Linear dynamical system identification

... basic elements and Labs guidelines

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Introduction

Course plan, overview and questions addressed

L1: Overview, signals construction, pre-treatment and non-parametric analysis

- Realization, transfer functions, ...
- Construct an experimental plan and signals.
- How to analyze signals, and derive some properties?

L2: Data-driven model construction in the time- and frequency-domain

- Construct a linear model from time- or frequency-domain data.
- How much is it valid? How to validate, discuss, amend it?

L3: L2 cont'd & Labs guidelines

- Experimental setup & numerical tools presentation.
- Methodology for the lab.

L2: Data-driven model construction in the time-domain & frequency

- Construct a linear model from time- or frequency-domain data.
- How much is it valid? How to validate, discuss, amend it?

Notions treated

- Time-domain:
  N4SID - Numerical algorithms for Subspace State-Space System IDentification (`n4sid` by Matlab)
  N4SID and Hankel
- Frequency-domain:
  LF - Loewner Framework (`lf.loewner_tgn` by pedagogical team)
  LF for data interpolation
  LF for minimal realization
  LF for identification

Introduction

Some history & references

Time-domain

- Deterministic SIMs (Ho, Kalman)
- Multivariable Output-Error State Space (Verhaegen, Dewilde)
- N4SID (Van Overschee, De Moor, Viberg)
- Pencil method (Antoulas, Ionita)

Frequency-domain

- N4SID applicable for frequency-domain (McKelvey)
- AAA (Trefethen, Nakasuka, Gugercin)
- Vector Fitting (...)
- Loewner framework (Antoulas, Ionita, Lefteriu, Gosea, Vuillemin, P-V.)

Time-domain identification (subspace method)

Basic state-space concepts reminder

Reachability & Observability

$$\mathcal{R}_n(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

$$\mathcal{O}_n(A,C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2 C^\top & \cdots & (A^\top)^n C^\top \end{bmatrix}^\top$$

Hankel matrix and Markov parameters

$$\mathcal{H}_n = \mathcal{O}_n \mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

where

$$H_0 = D \; , \; H_k = CA^{k-1}B \; (k \geq 1)$$

Time-domain identification (subspace method)

Identifying A, C (SISO)

Singular Value Decomposition & Rank

Compute the SVD

$$\begin{aligned}\mathcal{H}_n &= USV^\top \\ &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}\end{aligned}$$

then, estimate the rank as

$$\dim(S_1) = \mathbf{rank}\ (\mathcal{H}_n) = r$$

where $\mathbf{y} = [y_1, \cdots, y_{2N-1}]$ is the impulse response

$$\mathcal{H}_n = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & & \ddots & \vdots \\ y_N & y_{N+1} & \cdots & y_{2N-1} \end{bmatrix}$$

Time-domain identification (subspace method)

Identifying A, C (SISO)

Singular Value Decomposition & Rank

Compute the SVD

$$\mathcal{H}_n = USV^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

$$\dim(S_1) = \mathbf{rank}\ (\mathcal{H}_n) = r$$

Now define ($U_1 \in \mathbb{R}^{N \times r}$)

$$\hat{\mathcal{O}}_n = U_1 S_1^{1/2}$$

and set

$$\begin{aligned} J_1 &= \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \end{bmatrix} \\ J_2 &= \begin{bmatrix} \mathbf{0}_{N-1,1} & I_{N-1} \end{bmatrix} \\ J_3 &= \begin{bmatrix} 1 & \mathbf{0}_{1,N-1} \end{bmatrix} \end{aligned}$$

Thus,

$$\hat{A} = (J_1 U_1)^{-1} (J_2 U_1) \text{ and } \hat{C} = J_3 U_1$$

Time-domain identification (subspace method)

Identifying A, C (SISO)

Indeed

Recall, with

$$\begin{aligned} J_1 &= \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times N} \\ J_2 &= \begin{bmatrix} \mathbf{0}_{N-1,1} & I_{N-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times N} \\ J_3 &= \begin{bmatrix} 1 & \mathbf{0}_{1,N-1} \end{bmatrix} \in \mathbb{R}^{1\times N} \end{aligned}$$

we have

$$J_1 \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-2} \end{bmatrix} \text{ and } J_2 \begin{bmatrix} C \\ CA \\ \vdots \\ C^{N-1} \end{bmatrix} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ C^{N-1} \end{bmatrix}$$

and thus

$$J_1 \hat{\mathcal{O}}_n \hat{A} = J_2 \hat{\mathcal{O}}_n \text{ or equivalently } J_1 U_1 \hat{A} = J_2 U_1$$

Time-domain identification (subspace method)

Identifying B, D (SISO)

Estimate input matrices

$$\hat{B}, \hat{D} := \arg\min_{B,D} \sum_{k=1}^{N} ||\mathbf{y}(k) - \hat{\mathbf{y}}(k)||_F^2$$

In practice

$$\begin{aligned}
y_0 &= CB + D &&= \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} \\
y_1 &= CAB + D &&= \begin{bmatrix} CA & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} \\
&\vdots \\
y_N &= CA^N B + D &&= \begin{bmatrix} CA^N & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}
\end{aligned}$$

Time-domain identification (subspace method)

Identifying B, D (SISO)

Estimate input matrices

$$\hat{B}, \hat{D} := \arg\min_{B,D} \sum_{k=1}^{N} ||\mathbf{y}(k) - \hat{\mathbf{y}}(k)||_F^2$$

In practice

$$\mathbf{Y} = \left[\begin{array}{cc} \hat{\mathcal{O}}_n & \mathbf{1}_N \end{array} \right] \left[\begin{array}{c} B \\ D \end{array} \right]$$

Time-domain identification (subspace method)

Main properties

- Simple in parameterization
  No need for canonical forms for MIMO process models
  Subspace first, parameterization later
  Compact models in minimal realization
- Numerical property
  No nonlinear optimization techniques required
- Statistical property
  Simple Kalman filter framework

Time-domain identification (subspace method)

Algorithm & implementation

Input $\{t_i, \mathbf{u}_i, \mathbf{y}_i)\}_{i=1}^{N} \in \mathbb{R}$, $\texttt{tol} \in \mathbb{R}$

1. Compute Markov parameters $H_i \mathbb{R}$
2. Construct Hankel matrix $\mathcal{H}_n$
3. Estimate $r = \textbf{rank}\ (\mathcal{H}_n)$
4. Apply `svd` and obtain projector U_1 and singular scalings S_1
5. Construct the estimated observability matrix $\hat{\mathcal{O}}_n$
6. Estimate $\hat{A}$ and $\hat{C}$
7. Estimate $\hat{B}$ and $\hat{D}$

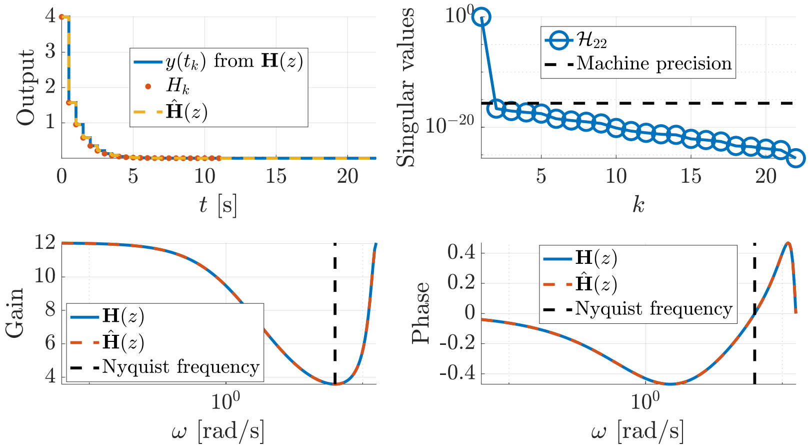
Output State-space model $\mathbf{H}$ and $\mathcal{S}$

Implementation

Subspace Framework, by MATLAB (with System Identification Toolbox)
`n4sid`

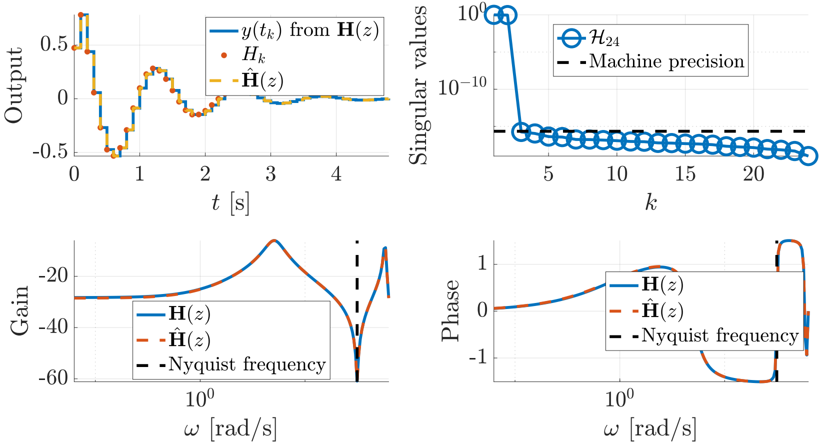
Time-domain identification (subspace method)

Identification examples



Time-domain identification (subspace method)

Identification examples



Frequency-domain identification (Loewner Framework)

Rational interpolation problem (SISO)

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \dots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \dots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \dots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

Frequency-domain identification (Loewner Framework)

Loewner pencil (SISO)

The right data can be expressed as:

$$\begin{array}{rcl} \mathbf{\Lambda} & = & \mathbf{diag}\,[\lambda_1, \ldots, \lambda_k] \in \mathbb{C}^{k\times k}, \\ \mathbf{R} & = & \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \in \mathbb{C}^{1\times k} \\ \mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{1\times k} \end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcl} \mathbf{M} & = & \mathbf{diag}\,[\mu_1, \ldots, \mu_q] \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top & = & \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \in \mathbb{C}^{1\times q} \\ \mathbf{V}^\top & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{1\times q} \end{array}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \ldots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \ldots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation: $\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

Frequency-domain identification (Loewner Framework)

Loewner pencil (SISO)

The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{L} - \mathbb{L}\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

The shifted Loewner matrix is:

$$\mathbb{S} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top - \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^\top - \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top - \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^\top - \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{S} - \mathbb{S}\mathbf{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\mathbf{\Lambda}$$

Frequency-domain identification (Loewner Framework)

Loewner pencil (SISO)

If data are sampled from $\mathbf{G}(s) = C(sE - A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} C(\mu_1 E - A)^{-1} \\ \vdots \\ C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1}B, \cdots, (\lambda_k E - A)^{-1}B,\right] \in \mathbb{C}^{n \times k}$$

be the generalized tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top - \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -C(\mu_j E - A)^{-1}E(\lambda_i E - A)^{-1}B \\ &= -[\mathcal{O}_q]_i E[\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{S}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top - \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -C(\mu_j E - A)^{-1}A(\lambda_i E - A)^{-1}B \\ &= -[\mathcal{O}_q]_i A[\mathcal{R}_k]_j \end{aligned}$$

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Frequency-domain identification (Loewner Framework)

Realization and minimality

Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{S}, \quad , \; B = \mathbf{V}, \quad C = \mathbf{W},$$

is **a descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}\,[\xi\mathbb{L} - \mathbb{S}] = \mathbf{rank}\,[\mathbb{L},\ \mathbb{S}] = \mathbf{rank}\,\begin{bmatrix}\mathbb{L}\\ \mathbb{S}\end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L},\ \mathbb{S}] = Y\Sigma_l\tilde{X}^\top, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{S}\end{bmatrix} = \tilde{Y}\Sigma_r X^H, \; Y, X \in \mathbb{C}^{N\times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{S}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

Frequency-domain identification (Loewner Framework)

Realization and minimality

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Frequency-domain identification (Loewner Framework)

Main properties

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Frequency-domain identification (Loewner Framework)

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$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$

- underlying rational (r) order

$$\begin{aligned} r &= \textbf{rank}\,(\xi\mathbb{L} - \mathbb{S}) \\ &= \textbf{rank}\,([\mathbb{L}, \mathbb{S}]) \\ &= \textbf{rank}\,([\mathbb{L}^H, \mathbb{S}^H]^H) \end{aligned}$$

- and McMillan (ν) order

$$\nu = \textbf{rank}\,(\mathbb{L})$$

- $\mathbb{L}$ and $\mathbb{S}$ are input-output independents.
- Minimal realization

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Frequency-domain identification (Loewner Framework)

Algorithm & implementation

Input $\{t_i, \mathbf{u}_i, \mathbf{y}_i)\}_{i=1}^{N} \in \mathbb{R}$, $\texttt{tol} \in \mathbb{R}$

1. Compute FRF $\{\imath\omega_i, \mathbf{G}(\imath\omega_i)\}_{i=1}^{N} \in \mathbb{C}$
2. Select the column and row variables: $\{\mathbf{\Lambda}, \mathbf{W}, \mathbf{R}\}$ and $\{\mathbf{M}, \mathbf{V}, \mathbf{L}\}$
3. Construct $\mathbb{L}$, $\mathbb{S}$, $\mathbf{V}$ and $\mathbf{W}$ (full realization)
4. Estimate $\nu = \mathbf{rank}\ (\mathbb{L})$, $r = \mathbf{rank}\ ([\mathbb{L}\, \mathbb{S}])$
5. Apply `svd` and obtain projectors X and Y
6. Project to get minimal or reduced-order realization

Output State-space model $\mathbf{H}$ and $\mathcal{S}$

Implementation

Loewner Framework (tangential version), by pedagogical team

`insapack.non_param_freq` $\rightarrow$ `insapack.data2loewner` $\rightarrow$ `insapack.loewner_tng`

Frequency-domain identification (Loewner Framework)

Identification examples (simple case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}, \mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}$$

Frequency-domain identification (Loewner Framework)

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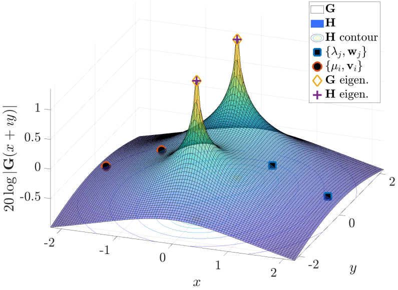

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$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{S} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Frequency-domain identification (Loewner Framework)

Identification examples (simple case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\textbf{rank } (\xi\mathbb{L} - \mathbb{S}) = r$$

$$\textbf{rank } (\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, $(\mathbb{S}, \mathbb{L})$ pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}, \mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

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Frequency-domain identification (Loewner Framework)

Identification examples (lot of data case)

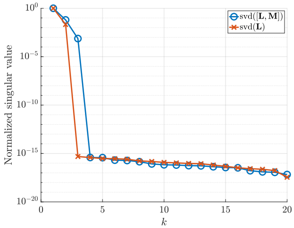
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Frequency-domain identification (Loewner Framework)

Identification examples (lot of data case)

Rational function satisfies

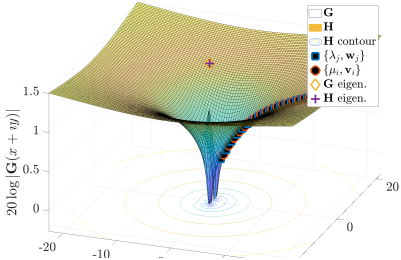
$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

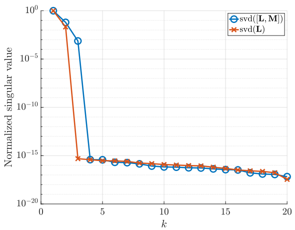
$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$

Realization $n = 20$:
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{S})^{-1}\mathbf{V}$





Frequency-domain identification (Loewner Framework)

Identification examples (lot of data case)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

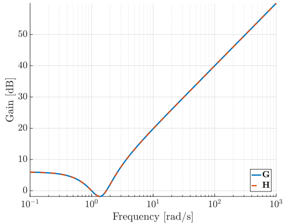
$$\textbf{rank }(\xi\mathbb{L} - \mathbb{S}) = r = 3$$

$$\textbf{rank }(\mathbb{L}) = \nu = 2$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots 20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots 20} = [1.5, 2.5, \ldots, 20.5]$$



Frequency-domain identification (Loewner Framework)

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$$\textbf{rank } (\xi\mathbb{L} - \mathbb{S}) = r = 3$$

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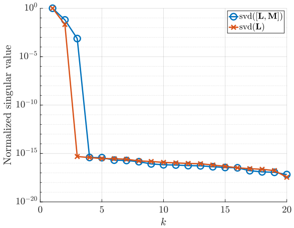
$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}X(-sY^\top\mathbb{L}X + Y^\top\mathbb{S}X)^{-1}Y^\top\mathbf{V} \\ &= \frac{s^2+s+2}{s+1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Evaluated at

$$\lambda_{1\ldots 20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots 20} = [1.5, 2.5, \ldots, 20.5]$$



Linear dynamical system identification

... basic elements and Labs guidelines

Charles Poussot-Vassal
February 18, 2026