

The Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality

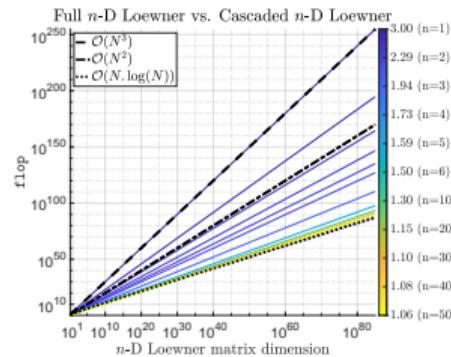
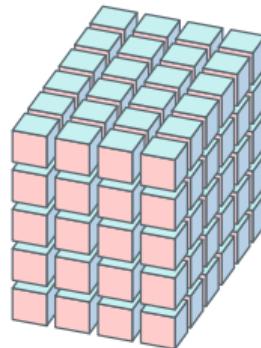
... from tensor to multivariate rational approximation and more

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September 23, 2025

<https://arxiv.org/abs/2405.00495> (to appear in SIAM Review - Research Spotlight)

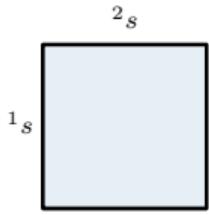
<https://github.com/cpoussot/mLF> (GitHub code)



Forewords

Starting (motivating) examples - toy case

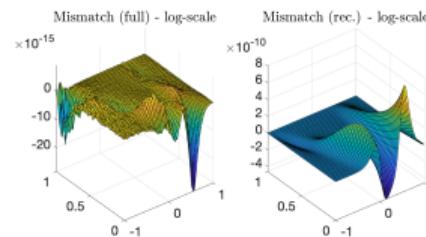
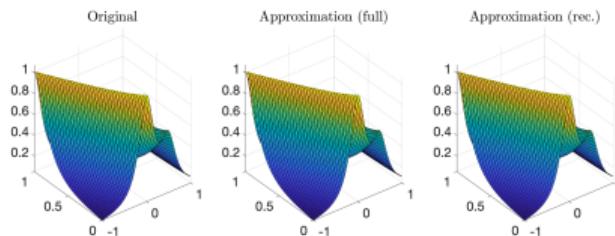
$$\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, p) = \frac{1}{1 + 25(s + p)^2} + \frac{0.5}{1 + 25(s - 0.5)^2} + \frac{0.1}{p + 25}$$



$$\begin{matrix} 1s & \times & 2s \\ [-1, 1] & \times & [0, 1] \end{matrix}$$

$$\mathbf{tab}_2 \in \mathbb{R}^{21 \times 21}$$

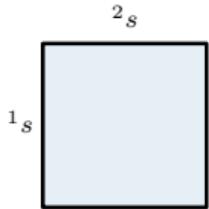
≈ 3.45 Ko ('real')



Forewords

Starting (motivating) examples - ReLU

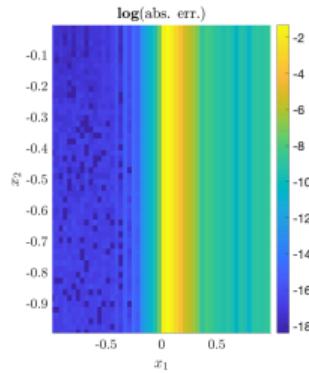
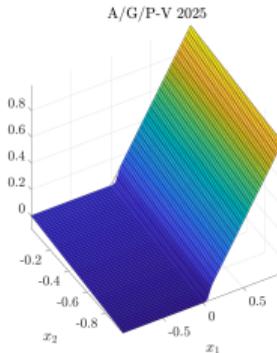
$$\mathbf{H}(^1s, ^2s) = \mathbf{ReLU}(^1s) + \frac{1}{100} {}^2s$$



$${}^1s \quad \times \quad {}^2s \\ [-1, 1] \quad \times \quad [0, 1]$$

$$\mathbf{tab}_2 \in \mathbb{R}^{20 \times 20}$$

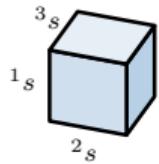
≈ 3.45 Ko ('real')



Forewords

Starting (motivating) examples - Airbus flutter case

$$\Sigma(1s, 2s, 3s) = \Sigma(s, m, v) : s^2 M(m) x(s) + s B(m) x(s) + K(m) x(s) - G(s, v) = u(s), \mathbf{y}(s) = C \mathbf{x}(s)$$



$$\begin{matrix} 1_s \\ \imath[10, 35] \end{matrix} \times \begin{matrix} 2_s \\ [\underline{m}, \bar{m}] \end{matrix} \times \begin{matrix} 3_s \\ [v, \bar{v}] \end{matrix}$$

$$\mathbf{tab}_3 \in \mathbb{C}^{300 \times 10 \times 10}$$

≈468.75 Ko ('complex')

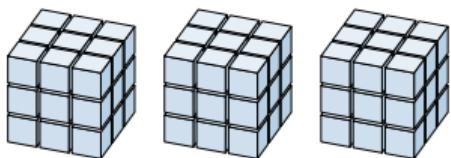


A. dos Reis de Souza et al., "Aircraft flutter suppression: from a parametric model to robust control", ECC, 2023.

Forewords

Starting (motivating) examples - Borehole function

$$\mathbf{H}(^1s, \dots, {}^8s) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right) + \frac{T_u}{T_l}}$$



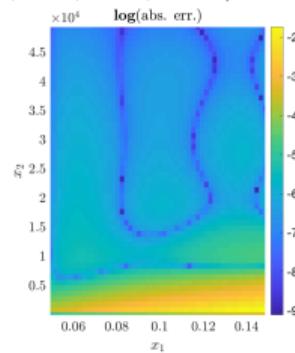
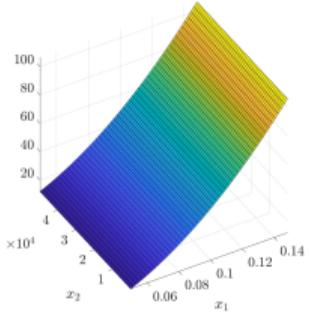
$${}^1s \quad \times \quad \dots \quad \times \quad {}^8s \\ [r_w, \overline{r_w}] \quad \times \quad \dots \quad \times \quad [\underline{K_w}, \overline{K_w}]$$

$\text{tab}_8 \in \mathbb{C}^{8 \times 8 \times \dots \times 8}$

$\approx 130 \text{ Mo ('real')}$

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	$1.02e + 04$	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	$1.02e+04$	39.1	0.00456	2.93e-09	0.0611

$$x_{3..8} = [93221.3376; 996.0245; 105.854; 765.3085; 1619.2701; 10834.7051] \\ \text{A/G/P-V 2025}$$



S. Surjanovic, "Borehole function", <https://www.sfu.ca/~ssurjano/borehole.html>.

Forewords

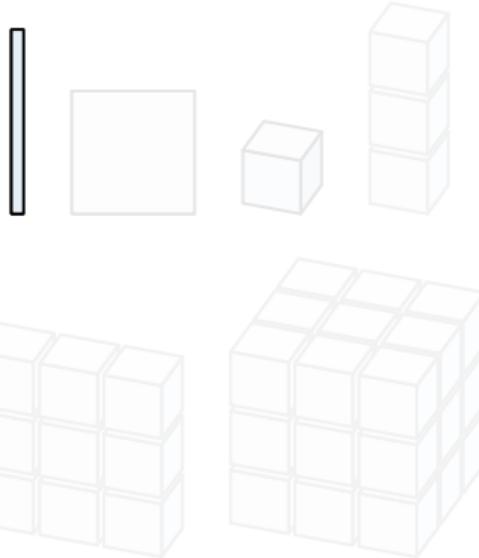
Data (and tensors)

Column / Row data

$$\left. \begin{array}{c} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \end{array} \right\} \xrightarrow{\mathbf{H}(^1s)} \left\{ \begin{array}{c} \mathbf{w}_{j_1}, \mathbf{v}_{i_1} \end{array} \right.$$

1s	
${}^1\lambda_{1,\dots,k_1}$	\mathbf{W}_{k_1}
${}^1\mu_{1,\dots,q_1}$	\mathbf{V}_{q_1}

Tensors (1-D) tab₁



Forewords

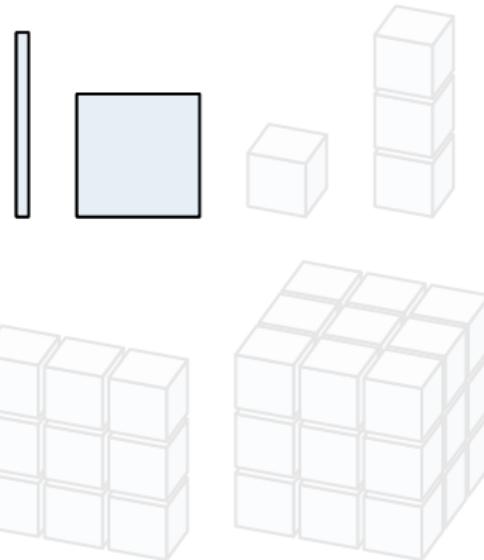
Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1s, {}^2s)} \left\{ \begin{array}{l} {}^2\lambda_{1,\dots,k_2} \\ {}^2\mu_{1,\dots,q_2} \end{array} \right.$$

1s	2s	
${}^1\lambda_{1,\dots,k_1}$	\mathbf{W}_{k_1,k_2}	ϕ_{cr}
${}^1\mu_{1,\dots,q_1}$	ϕ_{rc}	\mathbf{V}_{q_1,q_2}

Tensors (2-D) tab₂



Forewords

Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array} \right\} \xrightarrow{\mathbf{H}({}^1s, {}^2s, {}^3s)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2, j_3}, \mathbf{v}_{i_1, i_2, i_3} \end{array} \right.$$

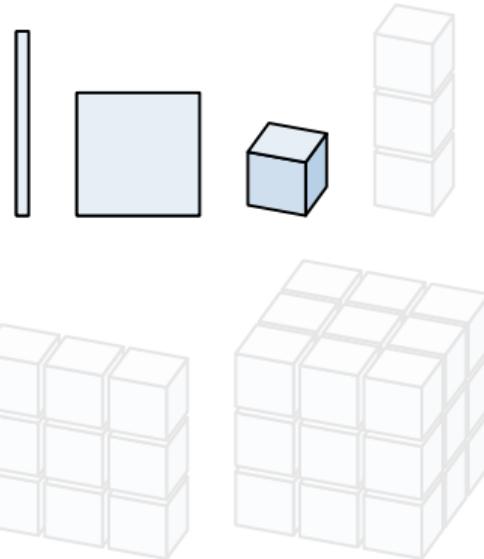
$${}^3s = {}^3\lambda_{1, \dots, k_3}$$

1s	2s	${}^2\lambda_{1, \dots, k_2}$	${}^2\mu_{1, \dots, q_2}$
${}^1\lambda_{1, \dots, k_1}$	$\mathbf{W}_{k_1, k_2, k_3}$	ϕ_{crc}	
${}^1\mu_{1, \dots, q_1}$	ϕ_{rcc}	ϕ_{rrc}	

$${}^3s = {}^3\mu_{1, \dots, q_3}$$

1s	2s	${}^2\lambda_{1, \dots, k_2}$	${}^2\mu_{1, \dots, q_2}$
${}^1\lambda_{1, \dots, k_1}$	ϕ_{crr}	ϕ_{crr}	
${}^1\mu_{1, \dots, q_1}$	ϕ_{rcr}	$\mathbf{V}_{q_1, q_2, q_3}$	

Tensors (3-D) tab₃



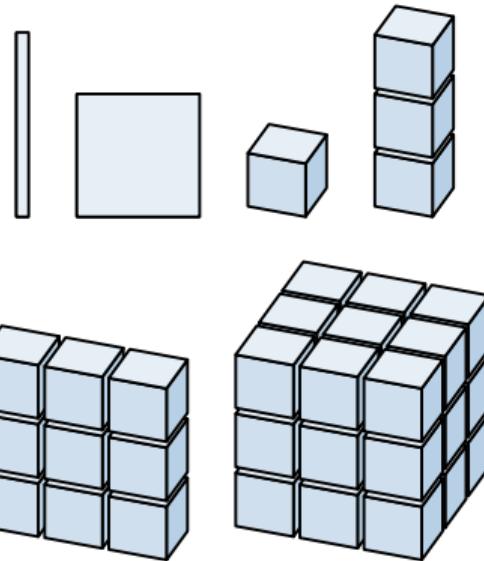
Forewords

Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\lambda_{j_3}, {}^3\mu_{i_3} \\ \vdots \\ {}^n\lambda_{j_n}, {}^n\mu_{i_n} \end{array} \right\} \xrightarrow{\mathbf{H}(^{1_s, \dots, n_s})} \left\{ \begin{array}{l} \mathbf{w}_{j_1, \dots, j_n}, \mathbf{v}_{i_1, \dots, i_n} \end{array} \right.$$

Tensors (n -D) tab_n



Forewords

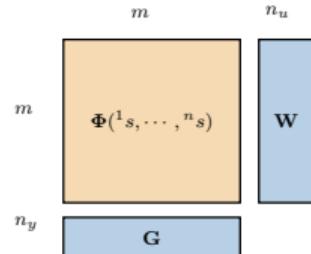
Problem description

Data-driven model approximation

Being given a n -dimensional tensor (data), we seek a multi-variate rational function $\hat{\mathbf{H}}$ and realization $(\mathbf{G}, \Phi, \mathbf{W})$

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^n s) = \mathbf{G}\Phi({}^1s, {}^2s, \dots, {}^n s)^{-1}\mathbf{W} \in \mathbb{C}$$

that interpolates the data.



Connection to standard dynamical system realization

A linear-in-state dynamical system parameterized in terms of parameters included in $\mathcal{S} = [{}^2s, \dots, {}^n s]^T \subset \mathbb{C}^{n-1}$

$$\begin{aligned} \mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t; \mathcal{S}) &= \mathbf{A}(\mathcal{S})\mathbf{x}(t; \mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t; \mathcal{S}) &= \mathbf{C}\mathbf{x}(t; \mathcal{S}) \end{aligned}$$

equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^n s) = \mathbf{C}(\mathcal{S}) \left[{}^1s \mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S}) \right]^{-1} \mathbf{B} \in \mathbb{C}.$$

Forewords

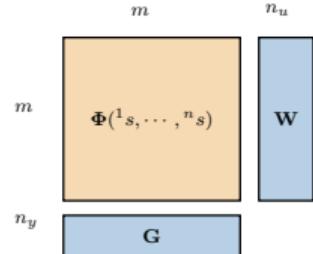
Problem description

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that interpolates the data.



Connection to standard dynamical system realization

A linear-in-state dynamical system parameterized in terms of parameters included in $\mathcal{S} = [{}^2s, \dots, {}^n s]^\top \subset \mathbb{C}^{n-1}$

$$\begin{aligned}\mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t; \mathcal{S}) &= \mathbf{A}(\mathcal{S})\mathbf{x}(t; \mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t; \mathcal{S}) &= \mathbf{C}\mathbf{x}(t; \mathcal{S})\end{aligned}$$

equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^n s) = \mathbf{C}(\mathcal{S}) \left[{}^1s \mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S}) \right]^{-1} \mathbf{B} \in \mathbb{C}.$$

Forewords

Where we stand (some references)

1-D Two-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ realization minimality
- ⇒ direct algorithm

1-D One-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ direct algorithm

1-D AAA (Adaptive Anderson Antoulas - one-sided)

- ⇒ (mixed interpolation LS) barycentric form
- ⇒ iterative algorithm

2-D Parametric one-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ realization (non-minimal)
- ⇒ direct algorithm

3-D Parametric AAA

- ⇒ (mixed interpolation LS) barycentric form
- ⇒ iterative algorithm

>3-D few results

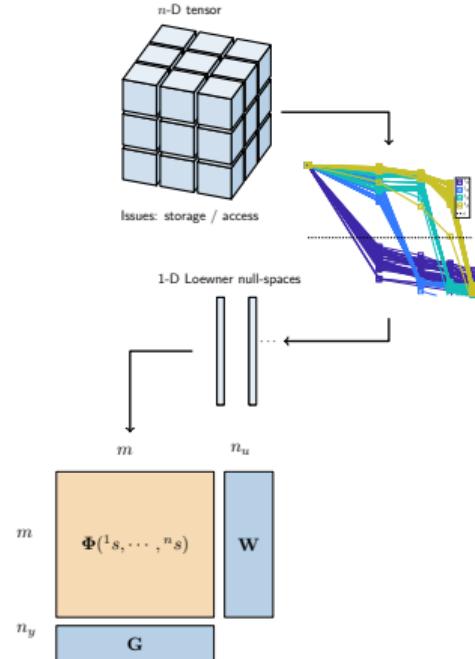
-
-  J-P. Berrut and N. Trefethen, "*Barycentric Lagrange Interpolation*", SIAM Review, 46(3), 2004.
 -  A.J. Mayo and A.C. Antoulas, "*A framework for the solution of the generalized realisation problem*", LAA, 425(2-3), 2007.
 -  A.C. Ionita and A.C. Antoulas, "*Data-Driven Parametrized Model Reduction in the Loewner Framework*", SIAM Journal on Scientific Computing, 36(3), 2014.
 -  T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, "*Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework*", SIAM Journal on Applied Dynamical Systems, 22(4), 2023.
 -  A.C. Rodriguez, L. Balicki and S. Gugercin, "*The p-AAA algorithm for data driven modeling of parametric dynamical systems*", SIAM Journal on Scientific Computing, 45(3), 2023.

Forewords

Contributions claim

List of contributions

- ▶ n -D tensor data to n -D Loewner matrix \mathbb{L}_n
- ▶ n -D recursive Sylvester equations
- ▶ n -variable transfer functions
- ▶ n -variable generalized realization
- ▶ Taming the curse of dimensionality
 - » in computation effort (flop)
 - » in storage needs (Bytes)
 - » in accuracy
- ▶ n -variable **decoupling**
 - » **KST** formulation for rational functions
 - » connection with **KAN**



Content

Forewords

Multi-variate data, function & Loewner matrix

Multi-variate realization

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

Conclusion

Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} &:= \left\{ \left({}^{\textcolor{brown}{1}} \lambda_{j_1}; \mathbf{w}_{j_1} \right), j_1 = 1, \dots, k_1 \right\} \\ P_r^{(1)} &:= \left\{ \left({}^{\textcolor{violet}{1}} \mu_{i_1}; \mathbf{v}_{i_1} \right), i_1 = 1, \dots, q_1 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{}^{\textcolor{violet}{1}} \mu_{i_1} - {}^{\textcolor{brown}{1}} \lambda_{j_1}$$

$$\mathbf{M}_1 \mathbb{L}_1 - \mathbb{L}_1 \boldsymbol{\Lambda}_1 = \mathbb{V}_1 \mathbf{R}_1 - \mathbf{L}_1 \mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}({}^{\textcolor{violet}{1}} s) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{}^{\textcolor{violet}{1}} s - {}^{\textcolor{brown}{1}} \lambda_{j_1}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{}^{\textcolor{violet}{1}} s - {}^{\textcolor{brown}{1}} \lambda_{j_1}}$$

Null space

$$\text{span } (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}(^1s) = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$ of complexity (2)

$$\left. \begin{array}{rcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [2, 4, 6, 8] \end{array} \right\} \xrightarrow{\mathbf{H}} \left\{ \begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array} \right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} &:= \left\{ (^1\lambda_{j_1}, ^2\lambda_{j_2}; \mathbf{w}_{j_1, j_2}), \ j_1 = 1, \dots, k_1 \quad j_2 = 1, \dots, k_2 \right\} \\ P_r^{(2)} &:= \left\{ (^1\mu_{i_1}, ^2\mu_{i_2}; \mathbf{v}_{i_1, i_2}), \ i_1 = 1, \dots, q_1 \quad i_2 = 1, \dots, q_2 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1, j_2}^{i_1, i_2} = \frac{\mathbf{v}_{i_1, i_2} - \mathbf{w}_{j_1, j_2}}{(^1\mu_{i_1} - ^1\lambda_{j_1})(^2\mu_{i_2} - ^2\lambda_{j_2})}$$

$$\begin{cases} \mathbf{M}_2 \mathbb{X} - \mathbb{X} \boldsymbol{\Lambda}_2 &= \mathbb{V}_2 \mathbf{R}_2 - \mathbf{L}_2 \mathbb{W}_2 \\ \mathbf{M}_1 \mathbb{L}_2 - \mathbb{L}_2 \boldsymbol{\Lambda}_1 &= \mathbb{X} \end{cases}$$

Lagrangian form

$$\mathbf{g}(^1s, ^2s) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{(^1s - ^1\lambda_{j_1})(^2s - ^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2}}{(^1s - ^1\lambda_{j_1})(^2s - ^2\lambda_{j_2})}}$$

Null space

$$\text{span } (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \begin{bmatrix} c_{1,1} \\ \vdots \\ \hline c_{1,k_2} \\ \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{bmatrix} \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{l} {}^1\lambda_{j_1} = [1, 3, 5] \\ {}^1\mu_{i_1} = [0, 2, 4] \\ {}^2\lambda_{j_2} = [-1, -3] \\ {}^2\mu_{i_2} = [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \hline \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \hline \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array} \right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array} \right]$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

$$\left. \begin{array}{lcl} {}^1\lambda_{j_1} & = & [1, 3, 5] \\ {}^1\mu_{i_1} & = & [0, 2, 4] \\ {}^2\lambda_{j_2} & = & [-1, -3] \\ {}^2\mu_{i_2} & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{ccc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

Lagrangian form

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s, t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \left\{ (^1\lambda_{j_1}, ^2\lambda_{j_2}, \dots, ^n\lambda_{j_n}; \mathbf{w}_{j_1, j_2, \dots, j_n}), j_l = 1, \dots, k_l, l = 1, \dots, n \right\} \\ P_r^{(n)} := \left\{ (^1\mu_{i_1}, ^2\mu_{i_2}, \dots, ^n\mu_{i_n}; \mathbf{v}_{i_1, i_2, \dots, i_n}), i_l = 1, \dots, q_l, l = 1, \dots, n \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n} = \frac{\mathbf{v}_{i_1, i_2, \dots, i_n} - \mathbf{w}_{j_1, j_2, \dots, j_n}}{(^1\mu_{i_1} - ^1\lambda_{j_1}) \cdots (^n\mu_{i_n} - ^n\lambda_{j_n})}$$

$$\begin{cases} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \boldsymbol{\Lambda}_n &= \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ \mathbf{M}_{n-1} \mathbb{X}_2 - \mathbb{X}_2 \boldsymbol{\Lambda}_{n-1} &= \mathbb{X}_1, \\ &\dots \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \boldsymbol{\Lambda}_1 &= \mathbb{X}_{n-1}. \end{cases}$$

Lagrangian form

$$\mathbf{g}(^1s, \dots, ^ns) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n} \mathbf{w}_{j_1, \dots, j_n}}{(^1s - ^1\lambda_{j_1}) \cdots (^n s - ^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n}}{(^1s - ^1\lambda_{j_1}) \cdots (^n s - ^n\lambda_{j_n})}}$$

Null space

$$\text{span } (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \begin{bmatrix} c_{1, \dots, 1} \\ \vdots \\ \hline c_{1, \dots, k_n} \\ \vdots \\ \hline c_{k_1, \dots, 1} \\ \vdots \\ c_{k_1, \dots, k_n} \end{bmatrix} \in \mathbb{C}^{k_1 \cdots k_n}$$

Content

Forewords

Multi-variate data, function & Loewner matrix

Multi-variate realization

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

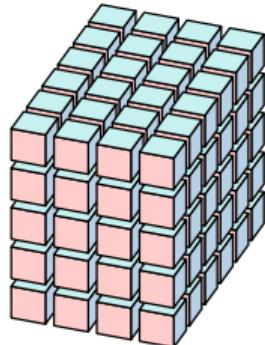
Conclusion

Taming the curse of dimensionality

Loewner matrix operator

$$\begin{aligned} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \dots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1) \times \dots \times (k_n+q_n)} &\longrightarrow \mathbb{C}^{Q \times K} \\ (\overset{1}{\lambda}_{j_1}, \overset{1}{\mu}_{i_1}, \dots, \overset{n}{\lambda}_{j_n}, \overset{n}{\mu}_{i_n}, \mathbf{tab}_n) &\longmapsto \mathbb{L}_n \end{aligned}$$

n -D tensor \mathbf{tab}_n

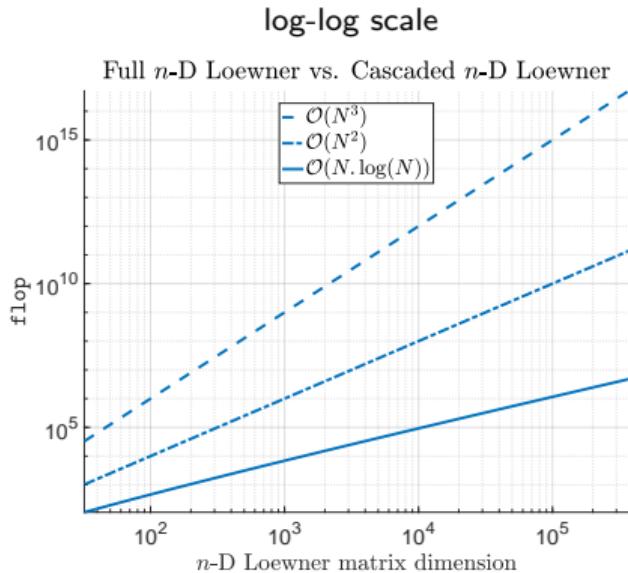


matrix \mathbb{L}_n



Taming the curse of dimensionality

Null space flop and memory issues



Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns)
 $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD flop estimation is

- ▶ QK^2 (if $Q > K$)
- ▶ N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- ▶ in real double
- ▶ in complex double

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

2s	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
1s				
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 \mathbb{L}_1 along 1s , for
 ${}^2s = {}^2\lambda_2 = -3$
- 3 \mathbb{L}_1 along 2s for
 ${}^2s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$

$$[{}_1^{^1\lambda_1} \cdot [{}^2\lambda_2]_1 \quad {}_1^{^1\lambda_2} \cdot [{}^2\lambda_2]_2 \quad {}_1^{^1\lambda_3} \cdot [{}^2\lambda_2]_3]^\top$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

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$${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

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${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
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$${}^2s = {}^2\lambda_2 = -3$$

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$${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}(^1s, ^2s) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$ of complexity (2, 1)

2s	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
1s				
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
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$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 \mathbb{L}_1 along 1s , for

$${}^2s = {}^2\lambda_2 = -3$$

- 3 \mathbb{L}_1 along 2s for

$${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$[\mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case

Theorem: 2-D to 1-D

Being given the tableau tab_2 tensor in response of the 2-variables $\mathbf{H}(^1s, ^2s)$ function, the null space of the corresponding 2-D Loewner matrix \mathbb{L}_2 , is spanned by

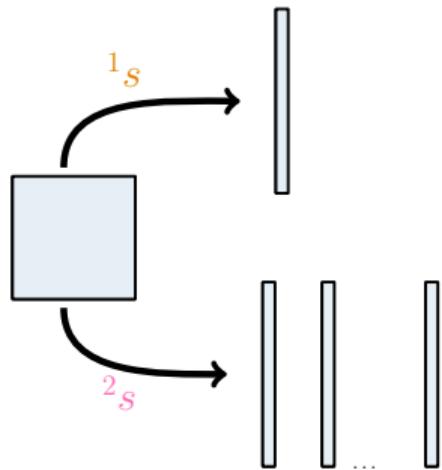
$$\mathcal{N}(\mathbb{L}_2) = \text{vec} \left[\mathbf{c}_1^{^2\lambda_1} \cdot \begin{bmatrix} ^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_1, \dots, \mathbf{c}_1^{^2\lambda_{k_2}} \cdot \begin{bmatrix} ^1\lambda_{k_1} \\ \mathbf{c}_1 \end{bmatrix}_{k_2} \right],$$

where

- ▶ $\mathbf{c}_1^{^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{^1\lambda_{k_1}})$,
i.e. the null space of the **1-D Loewner matrix** for frozen $^1s = ^1\lambda_{k_1}$, and
- ▶ $\mathbf{c}_1^{^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{^2\lambda_{j_2}})$,
i.e. the j_1 -th null space of the **1-D Loewner matrices** for frozen $^2s = \{^2\lambda_1, \dots, ^2\lambda_{k_2}\}$.

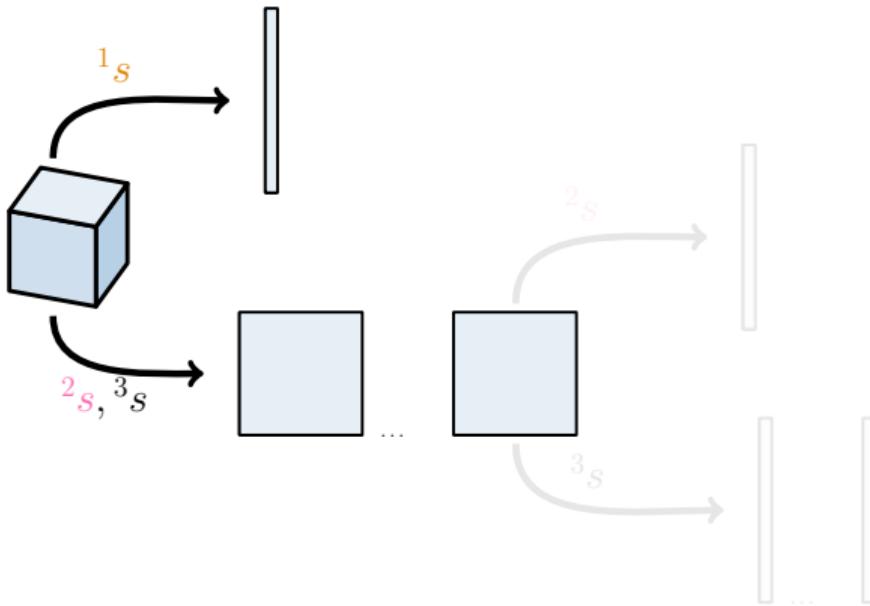
Taming the curse of dimensionality

2-D case



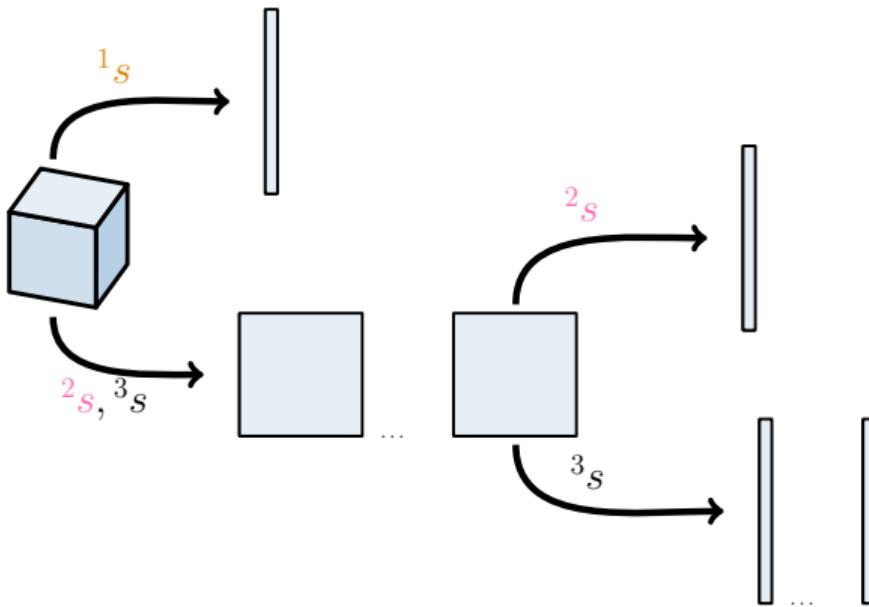
Taming the curse of dimensionality

3-D case



Taming the curse of dimensionality

3-D case



Taming the curse of dimensionality

n-D case

Theorem: *n*-D to (*n* − 1)-D

Being given the tableau tab_n tensor in response of the n -variables $\mathbf{H}({}^1s, \dots, {}^ns)$ function, the null space of the corresponding *n*-D Loewner matrix \mathbb{L}_n , is spanned by

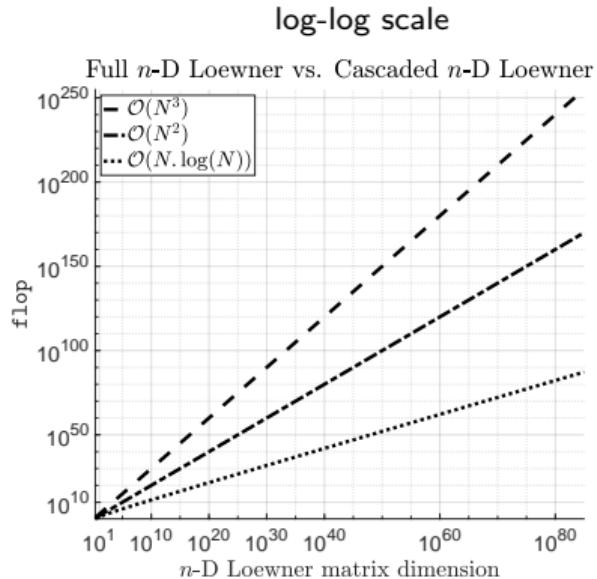
$$\mathcal{N}(\mathbb{L}_n) = \text{vec} \left[\mathbf{c}_{n-1}^{{}^1\lambda_1} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2}, {}^3\lambda_{k_3}, \dots, {}^n\lambda_{k_n})} \right]_1, \dots, \mathbf{c}_{n-1}^{{}^1\lambda_{k_1}} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2}, {}^3\lambda_{k_3}, \dots, {}^n\lambda_{k_n})} \right]_{k_1} \right],$$

where

- ▶ $\mathbf{c}_1^{({}^2\lambda_{k_2}, {}^3\lambda_{k_3}, \dots, {}^n\lambda_{k_n})}$ spans $\mathcal{N}(\mathbb{L}_1^{({}^2\lambda_{k_2}, {}^3\lambda_{k_3}, \dots, {}^n\lambda_{k_n})})$,
i.e. the null space of the **1-D Loewner matrix** for frozen $\{{}^2\lambda_{k_2}, {}^3\lambda_{k_3}, \dots, {}^n\lambda_{k_n}\}$, and
- ▶ $\mathbf{c}_{n-1}^{{}^1\lambda_{j_1}}$ spans $\mathcal{N}(\mathbb{L}_{n-1}^{{}^1\lambda_{j_1}})$,
i.e. the j_1 -th null space of the **(*n* − 1)-D Loewner matrix** for frozen ${}^1s_{j_1} = \{{}^1\lambda_1, \dots, {}^1\lambda_{k_1}\}$.

Taming the curse of dimensionality

Null space - flop complexity



Null space underlying problem

Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns) $K = k_1 k_2 \dots k_n$
 $\mathbb{L}_n \in \mathbb{C}^{Q \times K}$

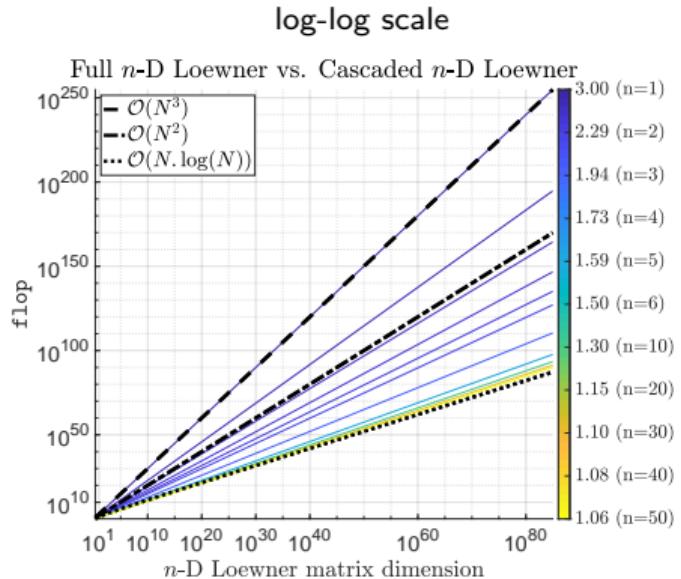
Note that $Q \times K$ matrix SVD flop estimation is

- ▶ QK^2 (if $Q > K$)
- ▶ N^3 (if $Q = K = N$)

⇒ The CURSE of dimensionality

Taming the curse of dimensionality

Null space - flop complexity



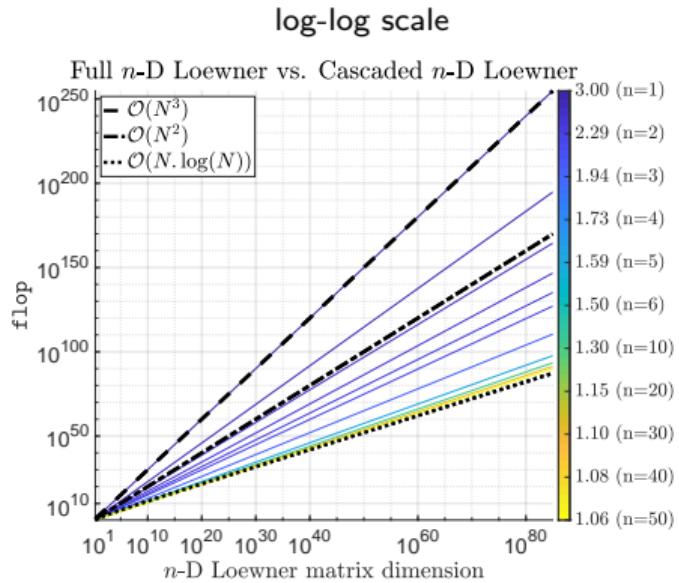
Theorem: Recursive complexity

$$\text{flop}_1(n) = \sum_{j=1}^n \left(k_j^3 \prod_{l=1}^j k_{l-1} \right) \text{ where } k_0 = 1.$$

⇒ The CURSE of dimensionality is TAMED

Taming the curse of dimensionality

Null space - flop complexity



Corollary: Worst case complexity

k interpolation points per variables.

$$\overline{\text{flop}_1} = k^3 \frac{1 - k^n}{1 - k} = k^3 \frac{1 - N}{1 - k},$$

which is a (n finite) geometric series of ratio k .

⇒ The CURSE of dimensionality is TAMED

$$\begin{aligned} \mathcal{O}(N^3) &\rightarrow \mathcal{O}(N^{2.29}) && \text{for } n = 2 \\ &\rightarrow \mathcal{O}(N^{1.94}) && \text{for } n = 3 \\ &\vdots && \\ &\rightarrow \mathcal{O}(N^{1.5}) && \text{for } n = 6 \\ &\vdots && \\ &\rightarrow \mathcal{O}(N^{1.06}) && \text{for } n = 50 \end{aligned}$$

Taming the curse of dimensionality

Null space - memory

With similar importance, the **data storage is a key element** in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB)}$$

Full n -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example: $N = 46,080$

Memory: **31.64 GB**

flop: $9.78 \cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory

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Full n -D Loewner

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Example: $N = 46,080$

Memory: **31.64 GB**

flop: $9.78 \cdot 10^{13}$

Cascaded n -D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\bar{k} \times \bar{k}}$$

where $\bar{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}} \bar{k}^2 \text{ MB}$$

Example: $\bar{k} = 20$

Memory: **6.25 KB**

flop: $8.13 \cdot 10^5$

Taming the curse of dimensionality

Numerical examples, 20-D example

$\mathbf{H}(^1s, ^2s, \dots, ^{20}s) =$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- ▶ 20-D tensor of dimension (≥ 48 TB in real double precision)
- ▶ Complexity: (3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)
- ▶ n -D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- ▶ Full SVD: $2.49 \cdot 10^{20}$ flop
Recursive SVD: $5.43 \cdot 10^7$ flop
- ▶ error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples, 20-D example

$$\mathbf{H}(^1s, ^2s, \dots, ^{20}s) =$$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- ▶ 20-D tensor of dimension (≥ 48 TB in real double precision)
- ▶ Complexity: (3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
- ▶ n -D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- ▶ Full SVD: $2.49 \cdot 10^{20}$ flop
Recursive SVD: $5.43 \cdot 10^7$ flop $\rightarrow 5.03 \cdot 10^7$ flop
- ▶ error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples (from 2 to 20 variables)

#4 Rational function

$$s_4^3 + \frac{s_1 s_3}{s_3^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3^2 + s_1 s_3 s_5^3}{s_1^3 + s_4 + s_2 s_3}$$

#6 Rational function

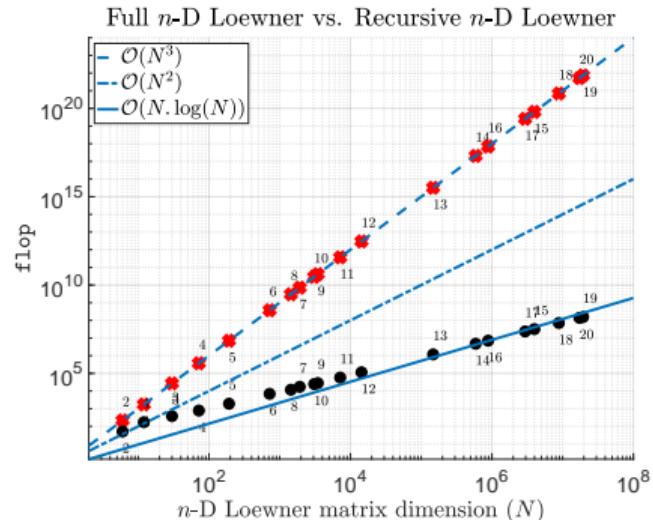
$$\frac{-\sqrt{2} s_6^2 + s_1 + s_3}{s_1^2 + s_4^3 + s_5^2 + s_6 + s_2 s_3}$$

#7 Rational function

$$\frac{s_3 s_2^3 + 1}{s_3 s_2^2 + s_4^2 + s_6^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3 s_1^3 + s_{18}^2 + 4 s_8 + s_{12} + s_{15} + s_{13} s_{14}}{s_3 s_2^2 + \pi s_{16}^3 + s_{17}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7 s_8 + s_9 s_{10} s_{11}}$$



Taming the curse of dimensionality

Numerical examples (rational and irrational)

#16 Arc-tangent function

$$\frac{\operatorname{atan}(x_1) + \operatorname{atan}(x_2) + \operatorname{atan}(x_3) + \operatorname{atan}(x_4)}{x_1^2 x_2^2 - x_1^2 - x_2^2 + 1}$$

#17 Exponential function

$$\frac{e^{x_1 x_2 x_3 x_4}}{x_1^2 + x_2^2 - x_3 x_4 + 3}$$

#18 Sinc function

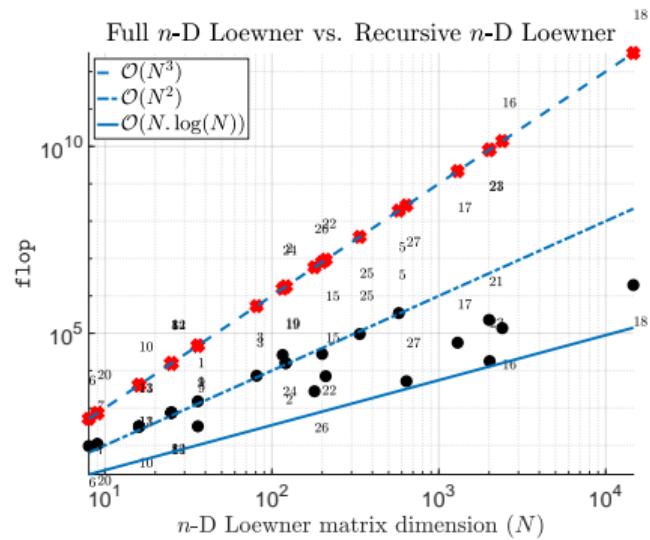
$$\frac{10 \sin(x_1) \sin(x_2) \sin(x_3) \sin(x_4)}{x_1 x_2 x_3 x_4}$$

#19 Sinc function

$$\frac{10 \sin(x_1) \sin(x_2)}{x_1 x_2}$$

#20 Polynomial function

$$x_1^2 + x_1 x_2 + x_2^2 - x_2 + 1$$



Content

Forewords

Multi-variate data, function & Loewner matrix

Multi-variate realization

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

Conclusion

Variables decoupling, KST and KANs

Variables decoupling

Variable decoupling

Given data tab_n , the (recursive) theorem achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{n_s}}_{\text{Bary}^{(n_s)}} \odot \underbrace{(\mathbf{c}^{n-1_s} \otimes \mathbf{1}_{k_n})}_{\text{Bary}^{(n-1_s)}} \odot \underbrace{(\mathbf{c}^{n-2_s} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\text{Bary}^{(n-2_s)}} \odot \cdots \odot \underbrace{(\mathbf{c}^1_s \otimes \mathbf{1}_{k_n \dots k_2})}_{\text{Bary}^{(1_s)}}.$$

where \mathbf{c}^j_s denotes the vectorized barycentric coefficients related to the j -th variable.

Variables decoupling, KST and KANs

Loewner, Kolmogorov Superposition Theorem and Hilbert's 13th problem

Variable decoupling

Given data tab_n , the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{n_s}}_{\text{Bary}^{(n_s)}} \odot \underbrace{(\mathbf{c}^{n-1_s} \otimes \mathbf{1}_{k_n})}_{\text{Bary}^{(n-1_s)}} \odot \underbrace{(\mathbf{c}^{n-2_s} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\text{Bary}^{(n-2_s)}} \odot \cdots \odot \underbrace{(\mathbf{c}^1_s \otimes \mathbf{1}_{k_n \dots k_2})}_{\text{Bary}^{(1_s)}}.$$

where \mathbf{c}^l_s denotes the vectorized barycentric coefficients related to the l -th variable.

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to its 13th problem) is wrong."



Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1s \cdot {}^2s$)

$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

KAN with rational activation functions

$$\left(\begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1s+1.0)({}^2s+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1s+1.0)({}^2s-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1s-1.0)({}^2s+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1s-1.0)({}^2s-1.0)} \end{array} \right)$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{c}^{{}^1s} \cdot \text{Lag}({}^1s) & \mathbf{c}^{{}^2s} \cdot \text{Lag}({}^2s) \\ \frac{1.0}{{}^1s+1.0} & \frac{1.0}{{}^2s+1.0} \\ \frac{1.0}{{}^1s+1.0} & \frac{1}{{}^2s-1.0} \\ \frac{1}{{}^1s-1.0} & \frac{1.0}{{}^2s+1.0} \\ \frac{1}{{}^1s-1.0} & \frac{1}{{}^2s-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\begin{aligned} \mathbf{c}^{{}^2s} &= \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\ \mathbf{c}^{{}^1s} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix} \end{aligned}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2s} \odot (\mathbf{c}^{{}^1s} \otimes \mathbf{1}_{k_2})$$

$$\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}$$

Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($H = {}^1s \cdot {}^2s$)

$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

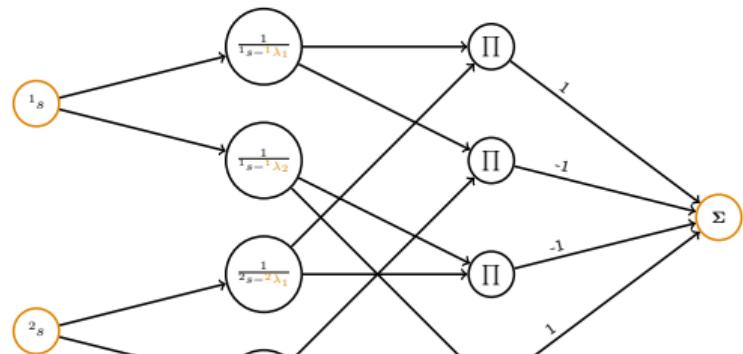
$$\left(\begin{array}{cccc} \mathbf{c} & \mathbf{w} & \mathbf{c} \cdot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1s+1.0)({}^2s+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1s+1.0)({}^2s-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{({}^1s-1.0)({}^2s+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{({}^1s-1.0)({}^2s-1.0)} \end{array} \right)$$

$$\mathbf{c}^{{}^2s} = \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix}$$

$$\mathbf{c}^{{}^1s} = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}$$

$$\mathbf{c}_2 = \mathbf{c}^{{}^2s} \odot (\mathbf{c}^{{}^1s} \otimes \mathbf{1}_{k_2})$$

KAN with rational activation functions



Variables decoupling, KST and KANs

Kolmogorov Superposition Theorem



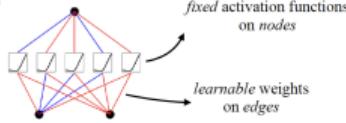
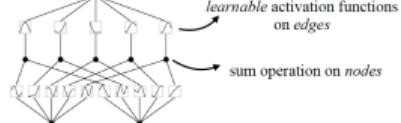
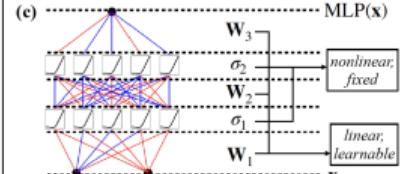
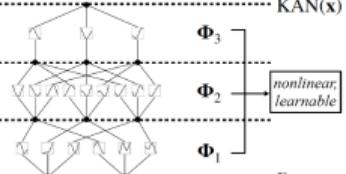
A.G. Vitushkin, "*On Hilbert's thirteenth problem and related questions*", Russian Math. Surveys 59:1, pp. 11–25.

Variables decoupling, KST and KANs

About KANs

KANs features

- ▶ Inspired by the Kolmogorov-Arnold representation theorem
- ▶ The model output is a composition of **sums** and **learnable activation functions** (e.g. splines)
- ▶ Alternate to Multi-Layer Perceptrons (MLP), having fixed activation functions (e.g. ReLU), **inspired by the universal approximation theorem**

Model	Multi-Layer Perceptron (MLP)	Kolmogorov-Arnold Network (KAN)
Theorem	Universal Approximation Theorem	Kolmogorov-Arnold Representation Theorem
Formula (Shallow)	$f(\mathbf{x}) \approx \sum_{i=1}^{N(c)} a_i \sigma(\mathbf{w}_i \cdot \mathbf{x} + b_i)$	$f(\mathbf{x}) = \sum_{q=1}^{2n+1} \Phi_q \left(\sum_{p=1}^n \phi_{q,p}(x_p) \right)$
Model (Shallow)	(a) 	(b) 
Formula (Deep)	$\text{MLP}(\mathbf{x}) = (\mathbf{W}_3 \circ \sigma_2 \circ \mathbf{W}_2 \circ \sigma_1 \circ \mathbf{W}_1)(\mathbf{x})$	$\text{KAN}(\mathbf{x}) = (\Phi_3 \circ \Phi_2 \circ \Phi_1)(\mathbf{x})$
Model (Deep)	(c) 	(d) 

Comparison between MLP and KAN (figure from Z. Liu et al.)



Variables decoupling, KST and KANs

KANs with splines

Building the Kolmogorov-Arnold model as follows

$$F(^1s, ^2s, \dots, ^ns) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^m f_{kj}(^js) \right)$$

$f_{kj} : [0, 1] \mapsto \mathbb{R}$ and $\Phi_k : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions.

The relation is approximated by $k = 1, \dots, d = 2m + 1$ as

$$\hat{F}(^1s, ^2s, \dots, ^ns) = \sum_{k=1}^d \Phi_k \underbrace{\left(\sum_{j=1}^m f_{kj}(^js_i) \right)}_{\theta_{ik}}$$

where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes splines

Variables decoupling, KST and KANs

KANs with splines

Building the Kolmogorov-Arnold model as follows

$$F(^1s, ^2s, \dots, ^ns) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^m f_{kj}(^js) \right)$$

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where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes **splines**

Variables decoupling, KST and KANs

KANs (via Loewner) with rational activation functions

$$\mathbf{H} = {}^1s \cdot {}^2s$$

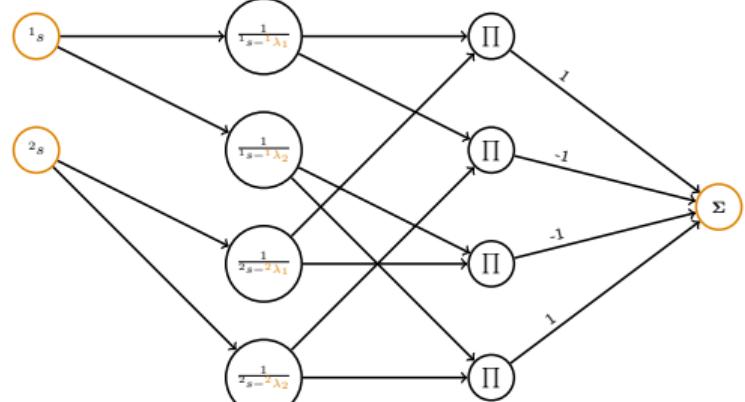
$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{D} = \begin{pmatrix} {}^1s \cdot \mathbf{Lag}({}^1s) & {}^2s \cdot \mathbf{Lag}({}^2s) \\ -\frac{1.0}{{}^1s+1.0} & -\frac{1.0}{{}^2s+1.0} \\ -\frac{1.0}{{}^1s+1.0} & \frac{1}{{}^2s-1.0} \\ \frac{1}{{}^1s-1.0} & -\frac{1.0}{{}^2s+1.0} \\ \frac{1}{{}^1s-1.0} & \frac{1}{{}^2s-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}$$

Equivalent KAN-like with rational activation functions (just \mathbf{D})



Content

Forewords

Multi-variate data, function & Loewner matrix

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Comparisons

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Comparisons

Some competitors

KAN (P/P 2025)

- ▶ Kolmogorov Arnold Network

Rat. app (B/G 2025)

- ▶ p-AAA
- ▶ Dense algebra

MLP (TensorFlow 2025)

- ▶ Multi Layer Perceptron
- ▶ Keras is the high-level API of TensorFlow (by Google)
<https://www.tensorflow.org/guide/keras?hl=en>
- ▶ Dense connected network / ReLU activation / ADAM optim. / 1000 iterations / random init.

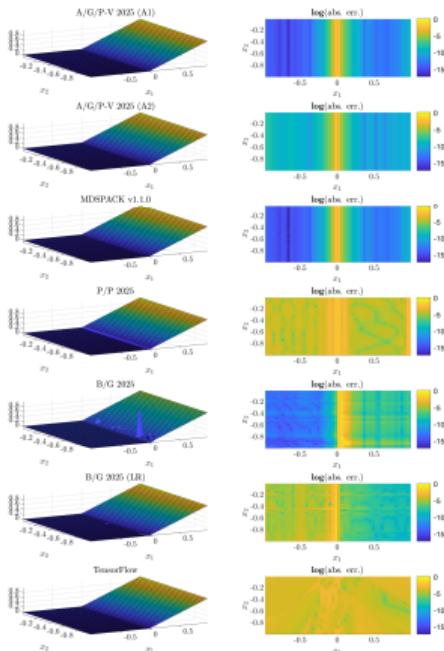
TensorFlow interface (Python code)

```
1 import numpy as np
2 import math
3 import matplotlib.pyplot as plt
4 import tensorflow as tf
5 from keras.models import Sequential
6 from matplotlib import cm
7
8 # Data
9 def f(x):
10     dy = pow(x[:,0],2)*x[:,1]
11     dy = 1/2*x[:,0] + np.abs(x[:,0])) + 1/10*x[:,1] #1
12     dy = x[:,0]*x[:,1] #3
13     dy = np.exp(x[:,0]*x[:,1])/( (pow(x[:,0],2)-1.44)*(pow(x[:,1],2)-1.44) ) #6
14     dy = np.tanh(4*(x[:,0]-x[:,1]))/3 #10
15     dy = pow(np.abs((x[:,0]-x[:,1])),3) #10
16     y = (pow(x[:,0],2) + pow(x[:,1],2)) + x[:,0] - x[:,1] + 1) / (pow(x[:,0],3) + pow(x[:,1],2) + 4) #15
17     return np.transpose(np.array([y]))
18
19 n   = 2
20 N1 = 40
21 N2 = 40
22 x1 = np.linspace(-1, 1, N1)
23 x2 = np.linspace(-1, 1, N2)
24 |
25 # IP
26 N = N1*N2
27 tab = np.zeros([N,1])
```

<https://arxiv.org/abs/2506.04791>
<https://github.com/cpoussot/mLF>

Comparisons

Irrational functions (example #1)



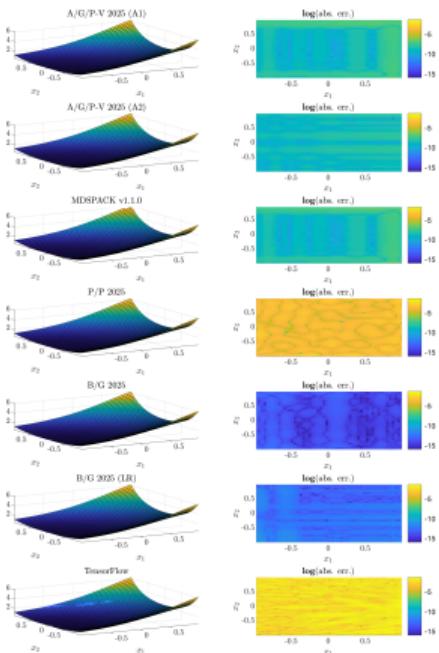
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
1	A/G/P-V 2025 (A1)	1e-11,3	144	0.0171	0.000699	1.39e - 17
	A/G/P-V 2025 (A2)	1e-15,3	160	0.0823	0.000389	4.49e-13
	MDSPACK v1.1.0	1e-11,1e-06	144	0.0623	0.000699	2.43e-17
	P/P 2025	1,0.95,50,0.01,4,6,9	130	0.282	0.0017	3.08e-07
	B/G 2025	0.001,20	612	0.353	0.0288	3.75e-16
	B/G 2025 (LR)	0.001,20,4	480	0.678	0.00147	7.82e-12
	TensorFlow		257	14.6	0.00074	6.31e-08

$$\text{ReLU}(^1s) + \frac{1}{100} {}^2s$$

- ▶ Reference: Personal communication, [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 12.5 KB (40² points)
- ▶ Bounds: $(\begin{array}{cc} -1 & 1 \end{array}) \times (\begin{array}{cc} -1 & -10^{-10} \end{array})$

Comparisons

Irrational functions (example #2)



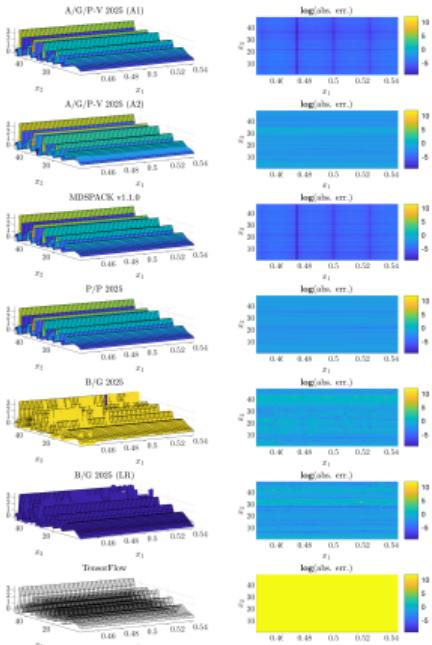
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
2	A/G/P-V 2025 (A1)	1e-06,1	168	0.0235	3.36e-08	4.2e-13	1.7e-08
	A/G/P-V 2025 (A2)	1e-15,1	196	0.0888	4.48e-09	2.58e-12	1.62e-08
	MDSPACK v1.1.0	1e-06,0.0001	168	0.0161	3.36e-08	2.09e-12	1.7e-08
	P/P 2025	1,1,50,0.01,6,6,13	238	0.591	0.000287	1.84e-07	0.001
	B/G 2025	1e-09,20	396	0.0665	4.88e - 14	0	4.32e-16
	B/G 2025 (LR)	1e-09,20,3	480	0.827	1.29e-12	2.22e-16	1.38e-16
	TensorFlow		257	14.6	0.00937	9.75e-05	0.036

$$\exp(\sin(^1 s) + ^2 s^2)$$

- ▶ Reference: Personal communication, [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 12.5 **KB** (40^2 points)
- ▶ Bounds: $(-1 \quad 1) \times (-1 \quad 1)$

Comparisons

Irrational functions (example #34)



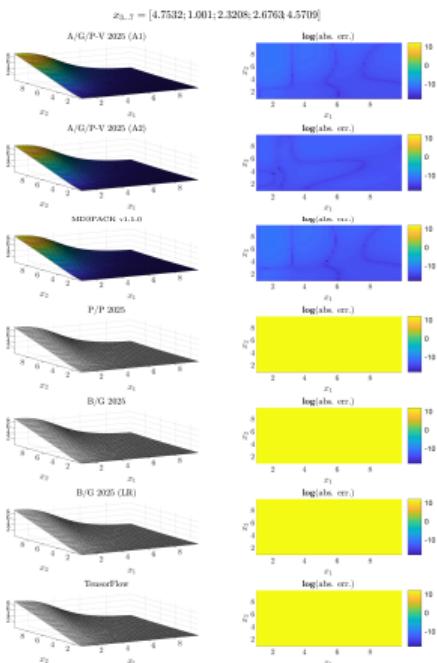
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
34	A/G/P-V 2025 (A1)	1e-10,2	2.32e+03	1.5	4.86e-05	2.74e -
	A/G/P-V 2025 (A2)	1e-15,2	864	0.654	0.137	1.51e-06
	MDSPACK v1.1.0	1e-10,1e-09	2.3e+03	1.58	3.21e - 05	1.87e-08
	P/P 2025	1,0.95,50,0.01,10,12,21	676	79	0.0254	1.36e-05
	B/G 2025	0.001,20	1.22e+03	85.6	2.53	0.000305
	B/G 2025 (LR)	1e-06,20,3	1.36e+03	16.9	4.74	1.23e-05
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\operatorname{Re}(\zeta(1s + i^2 s))$$

- ▶ Riemann ζ function (real part), [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: 1.22 MB (400² points)
- ▶ Bounds: $\left(\frac{9}{20}, \frac{11}{20} \right) \times (1, 50)$

Comparisons

Irrational functions (example #43)



#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
43	A/G/P-V 2025 (A1)	0.0001,1	1.73e + 04	5.58	1.41e-12	1.76e-16
	A/G/P-V 2025 (A2)	1e-15,1	1.73e+04	12.5	2.39e - 13	1.21e - 17
	MDSPACK v1.1.0	0.0001,1e-06	1.73e+04	5.72	1.4e-12	1.49e-16
	P/P 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025 (LR)	NaN	NaN	NaN	NaN	NaN
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\frac{3s^2s^3 + 1}{1s^4 + 2s^2\ 3s + 4s^2 + 5s + 6s^3 + 7s}$$

- ▶ Reference: Personal communication, [none]
- ▶ Domain: \mathbb{R}
- ▶ Tensor size: \mathbb{R}
- ▶ Tensor size: **76.3 MB** (10^7 points)
- ▶ Bounds: $(1 \quad 10)^7$

Content

Forewords

Multi-variate data, function & Loewner matrix

Multi-variate realization

Taming the curse of dimensionality

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Conclusion

Conclusion

Take home message

Main contributions

From any n -th order multi-variate transfer function / data tensor

- ▶ Construct a transfer function in barycentric form
- ▶ **Construct a realization with controlled complexity**
- ▶ **Tame the computational complexity**
 $\mathcal{O}(N^3) \rightarrow \approx \mathcal{O}(N^{2.29, 1.94, \dots})$
- ▶ Two algorithms (direct & iterative)
- ▶ Connection with Kolmogorov theorem

Side effects

[Theo.] Tame COD in KST

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with

A.C. Antoulas [Rice Univ.]

I.V. Goșea [MPI]

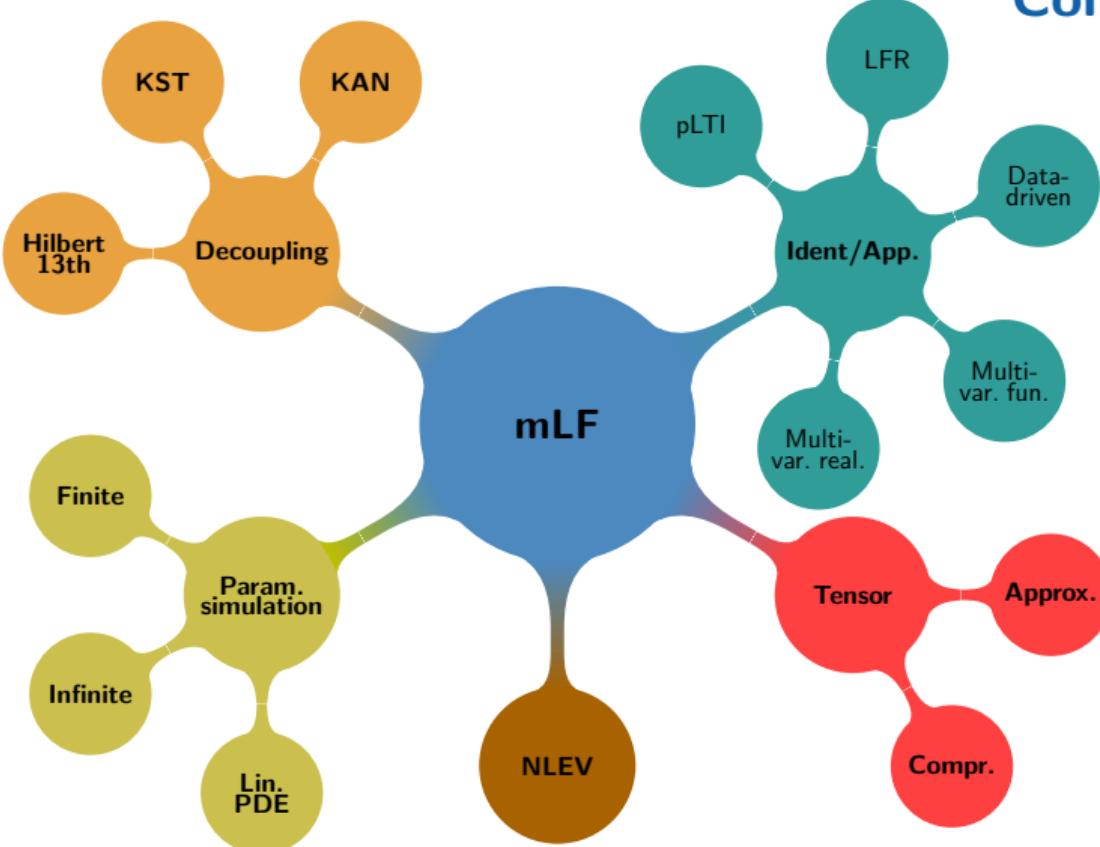
<https://arxiv.org/abs/2405.00495>

<https://github.com/cpoussot/mLF>



Conclusion

Big picture



Temporary page!

\LaTeX was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added to the final page this extra page has been added to receive it.
If you rerun the document (without altering it) this surplus page will go away, because \LaTeX now knows how many pages to expect for this document.