

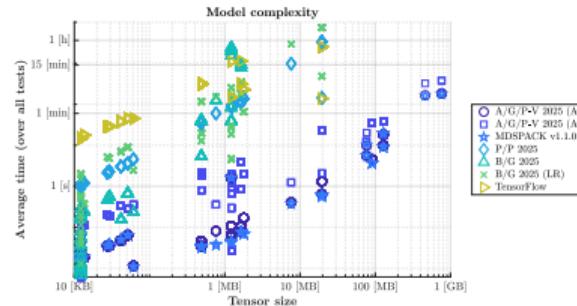
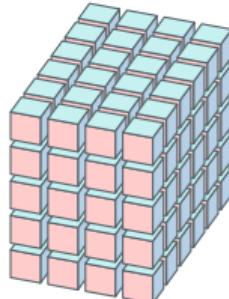
The Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality

... from tensor to multivariate rational approximation and more

C. Poussot-Vassal, in coll. with A.C. Antoulas [Rice Univ.], I.V. Goșea [MPI] and P. Vuillemin [ONERA]
Workshop on data-driven control and analysis of dynamical systems

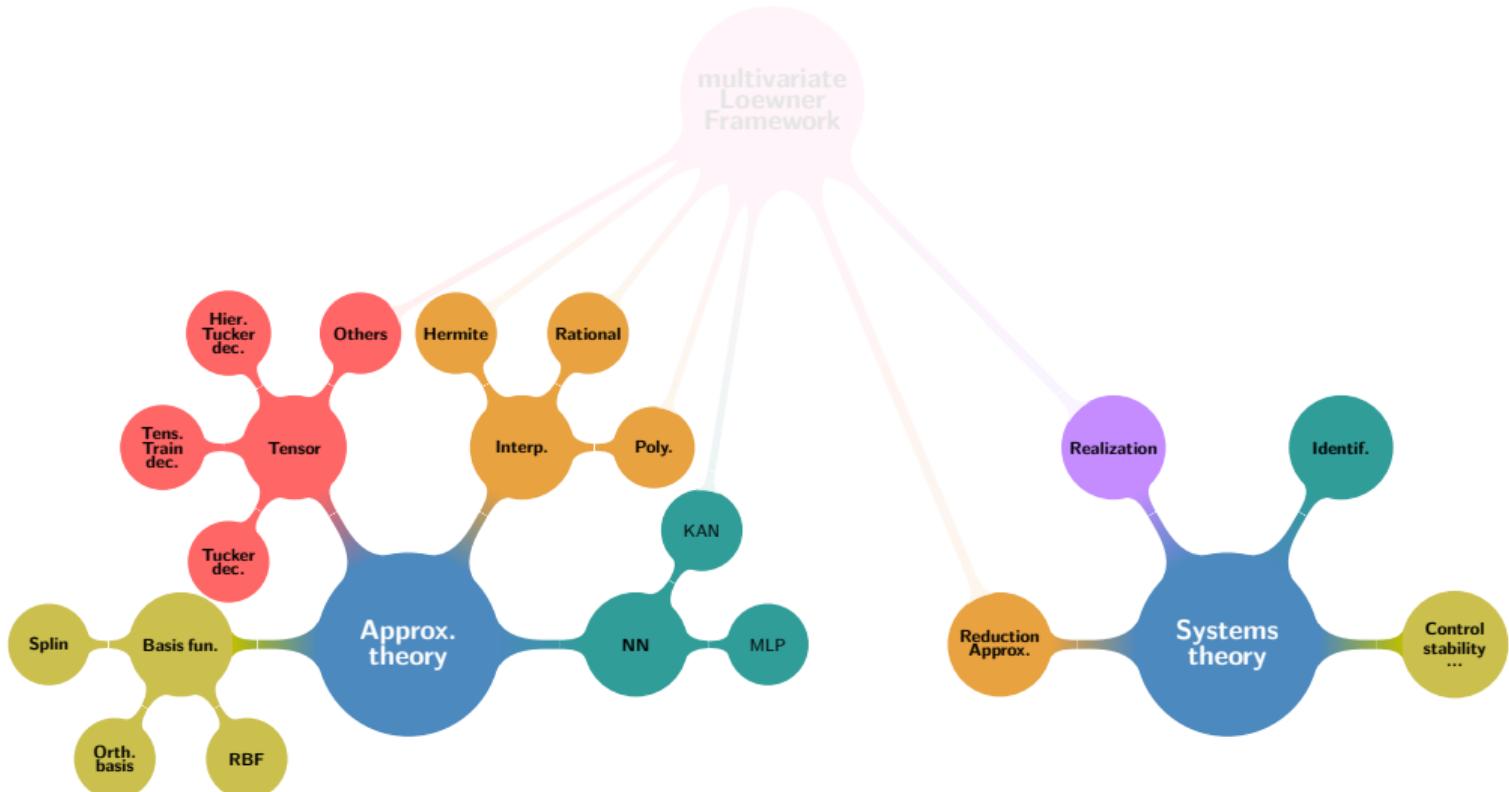
<https://arxiv.org/abs/2405.00495>
<https://arxiv.org/abs/2506.04791>
<https://github.com/cpoussot/mLF>

[in SIAM Review - Research Spotlight]
[extensive benchmark]
[research code package]



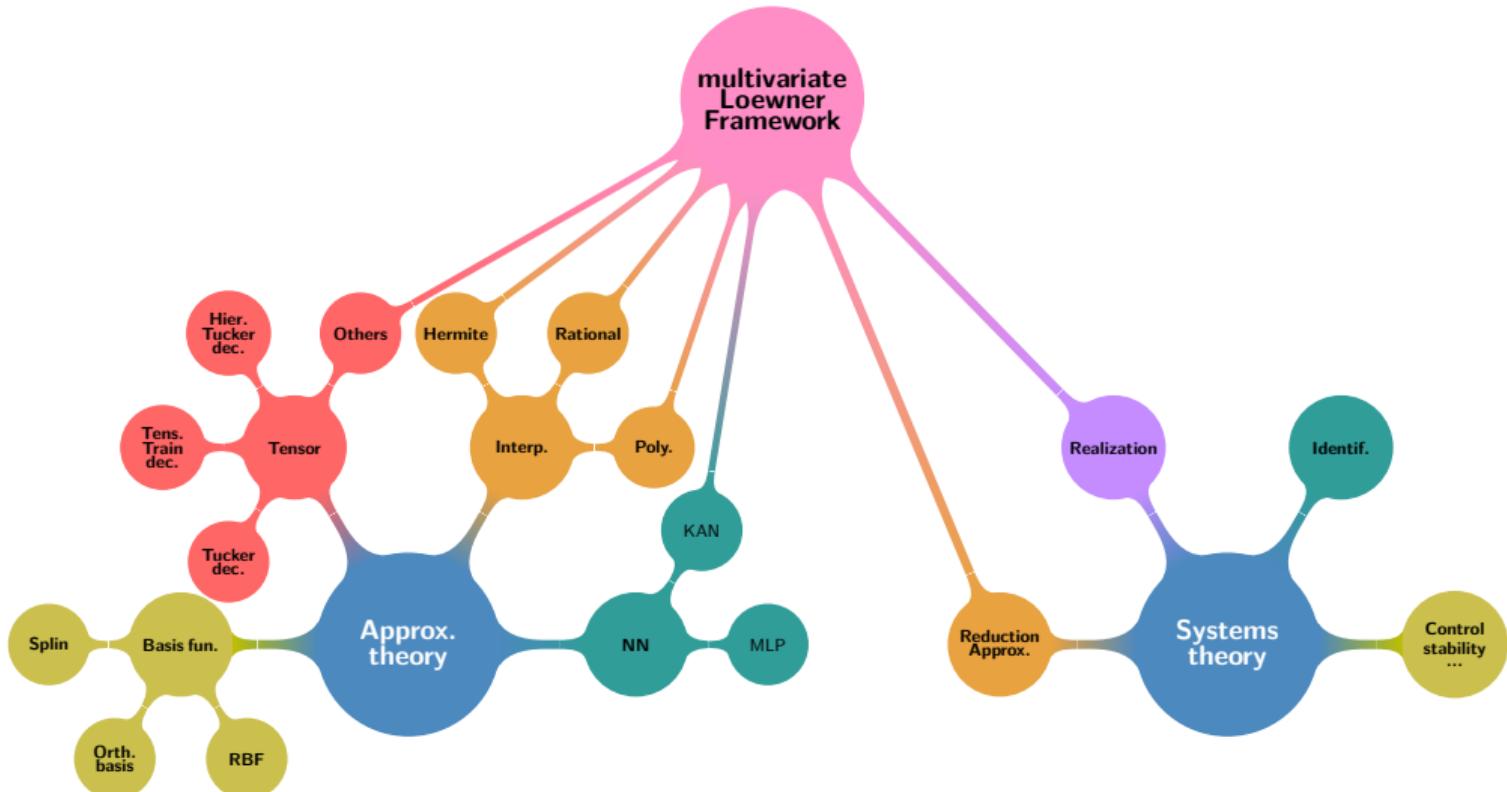
Forewords

Approximation & systems theory... where we stand



Forewords

Approximation & systems theory... where we stand



Forewords

Starting (motivating) examples - Borehole function

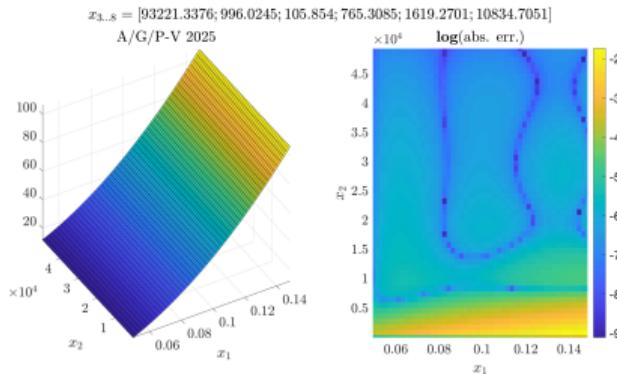
$$\mathbf{H}({}^1x, \dots, {}^8x) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$$\begin{matrix} {}^1x \\ [r_w, \overline{r_w}] \end{matrix} \times \cdots \times \begin{matrix} {}^8x \\ [K_w, \overline{K_w}] \end{matrix}$$

$\text{tab}_8 \in \mathbb{C}^{8 \times 8 \times \dots \times 8}$

$\approx 130 \text{ Mo ('real')}$



Forewords

Starting (motivating) examples - Borehole function

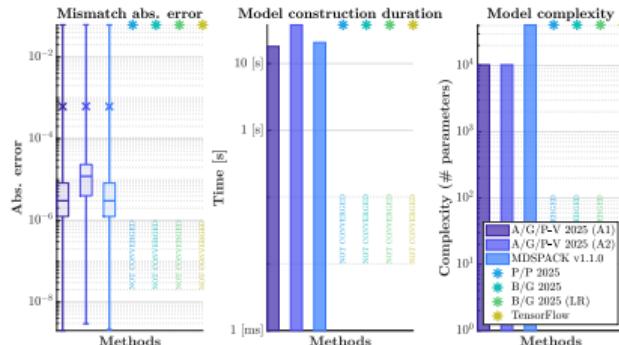
$$\mathbf{H}({}^1x, \dots, {}^8x) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$${}^1x \quad \times \quad \cdots \quad \times \quad {}^8x \\ [r_w, \overline{r_w}] \quad \times \quad \cdots \quad \times \quad [\overline{K_w}, \overline{K_w}]$$

$$\mathbf{tab}_8 \in \mathbb{C}^{8 \times 8 \times \cdots \times 8}$$

$\approx 130 \text{ Mo ('real')}$



Forewords

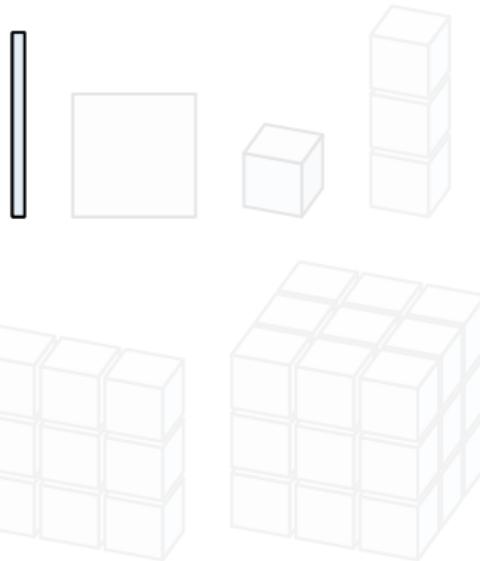
Data (and tensors)

Column / Row data

$${}^1\mathbf{x} = \left. {}^1\lambda_{j_1}, {}^1\mu_{i_1} \right\} \xrightarrow{\mathbf{H}({}^1x)} \left\{ \mathbf{w}_{j_1}, \mathbf{v}_{i_1} \right.$$

| | |
|-----------------------------|--------------------|
| 1x | |
| ${}^1\lambda_{1,\dots,k_1}$ | \mathbf{W}_{k_1} |
| ${}^1\mu_{1,\dots,q_1}$ | \mathbf{V}_{q_1} |

Tensors (1-D) tab₁



Forewords

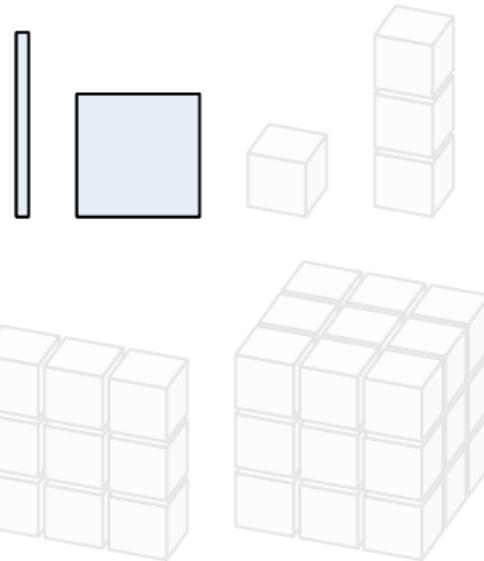
Data (and tensors)

Column / Row data

$$\begin{matrix} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{matrix} \left. \right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2}, \mathbf{v}_{i_1, i_2} \end{array} \right.$$

| | | |
|-------------------------------|-------------------------|-------------------------|
| 1x | 2x | |
| ${}^1\lambda_{1, \dots, k_1}$ | \mathbf{W}_{k_1, k_2} | ϕ_{cr} |
| ${}^1\mu_{1, \dots, q_1}$ | ϕ_{rc} | \mathbf{V}_{q_1, q_2} |

Tensors (2-D) tab₂



Forewords

Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\mathbf{x} = {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x, {}^3x)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2, j_3}, \mathbf{v}_{i_1, i_2, i_3} \end{array} \right.$$

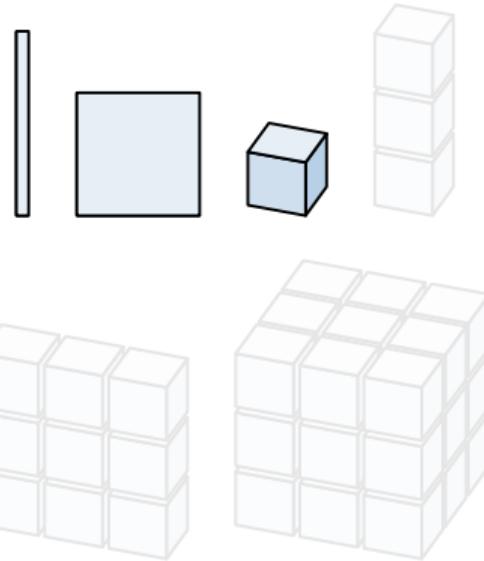
$${}^3x = {}^3\lambda_{1, \dots, k_3}$$

| 1x | 2x | ${}^2\lambda_{1, \dots, k_2}$ | ${}^2\mu_{1, \dots, q_2}$ |
|-------------------------------|------------------------------|-------------------------------|---------------------------|
| ${}^1\lambda_{1, \dots, k_1}$ | $\mathbf{W}_{k_1, k_2, k_3}$ | ϕ_{crc} | |
| ${}^1\mu_{1, \dots, q_1}$ | ϕ_{rcc} | ϕ_{rrc} | |

$${}^3x = {}^3\mu_{1, \dots, q_3}$$

| 1x | 2x | ${}^2\lambda_{1, \dots, k_2}$ | ${}^2\mu_{1, \dots, q_2}$ |
|-------------------------------|--------------|-------------------------------|---------------------------|
| ${}^1\lambda_{1, \dots, k_1}$ | ϕ_{crr} | ϕ_{crr} | |
| ${}^1\mu_{1, \dots, q_1}$ | ϕ_{rcr} | $\mathbf{V}_{q_1, q_2, q_3}$ | |

Tensors (3-D) tab_3



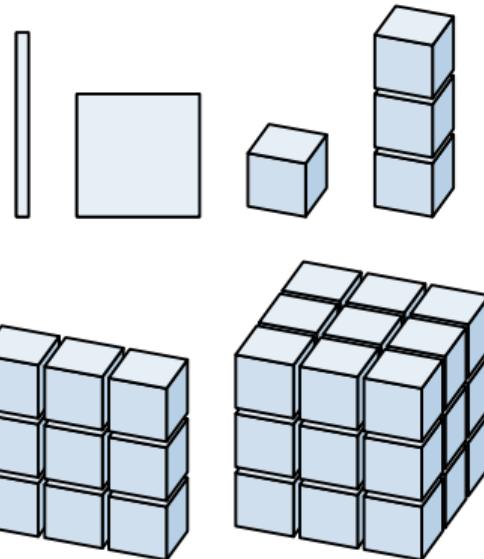
Forewords

Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\mathbf{x} = {}^3\lambda_{j_3}, {}^3\mu_{i_3} \\ \vdots \\ {}^n\mathbf{x} = {}^n\lambda_{j_n}, {}^n\mu_{i_n} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1x, \dots, {}^n x)} \left\{ \mathbf{w}_{j_1, \dots, j_n}, \mathbf{v}_{i_1, \dots, i_n} \right\}$$

Tensors (n -D) tab_n



Forewords

Contributions claim & trajectory of the presentation

List of contributions

- ▶ n -D tensor data to n -D Loewner matrix \mathbb{L}_n
- ▶ n -variable transfer functions
- ▶ Taming the curse of dimensionality
 - » in computation effort (flop)
 - » in storage needs (Bytes)
 - » in accuracy
- ▶ n -variable decoupling
 - » KST formulation for rational functions
 - » connection with KAN
- ▶ Comparison with MLP, KAN, AAA



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- 📚 A.C. Antoulas, I-V. Gosea and C. P-V., "[On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality](https://arxiv.org/abs/2405.00495)", SIAM Review, November, 2025 (<https://arxiv.org/abs/2405.00495>).
 - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "[Tensor-based multivariate function approximation: methods benchmarking and comparison](https://arxiv.org/abs/2506.04791)", June, 2025 <https://arxiv.org/abs/2506.04791>.
 - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "[mLF package](https://github.com/cpoussot/mLF)", <https://github.com/cpoussot/mLF>.

Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} &:= \left\{ \left({}^1\lambda_{j_1}; \mathbf{w}_{j_1} \right), j_1 = 1, \dots, k_1 \right\} \\ P_r^{(1)} &:= \left\{ \left({}^1\mu_{i_1}; \mathbf{v}_{i_1} \right), i_1 = 1, \dots, q_1 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{}_{^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

Lagrangian form

$$\mathbf{g}({}^1x) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{}_{^1x - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{}_{^1x - {}^1\lambda_{j_1}}}$$

Null space

$$\text{span } (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}(^1x) = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$ of complexity (2)

$$\left. \begin{array}{rcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [2, 4, 6, 8] \end{array} \right\} \xrightarrow{\mathbf{H}} \left\{ \begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array} \right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} &:= \left\{ (\textcolor{brown}{1}\lambda_{j_1}, \textcolor{brown}{2}\lambda_{j_2}; \mathbf{w}_{j_1, j_2}), \ j_1 = 1, \dots, k_1 \quad j_2 = 1, \dots, k_2 \right\} \\ P_r^{(2)} &:= \left\{ (\textcolor{violet}{1}\mu_{i_1}, \textcolor{violet}{2}\mu_{i_2}; \mathbf{v}_{i_1, i_2}), \ i_1 = 1, \dots, q_1 \quad i_2 = 1, \dots, q_2 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1, j_2}^{i_1, i_2} = \frac{\mathbf{v}_{i_1, i_2} - \mathbf{w}_{j_1, j_2}}{(\textcolor{violet}{1}\mu_{i_1} - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{violet}{2}\mu_{i_2} - \textcolor{brown}{2}\lambda_{j_2})}$$

Lagrangian form

$$\mathbf{g}(\textcolor{violet}{1}x, \textcolor{brown}{2}x) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{(\textcolor{violet}{1}x - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{brown}{2}x - \textcolor{brown}{2}\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2}}{(\textcolor{violet}{1}x - \textcolor{brown}{1}\lambda_{j_1})(\textcolor{brown}{2}x - \textcolor{brown}{2}\lambda_{j_2})}}$$

Null space

$$\text{span } (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \begin{bmatrix} c_{1,1} \\ \vdots \\ \hline c_{1,k_2} \\ \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{bmatrix} \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{l} {}^1\lambda_{j_1} = [1, 3, 5] \\ {}^1\mu_{i_1} = [0, 2, 4] \\ {}^2\lambda_{j_2} = [-1, -3] \\ {}^2\mu_{i_2} = [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \hline \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \hline \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array} \right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array} \right]$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [0, 2, 4] \\ {}^2\lambda_{j_2} & = & [-1, -3] \\ {}^2\mu_{i_2} & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Lagrangian form

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s, t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \left\{ (^1\lambda_{j_1}, ^2\lambda_{j_2}, \dots, ^n\lambda_{j_n}; \mathbf{w}_{j_1, j_2, \dots, j_n}), \quad j_l = 1, \dots, k_l, \quad l = 1, \dots, n \right\} \\ P_r^{(n)} := \left\{ (^1\mu_{i_1}, ^2\mu_{i_2}, \dots, ^n\mu_{i_n}; \mathbf{v}_{i_1, i_2, \dots, i_n}), \quad i_l = 1, \dots, q_l, \quad l = 1, \dots, n \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n} = \frac{\mathbf{v}_{i_1, i_2, \dots, i_n} - \mathbf{w}_{j_1, j_2, \dots, j_n}}{(^1\mu_{i_1} - ^1\lambda_{j_1}) \cdots (^n\mu_{i_n} - ^n\lambda_{j_n})}$$

Lagrangian form

$$\mathbf{g}(^1x, \dots, ^nx) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n} \mathbf{w}_{j_1, \dots, j_n}}{(^1x - ^1\lambda_{j_1}) \cdots (^n x - ^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n}}{(^1x - ^1\lambda_{j_1}) \cdots (^n x - ^n\lambda_{j_n})}}$$

Null space

$$\text{span } (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

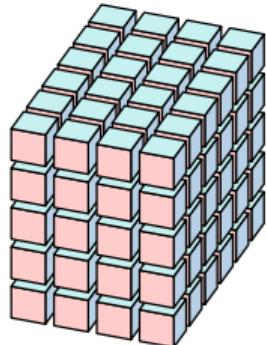
$$\mathbf{c}_n = \begin{bmatrix} c_{1, \dots, 1} \\ \vdots \\ \hline c_{1, \dots, k_n} \\ \vdots \\ \hline c_{k_1, \dots, 1} \\ \vdots \\ c_{k_1, \dots, k_n} \end{bmatrix} \in \mathbb{C}^{k_1 \cdots k_n}$$

Taming the curse of dimensionality

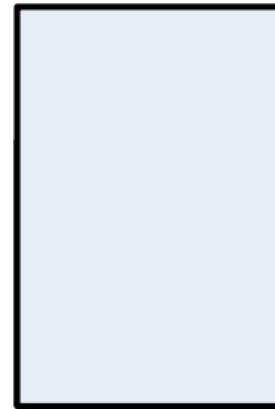
n-variable Loewner matrix operator

$$\begin{aligned} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \dots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1) \times \dots \times (k_n+q_n)} &\longrightarrow \mathbb{C}^{Q \times K} \\ (\textcolor{brown}{^1}\lambda_{j_1}, \textcolor{pink}{^1}\mu_{i_1}, \dots, \textcolor{brown}{^n}\lambda_{j_n}, \textcolor{pink}{^n}\mu_{i_n}, \mathbf{tab}_n) &\longmapsto \mathbb{L}_n \end{aligned}$$

n-D tensor \mathbf{tab}_n

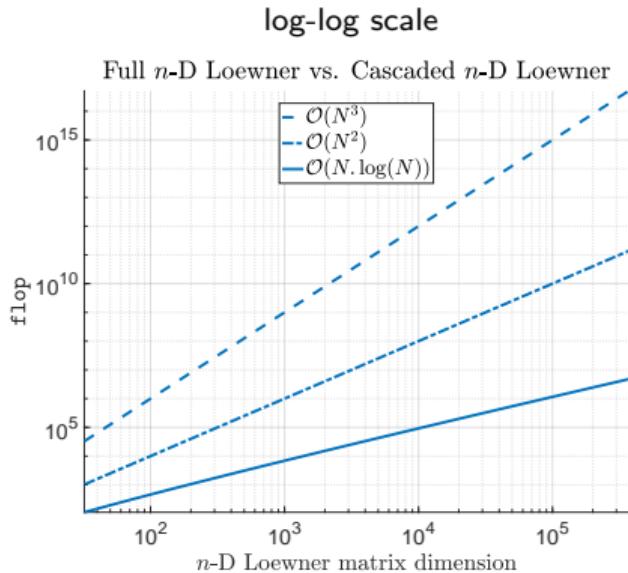


matrix \mathbb{L}_n



Taming the curse of dimensionality

Null space flop and memory issues



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD flop estimation is

- ▶ QK^2 (if $Q > K$)
- ▶ N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- ▶ in real double $QK \frac{8}{2^{20}}$ MB
- ▶ in complex double $2QK \frac{8}{2^{20}}$ MB

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

| 2x | ${}^2\lambda_1 = -1$ | ${}^2\lambda_2 = -3$ | ${}^2\mu_1 = -2$ | ${}^2\mu_2 = -4$ |
|---------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 1x | $h_{1,1} = -\frac{1}{3}$ | $h_{1,2} = -\frac{3}{5}$ | $h_{1,3} = -\frac{1}{2}$ | $h_{1,4} = -\frac{2}{3}$ |
| ${}^1\lambda_1 = 1$ | $h_{2,1} = -\frac{9}{5}$ | $h_{2,2} = -\frac{27}{7}$ | $h_{2,3} = -3$ | $h_{2,4} = -\frac{9}{2}$ |
| ${}^1\lambda_2 = 3$ | $h_{3,1} = -\frac{25}{7}$ | $h_{3,2} = -\frac{25}{3}$ | $h_{3,3} = -\frac{25}{4}$ | $h_{3,4} = -10$ |
| ${}^1\lambda_3 = 5$ | $h_{4,1} = 0$ | $h_{4,2} = 0$ | $h_{4,3} = 0$ | $h_{4,4} = 0$ |
| ${}^1\mu_1 = 0$ | $h_{5,1} = -1$ | $h_{5,2} = -2$ | $h_{5,3} = -\frac{8}{5}$ | $h_{5,4} = -\frac{16}{7}$ |
| ${}^1\mu_2 = 2$ | $h_{6,1} = -\frac{8}{3}$ | $h_{6,2} = -6$ | $h_{6,3} = -\frac{32}{7}$ | $h_{6,4} = -\frac{64}{9}$ |
| ${}^1\mu_3 = 4$ | | | | |

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 \mathbb{L}_1 along 1x , for
 ${}^2x = {}^2\lambda_2 = -3$
- 3 \mathbb{L}_1 along 2x for
 ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$

$$[{}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3]^\top$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

| 2x | ${}^2\lambda_1 = -1$ | ${}^2\lambda_2 = -3$ | ${}^2\mu_1 = -2$ | ${}^2\mu_2 = -4$ |
|---------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 1x | | | | |
| ${}^1\lambda_1 = 1$ | $h_{1,1} = -\frac{1}{3}$ | $h_{1,2} = -\frac{3}{5}$ | $h_{1,3} = -\frac{1}{2}$ | $h_{1,4} = -\frac{2}{3}$ |
| ${}^1\lambda_2 = 3$ | $h_{2,1} = -\frac{9}{5}$ | $h_{2,2} = -\frac{27}{7}$ | $h_{2,3} = -3$ | $h_{2,4} = -\frac{9}{2}$ |
| ${}^1\lambda_3 = 5$ | $h_{3,1} = -\frac{25}{7}$ | $h_{3,2} = -\frac{25}{3}$ | $h_{3,3} = -\frac{25}{4}$ | $h_{3,4} = -10$ |
| ${}^1\mu_1 = 0$ | $h_{4,1} = 0$ | $h_{4,2} = 0$ | $h_{4,3} = 0$ | $h_{4,4} = 0$ |
| ${}^1\mu_2 = 2$ | $h_{5,1} = -1$ | $h_{5,2} = -2$ | $h_{5,3} = -\frac{8}{5}$ | $h_{5,4} = -\frac{16}{7}$ |
| ${}^1\mu_3 = 4$ | $h_{6,1} = -\frac{8}{3}$ | $h_{6,2} = -6$ | $h_{6,3} = -\frac{32}{7}$ | $h_{6,4} = -\frac{64}{9}$ |

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 \mathbb{L}_1 along 1x , for

$${}^2x = {}^2\lambda_2 = -3$$

- 3 \mathbb{L}_1 along 2x for

$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[{}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1x, ^2x) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

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| 1x | $h_{1,1} = -\frac{1}{3}$ | $h_{1,2} = -\frac{3}{5}$ | $h_{1,3} = -\frac{1}{2}$ | $h_{1,4} = -\frac{2}{3}$ |
| ${}^1\lambda_1 = 1$ | $h_{2,1} = -\frac{9}{5}$ | $h_{2,2} = -\frac{27}{7}$ | $h_{2,3} = -3$ | $h_{2,4} = -\frac{9}{2}$ |
| ${}^1\lambda_2 = 3$ | $h_{3,1} = -\frac{25}{7}$ | $h_{3,2} = -\frac{25}{3}$ | $h_{3,3} = -\frac{25}{4}$ | $h_{3,4} = -10$ |
| ${}^1\lambda_3 = 5$ | $h_{4,1} = 0$ | $h_{4,2} = 0$ | $h_{4,3} = 0$ | $h_{4,4} = 0$ |
| ${}^1\mu_1 = 0$ | $h_{5,1} = -1$ | $h_{5,2} = -2$ | $h_{5,3} = -\frac{8}{5}$ | $h_{5,4} = -\frac{16}{7}$ |
| ${}^1\mu_2 = 2$ | $h_{6,1} = -\frac{8}{3}$ | $h_{6,2} = -6$ | $h_{6,3} = -\frac{32}{7}$ | $h_{6,4} = -\frac{64}{9}$ |
| ${}^1\mu_3 = 4$ | | | | |

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

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$${}^2x = {}^2\lambda_2 = -3$$

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$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[{}^1\lambda_1 \cdot [{}^2\lambda_2]_1 \quad {}^1\lambda_2 \cdot [{}^2\lambda_2]_2 \quad {}^1\lambda_3 \cdot [{}^2\lambda_2]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

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| ${}^1\lambda_1 = 1$ | $h_{2,1} = -\frac{9}{5}$ | $h_{2,2} = -\frac{27}{7}$ | $h_{2,3} = -3$ | $h_{2,4} = -\frac{9}{2}$ |
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| ${}^1\mu_3 = 4$ | | | | |

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

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- 3 \mathbb{L}_1 along 2x for

$${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$[\mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case

Theorem: 2-D to 1-D

Being given the tableau tab_2 tensor in response of the 2-variables $\mathbf{H}({}^1x, {}^2x)$ function, the null space of the corresponding 2-D Loewner matrix \mathbb{L}_2 , is spanned by

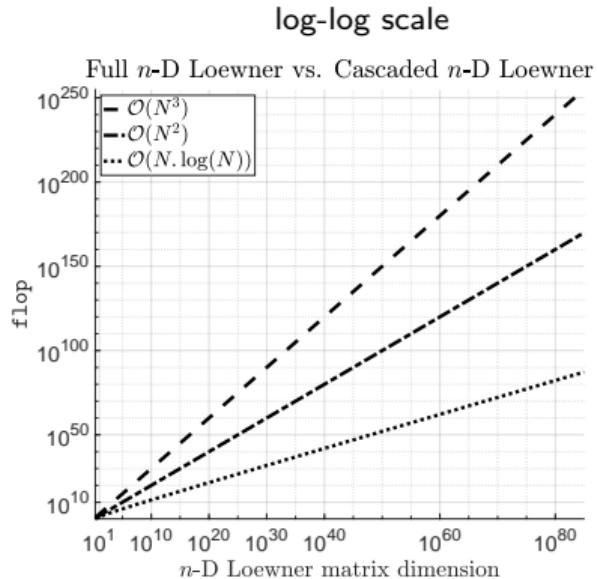
$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[\mathbf{c}_1^{{}^2\lambda_1} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \vdots \\ {}^1\lambda_{k_1} \end{bmatrix}_1, \dots, \mathbf{c}_1^{{}^2\lambda_{k_2}} \cdot \begin{bmatrix} {}^1\lambda_{k_1} \\ \vdots \\ {}^1\lambda_{k_1} \end{bmatrix}_{k_2} \right],$$

where

- ▶ $\mathbf{c}_1^{{}^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{{}^1\lambda_{k_1}})$,
i.e. the null space of the **1-D Loewner matrix** for frozen ${}^1x = {}^1\lambda_{k_1}$, and
- ▶ $\mathbf{c}_1^{{}^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{{}^2\lambda_{j_2}})$,
i.e. the j_1 -th null space of the **1-D Loewner matrices** for frozen ${}^2x = \{{}^2\lambda_1, \dots, {}^2\lambda_{k_2}\}$.

Taming the curse of dimensionality

Null space - flop complexity



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

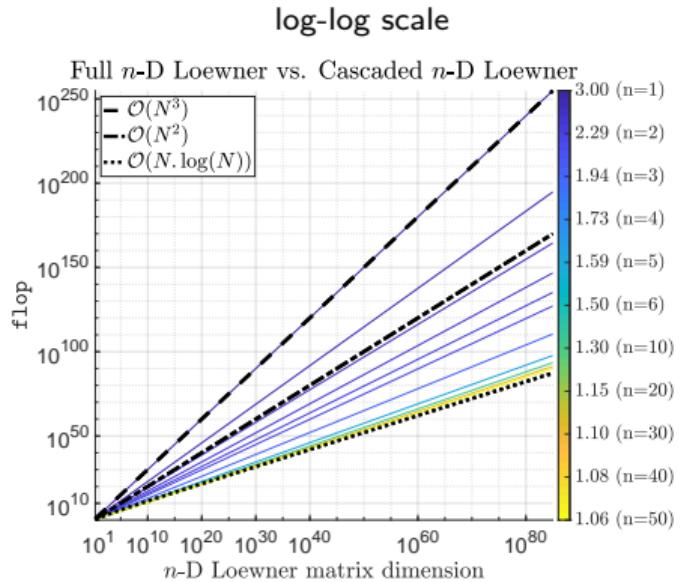
Note that $Q \times K$ matrix SVD flop estimation is

- ▶ QK^2 (if $Q > K$)
- ▶ N^3 (if $Q = K = N$)

⇒ The CURSE of dimensionality

Taming the curse of dimensionality

Null space - flop complexity



Theorem: Worst case complexity

k interpolation points per variables.

$$\overline{\text{flop}_1} = k^3 \frac{1 - k^n}{1 - k} = k^3 \frac{1 - N}{1 - k},$$

which is a (n finite) geometric series of ratio k .

⇒ The CURSE of dimensionality is TAMED

$$\begin{aligned} \mathcal{O}(N^3) &\rightarrow \mathcal{O}(N^{2.29}) && \text{for } n = 2 \\ &\rightarrow \mathcal{O}(N^{1.94}) && \text{for } n = 3 \\ &\vdots \\ &\rightarrow \mathcal{O}(N^{1.06}) && \text{for } n = 50 \end{aligned}$$

Taming the curse of dimensionality

Null space - memory

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB})$$

Full n -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example: $N = 46,080$

Memory: 31.64 GB

flop: $9.78 \cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \quad (\text{example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB})$$

Full n -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example: $N = 46,080$

Memory: 31.64 GB

flop: $9.78 \cdot 10^{13}$

Cascaded n -D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\bar{k} \times \bar{k}}$$

where $\bar{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}} \bar{k}^2 \text{ MB}$$

Example: $\bar{k} = 20$

Memory: 6.25 KB

flop: $8.13 \cdot 10^5$

Variables decoupling, KST and KANs

Kolmogorov Superposition Theorem and Hilbert's 13th problem

Kolmogorov, Arnold, Kahane, Lorentz and Sprecher

For every continuous function $f : \mathbb{I}^n \mapsto \mathbb{R}$ and any $n \in \mathbb{N}$, $n \geq 2$, there exist

- ▶ real numbers $\lambda_1, \dots, \lambda_n$;
- ▶ continuous functions $\Phi_k : \mathbb{I} \mapsto \mathbb{R}$, $k = 1, \dots, 2n + 1$;
- ▶ a continuous function $g : \mathbb{R} \mapsto \mathbb{R}$;

such that:

$$\forall (^1x, \dots, {}^nx) \in \mathbb{I}^n, \quad f(^1x, \dots, {}^nx) = \sum_{k=1}^{2n+1} g(\lambda_1 \Phi_k(^1x) + \dots + \lambda_n \Phi_k({}^nx))$$

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to its 13th problem) is wrong."



A.G. Vitushkin, "*On Hilbert's thirteenth problem and related questions*", Russian Math. Surveys 59:1, pp. 11-25.

Variables decoupling, KST and KANs

Loewner and KST

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[\mathbf{c}_1^{\frac{2}{\lambda_1}} \cdot \begin{bmatrix} 1 \\ \lambda_{k_1} \end{bmatrix}_1, \dots, \mathbf{c}_1^{\frac{2}{\lambda_{k_2}}} \cdot \begin{bmatrix} 1 \\ \lambda_{k_1} \end{bmatrix}_{k_2} \right],$$

Variable decoupling

Given data tab_n , the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{n_x}}_{\text{Bary}(^{n_x})} \odot \underbrace{(\mathbf{c}^{n-1_x} \otimes \mathbf{1}_{k_n})}_{\text{Bary}(^{n-1_x})} \odot \underbrace{(\mathbf{c}^{n-2_x} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\text{Bary}(^{n-2_x})} \odot \cdots \odot \underbrace{(\mathbf{c}^1_x \otimes \mathbf{1}_{k_n \dots k_2})}_{\text{Bary}(^1_x)}.$$

where \mathbf{c}^l_x denotes the vectorized barycentric coefficients related to the l -th variable.

This is decoupling !



A.G. Vitushkin, "On Hilbert's thirteenth problem and related questions", Russian Math. Surveys 59:1, pp. 11-25.

Variables decoupling, KST and KANs

Loewner and KST

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[\mathbf{c}_1^{\frac{2}{\lambda_1}} \cdot \begin{bmatrix} 1 \\ \lambda_{k_1} \end{bmatrix}_1, \dots, \mathbf{c}_1^{\frac{2}{\lambda_{k_2}}} \cdot \begin{bmatrix} 1 \\ \lambda_{k_1} \end{bmatrix}_{k_2} \right],$$

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Given data \mathbf{tab}_n , the latter achieves variables decoupling, and the null space can be equivalently written as:

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where $\mathbf{c}^l x$ denotes the vectorized barycentric coefficients related to the l -th variable.

This is decoupling !



A.G. Vitushkin, "On Hilbert's thirteenth problem and related questions", Russian Math. Surveys 59:1, pp. 11-25.

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($H = {}^1x \cdot {}^2x$)

$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\left(\begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\mathbf{c}^{{}^2x} = \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix}$$

$$\mathbf{c}^{{}^1x} = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1x \cdot {}^2x$)

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$$\left(\begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \mathbf{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\mathbf{c}^{{}^2x} = \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix}$$

$$\mathbf{c}^{{}^1x} = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

$$\mathbf{D} = \begin{pmatrix} \overbrace{\mathbf{c}^{{}^1x} \cdot \mathbf{Lag}^{{}^1x}}^{\mathbf{Bary}^{{}^1x}} & \overbrace{\mathbf{c}^{{}^2x} \cdot \mathbf{Lag}^{{}^2x}}^{\mathbf{Bary}^{{}^2x}} \\ -\frac{1.0}{{}^1x+1.0} & -\frac{1.0}{{}^2x+1.0} \\ -\frac{1.0}{{}^1x+1.0} & \frac{1}{{}^2x-1.0} \\ \frac{1}{{}^1x-1.0} & -\frac{1.0}{{}^2x+1.0} \\ \frac{1}{{}^1x-1.0} & \frac{1}{{}^2x-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\sum_{i-\text{th row}} \prod_{j-\text{th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i-\text{th row}} \mathbf{w} \cdot \prod_{j-\text{th col}} [\mathbf{D}]_{i,j}$$

Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($H = {}^1x \cdot {}^2x$)

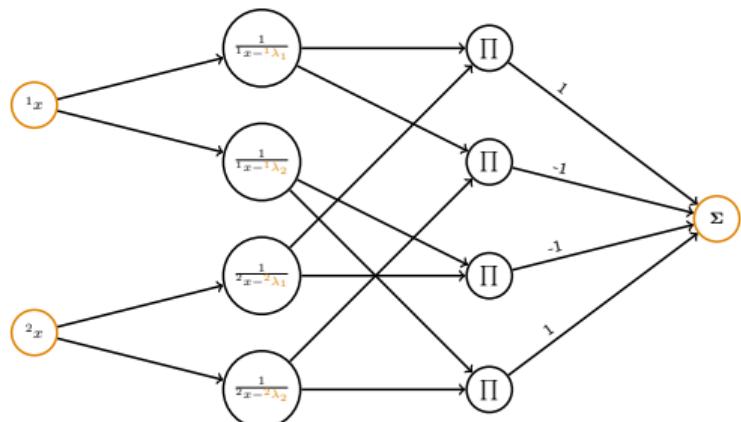
$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\left(\begin{array}{cccc} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)} \end{array} \right)$$

$$\begin{aligned} \mathbf{c}^{{}^2x} &= \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\ \mathbf{c}^{{}^1x} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix} \end{aligned}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})$$

Denominator Network view



Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1x \cdot {}^2x$)

KST via Loewner

$$\begin{aligned} &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{{}^c j_1, j_2 \mathbf{w}_{j_1, j_2}}{\binom{{}^1 x - 1}{\lambda_{j_1}} \binom{{}^2 x - 2}{\lambda_{j_2}}}} \\ &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{{}^c j_1, j_2}{\binom{{}^1 x - 1}{\lambda_{j_1}} \binom{{}^2 x - 2}{\lambda_{j_2}}}} \\ &= \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp \left(\log(\mathbf{w}_{j_1, j_2}) + \log(\mathbf{Bary}_{j_1}^{{}^1 x}) + \log(\mathbf{Bary}_{j_2}^{{}^2 x}) \right)}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \exp \left(\log(\mathbf{Bary}_{j_1}^{{}^1 x}) + \log(\mathbf{Bary}_{j_2}^{{}^2 x}) \right)} \end{aligned}$$

Decoupled barycentric weights

$$\begin{array}{ll} \overbrace{\mathbf{c}^{{}^1 x} \cdot \mathbf{Lag}({}^1 x)}^{\mathbf{Bary}({}^1 x)} & \overbrace{\mathbf{c}^{{}^2 x} \cdot \mathbf{Lag}({}^2 x)}^{\mathbf{Bary}({}^2 x)} \\ -\frac{1.0}{1x+1.0} & -\frac{1.0}{2x+1.0} \\ -\frac{1.0}{1x+1.0} & \frac{1}{2x-1.0} \\ \frac{1}{1x-1.0} & -\frac{1.0}{2x+1.0} \\ \frac{1}{1x-1.0} & \frac{1}{2x-1.0} \end{array}$$

This is the solution of KST for rational forms !

Comparisons

Some competitors

Rat. app [B/G 2025]

- ▶ Lagrangian interpolation theorem
- ▶ p-AAA

KAN [P/P 2025]

- ▶ Kolmogorov Arnold theorem
- ▶ Kolmogorov Arnold Network

MLP [TensorFlow by Google - Keras 2025]

- ▶ Universal approximation theorem
- ▶ Multi Layer Perceptron
- ▶ Dense connected / ReLU / ADAM / 1000 it. / rand. init.

TensorFlow interface (Python code)

```
1 import numpy as np
2 import math
3 import matplotlib.pyplot as plt
4 import tensorflow as tf
5 from keras.models import Sequential
6 from matplotlib import cm
7
8 # Data
9 def f(x):
10     y = pow(x[1], 0.2)*x[0]
11     y += 1/2*(x[1]**2) + np.abs(x[0]) + 0.5*abs(x[1])
12     y = x[1]*sin(x[1])
13     y = np.exp(x[0]*0.01*x[1]**3)/(log(x[0], 0.1)-1.44)*(pow(x[1], 2)-1.44)
14     y = sin(x[0]*0.01*x[1]**3)
15     y = pow(np.abs(x[0]-x[1]), 0.3)
16     y = pow(x[0], 2) + pow(x[1], 2) + x[0]*x[1] + 1 / (pow(x[0], 0.5) + pow(x[1], 0.5) + 4)
17     return np.transpose(np.array([y]))
18
19 # n = 2
20 N1 = 40
21 N2 = 40
22 x1 = np.linspace(-1, 1, N1)
23 x2 = np.linspace(0, 1, N2)
24
25 # If
26 N = N1*N2
27 lab = np.zeros(N, 1)
```



L. Balicki and S. Gugercin, "*Multivariate Rational Approximation via Low-Rank Tensors and the p-AAA Algorithm*", SISC, 2025.



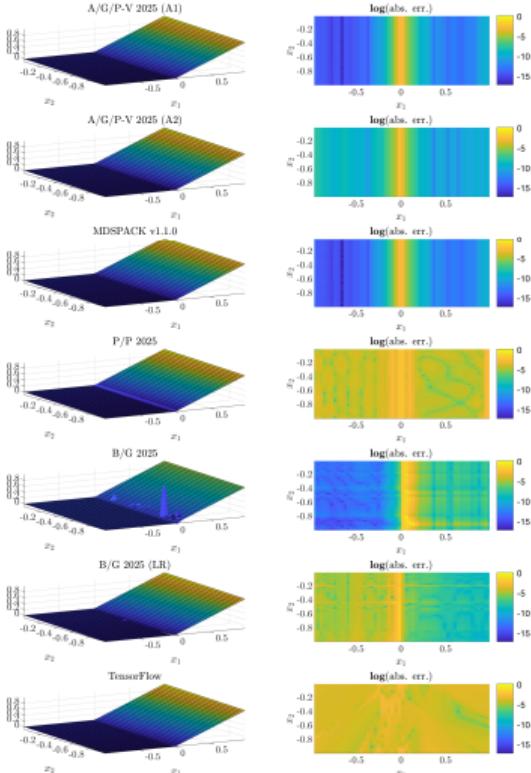
M. Poluektov and A. Polar, "*Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method*",
<https://arxiv.org/abs/2305.08194>.



M. Abadi et al., "*TensorFlow: Large-scale machine learning on heterogeneous systems, 2015*", Software available from tensorflow.org.

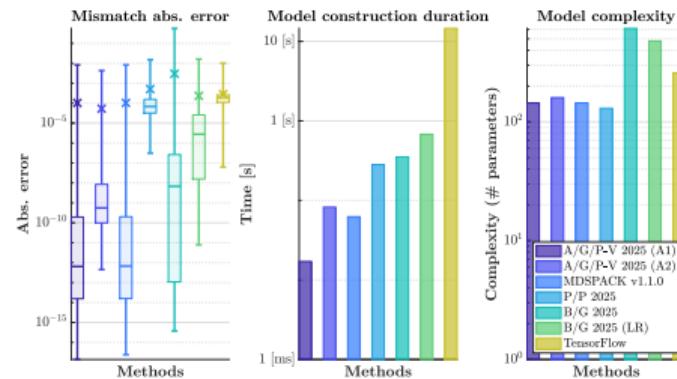
Comparisons

Irrational functions (example #1)



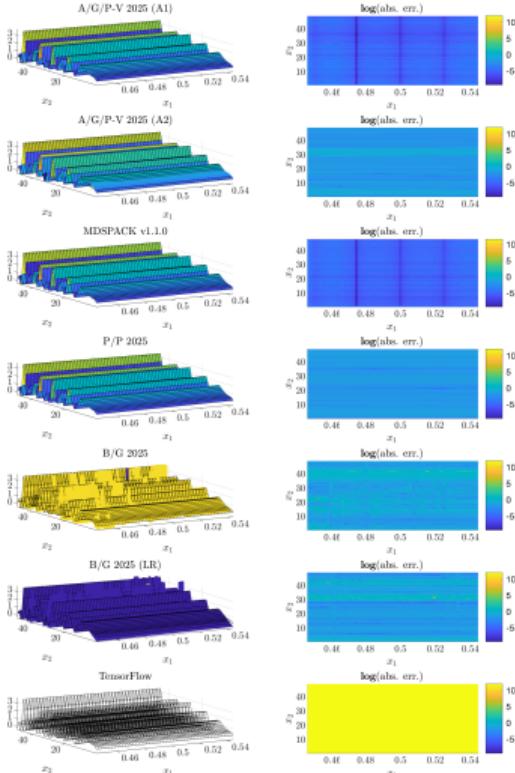
$$\text{ReLU}(\begin{pmatrix} 1 \\ x \end{pmatrix}) + \frac{1}{100} \begin{pmatrix} 2 \\ x \end{pmatrix}$$

- ▶ Reference: Personal communication, [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 12.5 KB (40^2 points)
- ▶ Bounds: $(-1, 1) \times (-1, -10^{-10})$



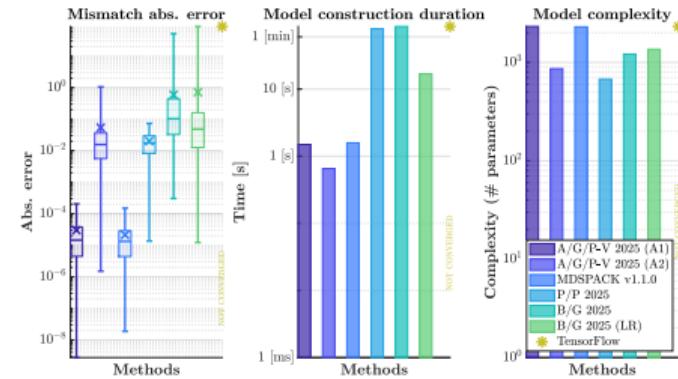
Comparisons

Irrational functions (example #34)



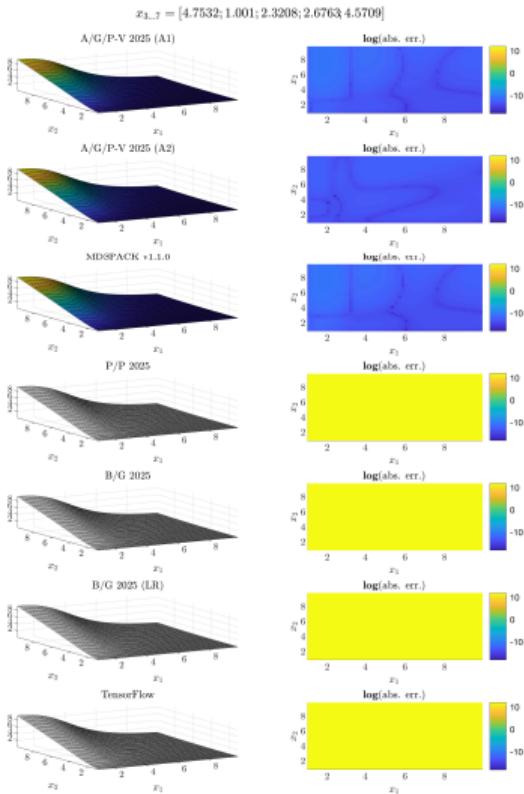
$$\operatorname{Re}(\zeta(1^1x + i^2x))$$

- ▶ Riemann ζ function (real part), [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 1.22 MB (400^2 points)
- ▶ Bounds: $\left(\frac{9}{20}, \frac{11}{20} \right) \times (1, 50)$



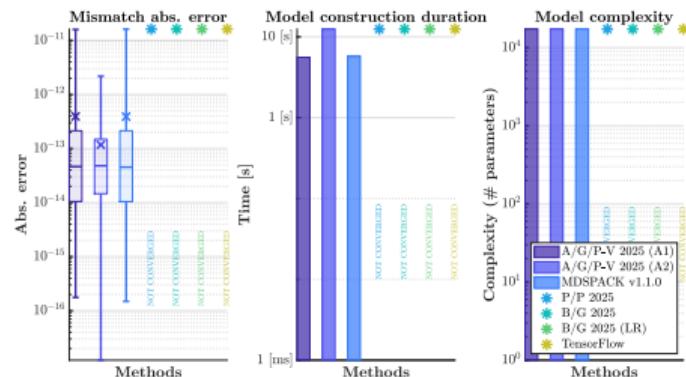
Comparisons

Irrational functions (example #43)



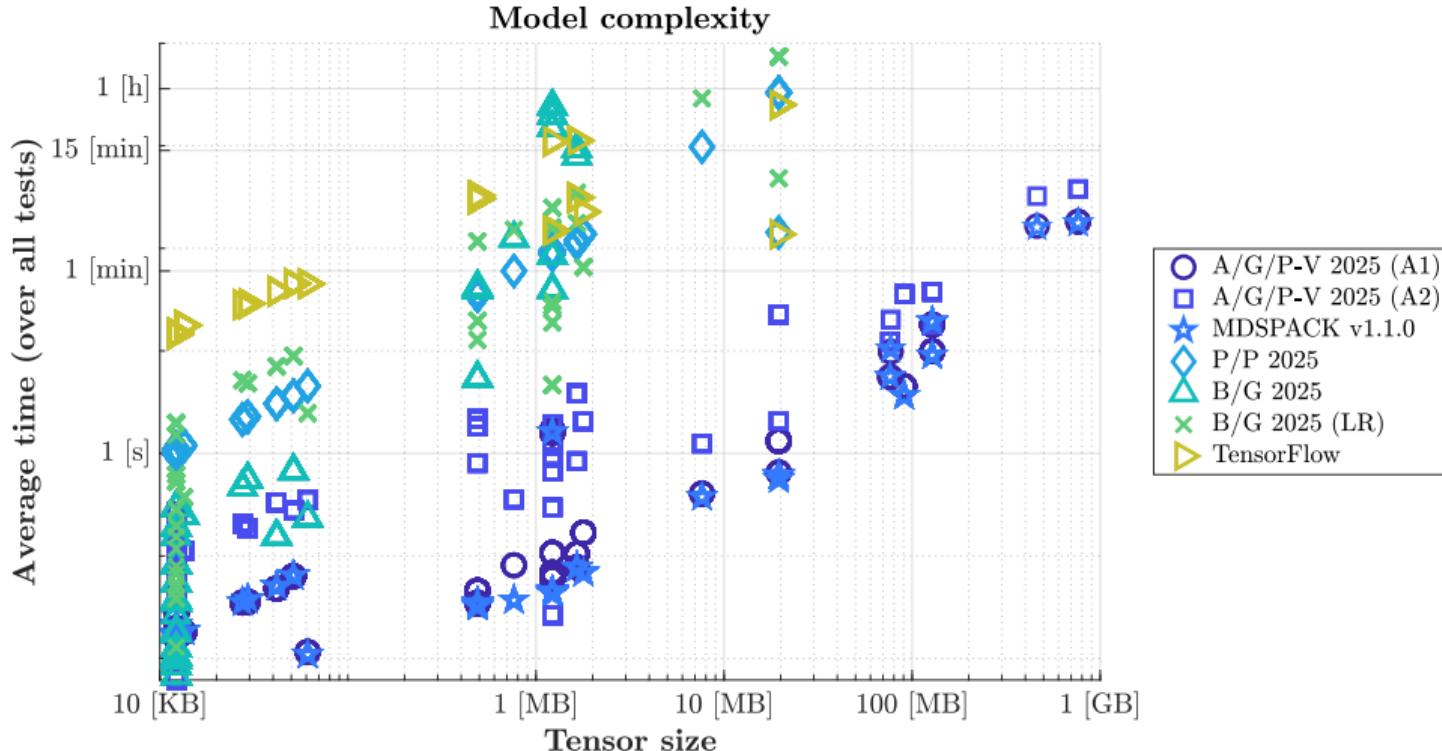
$$\frac{3x^2x^3 + 1}{1x^4 + 2x^2 \cdot 3x + 4x^2 + 5x + 6x^3 + 7x}$$

- ▶ Reference: Personal communication, [Riemann]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 76.3 MB (10^7 points)
- ▶ Bounds: $(1 \quad 10)^7$



Comparisons

Irrational functions (time, scalability)



Conclusion

Take home message

Main contributions

From any n -th order multi-variate transfer function / data tensor

- ▶ Construct a transfer function in barycentric form
- ▶ Construct a realization with controlled complexity
- ▶ Tame the computational complexity
- ▶ Two algorithms (direct & iterative)
- ▶ Connection with Kolmogorov theorem
- ▶ Connection with Kolmogorov networks

Side effects

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with

A.C. Antoulas [Rice Univ.]

I.V. Goșea [MPI]

P. Vuillemin [ONERA]

<https://arxiv.org/abs/2405.00495>

<https://arxiv.org/abs/2506.04791>

<https://github.com/cpoussot/mLF>

<https://cpoussot.github.io>



In parting... if enough time

Numerical examples, 20-D example

$\mathbf{H}(^1x, ^2x, \dots, ^{20}x) =$

$$\frac{3 \cdot {}^1x^3 + 4 \cdot {}^8x + {}^{12}x + {}^{13}x \cdot {}^{14}x + {}^{15}x}{{}^1x + {}^2x^2 \cdot {}^3x + {}^4x + {}^5x + {}^6x + {}^7x \cdot {}^8x + {}^9x \cdot {}^{10}x \cdot {}^{11}x + {}^{13}x + {}^{13}x^3 \cdot \pi + {}^{17}x + {}^{18}x \cdot {}^{19}x - {}^{20}x}$$

Statistics

- ▶ 20-D tensor of dimension (≥ 48 TB in real double precision)
- ▶ Complexity: (3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)
- ▶ n -D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- ▶ Full SVD: $2.49 \cdot 10^{20}$ flop
Recursive SVD: $5.43 \cdot 10^7$ flop
- ▶ error $\approx 10^{-11}$

In parting... if enough time

Numerical examples, 20-D example

$\mathbf{H}(^1x, ^2x, \dots, ^{20}x) =$

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Statistics

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- ▶ Complexity: (3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
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Recursive SVD: $5.43 \cdot 10^7$ flop $\rightarrow 5.03 \cdot 10^7$ flop
- ▶ error $\approx 10^{-11}$

In parting... if enough time

Numerical examples (from 2 to 20 variables)

#4 Rational function

$$s_4^3 + \frac{s_1 s_3}{s_3^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3^2 + s_1 s_3 s_5^3}{s_1^3 + s_4 + s_2 s_3}$$

#6 Rational function

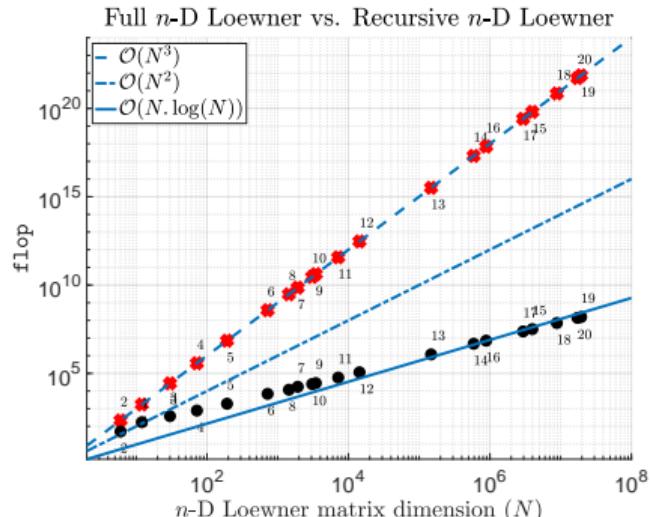
$$\frac{-\sqrt{2} s_6^2 + s_1 + s_3}{s_1^2 + s_4^3 + s_5^2 + s_6 + s_2 s_3}$$

#7 Rational function

$$\frac{s_3 s_2^3 + 1}{s_3 s_2^2 + s_4^2 + s_6^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3 s_1^3 + s_{18}^2 + 4 s_8 + s_{12} + s_{15} + s_{13} s_{14}}{s_3 s_2^2 + \pi s_{16}^3 + s_{17}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7 s_8 + s_9 s_{10} s_{11}}$$



In parting... if enough time

Numerical examples (rational and irrational)

#16 Arc-tangent function

$$\frac{\operatorname{atan}(x_1) + \operatorname{atan}(x_2) + \operatorname{atan}(x_3) + \operatorname{atan}(x_4)}{x_1^2 x_2^2 - x_1^2 - x_2^2 + 1}$$

#17 Exponential function

$$\frac{e^{x_1 x_2 x_3 x_4}}{x_1^2 + x_2^2 - x_3 x_4 + 3}$$

#18 Sinc function

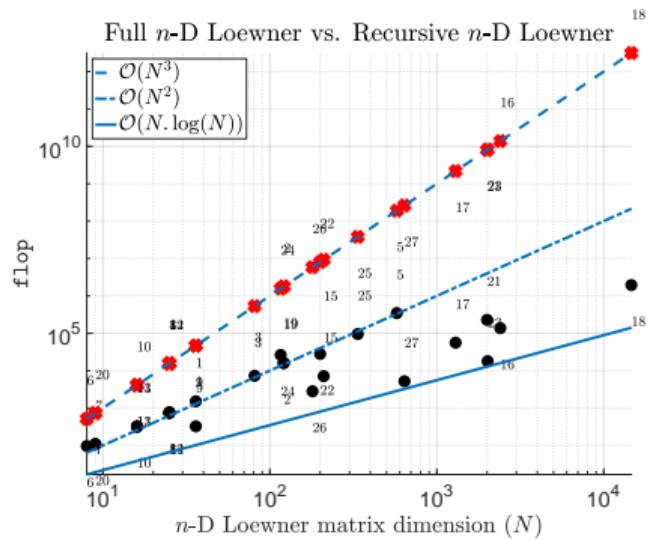
$$\frac{10 \sin(x_1) \sin(x_2) \sin(x_3) \sin(x_4)}{x_1 x_2 x_3 x_4}$$

#19 Sinc function

$$\frac{10 \sin(x_1) \sin(x_2)}{x_1 x_2}$$

#20 Polynomial function

$$x_1^2 + x_1 x_2 + x_2^2 - x_2 + 1$$

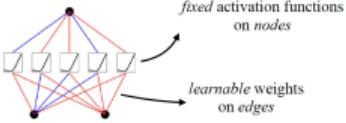
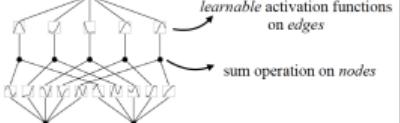
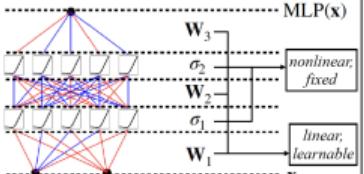
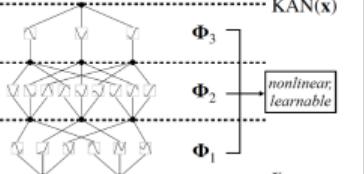


In parting... if enough time

About KANs

KANs features

- ▶ Inspired by the Kolmogorov-Arnold representation theorem
- ▶ The model output is a composition of **sums** and **learnable activation functions** (e.g. splines)
- ▶ Alternate to Multi-Layer Perceptrons (MLP), having fixed activation functions (e.g. ReLU), **inspired by the universal approximation theorem**

| Model | Multi-Layer Perceptron (MLP) | Kolmogorov-Arnold Network (KAN) |
|-------------------|---|---|
| Theorem | Universal Approximation Theorem | Kolmogorov-Arnold Representation Theorem |
| Formula (Shallow) | $f(\mathbf{x}) \approx \sum_{i=1}^{N(c)} a_i \sigma(\mathbf{w}_i \cdot \mathbf{x} + b_i)$ | $f(\mathbf{x}) = \sum_{q=1}^{2n+1} \Phi_q \left(\sum_{p=1}^n \phi_{q,p}(x_p) \right)$ |
| Model (Shallow) | (a)  | (b)  |
| Formula (Deep) | $\text{MLP}(\mathbf{x}) = (\mathbf{W}_3 \circ \sigma_2 \circ \mathbf{W}_2 \circ \sigma_1 \circ \mathbf{W}_1)(\mathbf{x})$ | $\text{KAN}(\mathbf{x}) = (\Phi_3 \circ \Phi_2 \circ \Phi_1)(\mathbf{x})$ |
| Model (Deep) | (c)  | (d)  |

Comparison between MLP and KAN (figure from Z. Liu et al.)



In parting... if enough time

KANs with splines

Building the Kolmogorov-Arnold model as follows

$$F(^1x, ^2x, \dots, ^nx) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^m f_{kj}(^jx) \right)$$

$f_{kj} : [0, 1] \mapsto \mathbb{R}$ and $\Phi_k : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions.

The relation is approximated by $k = 1, \dots, d = 2m + 1$ as

$$\hat{F}(^1x, ^2x, \dots, ^nx) = \sum_{k=1}^d \Phi_k \underbrace{\left(\sum_{j=1}^m f_{kj}(^jx_i) \right)}_{\theta_{ik}}$$

where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes splines

In parting... if enough time

KANs with splines

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