

Linear dynamical system identification

... basic elements and Labs guidelines

Charles Poussot-Vassal
February 19, 2026

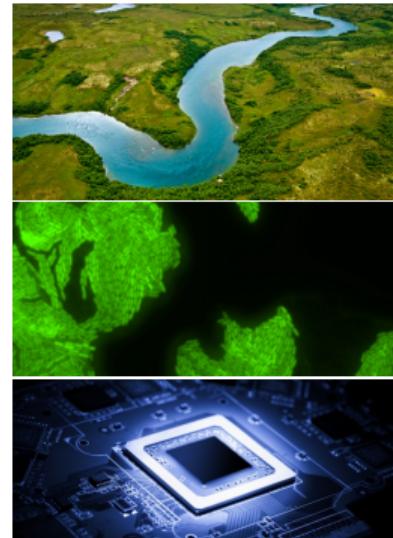


Introduction

Dynamical models and identification, what for?

Dynamical models are central tools in (control) engineering

- ▶ **for verification and validation**
(μ , \mathcal{H}_∞ -norm, pseudo-spectra, Monte Carlo)
- ▶ **for detection**
(fault isolation, param. & variables estim.)
- ▶ **for uncertainty propagation & detection**
(Multi Disc. Optim., robust optim.)
- ▶ **for feedback control synthesis**
($\mathcal{H}_\infty/\mathcal{H}_2$ -norm, MPC, adaptive)
- ▶ **for system optimization**
(many-query simulation-based)



Introduction

Motivations (antenna and telecommunication problem)

Antenna, cellular...

Antenna models

- ▶ to optimize parameters
- ▶ for polar computation

Blend physics from

- ▶ Maxwell equations
- ▶ Kirchoff equations
 - ▶ Replace costly simulations by accurate simple model
 - ▶ Preserve structure and properties (port-Hamiltonian)
 - ▶ Allows for geometry optimization

 M. Gouzien, C. P-V., G. Haine and D. Matignon, "[A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations](#)", journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Introduction

Motivations (Dassault-Aviation ground vibration experiments)

Aircraft models used

- ▶ for control design
- ▶ for life-cycle monitoring

Physics involved

- ▶ structure
- ▶ sensing

Raw data vs. Reduced model

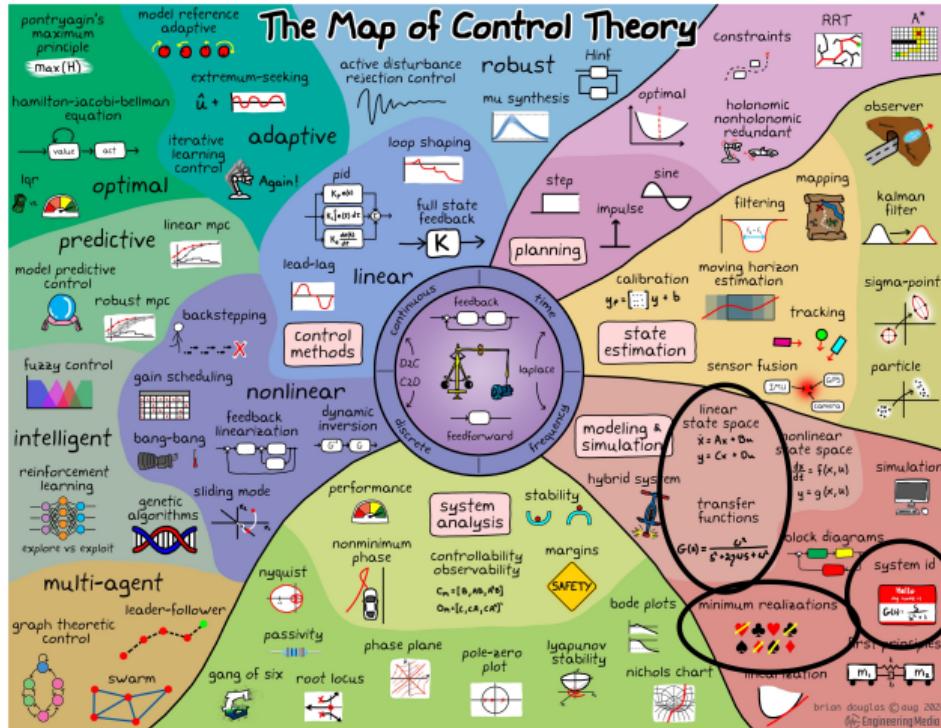
- ▶ Digital twin of the real aircraft with ≈ 100 outputs
- ▶ Process employed in ground & flight tests



C. P-V. and P. Vuillemin, "[MDSPACK](#)", MOR Digital Systems.

Introduction

The map of control theory (by Brian Douglas - <https://engineeringmedia.com/>)



Introduction

Course plan, overview and questions addressed

L1: Overview, signals construction, pre-treatment and non-parametric analysis

- ▶ Realization, transfer functions, ...
- ▶ Construct an experimental plan and signals.
- ▶ How to analyze signals, and derive some properties?

L2: Data-driven model construction in the time- and frequency-domain

- ▶ Construct a linear model from time- or frequency-domain data.
- ▶ How much is it valid? How to validate, discuss, amend it?

L3: L2 cont'd & Labs guidelines

- ▶ Illustration in practice
- ▶ Experimental setup & numerical tools presentation.
- ▶ Methodology for the lab.

Introduction

Lab sessions

TP: autonomous labs

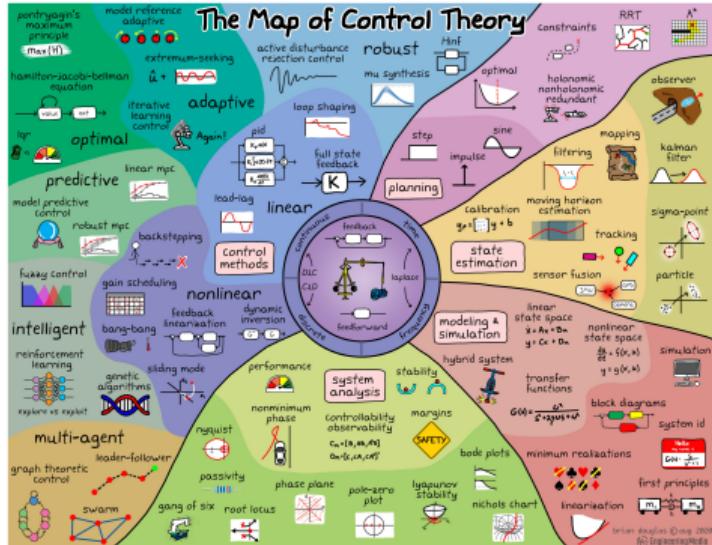
- ▶ 5 sessions, 2h45 each
- ▶ Group of 3 students

Motto

Explore dyn. systems & control engineering

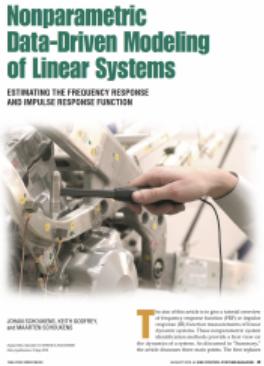
- ▶ Identification
Modeling
Analysis
Control
Performance optimisation
- ▶ Team group
Team work & presentation

The map of control theory, by Brian Douglas
<https://engineeringmedia.com/>



Introduction

References and material



<p>Chapter 8 A Tutorial Introduction to the Loewner Framework for Model Reduction</p> <p>Ashwini C. Antoulas, Sanda Lefteriu, and A. Constant Jea^{a,b}</p> <p>One of the main approaches to model reduction of both linear and nonlinear dynamical systems is by means of interpolation. Interpolatory model reduction methods are based on the Loewner framework. This chapter provides a tutorial introduction to the Loewner framework and its various variants applied to model reduction of linear and nonlinear systems. The emphasis is on the Loewner framework and its variants, and the reader is referred to other books for more details on the theory of model reduction.</p> <p>8.1 Introduction Model reduction techniques for large-scale systems have become a fundamental tool in the design and analysis of control systems. The need for such models arises from the complexity of the system under consideration. Models of this type are often used to predict the behavior of the system in such cases where exact mathematical analysis is difficult or impossible. In general, the model is derived from a set of differential equations that describe the system's behavior. These equations are often too complex to be solved exactly, so numerical methods are used to approximate the solution. Generally, large-scale systems are used in many applications, on the spatial dimension, time dimension, and frequency dimension. The Loewner framework is one of the most popular approaches for the approximation of distributed circuit elements. For some applications, the Loewner framework is also used to approximate the system's behavior.</p> <p>^aWith partial support to the authors for the model testing at the Institute of Fluid Mechanics of the TU Wien. ^bWork of A.C. Antoulas supported by the grants P09-01705 and F09-01904 as well as CRC 111.</p>	<p>Chapter 15 Data-driven modeling and control of large-scale dynamical systems in the Loewner framework</p> <p>Methodology and applications</p> <p>Ionut Gosea^{a,*}, Charles Poussot-Vassal^{a,b}, and Alessandro C. Antoulas^{a,c,d}</p> <p>Received 10 January 2018; revised 10 April 2019; accepted 10 April 2019 Handling Editor: Thomas Meurer Editorial review by: Michael Hinze, Christian Engwer, and Barbara Wohlmuth Published online 14 May 2019 © 2019 Society for Industrial and Applied Mathematics *Corresponding author. Email address: cpoussot@onera.fr ^aONERA, Paris, France ^bMOR Digital Systems, Paris, France ^cUniversity of Texas at Austin, Austin, TX, USA ^dUniversity of Michigan, Ann Arbor, MI, USA</p> <p>Contents</p> <table border="0"><tr><td>1. Introduction and motivation</td><td>389</td></tr><tr><td>2. The Loewner framework</td><td>393</td></tr><tr><td>3. Data-driven modeling and control</td><td>393</td></tr><tr><td>4. Examples</td><td>393</td></tr><tr><td>5. Conclusions</td><td>394</td></tr><tr><td>6. Acknowledgments</td><td>394</td></tr><tr><td>7. References</td><td>397</td></tr><tr><td>8. Appendix: Loewner framework for time-varying systems</td><td>399</td></tr><tr><td>9. Model order reduction</td><td>401</td></tr><tr><td>10. Identification</td><td>401</td></tr><tr><td>11. Control and optimization</td><td>401</td></tr><tr><td>12. Conclusion</td><td>402</td></tr><tr><td>13. Summary and conclusions</td><td>402</td></tr><tr><td>14. References</td><td>402</td></tr></table>	1. Introduction and motivation	389	2. The Loewner framework	393	3. Data-driven modeling and control	393	4. Examples	393	5. Conclusions	394	6. Acknowledgments	394	7. References	397	8. Appendix: Loewner framework for time-varying systems	399	9. Model order reduction	401	10. Identification	401	11. Control and optimization	401	12. Conclusion	402	13. Summary and conclusions	402	14. References	402
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<https://github.com/cpoussot/insapack>

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- J. Schoukens, K. Godfrey, M. Schoukens, "*Nonparametric data-driven modeling of linear systems*", IEEE Control System Magazine, 2018.
- A.C. Antoulas, A.C. Ionita, S. Lefteriu, "*A Tutorial Introduction to the Loewner Framework for Model Reduction*", Model Reduction and Approximation, Ch. 8.
- I.V. Gosea, C. P-V., A.C. Antoulas, "*Data-driven modeling and control of large-scale dynamical systems in the Loewner framework*", Handbook on Numerical Analysis, Ch. 11.
- Signals
- Non-parametric analysis
- Identification TD
- Identification FD
- Examples
- Codes

L1: Overview, signals construction, pre-treatment and non-parametric analysis

- ▶ Realization, transfer functions, ...
- ▶ Construct an experimental plan and signals.
- ▶ How to analyze signals, and derive some properties?

Notions treated

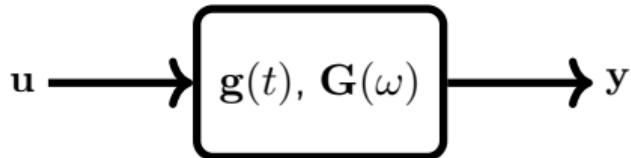
- ▶ Linear models: Transfer Function (TF), realization
- ▶ Observability, controllability, Hankel, minimality
- ▶ Fourier Transform (FT), Frequency Response Function (FRF), Impulse Response (IR)
- ▶ Measurement setup
- ▶ Binary sequence, periodic signals
- ▶ Averaging, mean, variance

Reminder on linear dynamical models

Presentation and notations

Model

(Internal representation - time-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$
(External representation - frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



(time-domain) $\{t_k, \mathbf{u}(t_k), \mathbf{y}(t_k)\}_{k=1}^N$
(frequency-domain) $\{\omega_k, \mathbf{U}(\omega_k), \mathbf{Y}(\omega_k)\}_{k=1}^N$

Data

Notations

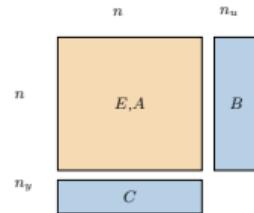
$\mathbf{u}(t_k)$ and $\mathbf{y}(t_k)$
 $\mathbf{U}(\omega_k)$ and $\mathbf{Y}(\omega_k)$
where $k = 1, \dots, N$

$$\begin{cases} t_k &= kT_s \\ \omega_k &= 2\pi f_k \\ f_k &= kf_0 = k\frac{f_s}{N} \\ f_s &= \frac{1}{T_s} \end{cases}$$

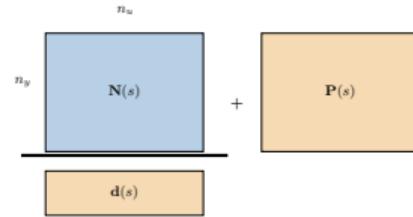
- ▶ f_0 : frequency resolution
- ▶ N : number of samples

Reminder on linear dynamical models

Realization and transfer function (generic form)



$$\mathbf{H} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$



$$\begin{aligned}\mathbf{H}(s) &= C(sE - A)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s)\end{aligned}$$

Realization

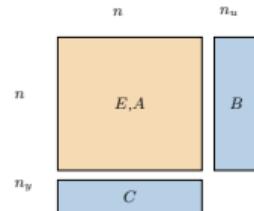
- E, A, B, C are matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Infinite number of realizations
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
 $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
 $\mathbf{x}(t) \in \mathbb{R}^n$

Transfer function

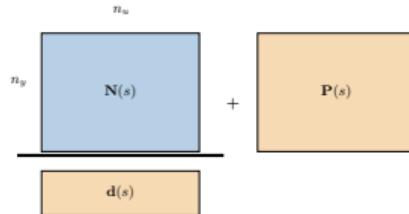
- \mathbf{H} is a complex function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Unique function
- $\mathbf{U}(s) \in \mathbb{C}^{n_u}$,
 $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

Reminder on linear dynamical models

Realization and transfer function (generic form)



$$\mathbf{H} : \begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{cases}$$



$$\begin{aligned}\mathbf{H}(s) &= C(sE - A)^{-1}B \\ &= \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s)\end{aligned}$$

Realization

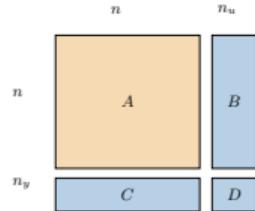
- ▶ E, A, B, C are matrices
- ▶ Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- ▶ Infinite number of realizations
- ▶ $\mathbf{u}(t) \in \mathbb{R}^{n_u},$
 $\mathbf{y}(t) \in \mathbb{R}^{n_y},$
 $\mathbf{x}(t) \in \mathbb{R}^n$

Transfer function

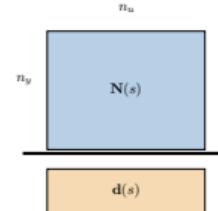
- ▶ \mathbf{H} is a complex function
- ▶ External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- ▶ Unique function
- ▶ $\mathbf{U}(s) \in \mathbb{C}^{n_u},$
 $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

Reminder on linear dynamical models

Realization and transfer function (ODE & bounded)



$$\mathbf{H} : \begin{cases} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{cases}$$



$$\begin{aligned} \mathbf{H}(s) &= C(sE - A)^{-1}B + D \\ &= \mathbf{N}(s)/\mathbf{d}(s) \end{aligned}$$

Realization

- A, B, C, D are **real** matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Infinite number of realizations
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$,
 $\mathbf{y}(t) \in \mathbb{R}^{n_y}$,
 $\mathbf{x}(t) \in \mathbb{R}^n$

Transfer function

- \mathbf{H} is a **rational** complex function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Unique function
- $\mathbf{U}(s) \in \mathbb{C}^{n_u}$,
 $\mathbf{Y}(s) \in \mathbb{C}^{n_y}$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s + 1}$$

ODE realization \mathbf{H}

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x\end{aligned}$$

Singularities λ and zeros z

$$\begin{aligned}\lambda_{\mathbf{H}} &= \text{eig}(\mathbf{A}, \mathbf{E}) \\ &= \Lambda(-1, 1) \\ &= \{-1\}\end{aligned}$$

$$\begin{aligned}z_{\mathbf{H}} &= \text{eig}([\mathbf{A} \ \mathbf{B}; \ \mathbf{C} \ \mathbf{D}], \text{blkdiag}(\mathbf{E}, \text{zeros}(ny, nu))) \\ &= \Lambda\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\ &= \{\infty, \infty\}\end{aligned}$$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization \mathbf{H}_1

$$\begin{aligned}\dot{x} &= -x + 2u \\ y &= x + 2u\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathbf{H}_1} = \Lambda(-1, 1) = \{-1\}$$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization \mathbf{H}_2

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2u \\ 0 &= -x_2 + 2u = x_2 - 2u \\ y &= x_1 + x_2\end{aligned}$$

Singularities of matrix pencil $(A, E)^a$

$$\lambda_{\mathbf{H}_2} = \Lambda \left(\left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & \\ \hline & 1 \end{array} \right] \right) = \{-1, \infty\}$$

$$^a B^\top = \left[\begin{array}{cc} 2 & -2 \end{array} \right] \text{ and } C = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization \mathbf{H}_2 (canonical form)

$$\left(\left[\begin{array}{c|c} A_1 = -1 & \\ \hline I_{n_2} = 1 & \end{array} \right], \left[\begin{array}{c|c} I_{n_1} = 1 & \\ \hline N = 0 & \end{array} \right] \right)$$

Index is the k -nilpotent degree of N

- ▶ Finite dynamic modes
 $n_1 = 1$
- ▶ Infinite dynamic (impulsive) modes
 $\text{rank } (E) - n_1 = 0$
- ▶ Non dynamic modes
 $n - \text{rank } (E) = 1$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2 + s + 2}{s+1}$$

DAE index-2 realization \mathbf{H}

$$\begin{aligned}\dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ x_2 &= -x_3 + u = x_3 - u \\ y &= x_1 + x_2 + 2x_3\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathbf{H}} = \{-1, \infty, \infty\}$$

Finite / impulsive & (non) dynamic modes $n_1 = 1$,
rank $(E) - n_1 = 1$, $n - \text{rank } (E) = 1$

Reminder on linear dynamical models

Finite dimensional... some differences

Structures

L-ODE

L-ODE / DAE-1

L-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2 + s + 2}{s+1}$$

DAE index-2 realization \mathbf{H}

$$\begin{aligned}\dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ x_2 &= -x_3 + u = x_3 - u \\ y &= x_1 + x_2 + 2x_3\end{aligned}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathbf{H}} = \{-1, \infty, \infty\}$$

Finite / impulsive & (non) dynamic modes $n_1 = 1$,
 $\text{rank } (E) - n_1 = 1$, $n - \text{rank } (E) = 1$

Reminder on linear dynamical models

Observability, controllability, minimality...

Structures

L-ODE

L-ODE / DAE-1

L-DAE

- Reachability matrix

$$\mathcal{R}_n(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

- Observability matrix

$$\mathcal{O}_n(A, C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2 C^\top & \cdots & (A^\top)^n C^\top \end{bmatrix}^\top$$

- Minimality: both controllable and observable.
- Connection with Markov parameters

$$H_0 = D, H_k = CA^{k-1}B \ (k \geq 1)$$

$$\mathcal{H}_n = \mathcal{O}_n \mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

Reminder on linear dynamical models

Example #1 (with D -term)

Continuous-time

$$\mathbf{H}(s) = \frac{2s + 4}{s + 1}$$

$$\mathcal{S} : \begin{cases} \dot{x}(t) &= -x(t) + 2u(t) \\ y(t) &= x(t) + 2u(t) \end{cases}$$

Sampled-time

$$\mathbf{H}(z) = \frac{2z - 0.4261}{z - 0.6065}$$

$$\mathcal{S} : \begin{cases} x(k+1) &= 0.6065x(k) + u(k) \\ y(k) &= 0.7869x(k) + 2u(k) \end{cases}$$

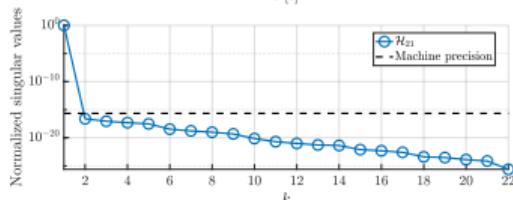
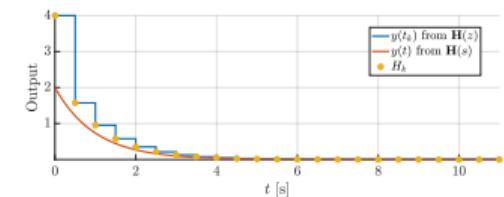
Reminder on linear dynamical models

Example #1 (with D -term)

Continuous-time

$$\mathbf{H}(s) = \frac{2s + 4}{s + 1}$$

$$\mathcal{S} : \begin{cases} \dot{x}(t) &= -x(t) + 2u(t) \\ y(t) &= x(t) + 2u(t) \end{cases}$$



Sampled-time

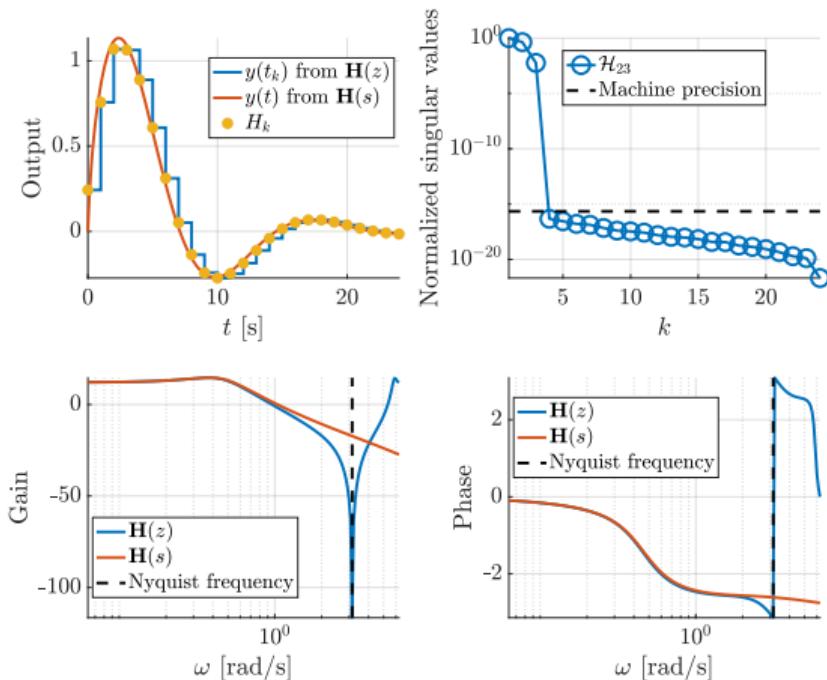
$$\mathbf{H}(z) = \frac{2z - 0.4261}{z - 0.6065}$$

$$\mathcal{S} : \begin{cases} x(k+1) &= 0.6065x(k) + u(k) \\ y(k) &= 0.7869x(k) + 2u(k) \end{cases}$$

- ▶ $T_s = 0.5$, 'zoh'
- ▶ $n = 1$
- ▶ $\text{rank } (\mathcal{O}_{n+20}) = 1$
- ▶ $\text{rank } (\mathcal{R}_{n+20}) = 1$
- ▶ $\text{rank } (\mathcal{H}_{n+20}) = 1$

Reminder on linear dynamical models

Example #2 (higher order)



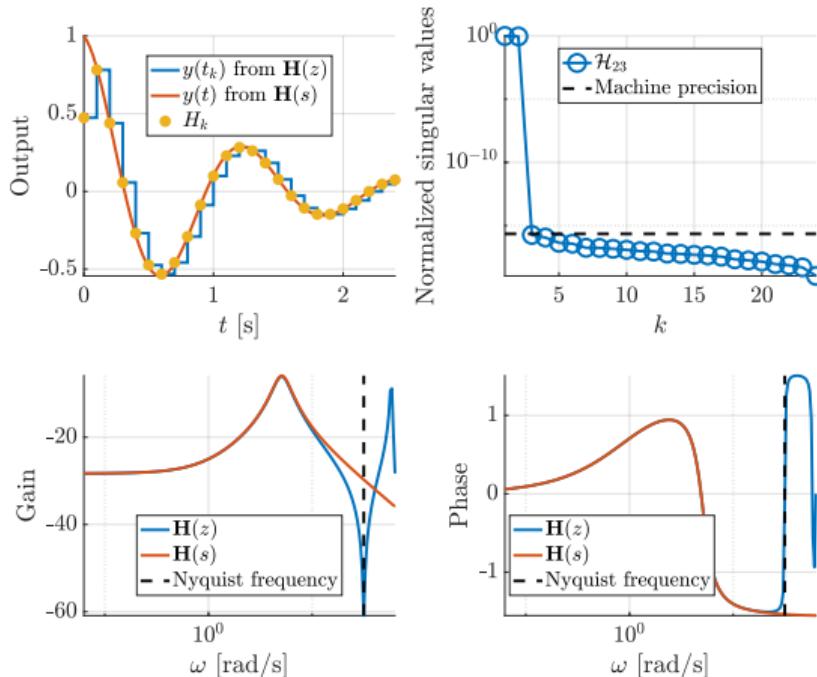
$$\mathbf{H}(s) = \frac{2s + 4}{s^3 + 5s^2 + 2s + 1}$$

- ▶ $T_s = 1$, 'tustin'
- ▶ $n = 3$

- ▶ $\text{rank } (\mathcal{O}_{n+20}) = 3$
- ▶ $\text{rank } (\mathcal{R}_{n+20}) = 3$
- ▶ $\text{rank } (\mathcal{H}_{n+20}) = 3$

Reminder on linear dynamical models

Example #2 (partially observable & minimality)



$$A = \begin{bmatrix} -1 & 5 \\ -5 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B, C^\top = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad -3$$

- ▶ $T_s = 0.1$, 'foh'
- ▶ $n = 3$
- ▶ $\text{rank } (\mathcal{O}_{n+20}) = 2$
- ▶ $\text{rank } (\mathcal{R}_{n+20}) = 3$
- ▶ $\text{rank } (\mathcal{H}_{n+20}) = 2$

Practical estimation from data

Impulse response estimation (g)

Time-domain response

The response of the dynamical system is:

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{g}(\tau) \mathbf{u}(t - \tau) d\tau = \mathbf{g}(t) * \mathbf{u}(t).$$

Estimate IR, $\mathbf{g}(t)$

- ▶ Solve the deconvolution problem, or
- ▶ Use cross-correlation functions

$$\begin{aligned} R_{yu}(\tau) &= \mathbf{g}(\tau) * R_{uu}(\tau) \\ &= \mathbf{g}(\tau) \sigma_u^2 \end{aligned}$$

with white random noise

Markov parameters and IR

$$\mathbf{y}(t_k) = \begin{cases} D & \text{if } k = 0 \\ CA^{k-1}B & \text{if } k > 0 \end{cases}$$

Cross-correlation

$$\begin{aligned} R_{yu}(\tau) &= E\{\mathbf{y}(t + \tau)\mathbf{u}(t)\} \\ R_{uu}(\tau) &= E\{\mathbf{u}(t + \tau)\mathbf{u}(t)\} \end{aligned}$$

Practical estimation from data

Frequency response estimation (G)

Frequency-domain response

The response of the dynamical system is:

$$\mathbf{Y}(k) = \mathbf{G}(k)\mathbf{U}(k)$$

Estimate FRF, $\mathbf{G}(k)$

- ▶ Apply (fails when \mathbf{U} close to zero)

$$\mathbf{G}(k) = \mathbf{Y}(k)/\mathbf{U}(k)$$

- ▶ Use cross-correlation spectrum functions

$$S_{\mathbf{yu}}(k) = \mathcal{F}(R_{\mathbf{yu}}) \text{ and } S_{\mathbf{uu}}(k) = \mathcal{F}(R_{\mathbf{uu}})$$

$$\mathbf{G}(k) = \frac{S_{\mathbf{yu}}(k)}{S_{\mathbf{uu}}(k)}$$

Fourier transform

Let $\mathbf{x}(t)$, $t = 0, \dots, N - 1$

$$\begin{aligned}\mathbf{X}(k) &= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \mathbf{x}(t) e^{-i2\pi tk/N} \\ \mathbf{x}(t) &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{X}(t) e^{i2\pi tk/N}\end{aligned}$$

$\mathbf{X}(k)$ is the Fourier coefficient of $\mathbf{x}(t)$ and frequency $f_k = kf_s/N$ ($f_s = 1/T_s$).

- ▶ FFT more accurate with power of 2

Measurement setup

From generator to measurement

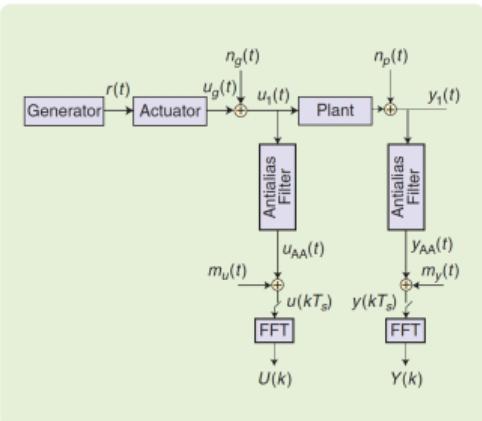


FIGURE 2 Measurement setup and notations. The generator signal is disturbed by generator noise $n_g(t)$, and the output of the system is disturbed by the process noise $n_p(t)$. The measured input and output signals are first low-pass filtered by the antialias filters. The measurement noise on the input and output is, respectively, $m_u(t)$ and $m_y(t)$. These signals are sampled at rate $f_s = 1/T_s$. The discrete Fourier transform of the measurements $u(kT_s), y(kT_s), k = 1, \dots, N$ is calculated using the fast Fourier transform (FFT) algorithm, and it is denoted as $U(k), Y(k)$. The frequency index k indicates the frequency kf_s/N .

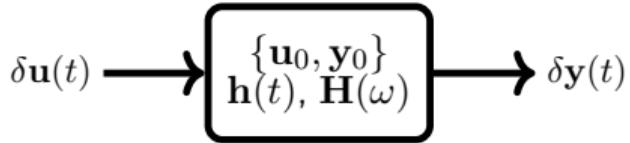
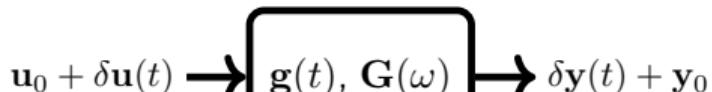
- ▶ Store:
 - $r(t)$: generated signal
 - $u_g(t)$: effectively sent signal
 - $u(t)$: plant input $y(t)$: plant output
- ▶ FFT: more accurate when signals of length $N = 2^k$
- ▶ Discuss with experimental team about AAF
 - IIR, FIR, delay, cut-off frequency?
- ▶ Discuss with experimental team about noises
 - knowledge on input/output?



J. Schoukens, K. Godfrey, M. Schoukens, "Nonparametric data-driven modeling of linear systems", IEEE Control System Magazine, 2018.

Measurement setup

Working point



The practical setup

A system evolves

- ▶ around an equilibrium point
- ▶ along trajectories

The theoretical model

A linear model \mathbf{H} is valid

- ▶ around an equilibrium point

$$\{\mathbf{u}_0, \mathbf{y}_0\}$$

- ▶ along trajectories^a

$$\{\mathbf{u}(t, p), \mathbf{y}(t, p)\}$$

^a t : time, p : parameter

Experiment design: choice of the exciting signal

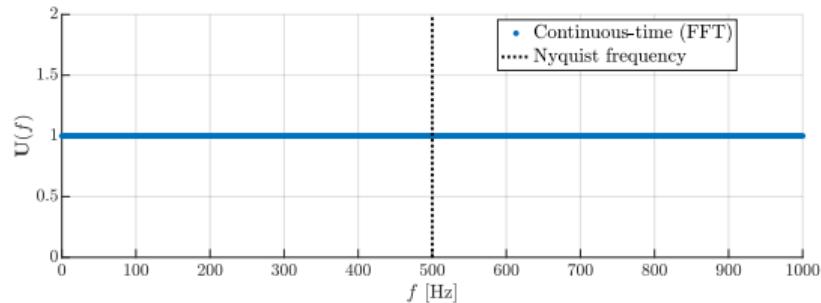
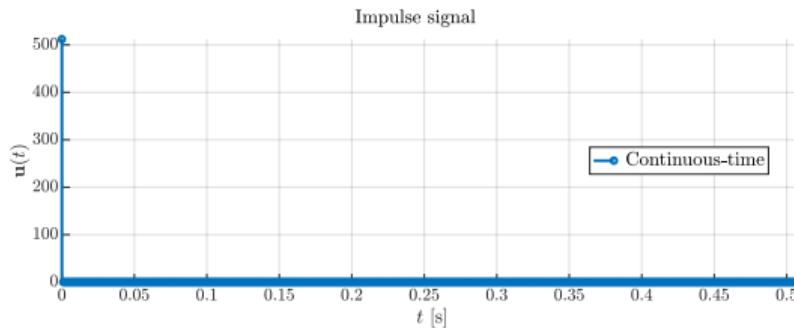
Impulse signals

$$u(t) = \begin{cases} N_s & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

where, $N_s = 2^9$

Characteristics

- + White noise
- Not easily applicable in practice



Experiment design: choice of the exciting signal

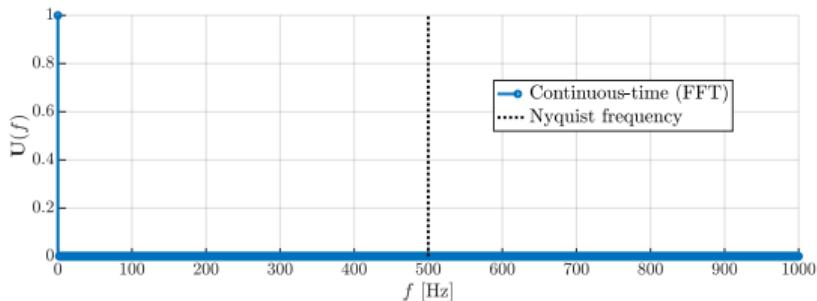
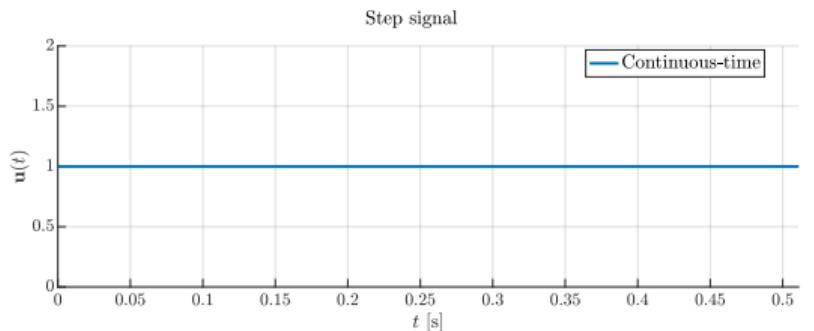
Step signals

$$u(t) = 1$$

where, $N_s = 2^9$

Characteristics

- + Easily applicable in practice
- Spectrum not rich



Experiment design: choice of the exciting signal

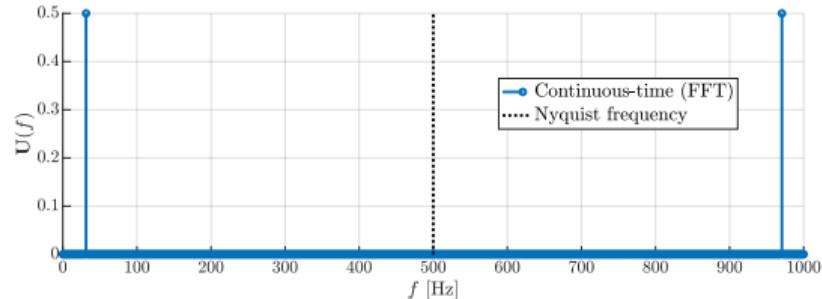
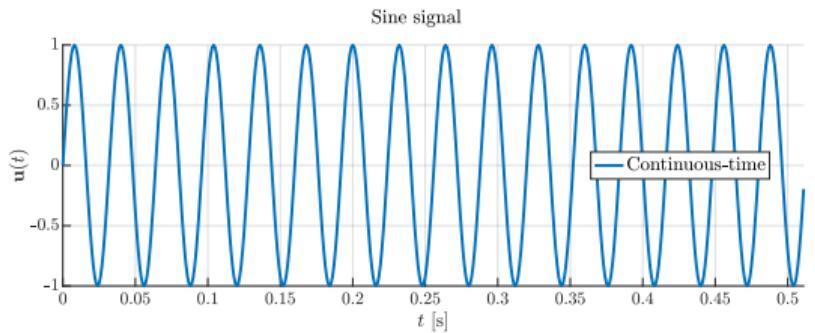
Sine signals

$$\mathbf{u}(t) = \sin(2\pi f_1 t)$$

where, $N_s = 2^9$

Characteristics

- + Easily applicable in practice
- + Provides local information
- Spectrum not rich



Experiment design: choice of the exciting signal

Maximum Length Binary Sequence signals

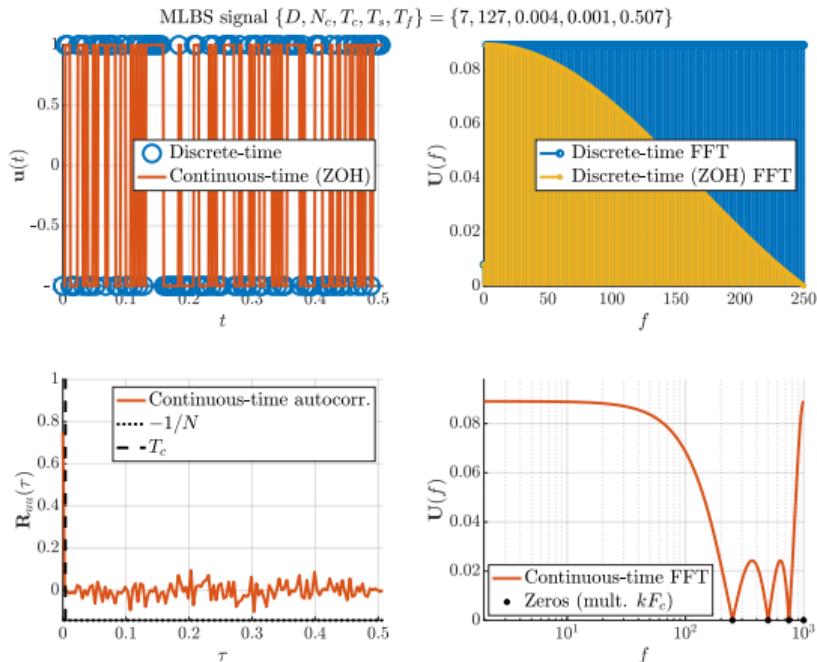
$$\mathbf{u}(t) = \{-1, +1\}$$

$$N_s = 2^9 \Rightarrow$$

$$f_c = 4f_{max} \text{ & } N_s = 508$$

Characteristics

- + Easily applicable in practice
- + Provides rich information
- Sensitive to nonlinear distortions (this can be attenuated by randomized realization or inverting $\{+1, -1\}$)
- ▶ `insapack.mlbs`



Experiment design: choice of the exciting signal

Maximum Length Binary Sequence signals

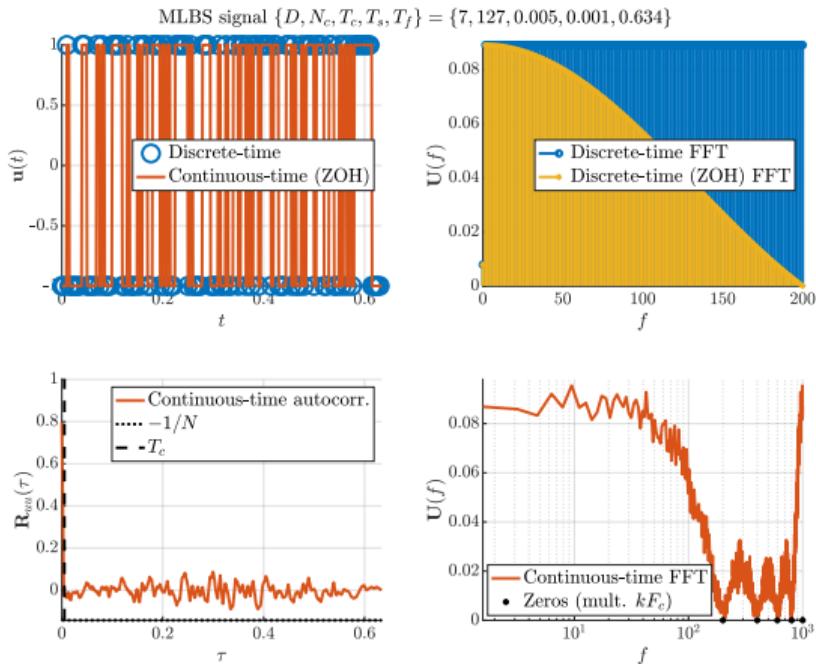
$$\mathbf{u}(t) = \{-1, +1\}$$

$$N_s = 2^9 \Rightarrow$$

$$f_c = 5f_{max} \text{ & } N_s = 635$$

Guidelines

- ▶ Use $f_c = \frac{1}{T_c} > 2.5f_{max}$ to obtain a sufficiently flat amplitude in the frequency range
- ▶ Signal length N_s , such that resolution $f_0 = \frac{1}{N_s}$ meets requirements
- ▶ Do not modify by zero padding, instead use generalized FFT should be used



Experiment design: choice of the exciting signal

Maximum Length Binary Sequence signals (insapack.mlbs)

```
1 % Description
2 % Computes a MLBS signal over FBND frequency range.
3 %
4 % Syntax
5 % [uc,tc,info] = insapack.mlbs(Ns,Ts,FBND,REV,SHOW)
6 %
7 % Input arguments
8 % - Ns : number of samples - estimated (integer)
9 %         Ns may be modified to ensure the number of MLBS cell generating
10 %        is a  $2^D-1$  sequence, where  $D = \text{round}(\log_2(Ns \cdot Ts / Tc))$ 
11 % - Ts : sampling period [s] (>0 real)
12 % - FBND : frequency range [Hz] (2x1 real vector)
13 % - REV : reverse signal (boolean)
14 %         - false: fmin to max
15 %         - true: fmax to fmin
16 % - SHOW : plot signal (boolean)
17 %
18 % Output arguments
19 % - uc : chirp signal in {-1,1}
20 % - tc : time samples
21 % - info : additional information
22 %
23 % See also for the Polynomial feedback sequence and explanations on MLBS
24 % and white noise generation:
25 % https://www.digikey.fr/fr/articles/techzone/2018/mar/use-ready-available-components-generate-binary-sequence
```

Experiment design: choice of the exciting signal

Chirp signals

$$\mathbf{u}(t) = A \sin \left(2\pi \left(\frac{\beta t^{1+p}}{1+p} + f_0 t \right) \right)$$

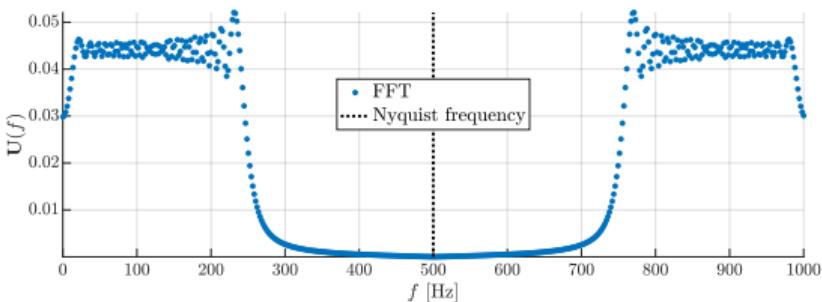
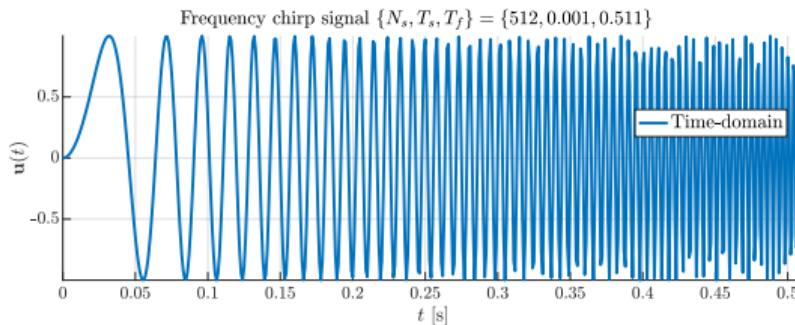
$$\beta = (f_1 - f_0)t^{-p},$$

$p = 1$ (linear),

$p = 2$ (quadratic).

Characteristics

- + Easily applicable in practice
- + Provides rich information
- May be long
- `insapack.chirp`



Experiment design: choice of the exciting signal

Chirp signals (`insapack.chirp`)

```
1 % Description
2 % Computes a chirp signal sweeping over FBND frequency range.
3 %
4 % Syntax
5 % [uc,tc,info] = insapack.chirp(Ns,Ts,FBND,REV,TYPE,SHOW)
6 %
7 % Input arguments
8 % - Ns : number of samples (integer)
9 % - Ts : sampling period [s] (>0 real)
10 % - FBND : frequency range [Hz] (2x1 real vector)
11 % - REV : reverse signal (boolean)
12 %       - false: fmin to max
13 %       - true: fmax to fmin
14 % - TYPE : variation of the chirp (string)
15 %       - 'linear'
16 %       - 'quadratic'
17 %       - 'logarithmic'
18 % - SHOW : plot signal (boolean)
19 %
20 % Output arguments
21 % - uc : chirp signal in [-1,1]
22 % - tc : time samples
23 % - info : additional information
```

Experiment design: choice of the exciting signal

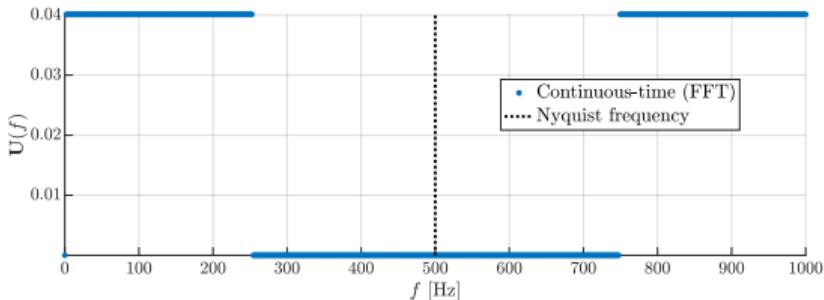
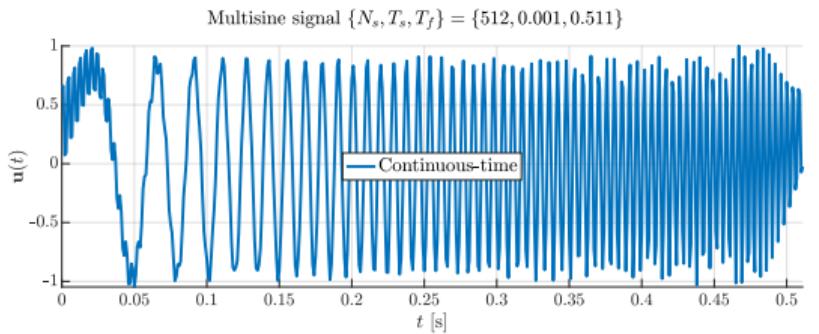
Multi-sine signals

$$\mathbf{u}(t) = \sum_{k=1}^F A_k \cos(2\pi k f_0 t + \phi_k)$$

$$\phi_k = -\frac{k(k-1)\pi}{F} \text{ (Schroder)}$$
$$\phi_k \in [0, 2\pi] \text{ (random).}$$

Characteristics

- + Easily applicable in practice
- + Provides very rich information
- + Can be adapted to deal with non-linearities
- ▶ `insapack.multisine`



Experiment design: choice of the exciting signal

Multi-sine signals (insapack.multisine)

```
1 % Description
2 % Computes a multi-sine signal sweeping over FBND frequency range.
3 %
4 % Syntax
5 % [uc,tc,info] = insapack.multisine(Ns,Ts,FBND,RPHI,ODD,REV,SHOW)
6 %
7 % Input arguments
8 % - Ns : number of samples (integer)
9 % - Ts : sampling period [s] (>0 real)
10 % - FBND : frequency range [Hz] (2x1 real vector)
11 % - RPHI : random phase (boolean)
12 %       - false: Schroeder multisine
13 %       - true: random multisine
14 % - ODD : frequency excited (used for nonlinear detection)
15 %       - 'all': all frequencies excited
16 %       - 'odd': odd frequencies excited (even discarded)
17 %       - 'odd-odd': odd-odd frequencies excited
18 %       - 'odd-rnd': odd frequencies excited (random block not)
19 % - REV : reverse signal (boolean)
20 %       - false: fmin to max
21 %       - true: fmax to fmin
22 % - SHOW : plot signal (boolean)
23 %
24 % Output arguments
25 % - uc : chirp signal in [-1,1]
26 % - tc : time samples
27 % - info : additional information
```

Some guidelines

Pre- and post-treatment

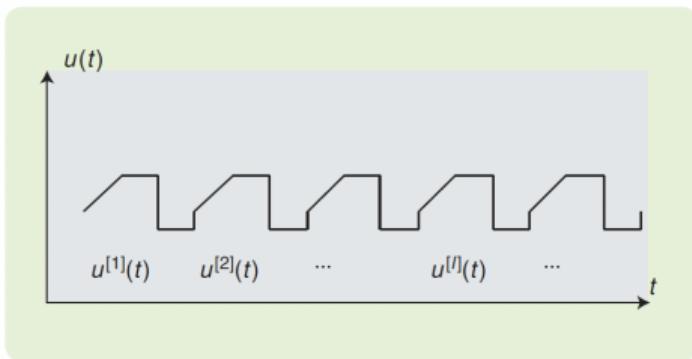
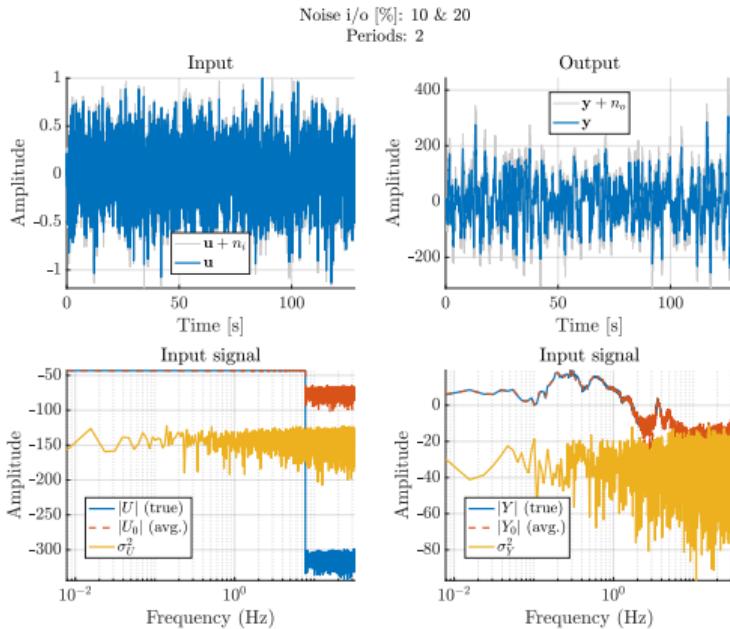


FIGURE 4 Calculating the sample mean and sample variance of a periodic signal starting from multiple measured periods. The input $u(t)$ and output $y(t)$ are measured over multiple periods and broken into subrecords of one period each, for example $u^{[l]}, l = 1, \dots, P$. The fast Fourier transform is applied to each subrecord. In the next step, the mean value and the (co)variance are calculated as a function of the frequency.

- ▶ Analyse the setup
- ▶ Verify anti-aliasing filters exist and have proper cut-off frequency
- ▶ Verify data synchronization
If not synchronized, re-sampling may be applied
- ▶ Pre-processing: mean and trend removal, missing data, pre-filtering
- ▶ Store reference/control/output signals
- ▶ Use data for "learning" and "validation"
- ▶ Use as much as possible periodic signal excitations
- ▶ **Average data (to account for noise)**
- ▶ Ask experimental team !

Some guidelines

Non-parametric analysis (over P experiments)



Signals variance

$$\mathbf{U}_0 = \frac{1}{P} \sum \mathbf{U}$$

$$\mathbf{Y}_0 = \frac{1}{P} \sum \mathbf{Y}$$

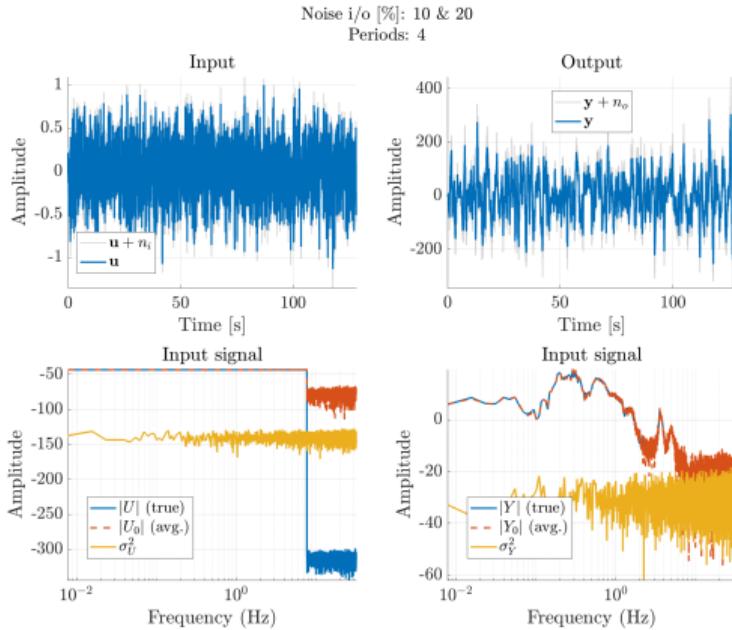
$$\sigma_{\mathbf{U}}^2 = \frac{1}{P-1} \sum |\mathbf{U} - \mathbf{U}_0|^2$$

$$\sigma_{\mathbf{Y}}^2 = \frac{1}{P-1} \sum |\mathbf{Y} - \mathbf{Y}_0|^2$$

$$\sigma_{\mathbf{U}\mathbf{Y}}^2 = \frac{1}{P-1} \sum (\mathbf{Y} - \mathbf{Y}_0)(\mathbf{U} - \mathbf{U}_0)$$

Some guidelines

Non-parametric analysis (over P experiments)



Signals variance

$$\mathbf{U}_0 = \frac{1}{P} \sum \mathbf{U}$$

$$\mathbf{Y}_0 = \frac{1}{P} \sum \mathbf{Y}$$

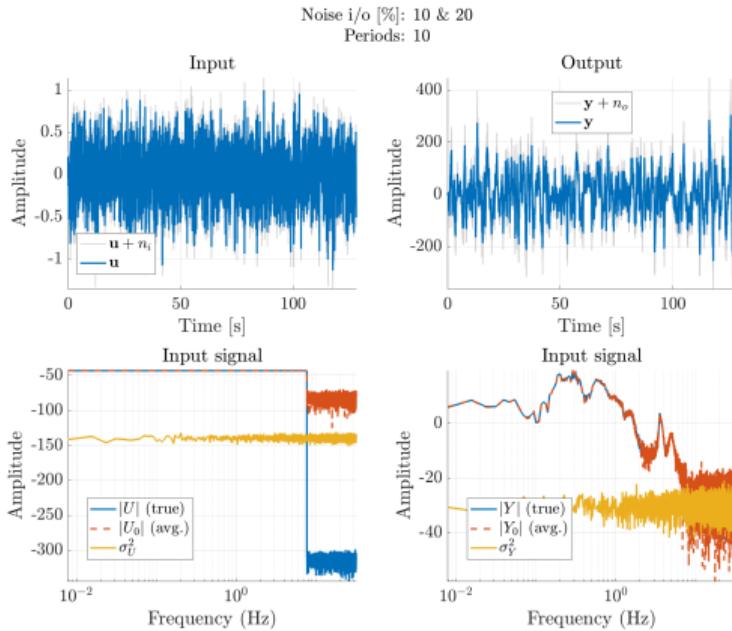
$$\sigma_{\mathbf{U}}^2 = \frac{1}{P-1} \sum |\mathbf{U} - \mathbf{U}_0|^2$$

$$\sigma_{\mathbf{Y}}^2 = \frac{1}{P-1} \sum |\mathbf{Y} - \mathbf{Y}_0|^2$$

$$\sigma_{\mathbf{U}\mathbf{Y}}^2 = \frac{1}{P-1} \sum (\mathbf{Y} - \mathbf{Y}_0)(\mathbf{U} - \mathbf{U}_0)$$

Some guidelines

Non-parametric analysis (over P experiments)

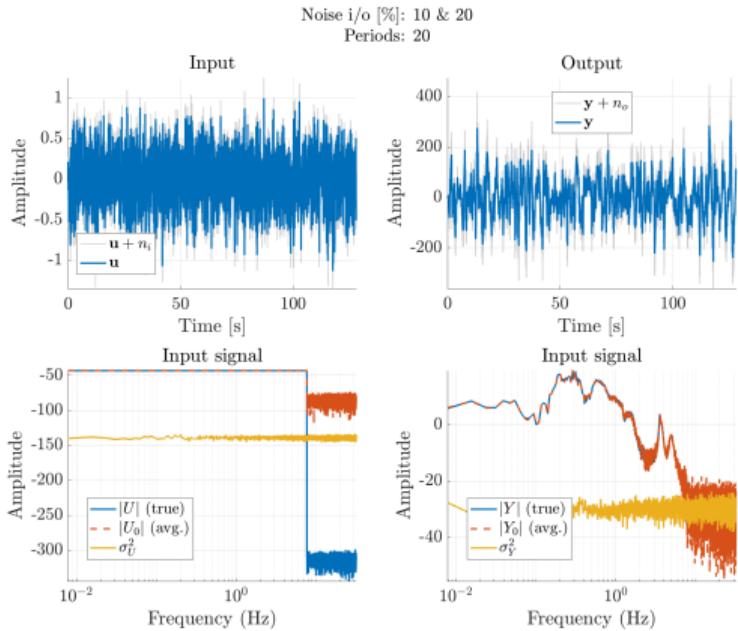


Signals variance

$$\mathbf{U}_0 = \frac{1}{P} \sum \mathbf{U}$$
$$\mathbf{Y}_0 = \frac{1}{P} \sum \mathbf{Y}$$
$$\sigma_{\mathbf{U}}^2 = \frac{1}{P-1} \sum |\mathbf{U} - \mathbf{U}_0|^2$$
$$\sigma_{\mathbf{Y}}^2 = \frac{1}{P-1} \sum |\mathbf{Y} - \mathbf{Y}_0|^2$$
$$\sigma_{\mathbf{U}\mathbf{Y}}^2 = \frac{1}{P-1} \sum (\mathbf{Y} - \mathbf{Y}_0)(\mathbf{U} - \mathbf{U}_0)$$

Some guidelines

Non-parametric analysis (over P experiments)

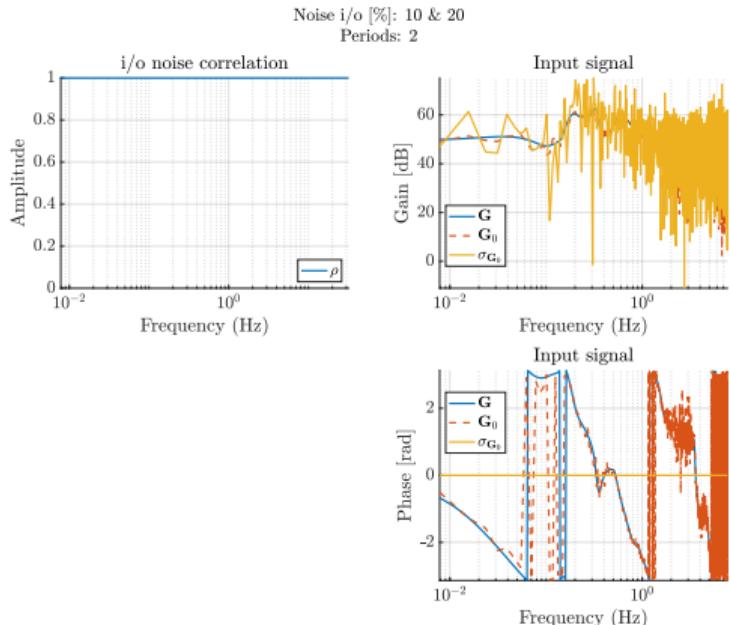


Signals variance

$$\begin{aligned}\mathbf{U}_0 &= \frac{1}{P} \sum \mathbf{U} \\ \mathbf{Y}_0 &= \frac{1}{P} \sum \mathbf{Y} \\ \sigma_{\mathbf{U}}^2 &= \frac{1}{P-1} \sum |\mathbf{U} - \mathbf{U}_0|^2 \\ \sigma_{\mathbf{Y}}^2 &= \frac{1}{P-1} \sum |\mathbf{Y} - \mathbf{Y}_0|^2 \\ \sigma_{\mathbf{U}\mathbf{Y}}^2 &= \frac{1}{P-1} \sum (\mathbf{Y} - \mathbf{Y}_0)(\mathbf{U} - \mathbf{U}_0)\end{aligned}$$

Some guidelines

Non-parametric analysis (over P experiments)



I/O noise correlation

$$\rho = \frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\sigma_{\mathbf{U}}\sigma_{\mathbf{Y}}}$$

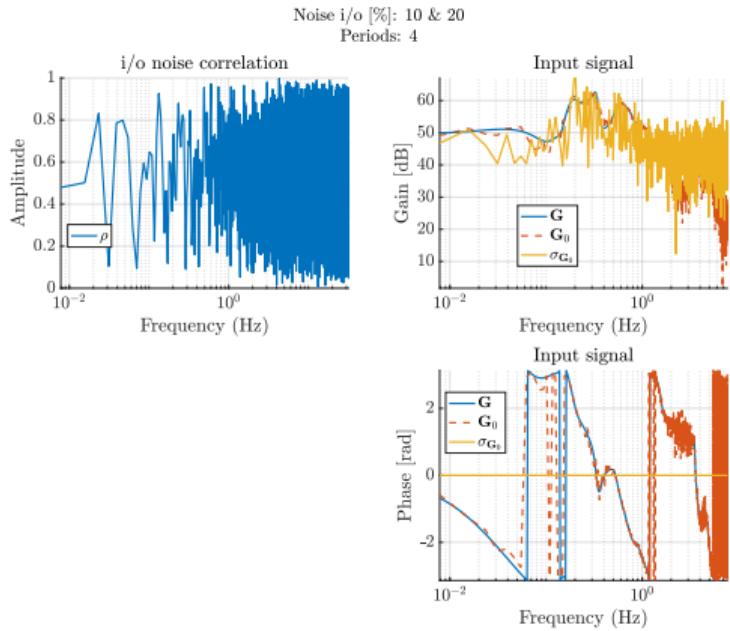
Model variance

The variability of $\mathbf{G}(k)$ around its expected value is characterized by the variance $\sigma_{\mathbf{G}}^2$. The larger, the wider the spread around the expected value. For a Gaussian pdf, the interval: $[-1.96\sigma_{\mathbf{G}}; +1.96\sigma_{\mathbf{G}}]$ corresponds to the 95% confidence interval.

$$\sigma_{\mathbf{G}}^2 = \frac{1}{P} |\mathbf{G}_0|^2 \left(\frac{\sigma_{\mathbf{Y}}^2}{|\mathbf{Y}_0|^2} + \frac{\sigma_{\mathbf{U}}^2}{|\mathbf{U}_0|^2} - 2\text{Re} \left(\frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\mathbf{Y}_0 \bar{\mathbf{U}}_0} \right) \right)$$

Some guidelines

Non-parametric analysis (over P experiments)



I/O noise correlation

$$\rho = \frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\sigma_{\mathbf{U}}\sigma_{\mathbf{Y}}}$$

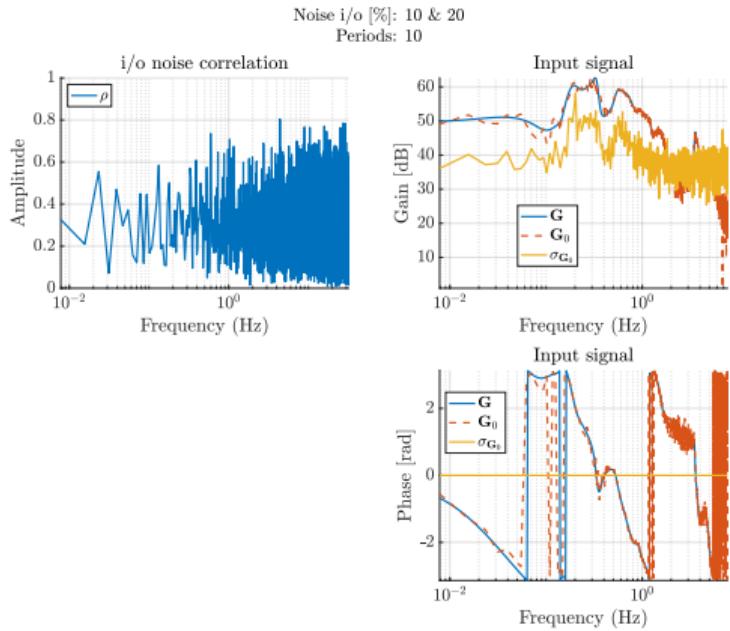
Model variance

The variability of $\mathbf{G}(k)$ around its expected value is characterized by the variance $\sigma_{\mathbf{G}}^2$. The larger, the wider the spread around the expected value. For a Gaussian pdf, the interval: $[-1.96\sigma_{\mathbf{G}}; +1.96\sigma_{\mathbf{G}}]$ corresponds to the 95% confidence interval.

$$\sigma_{\mathbf{G}}^2 = \frac{1}{P} |\mathbf{G}_0|^2 \left(\frac{\sigma_{\mathbf{Y}}^2}{|\mathbf{Y}_0|^2} + \frac{\sigma_{\mathbf{U}}^2}{|\mathbf{U}_0|^2} - 2\text{Re} \left(\frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\mathbf{Y}_0 \bar{\mathbf{U}}_0} \right) \right)$$

Some guidelines

Non-parametric analysis (over P experiments)



I/O noise correlation

$$\rho = \frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\sigma_{\mathbf{U}}\sigma_{\mathbf{Y}}}$$

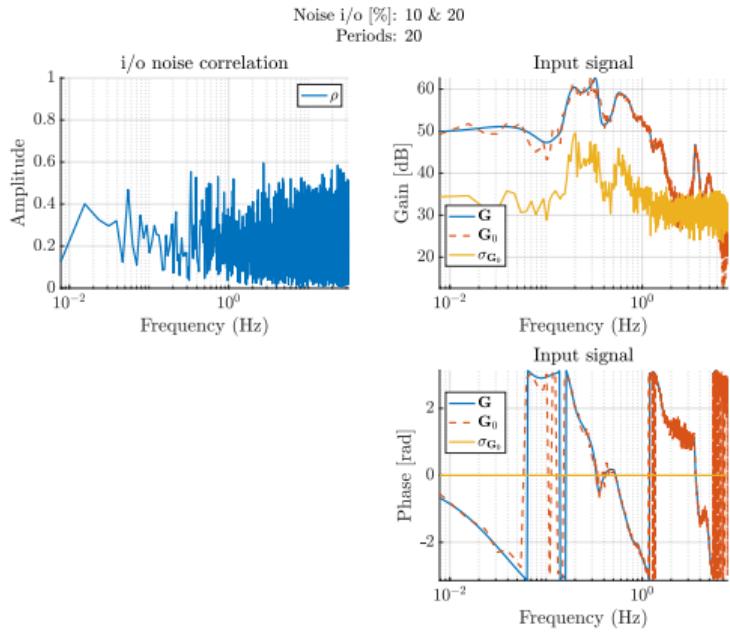
Model variance

The variability of $\mathbf{G}(k)$ around its expected value is characterized by the variance $\sigma_{\mathbf{G}}^2$. The larger, the wider the spread around the expected value. For a Gaussian pdf, the interval: $[-1.96\sigma_{\mathbf{G}}; +1.96\sigma_{\mathbf{G}}]$ corresponds to the 95% confidence interval.

$$\sigma_{\mathbf{G}}^2 = \frac{1}{P} |\mathbf{G}_0|^2 \left(\frac{\sigma_{\mathbf{Y}}^2}{|\mathbf{Y}_0|^2} + \frac{\sigma_{\mathbf{U}}^2}{|\mathbf{U}_0|^2} - 2\text{Re} \left(\frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\mathbf{Y}_0 \bar{\mathbf{U}}_0} \right) \right)$$

Some guidelines

Non-parametric analysis (over P experiments)



I/O noise correlation

$$\rho = \frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\sigma_{\mathbf{U}}\sigma_{\mathbf{Y}}}$$

Model variance

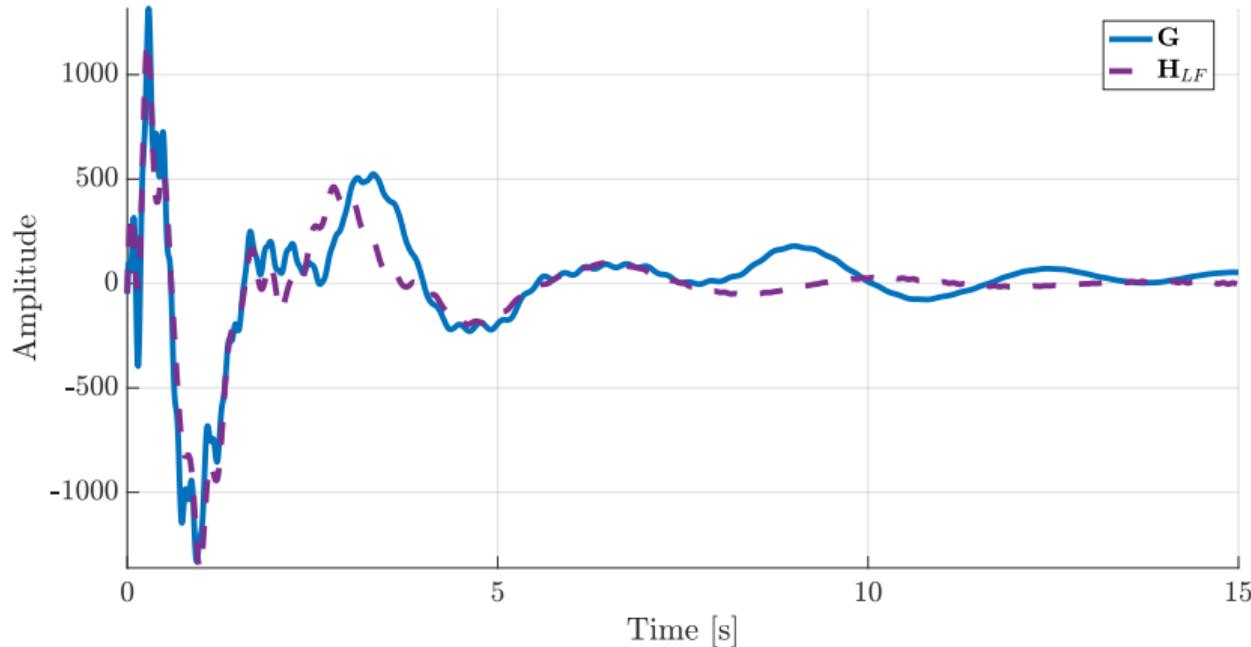
The variability of $\mathbf{G}(k)$ around its expected value is characterized by the variance $\sigma_{\mathbf{G}}^2$. The larger, the wider the spread around the expected value. For a Gaussian pdf, the interval: $[-1.96\sigma_{\mathbf{G}}; +1.96\sigma_{\mathbf{G}}]$ corresponds to the 95% confidence interval.

$$\sigma_{\mathbf{G}}^2 = \frac{1}{P} |\mathbf{G}_0|^2 \left(\frac{\sigma_{\mathbf{Y}}^2}{|\mathbf{Y}_0|^2} + \frac{\sigma_{\mathbf{U}}^2}{|\mathbf{U}_0|^2} - 2\text{Re} \left(\frac{\sigma_{\mathbf{U}\mathbf{Y}}^2}{\mathbf{Y}_0 \bar{\mathbf{U}}_0} \right) \right)$$

Some guidelines

Non-parametric analysis (over P experiments) - glimpse of identification ($r = 30$)

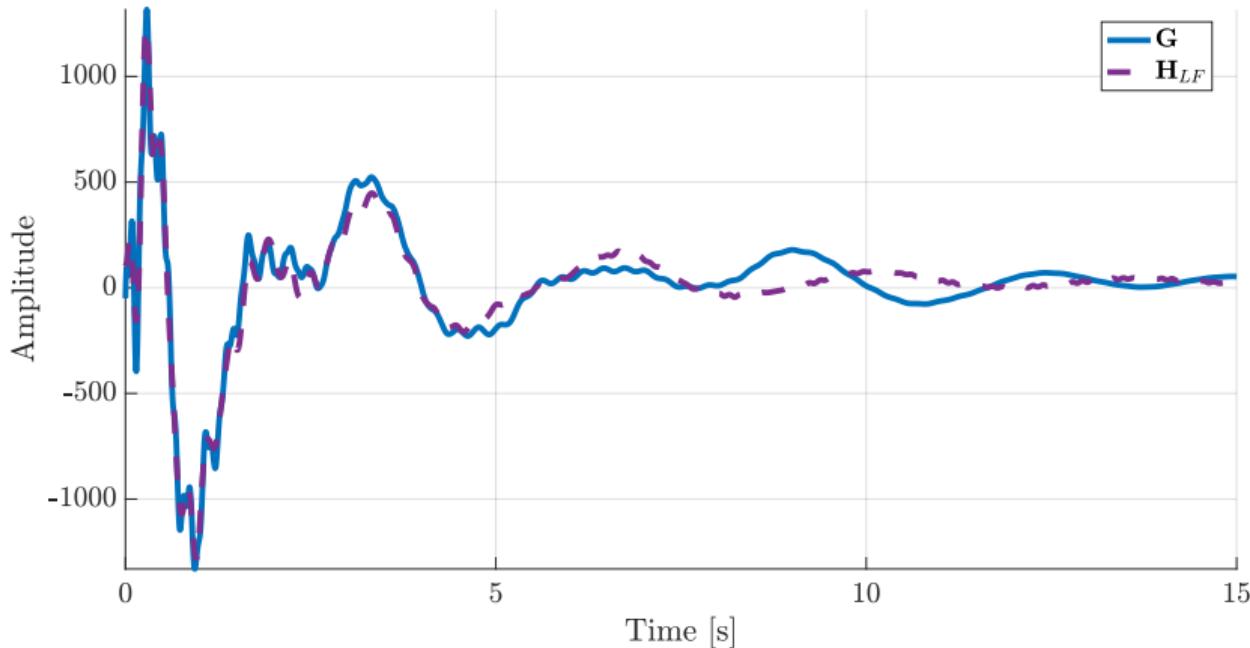
Noise i/o [%]: 10 & 20
Periods: 2



Some guidelines

Non-parametric analysis (over P experiments) - glimpse of identification ($r = 30$)

Noise i/o [%]: 10 & 20
Periods: 4

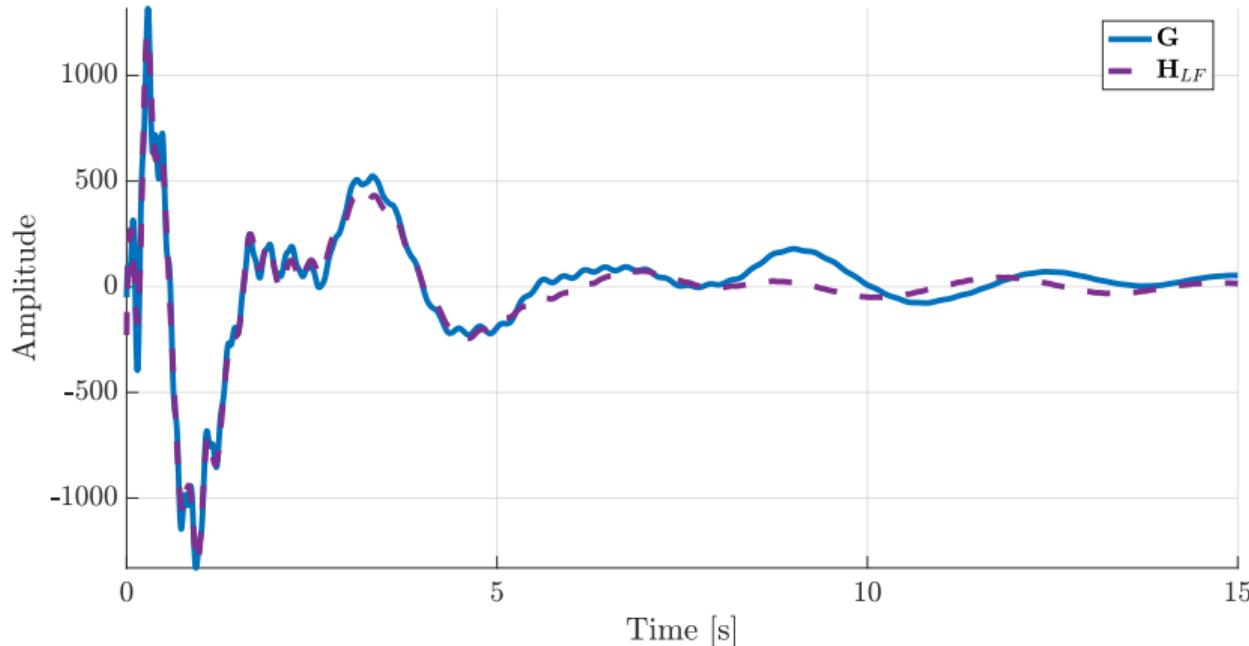


Some guidelines

Non-parametric analysis (over P experiments) - glimpse of identification ($r = 30$)

Noise i/o [%]: 10 & 20

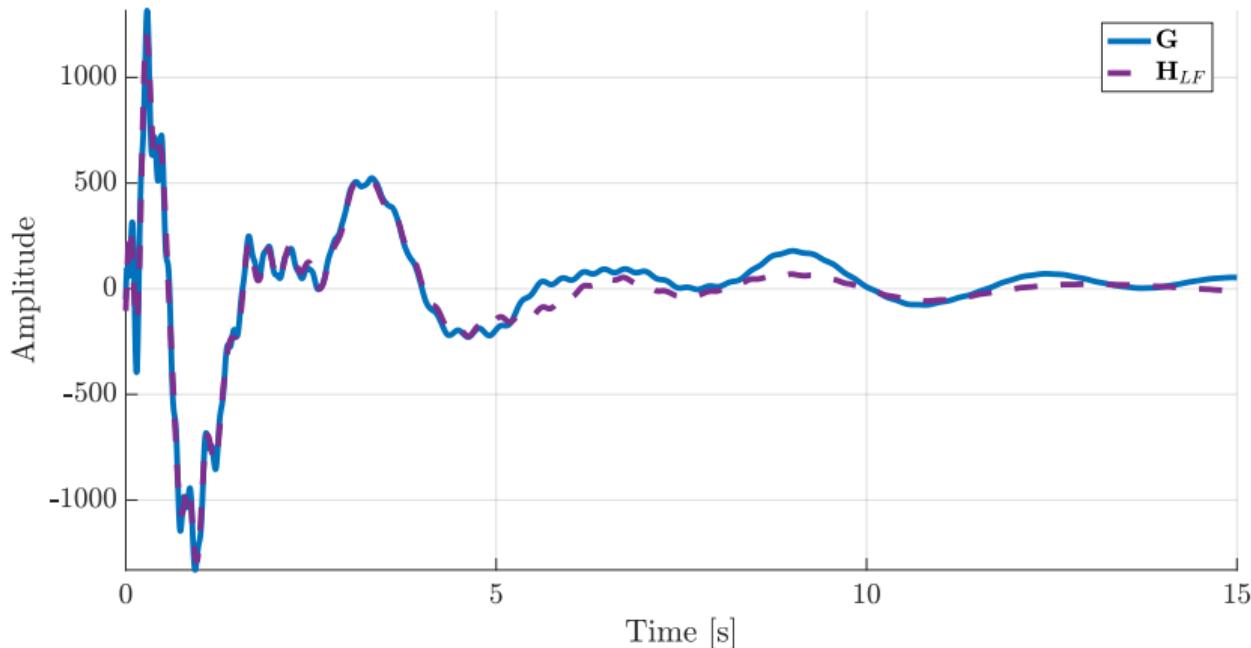
Periods: 10



Some guidelines

Non-parametric analysis (over P experiments) - glimpse of identification ($r = 30$)

Noise i/o [%]: 10 & 20
Periods: 20



Some guidelines

The closed-loop case

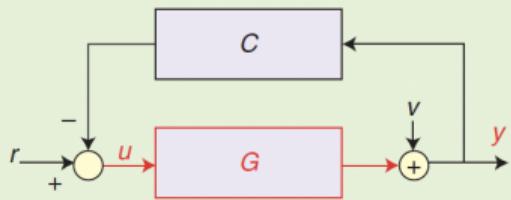


FIGURE S8 The frequency response function (FRF) measurement under closed-loop conditions requires special care. The FRF of the system G captured in a feedback loop is measured starting from the measured input $u(t)$ and output $y(t)$. This leads to a bias because the input u is correlated with the noise v through the feedback path C .

- ▶ Output $y(t)$ depends on both measured input $u(t)$ and disturbance source $v(t)$
- ▶ The FRF converges to

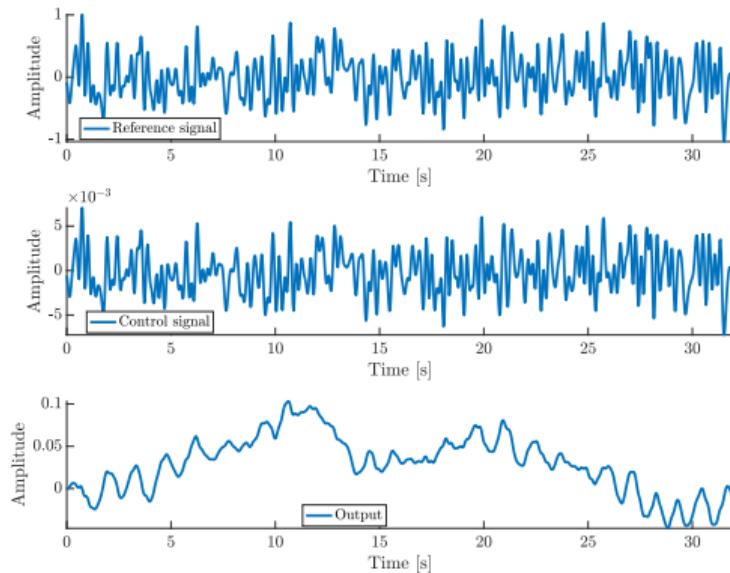
$$\hat{\mathbf{G}} = \frac{\mathbf{G}S_{rr} - \mathbf{C}S_{yy}}{S_{rr} + |\mathbf{C}|^2 S_{vv}}$$

- ▶ $\hat{\mathbf{G}} = \mathbf{G}$, r dominates over v ($S_{vv} = 0$)
- ▶ $\hat{\mathbf{G}} = \frac{-1}{\mathbf{C}}$, v dominates over r ($S_{rr} = 0$)
- ▶ If r is exactly known,

$$\hat{\mathbf{G}} = \frac{\mathbf{G}_{yr}}{\mathbf{G}_{ur}} = \frac{S_{yr}}{S_{ur}}$$

Some guidelines

The closed-loop case



Example

G dimension 50

PI controller

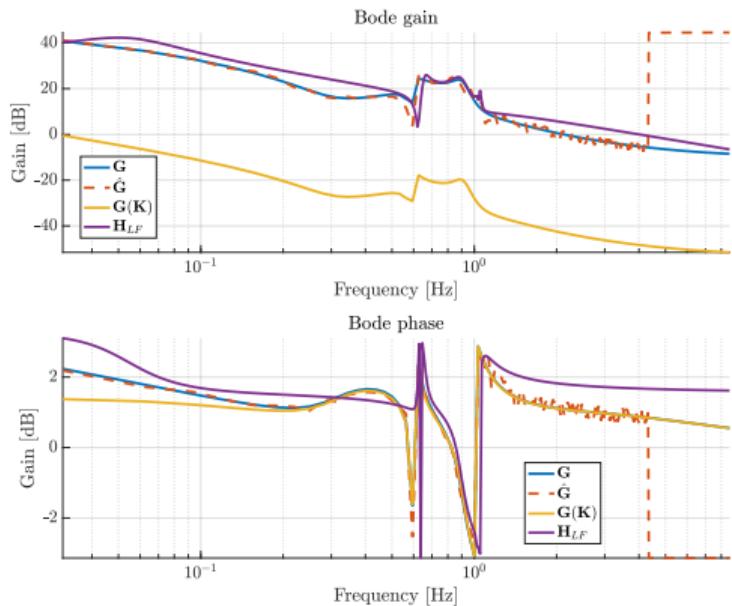
Some guidelines

The closed-loop case

Example

G dimension 50

PI controller



Some guidelines

Demonstration (if time)

Demo #1 (+insapack.mlbs)

Demo #2 (+insapack.multisine)

Linear dynamical system identification

... basic elements and Labs guidelines

Charles Poussot-Vassal
February 19, 2026

