On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality

... from tensor to multivariate model and realization

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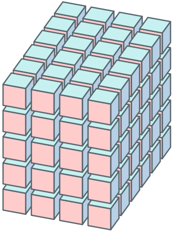
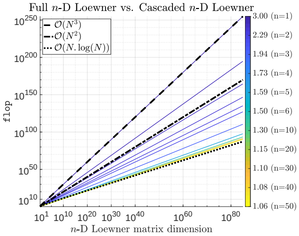
September 22, 2025
`https://arxiv.org/abs/2405.00495` (to appear in SIAM Review - Research Spotlight)
`https://github.com/cpoussot/mLF` (GitHub code)

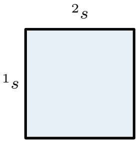




Forewords

Starting (motivating) examples - toy case

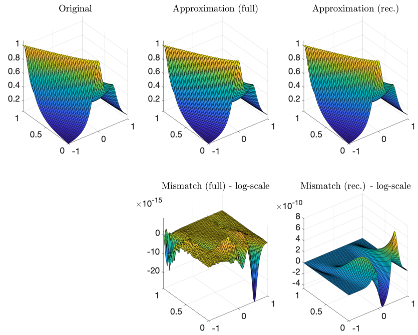
$$\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s, p) = \frac{1}{1 + 25(s+p)^2} + \frac{0.5}{1 + 25(s-0.5)^2} + \frac{0.1}{p + 25}$$



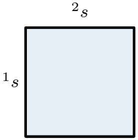
$$\begin{array}{ccc} {}^1s & \times & {}^2s \\ [-1, 1] & \times & [0, 1] \end{array}$$

$\textbf{tab}_2 \in \mathbb{R}^{21 \times 21}$

≈3.45 Ko ('real')



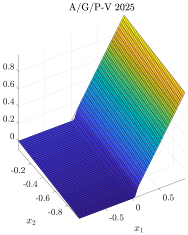
$$\mathbf{H}({}^1s, {}^2s) = \mathbf{ReLU}({}^1s) + \frac{1}{100}\,{}^2s$$

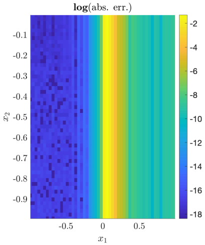


$$\begin{array}{ccc} {}^1s & \times & {}^2s \\ [-1,1] & \times & [0,1] \end{array}$$

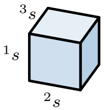
$\mathbf{tab}_2 \in \mathbb{R}^{20\times 20}$

≈3.45 Ko ('real')





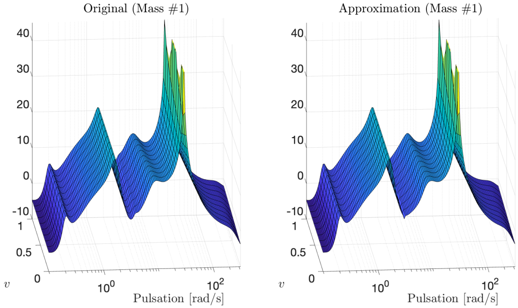
$$\mathbf{\Sigma}({}^1s, {}^2s, {}^3s) = \mathbf{\Sigma}(s, m, v) : s^2 M(m)x(s) + sB(m)x(s) + K(m)x(s) - G(s, v) = u(s),\ \mathbf{y}(s) = C\mathbf{x}(s)$$



$$\begin{array}{ccccc} {}^1s & \times & {}^2s & \times & {}^3s \\ \imath[10, 35] & \times & [\underline{m}, \overline{m}] & \times & [\underline{v}, \overline{v}] \end{array}$$

$\mathbf{tab}_3 \in \mathbb{C}^{300\times 10\times 10}$

≈468.75 Ko ('complex')

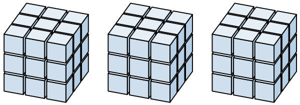


A. dos Reis de Souza et al., *"Aircraft flutter suppression: from a parametric model to robust control"*, ECC, 2023.

Forewords

Starting (motivating) examples - Borehole function

$$\mathbf{H}({}^1s,\cdots,{}^8s)=\mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w)=\frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$



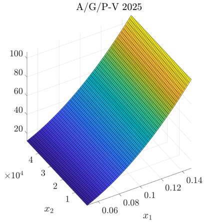
$$\begin{array}{ccccc} {}^1s & \times & \cdots & \times & {}^8s \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

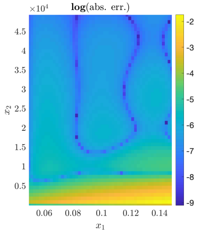
$$\textbf{tab}_8\in\mathbb{C}^{8\times 8\times\cdots\times 8}$$

≈130 `Mo ('real')`

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	$1.02e+04$	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	1.02e+04	39.1	0.00456	2.93e-09	0.0611

$x_{3\ldots 8}=[93221.3376; 996.0245; 105.854; 765.3085; 1619.2701; 10834.7051]$





S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

Forewords

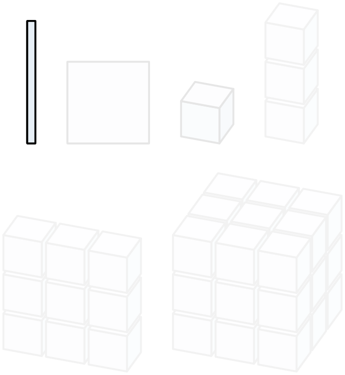
Data (and tensors)

Column / Row data

$$\left.{}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1}\right\} \xrightarrow{\mathbf{H}({}^{1}s)} \left\{ \mathbf{w}_{j_1}, \mathbf{v}_{i_1}\right.$$

${}^{1}s$	
${}^{1}\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1}$
${}^{1}\mu_{1,\cdots,q_1}$	$\mathbf{V}_{q_1}$

Tensors (1-D) tab_1



Forewords

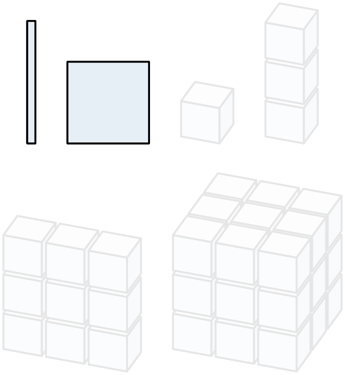
Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1s,{}^2s)} \left\{ \mathbf{w}_{j_1,j_2}, \mathbf{v}_{i_1,i_2} \right.$$

${}^1s$ \ ${}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2}$	ϕ_{cr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rc}	$\mathbf{V}_{q_1,q_2}$

Tensors (2-D) tab$_2$



Forewords

Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1s, {}^2s, {}^2s)} \left\{ \mathbf{w}_{j_1,j_2,j_3}, \mathbf{v}_{i_1,i_2,i_3} \right.$$

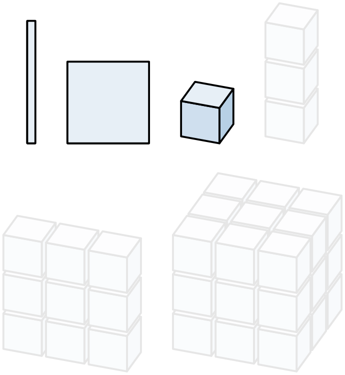
$^3s = {}^3\lambda_{1,\cdots,k_3}$

${}^1s$ \ ${}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2,k_3}$	ϕ_{crc}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcc}	ϕ_{rrc}

$^3s = {}^3\mu_{1,\cdots,q_3}$

${}^1s$ \ ${}^2s$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	ϕ_{crr}	ϕ_{crr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcr}	$\mathbf{V}_{q_1,q_2,q_3}$

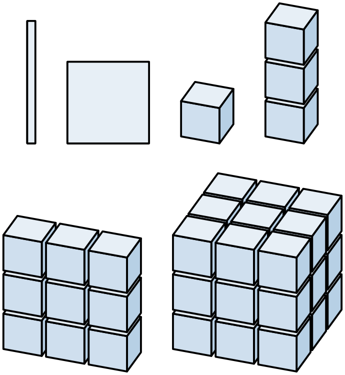
Tensors (3-D) tab_3



Column / Row data

$$\left.\begin{array}{c} {}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1} \\ {}^{2}\lambda_{j_2}, {}^{2}\mu_{i_2} \\ {}^{3}\lambda_{j_3}, {}^{3}\mu_{i_3} \\ \vdots \\ {}^{n}\lambda_{j_n}, {}^{n}\mu_{i_n} \end{array}\right\} \xrightarrow{\mathbf{H}({}^{1}s,\cdots,{}^{n}s)} \left\{ \mathbf{w}_{j_1,\cdots,j_n}, \mathbf{v}_{i_1,\cdots,i_n} \right.$$

Tensors (n-D) tab$_n$

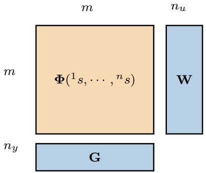


Data-driven model approximation

Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^ns) = \mathbf{G}\mathbf{\Phi}({}^1s, {}^2s, \dots, {}^ns)^{-1}\mathbf{W} \in \mathbb{C}$$

that interpolates the data.



Connection to standard dynamical system realization

A linear-in-state dynamical system parameterized in terms of parameters included in $\mathcal{S} = [{}^2s, \dots, {}^ns]^\top \subset \mathbb{C}^{n-1}$

$$\begin{aligned} \mathbf{E}(\mathcal{S})\dot{\mathbf{x}}(t;\mathcal{S}) &= \mathbf{A}(\mathcal{S})\mathbf{x}(t;\mathcal{S}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t;\mathcal{S}) &= \mathbf{C}\mathbf{x}(t;\mathcal{S}) \end{aligned}$$

equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^ns) = \mathbf{C}(\mathcal{S})\left[{}^1s\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1}\mathbf{B} \in \mathbb{C}.$$

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Being given a n-dimensional tensor (data), we seek a multi-variate **rational function** $\hat{\mathbf{H}}$ and **realization** $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$

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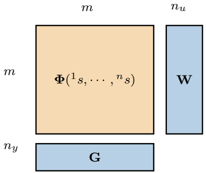

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equivalently

$$\hat{\mathbf{H}}({}^1s, {}^2s, \dots, {}^ns) = \mathbf{C}(\mathcal{S}) \left[{}^1s\mathbf{E}(\mathcal{S}) - \mathbf{A}(\mathcal{S})\right]^{-1} \mathbf{B} \in \mathbb{C}.$$

Forewords

Where we stand (some references)

1-D Two-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization minimality
$\Rightarrow$ direct algorithm

1-D One-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ direct algorithm

1-D AAA (Adaptive Anderson Antoulas - one-sided)
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

2-D Parametric one-sided Loewner
$\Rightarrow$ (interpolation) barycentric form
$\Rightarrow$ realization (non-minimal)
$\Rightarrow$ direct algorithm

3-D Parametric AAA
$\Rightarrow$ (mixed interpolation LS) barycentric form
$\Rightarrow$ iterative algorithm

$>$3-D **few results**

J-P. Berrut and N. Trefethen, *"Barycentric Lagrange Interpolation"*, SIAM Review, 46(3), 2004.

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realisation problem"*, LAA, 425(2-3), 2007.

A.C. Ionita and A.C. Antoulas, *"Data-Driven Parametrized Model Reduction in the Loewner Framework"*, SIAM Journal on Scientific Computing, 36(3), 2014.

T. Vojkovic, D. Quero, C. P-V and P. Vuillemin, *"Low-Order Parametric State-Space Modeling of MIMO Systems in the Loewner Framework"*, SIAM Journal on Applied Dynamical Systems, 22(4), 2023.

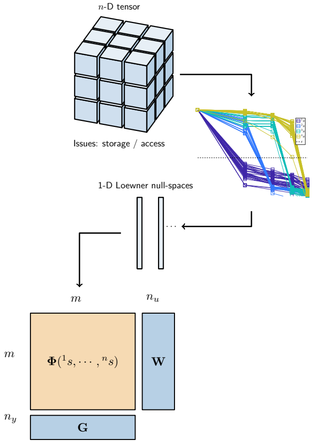
A.C. Rodriguez, L. Balicki and S. Gugercin, *"The p-AAA algorithm for data driven modeling of parametric dynamical systems"*, SIAM Journal on Scientific Computing, 45(3), 2023.

Forewords

Contributions claim

List of contributions

- n-D tensor data to n-D Loewner matrix $\mathbb{L}_n$
- n-D recursive Sylvester equations
- n-variable transfer functions
- n-variable generalized realization
- Taming the curse of dimensionality
  - in computation effort (`flop`)
  - in storage needs (`Bytes`)
  - in accuracy
- n-variable **decoupling**
  - **KST** formulation for rational functions
  - connection with **KAN**



Content

Multi-variate data, function & Loewner matrix

1-D case

$$\left\{\begin{array}{lll} P_c^{(1)} & := & \left\{\left({}^1\lambda_{j_1};\mathbf{w}_{j_1}\right),\ j_1=1,\dots,k_1\right\} \\ P_r^{(1)} & := & \left\{\left({}^1\mu_{i_1};\mathbf{v}_{i_1}\right),\ i_1=1,\dots,q_1\right\} \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1\times k_1}$$

$$(\mathbb{L}_1)_{i_1,j_1} = \frac{\mathbf{v}_{i_1}-\mathbf{w}_{j_1}}{{}^1\mu_{i_1}-{}^1\lambda_{j_1}}$$

$$\mathbf{M}_1\mathbb{L}_1 - \mathbb{L}_1\mathbf{\Lambda}_1 = \mathbb{V}_1\mathbf{R}_1 - \mathbf{L}_1\mathbb{W}_1$$

Lagrangian form

$$\mathbf{g}({}^1s) = \frac{\sum_{j_1=1}^{k_1}\frac{c_{j_1}\mathbf{w}_{j_1}}{{}^1s-{}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1}\frac{c_{j_1}}{{}^1s-{}^1\lambda_{j_1}}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}({}^1s) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{rcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\left\{\begin{array}{lcl} P_c^{(2)} & := & \left\{({}^1\lambda_{j_1}, {}^2\lambda_{j_2}; \mathbf{w}_{j_1,j_2}),\ j_1 = 1,\dots,k_1 \quad j_2 = 1,\dots,k_2\right\} \\ P_r^{(2)} & := & \left\{({}^1\mu_{i_1}, {}^2\mu_{i_2}; \mathbf{v}_{i_1,i_2}),\ i_1 = 1,\dots,q_1 \quad i_2 = 1,\dots,q_2\right\} \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

$$\left\{\begin{array}{rcl} \mathbf{M}_2\mathbb{X} - \mathbb{X}\mathbf{\Lambda}_2 & = & \mathbb{V}_2\mathbf{R}_2 - \mathbf{L}_2\mathbb{W}_2 \\ \mathbf{M}_1\mathbb{L}_2 - \mathbb{L}_2\mathbf{\Lambda}_1 & = & \mathbb{X} \end{array}\right.$$

Lagrangian form

$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \left[\begin{array}{c} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array}\right] \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1s,{}^2s)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} &=& [1,3,5] \\ {}^1\mu_{i_1} &=& [0,2,4] \\ {}^2\lambda_{j_2} &=& [-1,-3] \\ {}^2\mu_{i_2} &=& [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Loewner matrix

$$\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc} \frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\ \frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\ \frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\ \hline \frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\ \frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\ \frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9} \end{array}\right]$$

Null space

$$\mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [0,2,4] \\ {}^2\lambda_{j_2} & = & [-1,-3] \\ {}^2\mu_{i_2} & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Lagrangian form

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}} = \mathbf{H}(s,t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \left\{ ({}^1\lambda_{j_1}, {}^2\lambda_{j_2}, \cdots, {}^n\lambda_{j_n}; \mathbf{w}_{j_1,j_2,\cdots,j_n}),\ j_l = 1,\ldots,k_l, \quad l = 1,\cdots,n \right\} \\ P_r^{(n)} := \left\{ ({}^1\mu_{i_1}, {}^2\mu_{i_2}, \cdots, {}^n\mu_{i_n}; \mathbf{v}_{i_1,i_2,\cdots,i_n}),\ i_l = 1,\ldots,q_l, \quad l = 1,\cdots,n \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell^{i_1,i_2,\cdots,i_n}_{j_1,j_2,\cdots,j_n} = \frac{\mathbf{v}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1}) \cdots ({}^n\mu_{i_n} - {}^n\lambda_{j_n})}$$

$$\left\{ \begin{array}{rcl} \mathbf{M}_n \mathbb{X}_1 - \mathbb{X}_1 \mathbf{\Lambda}_n & = & \mathbb{V}_n \mathbf{R}_n - \mathbf{L}_n \mathbb{W}_n, \\ \mathbf{M}_{n-1} \mathbb{X}_2 - \mathbb{X}_2 \mathbf{\Lambda}_{n-1} & = & \mathbb{X}_1, \\ & \cdots & \\ \mathbf{M}_1 \mathbb{L}_n - \mathbb{L}_n \mathbf{\Lambda}_1 & = & \mathbb{X}_{n-1}. \end{array} \right.$$

Lagrangian form

$$\mathbf{g}({}^1s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1}) \cdots ({}^ns - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{({}^1s - {}^1\lambda_{j_1}) \cdots ({}^ns - {}^n\lambda_{j_n})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

$$\mathbf{c}_n = \left[\begin{array}{c} c_{1,\cdots,1} \\ \vdots \\ c_{1,\cdots,k_n} \\ \hline \vdots \\ \hline c_{k_1,\cdots,1} \\ \vdots \\ c_{k_1,\cdots,k_n} \end{array} \right] \in \mathbb{C}^{k_1 \cdots k_n}$$

Content

Multi-variate realization

1-D case (example cont'd)

Data generated from $\mathbf{H}({}^1 s) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}}$$

Lagrangian realization $\hat{\mathbf{H}}(s) = \mathbf{W}\mathbf{\Phi}(s)^{-1}\mathbf{G}$

$$\mathbf{\Phi}(s) = \left[\begin{array}{ccc} s-1 & 3-s & 0 \\ s-1 & 0 & 5-s \\ \hline -\frac{1}{3} & \frac{4}{3} & -1 \end{array}\right]$$

$$\mathbf{W} = \left[\begin{array}{ccc} \frac{5}{6} & -\frac{13}{3} & \frac{29}{6} \end{array}\right]$$

$$\mathbf{G}^\top = \left[\begin{array}{cc|c} 0 & 0 & -1 \end{array}\right]$$

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}({}^1s,{}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}(s,t) = \mathbf{W}\mathbf{\Phi}(s,t)^{-1}\mathbf{G}$

$$\mathbf{\Phi}(s,t) = \left[\begin{array}{ccc|c|cc} s-1 & 3-s & 0 & 0 & 0 & 0 \\ s-1 & 0 & 5-s & 0 & 0 & 0 \\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1 & 0 & 0 \\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 & 0 & 0 \\ \hline \frac{1}{9} & -2 & \frac{25}{9} & 0 & t+1 & \frac{1}{2} \\ -\frac{1}{3} & 6 & -\frac{25}{3} & 0 & -t-3 & -\frac{1}{2} \end{array}\right]$$

$$\mathbf{W} = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\mathbf{G}^\top = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1/2 & -1/2 & 0 & 0 \end{array}\right]$$

$\rightarrow$ (3,3) block is unimodular !

Multi-variate realization

2-D case (example cont'd)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

$$\mathbf{g}(s,t) = -\frac{\frac{1}{9(s-1)(t+1)} - \frac{1}{3(s-1)(t+3)} - \frac{2}{(s-3)(t+1)} + \frac{6}{(s-3)(t+3)} + \frac{25}{9(s-5)(t+1)} - \frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)} - \frac{5}{9(s-1)(t+3)} - \frac{10}{9(s-3)(t+1)} + \frac{14}{9(s-3)(t+3)} + \frac{7}{9(s-5)(t+1)} - \frac{1}{(s-5)(t+3)}}$$

Lagrangian realization $\hat{\mathbf{H}}_{\mathsf{c}}(s,t) = \mathbf{W}_{\mathsf{c}}(t)\mathbf{\Phi}_{\mathsf{c}}(s,t)^{-1}\mathbf{G}_{\mathsf{c}}$

$$\mathbf{\Phi}_{\mathsf{c}}(s,t) = \left[\begin{array}{ccc|c} s-1 & 3-s & 0 & 0 \\ s-1 & 0 & 5-s & 0 \\ \hline -\frac{1}{3} & \frac{10}{9} & -\frac{7}{9} & t+1 \\ \frac{5}{9} & -\frac{14}{9} & 1 & -t-3 \end{array}\right]$$

$$\mathbf{W}_{\mathsf{c}}(t) = \left[\begin{array}{ccc|c} -\frac{2\,t}{9} & 4\,t & -\frac{50\,t}{9} & 0 \end{array}\right]$$

$$\mathbf{G}_{\mathsf{c}}^\top = \left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\mathbf{g}({}^1s, {}^2s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}\mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1s-{}^1\lambda_{j_1})({}^2s-{}^2\lambda_{j_2})\cdots({}^ns-{}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}}{({}^1s-{}^1\lambda_{j_1})({}^2s-{}^2\lambda_{j_2})\cdots({}^ns-{}^n\lambda_{j_n})}},$$

Theorem: n-D Lagrangian realization

A $2\ell+\kappa-1 = m$-th order realization $(\mathbf{G}, \mathbf{\Phi}, \mathbf{W})$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\hat{\mathbf{H}}({}^1s, \cdots, {}^ns) = \mathbf{W}\mathbf{\Phi}({}^1s, {}^2s, \cdots, {}^ns)^{-1}\mathbf{G}$, is given by,

$$\begin{aligned}
\mathbf{\Phi}({}^1s, \cdots, {}^ns) &= \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m} \\
\mathbf{G} &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hdashline \mathbf{\Delta}(\ell,:)^\top \\ \hdashline \mathbf{0}_{\ell,1} \end{array}\right] \in \mathbb{C}^{m\times 1} \\
\mathbf{W} &= \left[\begin{array}{c|c|c} \mathbf{0}_{1,\kappa} & \mathbf{0}_{1,\ell-1} & -\mathbf{e}_\ell^\top \end{array}\right] \in \mathbb{C}^{1\times m}
\end{aligned}$$

where $\mathbb{A}^{\mathrm{Lag}}, \mathbb{B}^{\mathrm{Lag}} \in \mathbb{C}^{\ell\times\kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

Multi-variate realization

Generalized n-D Lagrangian realization (focus on left / right variable sets)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c:c:c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned}
\mathbf{\Gamma} &= {}^1\mathbb{X}^{\mathrm{Lag}} \otimes {}^2\mathbb{X}^{\mathrm{Lag}} \otimes \cdots \otimes {}^k\mathbb{X}^{\mathrm{Lag}} \in \mathbb{C}^{\kappa\times\kappa}[{}^1s,\cdots,{}^ks] \\
\mathbf{\Delta} &= {}^{k+1}\mathbb{X}^{\mathrm{Lag}} \otimes {}^{k+2}\mathbb{X}^{\mathrm{Lag}} \otimes \cdots \otimes {}^n\mathbb{X}^{\mathrm{Lag}} \in \mathbb{C}^{\ell\times\ell}[{}^{k+1}s,\cdots,{}^ns] \\
{}^j\mathbb{X}^{\mathrm{Lag}} &= \left[\begin{array}{ccccc} {}^j\mathbf{x}_1 & -{}^j\mathbf{x}_2 & 0 & \cdots & 0 \\ {}^j\mathbf{x}_1 & 0 & -{}^j\mathbf{x}_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}^j\mathbf{x}_1 & 0 & \cdots & 0 & -{}^j\mathbf{x}_{n_j} \\ \hline {}^jq_1 & {}^jq_2 & \cdots & {}^jq_{n_j-1} & {}^jq_{n_j} \end{array}\right] \in \mathbb{C}^{n_j\times n_j} \\
{}^j\mathbf{x}_i &= {}^js - {}^j\lambda_i
\end{aligned}$$

Facts

- Left / right variables splitting

$$\Gamma \text{ and } \Delta$$

- ${}^j\mathbb{X}^{\mathrm{Lag}}$ is unimodular, i.e.

$$\det({}^j\mathbb{X}^{\mathrm{Lag}}) = 1$$

- ...so are Γ and Δ

Multi-variate realization

Generalized n-D Lagrangian realization (focus on barycentric weights $\mathbb{A}^{\mathrm{Lag}}$ and $\mathbb{B}^{\mathrm{Lag}}$)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c:c:c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\mathrm{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\mathrm{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned} \mathbb{A}^{\mathrm{Lag}} &= \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m+1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,m+1} \end{bmatrix} \\ \mathbf{c}_n &= \text{vec}\left(\mathbb{A}^{\mathrm{Lag}}\right) \end{aligned}$$

Facts

- $\mathbb{A}^{\mathrm{Lag}}$ is simply some rearrangement of $\mathcal{N}(\mathbb{L}_n) = \mathbf{c}_n$
- $\mathbb{B}^{\mathrm{Lag}}$ follows

Multi-variate realization

Generalized n-D Lagrangian realization (control the complexity)

$$\mathbf{\Phi}({}^1s,\cdots,{}^ns) = \left[\begin{array}{c|c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} & \mathbf{0}_{\kappa-1,\ell} \\ \hdashline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top & \mathbf{0}_{\ell,\ell} \\ \hdashline \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} & \mathbf{\Delta}^\top \end{array}\right] \in \mathbb{C}^{m\times m}$$

$$\begin{aligned} m &= 2\ell+\kappa-1 \\ \kappa &= \prod_{j=1}^{k} n_j \\ \ell &= \prod_{j=k+1}^{n} n_j \end{aligned}$$

Facts

- Γ gathers the first group of parameters ${}^1s,\cdots,{}^ks$
- Δ gathers the second group of parameters ${}^{k+1}s,\cdots,{}^ns$

Complexity

Re-ordering allows complexity control, e.g. according to the order of each variable ${}^js$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$\begin{array}{rcl} \mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\ \mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right] \end{array}$$

Arrangement #1

$(s)-(t,p)$, one obtains a realization of dimension $m=13$:

$$\kappa = 2 \text{ and } \ell = 2\times 3$$

$$\boldsymbol{\Delta}(s) = {}^1\mathbb{X}^{\mathrm{Lag}}(s)$$

$$\boldsymbol{\Gamma}(t,p) = {}^2\mathbb{X}^{\mathrm{Lag}}(t) \otimes {}^3\mathbb{X}^{\mathrm{Lag}}(p)$$

Arrangement #2

$(s,t)-(p)$, one obtains a realization of dimension $m=9$:

$$\kappa = 2\times 2 \text{ and } \ell = 3$$

$$\boldsymbol{\Delta}(s,t) = {}^1\mathbb{X}^{\mathrm{Lag}}(s) \otimes {}^2\mathbb{X}^{\mathrm{Lag}}(t)$$

$$\boldsymbol{\Gamma}(p) = {}^3\mathbb{X}^{\mathrm{Lag}}(p)$$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$
\begin{aligned}
\mathbf{c}_3^\top &= \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\
\mathbb{W}_3 &= \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right]
\end{aligned}
$$

$$
\mathbf{\Phi} = \left[\begin{array}{cccc|cc|ccc}
(s-2)\ (t-1) & -(s-2)\ (t-3) & -(t-1)\ (s-4) & (s-4)\ (t-3) & 0 & 0 & 0 & 0 & 0 \\
1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 & 0 & 0 & 0 \\
-\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 & 0 & 0 & 0 \\
\frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p & 0 & 0 & 0 \\
\hline
\frac{1}{8} & -\frac{17}{56} & -\frac{9}{56} & \frac{19}{56} & 0 & 0 & p-5 & p-5 & \frac{1}{2} \\
-\frac{2}{7} & \frac{5}{7} & \frac{5}{14} & -\frac{11}{14} & 0 & 0 & 6-p & 0 & -1 \\
\frac{9}{56} & -\frac{23}{56} & -\frac{11}{56} & \frac{25}{56} & 0 & 0 & 0 & 7-p & \frac{1}{2}
\end{array}\right]
$$

$$
\mathbf{W} = -\mathbf{e}_9^\top \text{ and } \mathbf{G}^\top = \left[\begin{array}{c|ccc|c} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 & \mathbf{0}_{1,3} \end{array}\right]
$$

Multi-variate realization

3-D case (example)

Data generated from $\mathbf{H}({}^1s, {}^2s, {}^3s) = \mathbf{H}(s,t,p) = (s+pt)/(p^2+s+t)$ of complexity $(1,1,2)$

$$\begin{array}{rcl} \mathbf{c}_3^\top & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{2} & -\frac{39}{28} & \frac{13}{14} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & -\frac{15}{28} & \frac{41}{28} & -\frac{27}{28} & \frac{4}{7} & -\frac{43}{28} & 1 \end{array}\right] \\ \mathbb{W}_3 & = & \left[\begin{array}{ccc|ccc|ccc|ccc} \frac{1}{4} & \frac{8}{39} & \frac{9}{52} & \frac{17}{30} & \frac{20}{41} & \frac{23}{54} & \frac{3}{10} & \frac{10}{41} & \frac{11}{54} & \frac{19}{32} & \frac{22}{43} & \frac{25}{56} \end{array}\right] \end{array}$$

$$\mathbf{\Phi}_c = \left[\begin{array}{cccc|cc} (s-2)\,(t-1) & -(s-2)\,(t-3) & -(t-1)\,(s-4) & (s-4)\,(t-3) & 0 & 0 \\ 1-\frac{s}{2} & \frac{s}{2}-1 & \frac{s}{2}-2 & 2-\frac{s}{2} & 0 & 0 \\ \frac{1}{2}-\frac{t}{2} & \frac{t}{2}-\frac{3}{2} & \frac{t}{2}-\frac{1}{2} & \frac{3}{2}-\frac{t}{2} & 0 & 0 \\ \hline \frac{1}{2} & -\frac{15}{28} & -\frac{15}{28} & \frac{4}{7} & p-5 & p-5 \\ -\frac{39}{28} & \frac{41}{28} & \frac{41}{28} & -\frac{43}{28} & 6-p & 0 \\ \frac{13}{14} & -\frac{27}{28} & -\frac{27}{28} & 1 & 0 & 7-p \end{array}\right]$$

$$\mathbf{W}_c(p) = \left[\begin{array}{cccc|cc} \frac{p}{28}+\frac{1}{14} & -\frac{3\,p}{28}-\frac{1}{14} & -\frac{p}{28}-\frac{1}{7} & \frac{3\,p}{28}+\frac{1}{7} & 0 & 0 \end{array}\right] \text{ and } \mathbf{G}_c^\top = \left[\begin{array}{c|ccc} \mathbf{0}_{1,3} & 1/2 & -1 & 1/2 \end{array}\right]$$

Multi-variate realization

(Compressed) generalized n-D Lagrangian realization

$$\mathbf{g}({}^1s, {}^2s, \cdots, {}^ns) = \frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}\mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1s-{}^1\lambda_{j_1})({}^2s-{}^2\lambda_{j_2})\cdots({}^ns-{}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\frac{c_{j_1,j_2,\cdots,j_n}}{({}^1s-{}^1\lambda_{j_1})({}^2s-{}^2\lambda_{j_2})\cdots({}^ns-{}^n\lambda_{j_n})}},$$

Theorem: n-D Lagrangian compressed realization

A $\ell+\kappa-1=m$-th order realization $(\hat{\mathbf{G}}_c, \hat{\mathbf{\Phi}}_c, \hat{\mathbf{W}}_c)$ of the multi-variate function $\hat{\mathbf{H}}$ in barycentric form, satisfying $\mathbf{H}({}^1s,\cdots,{}^ns) = \hat{\mathbf{W}}_c\hat{\mathbf{\Phi}}_c({}^1s,{}^2s,\cdots,{}^ns)^{-1}\hat{\mathbf{G}}_c({}^{k+1}s,\cdots,{}^ns)$, is given by,

$$\begin{aligned}
\hat{\mathbf{\Phi}}_c({}^1s,\cdots,{}^ns) &= \left[\begin{array}{c|c} \mathbf{\Gamma}(1:\kappa-1,:) & \mathbf{0}_{\kappa-1,\ell-1} \\ \hline \mathbb{A}^{\text{Lag}} & \mathbf{\Delta}(1:\ell-1,:)^\top \end{array}\right] \in \mathbb{C}^{m\times m} \\
\hat{\mathbf{G}}_c({}^{k+1}s,\cdots,{}^ns) &= \left[\begin{array}{c} \mathbf{0}_{\kappa-1,1} \\ \hline \mathbf{\Delta}(\ell,:)^\top \end{array}\right] \in \mathbb{C}^{m\times 1} \\
\hat{\mathbf{W}}_c &= \mathbf{e}_\ell^\top \mathbf{\Delta}^{-\top} \left[\begin{array}{c|c} \mathbb{B}^{\text{Lag}} & \mathbf{0}_{\ell,\ell-1} \end{array}\right] \in \mathbb{C}^{1\times m}
\end{aligned}$$

where $\mathbb{A}^{\text{Lag}}, \mathbb{B}^{\text{Lag}} \in \mathbb{C}^{\ell\times\kappa}$ are appropriately chosen, according to the chosen pseudo-companion basis.

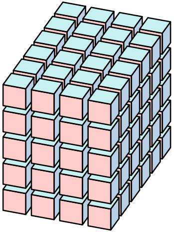
Content

Taming the curse of dimensionality

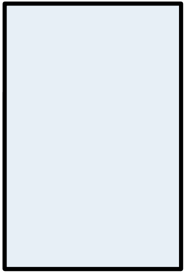
Loewner matrix operator

$$
\begin{array}{ccc}
\mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q\times K} \\
\left({}^1\lambda_{j_1}, {}^1\mu_{i_1}, \ldots, {}^n\lambda_{j_n}, {}^n\mu_{i_n}, \textbf{tab}_n\right) & \longmapsto & \mathbb{L}_n
\end{array}
$$

n-D tensor $\textbf{tab}_n$



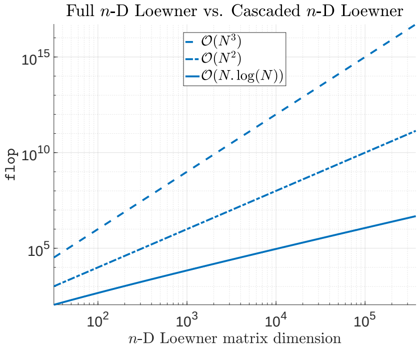
matrix $\mathbb{L}_n$



Taming the curse of dimensionality

Null space `flop` and memory issues

log-log scale



Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD `flop` estimation is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- in real double
- in complex double

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$ $\left[\begin{array}{ccc} \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 & \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 & \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \end{array}\right]^\top$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s$ \ ${}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top = \left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
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$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top = \left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1s \backslash {}^2s$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along ${}^1s$, for ${}^2s = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2s$ for ${}^1s = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\begin{array}{ccc} \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 & \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 & \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \end{array}\right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$ and $$\mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case

Theorem: 2-D to 1-D

Being given the tableau $\mathbf{tab}_2$ tensor in response of the 2-variables $\mathbf{H}({}^1s, {}^2s)$ function, the null space of the corresponding 2-D Loenwer matrix $\mathbb{L}_2$, is spanned by

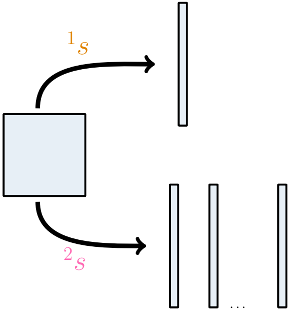
$$\mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{{}^2\lambda_1} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_1, \cdots, \mathbf{c}_1^{{}^2\lambda_{k_2}} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_{k_2}\right],$$

where

- $\mathbf{c}_1^{{}^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{{}^1\lambda_{k_1}})$,
  i.e. the null space of the **1-D Loewner matrix** for frozen ${}^1s = {}^1\lambda_{k_1}$, and
- $\mathbf{c}_1^{{}^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{{}^2\lambda_{j_2}})$,
  i.e. the j_1-th null space of the **1-D Loewner matrices** for frozen ${}^2s = \{{}^2\lambda_1, \cdots, {}^2\lambda_{k_2}\}$.

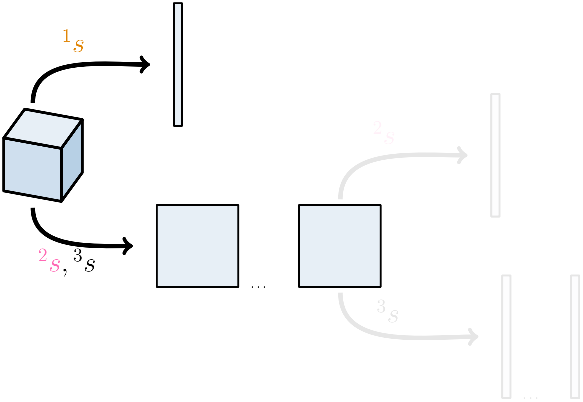
Taming the curse of dimensionality

2-D case



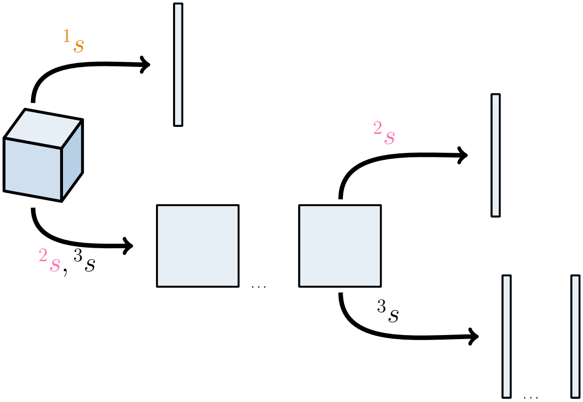
Taming the curse of dimensionality

3-D case



Taming the curse of dimensionality

3-D case



Taming the curse of dimensionality

n-D case

Theorem: n-D to $(n-1)$-D

Being given the tableau $\mathbf{tab}_n$ tensor in response of the n-variables $\mathbf{H}({}^1s,\cdots,{}^ns)$ function, the null space of the corresponding n-D Loenwer matrix $\mathbb{L}_n$, is spanned by

$$\mathcal{N}(\mathbb{L}_n) = \mathbf{vec}\left[c_{n-1}^{{}^1\lambda_1} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})} \right]_1, \cdots, \mathbf{c}_{n-1}^{{}^1\lambda_{k_1}} \cdot \left[\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})} \right]_{k_1} \right],$$

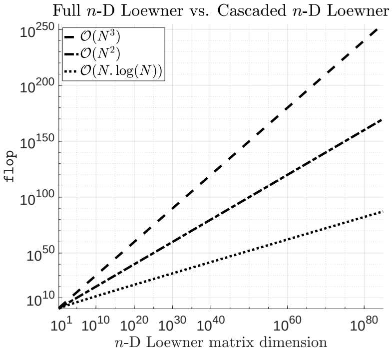
where

- $\mathbf{c}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})}$ spans $\mathcal{N}(\mathbb{L}_1^{({}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n})})$, i.e. the null space of the **1-D Loewner matrix** for frozen $\{{}^2\lambda_{k_2},{}^3\lambda_{k_3},\cdots,{}^n\lambda_{k_n}\}$, and
- $\mathbf{c}_{n-1}^{{}^1\lambda_{j_1}}$ spans $\mathcal{N}(\mathbb{L}_{n-1}^{{}^1\lambda_j})$, i.e. the j_1-th null space of the $(n-1)$**-D Loewner matrix** for frozen ${}^1s_{j_1} = \{{}^1\lambda_1,\cdots,{}^1\lambda_{k_1}\}$.

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Null space underlying problem

Let (rows) $Q = q_1 q_2 \dots q_n$ and (columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Note that $Q \times K$ matrix SVD `flop` estimation is

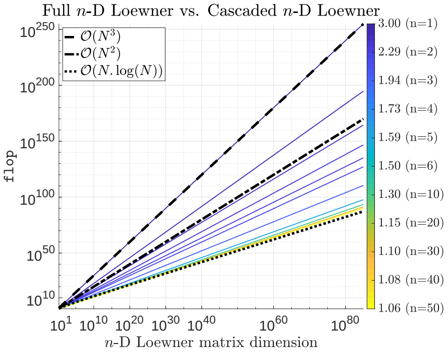
- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

$\Rightarrow$ **The CURSE of dimensionality**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Theorem: Recursive complexity

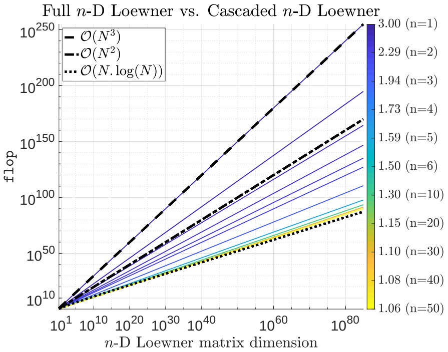
$$\texttt{flop}_1(n) = \sum_{j=1}^{n}\left(k_j^3 \prod_{l=1}^{j} k_{l-1}\right) \quad \text{where } k_0 = 1.$$

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Corollary: Worst case complexity

k interpolation points per variables.

$$\overline{\texttt{flop}_1} = k^3 \frac{1-k^n}{1-k} = k^3 \frac{1-N}{1-k},$$

which is a (n finite) geometric series of ratio k.

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

$$\begin{array}{lll} \mathcal{O}(N^3) & \rightarrow \ \mathcal{O}(N^{2.29}) & \text{for } n=2 \\ & \rightarrow \ \mathcal{O}(N^{1.94}) & \text{for } n=3 \\ & \vdots & \\ & \rightarrow \ \mathcal{O}(N^{1.5}) & \text{for } n=6 \\ & \vdots & \\ & \rightarrow \ \mathcal{O}(N^{1.06}) & \text{for } n=50 \end{array}$$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}} \prod_{l}^{n} q_l + k_l \;\; \texttt{MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs } 45 \texttt{ MB)}$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \;\; \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
flop: $9.78 \cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}}\prod_{l}^{n} q_l + k_l \ \ \texttt{MB} \text{ (example tableau } 2\cdot[20,6,4,6,8,2] = 2\cdot[k_1,k_2,k_3,k_4,k_5,k_6] \text{ needs } 45 \ \texttt{MB)}$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N\times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}}N^2 \ \ \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
flop: $9.78\cdot 10^{13}$

Cascaded n-D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\overline{k}\times\overline{k}}$$

where $\overline{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}}\overline{k}^2 \ \ \texttt{MB}$$

Example: $\overline{k} = 20$
Memory: 6.25 KB
flop: $8.13\cdot 10^{5}$

Taming the curse of dimensionality

Numerical examples, 20-D example

$$\mathbf{H}({}^1s, {}^2s, \ldots, {}^{20}s) =$$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- 20-D tensor of dimension (≥ 48 TB in real double precision)
- Complexity: $(3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)$
- n-D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- Full SVD: $2.49 \cdot 10^{20}$ `flop`
  Recursive SVD: $5.43 \cdot 10^7$ `flop`
- error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples, 20-D example

$$\mathbf{H}({}^1s, {}^2s, \ldots, {}^{20}s) =$$

$$\frac{3 \cdot {}^1s^3 + 4 \cdot {}^8s + {}^{12}s + {}^{13}s \cdot {}^{14}s + {}^{15}s}{{}^1s + {}^2s^2 \cdot {}^3s + {}^4s + {}^5s + {}^6s + {}^7s \cdot {}^8s + {}^9s \cdot {}^{10}s \cdot {}^{11}s + {}^{13}s + {}^{13}s^3 \cdot \pi + {}^{17}s + {}^{18}s \cdot {}^{19}s - {}^{20}s}$$

Statistics

- 20-D tensor of dimension (≥ 48 TB in real double precision)
- Complexity: $(3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
- n-D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- Full SVD: $2.49 \cdot 10^{20}$ `flop`
  Recursive SVD: $5.43 \cdot 10^{7}$ `flop` $\rightarrow 5.03 \cdot 10^{7}$ `flop`
- error $\approx 10^{-11}$

Taming the curse of dimensionality

Numerical examples (from 2 to 20 variables)

#4 Rational function

$$s_4{}^3 + \frac{s_1\, s_3}{s_3{}^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3{}^2 + s_1\, s_3\, s_5{}^3}{s_1{}^3 + s_4 + s_2\, s_3}$$

#6 Rational function

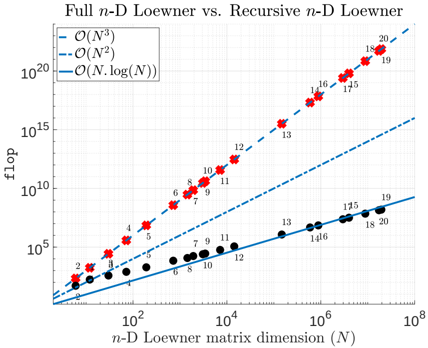
$$\frac{-\sqrt{2}\, s_6{}^2 + s_1 + s_3}{s_1{}^2 + s_4{}^3 + s_5{}^2 + s_6 + s_2\, s_3}$$

#7 Rational function

$$\frac{s_3\, s_2{}^3 + 1}{s_3\, s_2{}^2 + s_4{}^2 + s_6{}^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3\, s_1{}^3 + s_{18}{}^2 + 4\, s_8 + s_{12} + s_{15} + s_{13}\, s_{14}}{s_3\, s_2{}^2 + \pi\, s_{16}{}^3 + s_{17}{}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7\, s_8 + s_9\, s_{10}\, s_{11}}$$



Taming the curse of dimensionality

Numerical examples (rational and irrational)

#16 Arc-tangent function

$$\frac{\operatorname{atan}(x_1) + \operatorname{atan}(x_2) + \operatorname{atan}(x_3) + \operatorname{atan}(x_4)}{{x_1}^2\,{x_2}^2 - {x_1}^2 - {x_2}^2 + 1}$$

#17 Exponential function

$$\frac{e^{x_1\,x_2\,x_3\,x_4}}{{x_1}^2 + {x_2}^2 - x_3\,x_4 + 3}$$

#18 Sinc function

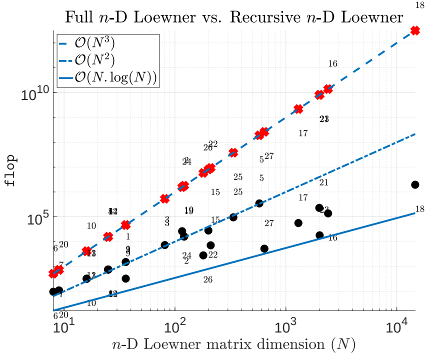
$$\frac{10\,\sin(x_1)\,\sin(x_2)\,\sin(x_3)\,\sin(x_4)}{x_1\,x_2\,x_3\,x_4}$$

#19 Sinc function

$$\frac{10\,\sin(x_1)\,\sin(x_2)}{x_1\,x_2}$$

#20 Polynomial function

$${x_1}^2 + x_1\,x_2 + {x_2}^2 - x_2 + 1$$



Content

Variable decoupling

Given data $\mathbf{tab}_n$, the (recursive) theorem achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{{}^n s}}_{\mathbf{Bary}({}^n s)} \odot \underbrace{(\mathbf{c}^{{}^{n-1} s} \otimes \mathbf{1}_{k_n})}_{\mathbf{Bary}({}^{n-1} s)} \odot \underbrace{(\mathbf{c}^{{}^{n-2} s} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\mathbf{Bary}({}^{n-2} s)} \odot \cdots \odot \underbrace{(\mathbf{c}^{{}^1 s} \otimes \mathbf{1}_{k_n \ldots k_2})}_{\mathbf{Bary}({}^1 s)}.$$

where $\mathbf{c}^{{}^j s}$ denotes the vectorized barycentric coefficients related to the j-th variable.

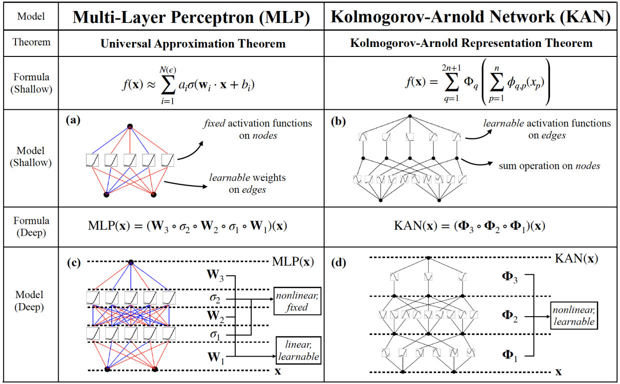
A.G. Vitushkin, *"On Hilbert's thirteenth problem and related questions"*, Russian Math. Surveys 59:1, pp. 11–25.

Variables decoupling, KST and KANs

About KANs

KANs features

- **Inspired by the Kolmogorov-Arnold representation theorem**
- The model output is a composition of **sums** and **learnable activation functions** (e.g. splines)
- Alternate to Multi-Layer Perceptrons (MLP), having fixed activation functions (e.g. ReLU), **inspired by the universal approximation theorem**



Comparison between MLP and KAN (figure from *Z. Liu et al.*)

Z. Liu et al., *"KAN: Kolmogorov–Arnold Networks"*, `https://arxiv.org/abs/2404.19756`.

Building the Kolmogorov-Arnold model as follows

$$F({}^1s, {}^2s, \cdots, {}^ns) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^{m} f_{kj}({}^js) \right)$$

$f_{kj} : [0,1] \mapsto \mathbb{R}$ and $\Phi_k : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions.

The relation is approximated by $k = 1, \cdots, d = 2m+1$ as

$$\hat{F}({}^1s, {}^2s, \cdots, {}^ns) = \sum_{k=1}^{d} \Phi_k \underbrace{\left(\sum_{j=1}^{m} f_{kj}({}^js_i) \right)}_{\theta_{ik}}$$

where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes **splines**

M. Poluektov and A. Polar, *"Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method"*, `https://arxiv.org/abs/2305.08194`.

Variables decoupling, KST and KANs

KANs with splines

Building the Kolmogorov-Arnold model as follows

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Variables decoupling, KST and KANs

KANs (via Loewner) with rational activation functions

$$\mathbf{H} = {}^1s \cdot {}^2s$$

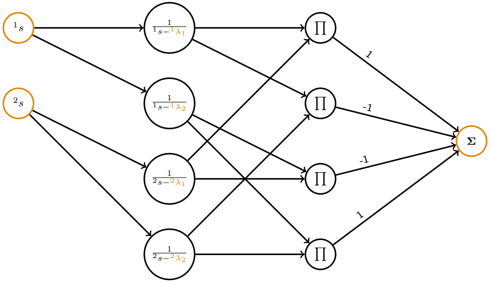
$$\begin{aligned} {}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\ {}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{c}^{{}^1s} \cdot \mathbf{Lag}({}^1s) & \mathbf{c}^{{}^2s} \cdot \mathbf{Lag}({}^2s) \\ -\frac{1.0}{{}^1s+1.0} & -\frac{1.0}{{}^2s+1.0} \\ -\frac{1.0}{{}^1s+1.0} & \frac{1}{{}^2s-1.0} \\ \frac{1}{{}^1s-1.0} & -\frac{1.0}{{}^2s+1.0} \\ \frac{1}{{}^1s-1.0} & \frac{1}{{}^2s-1.0} \end{pmatrix}$$

Equivalent denominator and numerator read:

$$\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \quad \text{and} \quad \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}$$

Equivalent KAN-like with rational activation functions (just $\mathbf{D}$)



Content

Comparison with Tensor Flow

About Keras via Tensor Flow (https://www.tensorflow.org/guide/keras?hl=en)

Some key figures

- Keras is the high-level API of the TensorFlow (by Google) platform.
- Multi Layer Perceptron
- Dense connected network
- ReLU activation functions
- Optim.: ADAM (Adaptive Moment Estimation)
- Random init. (with fixed seed)
- 1000 iterations

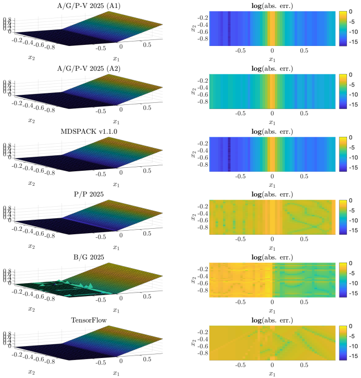
Python code

```python
import numpy as np
import math
import matplotlib.pyplot as plt
import tensorflow as tf
from keras.models import Sequential
from matplotlib import cm

# Data
def H(x):
    #y = pow(x[:,0],2)*x[:,1]
    #y = 1/2*(x[:,0] + np.abs(x[:,0])) + 1/10*x[:,1] #1
    #y = x[:,0]*x[:,1] #3
    #y = np.exp(x[:,0]*x[:,1])/( (pow(x[:,0],2)-1.44)*(pow(x[:,1],2)-1.44) ) #6
    #y = np.tanh(4*(x[:,0]-x[:,1])) #8
    #y = pow(np.abs((x[:,0]-x[:,1])),3) #10
    y = (pow(x[:,0],2) + pow(x[:,1],2) + x[:,0] - x[:,1] + 1) / (pow(x[:,0],3)  + pow(x[:,1],2) + 4) #15
    return np.transpose(np.array([y]))

n   = 2
N1  = 40
N2  = 40
x1  = np.linspace(-1, 1, N1)
x2  = np.linspace(-1, 1, N2)

# IP
N   = N1*N2
tab = np.zeros((N,1))
```

Comparison with Tensor Flow

Comparison (example #1)

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
1	A/G/P-V 2025 (A1)	1e-11,3	144	0.0171	0.000699	**1.39e − 17**	0.00841
	A/G/P-V 2025 (A2)	1e-15,3	160	0.0823	**0.000389**	4.49e-13	**0.00434**
	MDSPACK v1.1.0	1e-11,1e-06	144	**0.0108**	0.000699	2.43e-17	0.00841
	P/P 2025	1,0.95,50,0.01,4,6,9	**130**	0.282	0.0017	3.08e-07	0.0152
	B/G 2025	0.001,10	320	0.0753	0.0742	1.13e-09	1.58
	TensorFlow		257	25.3	0.000707	7.77e-08	0.0102



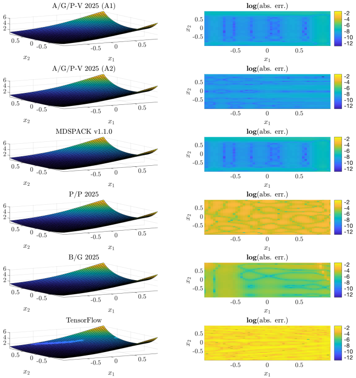
$$\mathrm{ReLU}({}^1s) + \frac{1}{100}{}^2s$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -\frac{1}{10000000000} \end{pmatrix}$

Comparison with Tensor Flow

Comparison (example #2)

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
1	A/G/P-V 2025 (A1)	1e-11,3	144	0.0171	0.000699	**1.39e − 17**	0.00841
	A/G/P-V 2025 (A2)	1e-15,3	160	0.0823	**0.000389**	4.49e-13	**0.00434**
	MDSPACK v1.1.0	1e-11,1e-06	144	**0.0108**	0.000699	2.43e-17	0.00841
	P/P 2025	1,0.95,50,0.01,4,6,9	**130**	0.282	0.0017	3.08e-07	0.0152
	B/G 2025	0.001,10	320	0.0753	0.0742	1.13e-09	1.58
	TensorFlow		257	25.3	0.000707	7.77e-08	0.0102



$$\exp(\sin({}^1s) + {}^2s^2)$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 1 \end{pmatrix}$

Content

Conclusion

Take home message

Main contributions

From any n-th order multi-variate transfer function / data tensor

- Construct a transfer function in barycentric form
- **Construct a realization with controlled complexity**
- **Tame the computational complexity**
  $\mathcal{O}(N^3) \rightarrow\approx \mathcal{O}(N^{2.29,1.94,\cdots})$
- Two algorithms (direct & iterative)
- Connection with Kolmogorov theorem

Side effects

[Theo.] Tame COD in KST

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with
A.C. Antoulas [Rice Univ.]
I.V. Goșea [MPI]

`https://arxiv.org/abs/2405.00495`
`https://github.com/cpoussot/mLF`

Conclusion

Big picture