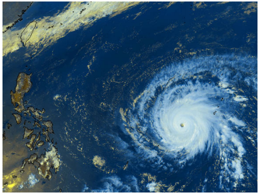
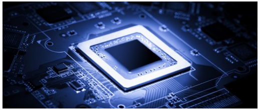
Loewner Framework for data-driven reduced order modeling

... a bridge between realization, approximation and identification

C. Poussot-Vassal
October 16, 2025

Dynamical models are centrals tools in engineering...





Digitalisation and computer-based modeling for

- simulation, optimisation, understanding
- control, estimation, analysis...

However

Finite machine precision, computational burden, memory management and actual solvers

- induces important time consumption
- generate inaccurate results
- limit the class of models to deal with
- limit the amount of data to treat

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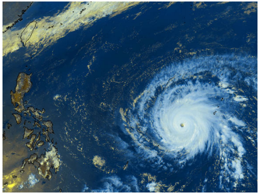

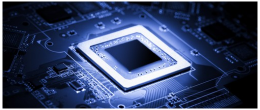

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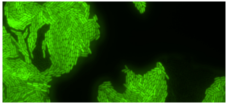
Develop robust and efficient tools to construct simplified dynamical models

... and in systems and control engineering

- **for verification and validation**
  (μ, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
- **for detection**
  (fault isolation, param. estim.)
- **for uncertainty propagation**
  (Multi Disc. Optim., robust optim.)
- **for feedback control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)



Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

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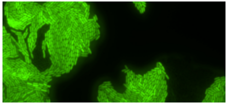


Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

$\Longrightarrow$

Model simplification

Simplified models

- reduced sim. time
- memory saving
- accurate results
- rational model

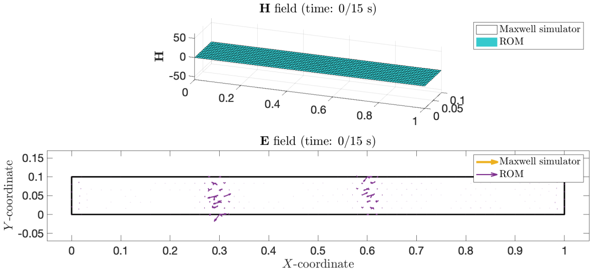
ANTENNA RESEARCH

Antenna models

- **to optimize parameters**
- **for polar computation**

Blend physics from

- **Maxwell equations**
- **Kirchoff equations**

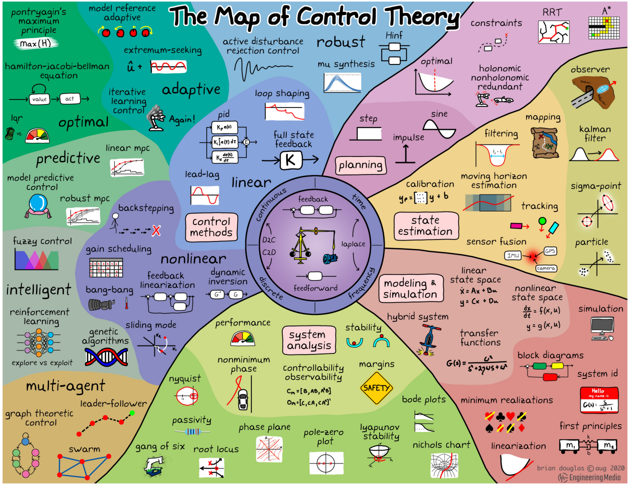


- Replace costly simulations by accurate simple model
- Preserve structure and properties (port-Hamiltonian)
- Allows for geometry optimization

M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, *"A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Forewords

The map of control theory (by Brian Douglas - `https://engineeringmedia.com/`)



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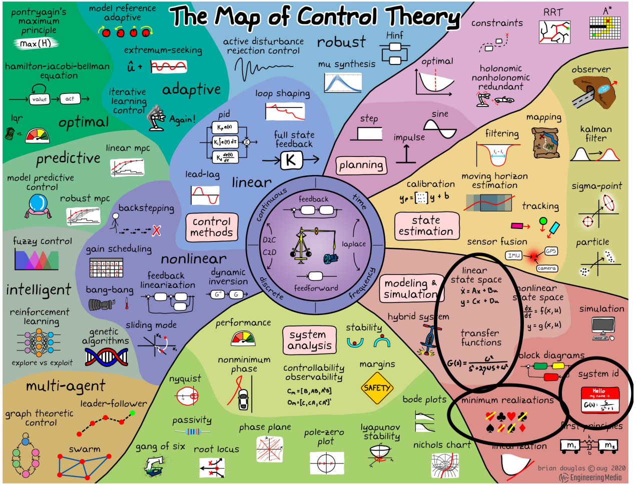

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Some references

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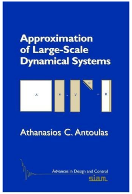
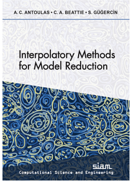


Antoulas/Beattie/Gugercin (2020)



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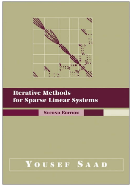

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- B-LTI, Q-LTI [Antoulas/Benner/Gosea/Karachalios/Pontes/Willcox/P-V./...]
- pHS [Van-Dooren/Beattie/Gugercin/Benner/Schwerdtner/Matignon/...]

side. Stability analysis [Vuillemin/P-V.]
side. Control [Kergus/Vuillemin/P-V.]
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appli. Aircraft gust, vibration & flutter [Quero/Vuillemin/Reis/P-V.]
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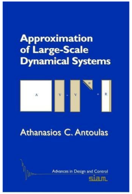

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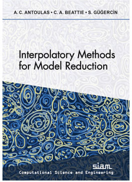

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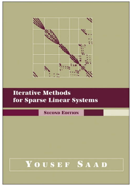

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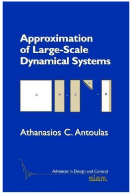

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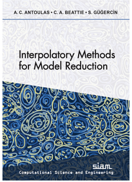

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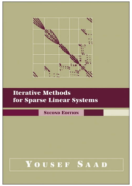

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Forewords

Today's talk

Part 1 (reminder)

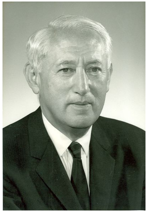
- Linear dynamical systems
- Realization and transfer functions

Part 2 (Loewner)

- Realization minimality
- Data-driven approximation
- Barycentric form

Part 3 (Loewner extended)

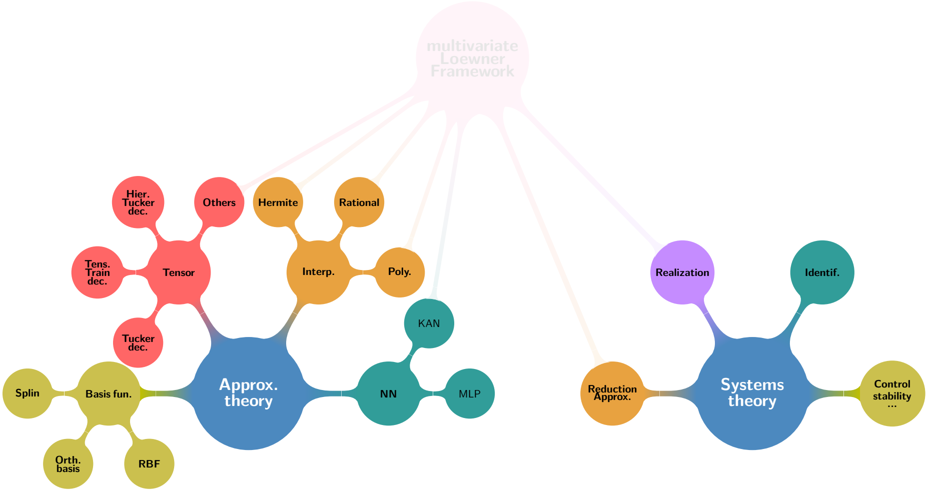
- Linear passive model (& pH)
- Linear parametric model
- Nonlinear model (quadratic & bilinear)



Karel Löwner (Czech)
1893 - 1968
Ph.D. advisor: G.A. Pick

Forewords

Approximation & systems theory... where we stand



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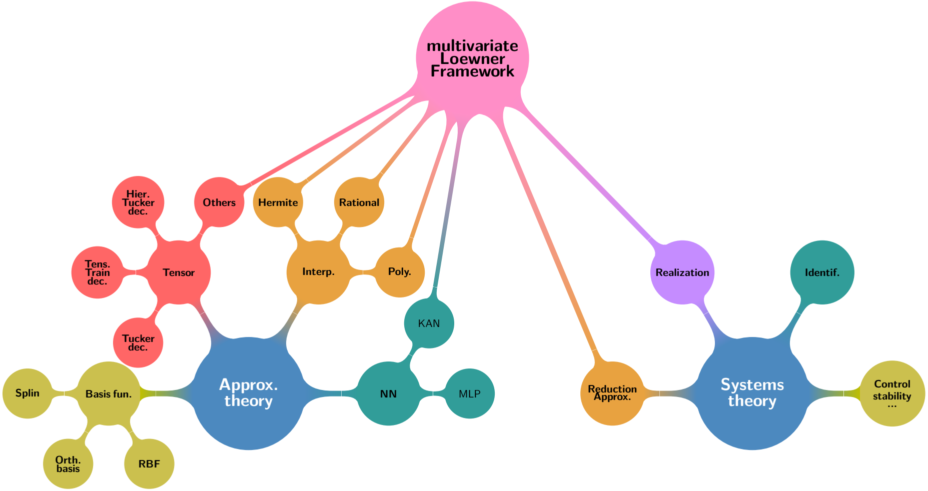
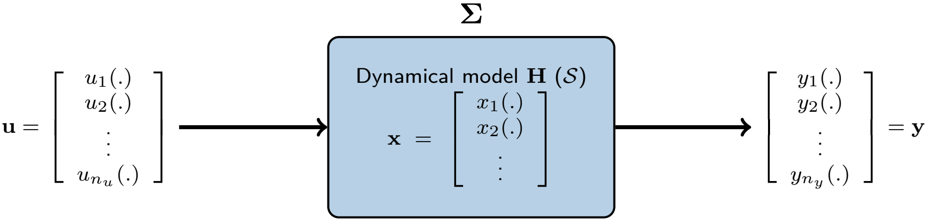
Approximation & systems theory... where we stand



Content

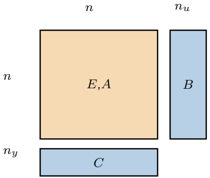
A dynamical model H (or $\mathcal{S}$) is a function mapping input u to output y signals of a system Σ



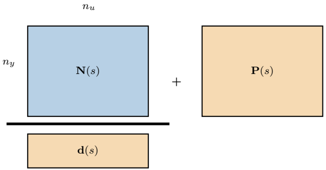
Let us stick (mainly) to linear systems only

Linear dynamical systems

Realization and transfer functions



$$\mathcal{S}:\left\{\begin{array}{rcl} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t)+B\mathbf{u}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) \end{array}\right.$$



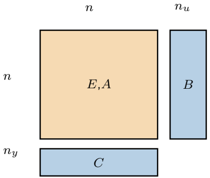
$$\begin{array}{rcl} \mathbf{H}(s) & = & C\left(sE-A\right)^{-1}B \\ & = & \mathbf{N}(s)/\mathbf{d}(s)+\mathbf{P}(s) \end{array}$$

Realizations

- E, A, B, C are (real) matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Realizations are infinite
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

Transfer functions

- $\mathbf{H}$ is a (complex) function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Transfer functions are unique
- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$
  $\mathbf{y}(s) \in \mathbb{C}^{n_y}$



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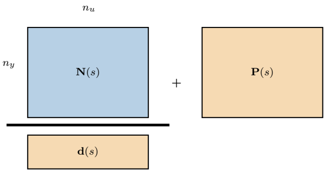

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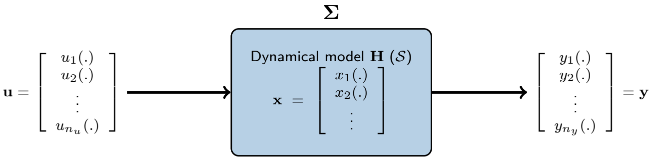
Linear dynamical systems
Model, data and structures

Structures
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



Structures

- **L-ODE**
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- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1}$$

ODE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x \end{array}$$

Singularities λ and zeros z

$$\begin{array}{rcl} \lambda_{\mathcal{S}} & = & \texttt{eig(A,E)} \\ & = & \Lambda(-1, 1) \\ & = & \{-1\} \\ z_{\mathcal{S}} & = & \texttt{eig([A B; C D],blkdiag(E,zeros(ny,nu))} \\ & = & \Lambda\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \\ & = & \{\infty, \infty\} \end{array}$$

Structures

- L-ODE
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Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}_1} = \Lambda(-1, 1) = \{-1\}$$

Structures

- L-ODE
- **L-ODE / DAE-1**
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- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + 2u \\ 0 & = & -x_2 + 2u = x_2 - 2u \\ y & = & x_1 + x_2 \end{array}$$

Singularities of matrix pencil (A, E)[a]

$$\lambda_{\mathcal{S}_2} = \Lambda\left(\left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]\right) = \{-1, \infty\}$$

[a] $B^\top = \begin{bmatrix} 2 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Linear dynamical systems

Linear finite dimensional models

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- L-PDE

Transfer function

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DAE index-1 realization $\mathcal{S}_2$ (canonical form)

$$\left(\left[\begin{array}{c|c} A_1 = -1 & \\ \hline & I_{n_2} = 1 \end{array} \right], \left[\begin{array}{c|c} I_{n_1} = 1 & \\ \hline & N = 0 \end{array} \right] \right)$$

Index is the k-nilpotent degree of N

- Finite dynamic modes
  $n_1 = 1$
- Infinite dynamic (impulsive) modes
  $\mathbf{rank}\ (E) - n_1 = \mathbf{rank}\ (N) = 1 - 1 = 0$
- Non dynamic modes
  $n - \mathbf{rank}\ (E) = 2 - 1 = 1$

Linear dynamical systems

Linear finite dimensional models

Structures

- L-ODE
- L-ODE / DAE-1
- **L-DAE**
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}} = \{-1, \infty, \infty\}$$

- Finite dynamic modes $n_1 = 1$
- Impulsive modes $\mathbf{rank}\,(E) - n_1 = 2 - 1 = 1$
- Non dynamic modes $n - \mathbf{rank}\,(E) = 3 - 2 = 1$

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Linear dynamical systems

Linear finite dimensional models (about observability, controllability, minimality...)

Structures

- **L-ODE**
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- L-PDE

- Reachability matrix

$$\mathcal{R}_n(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

- Observability matrix

$$\mathcal{O}_n(A,C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2 C^\top & \cdots & (A^\top)^n C^\top \end{bmatrix}^\top$$

- Minimality: both controllable and observable.
- Connection with Markov parameters

$$H_0 = D \; , \; H_k = CA^{k-1}B \; (k > 1)$$

$$\mathcal{O}_n\mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- **L-DDE**
- L-PDE

Transfer function

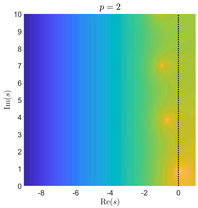
$$\mathbf{H}(s) = \frac{1}{s + e^{-ps}}$$

L-DDE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x(t-p) + u \\ y & = & x \end{array}$$

Singularities (periodic)

$$\lambda_{\mathcal{S}} = \{\omega \text{ s.t. } s + \cos(s) - \imath \sin(s) = 0\}$$



Structures

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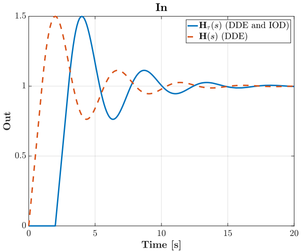
Transfer function

$$\mathbf{H}_\tau(s) = \frac{1}{s + e^{-ps}} e^{-2s}$$

L-DDE realization $\mathcal{S}$

$$\begin{aligned} \dot{x} &= -x(t-p) + u(t-2) \\ y &= x \end{aligned}$$



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- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$

L-PDE "realization" $\mathcal{S}$

$$\begin{array}{rcl} \frac{\partial \tilde{y}(x,t)}{\partial x} + 2x\frac{\partial \tilde{y}(x,t)}{\partial t} & = & 0 \\ \tilde{y}(x,0) & = & 0 \\ \tilde{y}(0,t) & = & \frac{1}{\sqrt{t}} \star \tilde{u}_f(0,t) \\ \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0,s) & = & u_f(0,s) \end{array}$$

Singularities

$$\lambda_{\mathcal{S}} = \{0, \lambda_1, \overline{\lambda_1}\}$$

Structures

- L-ODE
- L-ODE / DAE-1
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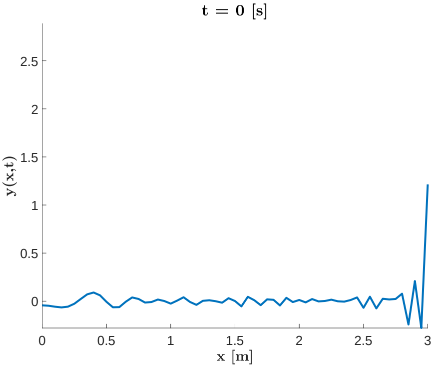
- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$



Linear dynamical systems

Why all this? What is common?

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$

Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$

$\boldsymbol{\Sigma}$

$$\mathbf{u} = \begin{bmatrix} u_1(.) \\ u_2(.) \\ \vdots \\ u_{n_u}(.) \end{bmatrix} \longrightarrow \boxed{\text{Dynamical model } \mathbf{H}\ (\mathcal{S}) \quad \mathbf{x} = \begin{bmatrix} x_1(.) \\ x_2(.) \\ \vdots \end{bmatrix}} \longrightarrow \begin{bmatrix} y_1(.) \\ y_2(.) \\ \vdots \\ y_{n_y}(.) \end{bmatrix} = \mathbf{y}$$

Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Data

- Time-domain
- Frequency-domain

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$

$\boldsymbol{\Sigma}$

$$\mathbf{u} = \begin{bmatrix} u_1(.) \\ u_2(.) \\ \vdots \\ u_{n_u}(.) \end{bmatrix} \longrightarrow \boxed{\text{Dynamical model } \mathbf{H}\ (\mathcal{S}) \quad \mathbf{x} = \begin{bmatrix} x_1(.) \\ x_2(.) \\ \vdots \end{bmatrix}} \longrightarrow \begin{bmatrix} y_1(.) \\ y_2(.) \\ \vdots \\ y_{n_y}(.) \end{bmatrix} = \mathbf{y}$$

(Time-domain) $\{t_i, \mathbf{G}(t_i)\}_{i=1}^{N}$

(Frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^{N}$

Data

Content

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \dots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \dots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \dots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

Loewner Matrix

Given a row / left array of pairs of complex numbers $\{\mu_i, \mathbf{v}_i\}$, $i = 1, \cdots, q$, and a column / right array of pairs of complex numbers $\{\lambda_j, \mathbf{w}_j\}$, $j = 1, \cdots, k$ with μ_i and λ_j distinct, the associated Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}.$$

If there is an underlying function $\mathbf{H}$ then

$$\mathbf{w}_j = \mathbf{H}(\lambda_j) \text{ and } \mathbf{v}_i = \mathbf{H}(\mu_i).$$

We will present the Loewner framework in connection with

(i) rational interpolation and consequently in connection with

(ii) reduced-order modeling of linear dynamical systems given frequency domain data.

The Loewner matrix rank encodes the minimal complexity of the solutions of the interpolation problem.

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

$$P := \{(x_i, y_i) \ : \ i = 1, \cdots, N, \ x_i \neq x_j, \ i \neq j\}$$

The idea behind the present approach to rational interpolation is to use a formula for rational interpolants which is similar to the one defining the Lagrange polynomial. First we partition the array P in two disjoint subarrays:

$$\begin{array}{lcl} P_c & := & \{(\lambda_j, \mathbf{w}_j) \ j = 1, \ldots, k\} \\ P_r & := & \{(\mu_i, \mathbf{w}_i) \ i = 1, \ldots, q\} \end{array}$$

e.g. $N = k + q$

$$\begin{array}{lll} \lambda_j = x_j & \mathbf{w}_j = y_j & j = 1, \ldots, k \\ \mu_i = x_{k+i} & \mathbf{v}_i = y_{k+i} & i = 1, \ldots, q \end{array}$$

Using array P_c , for constants α_j and $\mathbf{w}_j$, consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^{k} \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0 \ , \ \alpha_j \neq 0$$

For constants α_j and $\mathbf{w}_j$, consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^{k} \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0 \ , \ \alpha_j \neq 0$$

solving for $\mathbf{H}$, we get

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}}$$

It follows that

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j$$

ensures P_c constraints by construction. This is the barycentric (rational) Lagrange interpolation formula.

The free parameters α_j can be determined so that the additional constraints contained in array P_r are satisfied:

$$\mathbf{H}(\mu_i) = \mathbf{v}_i$$

Then it follows $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k.$$

Lemma

Given the rational function $\mathbf{H}$ and an array of points P, where $y_j = \mathbf{H}(x_j)$ and x_j is not a pole of $\mathbf{H}$, let $\mathbb{L}$ be a $q \times k$ Loewner matrix, for some partitioning P_c, P_r of P. Then

$$q, k \geq \mathbf{deg}(\mathbf{H}) \Rightarrow \mathbf{rank}\ \mathbb{L} = \mathbf{deg}(\mathbf{H})$$

Loewner

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

$$\begin{cases} P_c & := & \{(\lambda_j; \mathbf{w}_j)\,,\ j = 1, \dots, k\} \\ P_r & := & \{(\mu_i; \mathbf{v}_i)\,,\ i = 1, \dots, q\} \end{cases}$$

Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q \times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

$$M\mathbb{L} - \mathbb{L}\Lambda = V - W$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}}$$

Null space

$$\mathbf{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k$$

Data generated from $\mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2), with $k=3$, $q=4$

$$\left.\begin{array}{rcl} \lambda_j & = & [1,3,5] \\ \mu_i & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_j & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_i & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix} \in \mathbb{C}^{4\times 3}$$

Null space

$$\mathbf{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Consider system $\mathbf{H}$ in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{, where } \mathbf{q}_i(s) = \prod_{i'=1, i'\neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \ldots, \lambda_k\}, \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and solving

$$\mathbb{L}\mathbf{c} = 0$$

leads to $\mathbf{H}$ in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}^\top \mathbf{w}}_{C} \underbrace{\begin{bmatrix} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c}^\top \end{bmatrix}^{-1}}_{\Phi(s)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s-\lambda_1 & \lambda_2 - s & \\ s-\lambda_1 & & \lambda_3 - s \\ \vdots & & & \ddots \end{bmatrix} \in \mathbb{R}^{n\times(n+1)} \text{ and } \Phi(s) \in \mathbb{C}^{(n+1)\times(n+1)}$$

Consider system $\mathbf{H}$ in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{, where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \ldots, \lambda_k\}, \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and solving

$$\mathbb{L}\mathbf{c} = 0$$

leads to $\mathbf{H}$ in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}^\top \mathbf{w}}_{C} \underbrace{\begin{bmatrix} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c}^\top \end{bmatrix}^{-1}}_{\mathbf{\Phi}(s)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s - \lambda_1 & \lambda_2 - s & \\ s - \lambda_1 & & \lambda_3 - s \\ \vdots & & \ddots \end{bmatrix} \in \mathbb{R}^{n \times (n+1)} \text{ and } \mathbf{\Phi}(s) \in \mathbb{C}^{(n+1)\times(n+1)}$$

Loewner

Loewner barycentric examples (simple case, `CAS=1`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\mathbb{L} = \begin{bmatrix} \frac{\frac{2}{3}-1}{2-1} & \frac{\frac{2}{3}-\frac{1}{2}}{2-3} \\ \frac{\frac{2}{5}-1}{4-1} & \frac{\frac{2}{5}-\frac{1}{2}}{4-3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

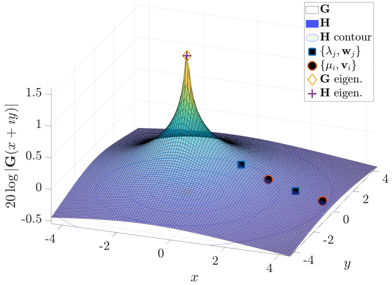
Loewner

Loewner barycentric examples (simple case, `CAS=1`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 2$, $\mathbf{H}(s) = C\mathbf{\Phi}(s)^{-1}B = \mathbf{G}(s)$



$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\ker(\mathbb{L}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{c}$$

$$C = \mathbf{c}^\top \mathbf{w} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \mathbf{L}_{s,\lambda,1} \\ \mathbf{c}^\top \end{bmatrix} = \begin{bmatrix} s-1 & 3-s \\ -1 & 2 \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\mathbb{L} = \begin{bmatrix} -\frac{1}{5}+\frac{3}{5}\imath & -\frac{1}{34}+\frac{13}{34}\imath & \frac{5}{17}+\frac{3}{17}\imath \\ \frac{17}{290}+\frac{59}{290}\imath & \frac{57}{986}+\frac{75}{986}\imath & \frac{31}{986}-\frac{63}{986}\imath \\ -\frac{46}{265}+\frac{108}{265}\imath & -\frac{37}{901}+\frac{209}{901}\imath & \frac{118}{901}+\frac{64}{901}\imath \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 3$

$$C = \begin{bmatrix} c_1\mathbf{w}_1 & c_2\mathbf{w}_2 & c_3\mathbf{w}_3 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} s-\imath & 2\imath - s & \\ s-\imath & & -2+\imath-s \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Resulting in

$$\frac{-\imath}{-s^2\imath + s(1-\imath) + (1-\imath)} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

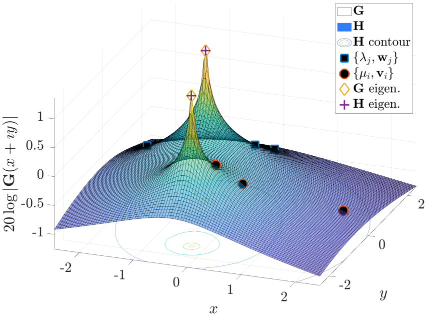
$$\ker(\mathbb{L}) = \begin{bmatrix} 13/17 - 16\imath/17 \\ -2+2\imath \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Loewner

Loewner barycentric examples (complex dynamical tippe top case, `CAS=3`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

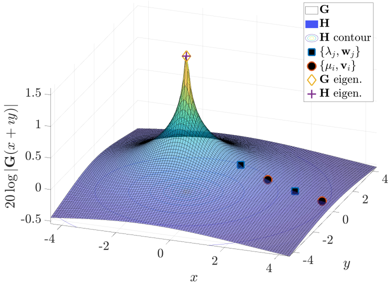
$$\ker(\mathbb{L}) = \begin{bmatrix} 13/17 - 16\imath/17 \\ -2+2\imath \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

```
% Model (CAS=1)
S     = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses
la    = [1 3]; k     = length(la);
mu    = [2 4]; q     = length(mu);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
LL    = lf.loewnerMatrix(la,mu,W,V);
c     = null(sym(LL));
% Realization
C     = c.'.*W(:).';
B     = zeros(k,1); B(end) = 1;
PHI = [s-la(1)*ones(k-1,1) diag(la(2:end))-s*eye(k-1); c.'];
Hr    = simplify(C*(PHI\B));
```

```
% Model (CAS=1)
S     = ss(tf(2,[1  1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses
la    = [1  3];  k     = length(la);
mu    = [2  4];  q     = length(mu);
for  ii  = 1:k;  W(1,1,ii)  = G(la(ii));  end
for  ii  = 1:q;  V(1,1,ii)  = G(mu(ii));  end

% Loewner
LL    = lf.loewnerMatrix(la,mu,W,V);
c     = null(sym(LL));
% Realization
C     = c.'.*W(:).';
B     = zeros(k,1);  B(end) = 1;
PHI  = [s-la(1)*ones(k-1,1)  diag(la(2:end))-s*eye(k-1);  c.'];
Hr    = simplify(C*(PHI\B));
```



SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$
\begin{array}{ll}
\{\lambda_j, \mathbf{w}_j\} & j = 1, \dots, k \\
\{\mu_i, \mathbf{v}_i^\top\} & i = 1, \dots, q
\end{array}
$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$
\begin{array}{rcll}
\mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \dots, k \\
\mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \dots, q
\end{array}
$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

MIMO tangential interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{cc} \{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{l}_i^\top, \mathbf{v}_i^\top\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rclc} \mathbf{H}(\lambda_j)\mathbf{r}_j & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{l}_i^\top \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \ldots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

The right data can be expressed as:

$$
\begin{array}{rcl}
\boldsymbol{\Lambda} & = & \mathbf{diag}\,[\lambda_1,\ldots,\lambda_k] \in \mathbb{C}^{k\times k},\\
\mathbf{R} & = & \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k\end{bmatrix} \in \mathbb{C}^{n_u\times k}\\
\mathbf{W} & = & \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k\end{bmatrix} \in \mathbb{C}^{n_y\times k}
\end{array}
$$

and the left data can be expressed as:

$$
\begin{array}{rcl}
\mathbf{M} & = & \mathbf{diag}\,[\mu_1,\ldots,\mu_q] \in \mathbb{C}^{q\times q}\\
\mathbf{L}^\top & = & \begin{bmatrix}\mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q\end{bmatrix} \in \mathbb{C}^{n_y\times q}\\
\mathbf{V}^\top & = & \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q\end{bmatrix} \in \mathbb{C}^{n_u\times q}
\end{array}
$$

The Loewner matrix in this case is

$$
\mathbb{L} = \begin{bmatrix}
\frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1-\lambda_k}\\
\vdots & \ddots & \vdots\\
\frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q-\lambda_k}
\end{bmatrix} \in \mathbb{C}^{q\times k}
$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L} - \mathbb{L}\boldsymbol{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

The right data can be expressed as:

$$\begin{array}{rcl}\mathbf{\Lambda} & = & \mathbf{diag}\,[\lambda_1,\dots,\lambda_k] \in \mathbb{C}^{k\times k},\\ \mathbf{R} & = & \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k\end{bmatrix} \in \mathbb{C}^{n_u\times k}\\ \mathbf{W} & = & \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k\end{bmatrix} \in \mathbb{C}^{n_y\times k}\end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcl}\mathbf{M} & = & \mathbf{diag}\,[\mu_1,\dots,\mu_q] \in \mathbb{C}^{q\times q}\\ \mathbf{L}^\top & = & \begin{bmatrix}\mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q\end{bmatrix} \in \mathbb{C}^{n_y\times q}\\ \mathbf{V}^\top & = & \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q\end{bmatrix} \in \mathbb{C}^{n_u\times q}\end{array}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix}\frac{\mathbf{v}_1^\top\mathbf{r}_1-\mathbf{l}_1^\top\mathbf{w}_1}{\mu_1-\lambda_1} & \dots & \frac{\mathbf{v}_1^\top\mathbf{r}_k-\mathbf{l}_1^\top\mathbf{w}_k}{\mu_1-\lambda_k}\\ \vdots & \ddots & \vdots\\ \frac{\mathbf{v}_q^\top\mathbf{r}_1-\mathbf{l}_q^\top\mathbf{w}_1}{\mu_q-\lambda_1} & \dots & \frac{\mathbf{v}_q^\top\mathbf{r}_k-\mathbf{l}_q^\top\mathbf{w}_k}{\mu_q-\lambda_k}\end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L}-\mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R}-\mathbf{L}\mathbf{W}$.

The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The shifted Loewner matrix is:

$$\mathbb{M} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{M} - \mathbb{M}\mathbf{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\mathbf{\Lambda}$$

If data are sampled from $\mathbf{G}(s) = C(sE-A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q\times n}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1} B\mathbf{r}_1, \dots, (\lambda_k E - A)^{-1} B\mathbf{r}_k,\right] \in \mathbb{C}^{n\times k}$$

be the generalized tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{M}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A (\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A [\mathcal{R}_k]_j \end{aligned}$$

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$$\begin{array}{rcl} [\mathbb{L}]_{ij} & = & \dfrac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ & = & -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B\mathbf{r}_j \\ & = & -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{array}$$

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$$\begin{aligned} [\mathbb{M}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A(\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A[\mathcal{R}_k]_j \end{aligned}$$

Loewner

Loewner realization

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is **a minimal descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}\,[\xi\mathbb{L} - \mathbb{M}] = \mathbf{rank}\,[\mathbb{L}, \;\mathbb{M}] = \mathbf{rank}\begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L}, \;\mathbb{M}] = Y\Sigma_l\tilde{X}^\top, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = \tilde{Y}\Sigma_r X^H, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{M}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

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$$[\mathbb{L},\ \mathbb{M}] = Y\Sigma_l\tilde{X}^\top, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = \tilde{Y}\Sigma_r X^H, \ Y, X \in \mathbb{C}^{N\times n}.$$

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$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{M}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

Being known the full state realization $\mathcal{S}$

$$
\begin{array}{rcl}
E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + B\mathbf{u}(t) \\
\mathbf{y}(t) & = & C\mathbf{x}(t) + D\mathbf{u}(t)
\end{array}
$$

The Petrov-Galerkin reduced order model $\hat{\mathcal{S}}$ is obtained with projectors $W, V \in \mathbb{C}^{n\times r}$ as

$$
\begin{array}{rcl}
W^\top E V \dot{\hat{\mathbf{x}}}(t) & = & W^\top A V \hat{\mathbf{x}}(t) + W^\top B\mathbf{u}(t) \\
\hat{\mathbf{y}}(t) & = & CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t)
\end{array}
$$

The above equations assume $W = \mathbf{span}\,(\mathcal{W}), V = \mathbf{span}\,(\mathcal{V})$ selected such that they meet the approximation objective:

$$
\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t), \quad , \hat{\mathbf{x}}_0 = W^\top \mathbf{x}_0
$$

Choosing two different bases V' and W' that respectively span the same subspaces $\mathcal{V}$ and $\mathcal{W}$ result in the same reconstructed solution $\mathbf{x}(t)$. Thus, subspaces are relevant, not basis.

Now recall,

$$\mathcal{O}E\mathcal{R} = \mathbb{L} \quad \text{and} \quad \mathcal{O}A\mathcal{R} = \mathbb{M},$$

where the generalized reachability and observability matrices are (complex conjugation trick for realness)

$$\mathcal{R} = \begin{bmatrix} \mathbf{r}_1^\top B^\top (\lambda_1 E - A)^{-\top} \\ \overline{\mathbf{r}}_1^T B^\top (\overline{\lambda}_1 E - A)^{-\top} \\ \vdots \\ \mathbf{r}_{k/2}^\top B^\top (\lambda_{k/2} E - A)^{-\top} \\ \overline{\mathbf{r}}_{k/2} B^\top (\overline{\lambda}_{k/2} E - A)^{-\top} \end{bmatrix}^\top \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{l}_1 C(\mu_1 E - A)^{-1} \\ \overline{\mathbf{l}}_1 C(\overline{\mu}_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_{q/2} C(\mu_{q/2} E - A)^{-1} \\ \overline{\mathbf{l}}_{q/2} C(\overline{\mu}_{q/2} E - A)^{-1} \end{bmatrix}.$$

$$(Y^\top J^H \mathcal{O})^\top E(\mathcal{R}JX) = \mathbb{L}, \quad \text{and} \quad (Y^\top J^H \mathcal{O})^\top A(\mathcal{R}JX) = \mathbb{M},$$

$$J_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ 1 & -\imath \end{bmatrix} \quad \text{and} \quad J = I_{k/2} \otimes (J_0 \otimes I_m)\,,$$

M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, *"APort-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Therefore,

$$\underbrace{(Y^\top J^H \mathcal{O})^\top}_{W^\top} E \underbrace{(\mathcal{R} J X)}_{V} = \mathbb{L}, \quad \text{and} \quad \underbrace{((Y^\top J^H \mathcal{O})^\top}_{W^\top} A \underbrace{(\mathcal{R} J X)}_{V} = \mathbb{M},$$

Therefore,

$$W = Y^\top J^H \mathcal{O} \quad \text{and} \quad V = \mathcal{R} J X.$$

and

$$\mathbf{x}(t) \approx V \hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R} J X.$$

can be recovered (approximated) solely from ROM simulation.

M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, *"APort-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \dots, k$; $i = 1, \dots, q$.

Rational interpolation

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

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$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

- underlying rational (r) order

$$\begin{aligned} r &= \mathbf{rank}\ (\xi\mathbb{L} - \mathbb{M}) \\ &= \mathbf{rank}\ ([\mathbb{L}, \mathbb{M}]) \\ &= \mathbf{rank}\ ([\mathbb{L}^H, \mathbb{M}^H]^H) \end{aligned}$$

- and McMillan (ν) order

$$\nu = \mathbf{rank}\ (\mathbb{L})$$

- $\mathbb{L}$ and $\mathbb{M}$ are input-output independents.
- Minimal realization
- If $\mathcal{S}$ is known, then internal state $\mathbf{x} \in \mathbb{R}^n$ is recovered by

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R}JX.$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

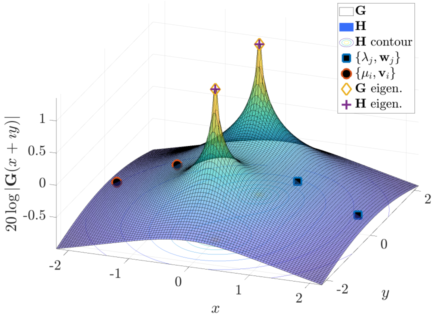
$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



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$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\textbf{rank } (\xi\mathbb{L} - \mathbb{M}) = r$$

$$\textbf{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, $(\mathbb{M}, \mathbb{L})$ pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

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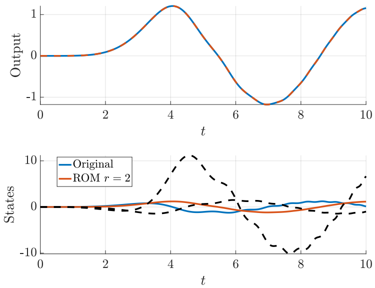
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal variables



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

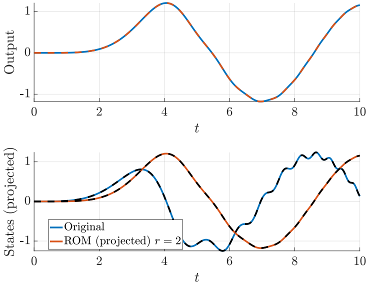
$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

> Rational function satisfies
>
> $$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal **projected** variables



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

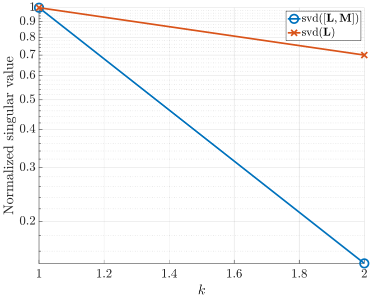
Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization rect.: $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^\dagger \mathbf{V}$



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\textbf{rank } (\xi\mathbb{L} - \mathbb{M}) = r$$

$$\textbf{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{\dagger}\mathbf{V} \\ &= \frac{1}{s^2 - 4.650e - 16s + 1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \end{bmatrix}$$

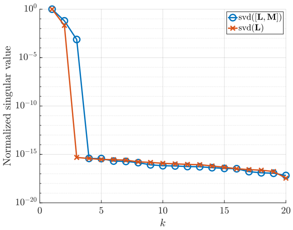
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

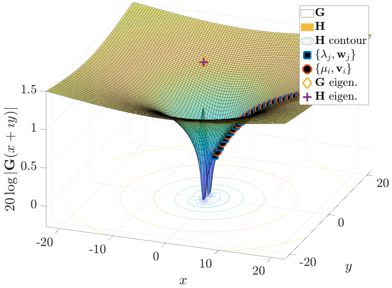
$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

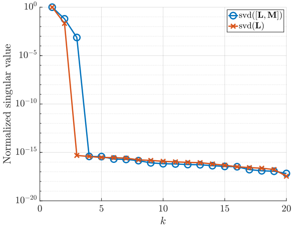
Realization $n = 20$: $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$



$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Loewner examples (lot of data case, `CAS=6`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

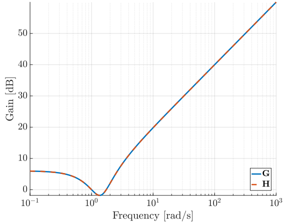
$$\textbf{rank}\ (\xi\mathbb{L} - \mathbb{M}) = r = 3$$

$$\textbf{rank}\ (\mathbb{L}) = \nu = 2$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots 20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots 20} = [1.5, 2.5, \ldots, 20.5]$$



Loewner

Loewner examples (lot of data case, `CAS=6`)

Rational function satisfies

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$$\textbf{rank } (\xi\mathbb{L} - \mathbb{M}) = r = 3$$

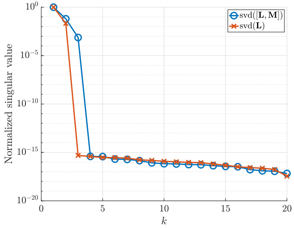
$$\textbf{rank } (\mathbb{L}) = \nu = 2$$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}X(-sY^\top\mathbb{L}X + Y^\top\mathbb{M}X)^{-1}Y^\top\mathbf{V} \\ &= \frac{s^2+s+2}{s+1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\textbf{rank } (\xi\mathbb{L} - \mathbb{M}) = r$$

$$\textbf{rank } (\mathbb{L}) = \nu$$

$$\mathbf{H}(s) = \frac{(1 + 2.22e - 16\imath)}{s^2 + (1+\imath)s + (1+\imath)}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2 + \imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\hat{\mathbb{L}} = \begin{bmatrix} -0.207 + 0.9568\imath & -0.1276 - 0.0294\imath \\ -0.0438 - 0.0818\imath & 0.039 - 0.1089\imath \end{bmatrix}$$

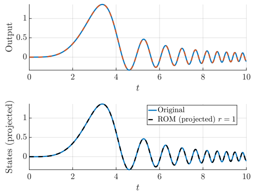
$$\hat{\mathbb{M}} = \begin{bmatrix} 0.4738 - 0.0560\imath & 0.0089 - 0.3392\imath \\ 0.2253 - 0.1607\imath & -0.0342 + 0.1308\imath \end{bmatrix}$$

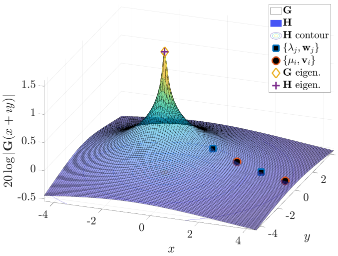
```
% Model (CAS = 1)
S = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses, tangent
la = [1 3];
mu = [2 4];
k = length(la);
q = length(mu);
R = ones(1,k);
L = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```

Loewner

Code break (https://github.com/cpoussot/lf)

```
% Model (CAS = 1)
S     = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses, tangent
la    = [1 3];
mu    = [2 4];
k     = length(la);
q     = length(mu);
R     = ones(1,k);
L     = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```





Content

Loewner extensions

More structures and properties

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE
- L-pH
- pL-DAE
- B-DAE
- Q-DAE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$
- (Q, J, R, G, P, N, S) and $\mathbf{H}(s)$
- (E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
- (A, B, C, N) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$
- (A, B, C, Q) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$

Loewner extensions

More structures and properties

Structures	Model
L-ODE	(A,B,C) and $\mathbf{H}(s)$
L-ODE / DAE-1	(A,B,C,D) and $\mathbf{H}(s)$
L-DAE	(E,A,B,C) and $\mathbf{H}(s)$
L-DDE	$(A_i\dots,B,C,\tau_i)$ and $\mathbf{H}(s)$
L-PDE	$\mathbf{H}(s)$
L-pH	(Q,J,R,G,P,N,S) and $\mathbf{H}(s)$
pL-DAE	(E_j,A_j,B_j,C_j) and $\mathbf{H}(s,p_j)$
B-DAE	(A,B,C,N) and $\mathbf{H}(s_1,s_2,\dots,s_k)$
Q-DAE	(A,B,C,Q) and $\mathbf{H}(s_1,s_2,\dots,s_k)$

Loewner extensions

Passive & port-Hamiltonian

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Passivity

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}\end{aligned}$$

$$\mathbf{\Phi_H}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$$

satisfy $\mathbf{\Phi_H}(\imath\omega) > 0$, $\texttt{Re}(\lambda_S) < 0$ and $D \succ 0$.

pH

$$\begin{aligned}\dot{\mathbf{x}} &= (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} &= (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{aligned}$$

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

Loewner extensions

Passive & port-Hamiltonian

Structures

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- B-DAE
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Passivity

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u}\end{array}$$

$$\mathbf{\Phi}_{\mathbf{H}}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$$

satisfy $\mathbf{\Phi}_{\mathbf{H}}(\imath\omega) > 0$, $\mathrm{Re}(\lambda_{\mathcal{S}}) < 0$ and $D \succ 0$.

pH

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{array}$$

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Loewner extensions

Passive & port-Hamiltonian

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- pL-DAE
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- Q-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Passivity

- $\mathtt{Re}(\Lambda(A)) = -1 < 0$
- $D = 2 \succ 0$
- $\mathbf{\Phi_H}(\imath\omega) = \mathbf{H}(\imath\omega) + \mathbf{H}(-\imath\omega)^\top > 0$

$$\begin{array}{rcl} \mathbf{\Phi_H}(s) & = & \frac{4\left(s^2-2\right)}{s^2-1} \\ \mathbf{\Phi_H}(\imath\omega) & = & \frac{4\left(\omega^2+2\right)}{\omega^2+1} \end{array}$$

Structures

- L-ODE
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$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

L-pH realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} = (0-1)1\mathbf{x} + (-2-\sqrt{2})\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u} = (-2+\sqrt{2})\mathbf{x} + (0+2)\mathbf{u} \end{array}$$

where $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

The transfer function $\mathbf{H}(s)$ is

- **strictly passive** if it is **strictly positive-real** and **asymptotically** stable.
- **(non strictly) passive** if it is **positive-real** and **stable**.

The rational transfer function $\mathbf{H}(s)$ is **strictly positive-real** if $\mathbf{\Phi_H}(\imath\omega) > 0$ and **positive-real** if $\mathbf{\Phi_H}(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$, $\mathbf{\Phi_H}(\imath\omega) = \mathbf{H}(\imath\omega)^H + \mathbf{H}(\imath\omega)$

Real $X \succ 0$ is a (strict) passivity certificate for realization $\mathcal{S} : (A, B, C, D)$ iff.

$$W(X, \mathcal{S}) = \begin{bmatrix} -A^\top X - XA & C^\top - XB \\ (\star)^\top & D + D^\top \end{bmatrix} \succeq (\succ) 0$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

LTI pH model

LTI pH system model of a proper transfer function $\mathbf{H}$, has the state-space form

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{array}$$

where

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

LTI strictly passive & LTI pH equivalence

If Q, $\mathcal{W}$ are invertible, $X = Q$ is a strict passivity certificate ($i.e.$ $W(X, \mathcal{S}) \succ 0$) of the normalized pH system, $\mathcal{S} : (A, B, C, D)$, it follows that $\mathcal{S}$, can always be transformed in the pH realization.

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The right data can be expressed as:

$$\begin{array}{rcl} \Lambda & = & \textbf{diag}\,[\lambda_1,\dots,\lambda_k] \in \mathbb{C}^{k\times k}, \\ \mathbf{R} & = & \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{m\times k} \\ \mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{m\times k} \end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcccl} \mathbf{M} & = & -\Lambda^H & = & \textbf{diag}\,[\mu_1,\dots,\mu_q] \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top & = & \mathbf{R}^H & = & \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{m\times q} \\ \mathbf{V}^\top & = & -\mathbf{W}^H & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{m\times q} \end{array}$$

Spectral zeros are $(\lambda_j, \mathbf{r}_j)$ from standard Loewner (n zeros in the open right half-plane)

$$\begin{bmatrix} 0 & A & B \\ A^\top & 0 & C^\top \\ B^\top & C & D+D^\top \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} 0 & E & 0 \\ E^\top & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$

A.C. Antoulas, *"A new result on passivity preserving model reduction"*, Systems & Control Letters, vol. 54, 2005.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

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\end{array}
$$

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Loewner extensions

Passive & port-Hamiltonian

Then $\mathbf{H}(s) \in \mathbb{C}^{m\times m}$ of McMillan degree n and $\mathcal{S}$ is a normalized pH form and satisfies

$$\mathbf{H}(\infty) = D \text{ , } \mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ , } \mathbf{r}_j^H \mathbf{H}(-\overline{\lambda_j}) = -\mathbf{w}_j^H \text{ , } D + D^\top \succ 0 \text{ and } \mathbb{L} \succ 0$$

By construction, one obtains an Hermitian $\mathbb{L} \in \mathbb{C}^{r\times r}$ and a skew symmetric $\mathbb{M} \in \mathbb{C}^{r\times r}$ matrix. By setting, $\mathbf{H}(\infty) = D$, one recovers an $m \times m$ real transfer function $\hat{\mathbf{H}}$. As $\mathbb{L} \succ 0$, one may apply the Cholesky decomposition $\mathbb{L} = T^\top T$. Then the *normalized pH model* is obtained as $\mathbf{\Sigma}_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, D)$,

$$\mathbf{S} := \begin{bmatrix} -T\hat{A}T^{-1} & -T\hat{B} \\ \hat{C}T^{-1} & D \end{bmatrix},$$

one obtains the equivalent pH-form by solving

$$\begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} := \frac{\mathbf{S} - \mathbf{S}^\top}{2} \text{ and } \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix} := \frac{\mathbf{S} + \mathbf{S}^\top}{2}.$$

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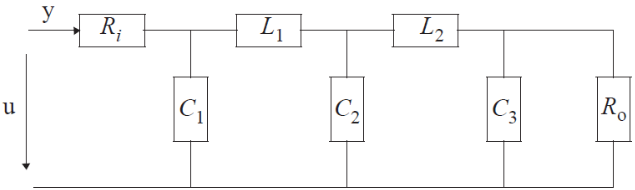
P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)

Variables & parameters

- x_1: voltage across C_1
- x_2: current across L_1
- x_3: voltage across C_2
- x_4: current across L_2
- x_5: voltage across C_3
- u: voltage
- y: current
- $n = 5$ internal variables
- $C_i = \frac{1}{10}$ F
- $L_i = \frac{1}{10}$ H
- $R_i = \frac{1}{2}\Omega$
- $R_o = 5\Omega$



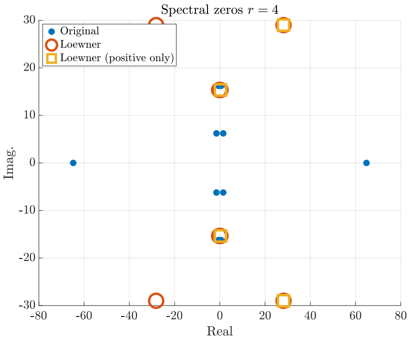
$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix}, B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

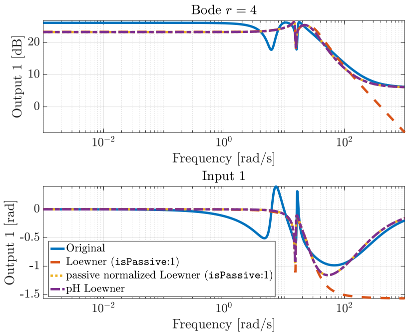
$$C = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \end{bmatrix}, D = 2$$

$$\mathbf{G}(s) = \frac{2\left(s^5 + 222\,s^4 + 840\,s^3 + 66600\,s^2 + 118000\,s + 2220000\right)}{s^5 + 22\,s^4 + 440\,s^3 + 6600\,s^2 + 38000\,s + 220000}$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)





Here **all models result stable and passive**; passive normalized allows pH reconstruction

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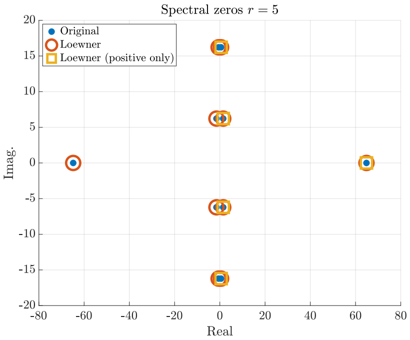
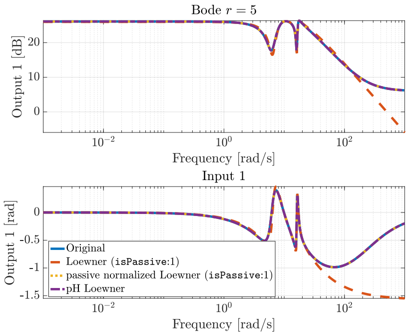
RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)





Here **all models result stable and passive**; passive normalized allows pH reconstruction

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)

Model

$$J = \begin{pmatrix} 0 & 6.05 & -1.45 & -9.45 & 2.96 \\ -6.05 & 0 & -16.1 & -1.59 & 0.497 \\ 1.45 & 16.1 & 0 & 0.682 & -0.213 \\ 9.45 & 1.59 & -0.682 & 0 & -6.3 \\ -2.96 & -0.497 & 0.213 & 6.3 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 20.7 \\ 1.94 \\ -0.465 \\ -3.02 \\ 0.946 \end{pmatrix} P = \begin{pmatrix} -6.25 \\ -1.05 \\ 0.451 \\ 1.77 \\ -0.686 \end{pmatrix}$$

$$R = \begin{pmatrix} 19.5 & 3.28 & -1.41 & -5.53 & 2.15 \\ 3.28 & 0.551 & -0.237 & -0.929 & 0.36 \\ -1.41 & -0.237 & 0.102 & 0.399 & -0.155 \\ -5.53 & -0.929 & 0.399 & 1.57 & -0.607 \\ 2.15 & 0.36 & -0.155 & -0.607 & 0.235 \end{pmatrix} \quad N = 0 \quad S = 2 \quad Q = I_5$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)

- $\mathcal{W} = \mathcal{W}^\top \succeq 0$

$$\mathcal{W} = \begin{pmatrix} 19.5 & 3.28 & -1.41 & -5.53 & 2.15 & -6.25 \\ 3.28 & 0.551 & -0.237 & -0.929 & 0.36 & -1.05 \\ -1.41 & -0.237 & 0.102 & 0.399 & -0.155 & 0.451 \\ -5.53 & -0.929 & 0.399 & 1.57 & -0.607 & 1.77 \\ 2.15 & 0.36 & -0.155 & -0.607 & 0.235 & -0.686 \\ -6.25 & -1.05 & 0.451 & 1.77 & -0.686 & 2.0 \end{pmatrix}$$

- $\mathcal{V} = -\mathcal{V}^\top$,

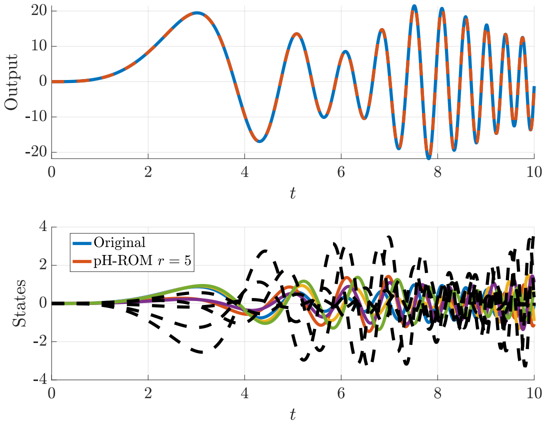
$$\mathcal{V} = \begin{pmatrix} 0 & -6.05 & 1.45 & 9.45 & -2.96 & -20.7 \\ 6.05 & 0 & 16.1 & 1.59 & -0.497 & -1.94 \\ -1.45 & -16.1 & 0 & -0.682 & 0.213 & 0.465 \\ -9.45 & -1.59 & 0.682 & 0 & 6.3 & 3.02 \\ 2.96 & 0.497 & -0.213 & -6.3 & 0 & -0.946 \\ 20.7 & 1.94 & -0.465 & -3.02 & 0.946 & 0 \end{pmatrix}$$

- $Q = Q^\top \succeq 0$

$$\Lambda(Q) = 1$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

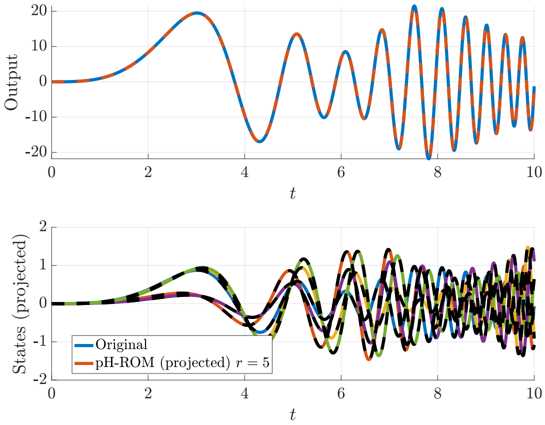
$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{array}{rcl} \mathcal{H}(t) & = & \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) \\ \mathcal{D}(t) & = & \hat{\mathbf{x}}(t)^\top (J-R)\hat{\mathbf{x}}(t) \\ \mathcal{E}(t) & = & \mathbf{u}(t)^\top \hat{\mathbf{y}}(t) \end{array}$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)



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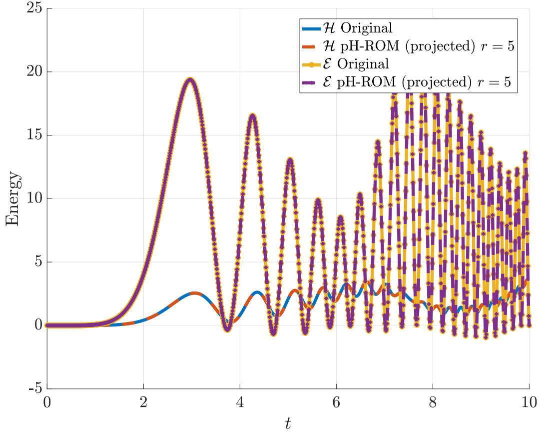
\mathcal{E}(t) & = & \mathbf{u}(t)^\top \hat{\mathbf{y}}(t)
\end{array}$$

Loewner extensions

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)



Time-domain lifting

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Loewner extensions

Parametric models

Structures

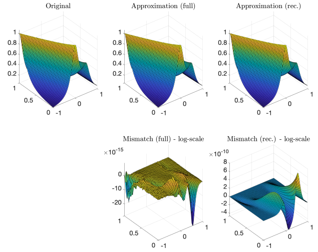
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- L-pH
- **pL-DAE**
- B-DAE
- Q-DAE

$$\mathbf{G}({}^1x, {}^2x) = \mathbf{H}(s, p) = \frac{1}{1+25(s+p)^2} + \frac{0.5}{1+25(s-0.5)^2} + \frac{0.1}{p+25}$$

Evaluated as

$$\begin{array}{ccc} {}^1x & \times & {}^2x \\ [-1,1] & \times & [0,1] \end{array} \quad \textbf{tab}_2 \in \mathbb{R}^{21\times 21}$$



Loewner extensions

1-D case

$$\left\{\begin{array}{lll} P_c^{(1)} & := & \left\{\left({}^1\lambda_{j_1}; \mathbf{w}_{j_1}\right),\ j_1 = 1,\dots,k_1\right\} \\ P_r^{(1)} & := & \left\{\left({}^1\mu_{i_1}; \mathbf{b}_{i_1}\right),\ i_1 = 1,\dots,q_1\right\} \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1,j_1} = \frac{\mathbf{b}_{i_1} - \mathbf{w}_{j_1}}{{}^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

$$\mathbf{M}_1\mathbb{L}_1 - \mathbb{L}_1\mathbf{\Lambda}_1 = \mathbb{V}_1\mathbf{R}_1 - \mathbf{L}_1\mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}({}^1x) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1}\mathbf{w}_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Data generated from $\mathbf{H}({}^1x) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{b}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Loewner extensions

2-D case

$$\begin{cases} P_c^{(2)} & := & \left\{ \left({}^1\lambda_{j_1}, {}^2\lambda_{j_2}; \mathbf{w}_{j_1,j_2} \right),\ j_1 = 1,\ldots,k_1 \quad j_2 = 1,\ldots,k_2 \right\} \\ P_r^{(2)} & := & \left\{ \left({}^1\mu_{i_1}, {}^2\mu_{i_2}; \mathbf{b}_{i_1,i_2} \right),\ i_1 = 1,\ldots,q_1 \quad i_2 = 1,\ldots,q_2 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{b}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{\left({}^1\mu_{i_1} - {}^1\lambda_{j_1}\right)\left({}^2\mu_{i_2} - {}^2\lambda_{j_2}\right)}$$

$$\begin{cases} \mathbf{M}_2\mathbb{X} - \mathbb{X}\mathbf{\Lambda}_2 & = & \mathbb{V}_2\mathbf{R}_2 - \mathbf{L}_2\mathbb{W}_2 \\ \mathbf{M}_1\mathbb{L}_2 - \mathbb{L}_2\mathbf{\Lambda}_1 & = & \mathbb{X} \end{cases}$$

Lagrangian form

$$\mathbf{g}({}^1x, {}^2x) = \frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1x - {}^1\lambda_{j_1})({}^2x - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{({}^1x - {}^1\lambda_{j_1})({}^2x - {}^2\lambda_{j_2})}}$$

Null space

$$\mathbf{span}\ (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \left[\begin{array}{c} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array}\right] \in \mathbb{C}^{k_1 k_2}$$

Loewner extensions

2-D case (example)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [0,2,4] \\ {}^2\lambda_{j_2} & = & [-1,-3] \\ {}^2\mu_{i_2} & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array}\right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

Loewner extensions

2-D case (example)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} &=& [1,3,5] \\ {}^1\mu_{i_1} &=& [0,2,4] \\ {}^2\lambda_{j_2} &=& [-1,-3] \\ {}^2\mu_{i_2} &=& [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Lagrangian form

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s,t)$$

Loewner extensions

n-D case

$$\left\{\begin{array}{l} P_c^{(n)} := \left\{({}^1\lambda_{j_1}, {}^2\lambda_{j_2}, \cdots, {}^n\lambda_{j_n}; \mathbf{w}_{j_1,j_2,\cdots,j_n}),\ j_l = 1,\ldots,k_l, \quad l = 1,\cdots,n\right\} \\ P_r^{(n)} := \left\{({}^1\mu_{i_1}, {}^2\mu_{i_2}, \cdots, {}^n\mu_{i_n}; \mathbf{b}_{i_1,i_2,\cdots,i_n}),\ i_l = 1,\ldots,q_l, \quad l = 1,\cdots,n\right\} \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \frac{\mathbf{b}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})\cdots({}^n\mu_{i_n} - {}^n\lambda_{j_n})}$$

$$\left\{\begin{array}{rcl} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \mathbf{\Lambda}_n & = & \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ & \cdots & \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \mathbf{\Lambda}_1 & = & \mathbb{X}_{n-1}. \end{array}\right.$$

Lagrangian form

$$\mathbf{g}({}^1x, \cdots, {}^nx) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{({}^1x - {}^1\lambda_{j_1})\cdots({}^nx - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{({}^1x - {}^1\lambda_{j_1})\cdots({}^nx - {}^n\lambda_{j_n})}}$$

Null space

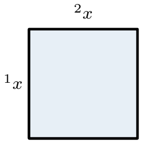
$$\textbf{span}\ (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \left[\begin{array}{c} c_{1,\cdots,1} \\ \vdots \\ c_{1,\cdots,k_n} \\ \hline \vdots \\ \hline c_{k_1,\cdots,1} \\ \vdots \\ c_{k_1,\cdots,k_n} \end{array}\right] \in \mathbb{C}^{k_1 \cdots k_n}$$

Loewner extensions

n-D case - ReLU

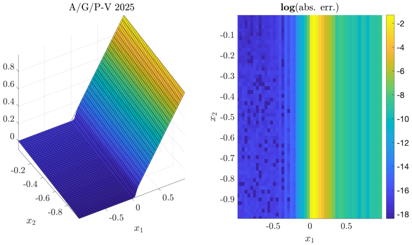
$$\mathbf{H}({}^1x, {}^2x) = \mathbf{ReLU}({}^1x) + \frac{1}{100}\,{}^2x$$



$$\begin{array}{ccc} {}^1x & \times & {}^2x \\ [-1,1] & \times & [0,1] \end{array}$$

$\textbf{tab}_2 \in \mathbb{R}^{20\times 20}$

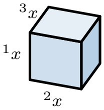
≈3.45 Ko ('real')



Loewner extensions

n-D case - Airbus flutter case

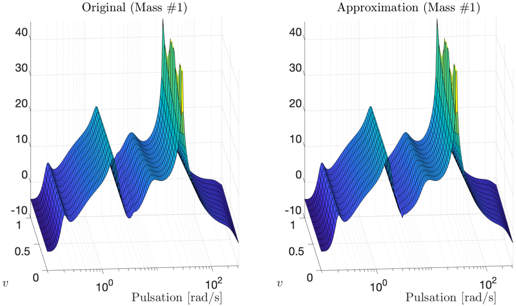
$$\mathbf{\Sigma}({}^1x, {}^2x, {}^3x) = \mathbf{\Sigma}(s, m, v) : s^2 M(m)x(s) + sB(m)x(s) + K(m)x(s) - G(s, v) = u(s),\ \mathbf{y}(s) = C\mathbf{x}(s)$$



$$\begin{array}{ccccc} {}^1x & \times & {}^2x & \times & {}^3x \\ \imath[10, 35] & \times & [\underline{m}, \overline{m}] & \times & [\underline{v}, \overline{v}] \end{array}$$

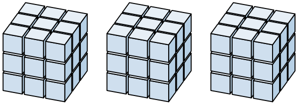
$\mathbf{tab}_3 \in \mathbb{C}^{300 \times 10 \times 10}$

$\approx$`468.75 Ko ('complex')`



A. dos Reis de Souza et al., *"Aircraft flutter suppression: from a parametric model to robust control"*, ECC, 2023.

$$\mathbf{H}({}^1x,\cdots,{}^8x) = \mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w) = \frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$

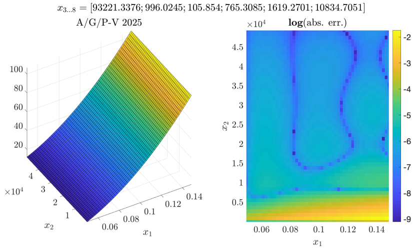


$$\begin{array}{ccccc} {}^1x & \times & \cdots & \times & {}^8x \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

$\textbf{tab}_8 \in \mathbb{C}^{8\times 8\times\cdots\times 8}$

≈130 `Mo ('real')`

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	$1.02e+04$	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	1.02e+04	39.1	0.00456	2.93e-09	0.0611



S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

Content

Conclusions

Loewner… a versatile tool

- solves the LTI realization problem
- solves data-driven model reduction
- solves data-driven model approximation
- … and pH, parametric
- … and also, bilinear, quadratic…

$\longrightarrow$ direct impact in engineers life

… still so much to do

- Technical references and slides at https://cpoussot.github.io/
- MOR Digital Systems numerical suite http://mordigitalsystems.fr/

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Loewner Framework for data-driven reduced order modeling

... a bridge between realization, approximation and identification

C. Poussot-Vassal
October 16, 2025

Conclusions

The map of mathematics (by Dominic Walliman)