Chapter 8

A Tutorial Introduction to the Loewner Framework for Model Reduction

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One of the main approaches to model reduction of both linear and nonlinear dynamical systems is by means of interpolation. Data-driven model reduction constitutes a special case. The Loewner matrix, originally developed for rational interpolation, has been recently extended to the Loewner framework and constitutes a versatile approach to data-driven model reduction. Its main attribute is that it provides a trade-off between accuracy of fit and complexity of the model. Furthermore, constructing models from the given data is quite natural. The purpose of this chapter is to present the fundamentals of the Loewner framework for data-driven reduction of linear systems.

8.1 ▪ Introduction

Model reduction seeks to replace a large-scale system described in terms of differential or difference equations by a system of much lower dimension that has nearly the same response characteristics.

Model (order) reduction (MOR) is commonly used to simulate and control complex physical processes. The systems that inevitably arise in such cases are often too complex to meet the expediency requirements of interactive design, optimization, or real-time control. MOR was devised as a means to reduce the dimensionality of these complex systems to a level that is amenable to such requirements. The ensuing methods are an indispensable tool for speeding up the simulations arising in various engineering applications involving large-scale dynamical systems.

Generally, large systems arise due to accuracy requirements on the spatial discretization of partial differential equations (PDEs) for fluids or structures or in the context of lumped-circuit approximations of distributed circuit elements. For some applications, see, e.g., [3].

[26]The authors wish to thank the referees for their careful reading of the manuscript and their useful suggestions.

The work of A. C. Antoulas was supported by NSF Grants CCF-1017401 and CCF-1320866 as well as DFG Grant AN-693/1-1.

Approaches to model reduction

Model reduction methods can be classified in two broad categories, namely, *SVD-based* and *Krylov-based* or *moment matching* methods. The former category derives its name from the fact that the corresponding reduction methods are related to SVD (singular value decomposition) and the associated 2-norm. The most prominent among them is *balanced truncation* (BT). These methods are based on computing controllability and observability *Gramians* (or generalized/empirical versions thereof), which leads to the elimination of states that are difficult to reach and observe. The bottleneck in applying them is due to the high computational cost required to obtain the Gramians by solving Lyapunov or Riccati equations. However, recent developments make it possible to obtain the approximate solution (i.e., approximate balancing) of realistic size problems; see, e.g., [10, 13, 14, 34] and references therein.

The reduction methods in the second category are based on *moment matching*, that is, matching of the coefficients of power series expansions of the transfer function at selected points in the complex plane; these coefficients are the values of the underlying transfer function, together with the values of its derivatives. The main underlying problem is *rational interpolation.* These methods are closely related to the so-called Krylov iteration, encountered in numerical linear algebra, as well as the Arnoldi or the Lanczos procedures, and multipoint (rational) versions thereof.

The advantages of balancing reduction methods include preservation of stability and an a priori computable error bound. Krylov-based methods are numerically efficient and have lower computational cost, but, in general, the preservation of other properties is not automatic and depends on the choice of the expansion points and the form of the projector (e.g., orthogonal or oblique).

For details on the above, as well as other issues in model reduction, we refer to the book [3].

8.1.1 ▪ Main notation

$\mathbb{R}, \mathbb{C}$	real numbers, complex numbers
$\imath$	$= \sqrt{-1}$
$(\cdot)^T$	transposition
$(\cdot)^*$	transposition and complex conjugation
$(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$	descriptor system realization
$(\mathbf{E}_\delta, \mathbf{A}_\delta, \mathbf{B}_\delta, \mathbf{C}_\delta)$	descriptor system realization as in (8.5)
$\mathbf{H}(s) \in \mathbb{R}^{p\times m}$	transfer function
$\mathbf{M} = \mathrm{diag}(\mu_j) \in \mathbb{C}^{q\times q}$	matrix of left interpolation frequencies
$\boldsymbol{\ell}_i \in \mathbb{C}^p$	left tangential directions
$\mathbf{L} \in \mathbb{C}^{q\times p}$	matrix of left directions
$\mathbf{v}_i \in \mathbb{C}^m$	left tangential values
$\mathbf{V} \in \mathbb{C}^{q\times m}$	matrix of left values
$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_j) \in \mathbb{C}^{k\times k}$	matrix of right interpolation frequencies

(continued on next page)

$\mathbf{r}_j \in \mathbb{C}^m$	right tangential directions
$\mathbf{R} \in \mathbb{C}^{m\times k}$	matrix of right directions
$\mathbf{w}_j \in \mathbb{C}^p$	right tangential values
$\mathbf{W} \in \mathbb{C}^{p\times k}$	matrix of right values
$\mathbb{L} \in \mathbb{C}^{q\times k}$	Loewner matrix
$\mathbb{L}_s \in \mathbb{C}^{q\times k}$	shifted Loewner matrix
$\phi(s)$	scalar rational interpolant
$\mathbf{c} = [\alpha_j] \in \mathbb{C}^k$	element in the null space of $\mathbb{L}$

8.1.2 ▪ Model reduction of linear descriptor systems

A linear time-invariant dynamical system Σ with m inputs, p outputs, and n internal variables in *descriptor-form representation* is given by a set of differential algebraic equations (DAEs):

$$\Sigma: \ \mathbf{E}\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{8.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the internal variable (the state if $\mathbf{E}$ is invertible); $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$ are the input and output functions, respectively; and

$$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \ \mathbf{B} \in \mathbb{R}^{n\times m}, \ \mathbf{C} \in \mathbb{R}^{p\times n}, \ \mathbf{D} \in \mathbb{R}^{p\times m}$$

are constant matrices. The matrix pencil $(\mathbf{A}, \mathbf{E})$ is *regular* if the matrix $\mathbf{A} - \lambda\mathbf{E}$ is nonsingular for some finite $\lambda \in \mathbb{C}$. In this case the *transfer function* of Σ is the $p \times m$ rational matrix function

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{8.2}$$

It is *proper* if its value at infinity is finite and *strictly proper* if that value is zero. The *poles* of the system are the eigenvalues of the matrix pencil $(\mathbf{A}, \mathbf{E})$. Σ is *stable* if all its finite poles are in the left half of the complex plane.

The quintuple $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is called a *descriptor realization* of $\mathbf{H}(s)$; realizations are not unique, and those with the smallest possible dimension n are called *minimal realizations*; furthermore, $\operatorname{rank}\mathbf{E}$ is the McMillan degree of Σ (regardless of the minimality of the realization) [28]. A realization is minimal if it is *completely controllable* and *observable*. A descriptor system with $(\mathbf{A}, \mathbf{E})$ regular is *completely controllable* [37] if $\operatorname{rank}[\mathbf{A} - \lambda\mathbf{E}, \mathbf{B}] = n$ for all finite $\lambda \in \mathbb{C}$ and $\operatorname{rank}[\mathbf{E}, \mathbf{B}] = n$. This is equivalent to the matrix

$$\mathscr{R}_n = \begin{bmatrix} (\lambda_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_n\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix} \tag{8.3}$$

having full rank n for any set of distinct $\lambda_i \in \mathbb{C}$ that are not eigenvalues of the pencil $(\mathbf{A}, \mathbf{E})$. It is called *completely observable* if $\operatorname{rank}\left[\mathbf{A}^T - \mu\mathbf{E}^T, \mathbf{C}^T\right] = n$ for all finite $\mu \in \mathbb{C}$ and $\operatorname{rank}\left[\mathbf{E}^T, \mathbf{C}^T\right] = n$ (where $(\cdot)^T$ denotes transpose); this is equivalent to the matrix

$$\mathscr{O}_n = \begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_n\mathbf{E} - \mathbf{A})^{-1} \end{bmatrix} \tag{8.4}$$

having full rank n for any set of distinct $\mu_i \in \mathbb{C}$ that are not eigenvalues of the pencil $(\mathbf{A}, \mathbf{E})$. For details on these concepts we refer to [15].

Remark 8.1. **(a) The D-term**. In description (8.1) we can eliminate $\mathbf{D}$ by incorporating it in the remaining matrices. To achieve this, we have to allow the dimension of the realization to increase by rank $\mathbf{D}$. Consider a rank-revealing factorization

$$\mathbf{D} = \mathbf{D}_1 \mathbf{D}_2 \quad \text{where} \quad \mathbf{D}_1 \in \mathbb{R}^{p \times \rho},\ \mathbf{D}_2 \in \mathbb{R}^{\rho \times m},$$

and $\rho = \operatorname{rank} \mathbf{D}$. It readily follows that

$$\mathbf{E}_\delta = \begin{bmatrix} \mathbf{E} & \\ & 0_{\rho\times\rho} \end{bmatrix},\ \mathbf{A}_\delta = \begin{bmatrix} \mathbf{A} & \\ & -\mathbf{I}_\rho \end{bmatrix},\ \mathbf{B}_\delta = \begin{bmatrix} \mathbf{B} \\ \mathbf{D}_2 \end{bmatrix},\ \mathbf{C}_\delta = \begin{bmatrix} \mathbf{C} & \mathbf{D}_1 \end{bmatrix} \tag{8.5}$$

is a descriptor realization of the same system with no $\mathbf{D}$-term (i.e., $\mathbf{D}_\delta = 0$). If the original realization is controllable and observable, this one is R-controllable and R-observable (see [15, 37] for a survey of these concepts).

The reason for introducing descriptor realizations where the $\mathbf{D}$*-term is incorporated in the remaining matrices is that the Loewner framework yields precisely such descriptor realizations.*

(b) Poles at infinity. The eigenvalues of the pencil $(\mathbf{A}_\delta, \mathbf{E}_\delta)$ at infinity are also referred to as the poles of the system Σ at infinity (under the assumption that the realization is minimal). A pole at infinity of the same algebraic and geometric multiplicity corresponds to the existence of a constant $\mathbf{D}$-term in $\mathbf{H}$, while a pole of algebraic multiplicity $\nu \geq 1$ and geometric multiplicity one implies the existence in $\mathbf{H}$ of a polynomial term of degree $\nu - 1$.

The *model reduction* problem consists of constructing reduced-order DAE systems of the form

$$\hat{\Sigma}:\ \hat{\mathbf{E}}\frac{d}{dt}\hat{\mathbf{x}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),\ \ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t), \tag{8.6}$$

where $\hat{\mathbf{x}}(t) \in \mathbb{R}^r$ is the internal variable (the state if $\hat{\mathbf{E}}$ is invertible), $\hat{\mathbf{y}}(t) \in \mathbb{R}^p$ is the output of $\hat{\Sigma}$ corresponding to the same input $\mathbf{u}(t)$, and

$$\hat{\mathbf{E}}, \hat{\mathbf{A}} \in \mathbb{R}^{r\times r},\ \ \hat{\mathbf{B}} \in \mathbb{R}^{r\times m},\ \ \hat{\mathbf{C}} \in \mathbb{R}^{p\times r},\ \ \hat{\mathbf{D}} \in \mathbb{R}^{p\times m}.$$

Thus, the number of inputs m and outputs p remains the same while $r \ll n$.

Some general *goals* for reduced-order models (ROMs) are as follows: (1) the reduced input-output map should be uniformly "close" to the original: for the same $\mathbf{u}$, $\mathbf{y} - \hat{\mathbf{y}}$ should be "small" in an appropriate sense; (2) critical system features and structure should be preserved, e.g., stability, passivity, Hamiltonian structure, subsystem interconnectivity, or second-order structure; (3) strategies for computing the reduced system should lead to robust, numerically stable algorithms and require minimal application-specific tuning.

8.1.3 ▪ Interpolatory reduction for linear systems

The exposition in this section follows [8]. See also [11]. These papers should be consulted for an overview of interpolatory reduction methods in general.

Consider the system Σ and its transfer function $\mathbf{H}(s)$ defined by (8.2). We are given *left interpolation points* $\{\mu_i\}_{i=1}^q \subset \mathbb{C}$, with *left tangential directions* $\{\boldsymbol{\ell}_i\}_{i=1}^q \subset \mathbb{C}^p$, and *right interpolation points* $\{\lambda_i\}_{i=1}^k \subset \mathbb{C}$, with *right tangential directions* $\{\mathbf{r}_i\}_{i=1}^k \subset \mathbb{C}^m$; for simplicity we assume that the left and right interpolation points are distinct. We seek a reduced-order system $\hat{\Sigma}$ such that the associated transfer function $\hat{\mathbf{H}}(s)$ is a *tangential interpolant* to $\mathbf{H}(s)$:

$$\left.\begin{array}{ccc} \boldsymbol{\ell}_j^T \hat{\mathbf{H}}(\mu_j) = \boldsymbol{\ell}_j^T \mathbf{H}(\mu_j) & & \hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{H}(\lambda_i)\mathbf{r}_i \\ \text{for } j=1,\ldots,q & \text{and} & \text{for } i=1,\ldots,k \end{array}\right\}. \tag{8.7}$$

Interpolation points and tangential directions are selected to realize the model reduction goals stated.

If, instead of descriptor-form data as in (8.1), we are given *input/output data* (measured or generated by DNS[27]), the resulting problem is modified as follows. Given a set of input-output response measurements specified by *left driving frequencies* $\{\mu_i\}_{i=1}^q \subset \mathbb{C}$, using *left input or tangential directions* $\{\boldsymbol{\ell}_i\}_{i=1}^q \subset \mathbb{C}^p$, producing *left responses* $\{\mathbf{v}_i\}_{i=1}^q \subset \mathbb{C}^m$, and *right driving frequencies* $\{\lambda_i\}_{i=1}^k \subset \mathbb{C}$, using *right input or tangential directions* $\{\mathbf{r}_i\}_{i=1}^k \subset \mathbb{C}^m$, producing *right responses* $\{\mathbf{w}_i\}_{i=1}^k \subset \mathbb{C}^p$, find a (low-order) system $\hat{\Sigma}$ such that the resulting transfer function, $\hat{\mathbf{H}}(s)$, is an (*approximate*) *tangential interpolant* to the data:

$$\left.\begin{array}{ccc} \boldsymbol{\ell}_j^T \hat{\mathbf{H}}(\mu_j) = \mathbf{v}_j^T & & \hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i \\ \text{for } j=1,\ldots,q & \text{and} & \text{for } i=1,\ldots,k \end{array}\right\}. \tag{8.8}$$

As before, interpolation points and tangential directions are determined by the problem. It should be noted that for SISO systems, i.e., systems with a single input and a single output ($m=p=1$), left and right directions can be taken equal to one ($\boldsymbol{\ell}_j = 1$, $\mathbf{r}_i = 1$), and hence conditions (8.7) become

$$\hat{\mathbf{H}}(\mu_j) = \mathbf{H}(\mu_j),\ j=1,\ldots,q, \quad \hat{\mathbf{H}}(\lambda_i) = \mathbf{H}(\lambda_i), i=1,\ldots,k, \tag{8.9}$$

while conditions (8.8) become

$$\hat{\mathbf{H}}(\mu_j) = \mathbf{v}_j,\ j=1,\ldots,q, \quad \hat{\mathbf{H}}(\lambda_i) = \mathbf{w}_i,\ i=1,\ldots,k. \tag{8.10}$$

In the following we will consider exclusively *interpolatory model reduction methods* for systems described either by (i) descriptor realizations or (ii) data, measured or computed via DNS. Roughly speaking, we will seek reduced models whose transfer function matches that of the original system at selected frequencies.

Remark 8.2. System identification from frequency response measurements is common in many engineering applications, for instance in electronic design, where S-(scattering), Y-(admittance), or Z-(impedance) *parameters* of chips, packages, or boards are considered, or in civil engineering, where frequency response functions (FRFs), of mechanical structures are given. For instance, given a Z-parameter representation $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$, $m=p$, the associated *S-parameter representation* is

$$\bar{\mathbf{Y}}(s) = \mathbf{S}(s)\bar{\mathbf{U}}(s) = [\mathbf{H}(s)+\mathbf{I}][\mathbf{H}(s)-\mathbf{I}]^{-1}\bar{\mathbf{U}}(s),$$

[27]DNS stands for direct numerical simulation.

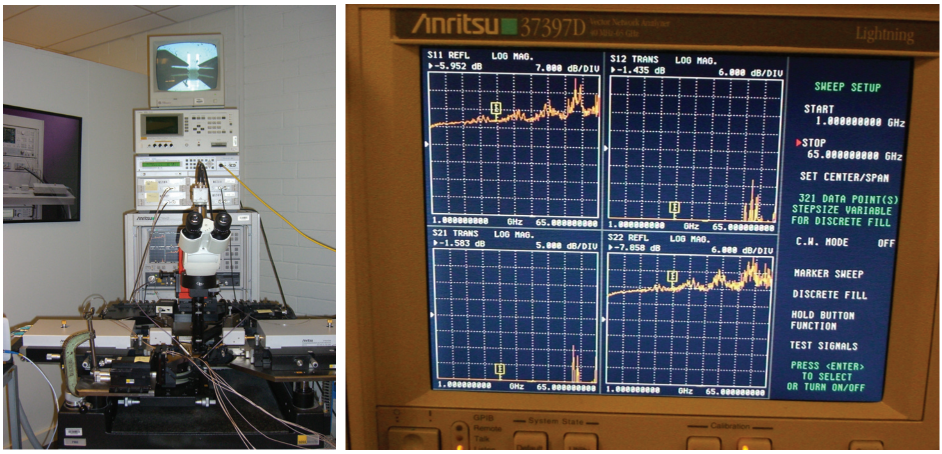

Figure 8.1. *Left pane: VNA. Right pane: screen showing the magnitude of the S-parameters for a two-port (two-input, two-output) device.*

where $\bar{\mathbf{Y}} = \frac{1}{2}(\mathbf{Y}+\mathbf{U})$ and $\bar{\mathbf{U}} = \frac{1}{2}(\mathbf{Y}-\mathbf{U})$ are the *transmitted and reflected waves*, respectively.[28] Important for our purposes is that S-parameters can be measured using *vector network analyzers* (VNAs), as shown in Figure 8.1.

In terms of the problems formulated above, (8.8) applies: given measured S-parameters at given frequencies λ_j, μ_i, we seek to construct a model of low order (complexity) for the underlying (unknown) system.

8.1.4 ▪ Content overview

This chapter starts with the definition of Loewner matrices $\mathbb{L}$, followed by a historical account thereof (Section 8.2.1). Section 8.2.2 discusses the basics in the scalar case. Our starting point is a Lagrange-type interpolation approach that is closely related to the so-called barycentric interpolation formula. Making use of the resulting degrees of freedom leads to the introduction of the Loewner matrix. Next we explore the relationship between rational functions and Loewner matrices, which leads to the fundamental result that given enough data, the rank of any Loewner matrix is equal to the complexity (McMillan degree) of the associated rational function. There follow three ways of constructing rational functions given a Loewner matrix. In Section 8.2.3 the framework is generalized to matrix and tangential interpolation. The new quantity introduced is the shifted Loewner matrix $\mathbb{L}_s$, which, together with $\mathbb{L}$, forms the Loewner pencil. Section 8.2.4 discusses the issues of positive real interpolation in the Loewner framework; the fact that projected systems satisfy interpolation conditions as well as pole and zero placement of interpolants; error expressions; and, last, how to work in real arithmetic. Section 8.2.5 presents several examples illustrating aspects of the Loewner framework for exact data. Section 8.3 introduces the model reduction problem for (approximate or noisy) measured data. The resulting data-driven reduction framework has the unique feature that it provides a trade-off between accuracy of fit and model complexity. Two numerical examples are presented, the second one treating measured data without a known underlying system.

[28]Given a function of time $\mathbf{f}(t)$, its Laplace transform is denoted by $\mathbf{F}(s)$.

What is not treated

Several topics are not covered in this chapter: (a) the multiple-point case (Hermite interpolation) [9, 33]; (b) the recursive framework, in which data are not provided all at once [29, 30]; (c) the generalization of the Loewner framework to linear parametric systems [9, 24, 25]; and (d) generalization to bilinear systems [23]. For the complete Loewner story, we refer the reader to the book [22].

8.2 ▪ The Loewner framework for linear systems

The main ingredient of our approach is the Loewner matrix, which was developed in a series of papers by one of the authors of this chapter (see, e.g., [2, 5, 6]). More recently, important contributions have been made in [33] as well as [9, 24, 25, 29, 30]. In the following, we provide an overview of the Loewner framework.

Definition 8.3 (The Loewner matrix). *Given a row array of pairs of complex numbers $(\mu_j, \mathbf{v}_j)$, $j = 1,\ldots,q$, and a column array of pairs of complex numbers $(\lambda_i, \mathbf{w}_i)$, $i = 1,\ldots,k$, with λ_i, μ_j distinct, the associated Loewner or divided-differences matrix is*

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1-\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1-\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q-\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q-\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}. \tag{8.11}$$

If there is a known underlying function $\boldsymbol{\phi}$, then $\mathbf{w}_i = \boldsymbol{\phi}(\lambda_i)$ and $\mathbf{v}_j = \boldsymbol{\phi}(\mu_j)$.

This matrix was introduced by Karel Löwner (later Charles Loewner) in his seminal paper [32]. For a biography of Loewner, see, e.g., [31].

In the following, we will present the Loewner framework in connection with rational interpolation and consequently in connection with reduced-order modeling of linear dynamical systems given frequency domain data (either measured or computed by DNS). The main property of the Loewner matrix is that its rank encodes information about the minimal admissible complexity of the solutions of the interpolation problem. In the case of measured data, the *numerical rank* of an appropriate Loewner matrix or (as we will see later) of a Loewner pencil needs to be determined.

8.2.1 ▪ Historical remarks

The Loewner matrix was introduced in [32] for the study of *operator convex functions*. The main contribution of [32] is the solution of the operator convexity problem involving the Loewner matrix $\mathbb{L}$ as the main tool. In the process of proving this result, Loewner established the connection of $\mathbb{L}$ with rational interpolation, also known as *Cauchy interpolation*. Since then, the connection of $\mathbb{L}$ with operator monotonicity has been studied extensively; we refer to papers by R. Bhatia and co-authors for details [18, 19]; see also [20]. The connection of $\mathbb{L}$ with rational interpolation (the Cauchy problem) was subsequently pursued by Belevitch [12]. The author rederives the result that, if we attach a large enough Loewner matrix to a given rational function, its rank is equal to the McMillan degree of the given rational function; as remarked by the same author, however, the opposite does not always hold true. This open problem was taken up later by Antoulas and Anderson, and their paper [5] provides the solution. All the above contributions construct interpolants based on computing determinants of submatrices of $\mathbb{L}$. An advance in this respect is constructing interpolants in state-space form in [2].

A breakthrough in the Loewner framework came with the publication of [33]. This paper introduces an additional quantity, the *shifted Loewner matrix*, denoted by $\mathbb{L}_s$. The advantage of this new component comes from the fact that the Loewner pencil $(\mathbb{L}_s, \mathbb{L})$ consisting of the Loewner and the shifted Loewner matrices constitutes a high-order realization of the underlying $(\mathbf{A}, \mathbf{E})$ pencil.

We would also like to refer the reader to the works by Meinguet [35] and Schneider and Werner [36] for a classical view of interpolation. Finally, the work of Amstutz [1] should be mentioned; an approach to rational interpolation based on the Loewner matrix is proposed therein. However, the author was apparently not aware of Loewner's work. More recently, Loewner matrices have also been studied by Fiedler; see, e.g., [21].

8.2.2 ▪ Scalar rational interpolation and the Loewner matrix

Consider the array of pairs of points

$$P = \{(x_i, y_i) : i = 1, \ldots, N,\ x_i \neq x_j,\ i \neq j\}. \tag{8.12}$$

We are looking for rational functions

$$\boldsymbol{\phi}(x) = \frac{\mathbf{n}(x)}{\mathbf{d}(x)}, \tag{8.13}$$

where the numerator and denominator polynomials $\mathbf{n}, \mathbf{d}$ have no common factors, that *interpolate* the points of the array P, i.e.,

$$\boldsymbol{\phi}(x_i) = y_i,\ i = 1, \ldots, N. \tag{8.14}$$

For what follows we will make use of this definition.

Definition 8.4. *The order or complexity of a scalar rational function $\boldsymbol{\phi}(s)$ is*

$$\deg \boldsymbol{\phi} = \max\{\deg \mathbf{n}, \deg \mathbf{d}\}.$$

This is sometimes referred to as the McMillan degree of $\boldsymbol{\phi}$.

Minimal degree solutions of the interpolation problem are of particular interest.

Remark 8.5. In array (8.12), the points x_i have been assumed *distinct*. In terms of the interpolation problem, this means that only the value of the underlying rational function is prescribed at each x_i. For the sake of presenting the main ideas as clearly as possible, in this section, only the scalar, distinct-point interpolation problem will be discussed.

A rational Lagrange-type formula

The idea behind the present approach to rational interpolation is to use a formula for rational interpolants that is similar to the one defining the Lagrange polynomial. First we partition the array P into two disjoint subarrays:

$$P_c = \{(\lambda_i, \mathbf{w}_i) : i = 1, \ldots, k\}, \quad P_r = \{(\mu_j, \mathbf{v}_j) : j = 1, \ldots, q\},$$

where, for simplicity of notation, the points have been redefined as

$$\left.\begin{array}{lll} \lambda_i = x_i, & \mathbf{w}_i = y_i, & i = 1,\dots,k \\ \mu_j = x_{k+j}, & \mathbf{v}_j = y_{k+j}, & j = 1,\dots,q \end{array}\right\} \quad \text{and} \quad k+q = N.$$

Thus, the first k pairs of points are denoted by λ_i and $\mathbf{w}_i$ and the rest by μ_j and $\mathbf{v}_j$. Next we define a *Lagrange basis* for polynomials of degree at most $k-1$: given $\lambda_i \in \mathbb{C}$, $i = 1,\dots,q$: $\lambda_i \neq \lambda_j$, $i \neq j$,

$$\mathbf{q}_i(s) = \prod_{\substack{i'=1 \\ i' \neq i}}^{i'=k} (s - \lambda_{i'}), \; i = 1,\dots,k.$$

For constants α_i, $\mathbf{w}_i$, $i = 1,\dots,k$, consider $\boldsymbol{\phi}(s)$ defined by

$$\sum_{i=1}^{k} \alpha_i \frac{\boldsymbol{\phi}(s) - \mathbf{w}_i}{s - \lambda_i} = 0, \;\; \alpha_i \neq 0. \tag{8.15}$$

Solving for $\boldsymbol{\phi}$, we obtain

$$\boldsymbol{\phi}(s) = \frac{\sum_{i=1}^{k} \frac{\alpha_i \mathbf{w}_i}{s-\lambda_i}}{\sum_{i=1}^{k} \frac{\alpha_i}{s-\lambda_i}} = \frac{\sum_{i=1}^{k} \alpha_i \mathbf{w}_i \, \mathbf{q}_i(s)}{\sum_{i=1}^{k} \alpha_i \, \mathbf{q}_i(s)}, \;\; \alpha_i \neq 0. \tag{8.16}$$

It follows that $\boldsymbol{\phi}(\lambda_i) = \mathbf{w}_i$. This is the *barycentric (rational) Lagrange interpolation* formula. The reference to Lagrange follows from the fact that, if we choose $\alpha_i = \frac{1}{\mathbf{q}_i(\lambda_i)}$, and since $\sum_{i=1}^{k} \frac{\mathbf{q}_i(s)}{\mathbf{q}_i(\lambda_i)} = 1$, we obtain

$$\boldsymbol{\phi}(s) = \boldsymbol{\phi}_{\text{lag}}(s) = \sum_{i=1}^{k} \mathbf{w}_i \frac{\mathbf{q}_i(s)}{\mathbf{q}_i(\lambda_i)};$$

in other words, the rational function $\boldsymbol{\phi}(s)$ becomes the *Lagrange polynomial* $\boldsymbol{\phi}_{\text{lag}}(s)$ interpolating the array P_c. For more information on barycentric rational interpolation, we refer to [16, 17] and references therein.

The free parameters α_i can be determined so that the additional constraints contained in array P_r are satisfied:

$$\boldsymbol{\phi}(\mu_j) = \mathbf{v}_j, \; j = 1,\dots,q.$$

After inserting these conditions into (8.15), the following equation results:

$$\mathbb{L}\mathbf{c} = 0, \tag{8.17}$$

where $\mathbb{L}$ is the Loewner matrix defined by (8.11) and the vector $\mathbf{c}$ contains the unknowns α_i:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}, \;\; \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k. \tag{8.18}$$

Here, the *Loewner matrix* is constructed by means of the *row array* $(\mu_j, \mathbf{v}_j)$, $j = 1,\ldots,q$, and the *column array* $(\lambda_i, \mathbf{w}_i)$, $i = 1,\ldots,k$.

Remark 8.6 (Hankel and Loewner matrices). If the value of successive derivatives at the same points is prescribed, we are dealing with the so-called *Hermite* or interpolation problem with *multiplicities*.

As shown in [5], the (generalized) Loewner matrix associated with an array P as in (8.12), but consisting of *one point with multiplicity* N, has *Hankel* structure. This matrix is actually the same as the Hankel matrix of the corresponding realization problem, hinting at the fact that the Loewner matrix is the right tool to generalize realization theory to rational interpolation.

The theory, part of which is presented in the following, has been worked out for the multiple-point as well as for the (more general) matrix and tangential interpolation problems; for details, the reader is referred to [2, 5, 30, 33].

From rational functions to Loewner matrices

Consider a rational function together with some samples. The key result in connection with the Loewner matrix is the following.

Lemma 8.7. Main property. *Given the rational function ϕ and an array of points P, where $y_i = \phi(x_i)$ and x_i is not a pole of ϕ, let $\mathbb{L}$ be a $q \times k$ Loewner matrix for some partitioning P_c, P_r of P. Then,*

$$q, k \geq \deg \phi \quad \Rightarrow \quad \operatorname{rank} \mathbb{L} = \deg \phi.$$

It follows that every square sub-Loewner matrix of size $\deg \phi$ *is nonsingular.*

This is a pivotal result in our approach. The proof presented next is based on the following known result.

Proposition 8.8. *Let* $(\mathbf{E}, \mathbf{A}, \mathbf{B})$ *be a controllable descriptor triple and λ_i, $i = 1,\ldots,r$, be distinct scalars that are not eigenvalues of the pencil of $n \times n$ matrices* $(\mathbf{A}, \mathbf{E})$. *It follows that*

$$\operatorname{rank} \left[(\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \quad \cdots \quad (\lambda_r \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \right] = n$$

provided that $r \geq n$.

For a proof of a similar version of this result, see [2]. Based on this proposition, we can now provide a state-space proof of Lemma 8.7.

Proof (Lemma 8.7). Let $(\mathbf{E}_\delta, \mathbf{A}_\delta, \mathbf{B}_\delta, \mathbf{C}_\delta)$ be a minimal descriptor realization of ϕ of order n (recall notation (8.5)):

$$\phi(s) = \mathbf{C}_\delta (s\mathbf{E}_\delta - \mathbf{A}_\delta)^{-1} \mathbf{B}_\delta.$$

This implies

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j} = -\mathbf{C}_\delta (\mu_i \mathbf{E}_\delta - \mathbf{A}_\delta)^{-1} \mathbf{E}_\delta (\lambda_j \mathbf{E}_\delta - \mathbf{A}_\delta)^{-1} \mathbf{B}_\delta$$

for $i = 1,\ldots,q$, $j = 1,\ldots,k$. Consequently, $\mathbb{L}$ can be factorized as

$$\mathbb{L} = -\mathcal{O}_q \, \mathbf{E}_\delta \, \mathcal{R}_k,$$

where matrices $\mathcal{R}_k$, $\mathcal{O}_q$ are defined in (8.3), (8.4), respectively. In analogy with the realization problem (where the Hankel matrix factors in a product of an observability times a controllability matrix), we will call $\mathcal{O}_q$ the *generalized observability matrix* and $\mathcal{R}_k$ the *generalized controllability matrix* associated with the sets of scalars λ_i and μ_j.

Because of Proposition 8.8, the rank of both $\mathcal{O}_q$ and $\mathcal{R}_k$ is n. This implies that the rank of $\mathbb{L}$ is equal to the rank of $\mathbf{E}_\delta$, which, in turn, is equal to the rank of $\mathbf{E}$ (recall (8.5)). Then, according to [28], the rank of $\mathbf{E}$ is equal to the McMillan degree of $\boldsymbol{\phi}(s)$.

Finally, since every square sub-Loewner matrix is a Loewner matrix for some subset of the row and column arrays, it has the same full-rank property. This completes the proof of the main lemma. □

The above result was derived by Löwner [32]; a different proof can be found in [1] and [7].

From Loewner matrices to interpolating functions

Given interpolation data (8.12), we are now ready to tackle the interpolation problem (8.13), (8.14). In this section, we construct interpolating functions. Three construction methods will be detailed: the first is based on the barycentric formula and constructs the numerator and denominator polynomials of interpolants, while the remaining two methods describe descriptor (generalized state-space) approaches to constructing interpolants.

Polynomial construction of interpolants. Given the array of points P defined by (8.12), the developments in this section follow the references cited earlier as well as [12]. The following definition will be needed.

Definition 8.9. *The rank of the array P is*

$$\textit{rank}\, P = \max_{\mathbb{L}} \{\textit{rank}\, \mathbb{L}\} = n,$$

where the maximum is taken over all possible Loewner matrices that can be formed from P.

A consequence of Lemma 8.7 is that the rank of all Loewner matrices with at least n rows and n columns is equal to n. Assume that $2n < N$. For any Loewner matrix with $\operatorname{rank} \mathbb{L} = n$ and n rows, there exists a column vector $\mathbf{c}$ satisfying

$$\mathbb{L}\mathbf{c} = 0, \ \ \mathbf{c} \in \mathbb{C}^k, \ \ k = N - n. \tag{8.19}$$

In this case, we can attach to $\mathbb{L}$ a rational function denoted by

$$\boldsymbol{\phi}_{\mathbb{L}}(s) = \frac{\mathbf{n}_{\mathbb{L}}(s)}{\mathbf{d}_{\mathbb{L}}(s)}, \tag{8.20}$$

using the barycentric formula (8.16), where $(\mathbf{c})_{j,1} = \alpha_j$, i.e.,

$$\mathbf{n}_{\mathbb{L}}(s) = \sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}, \quad \mathbf{d}_{\mathbb{L}}(s) = \sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}. \tag{8.21}$$

The rational function $\boldsymbol{\phi}_{\mathbb{L}}$ has the following properties.

Lemma 8.10. (a) $\deg \boldsymbol{\phi}_{\mathbb{L}} \leq n < N$.
(b) *There is a unique* $\boldsymbol{\phi}_{\mathbb{L}}$, *attached to all* $\mathbb{L}$ *and* $\mathbf{c}$ *satisfying* (8.19), *as long as* $\operatorname{rank} \mathbb{L} = n$.
(c) *The numerator and denominator polynomials* $\mathbf{n}_{\mathbb{L}}$, $\mathbf{d}_{\mathbb{L}}$ *have* $n - \deg \boldsymbol{\phi}_{\mathbb{L}}$ *common factors of the form* $s - \lambda_i$.
(d) $\boldsymbol{\phi}_{\mathbb{L}}$ *interpolates exactly* $N - n + \deg \boldsymbol{\phi}_{\mathbb{L}}$ *points of the array* P.

As a consequence of Lemma 8.10 and Lemma 8.7, we obtain the following.

Corollary 8.11. *The rational function* $\boldsymbol{\phi}_{\mathbb{L}}$ *interpolates all given points if and only if* $\deg \boldsymbol{\phi}_{\mathbb{L}} = n$ *if and only if all* $n \times n$ *Loewner matrices that can be formed from the data array* P *are nonsingular.*

We are now ready to state the main result in [5].

Theorem 8.12. *Given the array of* N *pairs of points* P, *let* $\operatorname{rank} P = n$.
(a) *If* $2n < N$ *and all square Loewner matrices of size* n *that can be formed from* P *are nonsingular, there is a unique interpolating function of minimal degree denoted by* $\boldsymbol{\phi}^{\min}(s)$ *and* $\deg \boldsymbol{\phi}^{\min} = n$.
(b) *Otherwise,* $\boldsymbol{\phi}^{\min}(s)$ *is not unique and* $\deg \boldsymbol{\phi}^{\min} = N - n$.

Corollary 8.13. *Under the assumptions of Theorem 8.12*(a), *the admissible degrees of interpolants are* n *and any integer larger than or equal to* $N - n$. *Otherwise, if Theorem 8.12*(b) *holds, i.e.,* $2n > N$, *the admissible degrees are all integers greater than or equal to* $N - n$.

The proof of the four results quoted above can be found in [5].

Remark 8.14. (a) If $2n = N$, the only solution $\mathbf{c}$ of (8.19) is $\mathbf{c} = 0$. Hence, $\boldsymbol{\phi}_{\mathbb{L}}$ defined by (8.20) does not exist, and part (b) of Theorem 8.12 applies.
(b) To distinguish between case (a) and case (b) of Theorem 8.12, we only need to check the nonsingularity of $2n + 1$ Loewner matrices. For details, see [5].
(c) Given the array P, let r be an admissible degree. For the polynomial construction, we need to form *any* Loewner matrix with $r + 1$ columns,

$$\mathbb{L}_r \in \mathbb{R}^{(N-r-1)\times(r+1)},$$

and determine a parametrization of all $\mathbf{c}_r$ such that $\mathbb{L}_r \mathbf{c}_r = 0$. A parametrization of all interpolating functions of degree r is then $\boldsymbol{\phi}_{\mathbb{L}_r}(s) = \frac{\mathbf{n}_{\mathbb{L}_r}(s)}{\mathbf{d}_{\mathbb{L}_r}(s)}$, where the numerator and denominator polynomials are defined by (8.21). If $r \geq N - n$, we have to make sure that there are no common factors between the numerator and denominator of $\boldsymbol{\phi}_{\mathbb{L}_r}$; this is the case for almost all $\mathbf{c}_r$. More precisely, the $2r + 1 - N$ (scalar) parameters that parametrize all $\mathbf{c}_r$ have to avoid the hypersurfaces defined by the equations

$$\mathbf{d}_{\mathbb{L}_r}(\lambda_i) = 0, \ i = 1, \ldots, r + 1.$$

Since we can always make sure that $\mathbf{c}_r$ depends affinely on these parameters, we are actually dealing with hyperplanes. For details and examples, see [5, 33].

Descriptor realization of interpolants based on c. We wish to derive a descriptor realization of the rational interpolants obtained above by means of the solution $\mathbf{c}$ of $\mathbb{L}\mathbf{c} = 0$. For details we refer to [9].

Assume that $k = q + 1$, and recall the rational function $\boldsymbol{\phi}(s)$ expressed in barycentric form (8.16):

$$\boldsymbol{\phi}(s) = \frac{\sum_{i=0}^{q} \beta_i \, \mathbf{q}_i(s)}{\sum_{i=0}^{q} \alpha_i \, \mathbf{q}_i(s)}, \quad \beta_i = \alpha_i \mathbf{w}_i.$$

We define

$$\mathbf{J}_{\text{lag}}(\xi; q) = \begin{bmatrix} \xi - x_1 & x_2 - \xi & & & \\ \xi - x_1 & 0 & x_3 - \xi & & \\ \vdots & & \ddots & \ddots & \\ \xi - x_1 & & & 0 & x_{q+1} - \xi \end{bmatrix} \in \mathbb{C}^{q \times (q+1)},$$

where, for simplicity, we assume that $x_i \neq x_j$, $i \neq j$, and

$$\mathbf{a} = [\alpha_0, \ \alpha_1, \ \ldots, \ \alpha_q], \ \ \mathbf{b} = [\beta_0, \ \beta_1, \ \ldots, \ \beta_q] \in \mathbb{R}^{1 \times (q+1)}.$$

Then, we have the following.

Lemma 8.15. *The following triple constitutes a descriptor realization of $\boldsymbol{\phi}$:*

$$\mathbf{C} = \mathbf{b}, \ \ \boldsymbol{\Phi}(s) = \begin{bmatrix} \mathbf{J}_{\text{lag}}(s; q) \\ \mathbf{a} \end{bmatrix}, \ \ \mathbf{B} = \mathbf{e}_{q+1}, \tag{8.22}$$

i.e., $\boldsymbol{\phi}(s) = \mathbf{C}\boldsymbol{\Phi}(s)^{-1}\mathbf{B}$, where $x_i = \lambda_i$, $i = 1, \ldots, q+1$. The dimension of this realization is $q + 1$ and it can represent arbitrary rational functions, including polynomials. Also, it is R-controllable and R-observable, that is, $[\boldsymbol{\Phi}(s), \ \mathbf{B}]$ and $[\boldsymbol{\Phi}^T(s), \ \mathbf{C}^T]$ have full rank for all $s \in \mathbb{C}$, provided that the numerator and denominator of $\boldsymbol{\phi}$ have no common factors.

Descriptor realization of interpolants without the use of c. Given the interpolation data left (row) array $(\mu_i, \mathbf{v}_i)$, $i = 1, \ldots, q$, and right (column) array $(\lambda_j, \mathbf{w}_j)$, $j = 1, \ldots, q+1$, let $\mathbb{L} \in \mathbb{C}^{q \times (q+1)}$ be the associated Loewner matrix. We will assume that all $q \times q$ Loewner matrices constructed from these data are nonsingular and hence there exists a unique interpolant of degree q. In this section we will show how to construct this interpolant *without* the explicit use of $\mathbf{c}$ satisfying $\mathbb{L}\mathbf{c} = 0$. Toward this goal, we partition the Loewner matrix as

$$\mathbb{L} = [\hat{\mathbb{L}} \quad \ell], \ \ \text{where} \ \ \mathbb{L} \in \mathbb{C}^{q \times (q+1)}, \ \hat{\mathbb{L}} \in \mathbb{C}^{q \times q}, \ \ell \in \mathbb{C}^{q}. \tag{8.23}$$

Define the matrix containing the factors making up the Lagrange basis as

$$\boldsymbol{\Pi}(s) = \begin{bmatrix} \mathbf{p}_1(s) & & \\ & \ddots & \\ & & \mathbf{p}_q(s) \\ -\mathbf{p}_{q+1}(s) & \cdots & -\mathbf{p}_{q+1}(s) \end{bmatrix} \in \mathbb{C}^{(q+1) \times q}, \quad \mathbf{p}_i(s) = s - \lambda_i.$$

It turns out that realization (8.24) below can be interpreted in terms of *one-sided shifted Loewner matrices*. We stress the *one-sided* aspect as, later on, we will encounter *two-sided* shifted Loewner matrices (see (8.30)). Explicitly, given $\mathbb{L} \in \mathbb{C}^{q\times k}$ defined by (8.11), the *one-sided shifted Loewner matrix* is

$$\mathbb{L}_\Lambda = \begin{bmatrix} \lambda_1 \frac{\mathbf{v}_1-\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \lambda_k \frac{\mathbf{v}_1-\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \lambda_1 \frac{\mathbf{v}_q-\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \lambda_k \frac{\mathbf{v}_q-\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix} = \mathbb{L}\boldsymbol{\Lambda} \in \mathbb{C}^{q\times k},$$

where $\boldsymbol{\Lambda} = \mathrm{diag}[\lambda_1, \ldots, \lambda_k]$. In this framework a realization is as follows.

Lemma 8.16. *For $k = q+1$, define the $(q+1)\times q$ matrix $\mathbb{J} = \begin{bmatrix} \mathbb{I} \\ -\mathbf{1} \end{bmatrix}$, where $\mathbb{I}$ is the identity of size q and $\mathbf{1} = ones(1,q)$. Partition $\mathbb{L}$ as in* (8.23). *With $\mathbf{W} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_q \ \ \mathbf{w}_{q+1}]$ and $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_q]^T$, the associated descriptor realization is*

$$\mathbf{H}(s) = \mathbf{w}_{q+1} - \mathbf{p}_{q+1}(s)(\mathbf{W}\mathbb{J})[(s\,\mathbb{L} - \mathbb{L}_\Lambda)\mathbb{J}]^{-1}\boldsymbol{\ell}. \tag{8.24}$$

If the interpolant is strictly proper rational (i.e., the sum of the entries of the vector $\mathbf{c}$ *satisfying $[\hat{\mathbb{L}} \ \ \boldsymbol{\ell}]\mathbf{c} = \mathbf{0}$ is not equal to zero), the interpolant can be expressed as*

$$\mathbf{H}(s) = (\mathbf{W}\mathbb{J})[(s\,\mathbb{L} - \mathbb{L}_\Lambda)\mathbb{J}]^{-1}\mathbf{V}. \tag{8.25}$$

Proof. First, recall that, given $\mathbb{L}$ and $\mathbf{c}$ such that $\mathbb{L}\mathbf{c} = \mathbf{0}$, according to the barycentric formula, the corresponding interpolant can be written as in (8.21), namely

$$\boldsymbol{\phi}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)} = \frac{\sum_{i=1}^{q+1} \frac{\alpha_i \mathbf{w}_i}{s-\lambda_i}}{\sum_{i=1}^{q+1} \frac{\alpha_i}{s-\lambda_i}}.$$

Assuming, without loss of generality, that $\alpha_{q+1} = 1$, we can write $[\hat{\mathbb{L}} \quad \boldsymbol{\ell}][\alpha_1, \ldots, \alpha_q, 1]^T = \mathbf{0}$. Hence, $[\alpha_1, \ldots, \alpha_q]^T = -\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}$. Thus, the numerator polynomial of $\boldsymbol{\phi}(s)$ times $(s-\lambda_{q+1})$ can be written as:

$$(s-\lambda_{q+1})\mathbf{n}(s) = \mathbf{w}_{q+1} - (s-\lambda_{q+1})\left[\mathbf{w}_1 \ \cdots \ \mathbf{w}_q\right]\left(\mathrm{diag}\left[s-\lambda_1, \ldots, s-\lambda_q\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell},$$

while the denominator polynomial times the same factor can be written as

$$(s-\lambda_{q+1})\mathbf{d}(s) = 1 - (s-\lambda_{q+1})[1 \ \cdots \ 1]\left(\mathrm{diag}\left[s-\lambda_1, \ldots, s-\lambda_q\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}.$$

Consequently, the quotient is

$$\boldsymbol{\phi}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$$

$$= \mathbf{w}_{q+1} - \frac{(s-\lambda_{q+1})\left[\mathbf{w}_1 - \mathbf{w}_{q+1} \ \cdots \ \mathbf{w}_q - \mathbf{w}_{q+1}\right]\left(\mathrm{diag}\left[s-\lambda_1, \ldots, s-\lambda_q\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}}{1 - (s-\lambda_{q+1})[1 \ \cdots \ 1]\left(\mathrm{diag}\left[s-\lambda_1, \ldots, s-\lambda_q\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}}. \tag{$*$}$$

The *Sherman–Morrison–Woodbury formula*[29] implies

$$\frac{\left(\mathrm{diag}\left[(s-\lambda_1),\cdots,(s-\lambda_q)\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}}{1-(s-\lambda_{q+1})[1\ \cdots\ 1]\left(\mathrm{diag}\left[(s-\lambda_1),\ldots,(s-\lambda_q)\right]\right)^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}}$$

$$=\left[\mathrm{diag}\left[(s-\lambda_1),\ldots,(s-\lambda_q)\right]-(s-\lambda_{q+1})\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}[1\cdots 1]\right]^{-1}\hat{\mathbb{L}}^{-1}\boldsymbol{\ell}=[(s\mathbb{L}-\mathbb{L}_\Lambda)\mathbb{J}]^{-1}\boldsymbol{\ell}.$$

Combining this with expression $(*)$ yields the desired (8.24). The second expression for $\mathbf{H}(s)$ is proved similarly. □

Remark 8.17. The method of constructing interpolants by means of the barycentric formula (8.21) and by means of the descriptor realization in Lemma 8.15 can be used to obtain any interpolant of admissible degree. The method presented in Lemma 8.16, howwever, applies only to minimal interpolants.

8.2.3 ▪ Matrix rational interpolation and the Loewner pencil

In this section, we will formulate the results outlined above for the more general *tangential interpolation* problem defined in (8.8). We are given, respectively, the *right* or *column data* and the *left* or *row data*:

$$(\lambda_i;\mathbf{r}_i,\mathbf{w}_i),\ i=1,\ldots,k,\quad (\mu_j;\boldsymbol{\ell}_j^T,\mathbf{v}_j^T),\ j=1,\ldots,q.$$

It is assumed for simplicity that all points, i.e., λ_i and μ_j, are distinct (for the general case, see [33]). The *right data* are organized as

$$\left.\begin{array}{l}\boldsymbol{\Lambda}=\mathrm{diag}\,[\lambda_1,\ \ldots,\ \lambda_k]\in\mathbb{C}^{k\times k}\\ \mathbf{R}=[\mathbf{r}_1\ \ \mathbf{r}_2\ \ \cdots\ \ \mathbf{r}_k]\in\mathbb{C}^{m\times k}\\ \mathbf{W}=[\mathbf{w}_1\ \ \mathbf{w}_2\ \ \cdots\ \ \mathbf{w}_k]\in\mathbb{C}^{p\times k}\end{array}\right\},\tag{8.26}$$

and the *left data* are organized as

$$\left.\begin{array}{l}\mathbf{M}=\mathrm{diag}\left[\mu_1,\ \ldots,\ \mu_q\right]\in\mathbb{C}^{q\times q}\\ \mathbf{L}^T=[\ \boldsymbol{\ell}_1\ \ \cdots\ \ \boldsymbol{\ell}_q\ ]\in\mathbb{C}^{p\times q}\\ \mathbf{V}^T=[\ \mathbf{v}_1\ \ \cdots\ \ \mathbf{v}_q\ ]\in\mathbb{C}^{m\times q}\end{array}\right\}.\tag{8.27}$$

The associated *Loewner* and *shifted Loewner* matrices, referred to as the *Loewner pencil*, are constructed next. The *Loewner matrix* for tangential data (first introduced in [33]) is

$$\mathbb{L}=\begin{bmatrix}\frac{\mathbf{v}_1^T\mathbf{r}_1-\boldsymbol{\ell}_1^T\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1^T\mathbf{r}_k-\boldsymbol{\ell}_1^T\mathbf{w}_k}{\mu_1-\lambda_k}\\ \vdots & \ddots & \vdots\\ \frac{\mathbf{v}_q^T\mathbf{r}_1-\boldsymbol{\ell}_q^T\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q^T\mathbf{r}_k-\boldsymbol{\ell}_q^T\mathbf{w}_k}{\mu_q-\lambda_k}\end{bmatrix}\in\mathbb{C}^{q\times k}.\tag{8.28}$$

[29]Namely $[\mathbf{A}-\mathbf{u}\mathbf{v}^T]^{-1}=\mathbf{A}^{-1}+\frac{1}{1-\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}\,\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\Rightarrow[\mathbf{A}-\mathbf{u}\mathbf{v}^T]^{-1}\mathbf{u}=[\frac{1}{1-\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}]\,\mathbf{A}^{-1}\mathbf{u}$.

Notice that the quantities $\mathbf{v}_i^T\mathbf{r}_j$ and $\ell_i^T\mathbf{w}_j$ are (complex) scalars. It is readily checked that $\mathbb{L}$ satisfies the Sylvester equation

$$\mathbf{M}\mathbb{L}-\mathbb{L}\mathbf{\Lambda}=\mathbf{V}\mathbf{R}-\mathbf{L}\mathbf{W}. \tag{8.29}$$

The *shifted Loewner matrix*, first introduced in [33] for both scalar and tangential (and consequently also matrix) data, is defined as

$$\mathbb{L}_s=\begin{bmatrix} \frac{\mu_1\mathbf{v}_1^T\mathbf{r}_1-\ell_1^T\mathbf{w}_1\lambda_1}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1^T\mathbf{r}_k-\ell_1^T\mathbf{w}_k\lambda_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q^T\mathbf{r}_1-\ell_q^T\mathbf{w}_1\lambda_1}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q^T\mathbf{r}_k-\ell_q^T\mathbf{w}_k\lambda_k}{\mu_q-\lambda_k} \end{bmatrix}\in\mathbb{C}^{q\times k}, \tag{8.30}$$

and it is straightforward to check that it satisfies the Sylvester equation

$$\mathbf{M}\mathbb{L}_s-\mathbb{L}_s\mathbf{\Lambda}=\mathbf{M}\mathbf{V}\mathbf{R}-\mathbf{L}\mathbf{W}\mathbf{\Lambda}. \tag{8.31}$$

If the data (8.26), (8.27) are sampled from a system with transfer function $\mathbf{H}(s)=\mathbf{C}_\delta(s\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{B}_\delta$[30] of size $p\times m$ with $\mathbf{A}_\delta,\mathbf{E}_\delta\in\mathbb{R}^{n\times n}$, we define

$$\left.\begin{aligned} \mathscr{O}_q&=\begin{bmatrix} \ell_1^T\mathbf{C}_\delta(\mu_1\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1} \\ \vdots \\ \ell_q^T\mathbf{C}_\delta(\mu_q\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1} \end{bmatrix} \\ \mathscr{R}_k&=\left[(\lambda_1\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{B}_\delta\mathbf{r}_1,\cdots,(\lambda_k\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{B}_\delta\mathbf{r}_k\right] \end{aligned}\right\} \tag{8.32}$$

of size $q\times n$, $n\times k$, respectively. Similarly to their scalar counterparts, these are called the *generalized tangential observability* and *generalized tangential controllability* matrices. It follows that the Loewner pencil constructed from tangential data has a system-theoretic interpretation in terms of the tangential controllability and observability matrices. Given left interpolation data $\mathbf{v}_j^T=\ell_j^T\mathbf{H}(\mu_j)$ and right interpolation data $\mathbf{w}_i=\mathbf{H}(\lambda_i)\mathbf{r}_i$,

$$\begin{aligned} (\mathbb{L})_{j,i}&=\frac{\mathbf{v}_j^T\mathbf{r}_i-\ell_j^T\mathbf{w}_i}{\mu_j-\lambda_i}=\frac{\ell_j^T\mathbf{H}(\mu_j)\mathbf{r}_i-\ell_j^T\mathbf{H}(\lambda_i)\mathbf{r}_i}{\mu_j-\lambda_i} \\ &=-\ell_j^T\mathbf{C}_\delta(\mu_j\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{E}_\delta(\lambda_i\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{B}_\delta\mathbf{r}_i \quad \text{and} \\ (\mathbb{L}_s)_{j,i}&=\frac{\mu_j\mathbf{v}_i^T\mathbf{r}_i-\lambda_i\ell_j^T\mathbf{w}_i}{\mu_j-\lambda_i}=\frac{\mu_j\ell_j^T\mathbf{H}(\mu_j)\mathbf{r}_i-\lambda_i\ell_j^T\mathbf{H}(\lambda_i)\mathbf{r}_i}{\mu_j-\lambda_i} \\ &=-\ell_j^T\mathbf{C}_\delta(\mu_j\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{A}_\delta(\lambda_i\mathbf{E}_\delta-\mathbf{A}_\delta)^{-1}\mathbf{B}_\delta\mathbf{r}_i. \end{aligned}$$

Thus, with notation (8.32), we obtain

$$\mathbb{L}=-\mathscr{O}_q\,\mathbf{E}_\delta\,\mathscr{R}_k \quad \text{and} \quad \mathbb{L}_s=-\mathscr{O}_q\,\mathbf{A}_\delta\,\mathscr{R}_k\,.$$

[30]Following Remark 8.1, we assume that a possible $\mathbf{D}$-term is incorporated in the remaining matrices, as in (8.5).

Lemma 8.18. *Given (tangential) samples of a rational function defined in terms of a minimal descriptor realization* $(\mathbf{E}_\delta, \mathbf{A}_\delta, \mathbf{B}_\delta, \mathbf{C}_\delta)$ *as in Remark* 8.1, *construct the associated Loewner and shifted Loewner matrices* $\mathbb{L}$, $\mathbb{L}_s$. *Assuming that we have enough samples, and that the left and right tangential directions* $\boldsymbol{\ell}_j$, $\mathbf{r}_i$ *are chosen so that* $\mathcal{O}_q$ *and* $\mathcal{R}_k$ *have full rank, we have the following:*

(a) $\mathit{rank}\,\mathbb{L} = \mathit{rank}\,\mathbf{E}_\delta = \mathit{rank}\,\mathbf{E} = \begin{cases} \text{McMillan degree of the} \\ \text{underlying rational function.} \end{cases}$

(b) $\mathit{rank}\,\mathbb{L}_s = \mathit{rank}\,\mathbf{A}_\delta = \mathit{rank}\,\mathbf{A} + \mathit{rank}\,\mathbf{D}$.

Remark 8.19. (a) In the SISO case, i.e., $m = p = 1$, in other words, the case where the associated transfer function is rational and scalar, the Loewner pencil contains $\mathbb{L}$ defined by (8.11) and

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1 - \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1 - \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q - \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q - \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix}. \tag{8.33}$$

(b) The issue of choice of tangential directions mentioned in Lemma 8.18 is illustrated in Example 8.36 below.
(c) Loewner matrices are sometimes referred to as *generalized Cauchy matrices*. See, e.g., [26, p. 343], and references therein. Such matrices possess what is known as *displacement structure*. In our case, this is a direct consequence of the fact that $\mathbb{L}$ and $\mathbb{L}_s$ satisfy the Sylvester equations (8.29), (8.31), where the right-hand side has low rank ($m + p$ is general).

We are now ready to state the main result concerning the construction of interpolants using the Loewner pencil.

Theorem 8.20 (Minimal amount of data). *Assume that* $k = q$ *and let* $(\mathbb{L}_s, \mathbb{L})$ *be a regular pencil with no* μ_i *or* λ_j *being an eigenvalue.*
(a) *The quadruple*

$$\mathbf{E}_\delta = -\mathbb{L}, \;\; \mathbf{A}_\delta = -\mathbb{L}_s, \;\; \mathbf{B}_\delta = \mathbf{V}, \;\; \mathbf{C}_\delta = \mathbf{W} \tag{8.34}$$

is a minimal descriptor realization of an interpolant of the data, i.e.,

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbf{V} \tag{8.35}$$

is a rational interpolant of the data.
(b) *If the solution is not unique, all solutions of the same McMillan degree are parametrized in terms of*

$$\mathbf{E} = -\mathbb{L}, \; \mathbf{A} = -(\mathbb{L}_s + \mathbf{LKR}), \; \mathbf{B} = \mathbf{V} - \mathbf{LK}, \; \mathbf{C} = \mathbf{W} - \mathbf{KR}, \; \mathbf{D} = \mathbf{K}, \tag{8.36}$$

where the parameter $\mathbf{K} \in \mathbb{C}^{p \times m}$.

Proof. [33] (a) Multiplying equation (8.29) by s and subtracting it from equation (8.31), we get

$$\mathbf{M}(\mathbb{L}_s - s\mathbb{L}) - (\mathbb{L}_s - s\mathbb{L})\mathbf{\Lambda} = (\mathbf{M} - s\mathbf{I})\mathbf{VR} - \mathbf{LW}(\mathbf{\Lambda} - s\mathbf{I}).$$

Multiplying this equation by $\mathbf{e}_i$ on the right and setting $s=\lambda_i$, we obtain

$$(\mathbf{M}-\lambda_i\mathbf{I})(\mathbb{L}_s-\lambda_i\mathbb{L})\mathbf{e}_i=(\mathbf{M}-\lambda_i\mathbf{I})\mathbf{V}\mathbf{r}_i \Rightarrow$$
$$(\mathbb{L}_s-\lambda_i\mathbb{L})\mathbf{e}_i=\mathbf{V}\mathbf{r}_i \Rightarrow \mathbf{W}\mathbf{e}_i=\mathbf{W}(\mathbb{L}_s-\lambda_i\mathbb{L})^{-1}\mathbf{V}\mathbf{r}_i.$$

Therefore $\mathbf{w}_i=\mathbf{H}(\lambda_i)\mathbf{r}_i$. This proves right tangential interpolation. To prove the left tangential interpolation property, we multiply the above equation by $\mathbf{e}_j^T$ on the left and set $s=\mu_j$:

$$\mathbf{e}_j^T(\mathbb{L}_s-\mu_j\mathbb{L})(\mathbf{\Lambda}-\mu_j\mathbf{I})=\mathbf{e}_j^T\mathbf{L}\mathbf{W}(\mathbf{\Lambda}-\mu_j\mathbf{I}) \Rightarrow$$
$$\mathbf{e}_j^T(\mathbb{L}_s-\mu_j\mathbb{L})=\boldsymbol{\ell}_j\mathbf{W} \Rightarrow \mathbf{e}_j^T\mathbf{V}=\boldsymbol{\ell}_j^T\mathbf{W}(\mathbb{L}_s-\mu_j\mathbb{L})^{-1}\mathbf{V}.$$

Therefore, $\mathbf{v}_j^T=\boldsymbol{\ell}_j^T\mathbf{H}(\mu_j)$, which proves the left property.

(b) With $\mathbf{K}\in\mathbb{C}^{p\times m}$, equations (8.29), (8.31) can be rewritten as

$$\mathbf{M}\mathbb{L}-\mathbb{L}\mathbf{\Lambda}=(\mathbf{V}-\mathbf{L}\mathbf{K})\mathbf{R}-\mathbf{L}(\mathbf{W}-\mathbf{K}\mathbf{R}),$$
$$\mathbf{M}(\mathbb{L}_s+\mathbf{L}\mathbf{K}\mathbf{R})-(\mathbb{L}_s+\mathbf{L}\mathbf{K}\mathbf{R})\mathbf{\Lambda}=\mathbf{M}(\mathbf{V}-\mathbf{L}\mathbf{K})\mathbf{R}-\mathbf{L}(\mathbf{W}-\mathbf{K}\mathbf{R})\mathbf{\Lambda}.$$

Repeating the procedure with the new quantities

$$\bar{\mathbb{L}}_s=\mathbb{L}_s+\mathbf{L}\mathbf{K}\mathbf{R},\ \ \bar{\mathbf{V}}=\mathbf{V}-\mathbf{L}\mathbf{K},\ \ \bar{\mathbf{W}}=\mathbf{W}-\mathbf{K}\mathbf{R}$$

(the Loewner matrix remains unchanged), the desired result follows. □

The case of redundant data

We will now consider the case where more data than absolutely necessary are provided, which is realistic for applications. As shown in [33], in this case, the problem has a solution provided that

$$\operatorname{rank}\,[\xi\,\mathbb{L}-\mathbb{L}_s]=\operatorname{rank}\,[\mathbb{L},\ \mathbb{L}_s]=\operatorname{rank}\begin{bmatrix}\mathbb{L}\\ \mathbb{L}_s\end{bmatrix}=r \tag{8.37}$$

for all $\xi\in\{\lambda_j\}\cup\{\mu_i\}$. Consider, then, the short SVDs

$$[\mathbb{L},\ \mathbb{L}_s]=\mathbf{Y}\Sigma_\ell\tilde{\mathbf{X}}^*,\ \ \begin{bmatrix}\mathbb{L}\\ \mathbb{L}_s\end{bmatrix}=\tilde{\mathbf{Y}}\Sigma_r\mathbf{X}^*, \tag{8.38}$$

where $\Sigma_\ell,\Sigma_r\in\mathbb{R}^{r\times r}$, $\mathbf{Y}\in\mathbb{C}^{q\times r}$, $\mathbf{X}\in\mathbb{C}^{k\times r}$.

Theorem 8.21 ([33]). *The quadruple* $(\mathbf{E}_\delta,\mathbf{A}_\delta,\mathbf{B}_\delta,\mathbf{C}_\delta)$ *of size* $r\times r$, $r\times r$, $r\times m$, $p\times r$, *respectively, given by*

$$\mathbf{E}_\delta=-\mathbf{Y}^*\mathbb{L}\mathbf{X},\ \ \mathbf{A}_\delta=-\mathbf{Y}^*\mathbb{L}_s\mathbf{X},\ \ \mathbf{B}_\delta=\mathbf{Y}^*\mathbf{V},\ \ \mathbf{C}_\delta=\mathbf{W}\mathbf{X}, \tag{8.39}$$

is a descriptor realization of an (approximate) interpolant of the data with McMillan degree $\nu=\operatorname{rank}\mathbb{L}$.

Next, in the case of exact data (i.e., the data are available with no measurement noise), we list four properties that shed light on the Loewner framework.

Proposition 8.22 ([33]). *From the above construction, we have*

$$\mathbf{Y}\mathbf{Y}^*\mathbb{L} = \mathbb{L}, \ \ \mathbf{Y}\mathbf{Y}^*\mathbb{L}_s = \mathbb{L}_s, \ \ \mathbf{Y}\mathbf{Y}^*\mathbf{V} = \mathbf{V}, \tag{8.40}$$

$$\mathbb{L}\mathbf{X}\mathbf{X}^* = \mathbb{L}, \ \ \mathbb{L}_s\mathbf{X}\mathbf{X}^* = \mathbb{L}_s, \ \ \mathbf{W}\mathbf{X}\mathbf{X}^* = \mathbf{W}. \tag{8.41}$$

Proposition 8.23. *The original pencil* $(\mathbb{L}_s, \mathbb{L})$ *and the projected pencil* $(\mathbf{A}, \mathbf{E})$ *have the same nontrivial eigenvalues.*

Proof. Let $(\mathbf{z}, \lambda)$ be a right eigenpair of $(\mathbb{L}_s, \mathbb{L})$. Then $\mathbb{L}_s\mathbf{z} = \lambda\mathbb{L}\mathbf{z} \Rightarrow$

$$\mathbb{L}_s\mathbf{X}\mathbf{X}^*\mathbf{z} = \lambda\mathbb{L}\mathbf{X}\mathbf{X}^*\mathbf{z} \ \Rightarrow \ \underbrace{\mathbf{Y}^*\mathbb{L}_s\mathbf{X}}_{\mathbf{A}}\mathbf{X}^*\mathbf{z} = \lambda\underbrace{\mathbf{Y}^*\mathbb{L}\mathbf{X}}_{\mathbf{E}}\mathbf{X}^*\mathbf{z}.$$

Thus, $(\mathbf{X}^*\mathbf{z}, \lambda)$ is an eigenpair of $(\mathbf{A}, \mathbf{E})$. Conversely, if $(\mathbf{z}, \lambda)$ is an eigenpair of $(\mathbf{A}, \mathbf{E})$, then $(\mathbf{X}\mathbf{z}, \lambda)$ is an eigenpair of the original pencil $(\mathbb{L}_s, \mathbb{L})$. We reason similarly for left eigenpairs. □

Furthermore, even though the pencil $(\mathbb{L}_s, \mathbb{L})$ is singular, a kind of generalized interpolation property holds.

Proposition 8.24 (Interpolation property). *Let* $\mathbf{z}_i$ *satisfy* $(\lambda_i\mathbb{L} - \mathbb{L}_s)\mathbf{z}_i = \mathbf{V}\mathbf{r}_i$. *It follows that* $\mathbf{z}_i = \mathbf{e}_i + \mathbf{z}_0$ *and* $\mathbf{W}\mathbf{z}_i = \mathbf{w}_i$, *where* $\mathbf{W}\mathbf{z}_0 = 0$.

The next result shows that the projection may indeed be chosen *arbitrarily*, instead of using $\mathbf{X}, \mathbf{Y}$ defined by (8.38).

Proposition 8.25. *Let* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Psi}$ *be such that* $\mathbf{X}^*\boldsymbol{\Phi}$ *and* $\boldsymbol{\Psi}^*\mathbf{Y}$ *are square and nonsingular. Then,*

$$(\mathbf{Y}^*\mathbb{L}\mathbf{X}, \ \mathbf{Y}^*\mathbb{L}_s\mathbf{X}, \ \mathbf{Y}^*\mathbf{V}, \ \mathbf{W}\mathbf{X}) \quad \text{and} \quad (\boldsymbol{\Phi}^*\mathbb{L}\boldsymbol{\Psi}, \ \boldsymbol{\Phi}^*\mathbb{L}_s\boldsymbol{\Psi}, \ \boldsymbol{\Phi}^*\mathbf{V}, \ \mathbf{W}\boldsymbol{\Psi})$$

are equivalent minimal descriptor realizations for the same system.

Proof. Given $\boldsymbol{\Phi} \in \mathbb{R}^{N\times k}$, (8.40) implies $\boldsymbol{\Phi}^*\mathbf{Y}\mathbf{Y}^*\mathbb{L} = \boldsymbol{\Phi}^*\mathbb{L}$. Given $\boldsymbol{\Psi} \in \mathbb{R}^{N\times k}$, (8.41) implies $\mathbb{L}\mathbf{X}\mathbf{X}^*\boldsymbol{\Psi} = \mathbb{L}\boldsymbol{\Psi}$. Combining these two relationships, we obtain

$$\underbrace{\boldsymbol{\Phi}^*\mathbf{Y}}_{\mathbf{T}_1}\mathbf{Y}^*\mathbb{L}\mathbf{X}\underbrace{\mathbf{X}^*\boldsymbol{\Psi}}_{\mathbf{T}_2} = \boldsymbol{\Phi}^*\mathbb{L}\boldsymbol{\Psi} \quad \Rightarrow \quad \mathbf{T}_1\mathbf{Y}^*\mathbb{L}\mathbf{X}\mathbf{T}_2 = \boldsymbol{\Phi}^*\mathbb{L}\boldsymbol{\Psi}.$$

Similarly, we have

$$\mathbf{T}_1\mathbf{Y}^*\mathbb{L}_s\mathbf{X}\mathbf{T}_2 = \boldsymbol{\Phi}^*\mathbb{L}_s\boldsymbol{\Psi}, \ \ \mathbf{T}_1\mathbf{Y}^*\mathbf{V} = \boldsymbol{\Phi}^*\mathbf{V}, \ \ \mathbf{W}\mathbf{X}\mathbf{T}_2 = \mathbf{W}\boldsymbol{\Psi}.$$

Hence, the nonsingularity of $\mathbf{T}_1$ and $\mathbf{T}_2$ implies that the two quadruples are equivalent. □

Remark 8.26. (a) The Loewner approach constructs a descriptor representation $(\mathbb{L}, \mathbb{L}_s, \mathbf{V}, \mathbf{W})$ of an underlying dynamical system exclusively from the data, with no further manipulations involved (i.e., matrix factorizations or inversions). In general, the pencil $(\mathbb{L}_s, \mathbb{L})$ is singular and needs to be projected to a regular pencil $(\mathbf{A}_\delta, \mathbf{E}_\delta)$ as in (8.39).

(b) In the Loewner framework, by construction, **D**-terms are absorbed in the other matrices of the realization (see also Remark 8.1). Extracting the **D**-term involves an eigenvalue decomposition of the pencil $(\mathbb{L}_s, \mathbb{L})$ (see Example 8.39 and Example 8.41).
(c) Notice the similarity between the components of formulas (8.25) and (8.39), where $\mathbf{Y} = \mathbf{I}$ and $\mathbf{X} = \mathbb{J}$.

Interpolation property of the projected systems

Given tangential interpolation data, the question arises whether projected quadruples as in (8.39) satisfy interpolation conditions as well. Toward this goal, recall the Sylvester equations (8.29) and (8.31). Adding $\mathbf{M}\mathbb{L}\boldsymbol{\Lambda}$ to and subtracting it from the second equation and collecting terms in $\boldsymbol{\Lambda}$ on the right and $\mathbf{M}$ on the left, we obtain

$$(\mathbb{L}_s - \mathbf{M}\mathbb{L} + \mathbf{L}\mathbf{W})\boldsymbol{\Lambda} - \mathbf{M}(\mathbb{L}_s - \mathbb{L}\boldsymbol{\Lambda} + \mathbf{V}\mathbf{R}) = \mathbb{0}.$$

Since $(\mathbb{L}_s - \mathbf{M}\mathbb{L} + \mathbf{L}\mathbf{W}) = (\mathbb{L}_s - \mathbb{L}\boldsymbol{\Lambda} + \mathbf{V}\mathbf{R}) =: \mathbf{Z}$, and assuming that $\boldsymbol{\Lambda}$ and $\mathbf{M}$ have no common eigenvalues, it follows that $\mathbf{Z} = \mathbb{0}$, and consequently

$$\mathbb{L}_s - \mathbf{M}\mathbb{L} = -\mathbf{L}\mathbf{W}, \quad \mathbb{L}_s - \mathbb{L}\boldsymbol{\Lambda} = -\mathbf{V}\mathbf{R}. \tag{8.42}$$

After projection, the following quantities (i.e., matrices of reduced dimension) are obtained:

$$\widehat{\mathbb{L}} = \mathbf{Y}^*\mathbb{L}\mathbf{X}, \quad \widehat{\mathbb{L}_s} = \mathbf{Y}^*\mathbb{L}_s\mathbf{X}, \quad \widehat{\mathbf{V}} = \mathbf{Y}^*\mathbf{V}, \quad \widehat{\mathbf{W}} = \mathbf{W}\mathbf{X}, \quad \widehat{\mathbf{L}} = \mathbf{Y}^*\mathbf{L}, \quad \widehat{\mathbf{R}} = \mathbf{R}\mathbf{X}.$$

Because of equations (8.42), the associated $\widehat{\boldsymbol{\Lambda}}$ and $\widehat{\mathbf{M}}$ must satisfy

$$\widehat{\mathbb{L}_s} - \widehat{\mathbf{M}}\widehat{\mathbb{L}} = -\widehat{\mathbf{L}}\widehat{\mathbf{W}}, \quad \widehat{\mathbb{L}_s} - \widehat{\mathbb{L}}\widehat{\boldsymbol{\Lambda}} = -\widehat{\mathbf{V}}\widehat{\mathbf{R}},$$

and hence

$$\widehat{\mathbf{M}} = (\mathbf{Y}^*\mathbb{L}_s\mathbf{X} + \mathbf{Y}^*\mathbf{L}\mathbf{W}\mathbf{X})(\mathbf{Y}^*\mathbb{L}\mathbf{X})^{-1}, \quad \widehat{\boldsymbol{\Lambda}} = (\mathbf{Y}^*\mathbb{L}\mathbf{X})^{-1}(\mathbf{Y}^*\mathbb{L}_s\mathbf{X} + \mathbf{Y}^*\mathbf{V}\mathbf{R}\mathbf{X}).$$

Finally, to recover the Loewner data for the projected system, we need to diagonalize $\widehat{\mathbf{M}}$ and $\widehat{\boldsymbol{\Lambda}}$. Let the EVD (eigenvalue decomposition) of $\widetilde{\boldsymbol{\Lambda}}$ and $\widetilde{\mathbf{M}}$ be $\widetilde{\boldsymbol{\Lambda}}\mathbf{T}_{\boldsymbol{\Lambda}} = \mathbf{T}_{\boldsymbol{\Lambda}}\mathbf{D}_{\boldsymbol{\Lambda}}$ and $\mathbf{T}_{\mathbf{M}}\tilde{\mathbf{M}} = \mathbf{D}_{\mathbf{M}}\mathbf{T}_{\mathbf{M}}$, respectively. Substituting in the above equations, we obtain the corresponding Loewner data for the reduced system:

$$\begin{aligned}
&\widetilde{\mathbb{L}} = \mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbb{L}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}}, \quad \widetilde{\mathbb{L}_s} = \mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbb{L}_s\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}},\\
&\widetilde{\mathbf{V}} = \mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{V}, \quad \widetilde{\mathbf{W}} = \mathbf{W}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}}, \quad \widetilde{\mathbf{L}} = \mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{L}, \quad \widetilde{\mathbf{R}} = \mathbf{R}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}}, \quad \widetilde{\boldsymbol{\Lambda}} = \mathbf{D}_{\boldsymbol{\Lambda}}, \quad \widetilde{\mathbf{M}} = \mathbf{D}_{\mathbf{M}}.
\end{aligned}$$

Thus, the *right* interpolation points are the generalized eigenvalues of the pencil

$$[(\mathbf{Y}^*\mathbb{L}_s\mathbf{X} + \mathbf{Y}^*\mathbf{V}\mathbf{R}\mathbf{X}), (\mathbf{Y}^*\mathbb{L}\mathbf{X})],$$

while the *left* interpolation points are the generalized eigenvalues of the pencil

$$[(\mathbf{Y}^*\mathbb{L}_s\mathbf{X} + \mathbf{Y}^*\mathbf{L}\mathbf{W}\mathbf{X}), (\mathbf{Y}^*\mathbb{L}\mathbf{X})].$$

Furthermore, the *right* and *left* interpolation values/directions are, respectively,

$$\widetilde{\mathbf{W}} = -(\mathbf{W}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}}), \quad \widetilde{\mathbf{R}} = (\mathbf{R}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}}), \quad \widetilde{\mathbf{V}} = -(\mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{V}), \quad \widetilde{\mathbf{L}} = (\mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{L}).$$

In the scalar case, the *right* and *left* interpolation values are (in MATLAB notation) $-(\mathbf{W}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}})./(\mathbf{R}\mathbf{X}\mathbf{T}_{\boldsymbol{\Lambda}})$, $-(\mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{V})./(\mathbf{T}_{\mathbf{M}}\mathbf{Y}^*\mathbf{L})$, respectively.

Parametrization of all interpolants

Parametrization of all interpolants is accomplished by means of the *generating system matrix* denoted by $\mathbf{\Theta}(s)$. The main property of $\mathbf{\Theta}$ is that any interpolant of a given set of data can be expressed as a linear combination of its columns. This general framework leads to recursive interpolation (that is, interpolation where the data are not provided all at once) [29, 30]. Interestingly, this framework has a system-theoretic interpretation as the feedback interconnection of linear dynamical systems. For a tutorial introduction to the generating system approach, we refer the reader to Section 4.5.3 of [3] and references therein. For the proofs of the results that follow, we refer to [29, 30].

In the present context, $\mathbf{\Theta}$ is explicitly defined in terms of the tangential interpolation data, namely $\mathbf{\Lambda}, \mathbf{M}, \mathbf{L}, \mathbf{R}, \mathbf{V}, \mathbf{W}$, defined by (8.26), (8.27), as well as the Loewner matrix $\mathbb{L}$ defined in (8.28). If $q = k$ and $\mathbb{L}$ is invertible, we define the $(p+m)\times(p+m)$ rational matrix $\mathbf{\Theta}(s)$ and its inverse $\bar{\mathbf{\Theta}}(s)$:

$$\mathbf{\Theta}(s) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{W} \\ -\mathbf{R} \end{bmatrix} (s\mathbb{L} - \mathbb{L}\mathbf{\Lambda})^{-1} \begin{bmatrix} \mathbf{L} & \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{\Theta}_{11}(s) & \mathbf{\Theta}_{12}(s) \\ \mathbf{\Theta}_{21}(s) & \mathbf{\Theta}_{22}(s) \end{bmatrix},$$

$$\bar{\mathbf{\Theta}}(s) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} + \begin{bmatrix} -\mathbf{W} \\ \mathbf{R} \end{bmatrix} (s\mathbb{L} - \mathbf{M}\mathbb{L})^{-1} \begin{bmatrix} \mathbf{L} & \mathbf{V} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Theta}}_{11}(s) & \bar{\mathbf{\Theta}}_{12}(s) \\ \bar{\mathbf{\Theta}}_{21}(s) & \bar{\mathbf{\Theta}}_{22}(s) \end{bmatrix}.$$

It should be mentioned that with appropriate changes, the above equations also hold for the case of interpolation with multiplicities.

First, we notice that the left and right interpolation conditions hold.

Proposition 8.27. *The following relationships are satisfied for j, $i = 1, \ldots, k$:*

$$\begin{bmatrix} \ell_j^T & \mathbf{v}_j^T \end{bmatrix} \mathbf{\Theta}(\mu_j) = 0_{1\times(p+m)} \quad \text{and} \quad \bar{\mathbf{\Theta}}(\lambda_i) \begin{bmatrix} -\mathbf{w}_i \\ \mathbf{r}_i \end{bmatrix} = 0_{(p+m)\times 1}.$$

Next, we assert that all interpolants can be obtained as linear matrix fractions constructed from the entries of $\mathbf{\Theta}$ or $\bar{\mathbf{\Theta}}$.

Lemma 8.28. *For any polynomial matrices $\mathrm{S}_1(s)$, $\mathrm{S}_2(s)$, $\bar{\mathrm{S}}_1(s)$, $\bar{\mathrm{S}}_2(s)$ of size $p\times m$, $m\times m$, $p\times p$, $p\times m$, respectively, that satisfy*

$$\mathrm{S}_1(s)\mathrm{S}_2(s)^{-1} = \bar{\mathrm{S}}_1(s)^{-1}\bar{\mathrm{S}}_2(s),$$

the following $p\times m$ rational function is an interpolant:

$$\mathbf{\Psi}(s) = \mathbf{\Psi}_1(s)\mathbf{\Psi}_2(s)^{-1} = [\mathbf{\Theta}_{11}(s)\mathrm{S}_1(s) - \mathbf{\Theta}_{12}(s)\mathrm{S}_2(s)][-\mathbf{\Theta}_{21}(s)\mathrm{S}_1(s) + \mathbf{\Theta}_{22}(s)\mathrm{S}_2(s)]^{-1},$$

provided that the denominator is nonsingular for all interpolation points λ_i and μ_j. Similarly, $\mathbf{\Psi}$ can also be written as

$$\mathbf{\Psi}(s) = \bar{\mathbf{\Psi}}_1(s)^{-1}\bar{\mathbf{\Psi}}_2(s) = [\bar{\mathrm{S}}_1(s)\bar{\mathbf{\Theta}}_{11}(s) + \bar{\mathrm{S}}_2(s)\bar{\mathbf{\Theta}}_{21}(s)]^{-1}[\bar{\mathrm{S}}_1(s)\bar{\mathbf{\Theta}}_{12}(s) + \bar{\mathrm{S}}_2(s)\bar{\mathbf{\Theta}}_{22}(s)].$$

The former is a right coprime factorization and the latter is a left coprime factorization of $\mathbf{\Psi}$. For right/left coprimeness of the factors, S_1, S_2 and $\bar{\mathrm{S}}_1$, $\bar{\mathrm{S}}_2$ must be right/left coprime, respectively. Conversely, any interpolant that satisfies the left and right tangential interpolation conditions, can be expressed as above for S_1, S_2, $\bar{\mathrm{S}}_1$, $\bar{\mathrm{S}}_2$ appropriately chosen.

Corollary 8.29. (a) *For* $\mathbf{S}_1(s)=0$ *and* $\mathbf{S}_2(s)=\mathbf{I}$, *we recover the minimal degree interpolant given by* (8.34): $\mathbf{\Psi}(s)=-\mathbf{\Theta}_{12}(s)\mathbf{\Theta}_{22}^{-1}(s)=\mathbf{C}_{\mathcal{S}}(s\mathbf{E}_{\mathcal{S}}-\mathbf{A}_{\mathcal{S}})^{-1}\mathbf{B}_{\mathcal{S}}$.
(b) *For* $\mathbf{S}_1(s)=\mathbf{K}$ *and* $\mathbf{S}_2(s)=\mathbf{I}$, *we recover all minimal degree interpolants defined by* (8.36): $\mathbf{\Psi}(s)=\mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$.

The Loewner algorithm

We are now ready to formulate a high-level algorithm that produces from given data an approximate model, possibly of reduced dimension.

- **Data:** $(\mu_i,\ell_i,\mathbf{v}_i)$, $i=1,\ldots,q$ and $(\lambda_j,\mathbf{r}_j,\mathbf{w}_j)$, $j=1,\ldots,k$, as defined in Section 8.1.3.

Algorithm 8.1 gives the procedure for obtaining (reduced) models.

ALGORITHM **8.1. The Loewner algorithm.**

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1. Build the Loewner matrix pencil $(\mathbb{L}_s,\mathbb{L})$ following (8.30) and (8.28). Let these matrices have size $q\times k$.

2. There are two important positive integers to compute, namely the dimension of minimal (approximate) descriptor realizations and their McMillan degree.

   (a) If condition (8.37) is satisfied, r is the dimension of the resulting minimal descriptor realizations.

   (b) In this case, $\nu=\operatorname{rank}\mathbb{L}$ is the McMillan degree of the corresponding realization.

   Both of these numbers can be computed as *numerical ranks*.

3. Following (8.38), matrices $\mathbf{X}\in\mathbb{C}^{k\times r}$, $\mathbf{Y}\in\mathbb{C}^{q\times r}$ are determined and a descriptor realization is given by (8.39):

$$(\mathbf{E}_{\mathcal{S}},\mathbf{A}_{\mathcal{S}},\mathbf{B}_{\mathcal{S}},\mathbf{C}_{\mathcal{S}})\in\mathbb{C}^{r\times r}\times\mathbb{C}^{r\times r}\times\mathbb{C}^{r\times m}\times\mathbb{C}^{p\times r}.$$

4. If $r=\nu$, the transfer function of the ensuing system is strictly proper rational.

5. If $r>\nu$, we need to decide whether there is a $\mathbf{D}$-term or a polynomial term. Toward this goal we compute the generalized eigenvalues of the pencil $(\mathbf{A}_{\mathcal{S}},\mathbf{E}_{\mathcal{S}})$.

   (a) If there is a multiple eigenvalue at infinity and the corresponding eigenspace has the same dimension, the system has a $\mathbf{D}$-term.

   (b) If there are Jordan blocks at infinity, the transfer function has a polynomial term.

8.2.4 ▪ Further issues

Positive real interpolation in the Loewner framework

The classical (scalar) *positive real (PR) interpolation* problem is as follows. Given pairs of points (λ_i, ϕ_i), $i = 1, \ldots, N$, we seek *PR* functions[31] $\boldsymbol{\phi}(s)$ such that

$$\boldsymbol{\phi}(\lambda_i) = \phi_i, \; i = 1, \ldots, N.$$

If $\mathscr{R}e(\lambda_i) > 0$, the classical condition for solution is that the associated *Pick matrix* $\boldsymbol{\Pi}$ be positive semidefinite:

$$\boldsymbol{\Pi} = \left[\frac{\phi_i^* + \phi_j}{\lambda_i^* + \lambda_j} \right]_{i,j=1,\ldots,N} = \boldsymbol{\Pi}^* \geq 0.$$

Observation. $\boldsymbol{\Pi}$ is a *Loewner matrix* with column array (λ_i, ϕ_i) and row array the *mirror-image* set $(-\lambda_j^*, -\phi_j^*)$. Thus, the interpolation data in the Loewner framework are

$$\mathbf{W} = [\phi_1 \cdots \phi_N], \;\; \mathbf{V} = -\mathbf{W}^*, \;\; \boldsymbol{\Lambda} = \mathrm{diag}[\lambda_1, \ldots, \lambda_N], \;\; \mathbf{M} = -\boldsymbol{\Lambda}^*,$$

and $\mathbf{R} = [1, \ldots, 1] = \mathbf{L}^*$ (the latter quantities do not appear in the classical framework). Consequently, the associated Loewner matrix pencil satisfies the Sylvester (Lyapunov) equations

$$\mathbb{L}\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^*\mathbb{L} = \mathbf{W}^*\mathbf{R} + \mathbf{R}^*\mathbf{W} \quad \text{and} \quad \mathbb{L}_s\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^*\mathbb{L}_s = \mathbf{R}^*\mathbf{W}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*\mathbf{W}^*\mathbf{R}.$$

Proposition 8.30 (Realization of PR interpolants). *Assuming that* $\boldsymbol{\Pi} = \mathbb{L} \geq 0$, *a minimal PR interpolant is*

$$\mathbf{H}(s) = \mathbf{W}(s\mathbb{L} - \mathbb{L}_s)^{-1}\mathbf{W}^*.$$

Proof. Making use of the fact that $\mathbb{L} = \mathbb{L}^* > 0$ (Hermitian positive definite) and that $\mathbb{L}_s^* = -\mathbb{L}_s$ (skew Hermitian), we obtain

$$\begin{aligned} \mathbf{H}(s) + \mathbf{H}^*(-s) \;\; &= \mathbf{W}(s\mathbb{L} - \mathbb{L}_s)^{-1}\mathbf{W}^* + \mathbf{W}(s^*\mathbb{L} - \mathbb{L}_s)^{-1}\mathbf{W}^* \\ &= \mathbf{W}(s\mathbb{L} - \mathbb{L}_s)^{-1}\left[(s + s^*)\mathbb{L} - \mathbb{L}_s - \mathbb{L}_s^*\right](s^*\mathbb{L} - \mathbb{L}_s)^{-1}\mathbf{W}^* \\ &= \underbrace{\mathbf{W}(s\mathbb{L} - \mathbb{L}_s)^{-1}}_{\mathbf{K}(s)}\left[(s + s^*)\mathbb{L}\right]\underbrace{(s^*\mathbb{L} - \mathbb{L}_s)^{-1}\mathbf{W}^*}_{\mathbf{K}^*(s^*)} \\ &= (s + s^*)\mathbf{K}(s)\mathbb{L}\mathbf{K}^*(s^*) \geq 0 \quad \text{for} \quad s + s^* \geq 0. \end{aligned}$$

This completes the proof. ☐

We will now turn our attention to the parametrization of minimal solutions. Following Theorem 8.20(b), from

$$\mathbb{L}_s\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^*\mathbb{L}_s \; = \mathbf{R}^*\mathbf{W}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*\mathbf{W}^*\mathbf{R},$$

[31] A function $\mathbf{f}$ of the complex variable s is *positive real* (abbreviated PR) if it is analytic in the (open) right half of the complex plane and, in addition, its real part is nonnegative in the same domain.

we have

$$[\mathbb{L}_s - \mathbf{R}^*\delta\mathbf{R}]\mathbf{\Lambda} + \mathbf{\Lambda}^*[\mathbb{L}_s - \mathbf{R}^*\delta\mathbf{R}] = \mathbf{R}^*[\mathbf{W} - \delta\mathbf{R}]\mathbf{\Lambda} - \mathbf{\Lambda}^*[\mathbf{W}^* + \mathbf{R}^*\delta]\mathbf{R}.$$

Hence, a parametrization of all solutions of degree N is given in terms of the (scalar) parameter δ. The associated realization of these interpolants is

$$\mathbf{H}(s) = \delta + [\mathbf{W} - \delta\mathbf{R}]\,[s\,\mathbb{L} - (\mathbb{L}_s - \mathbf{R}^*\,\delta\,\mathbf{R})]^{-1}[\mathbf{W}^* + \mathbf{R}^*\delta],\;\; \delta \geq 0. \tag{8.43}$$

Remark 8.31. Lemma 8.28 can be easily formulated and proved for the case $p = m > 1$, where $\mathbf{R} \in \mathbb{C}^{p\times N}$ contains (right) tangential directions and $\mathbf{W} \in \mathbb{C}^{p\times N}$ the associated (right) tangential values.

Connections with Port–Hamiltonian systems. We will conclude our considerations of the PR realization problem by pointing out that the above realizations are in *Port–Hamiltonian (PH)* form. According to [38] *PH systems* with feed-through terms are described as

$$\begin{aligned} \tfrac{d}{dt}\tilde{\mathbf{x}}(t) &= [\mathbf{J} - \hat{\mathbf{R}}]\mathbf{Q}\,\tilde{\mathbf{x}}(t) + [\mathbf{G} - \mathbf{P}]\mathbf{u}(t),\\ \mathbf{y}(t) &= [\mathbf{G}^* + \mathbf{P}^*]\mathbf{Q}\,\tilde{\mathbf{x}} + [\mathbf{M} + \mathbf{S}]\mathbf{u}(t), \end{aligned}$$

where

$$\mathbf{Z} = \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{P} \\ \mathbf{P}^* & \mathbf{S} \end{bmatrix} \geq 0,\;\; \mathbf{J} = -\mathbf{J}^*,\;\; \mathbf{M} = -\mathbf{M}^*.$$

This can be rewritten with $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$ as

$$\left.\begin{aligned} \mathbf{E}\tfrac{d}{dt}\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}\right\}\;\text{where}\;\left\{\begin{array}{c|c} \mathbf{E} = \mathbf{Q}^{-1},\ \mathbf{A} = \mathbf{J} - \hat{\mathbf{R}} & \mathbf{B} = \mathbf{G} - \mathbf{P} \\ \hline \mathbf{C} = \mathbf{G}^* + \mathbf{P}^* & \mathbf{D} = \mathbf{M} + \mathbf{S} \end{array}\right. .$$

Corollary 8.32. *It readily follows that the realizations in Proposition* 8.30 *and equation* (8.43) *are in PH form as* $\mathbf{E} = \mathbb{L}$, $\mathbf{J} = \mathbb{L}_s$, $\hat{\mathbf{R}} = \mathbf{R}^*\delta\mathbf{R}$, $\mathbf{G} = \mathbf{W}^*$, $\mathbf{P} = \delta\mathbf{R}^*$, $\mathbf{M} = 0$, $\mathbf{S} = \delta$.

Remark 8.33. PR functions and PH systems are important in model reduction as they represent passive resistor-inductor-capacitor electric circuits (closely related to *interconnect analysis* in VLSI synthesis), spring-mass-damper mechanical systems, etc. See also [4] for the construction of passive models from frequency response data.

Poles and zeros as interpolation points

Suppose that one of the values in (8.27) or (8.26) is infinite or zero. This means that the corresponding interpolation point is a pole or a zero of the interpolant that we wish to construct. Such conditions can be easily accounted for in the Loewner framework by appropriate definition of $\mathbf{r}_i$, $\mathbf{w}_i$ or $\boldsymbol{\ell}_j$, $\mathbf{v}_j$. In the former case (specified poles), either $\mathbf{r}_i = 0$ ($\mathbf{w}_i$ arbitrary) or $\boldsymbol{\ell}_j = 0$ ($\mathbf{v}_j$ arbitrary), in which case λ_i or μ_j is a pole, while in the latter, $\mathbf{w}_i = 0$ ($\mathbf{r}_i$ arbitrary) or $\mathbf{v}_j = 0$ ($\boldsymbol{\ell}_j$ arbitrary), in which case λ_i or μ_j is a zero.

Assume, for instance, that the given interpolation data satisfy $\mathbf{r}_i = 0$ for $i = 1,\ldots,k$, i.e., $\mathbf{R} = 0$. Then, combining (8.28) and (8.30), we obtain $\mathbb{L}_s = \mathbb{L}\mathbf{\Lambda}$. This, in turn, implies that λ_i, $i=1,\dots,k$, are the eigenvalues of the pencil $(\mathbb{L}_s,\mathbb{L})$ and, assuming that $\mathbb{L}$ is nonsingular, they are the poles of the interpolant

$$\mathbf{H}(s)=\mathbf{W}(\mathbb{L}_s-s\mathbb{L})^{-1}\mathbf{V}=\mathbf{W}\mathbb{L}^{-1}(s\mathbf{I}-\mathbf{\Lambda})^{-1}\mathbf{V}=\sum_{i=1}^{k}\frac{\mathbf{res}_i}{s-\lambda_i},$$

where $\mathbf{res}_i=(\mathbf{w}\mathbb{L}^{-1})_i\mathbf{v}_i^T$ is the residue corresponding to the pole λ_i.

Error expressions

We will now derive an expression for the error when interpolation is performed by means of an inexact barycentric formula (see relations (8.19), (8.20), (8.21)). In particular, let

$$\mathbb{L}\mathbf{c}=\mathbf{e},$$

where $\mathbf{e}$ is not necessarily zero. Using the barycentric formula, we attach to $\mathbf{c}=(\alpha_i)$ the rational function

$$\hat{\mathbf{H}}(s)=\frac{\sum_{j=1}^{q+1}\frac{\alpha_j\mathbf{w}_j}{s-\lambda_j}}{\sum_{j=1}^{q+1}\frac{\alpha_j}{s-\lambda_j}}.$$

It readily follows that $(\lambda_j,\mathbf{w}_j)$ are interpolated. Furthermore, it readily follows that

$$\mathbf{v}_i-\hat{\mathbf{H}}(\mu_i)=\frac{e_i}{\sum_{j=1}^{q+1}\frac{\alpha_j}{\mu_j-\lambda_j}}.$$

Therefore, if $\mathbf{c}$ is the left singular vector corresponding to the smallest singular value of $\mathbb{L}$, say σ_{q+1}, i.e., $\mathbb{L}\mathbf{c}=\sigma_{q+1}\mathbf{x}$, where $(\mathbf{x})_{i,1}=x_i$ is the corresponding left singular vector, this expression yields, for $i=1,\dots,k$,

$$\mathbf{v}_i-\hat{\mathbf{H}}(\mu_i)=\sigma_{q+1}\frac{x_i}{\sum_{j=1}^{q+1}\frac{\alpha_j}{\mu_j-\lambda_j}}\Rightarrow|\mathbf{v}_i-\hat{\mathbf{H}}(\mu_i)|\leq\frac{\sigma_{q+1}}{|\sum_{j=1}^{q+1}\frac{\alpha_j}{\mu_j-\lambda_j}|}.$$

Error bound. For the case when the full-order transfer function $\mathbf{H}(s)$ that generated the measurements $\mathbf{H}(s_i)$ is known, we want to bound the approximation error $\mathbf{H}(s)-\hat{\mathbf{H}}(s)$ over entire intervals, rather than just at the given points. The error can be bounded by the following formula [27]:

$$|\mathbf{H}(s)-\hat{\mathbf{H}}(s)|\leq\Delta(s)\max_s\left|\frac{d\mathbf{H}(s)}{ds}\right|,\ \ \Delta(s)=\frac{\sum_{i=1}^{q+1}|x_i|}{|\delta(s)|},\ \delta(s)=\sum_{i=1}^{q+1}\frac{\alpha_j}{s-\lambda_i}$$

for $\lambda_i\in\mathbb{R}$, $s\in[\lambda_1,\lambda_{q+1}]$, $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_{q+1}$, and x_i as defined above. The factor $\Delta(s)$ is given by the barycentric coefficients α_j of $\hat{\mathbf{H}}(s)$, while the derivative is given by $\frac{d\mathbf{H}(s)}{ds}=-\mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$. Usually, this bound correctly predicts interpolation at the Lagrange nodes because the Δ factor equals zero for λ_i. However, in general, this bound is loose over the interval, even though it may capture the general shape of the error.

Real Loewner matrices from complex data

In applications, assuming that the underlying system is real, to obtain solutions containing no complex quantities, it must be assumed that, given a certain set of data, the *complex conjugate data* are also provided. Then Λ, $\mathbf{M}$ as well as $\mathbf{W}$, $\mathbf{V}$ must contain complex conjugate data. For this purpose, we need the block diagonal matrix $\mathbf{J}$ with $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\imath \\ 1 & \imath \end{bmatrix}$ repeated on the diagonal.

The goal is to form $\mathbb{L}^{\mathbf{R}}$, $\mathbb{L}_s^{\mathbf{R}}$ with real entries. This can be done by first forming $\mathbb{L}^{\mathbf{C}}$, $\mathbb{L}_s^{\mathbf{C}}$ (with complex entries) and then multiplying by $\mathbf{J}, \mathbf{J}^*$:

$$\mathbb{L}^{\mathbf{R}} = \mathbf{J}^*\mathbb{L}^{\mathbf{C}}\mathbf{J},\ \ \mathbb{L}_s^{\mathbf{R}} = \mathbf{J}^*\mathbb{L}_s^{\mathbf{C}}\mathbf{J}.$$

Alternatively, $\mathbb{L}^{\mathbf{R}}$, $\mathbb{L}_s^{\mathbf{R}}$ can be obtained by solving the Sylvester equations

$$\mathbb{L}^{\mathbf{R}}\Lambda^{\mathbf{R}} - \mathbf{M}^{\mathbf{R}}\mathbb{L}^{\mathbf{R}} = \mathbf{V}^{\mathbf{R}}\mathbf{R}^{\mathbf{R}} - \mathbf{L}^{\mathbf{R}}\mathbf{W}^{\mathbf{R}},\ \ \mathbb{L}_s^{\mathbf{R}}\Lambda^{\mathbf{R}} - \mathbf{M}^{\mathbf{R}}\mathbb{L}_s^{\mathbf{R}} = \mathbf{M}^{\mathbf{R}}\mathbf{V}^{\mathbf{R}}\mathbf{R}^{\mathbf{R}} - \mathbf{L}^{\mathbf{R}}\mathbf{W}^{\mathbf{R}}\Lambda^{\mathbf{R}}, \tag{8.44}$$

where the quantities above are defined in terms of the following *real* blocks:

$$\Lambda^{\mathbf{R}} = \mathrm{blkdiag}\left[\ldots, \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}, \ldots\right],\ \ \mathbf{M}^{\mathbf{R}} = \mathrm{blkdiag}\left[\ldots, \begin{bmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{bmatrix}, \ldots\right],$$

and, assuming that the left and right directions are real vectors,

$$\mathbf{V}^{\mathbf{R}} = \begin{bmatrix} \vdots \\ \mathrm{Re}(\mathbf{v}_i^T) \\ -\mathrm{Im}(\mathbf{v}_i^T) \\ \vdots \end{bmatrix},\ \mathbf{L}^{\mathbf{R}} = \begin{bmatrix} \vdots \\ \ell_i^T \\ 0 \\ \vdots \end{bmatrix},$$

$$\mathbf{W}^{\mathbf{R}} = \left[\ldots, \mathrm{Re}(\mathbf{w}_j),\ \mathrm{Im}(\mathbf{w}_j), \ldots\right],\ \mathbf{R}_j = \left[\ldots, \mathbf{r}_j,\ 0, \ldots\right],$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary parts of $(\cdot)$.

8.2.5 ▪ Illustrative examples

To illustrate the above considerations, we consider several examples.

Example 8.34. We would like to recover the rational function $\mathbf{H}(s) = \frac{1}{s^2+1}$ from the following measurements: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$, $\mathbf{W} = [\frac{1}{2},\ \frac{1}{5},\ \frac{1}{10}] = \mathbf{V}^T$, and $\mathbf{R} = [1\ \ 1\ \ 1] = \mathbf{L}^T$; it follows that

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \\ \frac{1}{10} & \frac{1}{50} & 0 \end{bmatrix},\ \ \mathbb{L}_s = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \\ \frac{1}{5} & \frac{7}{50} & \frac{1}{10} \end{bmatrix}.$$

As condition (8.37) yields $r = 2$, we need to project to a two-dimensional realization. We (randomly) choose one projection matrix

$$\mathbf{X} = \begin{bmatrix} 5 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{Y}^T.$$

Thus, the projected quantities are

$$\tilde{\mathbb{L}} = \mathbf{Y}^T \mathbb{L} \mathbf{X} = \begin{bmatrix} 0 & -\frac{51}{50} \\ \frac{51}{50} & 0 \end{bmatrix}, \quad \tilde{\mathbb{L}}_s = \mathbf{Y}^T \mathbb{L}_s \mathbf{X} = \begin{bmatrix} \frac{157}{10} & -\frac{643}{50} \\ -\frac{643}{50} & \frac{53}{5} \end{bmatrix},$$

$$\tilde{\mathbf{W}} = \mathbf{W}\mathbf{X} = \left[\frac{27}{10}, -\frac{12}{5}\right], \tilde{\mathbf{V}} = \mathbf{Y}^T \mathbf{V} = \begin{bmatrix} \frac{27}{10} \\ -\frac{12}{5} \end{bmatrix},$$

$$\tilde{\mathbf{R}} = \mathbf{R}\mathbf{X} = [6, \quad -4], \tilde{\mathbf{L}} = \mathbf{Y}^T \mathbf{L} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}.$$

Therefore, it readily follows that we recover the original rational function:

$$(\mathbf{W}\mathbf{X})\left[(\mathbf{Y}^T \mathbb{L}_s \mathbf{X}) - s(\mathbf{Y}^T \mathbb{L} \mathbf{X})\right]^{-1} (\mathbf{Y}^T \mathbf{V}) = \frac{1}{s^2+1}.$$

Making use of the results of Section 8.2.3, we will determine the new interpolation data, which are compatible with the projected matrices $\tilde{\mathbb{L}}$, $\tilde{\mathbb{L}}_s$, $\tilde{\mathbf{W}}$, $\tilde{\mathbf{R}}$, $\tilde{\mathbf{V}}$, $\tilde{\mathbf{L}}$. The resulting new right interpolation points are the eigenvalues of the pencil $(\tilde{\mathbb{L}}_s + \tilde{\mathbf{V}}\tilde{\mathbf{R}}, \tilde{\mathbb{L}})$, while the new left interpolation points are the eigenvalues of the pencil $(\tilde{\mathbb{L}}_s + \tilde{\mathbf{L}}\tilde{\mathbf{W}}, \tilde{\mathbb{L}})$:

$$[\mathbf{T}, \mathbf{d}] = \mathrm{eig}(\tilde{\mathbb{L}}^{-1}(\tilde{\mathbb{L}}_s + \tilde{\mathbf{V}}\tilde{\mathbf{R}})) \quad \Rightarrow \quad -(\tilde{\mathbf{W}}\mathbf{T})./(\tilde{\mathbf{R}}\mathbf{T}) = \begin{bmatrix} 0.0686 & 0.9767 \end{bmatrix},$$

$$\mathrm{diag}(\mathbf{d}) = \begin{bmatrix} -3.684 \\ 0.154 \end{bmatrix} \Rightarrow \mathbf{H}(\mathrm{diag}(\mathbf{d})) = \begin{bmatrix} 0.0686 \\ 0.9767 \end{bmatrix},$$

$$[\mathbf{T}_1, \mathbf{d}_1] = \mathrm{eig}((\tilde{\mathbb{L}}_s + \tilde{\mathbf{L}}\tilde{\mathbf{W}})(\tilde{\mathbb{L}})^{-1}) \Rightarrow -(\mathbf{T}_1^{-1}\tilde{\mathbf{V}})./(\mathbf{T}_1^{-1}\tilde{\mathbf{L}}) = \begin{bmatrix} 0.9767 & 0.0686 \end{bmatrix},$$

$$\mathrm{diag}(\mathbf{d}_1) = \begin{bmatrix} -0.154 \\ 3.684 \end{bmatrix} \Rightarrow \mathbf{H}(\mathrm{diag}(\mathbf{d}_1)) = \begin{bmatrix} 0.9767 \\ 0.0686 \end{bmatrix}.$$

This example illustrates the interpolation property of the projected systems as described in Section 8.2.3. In particular, it shows that, starting from left and right interpolation points $(-1, -2, -3)$ and $(1, 2, 3)$, respectively, the projected system corresponds to left and right interpolation points $(-3.684, 0.154)$, $(3.684, -0.154)$. In other words, when the original left and right triples are compressed, the latter left and right pairs are obtained. ∎

Example 8.35. Next, we wish to recover the polynomial $\phi(s) = s^2$ by means of measurements. From

$$\Lambda = \mathrm{diag}(1, \ 2, \ 3), \ \mathbf{M} = \mathrm{diag}(-1, \ -2, \ -3), \ \mathbf{W} = \mathbf{V}^T = [1, \ 4, \ 9],$$

we calculate

$$\mathbb{L} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 4 & 7 \\ 7 & 7 & 9 \end{bmatrix}.$$

Since $\mathrm{rank}\,\mathbb{L} = 2$, while (8.37) yields $r = 3$, the minimal descriptor realizations have degree three and McMillan degree two. Therefore, $\mathbf{A} = -\mathbb{L}_s$, $\mathbf{E} = -\mathbb{L}$, $\mathbf{B} = \mathbf{V}$, $\mathbf{C} = \mathbf{W}$

is a desired realization, and the corresponding interpolant is $\boldsymbol{\phi}(s) = \mathbf{W}(\mathbb{L}_s - s\,\mathbb{L})^{-1}\mathbf{V} = s^2$.

We consider two additional points, namely $\lambda_4 = 4$, $\mu_4 = -4$: $\boldsymbol{\Lambda} = \mathrm{diag}[1,\ 2,\ 3,\ 4]$, $\mathbf{M} = -\boldsymbol{\Lambda}$. Then, $\mathbf{W} = \mathbf{V}^T = [1,\ 4,\ 9,\ 16]$. The Loewner pencil is updated by means of a new row and a new column:

$$\mathbb{L} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} 1 & 3 & 7 & 13 \\ 3 & 4 & 7 & 12 \\ 7 & 7 & 9 & 13 \\ 13 & 12 & 13 & 16 \end{bmatrix}.$$

Indeed, the pencil $(\mathbb{L}_s, \mathbb{L})$ has a (generalized) eigenvalue at infinity and a corresponding Jordan chain of length three:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 0 \end{bmatrix},$$

satisfying $\mathbb{L}\mathbf{v}_0 = 0$, $\mathbb{L}_s\mathbf{v}_0 = \mathbb{L}\mathbf{v}_1$, $\mathbb{L}_s\mathbf{v}_1 = \mathbb{L}\mathbf{v}_2$. Alternatively, one can check this statement by computing the QZ factorization of $(\mathbb{L}_s, \mathbb{L})$ to obtain an upper triangular pencil $(\mathbf{T}_{\mathbb{L}_s}, \mathbf{T}_{\mathbb{L}})$ with diagonal entries as follows:

$\mathrm{diag}(\mathbf{T}_{\mathbb{L}_s})$	$\mathrm{diag}(\mathbf{T}_{\mathbb{L}})$	
$-3.3467\,10^{-8} \;\; -1.8733\,10^{-1}\imath$	$1.4534\,10^{-8}$	$\leftarrow$ zero eig
$-2.0228\,10^{-7} \;\; +1.1323\,\imath$	$8.7842\,10^{-8}$	$\leftarrow$ zero eig
$5.3653\,10^{-15}$	$1.5794\,10^{-15}$	
3.9262	0	$\leftarrow$ zero eig

Consequently, the quotients of the first, second, and fourth entries of the diagonal yield the three infinite eigenvalues, while the third entry indicates an undetermined eigenvalue.

We will now project the quadruple $(\mathbb{L}_s, \mathbb{L}, \mathbf{V}, \mathbf{W})$ to get a minimal realization. The projectors are chosen randomly (use the command `round(randn(4,3))` in MATLAB):

$$\hat{\mathbf{X}} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 3 \\ -1 & -2 & 1 \end{bmatrix}, \hat{\mathbf{Y}}^T = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

satisfying the condition of Proposition 8.25. Consequently, the projected quadruple yields a minimal realization of the underlying rational function:

$$\hat{\mathbf{Y}}^T\mathbb{L}\hat{\mathbf{X}} = \begin{bmatrix} -5 & -4 & 11 \\ -19 & 16 & -5 \\ 7 & -4 & -1 \end{bmatrix}, \quad \hat{\mathbf{Y}}^T\mathbb{L}_s\mathbf{X} = \begin{bmatrix} 5 & -22 & 21 \\ 128 & 36 & -154 \\ -41 & -6 & 43 \end{bmatrix},$$

$\hat{\mathbf{Y}}^T\mathbf{V} = [-6,\ 47,\ 16]^T$, $\mathbf{W}\hat{\mathbf{X}} = [-13,\ -22,\ 39]$, and hence

$$\left[\mathbf{W}\hat{\mathbf{X}}\right] \cdot \left[(\hat{\mathbf{Y}}^T\mathbb{L}_s\mathbf{X}) - s \cdot (\hat{\mathbf{Y}}^T\mathbb{L}\mathbf{X})\right]^{-1} \cdot \left[\hat{\mathbf{Y}}^T\mathbf{V}\right] = s^2 = \boldsymbol{\phi}(s).$$

This example illustrates the following aspects. First, since the rank of the Loewner matrix is two, the McMillan degree of the underlying interpolant is two. Furthermore, (8.37) yields the minimal dimension $r = 2$; since the rank of the shifted Loewner matrix is three, according to Section 8.2.3 this means that the interpolant has a polynomial term. Finally, according to the same section, since there is a Jordan chain of length three at infinity, the interpolant is purely polynomial of order two. ∎

Example 8.36. Consider the descriptor system defined by

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ \mathbf{E}=\begin{bmatrix} 0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix} 0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{C}=\begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 1\end{bmatrix}.$$

This is a *minimal descriptor realization*, and the transfer function is

$$\mathbf{H}(s)=\begin{bmatrix} s & 1\\ 1 & \frac{1}{s}\end{bmatrix}.$$

The purpose of this example is to compare matrix interpolation with tangential interpolation. Notice that the McMillan degree of $\mathbf{H}(s)$ is two, while a minimal descriptor realization has degree three. We choose the following right/left data:

$\lambda_1=\imath,\quad \mathbf{w}_1=\mathbf{H}(\lambda_1)=\begin{bmatrix} 1 & -\imath\\ \imath & 1\end{bmatrix}$	$\mu_1=2\imath,\quad \mathbf{v}_1=\mathbf{H}(\mu_1)=\begin{bmatrix} 1 & -\imath/2\\ 2\imath & 1\end{bmatrix}$
$\lambda_2=-\imath,\quad \mathbf{w}_2=\mathbf{H}(\lambda_2)=\begin{bmatrix} 1 & \imath\\ -\imath & 1\end{bmatrix}$	$\mu_2=-2\imath,\quad \mathbf{v}_2=\mathbf{H}(\mu_2)=\begin{bmatrix} 1 & \imath/2\\ -2\imath & 1\end{bmatrix}$
$\lambda_3=1,\quad \mathbf{w}_3=\mathbf{H}(\lambda_3)=\begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix}$	$\mu_3=-1,\quad \mathbf{v}_3=\mathbf{H}(\mu_3)=\begin{bmatrix} 1 & -1\\ -1 & 1\end{bmatrix}$

$$\Rightarrow\ \mathbf{V}^T=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix},\quad \mathbf{W}=[\mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3];\quad \mathbf{R}=[\mathbf{I}_2\ \mathbf{I}_2\ \mathbf{I}_2]=\mathbf{L}^T.$$

It turns out that the associated Loewner and shifted Loewner matrices are

$$\mathbb{L}=\left[\begin{array}{cc|cc|cc} 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{\imath}{2}\\ 1 & 0 & 1 & 0 & 1 & 0\\ \hline 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{\imath}{2}\\ 1 & 0 & 1 & 0 & 1 & 0\\ \hline 0 & -\imath & 0 & \imath & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0\end{array}\right],\ \mathbb{L}_s=\left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0\\ 3\imath & 1 & \imath & 1 & 2\imath+1 & 1\\ \hline 1 & 0 & 1 & 0 & 1 & 0\\ -\imath & 1 & -3\imath & 1 & 1-2\imath & 1\\ \hline 1 & 0 & 1 & 0 & 1 & 0\\ \imath-1 & 1 & -\imath-1 & 1 & 0 & 1\end{array}\right].$$

∎

Remark 8.37. The above example shows that *matrix-valued* Loewner pencils are a special case of the general definition of Loewner pencils given by (8.28), (8.30) for appropriate $\mathbf{\Lambda}$, $\mathbf{M}$, $\mathbf{V}$, $\mathbf{L}$, $\mathbf{W}$, $\mathbf{R}$.

It follows that the rank of $\mathbb{L}$ and $\mathbb{L}_s$ is two, while that of $\xi\,\mathbb{L}-\mathbb{L}_s$ is three, for all interpolation points $\xi \in \{\lambda_j\}\cup\{\mu_i\}$. Hence, according to assumption (8.37), we cannot identify the original system from this data. We notice, however, that $\ker(\xi\,\mathbb{L}(1{:}3,1{:}3)-\mathbb{L}_s(1:3,1:3)) = 0$ for all interpolation points and therefore the corresponding matrices provide the realization

$$-[\mathbf{w}_1,\ \ \mathbf{w}_2(:,1)](s\mathbb{L}(1{:}3,1{:}3)-\mathbb{L}_s(1{:}3,1{:}3))^{-1}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2(1,:) \end{bmatrix} = \mathbf{H}(s).$$

Tangential interpolation for Example 8.36. If we choose the right and left directions as

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},\ \mathbf{L} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 1 \end{bmatrix},$$

the rank of $[\mathbb{L}\quad \mathbb{L}_s]$ is different from that of $\left[\begin{smallmatrix}\mathbb{L}\\ \mathbb{L}_s\end{smallmatrix}\right]$, and hence condition (8.37) is not satisfied. Therefore, we will choose new left directions

$$\widehat{\mathbf{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \widehat{\mathbf{V}} = \begin{bmatrix} \mathbf{L}(1,:)\mathbf{V}_1 \\ \mathbf{L}(2,:)\mathbf{V}_2 \\ \mathbf{L}(3,:)\mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\imath}{2} \\ -2\imath-1 & 1-\frac{\imath}{2} \\ -1 & 1 \end{bmatrix},$$

$$\widehat{\mathbb{L}} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{\imath}{2} \\ 1 & -\frac{1}{2} & -\frac{\imath}{2}-1 \\ 1 & 0 & -1 \end{bmatrix},\ \widehat{\mathbb{L}}_s = \begin{bmatrix} 1 & 0 & -1 \\ -\imath-1 & 1 & 2\imath-1 \\ \imath-1 & 1 & -1 \end{bmatrix}.$$

In this case,

$$\ker([\widehat{\mathbb{L}}^T,\quad \widehat{\mathbb{L}}_s^T]) = \ker([\widehat{\mathbb{L}},\quad \widehat{\mathbb{L}}_s]) = 3,$$

and $\det(s\widehat{\mathbb{L}}-\widehat{\mathbb{L}}_s) = s(2\imath+1) \neq 0$. Therefore, since this determinant is different from zero at the interpolation points, assumption (8.37) is satisfied and indeed $\mathbf{W}(\widehat{\mathbb{L}}_s - s\widehat{\mathbb{L}})^{-1}\widehat{\mathbf{V}} = \mathbf{H}(s)$. Notice also that this realization is complex as no provisions are made to obtain a real realization.

Example 8.38. Here, we will illustrate the relationship between the McMillan degree, the degree of minimal realizations, and the $\mathbf{D}$-term:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathbf{B} = \mathbf{I}_3,\ \mathbf{D} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \begin{bmatrix} \frac{1}{s}+1 & \frac{1}{s^2}+1 & \frac{1}{s^3}+1 \\ 1 & \frac{1}{s}+1 & \frac{1}{s^2}+1 \end{bmatrix}.$$

Let the interpolation points be

$$\mathbf{M} = \operatorname{diag}[1,\ 1,\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{4},\ -\tfrac{1}{4}],$$
$$\boldsymbol{\Lambda} = \operatorname{diag}[\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -1,\ -1,\ -1,\ 2,\ 2,\ 2].$$

The interpolation values $\mathbf{w}_i = \mathbf{H}(\lambda_i)$, $\mathbf{v}_i = \mathbf{H}(\mu_i)$, $i = 1,2,3$, are

$$\mathbf{w}_1 = \begin{bmatrix} 3 & 5 & 9 \\ 1 & 3 & 5 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3/2 & 5/4 & 9/8 \\ 1 & 3/2 & 5/4 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 & 5 & -7 \\ 1 & -1 & 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 & 17 & -63 \\ 1 & -3 & 17 \end{bmatrix}.$$

This corresponds to *matrix interpolation* (the values considered are matrices) as opposed to tangential interpolation (which will follow next):

$$\mathbf{V} = \left[\begin{array}{ccc} 2 & 2 & 2 \\ 1 & 2 & 2 \\ \hline -1 & 5 & -7 \\ 1 & -1 & 5 \\ \hline -3 & 17 & -63 \\ 1 & -3 & 17 \end{array}\right], \quad \mathbf{W} = \left[\begin{array}{ccc|ccc|ccc} 3 & 5 & 9 & 0 & 2 & 0 & \frac{3}{2} & \frac{5}{4} & \frac{9}{8} \\ 1 & 3 & 5 & 1 & 0 & 2 & 1 & \frac{3}{2} & \frac{5}{4} \end{array}\right],$$

$$\mathbf{R} = [\mathbf{I}_3 \;\; \mathbf{I}_3 \;\; \mathbf{I}_3], \quad \mathbf{L} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{I}_2 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\Rightarrow \quad \mathbb{L} = \left[\begin{array}{ccc|ccc|ccc} -2 & -6 & -14 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{3}{4} & -\frac{7}{8} \\ 0 & -2 & -6 & 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{4} \\ \hline 4 & 0 & 16 & -2 & 6 & -14 & 1 & -\frac{3}{2} & \frac{13}{4} \\ 0 & 4 & 0 & 0 & -2 & 6 & 0 & 1 & -\frac{3}{2} \\ \hline 8 & -16 & 96 & -4 & 20 & -84 & 2 & -7 & \frac{57}{2} \\ 0 & 8 & -16 & 0 & -4 & 20 & 0 & 2 & -7 \end{array}\right] \in \mathbb{R}^{6\times 9},$$

$$\text{and} \qquad \mathbb{L}_s = \left[\begin{array}{ccc|ccc|ccc} 1 & -1 & -5 & 1 & 2 & 1 & 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & -1 & 1 & 1 & 2 & 1 & 1 & \frac{1}{2} \\ \hline 1 & 5 & 1 & 1 & -1 & 7 & 1 & 2 & -\frac{1}{2} \\ 1 & 1 & 5 & 1 & 1 & -1 & 1 & 1 & 2 \\ \hline 1 & 9 & -15 & 1 & -3 & 21 & 1 & 3 & -6 \\ 1 & 1 & 9 & 1 & 1 & -3 & 1 & 1 & 3 \end{array}\right] \in \mathbb{R}^{6\times 9}.$$

It readily follows that, while $\operatorname{rank}\mathbb{L} = \operatorname{rank}\mathbb{L}_s = 3$, the rank of $[\mathbb{L};\ \mathbb{L}_s]$ is equal to the rank of $[\mathbb{L},\ \mathbb{L}_s]$, which is equal to four. Furthermore the rank of $\xi\mathbb{L} - \mathbb{L}_s$ is also four, for all $\xi \in \{\mu_i\} \cup \{\lambda_j\}$. Consequently, according to (8.37), the dimension of the minimal realization is $r = 4$. ∎

Tangential interpolation for Example 8.38. If we define the index set $I = [1\ 2\ 3\ 4]$, we get

$$\mathbf{\Lambda}(I,I) = \operatorname{diag}\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right], \ \mathbf{M}(I,I) = \operatorname{diag}\left[\begin{array}{cccc} 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right],$$

$$\mathbf{W}(:,I) = \begin{bmatrix} 3 & 5 & 9 & 0 \\ 1 & 3 & 5 & 1 \end{bmatrix}, \ \mathbf{V}(I,:) = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & 5 & -7 \\ 1 & -1 & 5 \end{bmatrix},$$

$$\mathbb{L}(I,I)=\begin{bmatrix} -2 & -6 & -14 & 1 \\ 0 & -2 & -6 & 0 \\ 4 & 0 & 16 & -2 \\ 0 & 4 & 0 & 0 \end{bmatrix},\ \mathbb{L}_s(I,I)=\begin{bmatrix} 1 & -1 & -5 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \end{bmatrix}.$$

Since condition (8.37) is satisfied with $r=4$ and the rank of the Loewner matrix is three, we recover a minimal descriptor realization with incorporated $\mathbf{D}$-term:

$$\mathbf{W}(:,I)(\mathbb{L}_s(I,I)-s\mathbb{L}(I,I))^{-1}\mathbf{V}(I,:)=\mathbf{H}(s).$$

An alternative way to obtain the interpolant is to project the original matrices by randomly generated projectors:

$$\mathbf{Y}^T=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix},\ \mathbf{X}=\begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{E}_\delta=\begin{bmatrix} -19 & 10 & 21 & 19 \\ -\frac{31}{2} & 3 & \frac{9}{2} & \frac{15}{2} \\ \frac{565}{8} & -\frac{103}{4} & -\frac{687}{8} & -\frac{641}{8} \\ \frac{45}{4} & -\frac{35}{2} & -\frac{95}{4} & -\frac{65}{4} \end{bmatrix},\ \mathbf{A}_\delta=\begin{bmatrix} 12 & 6 & 1 & -4 \\ 14 & 18 & 16 & 5 \\ -\frac{63}{4} & \frac{41}{2} & \frac{153}{4} & \frac{127}{4} \\ -\frac{25}{2} & -5 & \frac{5}{2} & \frac{15}{2} \end{bmatrix},$$

$$\mathbf{B}_\delta=\begin{bmatrix} 1 & -3 & 17 \\ 1 & 4 & 12 \\ -1 & 24 & -54 \\ 0 & 5 & -15 \end{bmatrix},\mathbf{C}_\delta=\begin{bmatrix} \frac{65}{8} & \frac{45}{4} & \frac{37}{8} & -\frac{13}{8} \\ \frac{29}{4} & \frac{17}{2} & \frac{25}{4} & \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow \mathbf{C}_\delta(\mathbf{A}_\delta-s\mathbf{E}_\delta)^{-1}\mathbf{B}_\delta=\mathbf{H}(s).$$

We thus obtain a different (equivalent) descriptor realization with incorporated $\mathbf{D}$-term.

Example 8.39. We consider the transfer function

$$\mathbf{H}(s)=\frac{1}{s(s+2)}\begin{bmatrix} s & 0 \\ 1 & s+2 \end{bmatrix}+\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

which is to be reconstructed from measurements. Let

$$\boldsymbol{\Lambda}=\mathrm{diag}[\imath,\ -\imath,\ 3\imath,\ -3\imath] \quad \text{and} \quad \mathbf{M}=\mathrm{diag}[2\imath,\ -2\imath,\ 4\imath,\ -4\imath].$$

The resulting matrix measurements are

$$\mathbf{w}_1 = \mathbf{H}(\imath) = \begin{bmatrix} \frac{7}{5} - \frac{\imath}{5} & 2 \\ -\frac{1}{5} - \frac{2\imath}{5} & -\imath \end{bmatrix}, \qquad \mathbf{w}_2 = \mathbf{H}(-\imath) = \overline{\mathbf{w}_1},$$

$$\mathbf{w}_3 = \mathbf{H}(3\imath) = \begin{bmatrix} \frac{15}{13} - \frac{3\imath}{13} & 2 \\ -\frac{1}{13} - \frac{2\imath}{39} & -\frac{\imath}{3} \end{bmatrix}, \quad \mathbf{w}_4 = \mathbf{H}(-3\imath) = \overline{\mathbf{w}_3},$$

$$\mathbf{v}_1 = \mathbf{H}(2\imath) = \begin{bmatrix} \frac{5}{4} - \frac{\imath}{4} & 2 \\ -\frac{1}{8} - \frac{\imath}{8} & -\frac{\imath}{2} \end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{H}(-2\imath) = \overline{\mathbf{v}_1},$$

$$\mathbf{v}_3 = \mathbf{H}(4\imath) = \begin{bmatrix} \frac{11}{10} - \frac{\imath}{5} & 2 \\ -\frac{1}{20} - \frac{\imath}{40} & -\frac{\imath}{4} \end{bmatrix}, \quad \mathbf{v}_4 = \mathbf{H}(-4\imath) = \overline{\mathbf{v}_3}.$$

With $\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathbf{L}^T$, the following tangential data result:

$$\begin{aligned} & k = 4, \\ & \{\lambda, \mathbf{r}, \mathbf{w}\} = \{(i, \mathbf{e}_1, \mathbf{H}(i)\mathbf{e}_1), (-\imath, \mathbf{e}_1, \mathbf{H}(-\imath)\mathbf{e}_1), (3\imath, \mathbf{e}_2, \mathbf{H}(3\imath)\mathbf{e}_2), (-3\imath, \mathbf{e}_2, \mathbf{H}(-3\imath)\mathbf{e}_2)\}, \\ & q = 4, \\ & \{\mu, \ell, \mathbf{v}\} = \left\{ \begin{array}{l} \left(2\imath, {\mathbf{e}_1}^T, {\mathbf{e}_1}^T \mathbf{H}(2\imath)\right), \left(-2\imath, {\mathbf{e}_1}^T, {\mathbf{e}_1}^T \mathbf{H}(-2\imath)\right), \\ \qquad\qquad \left(4\imath, {\mathbf{e}_2}^T, {\mathbf{e}_2}^T \mathbf{H}(4\imath)\right), \left(-4\imath, {\mathbf{e}_2}^T, {\mathbf{e}_2}^T \mathbf{H}(-4\imath)\right) \end{array} \right\}. \end{aligned}$$

We obtain the complex matrices $\mathbb{L}^{\mathbf{C}}$, $\mathbb{L}_s^{\mathbf{C}}$, $\mathbb{W}^{\mathbf{C}}$, and $\mathbf{V}^{\mathbf{C}}$. To obtain real matrices, we follow the procedure described in Section 8.2.4. In other words, we multiply by $\mathbf{J} = \text{blkdiag}[\hat{\mathbf{J}}, \hat{\mathbf{J}}]$, where $\hat{\mathbf{J}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\imath \\ 1 & \imath \end{bmatrix}$,

$$\mathbb{L}^{\mathbf{R}} = \mathbf{J}^* \mathbb{L}^{\mathbf{C}} \mathbf{J}, \ \mathbb{L}_s^{\mathbf{R}} = \mathbf{J}^* \mathbb{L}_s^{\mathbf{C}} \mathbf{J}, \ \mathbb{W}^{\mathbf{R}} = \mathbb{W}^{\mathbf{C}} \mathbf{J}, \ \mathbf{V}^{\mathbf{R}} = \mathbf{J}^* \mathbf{V}^{\mathbf{C}},$$

to obtain

$$\mathbb{L}^{\mathbf{R}} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} & 0 & 0 \\ -\frac{1}{5} & \frac{1}{10} & 0 & 0 \\ \frac{1}{25} & -\frac{1}{50} & 0 & 0 \\ \frac{2}{25} & \frac{21}{100} & 0 & \frac{1}{6} \end{bmatrix}, \qquad \mathbb{L}_s^{\mathbf{R}} = \begin{bmatrix} \frac{12}{5} & -\frac{1}{5} & 4 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ -\frac{2}{25} & \frac{1}{25} & 0 & 0 \\ -\frac{4}{25} & \frac{2}{25} & 0 & 0 \end{bmatrix},$$

$$\mathbb{W}^{\mathbf{R}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{14}{5} & -\frac{2}{5} & 4 & 0 \\ -\frac{2}{5} & -\frac{4}{5} & 0 & -\frac{2}{3} \end{bmatrix}, \quad \mathbf{V}^{\mathbf{R}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{5}{2} & 4 \\ \frac{1}{2} & 0 \\ -\frac{1}{10} & 0 \\ \frac{1}{20} & \frac{1}{2} \end{bmatrix}.$$

It readily follows that $\text{rank}\, \mathbb{L}^{\mathbf{R}} = 2$, $\text{rank}\, \mathbb{L}_s^{\mathbf{R}} = 2$, while the rank of both $[\mathbb{L}^{\mathbf{R}} \ \ \mathbb{L}_s^{\mathbf{R}}]$ and $[\mathbb{L}^{\mathbf{R}}; \ \mathbb{L}_s^{\mathbf{R}}]$ is three. Thus, the dimension of minimal descriptor realizations (including the $\mathbf{D}$-term) is three, while the McMillan degree is two. Hence, we have to project to a realization of order three. We (randomly) pick

$$\mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix}.$$

Thus, $\mathbf{E}_\mathcal{S} = -\mathbf{Y}^T \mathbb{L}^\mathbf{R}\mathbf{X}$, $\mathbf{A}_\mathcal{S} = -\mathbf{Y}^T \mathbb{L}_s^\mathbf{R}\mathbf{X}$, $\mathbf{B}_\mathcal{S} = \mathbf{Y}^T\mathbf{V}^\mathbf{R}$, $\mathbf{C}_\mathcal{S} = \mathbf{W}^\mathbf{R}\mathbf{X}$, where

$$\mathbf{E}_\mathcal{S} = \begin{bmatrix} \frac{8}{25} & -\frac{4}{25} & \frac{2}{25} \\ -\frac{71}{150} & -\frac{1}{75} & -\frac{173}{300} \\ \frac{7}{30} & \frac{2}{15} & \frac{31}{60} \end{bmatrix}, \quad \mathbf{A}_\mathcal{S} = \begin{bmatrix} -\frac{16}{25} & \frac{8}{25} & -\frac{4}{25} \\ \frac{32}{25} & -\frac{16}{25} & \frac{8}{25} \\ -\frac{24}{5} & \frac{12}{5} & -\frac{21}{5} \end{bmatrix},$$

$$\mathbf{B}_\mathcal{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{2}{5} & 0 \\ -\frac{21}{20} & -\frac{1}{2} \\ \frac{11}{4} & \frac{9}{2} \end{bmatrix}, \qquad \mathbf{C}_\mathcal{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{28}{5} & -\frac{14}{5} & \frac{22}{5} \\ -\frac{22}{15} & \frac{26}{15} & \frac{22}{15} \end{bmatrix}.$$

To extract the $\mathbf{D}$-term, we proceed as follows. The pencil $(\mathbf{A}_\mathcal{S}, \mathbf{E}_\mathcal{S})$ has two finite eigenvalues at -2, 0 and an infinite eigenvalue. The matrices of the associated right and left eigenvectors are

$$\mathbf{T}_1 = \begin{bmatrix} -13 & 1 & -7 \\ -14 & 2 & -11 \\ 6 & 0 & 6 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 4 \end{bmatrix}.$$

Therefore,

$$\mathbf{T}_2\mathbf{E}_\mathcal{S}\mathbf{T}_1 = \left[\begin{array}{cc|c} -\frac{36}{25} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \mathbf{T}_2\mathbf{A}_\mathcal{S}\mathbf{T}_1 = \left[\begin{array}{cc|c} \frac{72}{25} & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -72 \end{array}\right],$$

$$\mathbf{C}_\mathcal{S}\mathbf{T}_1 = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|c} -\frac{36}{5} & 0 & 18 \\ \frac{18}{5} & 2 & 0 \end{array}\right], \quad \mathbf{T}_2\mathbf{B}_\mathcal{S} = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} \frac{2}{5} & 0 \\ -\frac{1}{4} & -\frac{1}{2} \\ \hline 8 & 16 \end{array}\right].$$

Thus, a minimal realization of order two is

$$\mathbf{E} = -\mathrm{diag}\left[\frac{36}{25}, \;\; \frac{1}{2}\right], \quad \mathbf{A} = \mathrm{diag}\left[\frac{72}{25}, \;\; 0\right], \quad \mathbf{C} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{36}{5} & 0 \\ \frac{18}{5} & 2 \end{bmatrix},$$

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{2}{5} & 0 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{D} = \frac{1}{2} \begin{bmatrix} 18 \\ 0 \end{bmatrix} \frac{1}{72} [8 \quad 16] = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

■

8.3 ▪ Reduced-order modeling from data

We are now ready to address the problem of constructing reduced-order (linear) models from measurements in the Loewner framework. More precisely, we seek to construct models that fit the set of measurements defined in (8.7), (8.8), or (8.9), (8.10).

8.3.1 ▪ Solution in the Loewner framework

We assume that data sets contain samples of the frequency response of an underlying system; for instance, in the electronics industry, measurements of the S- (scattering-)

parameters may be provided. We denote these pairs of measurements by $(\imath\omega_j, \mathbf{S}^{(j)})$, $\omega_j \in \mathbb{R}$, $\mathbf{S}^{(j)} \in \mathbb{C}^{p\times m}$, $j = 1,\ldots,N$ (i.e., $\mathbf{S}^{(j)}$ is the measurement at frequency ω_j); see also Remark 8.2. As we did in Section 8.2.2, we arrange these measurements in two arrays, a *column* and a *row* array:

$$P_c = \{(\lambda_j, \mathbf{W}_j) : j = 1,\ldots,k\}, \quad P_r = \{(\mu_i, \mathbf{V}_i) : i = 1,\ldots,q\},$$

where we have redefined

$$\left.\begin{array}{lll} \lambda_j = \imath\omega_j, & \mathbf{W}_j = \mathbf{S}^{(j)}, & j = 1,\ldots,k \\ \mu_i = \imath\omega_{k+i}, & \mathbf{V}_i = \mathbf{S}^{(k+i)}, & i = 1,\ldots,q \end{array}\right\} \quad \text{and} \quad q + k = N.$$

To obtain a real system, the given set must be closed under conjugation, i.e., in addition to the above, the pairs $(-\imath\omega_j, \overline{\mathbf{S}^{(j)}})$, $j = 1,\ldots,N$, must also belong to the set of measurements. To generate *right and left tangential data sets*, we need the right and left directions $\mathbf{r}_i \in \mathbb{R}^m$, $\boldsymbol{\ell}_j \in \mathbb{R}^p$, which can be chosen either as columns of the identity matrix or as random vectors. Thus, the right data set corresponding to (8.7) is

$$\left\{\lambda_j, \overline{\lambda}_j; \mathbf{r}_j, \mathbf{r}_j; \mathbf{w}_j = \mathbf{W}_j\mathbf{r}_j, \overline{\mathbf{w}}_j = \overline{\mathbf{W}}_j\mathbf{r}_j\right\}, \ j = 1,\ldots,k,$$

while the left data set corresponding to (8.8) is

$$\left\{\mu_i, \overline{\mu}_i; \boldsymbol{\ell}_i, \boldsymbol{\ell}_i; \mathbf{v}_i^T = \boldsymbol{\ell}_i^T\mathbf{V}_i, \overline{\mathbf{v}}_i^T = \boldsymbol{\ell}_i^T\overline{\mathbf{V}}_i\right\}, \ i = 1,\ldots,q.$$

The associated Loewner and shifted Loewner matrices constructed as in (8.28), (8.30) have dimension $2q \times 2k$. To ensure real matrix entries, we follow the procedure described in Section 8.2.4. We define

$$\mathbf{J}_r = \text{blkdiag}\left[\hat{\mathbf{J}},\ldots,\hat{\mathbf{J}}\right] \in \mathbb{C}^{2r\times 2r}, \ \hat{\mathbf{J}} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & -\imath \\ 1 & \imath \end{array}\right]$$

and transform the quantities of the Loewner framework as follows:

$$\begin{array}{lllll} \mathbf{\Lambda}^{\mathbf{R}} & = & \mathbf{J}_k^*\mathbf{\Lambda}\mathbf{J}_k, & \mathbf{R}^{\mathbf{R}} = \mathbf{R}\mathbf{J}_k, & \mathbf{W}^{\mathbf{R}} = \mathbf{W}\mathbf{J}_k, \\ \mathbf{M}^{\mathbf{R}} & = & \mathbf{J}_q^*\mathbf{M}\mathbf{J}_q, & \mathbf{L}^{\mathbf{R}} = \mathbf{J}_q^*\mathbf{L}, & \mathbf{V}^{\mathbf{R}} = \mathbf{J}_q^*\mathbf{V}, \\ \mathbb{L}^{\mathbf{R}} & = & \mathbf{J}_q^*\mathbb{L}\mathbf{J}_k, & \mathbb{L}_s^{\mathbf{R}} = \mathbf{J}_q^*\mathbb{L}_s\mathbf{J}_k. & \end{array}$$

The transformed quantities have real entries. Then, computing the two SVDs of the Loewner pencil given in (8.38) and provided that condition (8.37) is satisfied, one obtains the projection defined by $\mathbf{X} \in \mathbb{R}^{2k\times r}$, $\mathbf{Y} \in \mathbb{R}^{2q\times r}$ as in (8.40), (8.41). The quantity r is the numerical rank of the corresponding matrices in (8.38) and depends on the application. Thus,

$$\mathbf{E}_\delta = -\mathbf{Y}^T\mathbb{L}^{\mathbf{R}}\mathbf{X}, \ \mathbf{A}_\delta = -\mathbf{Y}^T\mathbb{L}_s^{\mathbf{R}}\mathbf{X}, \ \mathbf{B}_\delta = \mathbf{Y}^T\mathbf{V}^{\mathbf{R}}, \ \mathbf{C}_\delta = \mathbf{W}^{\mathbf{R}}\mathbf{X},$$

with $\mathbf{D}$ incorporated in the other matrices as an approximate reduced-order descriptor realization of the data of order r.

8.3.2 ▪ Numerical results

The Loewner algorithm described in Section 8.2.3 is applied in two cases. The first consists of an a priori given system, while the second consists of frequency response measurements exclusively. During this procedure we monitor the accuracy and CPU time required to produce an RMO. The accuracy is assessed using

- the normalized H_∞-norm of the error system, defined as

$$H_\infty \text{ error} = \frac{\max_{i=1,\dots,N} \sigma_1\left(\mathbf{H}(\imath\omega_i) - \mathbf{S}^{(i)}\right)}{\max_{i=1,\dots,N} \sigma_1\left(\mathbf{S}^{(i)}\right)}, \tag{8.45}$$

  where $\sigma_1(\cdot)$ denotes the largest singular value of $(\cdot)$, and

- the normalized H_2-norm of the error system

$$(H_2 \text{ error})^2 = \frac{\sum_{i=1}^{N} \left\|\mathbf{H}(\imath\omega_i) - \mathbf{S}^{(i)}\right\|_F^2}{\sum_{i=1}^{N} \left\|\mathbf{S}^{(i)}\right\|_F^2}, \tag{8.46}$$

  where $\|\cdot\|_F^2$ stands for the (square of the) Frobenius norm (namely the sum of the squares of the magnitude of all entries).

The former error measure evaluates the maximum deviation in the singular values of $\mathbf{H}$ on the imaginary axis, while the latter evaluates the error in the magnitude of all entries, proving to be a good estimate of the overall performance of the model. For details on these norms, we refer to [3]. The experiments were performed on a Pentium Dual-Core at 2.2 GHz with 3 GB RAM.

Example 8.40 (Low-order given system). We consider the band-stop filter described in [22]; this system has two ports (i.e., two inputs and two outputs), state-space dimension 14, and a $\mathbf{D}$-term of size 2×2 and full rank; thus, if we incorporate $\mathbf{D}$ in the other matrices as described in Remark 8.1, we obtain a minimal descriptor realization of dimension 16.[32]

We take $k = 608$ samples of the transfer function on the imaginary axis (frequency response measurements) between 10^{-3} and 10^3 rad/sec (Figure 8.2, upper pane). Figure 8.2, lower pane, shows the first 30 normalized singular values of the resulting Loewner and shifted Loewner matrices (the rest are zero). The Loewner matrix has rank 14, while the shifted Loewner matrix has rank 16, so, based on the drop in singular values, we generate models of order 16 with $\mathbf{D} = 0$. Table 8.1 shows the CPU time and the errors for the resulting models. Two algorithms are compared: the complex SVD approach, where the Loewner pencil is left complex, followed by the real SVD approach, where the Loewner pencil is transformed to real form, as explained earlier.
This example shows that the Loewner approach recovers the original system with small error. Furthermore, it also shows that working in real arithmetic is more expensive timewise than working in complex arithmetic. ■

[32] A band-stop filter is a dynamical system that blocks signals with frequency in a given interval while it lets through, almost unchanged, signals with frequency outside the given interval. Such systems are characterized by a frequency response that is nearly zero in the given interval of frequencies while it is almost equal to one otherwise. The upper pane of Figure 8.2 shows that the system under consideration exhibits such a behavior between the first input and the second output, or vice versa (red curve).

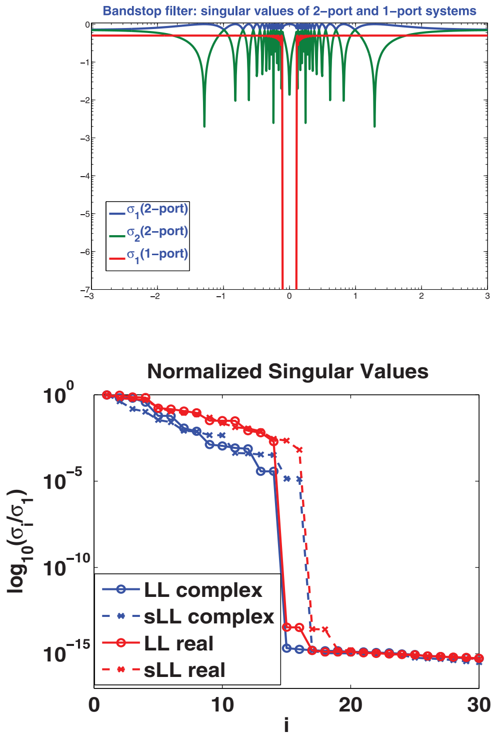

Figure 8.2. *Upper pane: the blue and green curves are the two singular values of the transfer function as a function of frequency; the red curve is the singular value (magnitude) of the* $(1,2)$ *entry of the transfer function, which exhibits band-stop behavior. Lower pane: singular values of the Loewner and the shifted Loewner matrices in both complex (blue curves) and real (red curves) form.*

Table 8.1. *Results for* $k = 608$ *noise-free measurements of an order-*14 *system with* $p = 2$ *ports (two inputs, two outputs).*

Algorithm	CPU (s)	H_∞-error	H_2-error
SVD Complex	0.88	1.3937e—010	2,4269e—011
SVD Real	1.82	1.3146e—012	3.0687e—013

Example 8.41 (Model reduction from measured data). Next, we apply the Loewner framework to a set of measured data containing $k = 100$ S-parameter measurements of an electronic device with $p = 50$ ports (50 inputs and 50 outputs). Measurements were performed using a VNA and were provided by CST AG. The accuracy of the models obtained was assessed by means of the norms (8.45) and (8.46). The frequency of the 100 measurements (100 matrices of size 50×50) ranges between 10 MHz and 1 GHz. For better conditioning, all frequencies were scaled by 10^{-6}.

MIMO case: Figure 8.3 shows the normalized singular values of the Loewner and shifted Loewner matrices constructed using all measurements in the MIMO case. The tangential directions have been chosen as columns of the identity matrix. According to the algorithm of Section 8.2.3, the drop in the singular values suggests that there is an underlying **D**-term. The singular values of the Loewner matrix decay about three

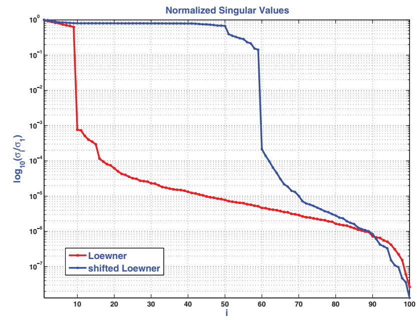

Figure 8.3. *Singular value drop of the Loewner and shifted Loewner matrices.*

Table 8.2. *Results for a device with $p = 50$ ports (50 inputs, 50 outputs).*

Algorithm	CPU time (sec)	$\mathcal{H}_\infty$-error	$\mathcal{H}_2$-error
Loewner ($n = 59$ with $\mathbf{D} = 0$)	0.05	5.3326e–3	4.5556e–4
Loewner ($n = 9$ with $\mathbf{D} \neq 0$)	0.15	6.1575e–3	5.9957e–4

orders of magnitude between the 9th and 10th, while the singular values of the shifted Loewner matrix decay several orders of magnitude between the 59th and 60th. The system can be decomposed into a strictly proper part of order nine plus a direct feed-through term $\mathbf{D}$ of size 50×50 and full rank.

Table 8.2 presents the errors for two models constructed with the Loewner approach: a model of order $n = 59$ obtained by projecting the Loewner pencil onto the subspace of singular vectors associated with the 59 dominant singular values of the Loewner pencil and a model of order $n = 9$ obtained after extracting the $\mathbf{D}$ matrix. The extraction of the $\mathbf{D}$ matrix is realized in two steps. First, the strictly proper part is extracted by projecting the $(\mathbf{A}, \mathbf{E})$ matrix pencil of the model onto the eigenvectors associated with eigenvalues with small magnitude. Subsequently, the $\mathbf{D}$ matrix is determined as the mean over all frequencies of the difference between the measurement and the evaluation of the strictly proper model at each frequency. The model obtained with the Loewner approach is shown in Figure 8.4. It shows small discrepancies in approximating the lowest singular values of the system function (due to the small scale on the y-axis).

It should be mentioned that state-of-the-art approaches are very expensive for such a large number of inputs and outputs. However, our approach is especially suited for such problems since the resulting Loewner matrices only depend on the number of measurements and not the number of ports. Finally, the very low CPU time should be stressed.

SISO case: Finally, we focus on the $(1,31)$ entry of the S-parameter matrix. As in the MIMO case, the drop in the singular values of the Loewner pencil (see Figure 8.5) shows that the data suggest a model of order four, together with a $\mathbf{D}$-term. Knowing the order of the system, we construct models of order five with $\mathbf{D} = 0$, as well as of order four with $\mathbf{D} \neq 0$. The errors of the constructed models are summarized in Table 8.3. The accuracy for the $\mathbf{D} \neq 0$ case deteriorates slightly because $\mathbf{D}$ was constructed by averaging over all frequencies. ■

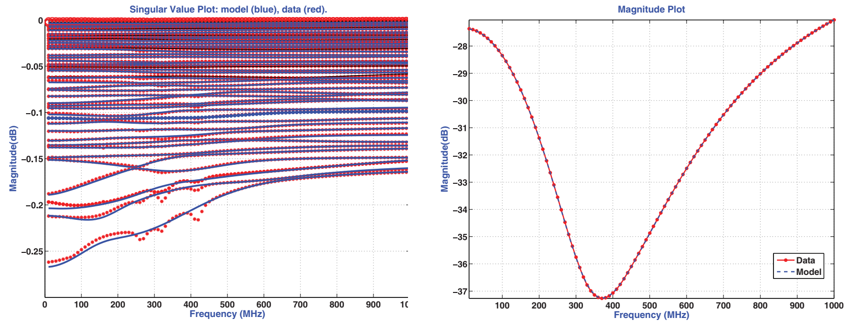

Figure 8.4. *Left pane: fit between the data and the* 50 *singular values (the device has* 50 *ports) of the transfer function of the model constructed from* $k = 100$ *samples. Right pane: data versus model for the* $(1,31)$ *entry.*

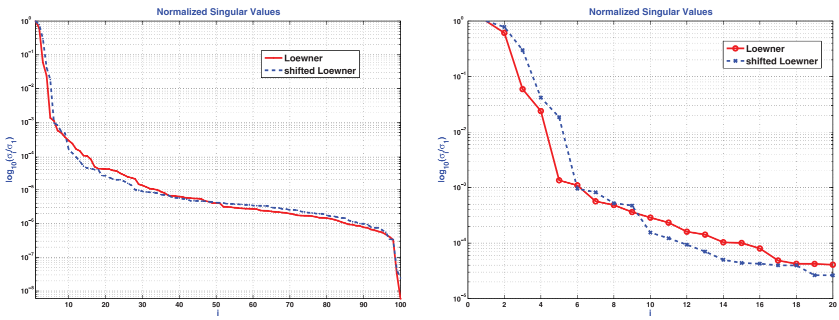

Figure 8.5. *Left pane: the singular value drop of the Loewner matrix pencil for the* $(1,31)$ *entry. Right pane: detail of left pane plot.*

Table 8.3. *Errors for a (scalar) model fitting the* $(1,31)$ *entry of the device with* $p = 50$ *ports (*50 *inputs and* 50 *outputs).*

Algorithm	Order 5, $\mathbf{D} = 0$		Order 4, $\mathbf{D} \neq 0$	
	$\mathcal{H}_\infty$-error	$\mathcal{H}_2$-error	$\mathcal{H}_\infty$-error	$\mathcal{H}_2$-error
Loewner	1.5128e−3	8.7226e−4	8.7645e−3	6.4048e−3

8.4 ▪ Summary

In this chapter we have presented a model reduction approach from data, be it measured (e.g., S-parameters) or computed by DNS. The main tool is the *Loewner pencil*, which is constructed exclusively from data. Its determination requires no computations like matrix factorizations or inversions and hence constitutes a natural approach to the problem of data-driven modeling. Furthermore, the singular values of the Loewner pencil provide a trade-off between accuracy of fit and model complexity. Here is a summary of the main features.

- Given input/output data, we can construct, with *no computation*, a singular high-order model in generalized state-space or descriptor form.

- In applications, the singular pencil $(\mathbb{L}_s, \mathbb{L})$ must be reduced to obtain a minimal- or low-order model.
- The approach is a *natural way* to construct full and reduced models because it
  - does not *force* the inversion of $\mathbf{E}$,
  - can deal with many inputs and outputs.
- In this framework, the *singular values of* $\mathbb{L}$, $\mathbb{L}_s$ offer a *trade-off between accuracy of fit and complexity of the reduced system*. The resulting (a posteriori computable) error is proportional to the first neglected singular value of $\mathbb{L}$.
- The Loewner framework has been extended to
  - *multiple-point (Hermite) interpolation* (i.e., values and derivatives at various points are provided) [9, 33];
  - *recursive modeling* (i.e., data become available successively) [29, 30];
  - *linear parametrized systems* [9, 24, 25]; and
  - *bilinear systems* by means of *generalized Loewner pencils* [23].
- The underlying philosophy is therefore to *collect data and extract the desired information*.

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