Math. Surveys 59:1, pp. 11-25.

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{\lambda_2(1)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_1, \cdots, \mathbf{c}_1^{\lambda_2(k_2)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_{k_2}\right],$$

Variable decoupling

Given data $\mathbf{tab}_n$, the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{x_n}}_{\mathbf{Bary}(x_n)} \odot \underbrace{(\mathbf{c}^{x_{n-1}} \otimes \mathbf{1}_{k_n})}_{\mathbf{Bary}(x_{n-1})} \odot \underbrace{(\mathbf{c}^{x_{n-2}} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\mathbf{Bary}(x_{n-2})} \odot \cdots \odot \underbrace{(\mathbf{c}^{x_1} \otimes \mathbf{1}_{k_n \ldots k_2})}_{\mathbf{Bary}(x_1)}.$$

where $\mathbf{c}^{x_l}$ denotes the vectorized barycentric coefficients related to the l-th variable.

This is decoupling !

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{\lambda_2(1)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_1, \cdots, \mathbf{c}_1^{\lambda_2(k_2)} \cdot \left[\mathbf{c}_1^{\lambda_1(k_1)}\right]_{k_2}\right],$$

Variable decoupling

Given data $\mathbf{tab}_n$, the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{x_n}}_{\mathbf{Bary}(x_n)} \odot \underbrace{(\mathbf{c}^{x_{n-1}} \otimes \mathbf{1}_{k_n})}_{\mathbf{Bary}(x_{n-1})} \odot \underbrace{(\mathbf{c}^{x_{n-2}} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\mathbf{Bary}(x_{n-2})} \odot \cdots \odot \underbrace{(\mathbf{c}^{x_1} \otimes \mathbf{1}_{k_n \dots k_2})}_{\mathbf{Bary}(x_1)}.$$

where $\mathbf{c}^{x_l}$ denotes the vectorized barycentric coefficients related to the l-th variable.

This is decoupling !

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = x_1 \cdot x_2$)

$$
\begin{array}{rcl}
\lambda_1(j_1) & = & \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\lambda_2(j_2) & = & \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\mu_1(i_1) & = & \lambda_1(j_1) + 0.5 \\
\mu_2(i_2) & = & \lambda_2(j_2) + 0.5
\end{array}
$$

$$
\textbf{tab}_2 = \left(\begin{array}{cc|cc}
1 & -1 & \frac{1}{2} & -\frac{3}{2} \\
-1 & 1 & -\frac{1}{2} & \frac{3}{2} \\
\hline
\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\
-\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4}
\end{array}\right)
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)\,(x_2+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)\,(x_2-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)\,(x_2+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)\,(x_2-1.0)}
\end{pmatrix}
$$

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = x_1 \cdot x_2$)

$$
\begin{aligned}
\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \end{pmatrix}\\
\lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \end{pmatrix}\\
\mu_1(i_1) &= \lambda_1(j_1) + 0.5\\
\mu_2(i_2) &= \lambda_2(j_2) + 0.5
\end{aligned}
$$

$$
\textbf{tab}_2 = \left(\begin{array}{cc|cc}
1 & -1 & \frac{1}{2} & -\frac{3}{2}\\
-1 & 1 & -\frac{1}{2} & \frac{3}{2}\\
\hline
\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4}\\
-\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4}
\end{array}\right)
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag}\\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2+1.0)}\\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2-1.0)}\\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2+1.0)}\\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2-1.0)}
\end{pmatrix}
$$

Decoupling theorem

$$
\begin{aligned}
\mathbf{c}^{x_2} &= \textbf{vec}\begin{pmatrix} -1.0 & -1.0\\ 1.0 & 1.0 \end{pmatrix}\\
\mathbf{c}^{x_1} &= \begin{pmatrix} -1.0\\ 1.0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_2 &= \mathbf{c}^{x_2} \odot (\mathbf{c}^{x_1} \otimes \mathbf{1}_{k_2})\\
&= \begin{pmatrix} -1\\ 1\\ -1\\ 1 \end{pmatrix} \odot \begin{pmatrix} -1\\ -1\\ 1\\ 1 \end{pmatrix}
\end{aligned}
$$

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = x_1 \cdot x_2$)

$$
\begin{aligned}
\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\mu_1(i_1) &= \lambda_1(j_1) + 0.5 \\
\mu_2(i_2) &= \lambda_2(j_2) + 0.5
\end{aligned}
$$

$$
\mathbf{tab}_2 = \left(\begin{array}{cc|cc}
1 & -1 & \frac{1}{2} & -\frac{3}{2} \\
-1 & 1 & -\frac{1}{2} & \frac{3}{2} \\
\hline
\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\
-\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4}
\end{array}\right)
$$

$$
\left(\begin{array}{cccc}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)\,(x_2+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)\,(x_2-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)\,(x_2+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)\,(x_2-1.0)}
\end{array}\right)
$$

Denominator / Numerator

$$
\mathbf{D} = \begin{pmatrix}
\overbrace{\mathbf{c}^{x_1} \cdot \mathbf{Lag}(x_1)}^{\mathsf{Bary}(x_1)} & \overbrace{\mathbf{c}^{x_2} \cdot \mathbf{Lag}(x_2)}^{\mathsf{Bary}(x_2)} \\
-\frac{1.0}{x_1+1.0} & -\frac{1.0}{x_2+1.0} \\
-\frac{1.0}{x_1+1.0} & \frac{1}{x_2-1.0} \\
\frac{1}{x_1-1.0} & -\frac{1.0}{x_2+1.0} \\
\frac{1}{x_1-1.0} & \frac{1}{x_2-1.0}
\end{pmatrix}
$$

Equivalent denominator and numerator read:

$$
\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \quad \text{and} \quad \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}
$$

Variables decoupling, KST and KANs

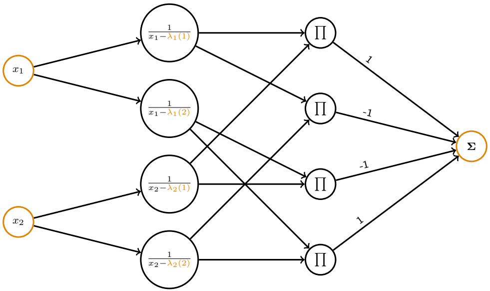
Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = x_1 \cdot x_2$)

$$
\begin{aligned}
\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
\mu_1(i_1) &= \lambda_1(j_1) + 0.5 \\
\mu_2(i_2) &= \lambda_2(j_2) + 0.5
\end{aligned}
$$

$$
\textbf{tab}_2 = \left(\begin{array}{cc|cc}
1 & -1 & \frac{1}{2} & -\frac{3}{2} \\
-1 & 1 & -\frac{1}{2} & \frac{3}{2} \\
\hline
\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\
-\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4}
\end{array}\right)
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2-1.0)}
\end{pmatrix}
$$

Denominator Network view



Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = x_1 \cdot x_2$)

KST via Loewner

$$
= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2} \mathbf{w}_{j_1,j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}
= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp\left(\log(\mathbf{w}_{j_1,j_2}) + \log(\mathbf{Bary}_{j_1}^{x_1}) + \log(\mathbf{Bary}_{j_2}^{x_2})\right)}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp\left(\log(\mathbf{Bary}_{j_1}^{x_1}) + \log(\mathbf{Bary}_{j_2}^{x_2})\right)}
$$

Decoupled barycentric weights

$$
\overbrace{\mathbf{c}^{x_1} \cdot \mathbf{Lag}(x_1)}^{\mathbf{Bary}(x_1)} \qquad \overbrace{\mathbf{c}^{x_2} \cdot \mathbf{Lag}(x_2)}^{\mathbf{Bary}(x_2)}
$$

$$
\begin{array}{cc}
-\frac{1.0}{x_1+1.0} & -\frac{1.0}{x_2+1.0} \\
-\frac{1.0}{x_1+1.0} & \frac{1}{x_2-1.0} \\
\frac{1}{x_1-1.0} & -\frac{1.0}{x_2+1.0} \\
\frac{1}{x_1-1.0} & \frac{1}{x_2-1.0}
\end{array}
$$

This is the solution of KST for rational forms !

Content

Comparisons

Some competitors

Rat. app [B/G 2025]

- Lagrangian interpolation theorem
- p-AAA

KAN [P/P 2025]

- Kolmogorov Arnold theorem
- Kolmogorov Arnold Network

MLP [TensorFlow by Google - Keras 2025]

- Universal approximation theorem
- Multi Layer Perceptron
- Dense connected / ReLU / ADAM / 1000 it. / rand. init.

TensorFlow interface (Python code)

```python
import numpy as np
import math
import matplotlib.pyplot as plt
import tensorflow as tf
from keras.models import Sequential
from matplotlib import cm

# Data
def H(x):
    #y = pow(x[:,0],2)*x[:,1]
    #y = 1/2*(x[:,0] + np.abs(x[:,0])) + 1/10*x[:,1] #1
    #y = x[:,0]*x[:,1] #3
    #y = np.exp(x[:,0]*x[:,1])/( (pow(x[:,0],2)-1.44)*(pow(x[:,1],2)-1.44) ) #6
    #y = np.tanh(4*(x[:,0]-x[:,1])) #8
    #y = pow(np.abs((x[:,0]-x[:,1])),3) #10
    y = (pow(x[:,0],2) + pow(x[:,1],2) + x[:,0] - x[:,1] + 1) / (pow(x[:,0],3)  + pow(x[:,1],2) + 4) #15
    return np.transpose(np.array([y]))

n   = 2
N1  = 40
N2  = 40
x1  = np.linspace(-1, 1, N1)
x2  = np.linspace(-1, 1, N2)

# IP
N   = N1*N2
tab = np.zeros((N,1))
```

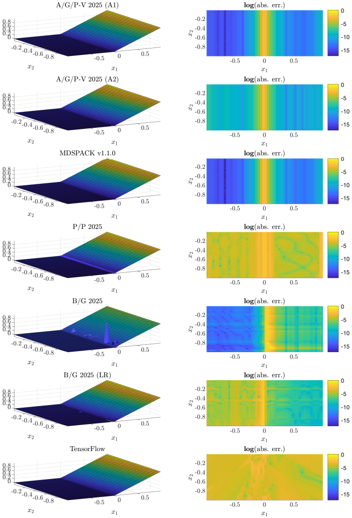

L. Balicki and S. Gugercin, *"Multivariate Rational Approximation via Low-Rank Tensors and the p-AAA Algorithm"*, SISC, 2025.

M. Poluektov and A. Polar, *"Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method"*, https://arxiv.org/abs/2305.08194.

M. Abadi et al., *"TensorFlow: Large-scale machine learning on heterogeneous systems, 2015"*, Software available from tensorflow.org.

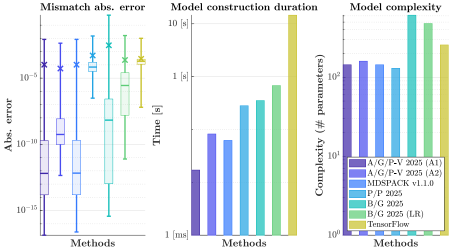
Comparisons

Irrational functions (example #1)



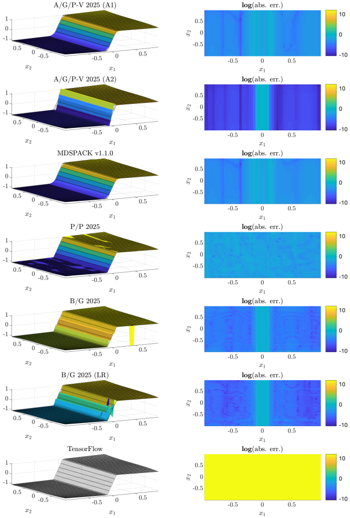
$$\mathrm{ReLU}(x_1) + \frac{1}{100}x_2$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -10^{-10} \end{pmatrix}$



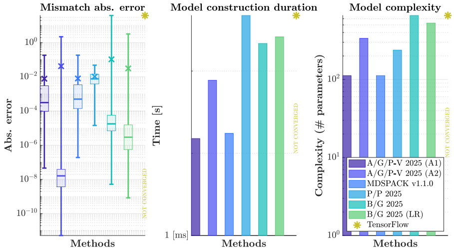
Comparisons

Irrational functions (example #29)



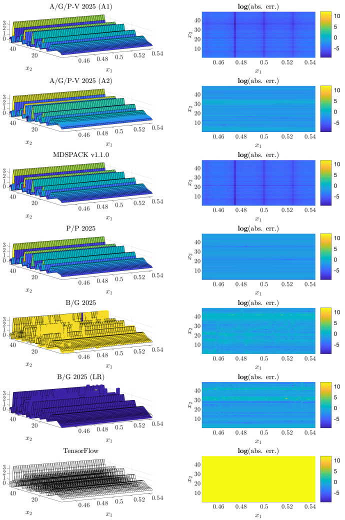
$$\min(10|x_1|,1)\mathrm{sign}(x_1)+\frac{x_1x_2^3}{10}$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 1 \end{pmatrix}$



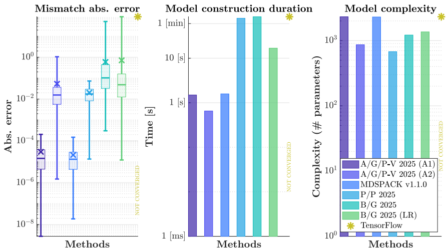
Comparisons

Irrational functions (example #34)



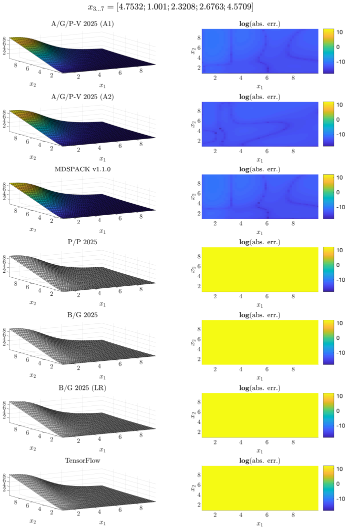
$$\mathrm{Re}(\zeta(x_1 + \imath x_2))$$

- Riemann ζ function (real part), [Riemann]
- Domain: $\mathbb{R}$
- Tensor size: 1.22 **MB** (400^2 points)
- Bounds: $\begin{pmatrix} \frac{9}{20} & \frac{11}{20} \end{pmatrix} \times \begin{pmatrix} 1 & 50 \end{pmatrix}$



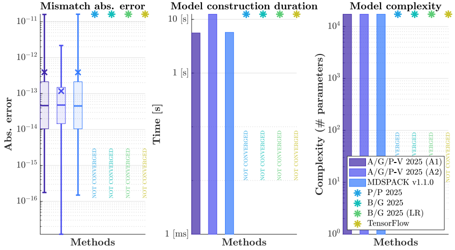
Comparisons

Irrational functions (example #43)



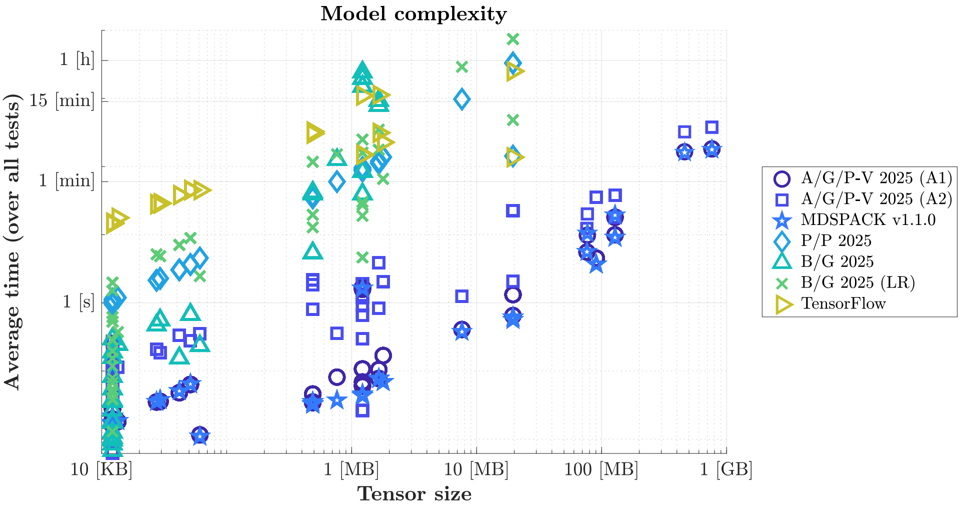
$$\frac{x_3\ x_2^3 + 1}{x_1^4 + x_2^2\ x_3 + x_4^2 + x_5 + x_6^3 + x_7}$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 76.3 **MB** (10^7 points)
- Bounds: $\begin{pmatrix} 1 & 10 \end{pmatrix}^7$



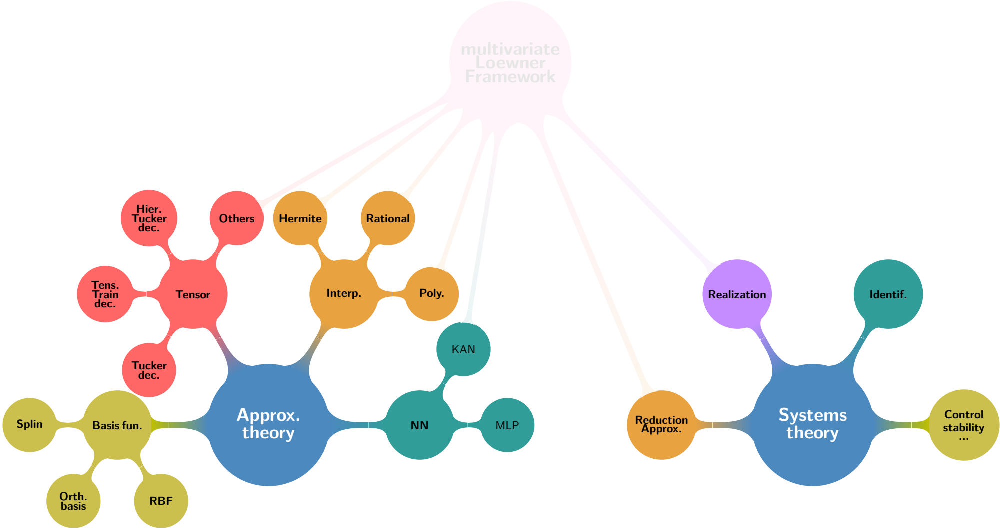
Comparisons

Irrational functions (time, scalability)



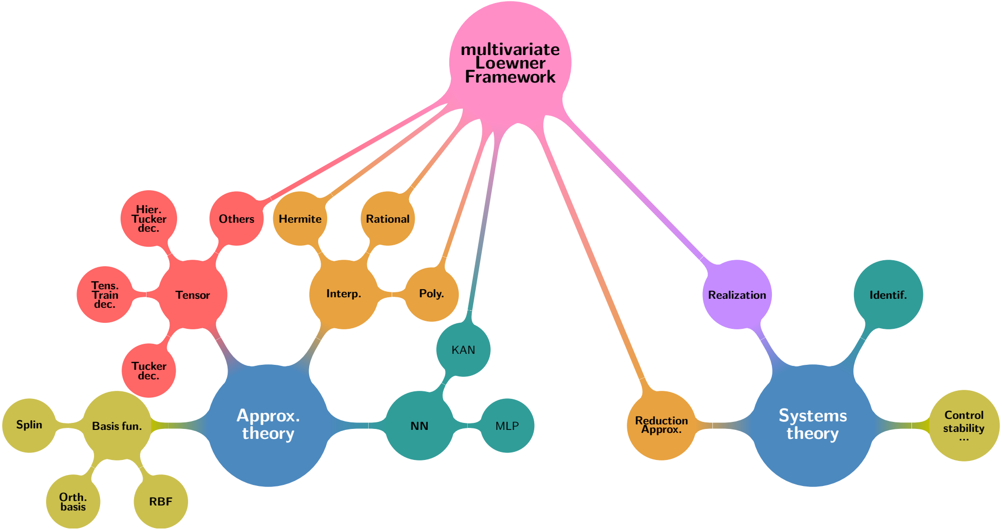
Content

Conclusion
Multivariate Loewner

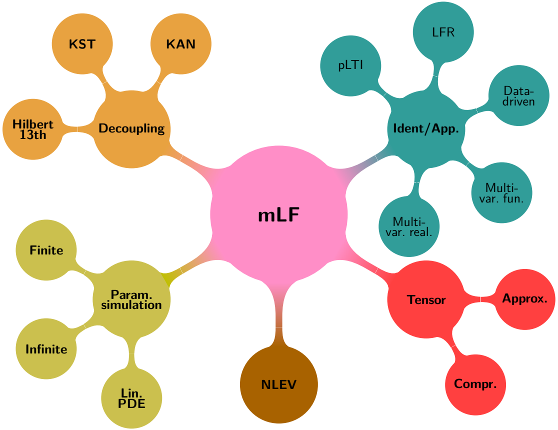


Conclusion

Multivariate Loewner



Conclusion
Multivariate Loewner



Conclusion

Take home message

Main contributions

From any n-th order multi-variate transfer function / data tensor

- Construct a transfer function in barycentric form
- **Construct a realization with controlled complexity**
- **Tame the computational complexity**
- Two algorithms (direct & iterative)
- Connection with Kolmogorov theorem
- Connection with Kolmogorov networks

Side effects

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with
A.C. Antoulas [Rice Univ.]
I.V. Goșea [MPI]
P. Vuillemin [ONERA]

`https://arxiv.org/abs/2405.00495`
`https://arxiv.org/abs/2506.04791`
`https://github.com/cpoussot/mLF`
`https://cpoussot.github.io`