

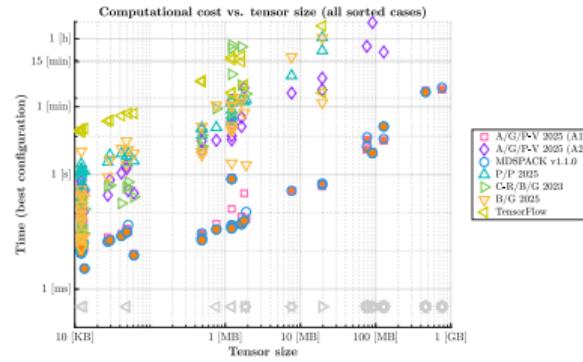
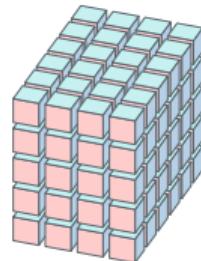
# The Loewner framework, in the eye of the tensor...

The Kolmogorov superposition theorem, the curse of dimensionality & benchmark

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February 17, 2026

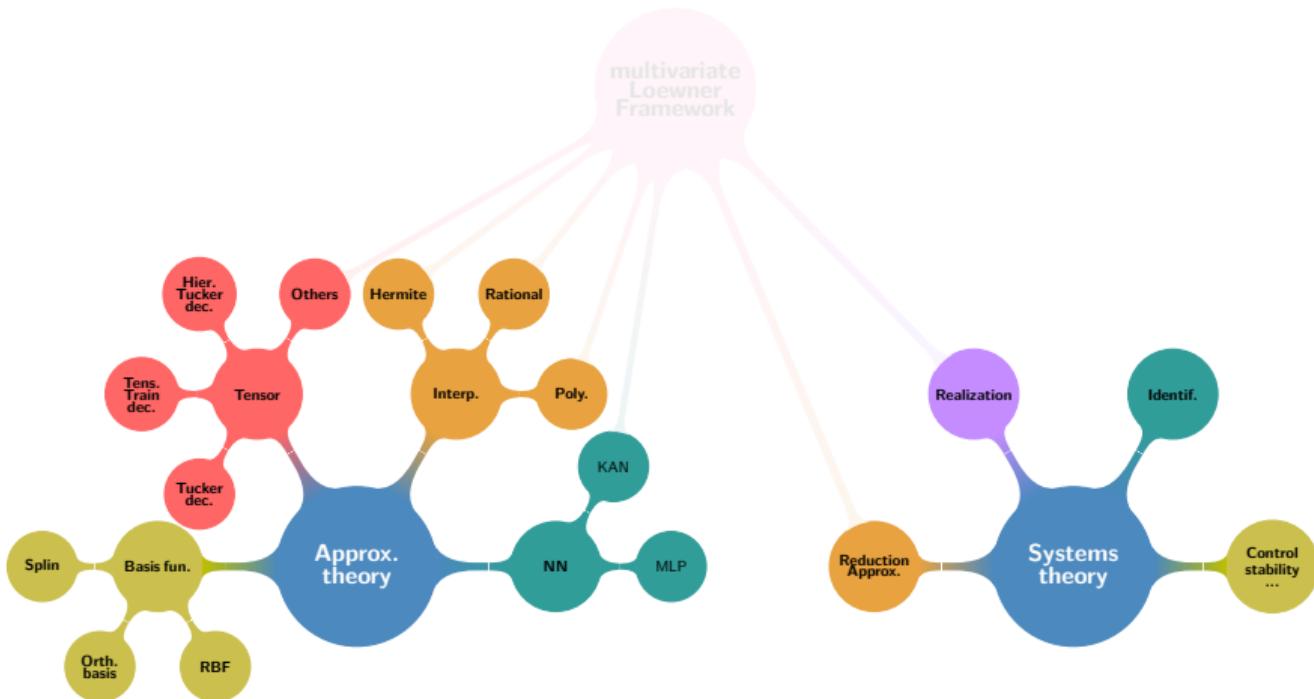
<https://doi.org/10.1137/24M1656657>  
<https://arxiv.org/abs/2506.04791>

[in SIAM Review - Research Spotlight]  
[extensive benchmark]



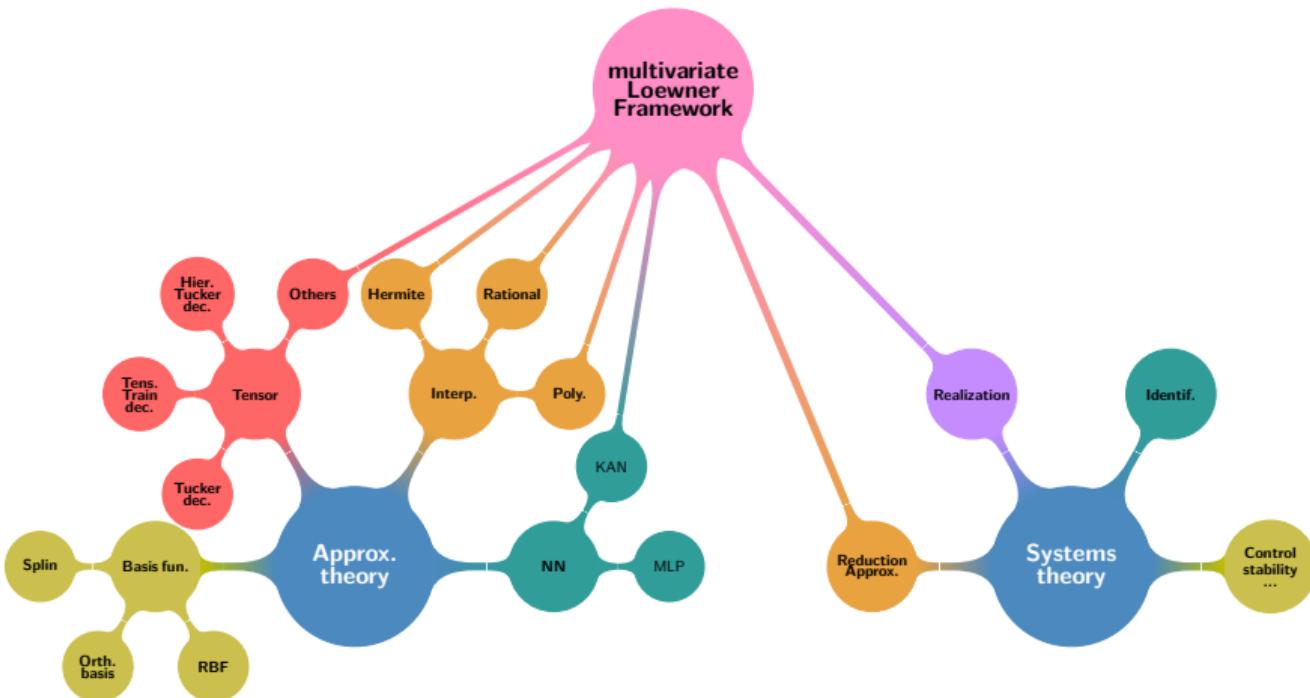
# Forewords

Approximation & systems theory... where we stand



# Forewords

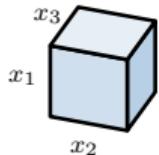
Approximation & systems theory... where we stand



# Forewords

Starting (motivating) examples - dynamical Airbus flutter problem

$$\Sigma(x_1, x_2, x_3) = \Sigma(s, m, v) : s^2 M(m) \mathbf{x}(s) + s B(m) \mathbf{x}(s) + K(m) \mathbf{x}(s) - G(s, v) = \mathbf{u}(s), \mathbf{y}(s) = C \mathbf{x}(s)$$



$$x_1 \quad \times \quad x_2 \quad \times \quad x_3 \\ i[10, 35] \quad \times \quad [\underline{m}, \bar{m}] \quad \times \quad [\underline{v}, \bar{v}]$$

$$\mathcal{T}_3^\otimes \in \mathbb{C}^{300 \times 10 \times 10}$$

≈468.75 Ko ('complex')



A. dos Reis de Souza et al., "Aircraft flutter suppression: from a parametric model to robust control", ECC, 2023.

# Forewords

## Starting (motivating) examples - static function #30 (Borehole)

$$\mathbf{H}(x_1, \dots, x_8) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$$x_1 \quad \times \quad \cdots \quad \times \quad x_8 \\ [\underline{r_w}, \overline{r_w}] \quad \times \quad \cdots \quad \times \quad [\underline{K_w}, \overline{K_w}]$$

$$\mathcal{T}_8^\otimes \in \mathbb{C}^{8 \times 8 \times \cdots \times 8}$$

$\approx 130$  Mo ('real')

$r_w \in [0.05, 0.15]$	radius of borehole (m)
$r \in [100, 50\,000]$	radius of influence (m)
$T_u \in [63\,070, 115\,600]$	transmissivity of upper aquifer ( $m^2/yr$ )
$H_u \in [990, 1110]$	potentiometric head of upper aquifer (m)
$T_l \in [63.1, 116]$	transmissivity of lower aquifer ( $m^2/yr$ )
$H_l \in [700, 820]$	potentiometric head of lower aquifer (m)
$L \in [1120, 1680]$	length of borehole (m)
$K_w \in [9855, 12\,045]$	hydraulic conductivity of borehole (m/yr)



# Forewords

## Starting (motivating) examples - static function #30 (Borehole)

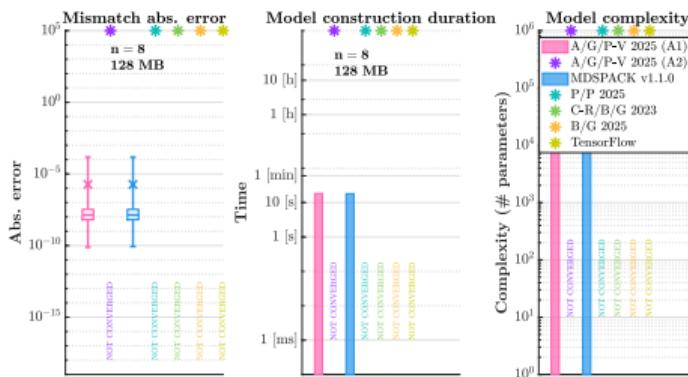
$$\mathbf{H}(x_1, \dots, x_8) = \mathbf{H}(r_w, r, T_u, H_u, T_l, H_l, L, K_w) = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln(r/r_w)r_w^2 K_w}\right) + \frac{T_u}{T_l}}$$



$$x_1 \quad \times \quad \cdots \quad \times \quad x_8 \\ [r_w, \overline{r_w}] \quad \times \quad \cdots \quad \times \quad [K_w, \overline{K_w}]$$

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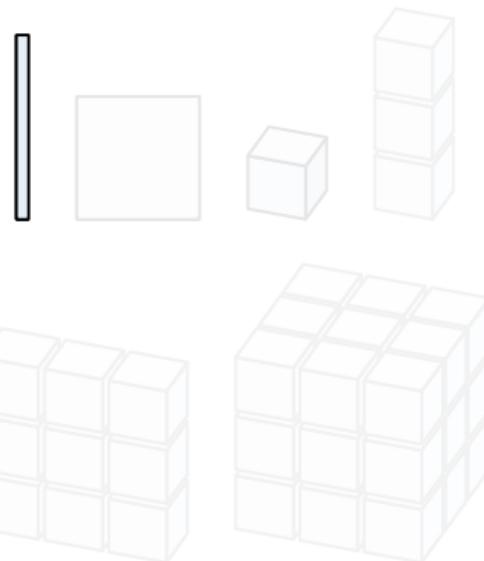
## Data (and tensors)

Column / Row data

$$\mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \quad \} \xrightarrow{\mathbf{H}(x_1)} \{ \quad \mathbf{w}_{j_1}, \mathbf{v}_{i_1}$$

$x_1$	
$\lambda_1(1, \dots, k_1)$	$\mathbf{W}_{k_1}$
$\mu_1(1, \dots, q_1)$	$\mathbf{V}_{q_1}$

Tensors (1-D)  $\mathcal{T}_1^{\otimes}$



# Forewords

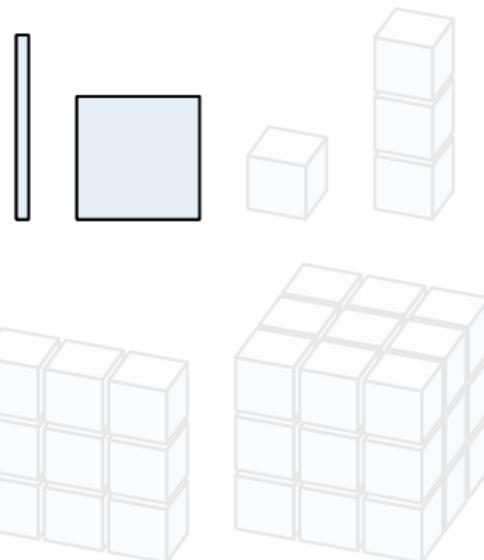
## Data (and tensors)

### Column / Row data

$$\left. \begin{array}{l} \mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \\ \mathbf{x}_2 = \lambda_2(j_2), \mu_2(i_2) \end{array} \right\} \xrightarrow{\mathbf{H}(x_1, x_2)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2} \\ \mathbf{v}_{i_1, i_2} \end{array} \right.$$

$x_2$	$\lambda_2(1, \dots, k_2)$	$\mu_2(1, \dots, q_2)$
$x_1$	$\mathbf{W}_{k_1, k_2}^\otimes$	$\phi_{cr}$
$\lambda_1(1, \dots, k_1)$	$\phi_{rc}$	$\mathbf{V}_{q_1, q_2}^\otimes$
$\mu_1(1, \dots, q_1)$		

### Tensors (2-D) $\mathcal{T}_2^\otimes$



# Forewords

## Data (and tensors)

### Column / Row data

$$\left. \begin{array}{l} \mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \\ \mathbf{x}_2 = \lambda_2(j_2), \mu_2(i_2) \\ \mathbf{x}_3 = \lambda_3(j_3), \mu_3(i_3) \end{array} \right\} \xrightarrow{\mathbf{H}(x_1, x_2, x_3)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, j_2, j_3} \\ \mathbf{v}_{i_1, i_2, i_3} \end{array} \right.$$

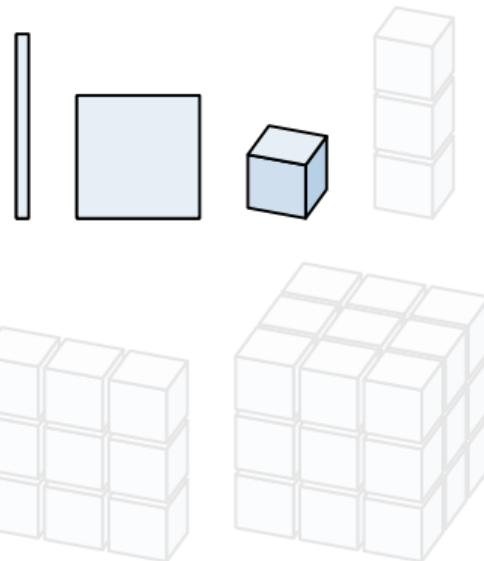
$$x_3 = \lambda_3(1, \dots, k_3)$$

$x_2$	$\lambda_2(1, \dots, k_2)$	$\mu_2(1, \dots, q_2)$
$x_1$		
$\lambda_1(1, \dots, k_1)$	$\mathbf{W}_{k_1, k_2, k_3}^\otimes$	$\phi_{crc}$
$\mu_1(1, \dots, q_1)$	$\phi_{rcc}$	$\phi_{rrc}$

$$x_3 = \mu_3(1, \dots, q_3)$$

$x_2$	$\lambda_2(1, \dots, k_2)$	$\mu_2(1, \dots, q_2)$
$x_1$		
$\lambda_1(1, \dots, k_1)$	$\phi_{crr}$	$\phi_{crr}$
$\mu_1(1, \dots, q_1)$	$\phi_{rcr}$	$\mathbf{V}_{q_1, q_2, q_3}^\otimes$

### Tensors (3-D) $\mathcal{T}_3^\otimes$



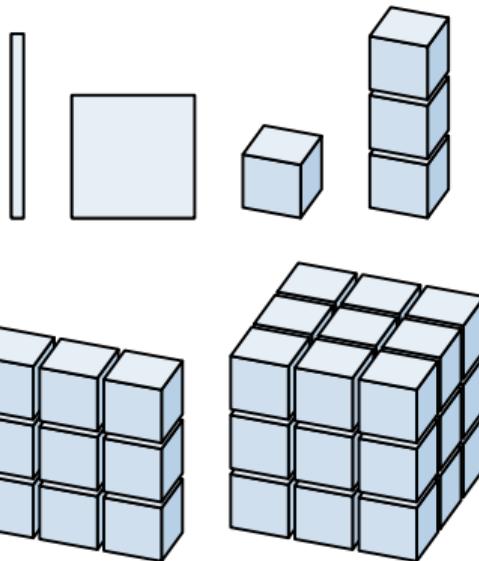
# Forewords

## Data (and tensors)

Column / Row data

$$\left. \begin{array}{l} \mathbf{x}_1 = \lambda_1(j_1), \mu_1(i_1) \\ \mathbf{x}_2 = \lambda_2(j_2), \mu_2(i_2) \\ \mathbf{x}_3 = \lambda_3(j_3), \mu_3(i_3) \\ \vdots \\ \mathbf{x}_n = \lambda_n(j_n), \mu_n(i_n) \end{array} \right\} \xrightarrow{\mathbf{H}(x_1, \dots, x_n)} \left\{ \begin{array}{l} \mathbf{w}_{j_1, \dots, j_n} \\ \mathbf{v}_{i_1, \dots, i_n} \end{array} \right.$$

Tensors ( $n$ -D)  $\mathcal{T}_n^{\otimes}$



# Forewords

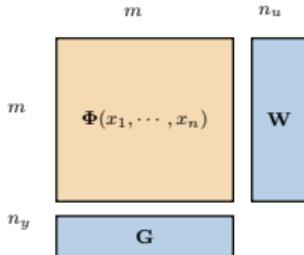
## Problem description

### Data-driven model approximation

Being given a  $n$ -dimensional tensor (data), we seek a multi-variate rational function  $\hat{\mathbf{H}}$  and realization  $(\mathbf{G}, \Phi, \mathbf{W})$

$$\hat{\mathbf{H}}(x_1, x_2, \dots, x_n) = \mathbf{G}\Phi(x_1, x_2, \dots, x_n)^{-1}\mathbf{W} \in \mathbb{C}$$

interpolating the data.



### Connection to standard dynamical system realization

A linear-in-state dynamical system  $x_1 := s$  parameterized with  $\mathcal{S} := [x_2, \dots, x_n]^\top \subset \mathbb{C}^{n-1}$

$$\begin{aligned} \mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t; \mathcal{S}) &= \mathbf{A}(\mathcal{S})\mathbf{x}(t; \mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t; \mathcal{S}) &= \mathbf{C}\mathbf{x}(t; \mathcal{S}) \end{aligned}$$

equivalently

$$\hat{\mathbf{H}}(x_1, x_2, \dots, x_n) = \mathbf{C}(\mathcal{S})[x_1 \mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})]^{-1} \mathbf{B} \in \mathbb{C}.$$

# Forewords

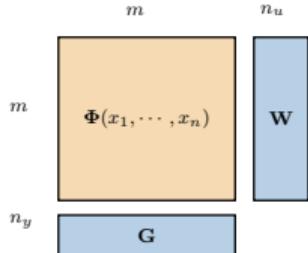
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# Forewords

## Where we stand (some references)

### 1-D Two-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ realization minimality
- ⇒ direct algorithm

### 1-D One-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ direct algorithm

### 1-D AAA (Adaptive Anderson Antoulas - one-sided)

- ⇒ (mixed interpolation LS) barycentric form
- ⇒ iterative algorithm

### 2-D Parametric one-sided Loewner

- ⇒ (interpolation) barycentric form
- ⇒ realization (non-minimal)
- ⇒ direct algorithm

### 3-D Parametric AAA

- ⇒ (mixed interpolation LS) barycentric form
- ⇒ iterative algorithm

### >3-D few results

- 
-  J-P. Berrut and N. Trefethen, "*Barycentric Lagrange Interpolation*", SIAM Review, 46(3), 2004.
  -  A.J. Mayo and A.C. Antoulas, "*A framework for the solution of the generalized realisation problem*", LAA, 425(2-3), 2007.
  -  A.C. Ionita and A.C. Antoulas, "*Data-Driven Parametrized Model Reduction in the Loewner Framework*", SIAM Journal on Scientific Computing, 36(3), 2014.
  -  T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, "*Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework*", SIAM Journal on Applied Dynamical Systems, 22(4), 2023.
  -  A.C. Rodriguez, L. Balicki and S. Gugercin, "*The p-AAA algorithm for data driven modeling of parametric dynamical systems*", SIAM Journal on Scientific Computing, 45(3), 2023.

# Forewords

Contributions claim & trajectory of the presentation

- ▶  $n$ -D tensor data to  $n$ -D Loewner matrix  $\mathbb{L}_n$
- ▶  $n$ -variable transfer functions
- ▶  $n$ -variable generalized realization
- ▶ Taming the curse of dimensionality (flop/Bytes/accuracy)
- ▶  $n$ -variable **decoupling**
- ▶ **KST** formulation for rational functions
- ▶ Comparison with **MLP**, **KAN**, **pAAA**



- 
- 📚 A.C. Antoulas, I-V. Gosea and C. P-V., "*On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality*", SIAM Review, November, 2025 (<https://arxiv.org/abs/2405.00495>).
  - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "*Tensor-based multivariate function approximation: methods benchmarking and comparison*", arXiv, June, 2025 (<https://arxiv.org/abs/2506.04791>).
  - 📚 A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, "*mLF package*", GitHub (<https://github.com/cpoussot/mLF>).

# Content

Forewords

**Multi-variate data, function & Loewner matrix**

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

Multi-variate realization

The MATLAB mLF package

Conclusion

# Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} &:= \{(\lambda_1(j_1); \mathbf{w}_{j_1}), j_1 = 1, \dots, k_1\} \\ P_r^{(1)} &:= \{(\mu_1(i_1); \mathbf{v}_{i_1}), i_1 = 1, \dots, q_1\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{\mu_1(i_1) - \lambda_1(j_1)}$$

$$\mathbf{M}_1 \mathbb{L}_1 - \mathbb{L}_1 \boldsymbol{\Lambda}_1 = \mathbb{V}_1 \mathbf{R}_1 - \mathbf{L}_1 \mathbb{W}_1$$

## Lagrangian form

$$\mathbf{g}(x_1) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1} \mathbf{w}_{j_1}}{x_1 - \lambda_1(j_1)}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{x_1 - \lambda_1(j_1)}}$$

## Null space

$$\text{span } (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

# Multi-variate data, function & Loewner matrix

## 1-D case (example)

Data generated from  $\mathbf{H}(x_1) = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$  of complexity (2)

$$\left. \begin{array}{rcl} \lambda_1(j_1) & = & [1, 3, 5] \\ \mu_1(i_1) & = & [2, 4, 6, 8] \end{array} \right\} \xrightarrow{\mathbf{H}} \left\{ \begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array} \right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

# Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} &:= \{(\lambda_1(j_1), \lambda_2(j_2); \mathbf{w}_{j_1,j_2}), \ j_1 = 1, \dots, k_1 \quad j_2 = 1, \dots, k_2\} \\ P_r^{(2)} &:= \{(\mu_1(i_1), \mu_2(i_2); \mathbf{v}_{i_1,i_2}), \ i_1 = 1, \dots, q_1 \quad i_2 = 1, \dots, q_2\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1, j_2}^{i_1, i_2} = \frac{\mathbf{v}_{i_1, i_2} - \mathbf{w}_{j_1, j_2}}{(\mu_1(i_1) - \lambda_1(j_1))(\mu_2(i_2) - \lambda_2(j_2))}$$

$$\begin{cases} \mathbf{M}_2 \mathbb{X} - \mathbb{X} \boldsymbol{\Lambda}_2 &= \mathbb{V}_2 \mathbf{R}_2 - \mathbf{L}_2 \mathbb{W}_2 \\ \mathbf{M}_1 \mathbb{L}_2 - \mathbb{L}_2 \boldsymbol{\Lambda}_1 &= \mathbb{X} \end{cases}$$

## Lagrangian form

$$g(x_1, x_2) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1, j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}$$

## Null space

$$\text{span } (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \left[ \begin{array}{c} c_{1,1} \\ \vdots \\ \hline c_{1,k_2} \\ \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array} \right] \in \mathbb{C}^{k_1 k_2}$$

# Multi-variate data, function & Loewner matrix

## 2-D case (example)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\left. \begin{array}{lcl} \lambda_1(j_1) & = & [1, 3, 5] \\ \mu_1(i_1) & = & [0, 2, 4] \\ \lambda_2(j_2) & = & [-1, -3] \\ \mu_2(i_2) & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[ \begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

### Loewner matrix

$$\mathbb{L}_2 = \left[ \begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array} \right]$$

### Null space

$$\mathbf{c}_2 = \left[ \begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array} \right]$$

# Multi-variate data, function & Loewner matrix

## 2-D case (example)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\left. \begin{array}{lcl} \lambda_1(j_1) & = & [1, 3, 5] \\ \mu_1(i_1) & = & [0, 2, 4] \\ \lambda_2(j_2) & = & [-1, -3] \\ \mu_2(i_2) & = & [-2, -4] \end{array} \right\} \xrightarrow{\mathbf{H}} \left[ \begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array} \right]$$

### Lagrangian form

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s, t)$$

# Multi-variate data, function & Loewner matrix

*n-D case*

$$\begin{cases} P_c^{(n)} := \{(\lambda_1(j_1), \lambda_2(j_2), \dots, \lambda_n(j_n); \mathbf{w}_{j_1, j_2, \dots, j_n}), \quad j_l = 1, \dots, k_l, \quad l = 1, \dots, n\} \\ P_r^{(n)} := \{(\mu_1(i_1), \mu_2(i_2), \dots, \mu_n(i_n); \mathbf{v}_{i_1, i_2, \dots, i_n}), \quad i_l = 1, \dots, q_l, \quad l = 1, \dots, n\} \end{cases}$$

## Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1, j_2, \dots, j_n}^{i_1, i_2, \dots, i_n} = \frac{\mathbf{v}_{i_1, i_2, \dots, i_n} - \mathbf{w}_{j_1, j_2, \dots, j_n}}{(\mu_1(i_1) - \lambda_1(j_1)) \cdots (\mu_n(i_n) - \lambda_n(j_n))}$$

$$\begin{cases} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \boldsymbol{\Lambda}_n &= \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ &\vdots \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \boldsymbol{\Lambda}_1 &= \mathbb{X}_{n-1}. \end{cases}$$

## Lagrangian form

$$\mathbf{g}(x_1, \dots, x_n) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n} \mathbf{w}_{j_1, \dots, j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, \dots, j_n}}{(x_1 - \lambda_1(j_1)) \cdots (x_n - \lambda_n(j_n))}}$$

## Null space

$$\text{span } (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \left[ \begin{array}{c} c_{1, \dots, 1} \\ \vdots \\ \hline c_{1, \dots, k_n} \\ \hline \vdots \\ \hline c_{k_1, \dots, 1} \\ \vdots \\ \hline c_{k_1, \dots, k_n} \end{array} \right] \in \mathbb{C}^{k_1 \cdots k_n}$$

# Content

Forewords

Multi-variate data, function & Loewner matrix

## Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

Multi-variate realization

The MATLAB `mLF` package

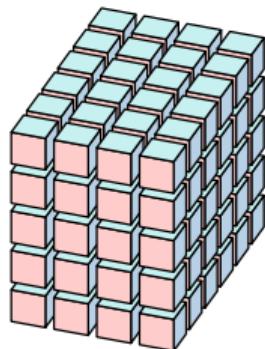
Conclusion

# Taming the curse of dimensionality

*n*-variable Loewner matrix operator<sup>1</sup>

$$\begin{aligned} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \dots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1) \times \dots \times (k_n+q_n)} &\longrightarrow \mathbb{C}^{\textcolor{magenta}{Q} \times \textcolor{brown}{K}} \\ (\lambda_1(j_1), \mu_1(i_1), \dots, \lambda_n(j_n), \mu_n(i_n), \mathcal{T}_n^\otimes) &\longmapsto \mathbb{L}_n \end{aligned}$$

*n*-D tensor  $\mathcal{T}_n^\otimes$



matrix  $\mathbb{L}_n$

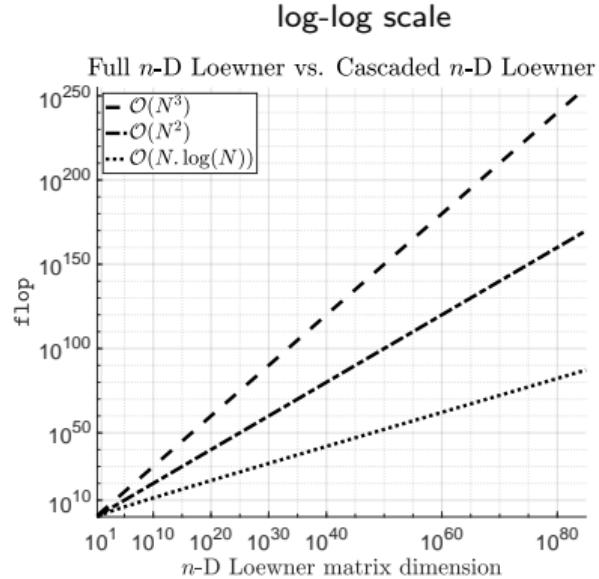


---

<sup>1</sup>where  $Q = q_1 q_2 \cdots q_n$  and  $K = k_1 k_2 \cdots k_n$ .

# Taming the curse of dimensionality

Null space flop and memory issues



(rows)  $Q = q_1 q_2 \dots q_n$  and  
(columns)  $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

## Computational issue

Note that  $Q \times K$  matrix SVD flop estimation is

- ▶  $QK^2$  (if  $Q > K$ )
- ▶  $N^3$  (if  $Q = K = N$ )

## Storage issue

Note that  $Q \times K$  matrix storage estimation is

- ▶ in real double  $QK \frac{8}{2^{20}}$  MB
- ▶ in complex double  $2QK \frac{8}{2^{20}}$  MB

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$x_1 \backslash x_2$	$\lambda_2(1) = -1$	$\lambda_2(2) = -3$	$\mu_2(1) = -2$	$\mu_2(2) = -4$
$\lambda_1(1) = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
$\lambda_1(2) = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
$\lambda_1(3) = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
$\mu_1(1) = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
$\mu_1(2) = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
$\mu_1(3) = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- ▶ 1  $\mathbb{L}_1$  along  $x_1$ , for  
 $x_2 = \lambda_2(2) = -3$
- ▶ 3  $\mathbb{L}_1$  along  $x_2$  for  
 $x_1 = \{\lambda_1(1), \lambda_1(2), \lambda_1(3)\}$
- ▶ Scaled null space  $\mathbf{c}_2^\top =$   

$$[ \mathbf{c}_1^{\lambda_1(1)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)\top}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)\top}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)\top}]_3 ]^\top$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$x_1 \backslash x_2$	$\lambda_2(1) = -1$	$= -3$	$\mu_2(1) = -2$	$\mu_2(2) = -4$
$\lambda_1(1) = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
$\lambda_1(2) = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
$\lambda_1(3) = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
$\mu_1(1) = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
$\mu_1(2) = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
$\mu_1(3) = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  $x_1$ , for

$$x_2 = \lambda_2(2) = -3$$

- 3  $\mathbb{L}_1$  along  $x_2$  for

$$x_1 = \{\lambda_1(1), \lambda_1(2), \lambda_1(3)\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$\left[ \mathbf{c}_1^{\lambda_1(1)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_3 \right]^\top$$

$$\mathbf{c}_1^{\lambda_2(2)} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$x_1 \backslash x_2$	$\lambda_2(1) = -1$	$\lambda_2(2) = -3$	$\mu_2(1) = -2$	$\mu_2(2) = -4$
$\lambda_1(1) = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
$\lambda_1(2) = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
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$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  $x_1$ , for

$$x_2 = \lambda_2(2) = -3$$

- 3  $\mathbb{L}_1$  along  $x_2$  for

$$x_1 = \{\lambda_1(1), \lambda_1(2), \lambda_1(3)\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$\left[ \mathbf{c}_1^{\lambda_1(1)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_3 \right]^\top$$

$$\mathbf{c}_1^{\lambda_2(2)} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(1)} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(2)} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(3)} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$x_1 \backslash x_2$	$\lambda_2(1) = -1$	$\lambda_2(2) = -3$	$\mu_2(1) = -2$	$\mu_2(2) = -4$
$\lambda_1(1) = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
$\lambda_1(2) = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
$\lambda_1(3) = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
$\mu_1(1) = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
$\mu_1(2) = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
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$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1  $\mathbb{L}_1$  along  $x_1$ , for

$$x_2 = \lambda_2(2) = -3$$

- 3  $\mathbb{L}_1$  along  $x_2$  for

$$x_1 = \{\lambda_1(1), \lambda_1(2), \lambda_1(3)\}$$

- Scaled null space  $\mathbf{c}_2^\top =$

$$\left[ \mathbf{c}_1^{\lambda_1(1)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_1 \quad \mathbf{c}_1^{\lambda_1(2)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_2 \quad \mathbf{c}_1^{\lambda_1(3)\top} \cdot [\mathbf{c}_1^{\lambda_2(2)}]_3 \right]^\top$$

$$\mathbf{c}_1^{\lambda_2(2)} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(1)} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(2)} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{\lambda_1(3)} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

# Taming the curse of dimensionality

2-D case

## Theorem: 2-D to 1-D

Being given the tableau  $\mathcal{T}_2^{\otimes}$  tensor in response of the 2-variables  $\mathbf{H}(x_1, x_2)$  function, the null space of the corresponding 2-D Loewner matrix  $\mathbb{L}_2$ , is spanned by

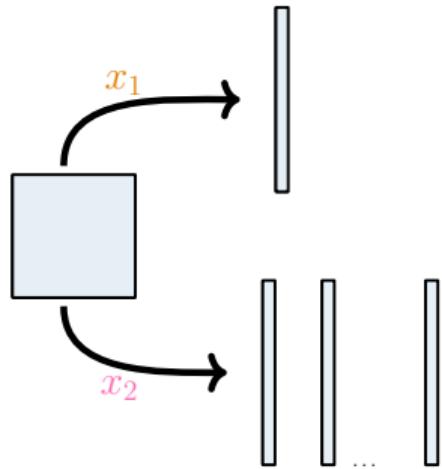
$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[ \mathbf{c}_1^{\lambda_2(1)} \cdot \left[ \mathbf{c}_1^{\lambda_1(k_1)} \right]_1, \dots, \mathbf{c}_1^{\lambda_2(k_2)} \cdot \left[ \mathbf{c}_1^{\lambda_1(k_1)} \right]_{k_2} \right],$$

where

- ▶  $\mathbf{c}_1^{\lambda_1(k_1)} = \mathcal{N}(\mathbb{L}_1^{\lambda_1(k_1)})$ ,  
i.e. the null space of the **1-D Loewner matrix** for frozen  $x_1 = \lambda_1(k_1)$ , and
- ▶  $\mathbf{c}_1^{\lambda_2(j_2)} = \mathcal{N}(\mathbb{L}_1^{\lambda_2(j_2)})$ ,  
i.e. the  $j_2$ -th null space of the **1-D Loewner matrices** for frozen  $x_2 = \{\lambda_2(1), \dots, \lambda_2(k_2)\}$ .

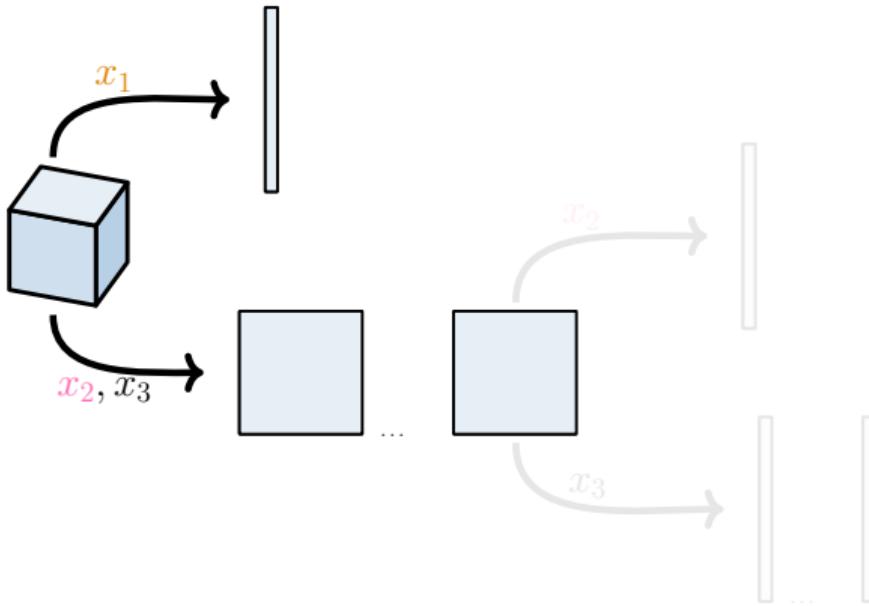
# Taming the curse of dimensionality

2-D case



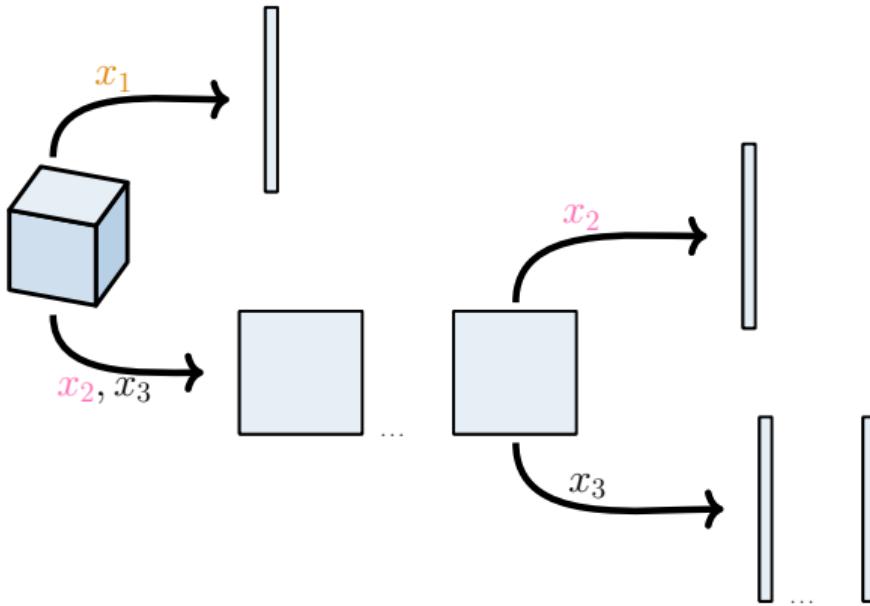
# Taming the curse of dimensionality

3-D case



# Taming the curse of dimensionality

3-D case



# Taming the curse of dimensionality

*n-D case*

## Theorem: *n*-D to $(n - 1)$ -D

Being given the tableau  $\mathcal{T}_n^{\otimes}$  tensor in response of the  $n$ -variables  $\mathbf{H}(x_1, \dots, x_n)$  function, the null space of the corresponding  $n$ -D Loewner matrix  $\mathbb{L}_n$ , is spanned by

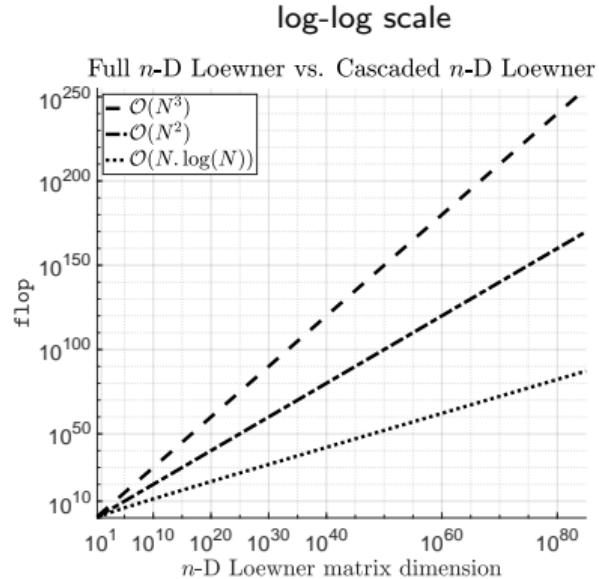
$$\mathcal{N}(\mathbb{L}_n) = \text{vec} \left[ \mathbf{c}_{n-1}^{\lambda_1(1)} \cdot \left[ \mathbf{c}_1^{(\lambda_2(k_2), \lambda_3(k_3), \dots, \lambda_n(k_n))} \right]_1, \dots, \mathbf{c}_{n-1}^{\lambda_1(k_1)} \cdot \left[ \mathbf{c}_1^{(\lambda_2(k_2), \lambda_3(k_3), \dots, \lambda_n(k_n))} \right]_{k_1} \right],$$

where

- ▶  $\mathbf{c}_1^{(\lambda_2(k_2), \lambda_3(k_3), \dots, \lambda_n(k_n))}$  spans  $\mathcal{N}(\mathbb{L}_1^{(\lambda_2(k_2), \lambda_3(k_3), \dots, \lambda_n(k_n))})$ ,  
i.e. the null space of the **1-D Loewner matrix** for frozen  $\{\lambda_{k_2 2}, \lambda_{k_3 3}, \dots, \lambda_{k_n n}\}$ , and
- ▶  $\mathbf{c}_{n-1}^{\lambda_1(j_1)}$  spans  $\mathcal{N}(\mathbb{L}_{n-1}^{\lambda_1(j_1)})$ ,  
i.e. the  $j_1$ -th null space of the  $(n - 1)$ -D **Loewner matrix** for frozen  $x_1 = \{\lambda_1(1), \dots, \lambda_1(k_1)\}$ .

# Taming the curse of dimensionality

Null space - flop complexity



(rows)  $Q = q_1 q_2 \dots q_n$  and  
(columns)  $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

## Computational issue

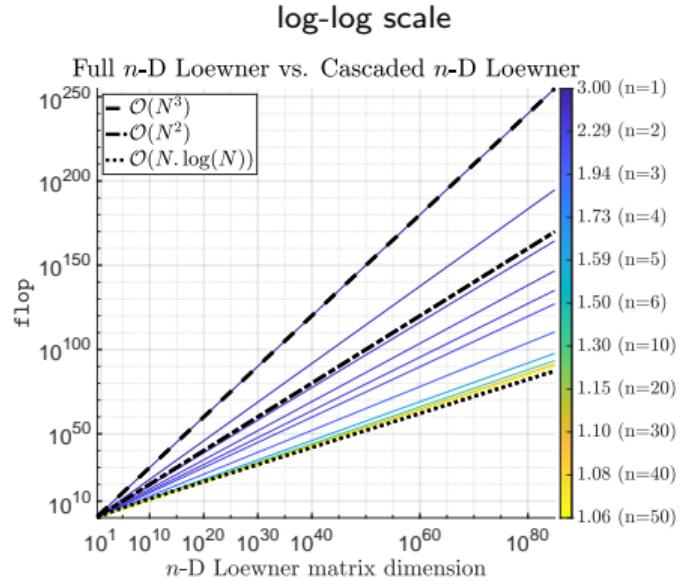
Note that  $Q \times K$  matrix SVD flop estimation is

- ▶  $QK^2$  (if  $Q > K$ )
- ▶  $N^3$  (if  $Q = K = N$ )

⇒ The CURSE of dimensionality

# Taming the curse of dimensionality

Null space - flop complexity



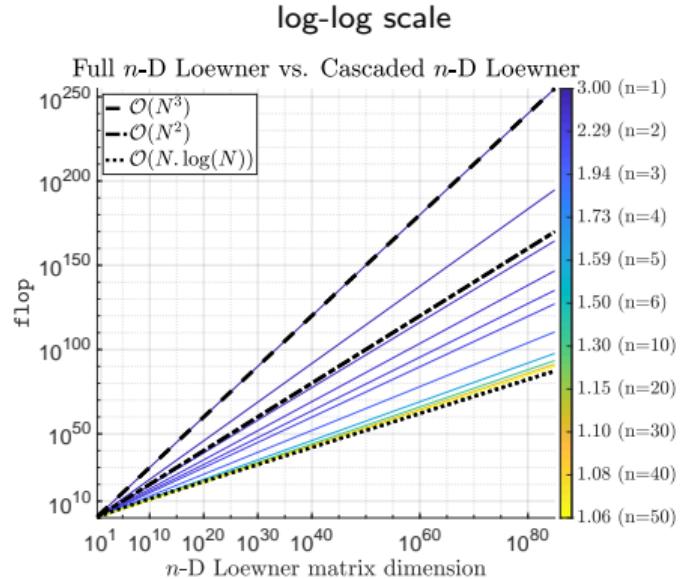
**Theorem: Recursive complexity**

$$\text{flop}_1(n) = \sum_{j=1}^n \left( k_j^3 \prod_{l=1}^j k_{l-1} \right) \text{ where } k_0 = 1.$$

⇒ The CURSE of dimensionality is TAMED

# Taming the curse of dimensionality

Null space - flop complexity



## Theorem: Worst case complexity

$k$  interpolation points per variables.

$$\overline{\text{flop}_1} = k^3 \frac{1 - k^n}{1 - k} = k^3 \frac{1 - N}{1 - k},$$

which is a ( $n$  finite) geometric series of ratio  $k$ .

⇒ The CURSE of dimensionality is TAMED

$$\begin{aligned}\mathcal{O}(N^3) &\rightarrow \mathcal{O}(N^{2.29}) && \text{for } n = 2 \\ &\rightarrow \mathcal{O}(N^{1.94}) && \text{for } n = 3 \\ &\vdots \\ &\rightarrow \mathcal{O}(N^{1.06}) && \text{for } n = 50\end{aligned}$$

# Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB})$$

## Full $n$ -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where  $N = k_1 k_2 \cdots k_n$ , needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example:  $N = 46,080$

Memory: 31.64 GB

flop:  $9.78 \cdot 10^{13}$

# Taming the curse of dimensionality

Null space - memory and storage

The data (tableau) storage is (in complex and double precision)

$$\frac{8}{2^{20}} \prod_l^n q_l + k_l \text{ MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs 45 MB})$$

## Full $n$ -D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where  $N = k_1 k_2 \cdots k_n$ , needs

$$\frac{8}{2^{20}} N^2 \text{ MB}$$

Example:  $N = 46,080$

Memory: 31.64 GB

flop:  $9.78 \cdot 10^{13}$

## Cascaded $n$ -D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\bar{k} \times \bar{k}}$$

where  $\bar{k} = \max_j k_j$ , needs

$$\frac{8}{2^{20}} \bar{k}^2 \text{ MB}$$

Example:  $\bar{k} = 20$

Memory: 6.25 KB

flop:  $8.13 \cdot 10^5$

# Content

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# Variables decoupling, KST and KANs

Kolmogorov Superposition Theorem and Hilbert's 13th problem

## Kolmogorov, Arnold, Kahane, Lorentz and Sprecher

For every continuous function  $f : \mathbb{I}^n \mapsto \mathbb{R}$  and any  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist

- ▶ real numbers  $\lambda_1, \dots, \lambda_n$ ;
- ▶ continuous functions  $\phi_k : \mathbb{I} \mapsto \mathbb{R}$ ,  $k = 1, \dots, 2n + 1$ ;
- ▶ a continuous function  $\Phi : \mathbb{R} \mapsto \mathbb{R}$ ;

such that  $\forall(x_1, \dots, x_n) \in \mathbb{I}^n$ :

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} \Phi \left( \sum_{p=1}^n \phi_{q,p}(x_p) \right) = \sum_{k=1}^{2n+1} \Phi(\lambda_1 \phi_k(x_1) + \dots + \lambda_n \phi_k(x_n))$$

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to its 13th problem) is wrong."



A.G. Vitushkin, "*On Hilbert's thirteenth problem and related questions*", Russian Math. Surveys 59:1, pp. 11-25.

# Variables decoupling, KST and KANs

Loewner and KST

Remember that (in 2-D)

$$\mathbf{c}_2 = \mathcal{N}(\mathbb{L}_2) = \text{vec} \left[ \mathbf{c}_1^{\lambda_2(1)} \cdot \left[ \mathbf{c}_1^{\lambda_1(k_1)} \right]_1, \dots, \mathbf{c}_1^{\lambda_2(k_2)} \cdot \left[ \mathbf{c}_1^{\lambda_1(k_1)} \right]_{k_2} \right],$$

## Variable decoupling

Given data  $\mathcal{T}_n^\otimes$ , the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{x_n}}_{\text{Bary}(x_n)} \odot \underbrace{(\mathbf{c}^{x_{n-1}} \otimes \mathbf{1}_{k_n})}_{\text{Bary}(x_{n-1})} \odot \underbrace{(\mathbf{c}^{x_{n-2}} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\text{Bary}(x_{n-2})} \odot \cdots \odot \underbrace{(\mathbf{c}^{x_1} \otimes \mathbf{1}_{k_n \dots k_2})}_{\text{Bary}(x_1)}.$$

where  $\mathbf{c}^{x_l}$  denotes the vectorized barycentric coefficients related to the  $l$ -th variable.

This is decoupling !

# Variables decoupling, KST and KANs

Loewner and KST

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# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $H = x_1 \cdot x_2$ )

$$\begin{aligned}\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \mu_1(i_1) &= \lambda_1(j_1) + 0.5 \\ \mu_2(i_2) &= \lambda_2(j_2) + 0.5\end{aligned}$$

$$\mathcal{T}_2^{\otimes} = \left( \begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4} \end{array} \right)$$

$$= \frac{\sum_{j_1=1}^2 \sum_{j_n=1}^2 \frac{\mathbf{C}_{2,2}^{\otimes}(j_1,j_2) \mathbf{W}_{2,2}^{\otimes}(j_1,j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^2 \sum_{j_2=1}^2 \frac{\mathbf{C}_{2,2}^{\otimes}(j_1,j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}$$

where  $\mathbf{C}_{2,2}^{\otimes}$  and  $\mathbf{W}_{2,2}^{\otimes}$  are the tensorized forms of  $\mathbf{c}$  and  $\mathbf{w}$ .

<b>c</b>	<b>w</b>	<b>c ⊙ w</b>	<b>Lag</b>
1.0	1.0	1.0	$\frac{1}{(x_1+1.0)(x_2+1.0)}$
-1.0	-1.0	1.0	$\frac{1}{(x_1+1.0)(x_2-1.0)}$
-1.0	-1.0	1.0	$\frac{1}{(x_1-1.0)(x_2+1.0)}$
1.0	1.0	1.0	$\frac{1}{(x_1-1.0)(x_2-1.0)}$

# Variables decoupling, KST and KANs

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$$\mathcal{T}_2^{\otimes} = \left( \begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4} \end{array} \right)$$

$$\left( \begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \odot \mathbf{w} & \text{Lag} \\ \hline 1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2-1.0)} \end{array} \right)$$

**Decoupling theorem**

$$\begin{aligned}\mathbf{c}^{x_2} &= \text{vec} \begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\ \mathbf{c}^{x_1} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{c}_2 &= \underbrace{\mathbf{c}^{x_2}}_{\mathbf{Bary}(x_1)} \odot (\mathbf{c}^{x_1} \otimes \mathbf{1}_{k_2}) \\ &= \underbrace{\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}_{\mathbf{Bary}(x_1)} \odot \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}_{\mathbf{Bary}(x_2)}\end{aligned}$$

# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $H = x_1 \cdot x_2$ )

$$\begin{aligned}\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \mu_1(i_1) &= \lambda_1(j_1) + 0.5 \\ \mu_2(i_2) &= \lambda_2(j_2) + 0.5\end{aligned}$$

$$\mathcal{T}_2^{\otimes} = \left( \begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{3}{2} & -\frac{1}{4} & \frac{9}{4} \end{array} \right)$$

$$\left( \begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \odot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2-1.0)} \end{array} \right)$$

Denominator / Numerator

$$\mathbf{D} = \left( \begin{array}{cc} \overbrace{\Phi_1(x_1)}^{\text{Bary}(x_1) \cdot \text{Lag}(x_1)} & \overbrace{\Phi_2(x_2)}^{\text{Bary}(x_2) \cdot \text{Lag}(x_2)} \\ \frac{1.0}{x_1+1.0} & \frac{1.0}{x_2+1.0} \\ -\frac{1.0}{x_1+1.0} & -\frac{1.0}{x_2+1.0} \\ \frac{1}{x_1-1.0} & \frac{1}{x_2-1.0} \\ \frac{1}{x_1-1.0} & \frac{1}{x_2-1.0} \end{array} \right)$$

Equivalent denominator and numerator read:

$$\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \quad \text{and} \quad \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}$$

# Variables decoupling, KST and KANs

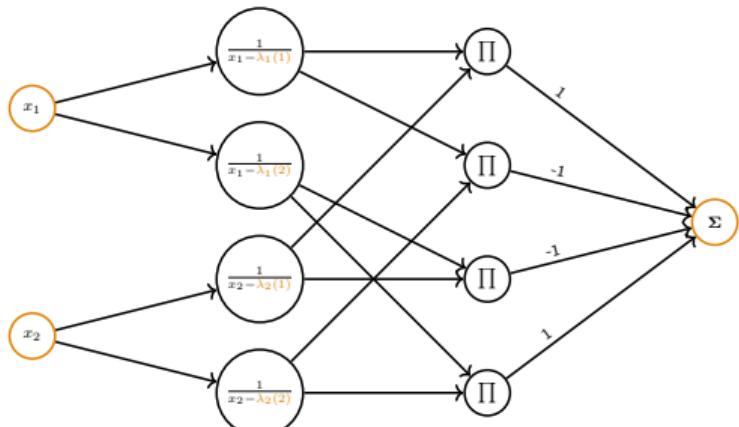
Decoupling, KST and KANs via Loewner with rational activation functions ( $H = x_1 \cdot x_2$ )

$$\begin{aligned}\lambda_1(j_1) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \lambda_2(j_2) &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \mu_1(i_1) &= \lambda_1(j_1) + 0.5 \\ \mu_2(i_2) &= \lambda_2(j_2) + 0.5\end{aligned}$$

$$\mathcal{T}_2^{\otimes} = \left( \begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & \frac{9}{4} \end{array} \right)$$

$$\left( \begin{array}{ccc|c} \mathbf{c} & \mathbf{w} & \mathbf{c} \odot \mathbf{w} & \text{Lag} \\ 1.0 & 1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2+1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1+1.0)(x_2-1.0)} \\ -1.0 & -1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2+1.0)} \\ 1.0 & 1.0 & 1.0 & \frac{1}{(x_1-1.0)(x_2-1.0)} \end{array} \right)$$

Denominator Network view



# Variables decoupling, KST and KANs

Decoupling, KST and KANs via Loewner with rational activation functions ( $\mathbf{H} = x_1 \cdot x_2$ )

## KST via Loewner

$$\mathbf{H}_{\text{lag}}(x_1, x_2) = \frac{\sum_{j_1=1}^2 \sum_{j_2=1}^2 \frac{c_{j_1, j_2} \mathbf{w}_{j_1, j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^2 \sum_{j_2=1}^2 \frac{c_{j_1, j_2}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}$$

$\Updownarrow$

$$\begin{aligned}\mathbf{H}_{\text{kst}}(x_1, x_2) &= \frac{\sum_{\text{rows}} \mathbf{w} \odot \Phi_1(x_1) \odot \Phi_2(x_2)}{\sum_{\text{rows}} \Phi_1(x_1) \odot \Phi_2(x_2)} \\ &= \frac{\sum_{\text{rows}} \exp(\log(\mathbf{w}) + \log(\Phi_1(x_1)) + \log(\Phi_2(x_2)))}{\sum_{\text{rows}} \exp(\log(\Phi_1(x_1)) + \log(\Phi_2(x_2)))}\end{aligned}$$

This is the solution of KST for rational forms !

# Content

Forewords

Multi-variate data, function & Loewner matrix

Taming the curse of dimensionality

Variables decoupling, KST and KANs

## Comparisons

Multi-variate realization

The MATLAB `mLF` package

Conclusion

# Comparisons

## Some competitors

Rat. app [B/G 2025]

- ▶ Lagrangian interpolation theorem
- ▶ p-AAA

KAN [P/P 2025]

- ▶ Kolmogorov Arnold theorem
- ▶ Kolmogorov Arnold Network

MLP [TensorFlow by Google, via Keras 2025]

- ▶ Universal approximation theorem
- ▶ Multi Layer Perceptron
- ▶ Dense connected / ReLU / ADAM / 1000 it. / rand. init.



L. Balicki and S. Gugercin, "[Multivariate Rational Approximation via Low-Rank Tensors and the p-AAA Algorithm](#)", SISC, 2025.



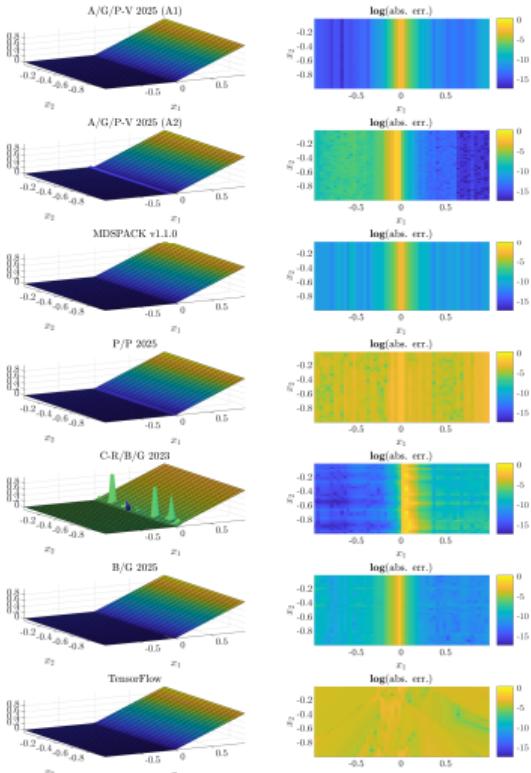
M. Poluektov and A. Polar, "[Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method](#)",  
<https://arxiv.org/abs/2305.08194>.



M. Abadi et al., "[TensorFlow: Large-scale machine learning on heterogeneous systems, 2015](#)", Software available from [tensorflow.org](http://tensorflow.org).

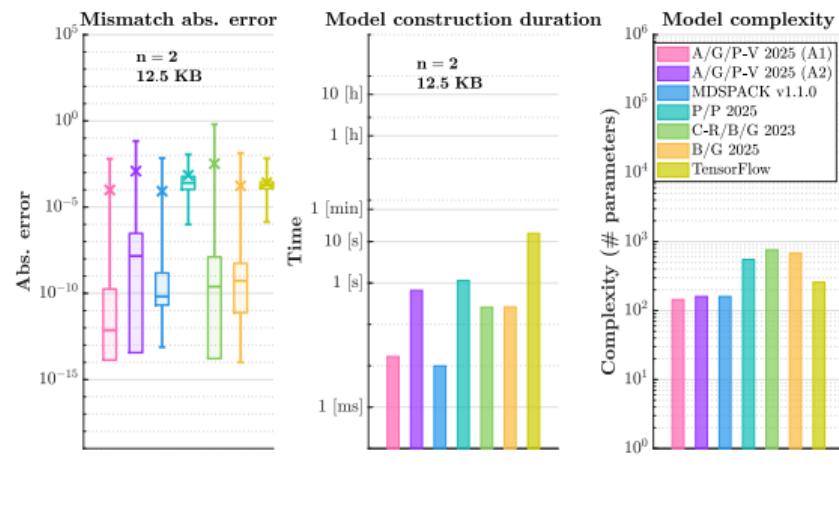
# Comparisons

## Function #1



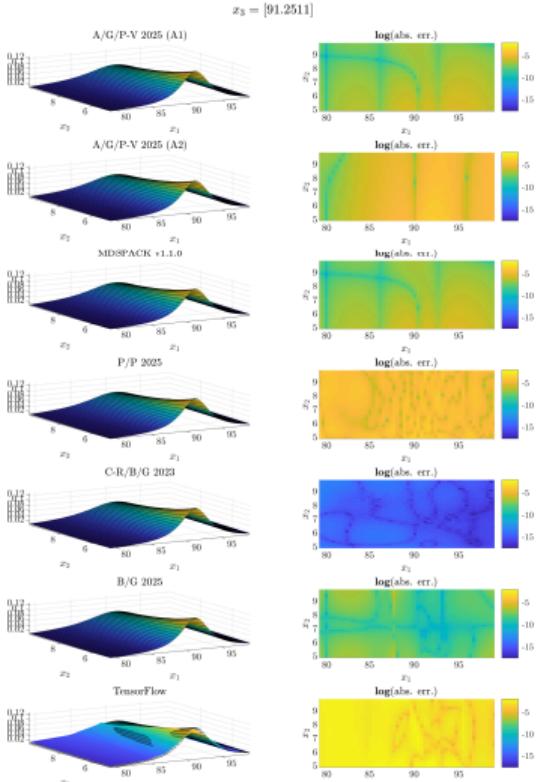
$$\text{ReLU}(x_1) + \frac{1}{100}x_2$$

- ▶ Reference: Personal communication
- ▶ Tensor size: **12.5 KB**



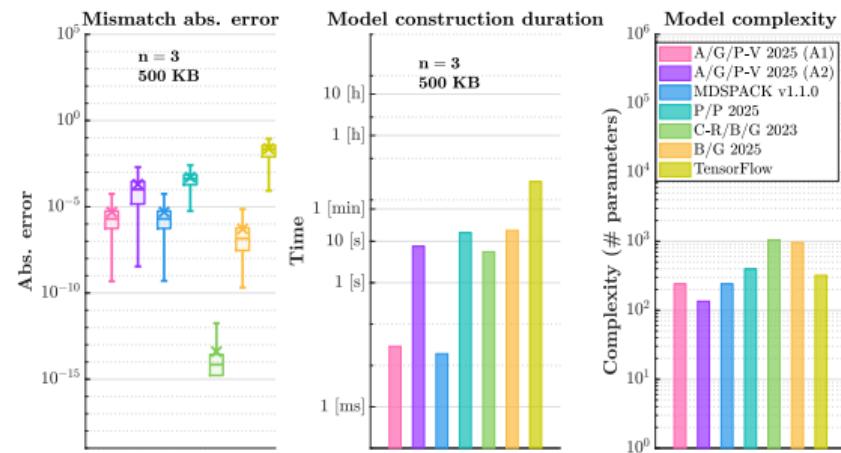
# Comparisons

## Function #20



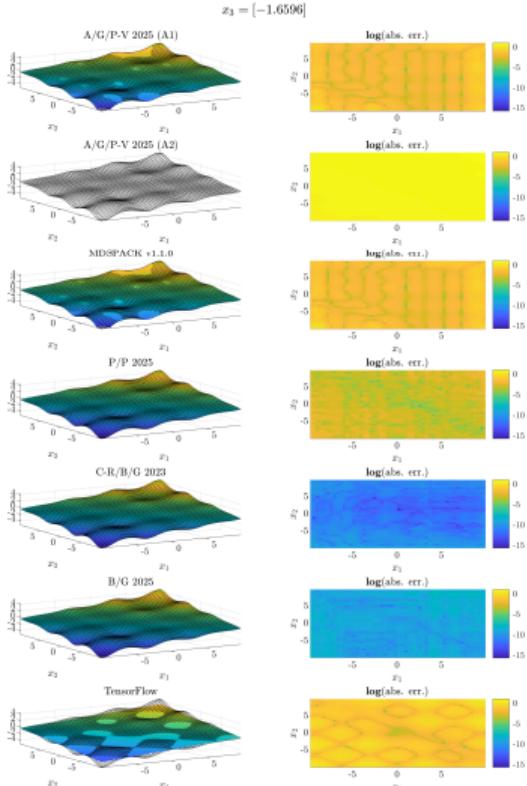
### Breit Wigner function

- ▶ Reference: A/al. 2021 (A.5.15)
- ▶ Tensor size: **500 KB**



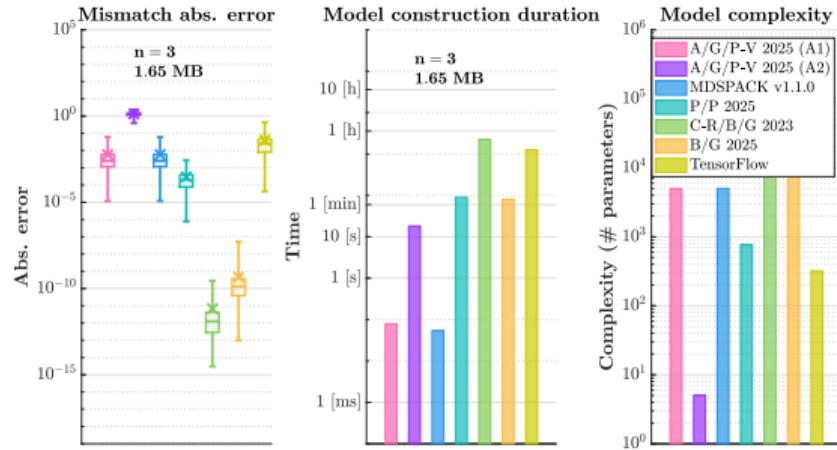
# Comparisons

## Function #26



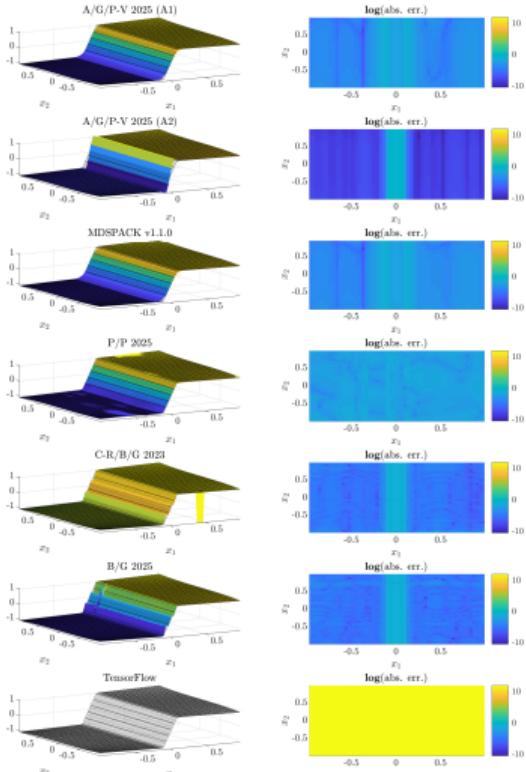
$$\frac{x_1 + x_2 + x_3}{6 + \cos(x_1) + \cos(x_2) + \cos(x_3)}$$

- ▶ Reference: B/G 2025
- ▶ Tensor size: **1.65 MB**



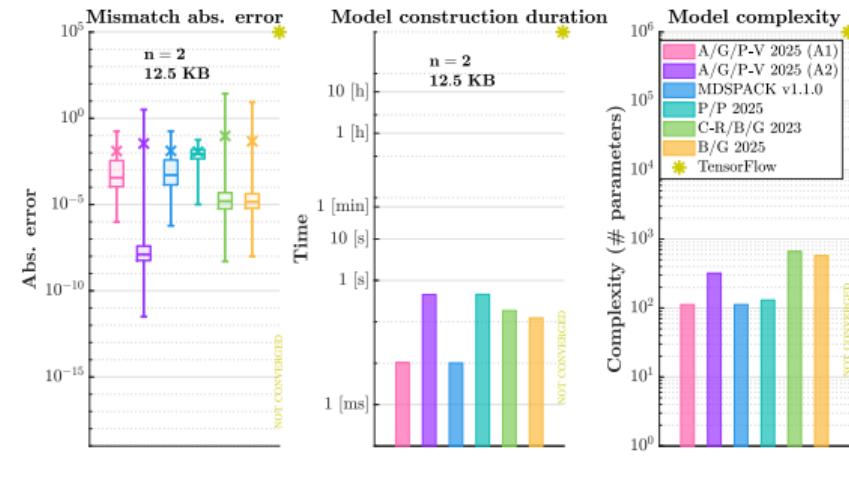
# Comparisons

## Function #29



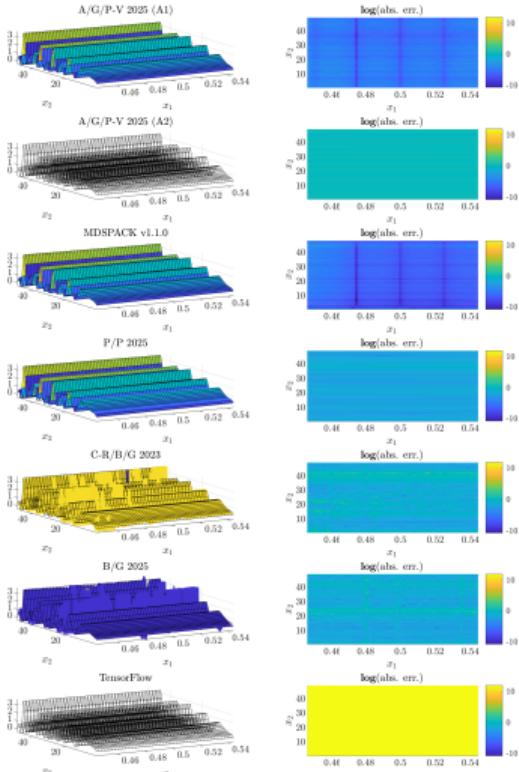
$$\min(10|x_1|, 1)\text{sign}(x_1) + \frac{x_1 x_2^3}{10}$$

- ▶ Reference: Personal communication
- ▶ Tensor size: **12.5 KB**



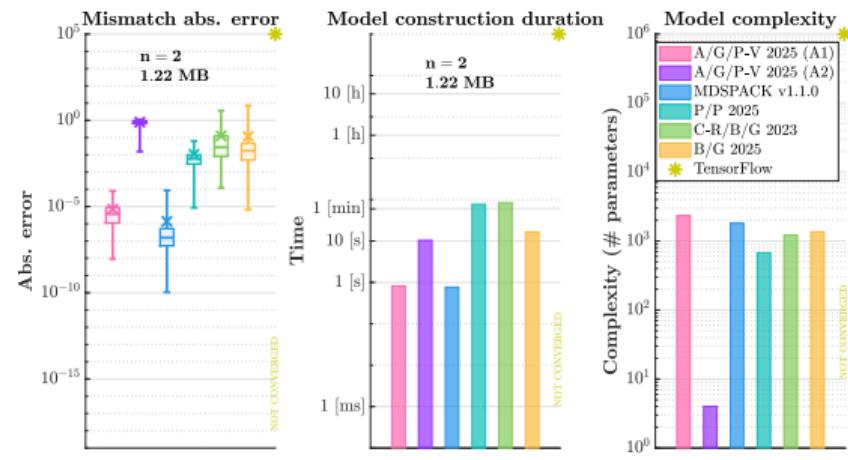
# Comparisons

## Function #34



Riemann  $\zeta$  function (real part)

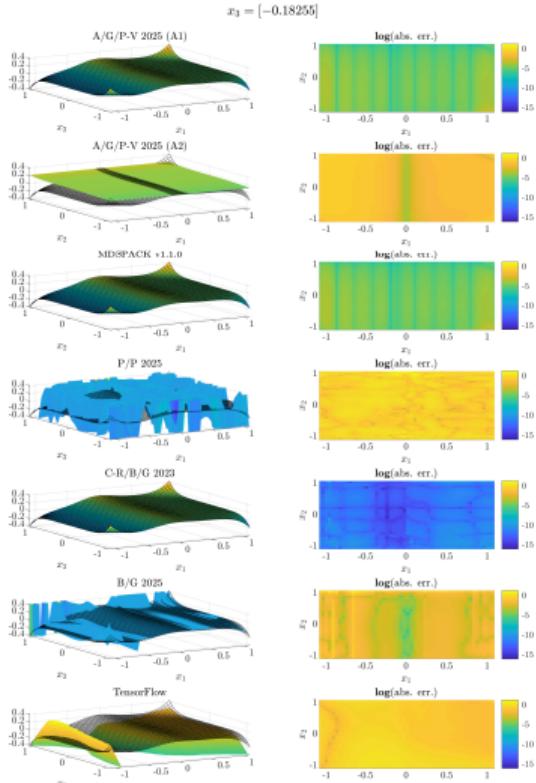
- ▶ Reference: Riemann  $\zeta$  function (real part)
- ▶ Tensor size: **1.22 MB**



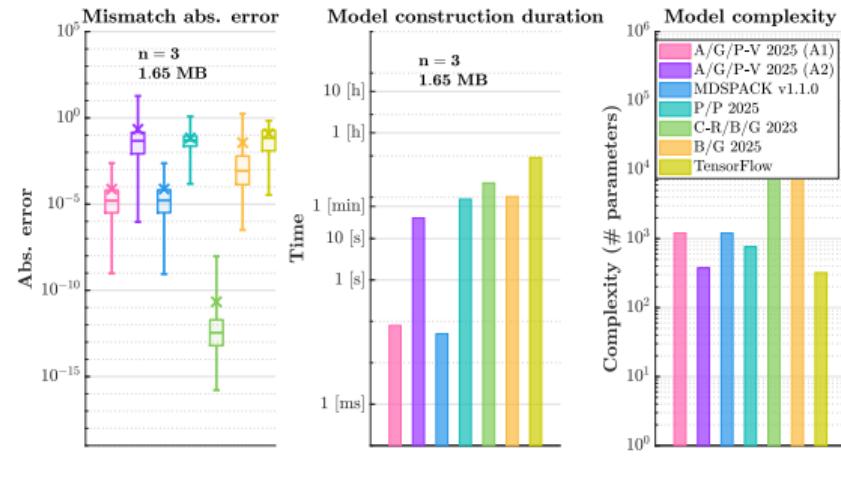
# Comparisons

## Function #38

$$\frac{x_1^9 x_2^7 + x_1^3 + 5x_3^2}{5x_1^4 + 4x_1^2 + x_3 x_2^3 + 1}$$

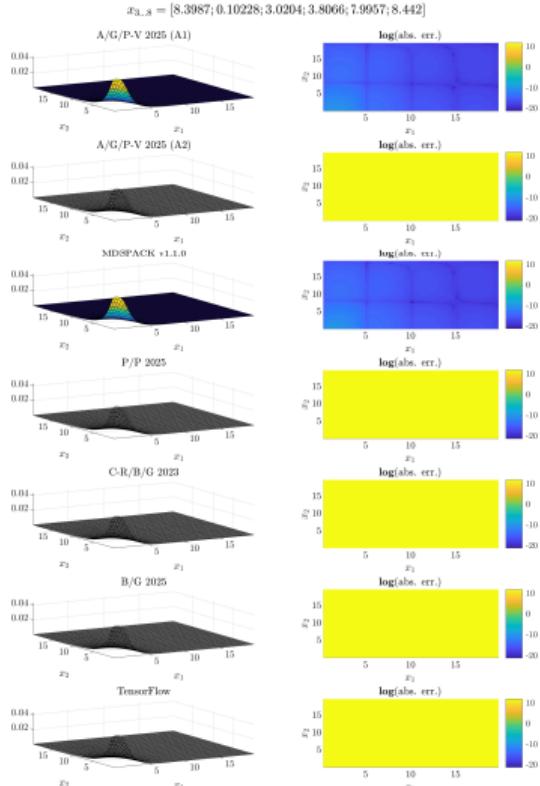


- ▶ Reference: A.C. Antoulas presentation
- ▶ Tensor size: **1.65 MB**



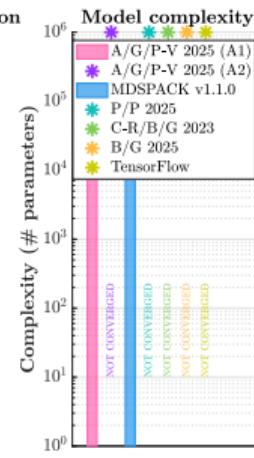
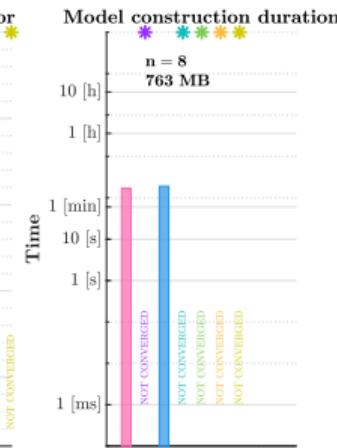
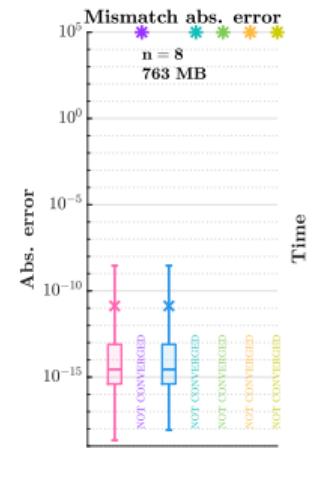
# Comparisons

## Function #44



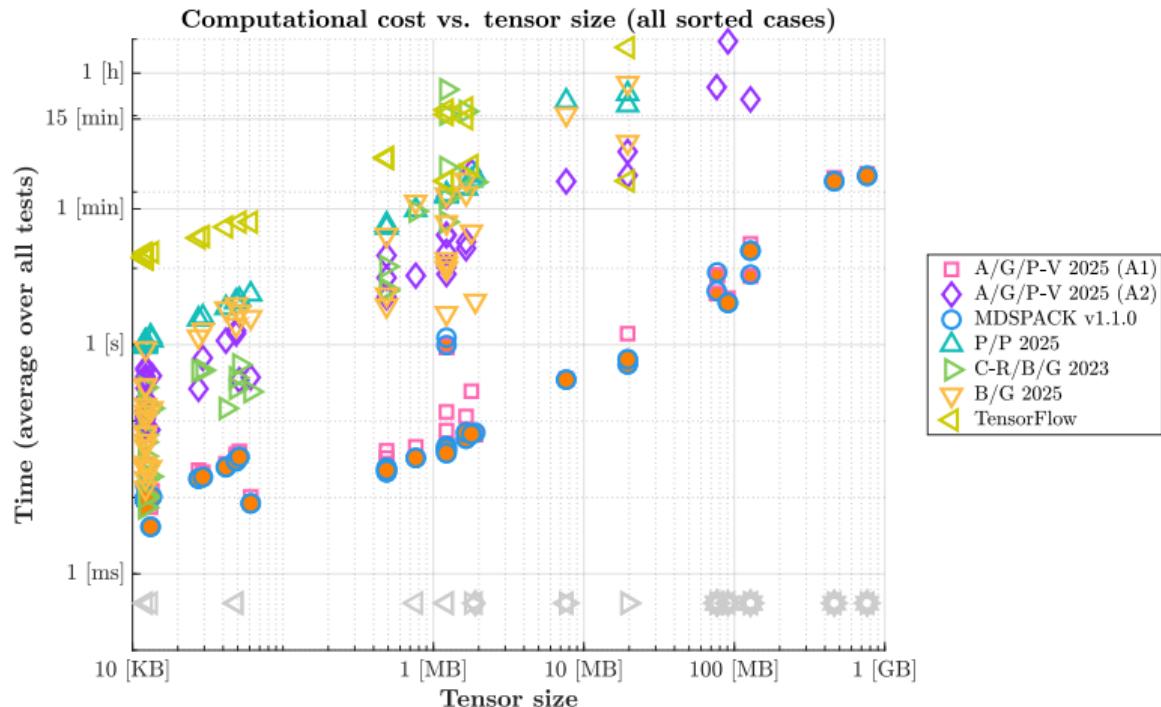
$$\frac{1}{x_1^4 + x_2^2 x_3 + x_4^2 + x_5 + x_6 + x_7 + x_8}$$

- ▶ Reference: Personal communication
- ▶ Tensor size: **763 MB**



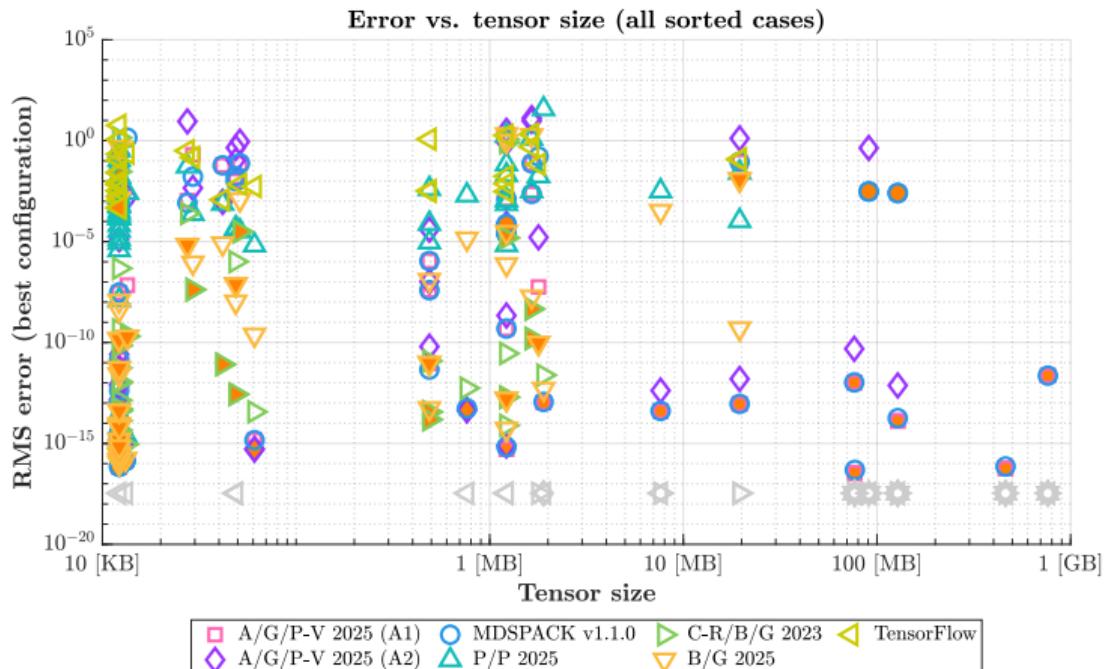
# Comparisons

## Size vs. computation time



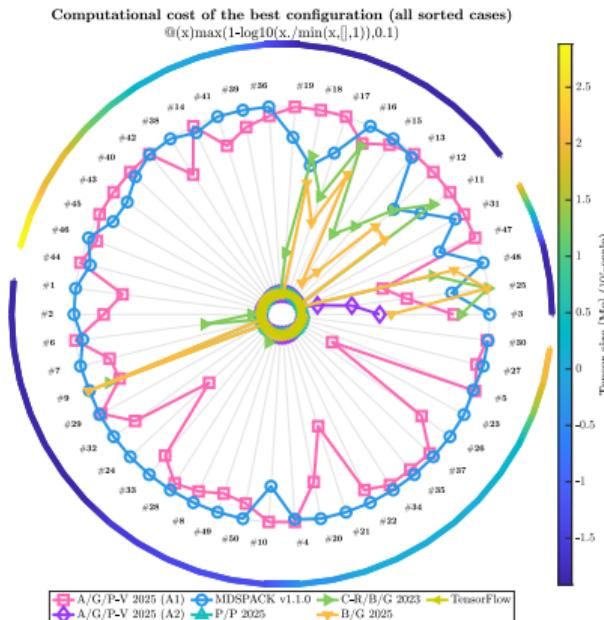
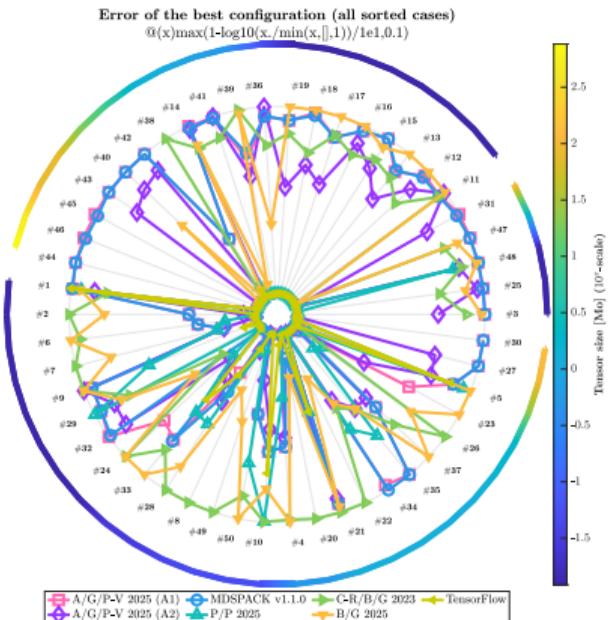
# Comparisons

## Size vs. error



# Comparisons

## Radar error & computational time



# Content

Forewords

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**Multi-variate realization**

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# Multi-variate realization

1-D case (example cont'd)

Data generated from  $\mathbf{H}(x_1) = \mathbf{H}(s) = (s^2 + 4)/(s + 1)$  of complexity (2)

$$g(s) = \frac{\frac{5}{6(s-1)} - \frac{13}{3(s-3)} + \frac{29}{6(s-5)}}{\frac{1}{3(s-1)} - \frac{4}{3(s-3)} + \frac{1}{s-5}}$$

Lagrangian realization  $\hat{\mathbf{H}}(s) = \mathbf{W}\Phi(s)^{-1}\mathbf{G}$

$$\Phi(s) = \begin{bmatrix} s-1 & 3-s & 0 \\ s-1 & 0 & 5-s \\ -\frac{1}{3} & \frac{4}{3} & -1 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} \frac{5}{6} & -\frac{13}{3} & \frac{29}{6} \end{bmatrix}$$

$$\mathbf{G}^\top = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

# Multi-variate realization

2-D case (example cont'd)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}}$$

Lagrangian realization  $\hat{\mathbf{H}}(s, t) = \mathbf{W}\Phi(s, t)^{-1}\mathbf{G}$

$$\Phi(s, t) = \begin{bmatrix} s-1 & 3-s & 0 & | & 0 & | & 0 & 0 \\ s-1 & 0 & 5-s & | & 0 & | & 0 & 0 \\ -\frac{1}{3} & -\frac{10}{9} & -\frac{7}{9} & | & t+1 & | & 0 & 0 \\ \frac{5}{9} & -\frac{14}{9} & 1 & | & -t-3 & | & 0 & 0 \\ \frac{1}{9} & -2 & -\frac{25}{9} & | & 0 & | & t+1 & -\frac{1}{2} \\ -\frac{1}{3} & 6 & -\frac{25}{3} & | & 0 & | & -t-3 & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & | & -1 \end{bmatrix}$$

$$\mathbf{G}^\top = \begin{bmatrix} 0 & 0 & | & 1/2 & -1/2 & | & 0 & 0 \end{bmatrix}$$

→ (3,3) block is unimodular !

# Multi-variate realization

2-D case (example cont'd)

Data generated from  $\mathbf{H}(x_1, x_2) = \mathbf{H}(s, t) = (s^2t)/(s - t + 1)$  of complexity (2, 1)

$$\mathbf{g}(s, t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}}$$

Lagrangian realization  $\hat{\mathbf{H}}_{\mathbf{c}}(s, t) = \mathbf{W}_{\mathbf{c}}(t)\Phi_{\mathbf{c}}(s, t)^{-1}\mathbf{G}_{\mathbf{c}}$

$$\Phi_{\mathbf{c}}(s, t) = \left[ \begin{array}{ccc|c} s-1 & 3-s & 0 & 0 \\ s-1 & 0 & 5-s & 0 \\ -\frac{1}{3} & -\frac{10}{9} & -\frac{7}{9} & t+1 \\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 \end{array} \right]$$

$$\mathbf{W}_{\mathbf{c}}(t) = \left[ \begin{array}{ccc|c} -\frac{2t}{9} & 4t & -\frac{50t}{9} & 0 \end{array} \right]$$

$$\mathbf{G}_{\mathbf{c}}^T = \left[ \begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

# Multi-variate realization

Generalized  $n$ -D Lagrangian realization

$$g(x_1, x_2, \dots, x_n) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, j_2, \dots, j_n} w_{j_1, j_2, \dots, j_n}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2)) \cdots (x_n - \lambda_n(j_n))}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, j_2, \dots, j_n}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2)) \cdots (x_n - \lambda_n(j_n))}},$$

## Theorem: $n$ -D Lagrangian realization

A  $2\ell + \kappa - 1 = m$ -th order realization  $(G, \Phi, W)$  of the multi-variate function  $\hat{H}$  in barycentric form, satisfying  $\hat{H}(x_1, \dots, x_n) = W\Phi(x_1, x_2, \dots, x_n)^{-1}G$ , is given by,

$$\begin{aligned}\Phi(x_1, \dots, x_n) &= \begin{bmatrix} \Gamma(1 : \kappa - 1, :) & | & \mathbf{0}_{\kappa-1, \ell-1} & | & \mathbf{0}_{\kappa-1, \ell} \\ \bar{A}^{\text{Lag}} & | & \Delta(\bar{1} : \bar{\ell} - 1, :)^\top & | & \mathbf{0}_{\ell, \ell} \\ \bar{B}^{\text{Lag}} & | & \mathbf{0}_{\ell, \ell-1} & | & \Delta^\top \end{bmatrix} \in \mathbb{C}^{m \times m} \\ G &= \begin{bmatrix} \mathbf{0}_{\kappa-1, 1} \\ \bar{\Delta}(\bar{\ell}, :)^\top \\ \mathbf{0}_{\ell, 1} \end{bmatrix} \in \mathbb{C}^{m \times 1} \\ W &= \begin{bmatrix} \mathbf{0}_{1, \kappa} & | & \mathbf{0}_{1, \ell-1} & | & -\mathbf{e}_\ell^\top \end{bmatrix} \in \mathbb{C}^{1 \times m}\end{aligned}$$

where  $\bar{A}^{\text{Lag}}, \bar{B}^{\text{Lag}} \in \mathbb{C}^{\ell \times \kappa}$  are appropriately chosen, according to the chosen pseudo-companion basis.

# Multi-variate realization

Generalized  $n$ -D Lagrangian realization (focus on left / right variable sets)

$$\Phi(x_1, \dots, x_n) = \begin{bmatrix} \Gamma(1 : \kappa - 1, :) & | & \mathbf{0}_{\kappa-1, \ell-1} & | & \mathbf{0}_{\kappa-1, \ell} \\ \bar{\mathbb{A}}^{\text{Lag}} & | & \Delta(\bar{1} : \bar{\ell} - 1, :)^\top & | & \mathbf{0}_{\ell, \ell} \\ \bar{\mathbb{B}}^{\text{Lag}} & | & \mathbf{0}_{\ell, \ell-1} & | & \Delta^\top \end{bmatrix} \in \mathbb{C}^{m \times m}$$

$$\Gamma = \mathbb{X}^{\text{Lag}}_1 \otimes \mathbb{X}^{\text{Lag}}_2 \otimes \cdots \otimes \mathbb{X}^{\text{Lag}}_k \in \mathbb{C}^{\kappa \times \kappa}[x_1, \dots, x_k]$$

$$\Delta = \mathbb{X}^{\text{Lag}}_{k+1} \otimes \mathbb{X}^{\text{Lag}}_{k+2} \otimes \cdots \otimes \mathbb{X}^{\text{Lag}}_n \in \mathbb{C}^{\ell \times \ell}[x_{k+1}, \dots, x_n]$$

$$\begin{aligned} \mathbb{X}^{\text{Lag}}_j &= \begin{bmatrix} \mathbf{x}_{1j} & -\mathbf{x}_{2j} & 0 & \cdots & 0 \\ \mathbf{x}_{1j} & 0 & -\mathbf{x}_{3j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{x}_{1j} & 0 & \cdots & 0 & -\mathbf{x}_{nj,j} \\ q_{1j} & q_{2j} & \cdots & q_{nj-1,j} & q_{nj,j} \end{bmatrix} \in \mathbb{C}^{n_j \times n_j} \\ \mathbf{x}_{ij} &= x_j - \lambda_j(i) \end{aligned}$$

## Facts

- Left / right variables splitting

$\Gamma$  and  $\Delta$

- $\mathbb{X}^{\text{Lag}}_j$  is unimodular, i.e.

$$\det(\mathbb{X}^{\text{Lag}}_j) = 1$$

- ... so are  $\Gamma$  and  $\Delta$

# Multi-variate realization

Generalized  $n$ -D Lagrangian realization (focus on barycentric weights  $\mathbb{A}^{\text{Lag}}$  and  $\mathbb{B}^{\text{Lag}}$ )

$$\Phi(x_1, \dots, x_n) = \begin{bmatrix} \Gamma(1 : \kappa - 1, :) & \mathbf{0}_{\kappa-1, \ell-1} & \mathbf{0}_{\kappa-1, \ell} \\ \bar{\mathbb{A}}^{\text{Lag}} & \Delta(1 : \ell - 1, :)^\top & \mathbf{0}_{\ell, \ell} \\ \bar{\mathbb{B}}^{\text{Lag}} & \mathbf{0}_{\ell, \ell-1} & \Delta^\top \end{bmatrix} \in \mathbb{C}^{m \times m}$$

$$\begin{aligned}\mathbb{A}^{\text{Lag}} &= \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m+1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,m+1} \end{bmatrix} \\ \mathbf{c}_n &= \text{vec}(\mathbb{A}^{\text{Lag}})\end{aligned}$$

## Facts

- $\mathbb{A}^{\text{Lag}}$  is simply some rearrangement of  $\mathcal{N}(\mathbb{L}_n) = \mathbf{c}_n$
- $\mathbb{B}^{\text{Lag}}$  follows

# Multi-variate realization

Generalized  $n$ -D Lagrangian realization (control the complexity)

$$\Phi(x_1, \dots, x_n) = \begin{bmatrix} \Gamma(1 : \kappa - 1, :) & \mathbf{0}_{\kappa-1, \ell-1} & \mathbf{0}_{\kappa-1, \ell} \\ \bar{\mathbb{A}}^{\text{Lag}} & \Delta(1 : \ell - 1, :)^\top & \mathbf{0}_{\ell, \ell} \\ \bar{\mathbb{B}}^{\text{Lag}} & \mathbf{0}_{\ell, \ell-1} & \Delta^\top \end{bmatrix} \in \mathbb{C}^{m \times m}$$

$$m = 2\ell + \kappa - 1$$

$$\begin{aligned} \kappa &= \prod_{j=1}^k n_j \\ \ell &= \prod_{j=k+1}^n n_j \end{aligned}$$

## Facts

- ▶  $\Gamma$  gathers the first group of parameters  $x_1, \dots, x_k$
- ▶  $\Delta$  gathers the second group of parameters  $x_{k+1}, \dots, x_n$

## Complexity

Re-ordering allows complexity control, e.g. according to the order of each variable  $x_j$

# Multi-variate realization

## 3-D case (example)

Data generated from  $\mathbf{H}(x_1, x_2, x_3) = \mathbf{H}(s, t, p) = (s + pt)/(p^2 + s + t)$  of complexity  $(1, 1, 2)$

$$\begin{aligned}\mathbf{c}_3^\top &= \left[ \frac{1}{2} \quad -\frac{39}{28} \quad \frac{13}{14} \mid -\frac{15}{28} \quad \frac{41}{28} \quad -\frac{27}{28} \mid -\frac{15}{28} \quad \frac{41}{28} \quad -\frac{27}{28} \mid \frac{4}{7} \quad -\frac{43}{28} \quad 1 \right] \\ \mathbb{W}_3 &= \left[ \frac{1}{4} \quad \frac{8}{39} \quad \frac{9}{52} \mid \frac{17}{30} \quad \frac{20}{41} \quad \frac{23}{54} \mid \frac{3}{10} \quad \frac{10}{41} \quad \frac{11}{54} \mid \frac{19}{32} \quad \frac{22}{43} \quad \frac{25}{56} \right]\end{aligned}$$

### Arrangement #1

$(s) - (t, p)$ , one obtains a realization of dimension  $m = 13$ :

$$\kappa = 2 \text{ and } \ell = 2 \times 3$$

$$\Delta(s) = \mathbb{X}^{\text{Lag}}_1(s)$$

$$\Gamma(t, p) = \mathbb{X}^{\text{Lag}}_2(t) \otimes \mathbb{X}^{\text{Lag}}_3(p)$$

### Arrangement #2

$(s, t) - (p)$ , one obtains a realization of dimension  $m = 9$ :

$$\kappa = 2 \times 2 \text{ and } \ell = 3$$

$$\Delta(s, t) = \mathbb{X}^{\text{Lag}}_1(s) \otimes \mathbb{X}^{\text{Lag}}_2(t)$$

$$\Gamma(p) = \mathbb{X}^{\text{Lag}}_3(p)$$

# Multi-variate realization

## 3-D case (example)

Data generated from  $\mathbf{H}(x_1, x_2, x_3) = \mathbf{H}(s, t, p) = (s + pt)/(p^2 + s + t)$  of complexity (1, 1, 2)

$$\begin{aligned}\mathbf{c}_3^\top &= \left[ \begin{array}{ccc|ccc|ccc|cc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array} \right] \\ \mathbb{W}_3 &= \left[ \begin{array}{ccc|ccc|ccc|cc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array} \right]\end{aligned}$$

$$\Phi = \left[ \begin{array}{cccc|cccc|cccc} (s-2)(t-1) & -(s-2)(t-3) & -(t-1)(s-4) & (s-4)(t-3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 - \frac{s}{2} & \frac{s}{2} - 1 & \frac{s}{2} - 2 & 2 - \frac{s}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} - \frac{t}{2} & \frac{t}{2} - \frac{3}{2} & \frac{t}{2} - \frac{1}{2} & \frac{3}{2} - \frac{t}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{8} & -\frac{17}{56} & -\frac{9}{56} & \frac{19}{56} & 0 & 0 & 0 & p-5 & p-5 & \frac{1}{2} & 0 & 0 \\ -\frac{2}{7} & \frac{5}{7} & \frac{5}{14} & -\frac{11}{14} & 0 & 0 & 6-p & 0 & 0 & 0 & 0 & -1 \\ \frac{9}{56} & -\frac{23}{56} & -\frac{11}{56} & \frac{25}{56} & 0 & 0 & 0 & 7-p & \frac{1}{2} & 0 & 0 & 0 \end{array} \right]$$

$$\mathbf{W} = -\mathbf{e}_9^\top \text{ and } \mathbf{G}^\top = \left[ \begin{array}{ccccc} \mathbf{0}_{1,3} & | & 1/2 & -1 & 1/2 & | & \mathbf{0}_{1,3} \end{array} \right]$$

# Multi-variate realization

## 3-D case (example)

Data generated from  $\mathbf{H}(x_1, x_2, x_3) = \mathbf{H}(s, t, p) = (s + pt)/(p^2 + s + t)$  of complexity (1, 1, 2)

$$\begin{aligned}\mathbf{c}_3^\top &= \left[ \begin{array}{ccc|ccc|ccc|cc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array} \right] \\ \mathbb{W}_3 &= \left[ \begin{array}{ccc|ccc|ccc|cc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array} \right]\end{aligned}$$

$$\Phi_c = \left[ \begin{array}{cccc|cc} (s-2)(t-1) & -(s-2)(t-3) & -(t-1)(s-4) & (s-4)(t-3) & 0 & 0 \\ 1 - \frac{s}{2} & \frac{s}{2} - 1 & \frac{s}{2} - 2 & 2 - \frac{s}{2} & 0 & 0 \\ \frac{1}{2} - \frac{t}{2} & \frac{t}{2} - \frac{3}{2} & \frac{t}{2} - \frac{1}{2} & \frac{3}{2} - \frac{t}{2} & 0 & 0 \\ \hline -\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p - 5 & p - 5 \\ -\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6 - p & 0 \\ \frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7 - p \end{array} \right]$$

$$\mathbf{W}_c(p) = \left[ \begin{array}{ccccc} \frac{p}{28} + \frac{1}{14} & -\frac{3p}{28} - \frac{1}{14} & -\frac{p}{28} - \frac{1}{7} & \frac{3p}{28} + \frac{1}{7} & 0 \end{array} \right] \text{ and } \mathbf{G}_c^\top = \left[ \begin{array}{ccccc} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 \end{array} \right]$$

# Multi-variate realization

(Compressed) generalized  $n$ -D Lagrangian realization

$$g(x_1, x_2, \dots, x_n) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, j_2, \dots, j_n} w_{j_1, j_2, \dots, j_n}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2)) \cdots (x_n - \lambda_n(j_n))}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1, j_2, \dots, j_n}}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2)) \cdots (x_n - \lambda_n(j_n))}},$$

## Theorem: $n$ -D Lagrangian compressed realization

A  $\ell + \kappa - 1 = m$ -th order realization  $(\hat{\mathbf{G}}_c, \hat{\Phi}_c, \hat{\mathbf{W}}_c)$  of the multi-variate function  $\hat{\mathbf{H}}$  in barycentric form, satisfying  $\mathbf{H}(x_1, \dots, x_n) = \hat{\mathbf{W}}_c \hat{\Phi}_c(x_1, x_2, \dots, x_n)^{-1} \hat{\mathbf{G}}_c(x_{k+1}, \dots, x_n)$ , is given by,

$$\begin{aligned}\hat{\Phi}_c(x_1, \dots, x_n) &= \begin{bmatrix} \Gamma(1 : \kappa - 1, :) \\ \vdots \\ \mathbb{A}^{\text{Lag}} \end{bmatrix}^\top \begin{bmatrix} \mathbf{0}_{\kappa-1, \ell-1} \\ \Delta(\bar{1} : \ell - 1, :)^\top \end{bmatrix} \in \mathbb{C}^{m \times m} \\ \hat{\mathbf{G}}_c(x_{k+1}, \dots, x_n) &= \begin{bmatrix} \mathbf{0}_{\kappa-1, 1} \\ \Delta(\bar{\ell}, :)^\top \end{bmatrix} \in \mathbb{C}^{m \times 1} \\ \hat{\mathbf{W}}_c &= \mathbf{e}_\ell^\top \Delta^{-\top} \begin{bmatrix} \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell, \ell-1} \end{bmatrix} \in \mathbb{C}^{1 \times m}\end{aligned}$$

where  $\mathbb{A}^{\text{Lag}}, \mathbb{B}^{\text{Lag}} \in \mathbb{C}^{\ell \times \kappa}$  are appropriately chosen, according to the chosen pseudo-companion basis.

# Content

Forewords

Multi-variate data, function & Loewner matrix

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

Multi-variate realization

**The MATLAB mLF package**

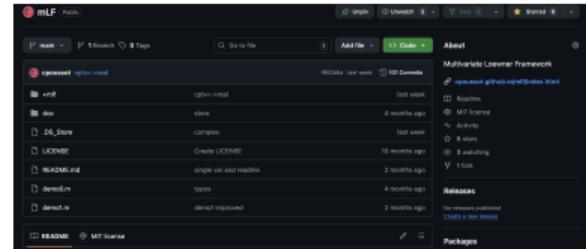
Conclusion

# The MATLAB mLF package

[GitHub page](https://github.com/cpoussot/mLF)

<https://github.com/cpoussot/mLF>

```
%% Chose model
[H, infoCas] = mlf.examples(34);
n = infoCas.n;
p_c = infoCas.p_c;
p_r = infoCas.p_r;
%% Data tensor/rand
[y, x, dim] = mlf.make_tab_vec(H, p_c, p_r);
tab = mlf.vec2mat(y, dim);
%% Alg. 1: direct pLoe [A/G/P-V, 2025]
opt.ord_tol = 1e-9;
opt.method_null = 'svd0';
opt.method = 'rec';
opt.ord_obj = [];
opt.ord_N = 10;
opt.ord_show = false;
opt.data_min = true;
[g, iloe1] = mlf.alg1(tab, p_c, p_r, opt);
```



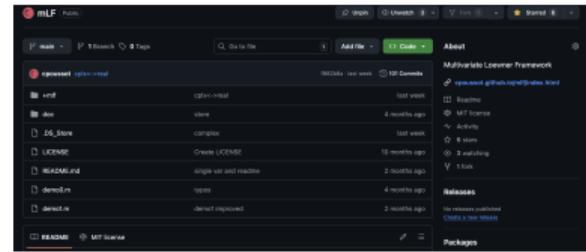
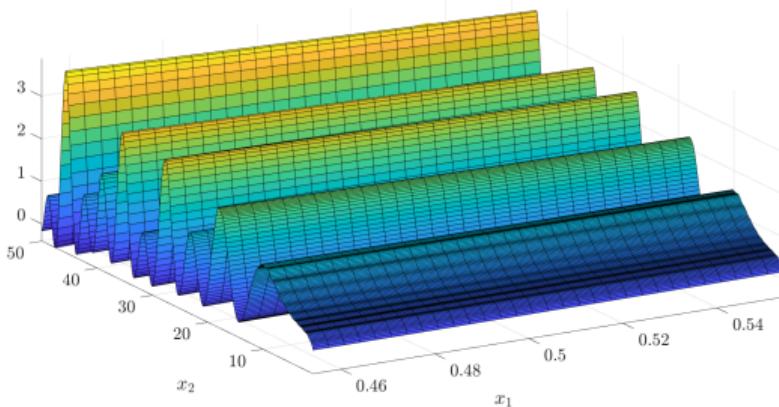
- ▶ Generic algorithm
  - mlf.alg1
  - mlf.alg2
- ▶ Specific steps
  - mlf.compute\_order,
  - mlf.point\_selection,
  - mlf.loewner\_null\_rec,
  - mlf.tf\_lagrangian, ...

# The MATLAB mLF package

[GitHub page](https://github.com/cpoussot/mLF)

<https://github.com/cpoussot/mLF>

mLF alg. 1 (direct),  $r = [4, 113]$



- ▶ Generic algorithm
  - `mlf.alg1`
  - `mlf.alg2`
- ▶ Specific steps
  - `mlf.compute_order`,
  - `mlf.point_selection`,
  - `mlf.loewner_null_rec`,
  - `mlf.tf_lagrangian`, ...

# The MATLAB mLF package

## Detailed example 2-D (define the problem)

---

```
%// Define a multivariate problem
syms x1 x2;
Hsym = x1*x2^3+2*x1*x2-1;
Hf = matlabFunction(Hsym);
H = @(x) Hf(x(:,1),x(:,2));
n = 2; % number of variables
xbnd = [-1 1]; % bounds of both variables
Nip = 10; % number of interpolation points
X1 = linspace(xbnd(1),xbnd(2),Nip);
X2 = linspace(xbnd(1),xbnd(2),Nip);
```

---

$$\mathbf{H}(x_1, x_2) = x_1 x_2^3 + 2 x_1 x_2 - 1$$

- ▶  $n = 2$ , bounds  $\mathcal{X}_l := \begin{bmatrix} -1 & 1 \end{bmatrix}$ , discretized with  $N_l = 10$ ,  $l = \{1, 2\}$
- ▶  $\mathbf{x}_l = \begin{bmatrix} -1 & -\frac{7}{9} & -\frac{5}{9} & -\frac{1}{3} & -\frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{5}{9} & \frac{7}{9} & 1 \end{bmatrix} \in \mathcal{X}_l^{10}$

# The MATLAB mLF package

## Detailed example 2-D (select interpolation points and construct the tensor)

```
%%% Interpolation points
p_c{1} = X1(2:2:end);
p_r{1} = X1(1:2:end);
p_c{2} = X2(2:2:end);
p_r{2} = X2(1:2:end);
```

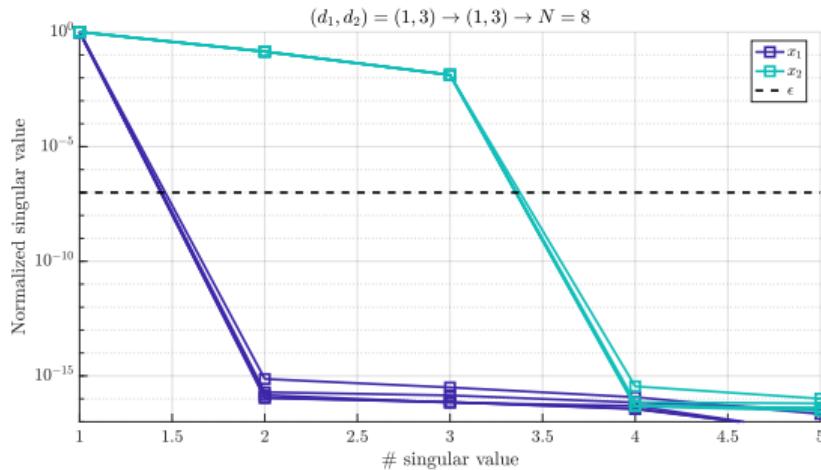
```
%%% Construct tensor
% Check that column/row IP are disjoint
ok = mlf.check(p_c, p_r);
% Construct tab_n
[y, x, dim] = mlf.make_tab_vec(H, p_c, p_r);
tab = mlf.vec2mat(y, dim);
```

- ▶ columns  $\lambda_l(j_l)$  (denoted  $p_c\{1\}$  and  $p_c\{2\}$ ) and
- ▶ rows  $\mu_l(i_l)$  (denoted  $p_r\{1\}$  and  $p_r\{2\}$ )
- ▶ we chose equal dimensions (i.e.  $k_l = 5$  and  $q_l = 5$ )
- ▶ IP are disjoints, and 2-D tensor  $T_2^\otimes \in \mathbb{R}^{(q_1+q_2) \times (k_1+k_2)}$

# The MATLAB mLF package

## Detailed example 2-D (estimate the orders)

```
%// Estimate orders along each variables and select a subset of IP  
tol          = 1e-7;  
ord          = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );  
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord,true );  
w            = mlf.mat2vec(W);
```



# The MATLAB mLF package

## Detailed example 2-D (estimate the orders)

```
%// Estimate orders along each variables and select a subset of IP  
tol          = 1e-7;  
ord          = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );  
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord,true );  
w            = mlf.mat2vec(W);
```

- ▶  $(k_1, k_2) = (2, 4)$  column IP
- ▶ Barycentric dimension  $2 \times 4 = 8 = N$
- ▶  $\lambda_1(j_1) = \begin{bmatrix} -\frac{7}{9} & 1 \end{bmatrix} (pc\{1\})$
- ▶  $\lambda_2(j_2) = \begin{bmatrix} -\frac{7}{9} & -\frac{1}{3} & \frac{1}{9} & 1 \end{bmatrix} (pc\{2\})$
- ▶  $\mu_1(i_1) = \begin{bmatrix} -1 & \frac{7}{9} \end{bmatrix} (pr\{1\})$
- ▶  $\mu_2(i_2) = \begin{bmatrix} -1 & -\frac{5}{9} & -\frac{1}{9} & \frac{7}{9} \end{bmatrix} (pr\{2\})$

# The MATLAB mLF package

## Detailed example 2-D (compute null space)

---

```
%% Compute rec. LL null
[c,lag] = mlf.loewner_null_rec(pc,pr,tabr,'svd0');
```

---

- ▶ Recursive: 136 flop and 16 Bytes
- ▶ Full: 512 flop and 64 Bytes
- ▶  $\mathbf{W}_{2,4}^{\otimes} = \begin{bmatrix} \frac{3778}{6561} & -\frac{110}{243} & -\frac{7702}{6561} & -\frac{10}{3} \\ -\frac{2206}{729} & -\frac{46}{27} & -\frac{566}{729} & 2 \end{bmatrix}$
- ▶  $\mathbf{C}_{2,4}^{\otimes} = \begin{bmatrix} 3 & -8 & 6 & -1 \\ -3 & 8 & -6 & 1 \end{bmatrix}$

# The MATLAB mLF package

## Detailed example 2-D (Lagrange form)

---

```
% Transfer function in Lagrangian form
[glag, ilag] = mlf.tf_lagrangian(pc, w, c, false);
```

---

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{lag}}(x_1, x_2) & \mathbf{N}_{\text{lag}} & \mathbf{D}_{\text{lag}} \\ \hline \left(x_1 + \frac{7}{9}\right) \left(x_2 + \frac{7}{9}\right) & \frac{3778}{2187} & 3 \\ \left(x_2 + \frac{1}{3}\right) \left(x_1 + \frac{7}{9}\right) & \frac{880}{243} & -8 \\ \left(x_2 - \frac{1}{9}\right) \left(x_1 + \frac{7}{9}\right) & -\frac{15404}{2187} & 6 \\ \left(x_2 - 1\right) \left(x_1 + \frac{7}{9}\right) & \frac{10}{3} & -1 \\ \left(x_1 - 1\right) \left(x_2 + \frac{7}{9}\right) & \frac{2206}{243} & -3 \\ \left(x_1 - 1\right) \left(x_2 + \frac{1}{3}\right) & -\frac{368}{27} & 8 \\ \left(x_1 - 1\right) \left(x_2 - \frac{1}{9}\right) & \frac{1132}{243} & -6 \\ \left(x_1 - 1\right) \left(x_2 - 1\right) & 2 & 1 \end{array} \right)$$

$$\mathbf{G}_{\text{lag}}(x_1, x_2) = \frac{\sum_{j_1=1}^2 \sum_{j_2=1}^4 \frac{\mathbf{C}_{2,4}^{\otimes}(j_1, j_2) \mathbf{W}_{2,4}^{\otimes}(j_1, j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^2 \sum_{j_2=1}^4 \frac{\mathbf{C}_{2,4}^{\otimes}(j_1, j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}},$$

where

$$\mathbf{W}_{2,4}^{\otimes} = \begin{bmatrix} \frac{3778}{6561} & -\frac{110}{243} & -\frac{7702}{6561} & -\frac{10}{3} \\ -\frac{2206}{729} & -\frac{46}{27} & -\frac{566}{729} & 2 \end{bmatrix}$$

$$\mathbf{C}_{2,4}^{\otimes} = \begin{bmatrix} 3 & -8 & 6 & -1 \\ -3 & 8 & -6 & 1 \end{bmatrix}$$

# The MATLAB mLF package

## Detailed example 2-D (Monomial form)

---

```
%% Transfer function in Monomial form
[gmon,imon] = mlf.tf_monomial(pc,w,c,false);
```

---

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{mon}}(x_1, x_2) & \parallel & \mathbf{N}_{\text{mon}} \mid \mathbf{D}_{\text{mon}} \end{array} \right)$$

$$\left( \begin{array}{c|cc} x_1 x_2^3 & 1 & 0 \\ x_1 x_2^2 & 0 & 0 \\ x_1 x_2 & 2 & 0 \\ x_1 & 0 & 0 \\ x_2^3 & 0 & 0 \\ x_2^2 & 0 & 0 \\ x_2 & 0 & 0 \\ 1 & -1 & 1 \end{array} \right)$$

$$\mathbf{G}_{\text{mon}}(x_1, x_2) = \frac{\sum_{j_1=1}^2 \sum_{j_2=1}^4 \mathbf{N}_{2,4}^{\otimes}(j_1, j_2) \left( x_1^{(j_1-1)} x_2^{(j_2-1)} \right)}{\sum_{j_1=1}^2 \sum_{j_2=1}^4 \mathbf{D}_{2,4}^{\otimes}(j_1, j_2) \left( x_1^{(j_1-1)} x_2^{(j_2-1)} \right)},$$

where

$$\mathbf{W}_{2,4}^{\otimes} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{C}_{2,4}^{\otimes} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# The MATLAB mLF package

## Detailed example 2-D (KST form)

---

**%% Decoupling toward KST**

```
[Bary ,Lag ,Cx] = mlf.decoupling( pc , lag );
```

---

$$\begin{bmatrix} \mathbf{Bary}(x_1) & \mathbf{Bary}(x_2) \end{bmatrix} =$$

$$\begin{bmatrix} -1 & -3 \\ -1 & 8 \\ -1 & -6 \\ -1 & 1 \\ 1 & -3 \\ 1 & 8 \\ 1 & -6 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Lag}(x_1) & \mathbf{Lag}(x_2) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{x_1 + \frac{7}{9}} & \frac{1}{x_2 + \frac{7}{9}} \\ \frac{1}{x_1 + \frac{7}{9}} & \frac{1}{x_2 + \frac{1}{3}} \\ \frac{1}{x_1 + \frac{7}{9}} & \frac{x_2 - \frac{1}{9}}{x_2 - 1} \\ \frac{1}{x_1 + \frac{7}{9}} & \frac{1}{x_2 - 1} \\ \frac{1}{x_1 - 1} & \frac{1}{x_2 + \frac{7}{9}} \\ \frac{1}{x_1 - 1} & \frac{1}{x_2 + \frac{1}{3}} \\ \frac{1}{x_1 - 1} & \frac{x_2 - \frac{1}{9}}{x_2 - 1} \\ \frac{1}{x_1 - 1} & \frac{1}{x_2 - 1} \end{bmatrix}$$

Univariate vector functions  $\Phi_1(x_1)$  and  $\Phi_2(x_2)$

$$\left\{ \begin{array}{l} \mathbf{Bary}(x_1) \odot \mathbf{Lag}(x_1) \\ \mathbf{Bary}(x_2) \odot \mathbf{Lag}(x_2) \end{array} \right.$$

KST rational interpolant

$$\mathbf{H}_{\text{kst}}(x_1, x_2) = \frac{\mathbf{n}_{\text{kst}}(x_1, x_2)}{\mathbf{d}_{\text{kst}}(x_1, x_2)}$$

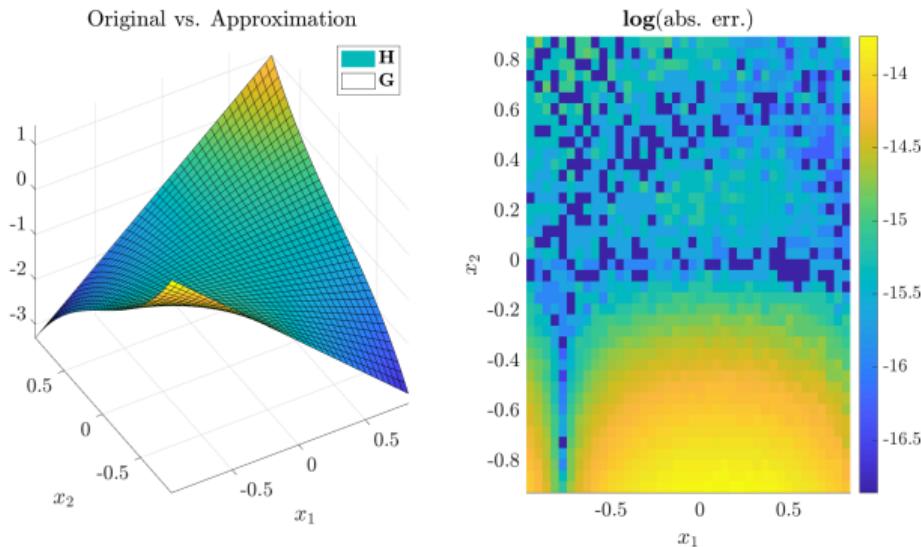
where numerator and denominator are

$$\left\{ \begin{array}{l} \sum_{\text{rows}} \mathbf{w} \odot \Phi_1(x_1) \odot \Phi_2(x_2) \\ \sum_{\text{rows}} \Phi_1(x_1) \odot \Phi_2(x_2) \end{array} \right.$$

# The MATLAB mLF package

## Detailed example 2-D (KST form)

$$\mathbf{H}(x_1, x_2) = x_1 x_2^3 + 2 x_1 x_2 - 1$$



# The MATLAB mLF package

## Detailed example 2-D (define the problem)

```
%% Define a multivariate problem
syms x1 x2;
Hsym = (sin(.01*x2+pi)+1)/(tanh(x1)*x2+3);
Hf = matlabFunction(Hsym);
H = @(x) Hf(x(:,1),x(:,2));
n = 2; % number of variables
xbnd = [-2 2]; % bounds of both variables
Nip = 12; % number of interpolation points
X1 = linspace(xbnd(1),xbnd(2),Nip);
X2 = linspace(xbnd(1),xbnd(2),Nip);
```

$$\mathbf{H}(x_1, x_2) = -\frac{\sin\left(\frac{x_2}{100}\right) - 1}{x_2 \tanh(x_1) + 3}$$

- ▶  $n = 2$ , bounds  $\mathcal{X}_l := \begin{bmatrix} -2.0 & 2.0 \end{bmatrix}$ , discretized with  $N_l = 12$ ,  $l = \{1, 2\}$
- ▶  $\mathbf{x}_l = \begin{bmatrix} -2.0 & -1.64 & -1.27 & -0.909 & -0.545 & -0.182 & 0.182 & 0.545 & 0.909 & 1.27 & 1.64 & 2.0 \end{bmatrix} \in \mathcal{X}_l^{12}$

# The MATLAB mLF package

## Detailed example 2-D (select interpolation points and construct the tensor)

```
%// Interpolation points
p_c{1} = X1(2:2:end);
p_r{1} = X1(1:2:end);
p_c{2} = X2(2:2:end);
p_r{2} = X2(1:2:end);
```

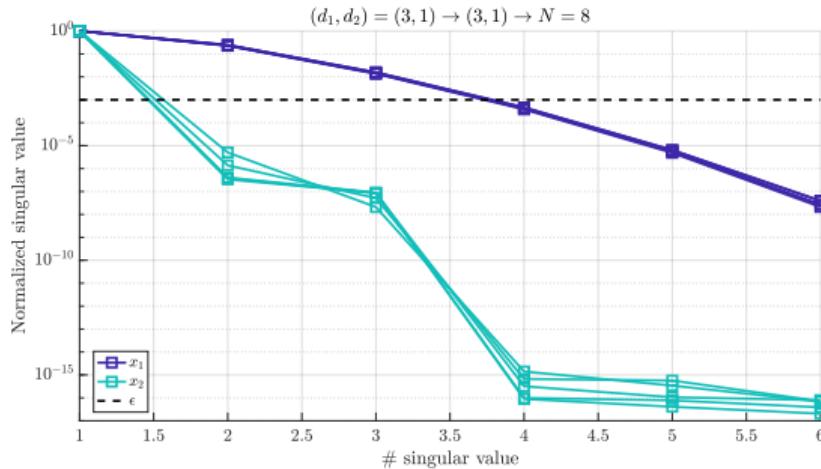
```
%// Construct tensor
% Check that column/row IP are disjoint
ok = mlf.check(p_c, p_r);
% Construct tab_n
[y, x, dim] = mlf.make_tab_vec(H, p_c, p_r);
tab = mlf.vec2mat(y, dim);
```

- ▶ columns  $\lambda_l(j_l)$  (denoted  $p_c\{1\}$  and  $p_c\{2\}$ ) and
- ▶ rows  $\mu_l(i_l)$  (denoted  $p_r\{1\}$  and  $p_r\{2\}$ )
- ▶ we chose equal dimensions (i.e.  $k_l = 6$  and  $q_l = 6$ )
- ▶ IP are disjoint, and 2-D tensor  $T_2^\otimes \in \mathbb{R}^{(q_1+q_2) \times (k_1+k_2)}$

# The MATLAB mLF package

## Detailed example 2-D (estimate the orders)

```
%% Estimate orders along each variables and select a subset of IP  
tol = 1e-3;  
ord = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );  
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord,true );  
w = mlf.mat2vec(W);
```



# The MATLAB mLF package

## Detailed example 2-D (estimate the orders)

```
%// Estimate orders along each variables and select a subset of IP  
tol          = 1e-3;  
ord          = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );  
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord,true );  
w            = mlf.mat2vec(W);
```

- ▶  $(k_1, k_2) = (4, 2)$  column IP
- ▶ Barycentric dimension  $4 \times 2 = 8 = N$
- ▶  $\lambda_1(j_1) = \begin{bmatrix} -1.64 & -0.909 & 0.545 & 2.0 \end{bmatrix}$  ( $pc\{1\}$ )
- ▶  $\lambda_2(j_2) = \begin{bmatrix} -1.64 & 2.0 \end{bmatrix}$  ( $pc\{2\}$ )
- ▶  $\mu_1(i_1) = \begin{bmatrix} -2.0 & -1.27 & 0.182 & 1.64 \end{bmatrix}$  ( $pr\{1\}$ )
- ▶  $\mu_2(i_2) = \begin{bmatrix} -2.0 & 1.64 \end{bmatrix}$  ( $pr\{2\}$ )

# The MATLAB mLF package

## Detailed example 2-D (compute null space)

---

```
%% Compute rec. LL null
[c,lag] = mlf.loewner_null_rec(pc,pr,tabr,'svd0');
```

---

- ▶ Recursive: 96 flop and 16.0 Bytes
- ▶ Full: 512 flop and 64.0 Bytes

- ▶  $\mathbf{W}_{4,2}^{\otimes} = \begin{bmatrix} 0.225 & 0.855 \\ 0.243 & 0.629 \\ 0.465 & 0.245 \\ 0.714 & 0.199 \end{bmatrix}$

- ▶  $\mathbf{C}_{4,2}^{\otimes} = \begin{bmatrix} -1.79 & 0.453 \\ 1.97 & -0.735 \\ -0.547 & 1.0 \\ 0.233 & -0.808 \end{bmatrix}$

# The MATLAB mLF package

## Detailed example 2-D (Lagrange form)

```
%>% Transfer function in Lagrangian form
```

```
[glag,ilag] = mlf.tf_lagrangian(pc,w,c,false);
```

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{lag}}(x_1, x_2) & \mathbf{N}_{\text{lag}} & \mathbf{D}_{\text{lag}} \end{array} \right)$$
$$\left( \begin{array}{c|cc} (x_1 + 1.64)(x_2 + 1.64) & -0.402 & -1.79 \\ (x_2 - 2.0)(x_1 + 1.64) & 0.387 & 0.453 \\ (x_2 + 1.64)(x_1 + 0.909) & 0.479 & 1.97 \\ (x_2 - 2.0)(x_1 + 0.909) & -0.462 & -0.735 \\ (x_1 - 0.545)(x_2 + 1.64) & -0.254 & -0.547 \\ (x_2 - 2.0)(x_1 - 0.545) & 0.245 & 1.0 \\ (x_1 - 2.0)(x_2 + 1.64) & 0.167 & 0.233 \\ (x_1 - 2.0)(x_2 - 2.0) & -0.161 & -0.808 \end{array} \right)$$

$$\mathbf{G}_{\text{lag}}(x_1, x_2) = \frac{\sum_{j_1=1}^4 \sum_{j_2=1}^2 \frac{\mathbf{C}_{4,2}^{\otimes}(j_1, j_2) \mathbf{W}_{4,2}^{\otimes}(j_1, j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}}{\sum_{j_1=1}^4 \sum_{j_2=1}^2 \frac{\mathbf{C}_{4,2}^{\otimes}(j_1, j_2)}{(x_1 - \lambda_1(j_1))(x_2 - \lambda_2(j_2))}},$$

where

$$\mathbf{W}_{4,2}^{\otimes} = \begin{bmatrix} 0.225 & 0.855 \\ 0.243 & 0.629 \\ 0.465 & 0.245 \\ 0.714 & 0.199 \end{bmatrix}$$

$$\mathbf{C}_{4,2}^{\otimes} = \begin{bmatrix} -1.79 & 0.453 \\ 1.97 & -0.735 \\ -0.547 & 1.0 \\ 0.233 & -0.808 \end{bmatrix}$$

# The MATLAB mLF package

## Detailed example 2-D (Monomial form)

```
%%% Transfer function in Monomial form
[gmon,imon] = mlf.tf_monomial(pc,w,c,false);
```

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{mon}}(x_1, x_2) & \parallel & \mathbf{N}_{\text{mon}} \mid \mathbf{D}_{\text{mon}} \end{array} \right)$$

$$\left( \begin{array}{c|cc} x_1^3 x_2 & -3.27e - 5 & -0.019 \\ x_1^3 & 0.00328 & 0.00983 \\ x_1^2 x_2 & 0.00129 & 0.0134 \\ x_1^2 & -0.129 & -0.386 \\ x_1 x_2 & -1.48e - 4 & -0.333 \\ x_1 & 0.0148 & 0.0444 \\ x_2 & 0.00333 & 2.87e - 4 \\ 1.0 & -0.333 & -1.0 \end{array} \right)$$

$$\mathbf{G}_{\text{mon}}(x_1, x_2) = \frac{\sum_{j_1=1}^4 \sum_{j_2=1}^2 \mathbf{N}_{4,2}^{\otimes}(j_1, j_2) \left( x_1^{(j_1-1)} x_2^{(j_2-1)} \right)}{\sum_{j_1=1}^4 \sum_{j_2=1}^2 \mathbf{D}_{4,2}^{\otimes}(j_1, j_2) \left( x_1^{(j_1-1)} x_2^{(j_2-1)} \right)},$$

where

$$\mathbf{W}_{4,2}^{\otimes} = \begin{bmatrix} -3.27e - 5 & 0.00328 \\ 0.00129 & -0.129 \\ -1.48e - 4 & 0.0148 \\ 0.00333 & -0.333 \end{bmatrix}$$

$$\mathbf{C}_{4,2}^{\otimes} = \begin{bmatrix} -0.019 & 0.00983 \\ 0.0134 & -0.386 \\ -0.333 & 0.0444 \\ 2.87e - 4 & -1.0 \end{bmatrix}$$

# The MATLAB mLF package

## Detailed example 2-D (KST form)

---

**%% Decoupling toward KST**

```
[Bary ,Lag ,Cx] = mlf.decoupling( pc , lag );
```

---

$$\begin{bmatrix} \mathbf{Bary}(x_1) & \mathbf{Bary}(x_2) \end{bmatrix} =$$

$$\begin{bmatrix} 0.453 & -3.94 \\ 0.453 & 1.0 \\ -0.735 & -2.68 \\ -0.735 & 1.0 \\ 1.0 & -0.547 \\ 1.0 & 1.0 \\ -0.808 & -0.289 \\ -0.808 & 1.0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Lag}(x_1) & \mathbf{Lag}(x_2) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{x_1+1.64} & \frac{1}{x_2+1.64} \\ \frac{1}{x_1+1.64} & \frac{1}{x_2-2.0} \\ \frac{1}{x_1+0.909} & \frac{1}{x_2+1.64} \\ \frac{1}{x_1+0.909} & \frac{1}{x_2-2.0} \\ \frac{1}{x_1-0.545} & \frac{1}{x_2+1.64} \\ \frac{1}{x_1-0.545} & \frac{1}{x_2-2.0} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2+1.64} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2-2.0} \end{bmatrix}$$

Univariate vector functions  $\Phi_1(x_1)$  and  $\Phi_2(x_2)$

$$\left\{ \begin{array}{l} \mathbf{Bary}(x_1) \odot \mathbf{Lag}(x_1) \\ \mathbf{Bary}(x_2) \odot \mathbf{Lag}(x_2) \end{array} \right.$$

KST rational interpolant

$$\mathbf{H}_{\text{kst}}(x_1, x_2) = \frac{\mathbf{n}_{\text{kst}}(x_1, x_2)}{\mathbf{d}_{\text{kst}}(x_1, x_2)}$$

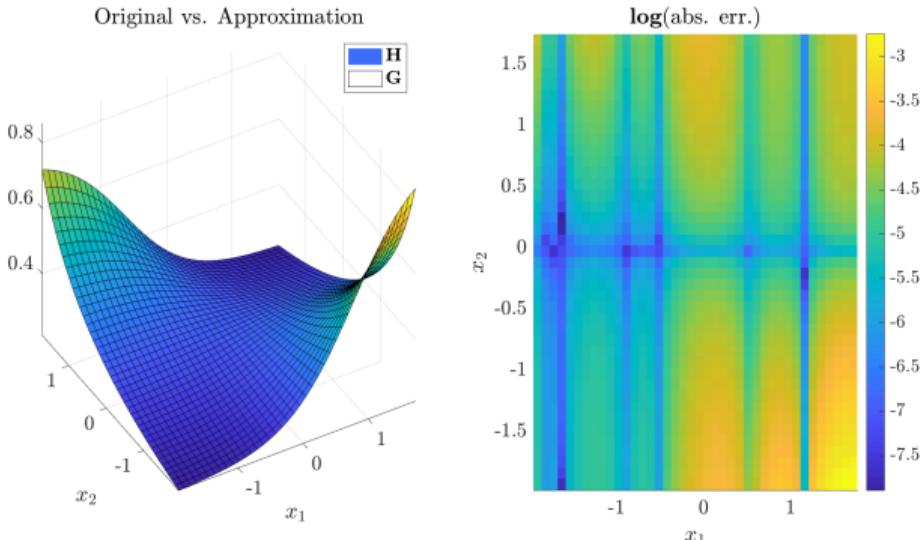
where numerator and denominator are

$$\left\{ \begin{array}{l} \sum_{\text{rows}} \mathbf{w} \odot \Phi_1(x_1) \odot \Phi_2(x_2) \\ \sum_{\text{rows}} \Phi_1(x_1) \odot \Phi_2(x_2) \end{array} \right.$$

# The MATLAB mLF package

## Detailed example 2-D (KST form)

$$\mathbf{H}(x_1, x_2) = -\frac{\sin\left(\frac{x_2}{100}\right) - 1}{x_2 \tanh(x_1) + 3}$$



# The MATLAB mLF package

## Detailed example 3-D (define the problem)

```
%// Define a multivariate problem
syms x1 x2 x3;
Hsym = (x1+x2^2*x3)/(x2*x1-10);
Hf = matlabFunction(Hsym);
H = @(x) Hf(x(:,1),x(:,2),x(:,3));
n = 3; % number of variables
xbnd = [-2 2]; % bounds of both variables
Nip = 6; % number of interpolation points
X1 = linspace(xbnd(1),xbnd(2),Nip);
X2 = linspace(xbnd(1),xbnd(2),Nip);
X3 = linspace(xbnd(1),xbnd(2),Nip);
```

$$\mathbf{H}(x_1, x_2, x_3) = \frac{x_3 x_2^2 + x_1}{x_1 x_2 - 10}$$

- ▶  $n = 3$ , bounds  $\mathcal{X}_l := \begin{bmatrix} -2.0 & 2.0 \end{bmatrix}$ , discretized with  $N_l = 6$ ,  $l = \{1, 2\}$
- ▶  $\mathbf{x}_l = \begin{bmatrix} -2.0 & -1.2 & -0.4 & 0.4 & 1.2 & 2.0 \end{bmatrix} \in \mathcal{X}_l^6$

# The MATLAB mLF package

## Detailed example 3-D (select interpolation points and construct the tensor)

```
%% Interpolation points
p_c{1} = X1(2:2:end);
p_r{1} = X1(1:2:end);
p_c{2} = X2(2:2:end);
p_r{2} = X2(1:2:end);
p_c{3} = X3(2:2:end);
p_r{3} = X3(1:2:end);
```

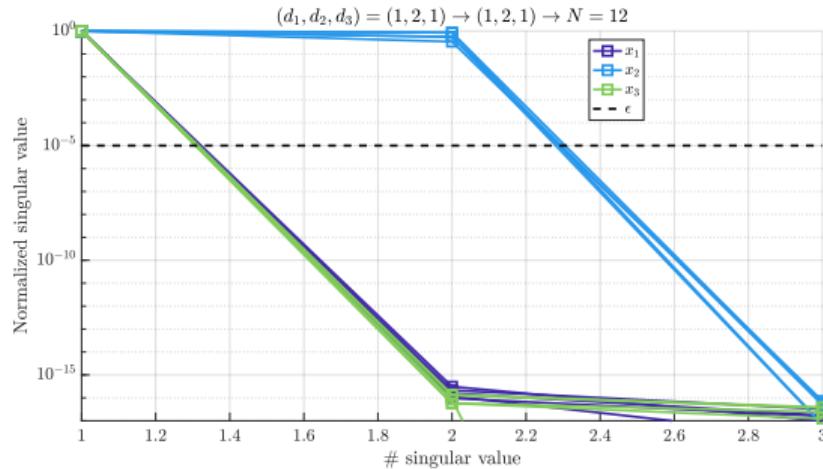
```
%% Construct tensor
% Check that column/row IP are disjoint
ok = mlf.check(p_c, p_r);
% Construct tab_n
[y, x, dim] = mlf.make_tab_vec(H, p_c, p_r);
tab = mlf.vec2mat(y, dim);
```

- ▶ columns  $\lambda_l(j_l)$  (denoted  $p_c\{1\}$ ,  $p_c\{2\}$  and  $p_c\{3\}$ ) and
- ▶ rows  $\mu_l(i_l)$  (denoted  $p_r\{1\}$ ,  $p_r\{2\}$  and  $p_r\{3\}$ )
- ▶ we chose equal dimensions (i.e.  $k_l = 3$  and  $q_l = 3$ )
- ▶ IP are disjoint, and 2-D tensor  $T_2^\otimes \in \mathbb{R}^{(q_1+q_2+q_3) \times (k_1+k_2+k_3)}$

# The MATLAB mLF package

## Detailed example 3-D (estimate the orders)

```
%// Estimate orders along each variables and select a subset of IP  
tol          = 1e-5;  
ord          = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );  
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord ,true );  
w            = mlf.mat2vec(W);
```



# The MATLAB mLF package

## Detailed example 3-D (estimate the orders)

---

```
%// Estimate orders along each variables and select a subset of IP
tol = 1e-5;
ord = mlf.compute_order(p_c,p_r,tab,[] ,5 ,true );
[pc,pr,W,V,tabr] = mlf.points_selection(p_c,p_r,tab,ord,true );
w = mlf.mat2vec(W);
```

---

- ▶  $(k_1, k_2, k_3) = (2, 3, 2)$  column IP
- ▶ Barycentric dimension  $2 \times 3 \times 2 = 12 = N$
- ▶  $\lambda_1(j_1) = \begin{bmatrix} -1.2 & 2.0 \end{bmatrix}$  (pc{1})
- ▶  $\lambda_2(j_2) = \begin{bmatrix} -1.2 & 0.4 & 2.0 \end{bmatrix}$  (pc{2})
- ▶  $\lambda_3(j_3) = \begin{bmatrix} -1.2 & 2.0 \end{bmatrix}$  (pc{3})
- ▶  $\mu_1(i_1) = \begin{bmatrix} -2.0 & 1.2 \end{bmatrix}$  (pr{1})
- ▶  $\mu_2(i_2) = \begin{bmatrix} -2.0 & -0.4 & 1.2 \end{bmatrix}$  (pr{2})
- ▶  $\mu_3(i_3) = \begin{bmatrix} -2.0 & 1.2 \end{bmatrix}$  (pr{3})

# The MATLAB mLF package

## Detailed example 3-D (compute null space)

---

```
%% Compute rec. LL null
[c,lag] = mlf.loewner_null_rec(pc,pr,tabr,'svd0');
```

---

- ▶ Recursive: 110 flop and 9.0 Bytes
- ▶ Full: 1728 flop and 144.0 Bytes
- ▶  $\mathbf{W}_{2,3,2}^{\otimes} = \begin{bmatrix} 0.342 & 0.133 & 0.484 \\ -0.0219 & -0.197 & 0.467 \end{bmatrix}, \begin{bmatrix} -0.196 & 0.084 & -0.548 \\ -0.394 & -0.252 & -1.67 \end{bmatrix}$
- ▶  $\mathbf{C}_{2,3,2}^{\otimes} = \begin{bmatrix} -0.69 & 1.69 & -1.0 \\ 1.0 & -1.48 & 0.484 \end{bmatrix}, \begin{bmatrix} 0.69 & -1.69 & 1.0 \\ -1.0 & 1.48 & -0.484 \end{bmatrix}$

# The MATLAB mLF package

## Detailed example 3-D (Lagrange form)

```
%>% Transfer function in Lagrangian form
```

```
[glag,ilag] = mlf.tf_lagrangian(pc,w,c,false);
```

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{lag}}(x_1, x_2, x_3) & \parallel & \mathbf{N}_{\text{lag}} \mid \mathbf{D}_{\text{lag}} \end{array} \right) \quad \mathbf{G}_{\text{lag}}(x_1, x_2, x_3) =$$
$$\frac{\sum_{j_1=1}^2 \sum_{j_2=1}^3 \sum_{j_3=1}^2 \frac{\mathbf{C}_{2,3,2}^{\otimes}(j_1, j_2, j_3) \mathbf{W}_{2,3,2}^{\otimes}(j_1, j_2, j_3)}{(x_1 - \lambda_1(j_1)) \cdots (x_3 - \lambda_3(j_3))}}{\sum_{j_1=1}^2 \sum_{j_2=1}^3 \sum_{j_3=1}^2 \frac{\mathbf{C}_{2,3,2}^{\otimes}(j_1, j_2, j_3)}{(x_1 - \lambda_1(j_1)) \cdots (x_3 - \lambda_3(j_3))}},$$

$(x_1 + 1.2)(x_2 + 1.2)(x_3 + 1.2)$	-0.236	-0.69
$(x_3 - 2.0)(x_1 + 1.2)(x_2 + 1.2)$	-0.135	0.69
$(x_1 + 1.2)(x_3 + 1.2)(x_2 - 0.4)$	0.225	1.69
$(x_3 - 2.0)(x_1 + 1.2)(x_2 - 0.4)$	-0.142	-1.69
$(x_2 - 2.0)(x_1 + 1.2)(x_3 + 1.2)$	-0.484	-1.0
$(x_2 - 2.0)(x_3 - 2.0)(x_1 + 1.2)$	-0.548	1.0
$(x_1 - 2.0)(x_2 + 1.2)(x_3 + 1.2)$	-0.0219	1.0
$(x_1 - 2.0)(x_3 - 2.0)(x_2 + 1.2)$	0.394	-1.0
$(x_1 - 2.0)(x_3 + 1.2)(x_2 - 0.4)$	0.292	-1.48
$(x_1 - 2.0)(x_3 - 2.0)(x_2 - 0.4)$	-0.374	1.48
$(x_1 - 2.0)(x_2 - 2.0)(x_3 + 1.2)$	0.226	0.484
$(x_1 - 2.0)(x_2 - 2.0)(x_3 - 2.0)$	0.806	-0.484

# The MATLAB mLF package

## Detailed example 3-D (Monomial form)

```
%% Transfer function in Monomial form  
[gmon,imon] = mlf.tf_monomial(pc,w,c,false);
```

$$\left( \begin{array}{c|cc} \mathcal{B}_{\text{mon}}(x_1, x_2, x_3) & \parallel & \mathbf{N}_{\text{mon}} \mid \mathbf{D}_{\text{mon}} \end{array} \right)$$
$$\left( \begin{array}{c|cc} x_1 x_2^2 x_3 & 0 & 0 \\ x_1 x_2^2 & 0 & 0 \\ x_1 x_2 x_3 & 0 & 0 \\ x_1 x_2 & 0 & 0.1 \\ x_1 x_3 & 0 & 0 \\ x_1 & 0.1 & 0 \\ x_2^2 x_3 & 0.1 & 0 \\ x_2^2 & 0 & 0 \\ x_2 x_3 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \\ 1.0 & 0 & -1.0 \end{array} \right)$$

$$\mathbf{G}_{\text{mon}}(x_1, x_2, x_3) =$$
$$\frac{\sum_{j_1=1}^2 \sum_{j_2=1}^3 \sum_{j_3=1}^2 \mathbf{N}_{2,3,2}^{\otimes}(j_1, j_2, j_3) (x_1^{(j_1-1)} \cdots x_3^{(j_3-1)})}{\sum_{j_1=1}^2 \sum_{j_2=1}^3 \sum_{j_3=1}^2 \mathbf{D}_{2,3,2}^{\otimes}(j_1, j_2, j_3) (x_1^{(j_1-1)} \cdots x_3^{(j_3-1)})}$$

# The MATLAB mLF package

## Detailed example 3-D (KST form)

```
%% Transfer function in Monomial form  
[gmon,imon] = mlf.tf_monomial(pc,w,c,false);
```

$$[\mathbf{Bary}(x_1), \mathbf{Bary}(x_2), \mathbf{Bary}(x_3)] =$$

$$\begin{bmatrix} 1.0 & 0.69 & -1.0 \\ 1.0 & 0.69 & 1.0 \\ 1.0 & -1.69 & -1.0 \\ 1.0 & -1.69 & 1.0 \\ 1.0 & 1.0 & -1.0 \\ 1.0 & 1.0 & 1.0 \\ -0.484 & 2.07 & -1.0 \\ -0.484 & 2.07 & 1.0 \\ -0.484 & -3.07 & -1.0 \\ -0.484 & -3.07 & 1.0 \\ -0.484 & 1.0 & -1.0 \\ -0.484 & 1.0 & 1.0 \end{bmatrix}$$

$$[\mathbf{Lag}(x_1), \mathbf{Lag}(x_2), \mathbf{Lag}(x_3)] =$$

$$\begin{bmatrix} \frac{1}{x_1+1.2} & \frac{1}{x_2+1.2} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1+1.2} & \frac{1}{x_2+1.2} & \frac{1}{x_3-2.0} \\ \frac{1}{x_1+1.2} & \frac{1}{x_2-0.4} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1+1.2} & \frac{1}{x_2-0.4} & \frac{1}{x_3-2.0} \\ \frac{1}{x_1+1.2} & \frac{1}{x_2-2.0} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1+1.2} & \frac{1}{x_2-2.0} & \frac{1}{x_3-2.0} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2+1.2} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2+1.2} & \frac{1}{x_3-2.0} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2-0.4} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2-0.4} & \frac{1}{x_3-2.0} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2-2.0} & \frac{1}{x_3+1.2} \\ \frac{1}{x_1-2.0} & \frac{1}{x_2-2.0} & \frac{1}{x_3-2.0} \end{bmatrix}$$

Univariate vector functions  $\Phi_1(x_1) \dots \Phi_3(x_3)$

$$\left\{ \begin{array}{l} \mathbf{Bary}(x_1) \odot \mathbf{Lag}(x_1) \\ \mathbf{Bary}(x_2) \odot \mathbf{Lag}(x_2) \\ \mathbf{Bary}(x_3) \odot \mathbf{Lag}(x_3) \end{array} \right.$$

KST rational interpolant

$$\mathbf{H}_{\text{kst}}(x_1, x_2, x_3) = \frac{\mathbf{n}_{\text{kst}}(x_1, x_2, x_3)}{\mathbf{d}_{\text{kst}}(x_1, x_2, x_3)}$$

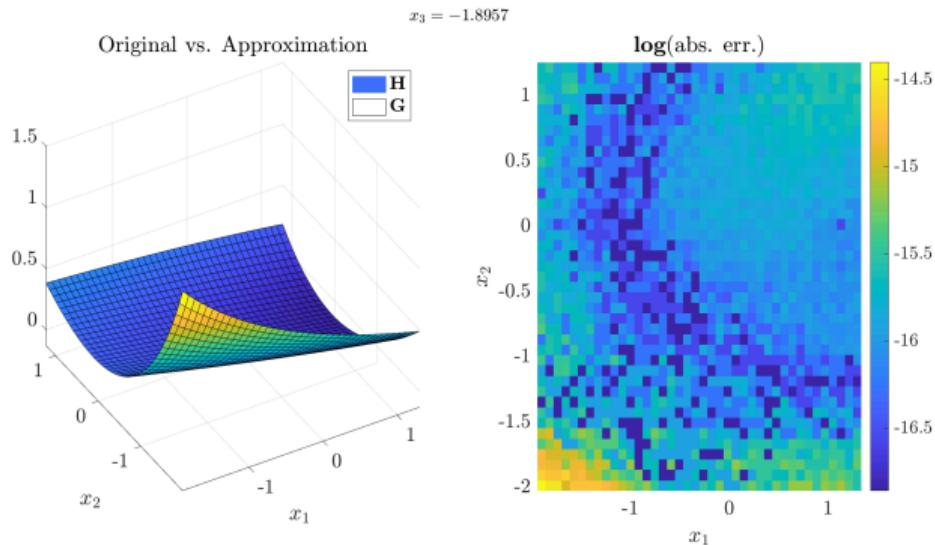
where numerator and denominator are

$$\left\{ \begin{array}{l} \sum_{\text{rows}} \mathbf{w} \odot \Phi_1(x_1) \odot \Phi_2(x_2) \odot \Phi_2(x_3) \\ \sum_{\text{rows}} \Phi_1(x_1) \odot \Phi_2(x_2) \odot \Phi_2(x_3) \end{array} \right.$$

# The MATLAB mLF package

## Detailed example 3-D (KST form)

$$\mathbf{H}(x_1, x_2) = \frac{x_3 x_2^2 + x_1}{x_1 x_2 - 10}$$



# Content

Forewords

Multi-variate data, function & Loewner matrix

Taming the curse of dimensionality

Variables decoupling, KST and KANs

Comparisons

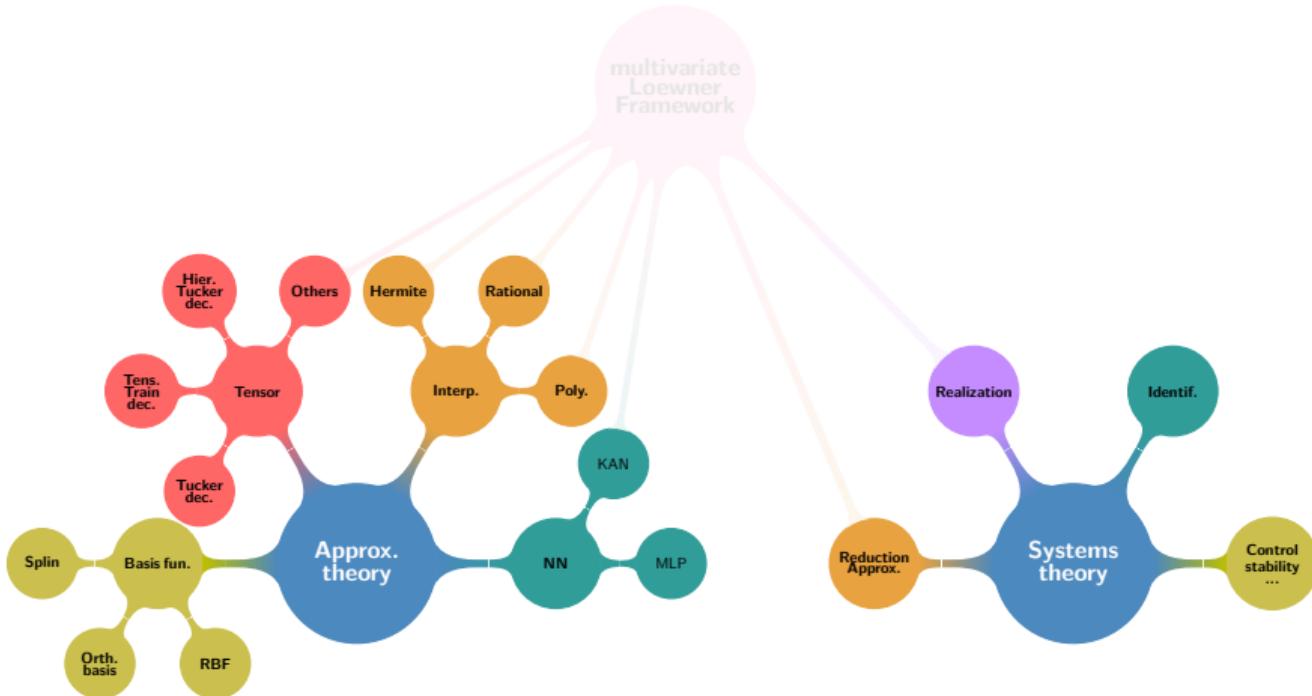
Multi-variate realization

The MATLAB `mLF` package

**Conclusion**

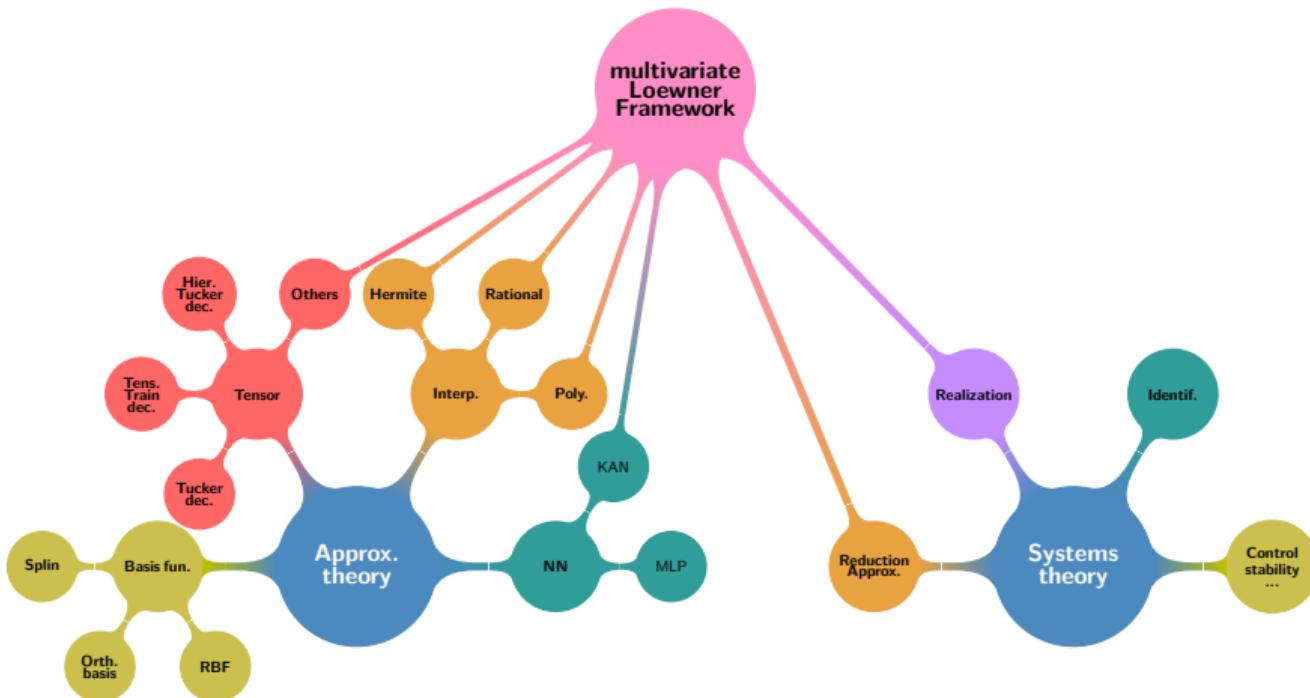
# Conclusion

## Multivariate Loewner



# Conclusion

## Multivariate Loewner



# Conclusion

## Take home message

### Main contributions (from $n$ -D tensor)

- ▶ Transfer function in barycentric form
- ▶ Realization with controlled complexity
- ▶ Tame the computational complexity
- ▶ Two algorithms (direct & iterative)
- ▶ Solution of rational KST
- ▶ Connection with KAN

### Side effects

- [Sci. con.] Tensor approximation
- [Sci. con.] NEVP multi-linearization
- [Sci. con.] Exact (Loewner) matrix null space
- [Dyn. sys.] Parametric realization
- [Dyn. sys.] Uncertainty propagation
- [Dyn. sys.] LFR connections

Collaboration with

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<https://cpoussot.github.io>

**Theory (paper & code)**

<https://doi.org/10.1137/24M1656657>

<https://github.com/cpoussot/mLF>

**Benchmark (paper & code)**

<https://arxiv.org/abs/2506.04791>

[https://github.com/cpoussot/benchmark\\_tensor](https://github.com/cpoussot/benchmark_tensor)



# Content

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# Appendix & open research

## Algorithm and heuristics (direct)

**Require:**  $\mathcal{T}_n^{\otimes}$

- 1: Check that interpolation points are disjoints.
- 2: Compute  $d_j = \max_k \text{rank } \mathbb{L}_1^{(k)}$ , the order along variable  $x_j$  ( $k$  being all possible combinations for frozen the variables  $\{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n\}$ ).
- 3: Construct data set, a sub-selection  $P_c^{(n)}$  where  $(k_1, k_2 \dots, k_n) = (d_1, d_2, \dots, d_n) + 1$ ; and  $P_r^{(n)}$  where  $(q_1, q_2 \dots, q_n)$  gather the rest of the data.
- 4: Compute  $\mathbf{c}_n$ , the  $n$ -D Loewner matrix null space.
- 5: Construct  $\mathbb{A}^{\text{Lag}}$ ,  $\mathbb{B}^{\text{Lag}}$ ,  $\boldsymbol{\Gamma}$  and  $\Delta$  with any left/right separation.
- 6: Construct multivariate realization.

**Ensure:**  $\hat{\mathbf{H}}(x_1, \dots, x_n) = \mathbf{W}\Phi(x_1, x_2, \dots, x_n)^{-1}\mathbf{G}$  interpolates  $\mathbf{H}(x_1, x_2, \dots, x_n)$  along  $P_c^{(n)}$ .



# Appendix & open research

## Algorithm and heuristics (iterative)

**Require:**  $\mathcal{T}_n^{\otimes}$  and tolerance  $\text{tol} > 0$

- 1: Check that interpolation points are disjoints.
- 2: **while**  $\text{error} > \text{tol}$  **do**
- 3:   Search the point indexes with maximal error (first iteration: pick any set).
- 4:   Add points in  $P_c^{(n)}$  and put the remaining ones in  $P_r^{(n)}$ , obtain data set.
- 5:   Compute  $\mathbf{c}_n$ , the  $n$ -D Loewner matrix null space.
- 6:   Construct  $\mathbb{A}^{\text{Lag}}$ ,  $\mathbb{B}^{\text{Lag}}$ ,  $\boldsymbol{\Gamma}$  and  $\Delta$  with any left/right separation.
- 7:   Construct multivariate realization.
- 8:   Evaluate  $\text{error} = \max ||\widehat{\mathcal{T}}_n^{\otimes} - \mathcal{T}_n^{\otimes}||$  where  $\widehat{\mathcal{T}}_n^{\otimes}$  is the evaluation of  $\hat{\mathbf{H}}(x_1, \dots, x_n)$  along the support points.
- 9: **end while**

**Ensure:**  $\hat{\mathbf{H}}(x_1, \dots, x_n) = \mathbf{W}\Phi(x_1, x_2, \dots, x_n)^{-1}\mathbf{G}$  interpolates  $\mathbf{H}(x_1, x_2, \dots, x_n)$  along  $P_c^{(n)}$ .

# Appendix & open research

Nonlinear eigenvalues approximation

$$\mathbf{H}(\lambda)\mathbf{v} = 0$$

## Linear model

$$\mathbf{H}(s, p) = \mathbf{1}^\top \left( sI_3 - \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & 0 \\ 0 & 1 & p \end{bmatrix} \right) \mathbf{1}$$

- ▶ `mlf.alg1`
- ▶ `mlf.make_realization_lag`



$$\mathbf{H}(\lambda)\mathbf{v} = 0$$

## Delay model

Let  $E = \text{diag}([10^{-4}, \dots, 10^{10}])$ ,  $n = 10$ ,

$$\mathbf{x} = -E\mathbf{x}(t) - 0.001\mathbf{x}(t-p)$$

or

$$\mathbf{H}(s, p) = \mathbf{1}_n^\top \left( (s - 0.001e^{-ps})I - E \right) \mathbf{1}_n$$

- ▶ `mlf.alg1`
- ▶ `mlf.make_realization_lag`



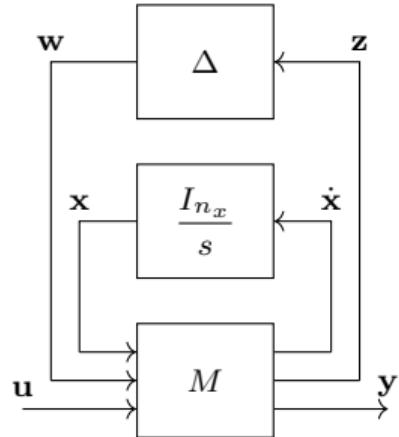
# Appendix & open research

## Linear fractional representation (the 2-D case)

$$\Phi(x_1, \dots, x_n) = \left[ \begin{array}{c|c|c} \Gamma(1 : \kappa - 1, :) & \mathbf{0}_{\kappa-1, \ell-1} & \mathbf{0}_{\kappa-1, \ell} \\ \hline \mathbb{A}^{\text{Lag}} & \Delta(1 : \ell - 1, :)^\top & \mathbf{0}_{\ell, \ell} \\ \hline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell, \ell-1} & \Delta^\top \end{array} \right] \in \mathbb{C}^{m \times m}$$

$$\left\{ \begin{array}{lcl} \Phi_0 & = & \Phi(0, 0) \\ \Phi_{x_1} & = & \Phi(1, 0) - \Phi_0 \\ \Phi_{x_2} & = & \Phi(0, 1) - \Phi_0 \\ n_x & = & \kappa \\ n_w & = & 2(\ell - 1) \end{array} \right.$$

$$\left\{ \begin{array}{lcl} E & = & \Phi_{x_1} \\ A & = & -\Phi_0 \\ B & = & \left[ \begin{array}{c|c} -\Phi_{x_2}(:, n_x + 2 : m - 1) & \mathbf{G} \end{array} \right] \\ C & = & \left[ \begin{array}{c|c} \mathbf{0}_{n_w, n_x + 1} & I_{n_w} \\ \hline \mathbf{W} & \mathbf{0}_{n_w} \end{array} \right] \\ D & = & \mathbf{0} \\ \Delta & = & x_2 I_{n_w} \end{array} \right.$$



# Appendix & open research

On-going

- ▶ Parametric and uncertain modeling
- ▶ Uncertainty propagation

