The Applicability of Darcy's Law[1, 3]

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ABSTRACT

Deviations from Darcian proportionality between flow velocity v and hydraulic gradient i are considered in terms of a gradient-dependent hydraulic conductivity $K(i) = v/i$. Published greater-than-proportional data for saturated flow indicate measured variations in $K(i)$ of 2- to 4-fold, with a potentiality for 5- to 15-fold variations inferred from fitted non-Darcian flow equations. Non-Darcian effects for two-dimensional radial flow are also analyzed. The hydraulic head distribution is shifted from the Darcian-derived logarithmic pattern toward the characteristic linear distribution for one-dimensional flow, and the amount of the shift is velocity dependent. This implies that the flow net for two-dimensional non-Darcian flow would generally change with flow velocity, and that the high-gradient flow regions would be more permeable than for Darcian flow.

For a particular unsaturated soil, the soil-water diffusivity D appears to be a gradient-dependent quantity $D(\theta, w)$, where θ is the soil-water content, and w is the water-content gradient. A maximum 8-fold variation in D at given low water contents is associated with a 40% discrepancy between experimental water-absorption rates and those calculated from proportional flow theory. Similarly, a 115% discrepancy is found for the vertically downward rate of infiltration into a mixture of quartz sand and ground silica, but the deviating behavior can be precisely accommodated with a non-Darcian infiltration equation derived by a simplified analysis. Various implications of nonproportional effects are discussed and enumerated briefly.

Additional Key Words for Indexing: nonproportional flow, saturated flow, hydraulic conductivity, unsaturated flow, water diffusivity, infiltration.

THE USE of Darcy's (2) equation for describing soil-water flow has become almost axiomatic, except for the less-than-proportional increase of velocity with gradient generally attributed to turbulence and kinetic effects. The traditionally accepted verifications for saturated flow are discussed in detail elsewhere (27, 29, 30). But the classic confirmatory data of Fancher and Lewis (5) were for sands and sandstones, and specific mention was made of the absence of clays. Recent work of Olsen (18, 19) with compacted kaolinite has shown excellent proportional conformity, thereby demonstrating that good adherence to Darcy's equation is obtainable for nonswelling silt-sized particles down to the 2-μ size generally considered the upper limit of the clay range.

When, however, the porous medium contains particles within the clay range, greater-than-proportional response of flow velocity to gradient has been observed. This was especially evident in the careful experiments of Von Engelhardt and Tunn (34) with sandstones claimed to contain only nonswelling clays. Work of Lutz and Kemper (15) and Paez[4] revealed reasonable velocity-gradient proportionality for calcium- and certain hydrogen-saturated clays even

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[4] Paez, J. A. 1962. The effect of gradient on the permeability of saturated porous media. M.S. Thesis, University of California, Davis, Calif.

of swelling types, but both of these studies are particularly noteworthy for the pronounced greater-than-proportional flow responses also observed, especially for sodium- and certain other hydrogen-saturated clays. Kutilek (12, 13), however, found greater-than-proportional responses even in some of his calcium-saturated clays.

As discussed recently (29, 30), various tentative explanations of greater-than-proportional behavior have been proposed, and range from Low's (14) modified-water hypothesis on one hand to suggested artifacts of experimental procedure on the other [see Olsen (18)]. Until the causes are better understood, it will be difficult to provide positive *a priori* identification of the particular combinations of porous medium and flow liquid for which greater-than-proportional responses will be encountered. Hence, rather than treating of causes, the approach of the present paper will be to consider the magnitudes of various reported deviations from Darcy's equation, and to assess their importance in flow processes and problems. Both saturated and unsaturated flow will be considered.

FLOW THROUGH SATURATED MEDIA

Gradient-Dependent Hydraulic Conductivity

Deviations from Darcy's equation may still be viewed from a Darcian standpoint by considering the hydraulic conductivity $K = v/i$, where v is the flow velocity and i is the hydraulic gradient. Instead of being constant, K now becomes the gradient-dependent function $K(i)$. This is illustrated for two non-Darcian velocity-gradient curves in Fig. 1. Confining attention to curve I, observe that $K(i)$ takes on a minimum value, K_{min}, at $i = 0$, and increases to a maximum value, K_{max}, as i becomes large. The finite K_{max} is a consequence of the velocity-gradient curve's approach to linearity. Since K_{max}/K_{min} would equal unity if $v(i)$ were a straight line through the origin (Darcian proportionality), the extent to which this ratio exceeds unity becomes a measure of greater-than-proportional non-Darcian behavior. Also, K_{max}/K_{min} indicates the maximum relative variation induced in $K(i)$ by gradient dependence.

If $v(i)$ can be expressed by the equation of Swartzendruber (27)

$$v = B[i - J(1 - e^{-Ci})] \quad [1]$$

where B, J, and C are constants, then $K_{max}/K_{min} = 1/(1 - JC)$. A form of $v(i)$ having qualitatively similar characteristics to equation [1] has been suggested by Kutilek (12, 13), which, after some modification of arrangement and symbols, is

$$v = E[\ln(A + e^{Mi/E}) - \ln(A + 1)] \quad [2]$$

where A, E, and M are constants. Using equation [2], $K_{max}/K_{min} = A + 1$. Equations [1] and [2] become asymptotically linear for large i, and each contains Darcian proportionality as a special case. Choosing between these two equations would seem largely a matter of convenience or preference, since each appears to fit the data for which its use has been attempted.

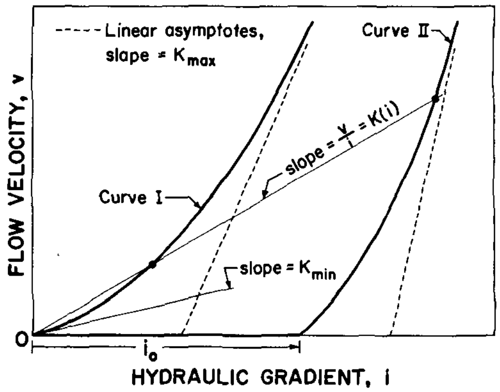

Fig. 1—Velocity-gradient relationships for non-Darcian behavior, without threshold gradient (curve I) and with threshold gradient (curve II).

K_{max}/K_{min} was calculated for data from five investigations as listed in Table 1. Also shown are the mean $K(i_2)$ and the ratio $K(i_2)/K(i_1)$, where i_1 and i_2 are, respectively, the smallest and largest gradients applied experimentally to a given porous medium. For inclusion in Table 1, the primary basis of selecting a porous medium was that it show greater-than-proportional flow behavior for pure water. Paez[4] and Hadas (7), however, had not used water alone as a flow liquid. Also, the data of Hadas were obtained at 10 cm H_2O suction, but are included in the category of saturated flow since measurements indicated essential liquid saturation at this low suction. The data of Hansbo (9) have not been included, since his deviations are no greater than could be accounted for by Olsen's (18) bubble-error correction.

Inspection of Table 1 shows that $K(i_2)/K(i_1)$, the experimentally observed variation in $K(i)$, averages from 1.2 to 4.9, with individual values going as high as 12.7. For K_{max}/K_{min}, the means reach 16.8, while the individual values reach 84.0. Even if only the means are considered, the ignoring of non-Darcian behavior would involve 2- to 4-fold errors, with potentialities of 5- to 15-fold as inferred from the fitted equations.

It must be granted that many of the data in Table 1 are for clay materials of mean $K(i_2) = 0.03$ cm/hr or less, which, for some purposes such as drainage, might have to be regarded as essentially impermeable. However, the soils data of Paez and of Hadas, as well as the sandstone data of Von Engelhardt and Tunn, contain mean $K(i_2)$ values on the substantial order of 1 cm/hr. Even so, 2-fold variations in $K(i)$ are found, and the variation potentiality is still 5- to 15-fold on the basis of the fitted flow equation. This implies that non-Darcian behavior should not be summarily dismissed on the premise that it is only of appreciable magnitude for materials which are practically impermeable in the first place.

If the slope of the velocity-gradient curve is zero over a threshold-gradient range $0 \leq i \leq i_o$ as in curve II of Fig. 1, or even is zero just at the origin, then $K(i)$ varies from zero to K_{max}, and the resulting zero K_{min} produces an infinite K_{max}/K_{min} which no longer can serve as a quantitative characterization of non-Darcian behavior. Data on velocity-gradient relationships exhibiting a threshold gradi-

Table 1—Conductivity ratio and largest measured hydraulic conductivity for water and various solutions of electrolyte.

Investigators and porous materials	Number of deter- minations	Electro- lyte conc.	Mean $K(i_2)$* 10^{-2}cm/hr	Ratio of hydraulic conductivity: $K(i_2)/K(i_1)$* Smallest	Largest	Mean	K_{max}/K_{min}, from fitted equation† Smallest	Largest	Mean
Von Engelhardt and Tunn (34), claybearing sandstones.	5	0.00N	84.	1.4	5.1	2.5	3.5	16.7	9.0
	3	0.88N	80.	1.7	2.7	2.3	3.2	37.0	16.8
	2	1.83N	105.	1.4	1.6	1.5	2.2	2.4	2.3
Lutz and Kemper (15), clays.	6	0.000N	0.18	1.1	2.4	1.6	1.3	22.2	7.1
	6	0.005N	0.21	1.0	2.3	1.4	1.0	13.0	3.9
	2	0.050N	0.24	1.0	1.3	1.2	1.0	1.8	1.4
	3	0.500N	0.51	1.1	1.4	1.3	1.3	2.8	2.2
Kutilek (12, 13), clays.	9	0.000N	0.52	1.5	11.1	4.9	3.0	35.0	12.9
	2	0.001N	0.96	2.2	3.6	2.9	1.6	4.0	2.8
	10	0.010N	1.52	1.0	12.7	3.6	1.0	84.0	14.0
	8	0.100N	2.33	1.0	5.0	2.5	1.0	11.6	5.5
	4	1.000N	3.06	1.7	4.6	3.0	4.9	5.9	5.5
Paez[4], soils.	6	0.01N	25.	1.2	2.6	1.7	1.4	10.0	5.3
Hadas (7), soils.	3	200 ppm	140.	1.2	2.4	1.9	1.4	10.0	5.9

* Where i_2 and i_1 are, respectively, the largest and smallest gradients employed experimentally. † Equation [2] for Kutilek's (12, 13) data, where $K_{max}/K_{min} = A + 1$; equation [1] for all other data, where $K_{max}/K_{min} = 1/(1-JC)$.

ent are meager. Those of Miller and Low (16) indicate i_o values as high as 40 to 55 for sodium- and lithium-saturated clays, even if Olsen's (18) bubble-error correction were accepted. However, the observed $K(i_2)$ values range from 2.5×10^{-7} to 3.5×10^{-6} cm/hr, which is three orders of magnitude smaller than the smallest mean $K(i_2)$ in Table 1. Thus, from a practical drainage point of view such clays would be considered impermeable, so that the non-Darcian behavior would become irrelevant.

A threshold gradient, however, or even non-Darcian behavior as in curve I of Fig. 1, has important implications for the consolidation of saturated soils, since additional bearing strength is bestowed thereby. The threshold gradient also has relevance for the water-sealing properties of clays. Consider surface sealing during raindrop splash as an example. If $i_o = 55$, the development of a clay film even 0.1 mm thick over the soil surface could support a water depth of 5.4 mm before water movement through the soil could begin.

If existing greater-than-proportional behavior were ignored when determining hydraulic conductivity by a laboratory test, Fig. 1 illustrates the resulting implications. If the test were conducted at high gradient and the flow condition of interest occurred at low gradient, the effective value of K under the flow condition of interest would be overestimated. The reverse would be true if the test gradient were less than the gradient at the flow condition of interest. The magnitude of the discrepancy would depend on the $v(i)$ relationship and on the spread between the test gradient and the gradient under the flow condition of interest.

Flow in More than One Dimension

For non-Darcian behavior, the combination of such as equation [1] with the equation of continuity for steady-state flow would yield nonlinear partial differential equations more difficult to solve than the Laplace equation which governs Darcian flow. Some simplification occurs if the exponential in equation [1] is expanded in series, and terms beyond the second degree are ignored; the resulting expression is

$$v = ai + bi^2 \qquad [3]$$

where $a = B(1 - JC)$ and $b = BJC^2/2$. Equation [3] is not asymptotically linear for large i, and is an acceptable approximation for equation [1] only in the curvilinear region.

As Raats (23) has pointed out, earlier suggestions (26, 27) of a straightforward application of such as equations [1] and [3] for any flow direction are not quite correct, since the general flow vector is not necessarily independent of the coordinate system. This difficulty does not arise if only radial flow is considered. We therefore choose two-dimensional radial flow as the simplest illustrative example, based on flow equation [3]. The hydraulic head ϕ is then a function only of the radial distance coordinate r. Take $\phi = \phi_1$ over the circular inlet boundary $r = r_1$ where $v = v_1$, and take $\phi = \phi_2$ over the circular outlet boundary $r = r_2$; this means that $r_2 > r_1$ and $\phi_1 > \phi_2$. The mathematical analysis, given in detail in the APPENDIX, provides the fractional head difference, $(\phi - \phi_2)/(\phi_1 - \phi_2)$, by the expression

$$\frac{\phi - \phi_2}{\phi_1 - \phi_2} = \frac{\dfrac{(r - r_2 + r_2R_2 - rR)}{2(b/a)} + \dfrac{r_1v_1}{a}\ln\dfrac{(R-1)(R_2+1)}{(R+1)(R_2-1)}}{\dfrac{(r_1 - r_2 + r_2R_2 - r_1R_1)}{2(b/a)} + \dfrac{r_1v_1}{a}\ln\dfrac{(R_1-1)(R_2+1)}{(R_1+1)(R_2-1)}} \qquad [4]$$

with R, R_1, and R_2 defined in the APPENDIX. If v_1 is very small, equation [4] reduces to

$$\frac{\phi - \phi_2}{\phi_1 - \phi_2} = \frac{\ln(r/r_2)}{\ln(r_1/r_2)} \qquad [5]$$

which is the familiar logarithmic head distribution for Darcian flow (11); it should and can be obtained from equation [4] by letting b approach zero for constant nonzero v_1.

To obtain a and b values, needed along with v_1, r_1, and r_2 in equation [4] to compute $(\phi - \phi_2)/(\phi_1 - \phi_2)$ as a function of the fractional distance $(r - r_1)/(r_2 - r_1)$, observe from equation [3] that a plot of v/i vs. i should yield a straight line of slope b and intercept a. Using the data of Hadas (7) as scaled from his Fig. 5 for Gilat loess at 10-cm suction, since at this low suction the soil was still essentially saturated, the plot of v/i vs. i is shown in the upper graph of Fig. 2. The lower graph shows the curve of v vs. i calculated from the a and b evaluated in the upper graph, the fit to the data being quite satisfactory.

These values of a and b, along with $r_1 = 1$ cm and $r_2 = 5$ cm, were used in equation [4] to prepare the fractional head distribution curve shown in Fig. 3 for $v_1 = 12$ cm/hr, which is the greatest velocity that can be used without exceeding the range of the data points in Fig. 2. The logarithmic distribution, for Darcian flow in general and non-Darcian flow at very small v_1, was computed from equation [5] and also plotted in Fig. 3. The linear distribution shown (broken line, Fig. 3) is that which obtains for one-dimensional flow in a uniform medium regardless of the nature of the relationship between velocity and gradient.

Non-Darcian behavior at a v_1 of 12 cm/hr (Fig. 3) causes the head distribution curve to shift in the direction of that for one-dimensional flow. In comparison with the Darcian (logarithmic) distribution curve, the non-Darcian curve is flatter near the flow inlet and steeper near the flow outlet. This is a consequence of $K(i)$ being greater at large gradients than at small gradients. The figure also illustrates the velocity dependence of the head distribution for non-Darcian flow, in contrast with the constant Darcian (logarithmic) distribution which is independent of velocity.

A two-dimensional non-Darcian flow problem less restrictive and of more interest than the radial case would be the flow of ponded water into a tile line. On the basis of the non-Darcian behavior in Fig. 3, it appears that the head distribution, and hence the flow net, would be velocity-dependent for the tile problem also. The region immediately above the tile line contains the highest gradients in Darcian flow [see Kirkham (11)]. For non-Darcian flow, this same region should then become relatively more permeable, by virtue of $K(i)$ being greater at large gradients than at small. Thus, when compared with its Darcian counterpart, a tile line in a non-Darcian medium might be expected to function more like an open ditch.

FLOW THROUGH UNSATURATED MEDIA

Basic Flow Equations

Assuming for unsaturated soil that the hydraulic conductivity is a unique function of the volumetric water content θ, Darcian proportionality is written $v = K(\theta)i$. If the total hydraulic head consists only of gravity and suction components, the horizontal flow velocity can be written in the diffusivity form

$$v = -D(\theta)\frac{\partial\theta}{\partial x} = D(\theta)w \qquad [6]$$

where x is the horizontal distance coordinate, $w = -\partial\theta/\partial x$ is the water-content gradient, $D(\theta) = [-d\tau/d\theta]K(\theta)$ is the soil-water (or moisture) diffusivity, and $\tau(\theta)$ is the assumed unique suction function. The derivation leading to equation [6] can explicitly use τ in place of θ, with the result

$$v = K(\tau)\frac{\partial\tau}{\partial x} \qquad [7]$$

where K is considered dependent on τ, and $\partial\tau/\partial x$ is the suction gradient. It is to be emphasized that the validity of *both* $K = K(\theta)$ and $\tau = \tau(\theta)$ is crucial to equations [6] and [7], and any breakdown in them, or in solutions based upon them, could be caused by failure in $K(\theta)$, $\tau(\theta)$, or both. The approach of the present paper is to investigate the implications of $K(\theta)$ breakdown while $\tau(\theta)$ is pre-

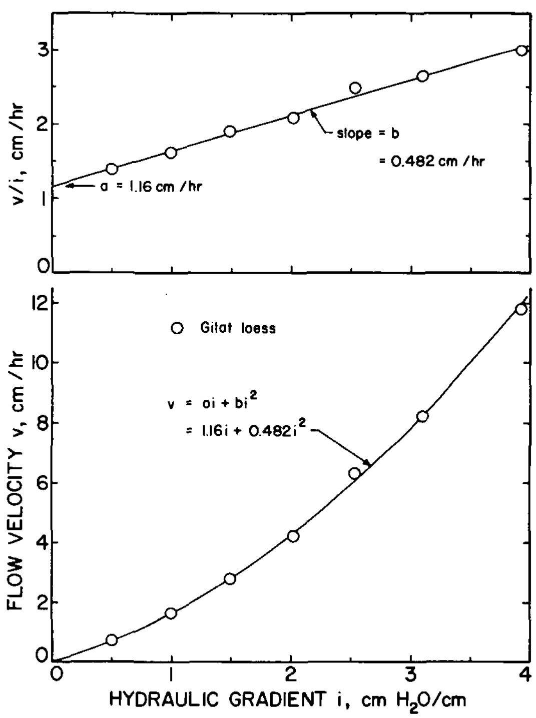

Fig. 2—Determination of constants a and b (equation [3]) for Hadas' (7) Gilat loess (*upper graph*), and comparison of resulting velocity-gradient curve with experimental values (*lower graph*).

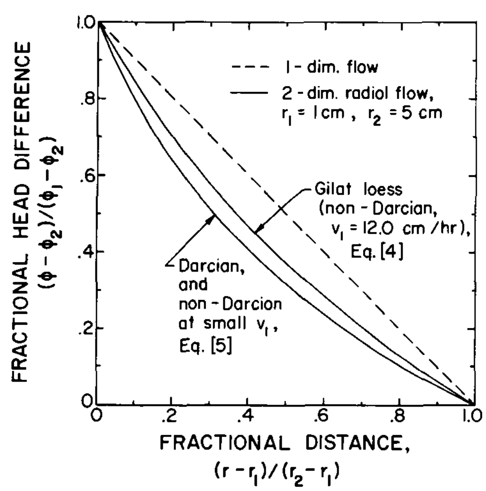

Fig. 3—Distribution of hydraulic head for both two-dimensional radial and one-dimensional flow, for the Gilat loess of Hadas (7), using the a and b of Fig. 2.

sumed uniquely valid. Non-hysteretic breakdown in $\tau(\theta)$, however, has been proposed by Gardner and Gardner (6), and experimental suggestions of such aberrancy have been given by Bazargani (J. I. Bazargani, 1964. Transient water flow characteristics of unsaturated soils. *Ph.D. Thesis. Purdue University, Lafayette, Ind.*), Davidson, Nielsen and Biggar (4), and Topp, Klute, and Peters (32).

Direct or implied experimental confirmation of velocity-gradient proportionality for unsaturated flow has been reported for coarse sand by Childs and Collis-George (1) and Watson (35), glass beads and slate dust by Youngs (36), and soils by Richards and Weeks (25) and Vachaud (33). In other soils or situations, however, Nielsen, Biggar and Davidson (17), Rawlins and Gardner (24), and Davidson, Nielsen, and Biggar (3) found that the water-content profiles for horizontal absorption did not advance as rapidly as they should have on the basis of square-root-of-time dependence required by the mathematical solution based ultimately on equation [6]. The equivalent of greater-than-proportional flow response has also been found for vertical infiltration into two different soils (J. L. Thames, 1966. Flow of water under transient conditions in unsaturated soils. *Ph.D. Thesis. University of Arizona, Tucson*).

The Rawlins and Gardner (24) data were further analyzed (28) in the form of an explicit test of equation [6], which asserts that a plot of velocity v vs. gradient w should be a straight line through the origin at a given water content θ. Proportionality did hold at the higher water contents but not at the lower ones, failing in the sense that the velocity increased more than proportionally with gradient. Since such deviations were qualitatively similar to those described previously for saturated flow, they were interpreted (28) as consistent with the hypothesis of a gradient-dependent diffusivity $D(\theta, w)$ as the cause of the various observed departures (17, 24) from proportional flow theory.

The deviating data were accommodated by a modification of equation [1] in which B, J, and C were taken as functions only of θ, along with $\tau(\theta)$. The final modified form was $v = \beta[w - \alpha(1 - e^{-\gamma w})]$, where $\beta = \beta(\theta)$, $\alpha = \alpha(\theta)$, and $\gamma = \gamma(\theta)$ are constants at a given θ. Using the modified form to evaluate $D(\theta, w) = v/w$ yields

$$D(\theta, w) = \beta[1 - \alpha(1 - e^{-\gamma w})/w] \qquad [8]$$

Note that $D(\theta, w)$, the gradient-dependent diffusivity, is the unsaturated-flow counterpart of the gradient-dependent hydraulic conductivity, $K(i)$, considered for saturated flow earlier in the present paper. Using the fitted values of β, α, and γ reported in (28), $D = D(\theta, w)$ was calculated from equation [8]. Curves of D vs. θ for selected values of w are shown in Fig. 4. The coalescing curves for $\theta > 0.35$ show that equation [6] holds in that region. However, the curves diverge markedly as the water content decreases, with the maximum discrepancy approaching 8-fold at $\theta = 0.15$.

Whether velocity v is proportional to suction gradient $\partial\tau/\partial x$ at given suction τ, as asserted by equation [7], was investigated for steady-state flow in unsaturated soils by Hadas (7). Pronounced greater-than-proportional responses were found, and could be described by equation [1]. The

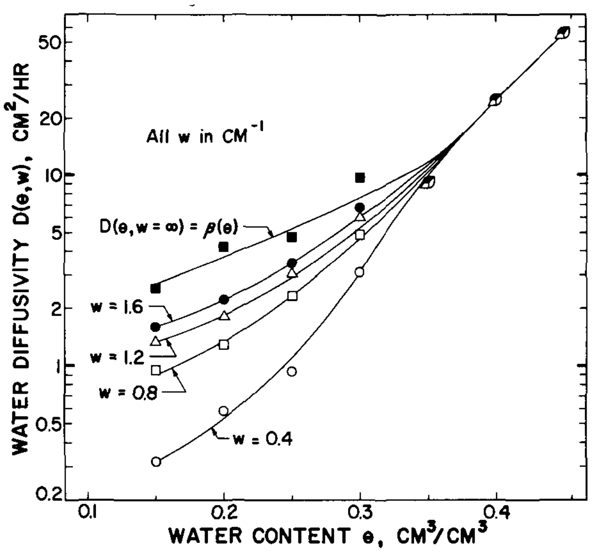

Fig. 4—Diffusivity vs. water content for specified values of water-content gradient w, calculated from equation [8] for the Salkum silty clay loam of Rawlins and Gardner (24).

deviations increased progressively as the suction increased. However, in a similar study conducted with several experimental refinements, satisfactory proportionality has recently been found for nonswelling artificial soils (T. C. Olson, 1966. Velocity-gradient relationships for steady-state flow of water in partially saturated soil. *Ph.D. Thesis. Purdue University, Lafayette, Ind.*).

One-Dimensional Water Absorption

Horizontal Flow—Consider a uniform horizontal column of soil, initially at a constant water content θ_n, to which a higher constant water content θ_o is applied at time zero. The mathematical solution (22, 30) based on equation [6] gives the volume y of water absorbed per unit soil cross sectional area by

$$y = St^{1/2} \qquad [9]$$

where t is time, and S is the constant sorptivity of Philip (22). A time differentiation of equation [9], along with its use to eliminate t itself from the time derivative, yields

$$dy/dt = S^2/2y \qquad [10]$$

which asserts the absorption rate dy/dt to be inversely proportional to y, the quantity of water absorbed.

Using the data of Rawlins and Gardner (24), y was computed from areas under the water-content profiles at various times; dy/dt was then determined from the slope of the $y(t)$ curve by the prism method (31). Fig. 5 shows a log-log plot of dy/dt vs. y, which, according to equation [10], should be a straight line of slope -1. Such a line is drawn through the first point in the figure, and successive points fall progressively below the line. At $y = 17.3$ cm, the theoretical value specified by the straight line is 40% above the observed point. This discrepancy, however, is considerably less than the 8-fold variation observed in the $D(\theta, w)$ curves of Fig. 4 at low water contents. Apparently, the single-curve, wet-end diffusivity tends to dominate the flow process for the high θ_o ($= 0.557$) used by Rawlins and Gardner (24). Greater discrepancies in Fig. 5 would be expected for a smaller θ_o if it were near or within the region of diverging and separated $D(\theta, w)$ curves in Fig. 4. The failure of the 8-fold reduction in dry-end diffusivity to produce a comparable reduction in absorption rate is an effect qualitatively similar to the computer-analysis results of Hanks and Bowers (8) based on velocity-gradient proportionality. Their work, however, implies an effect of substantially smaller magnitude than the maximum discrepancy shown in Fig. 5.

Vertical Flow (Infiltration)—The simplified theoretical approach of Philip (21) can be utilized with equation [1] to derive an infiltration equation on the basis of non-Darcian flow behavior. The APPENDIX contains the mathematical detail, which, for a uniform soil at constant initial water content θ_n, yields

$$\frac{dy}{B_h dt} = 1 + 1/Fy - J_h[1 - e^{-(1+1/Fy)/J_h}] \qquad [11]$$

where B_h is a constant, $F = [(\bar{\theta} - \theta_n)(H + P)]^{-1}$, $\bar{\theta}$ is the constant mean water content behind the wetting front, H is the constant depth of water on the soil surface, and P is a constant-head component arising from the capillarity of the soil. J_h is the measure of deviation from Darcian-type proportionality, and, if $J_h = 0$, equation [11] reduces to

$$\frac{dy}{B_h dt} = 1 + 1/Fy \qquad [12]$$

which in essence is Philip's (21) result.

Later, a short-time approximation of a rigorous mathematical solution based on Darcian proportionality was given by Philip (22) as

$$y = St^{1/2} + Gt \qquad [13]$$

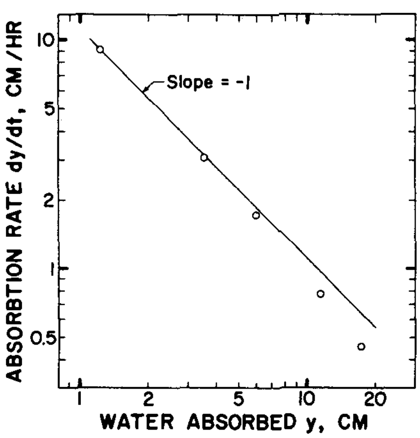

Fig. 5—Log-log plot (circles) of water absorption rate vs. quantity of water absorbed, for the Salkum silty clay loam of Rawlins and Gardner (24). The straight line represents equation [10].

where G is a constant. Solving equation [13] for $t^{1/2}$, by using the standard quadratic formula and choosing the positive root, yields $t^{1/2} = [(S^2 + 4Gy)^{1/2} - S]/2G$. If this expression is substituted for $t^{1/2}$ in the dy/dt obtained by differentiating equation [13], we eventually get

$$\frac{dy}{Gdt} = \frac{1 + 2Iy + (1 + 2Iy)^{1/2}}{Iy} \qquad [14]$$

where the constant $I = 2G/S^2$.

A convenient way of visualizing equations [11], [12], and [14] is to prepare dimensionless plots on log-log graph paper. From equations [11] and [12], $dy/B_h dt$ is computed as a function of Fy. From equation [14], dy/Gdt is computed as a function of Iy. Equations [12] and [14] result in one curve each, but equation [11] yields a curve for each selected value of J_h. All of the various theoretical curves are plotted in Fig. 6. Note that the two Darcian curves (equations [12] and [14]) are almost alike, as has already been pointed out by Philip (22).

Using a second sheet of semitransparent log-log paper of the same scale size as that of the theoretical curves, experimental values of dy/dt can be plotted against y. Comparison with the theoretical curves is made by placing this second sheet, called the overlay, upon the theoretical curves with the coordinate axes of both graphs aligned. The overlay is then translated vertically and horizontally without rotation until a satisfactory fit with a theoretical curve is found. The process will be illustrated with data obtained for downward infiltration of constant-depth ponded water into vertical laboratory columns of a uniformly-packed weight-basis mixture of 93.75% Ottawa[5] banding sand and 6.25% Ottawa[5] no. 290 ground silica. Each data point is the average of duplicate determinations on separate columns.

[5] Ottawa Silica Company, Ottawa, Illinois.

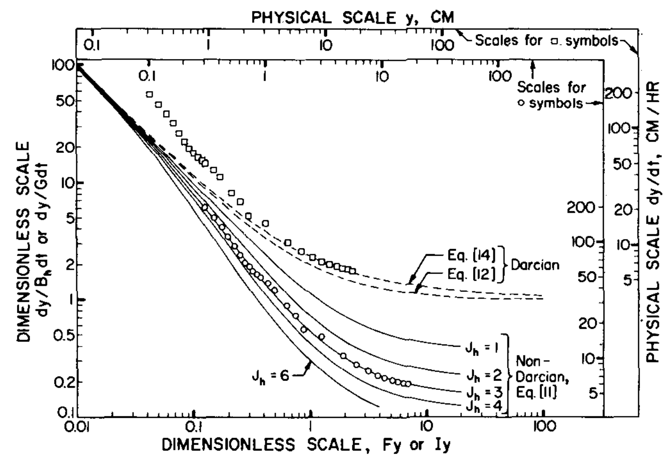

Fig. 6—Log-log plots of infiltration rate vs. quantity of water infiltrated, including theoretical curves (solid and broken lines), and experimental values (circles and squares) for a weight-basis mixture of 6.25% ground silica[5] and 93.75% banding sand.[5]

The square symbols in Fig. 6 represent an attempt to match the data with a Darcian curve (equation [14]), which is successful only for the higher values of y. As fitted, the initial experimental value of dy/dt is 2.15 times higher than the corresponding value on the theoretical curve. If the experimental points were all translated vertically downward until the initial point agreed with the theoretical curve, the later theoretical values would eventually exceed their experimental counterparts by 115%. This would be in qualitative agreement with Fig. 5 for horizontal flow, although the magnitude of the discrepancy is substantially larger than in Fig. 5.

The circular symbols in Fig. 6 demonstrate the precise manner in which the same data conform to the non-Darcian curve for $J_h = 3$. It must be granted that the theoretical development of equation [11] is oversimplified on several grounds, including the exclusion of possible deviating behavior from the gradient term i_h (see APPENDIX). This exclusion is somewhat analogous to assuming that deviating behavior in unsaturated flow is caused by a breakdown in $K(\theta)$ rather than in $\tau(\theta)$. Regardless of the reason, however, equation [11] provides a distinctly improved fit for the data here considered.

CONCLUDING COMMENTS

The several foregoing illustrations of greater-than-proportional behavior in both saturated and unsaturated flow should alert the soil-water scientist to the possible existence of such effects in various flow problems of interest. This is not to imply that Darcian or proportional-type approaches and analyses are to be summarily dismissed or abandoned. As pointed out in the introduction, Darcian behavior may be closer to the rule than the exception in saturated sands and silts. Also, proportional analysis provides a standard against which nonproportionalities can be evaluated. But it would also seem that the presence of clay in a porous medium could well reduce Darcian or proportional validity to the level of a working-hypothesis type of first approximation, requiring individual validation for the particular case in point. Admittedly, this validity must be judged in terms of the overall experimental precision involved; that is, there is obviously no point in showing concern about 2- to 4-fold variations in hydraulic conductivity or water diffusivity if the overall error of measurement is 4-fold or greater.

For saturated flow, it should not be too difficult for an investigator to conduct several quick checks to determine the extent of applicability of Darcian proportionality. The corresponding check for unsaturated flow is obviously much more difficult and involved, but it is here also that more investigation is still needed. Desaturation conceivably could enhance nonproportionalities, especially if they were due to modified water properties induced by particle-water interaction, since the water remaining in the unsaturated material would be moving in closer proximity to particle surfaces.

Areas where nonproportionalities might have relevant implications will now be enumerated briefly. Flow of water to plant roots frequently occurs at low water contents, and hence could involve the gradient-induced shifts in the low-water-content range of Fig. 4. Soil-water redistribution and field-capacity phenomena could receive new explanation if a threshold gradient occurs at some critical water content, so that water at contents below this critical value would be almost immobilized in the soil, except for removal by plant roots or vapor transfer. Threshold gradients in porous plates or membranes might also affect suction measurements, as considered by Hillel and Mottes (10). Present difficulties with the outflow method of measuring soil-water diffusivity might have some origin in nonproportionalities, as discussed recently (29, 30). Finally, miscible displacement of ions or solutions in the soil water may be affected by nonproportionality of flow, the process having conceivable bearing on salt leaching or soluble fertilizer movement.

APPENDIX

Two-Dimensional Radial-Flow Solution

Let v be the flow velocity along the radial distance coordinate r, while ϕ is the hydraulic head. Then, $i = -d\phi/dr$, so that equation [3] becomes

$$v = -ad\phi/dr + b(d\phi/dr)^2. \qquad [15]$$

The equation of continuity for steady-state, incompressible, two-dimensional radial flow is $d(rv)/dr = 0$, which integrates to $rv = N' = \text{const}$. Evaluating N' at the flow inlet boundary, where $r = r_1$ and $v = v_1$, yields $N' = r_1v_1 = rv$, so that $v = r_1v_1/r$. Substituting r_1v_1/r for v in equation [15] and using the standard quadratic formula to solve for $d\phi/dr$, yields

$$\frac{d\phi}{dr} = \frac{a}{2b} - \frac{a(r^2 + 4br_1v_1r/a^2)^{1/2}}{2br}. \qquad [16]$$

The negative sign in equation [16] arises from choosing the negative root in the standard quadratic formula, and is needed to make ϕ decrease with r.

Separating variables and integrating, using formulas no. 187 and no. 160 in Peirce (20) for the radical portion in equation [16], yields, after suitable manipulation

$$\phi = \frac{ar(1-R)}{2b} + \frac{r_1v_1}{a}\ln\frac{R-1}{R+1} + N \qquad [17]$$

where N is the constant of integration, and

$$R = (1 + 4br_1v_1/a^2r)^{1/2}. \qquad [18]$$

At the flow outlet boundary $r = r_2$ we have $\phi = \phi_2$, which is used in equations [17] and [18] to express ϕ_2. If this expression for ϕ_2 is subtracted from ϕ of equation [17], the result is

$$\phi - \phi_2 = \frac{r - r_2 + r_2R_2 - rR}{2(b/a)} + \frac{r_1v_1}{a}\ln\frac{(R-1)(R_2+1)}{(R+1)(R_2-1)} \qquad [19]$$

where

$$R_2 = (1 + 4br_1v_1/a^2r_2)^{1/2}$$

from equation [18]. At the flow inlet boundary $r = r_1$, $\phi = \phi_1$, and, from equation [18],

$$R = R_1 = (1 + 4bv_1/a^2)^{1/2}.$$

These conditions are used in equation [19] to evaluate $\phi_1 - \phi_2$, the total difference in hydraulic head, in turn used with equation [19] to form the ratio $(\phi - \phi_2)/(\phi_1 - \phi_2)$, which yields equation [4] of the text.

Non-Darcian Infiltration Equation

In a simplified and approximate theoretical approach to the infiltration process, Philip (21) takes $\bar{\theta}$, the mean water content from the soil surface to the wetting front, to be a constant. Hence

$$y = (\bar{\theta} - \theta_n) z_n \qquad [20]$$

where y is the surface-depth quantity of water infiltrated, θ_n is the constant initial water content, and z_n is the depth of the wetting front below the soil surface. The infiltration velocity dy/dt is taken to be of the Darcian form

$$dy/dt = K_h i_h,$$

with K_h a constant and $i_h = (H + P + z_n)/z_n$, where H is the depth of ponded water on the soil surface, and P is a constant-head component arising from the capillarity of the soil. Making use of equation [20] expresses i_h as

$$i_h = [(\bar{\theta} - \theta_n)(H + P) + y]/y. \qquad [21]$$

To modify the approach for non-Darcian behavior, use equation [1] but with $C = 1/J$, which results in a two-parameter equation of flow (26). Then, in this modification of equation [1], write B, J, and i with subscript h and set $v = dy/dt$. Using equation [21] to express i_h results in equation [11] of the text.

LITERATURE CITED

1. Childs, E. C., and N. Collis-George. 1950. The permeability of porous materials. Proc. Roy. Soc. Lond. 201 A: 392–405.
2. Darcy, H. 1856. Les Fontaines Publique de la Ville de Dijon, p. 592. Victor Dalmont, Paris.
3. Davidson, J. M., D. R. Nielsen, and J. W. Biggar. 1963. The measurement and description of water flow through Columbia silt loam and Hesperia sandy loam. Hilgardia 34:601–617.
4. Davidson, J. M., D. R. Nielsen, and J. W. Biggar. 1966. The dependence of soil-water uptake and release upon the applied pressure increment. Soil Sci. Soc. Amer. Proc. 30:298–304.
5. Fancher, G. H., and J. A. Lewis. 1933. Flow of simple fluids through porous materials. Ind. Eng. Chem. 25:1139–1147.
6. Gardner, W. H., and W. Gardner. 1951. Flow of soil moisture in the unsaturated state. Soil Sci. Soc. Amer. Proc. (1950) 15:42–50.
7. Hadas, A. 1964. Deviations from Darcy's law for the flow of water in unsaturated soils. Israel J. Agr. Res. 14:159–168.
8. Hanks, R. J., and S. A. Bowers. 1963. Influence of variations in the diffusivity-water content relation on infiltration. Soil Sci. Soc. Amer. Proc. 27:263–265.
9. Hansbo, S. 1960. Consolidation of clay, with special reference to influence of vertical sand drains. Swed. Geotech. Inst. Proc. 18, Stockholm.
10. Hillel, D., and J. Mottes. 1966. Effect of plate impedance, wetting method, and aging on soil moisture retention. Soil Sci. 102:135–139.
11. Kirkham, D. 1957. The ponded water case. *In* J. N. Luthin [ed.] Drainage of agricultural lands. Agronomy 7:139–181.
12. Kutilek, M. 1964. The filtration of water in soils in the region of the laminar flow. Int. Congr. Soil Sci., Trans. 8th (Bucharest, Romania) II:45–52.
13. Kutilek, M. 1965. Influence de l'interface sur la filtration de l'eau dans les sols. Science du Sol 1:3–14.
14. Low, P. F. 1961. Physical chemistry of clay-water interaction. *In* A. G. Norman, ed. Advances. Agron. 13:269–327.
15. Lutz, J. F., and W. D. Kemper. 1959. Instrinsic permeability of clay as affected by clay-water interaction. Soil Sci. 88:83–90.
16. Miller, R. J., and P. F. Low. 1963. Threshold gradient for water flow in clay systems. Soil Sci. Soc. Amer. 27:605–609.
17. Nielsen, D. R., J. W. Biggar, and J. M. Davidson. 1962. Experimental consideration of diffusion analysis in unsaturated flow problems. Soil Sci. Soc. Amer. Proc. 26:107–111.
18. Olsen, H. W. 1965. Deviations from Darcy's law in saturated clays. Soil Sci. Soc. Amer. Proc. 29:135–140.
19. Olsen, H. W. 1966. Darcy's law in saturated kaolinite. Water Resources Res. 2:287–295.
20. Peirce, B. O. 1929. A short table of integrals. 3rd rev. ed. Ginn & Company, Boston.
21. Philip, J. R. 1954. An infiltration equation with physical significance. Soil Sci. 77:153–157.
22. Philip, J. R. 1957. The theory of infiltration: 4. Sorptivity and algebraic infiltration equations. Soil Sci. 84:257–264.
23. Raats, P. A. C. 1967. Non-Darcy flow in soils. Proc. Int. Soil Water Symp. (Prague, Czechoslovakia) I:1–8.
24. Rawlins, S. L., and W. H. Gardner. 1963. A test of the validity of the diffusion equation for unsaturated flow of soil water. Soil Sci. Soc. Amer. Proc. 27:507–511.
25. Richards, S. J., and L. V. Weeks. 1953. Capillary conductivity values from moisture yield and tension measurements on soil columns. Soil Sci. Soc. Amer. Proc. 17:206–209.
26. Swartzendruber, D. 1962. Modification of Darcy's law for the flow of water in soils. Soil Sci. 93:22–29.
27. Swartzendruber, D. 1962. Non-Darcy flow behavior in liquid-saturated porous media. J. Geophys. Res. 67:5205–5213.
28. Swartzendruber, D. 1963. Non-Darcy behavior and the flow of water in unsaturated soils. Soil Sci. Soc. Amer. Proc. 27:491–495.
29. Swartzendruber, D. 1966. Soil-water behavior as described by transport coefficients and functions, Advance Agron. 18:327–370.
30. Swartzendruber, D. 1968. The flow of water in unsaturated soils. *In* R. J. M. DeWiest [ed.] Flow through porous media. Academic Press, New York (in press).
31. Swartzendruber, D., and T. C. Olson. 1963. Rate of change as determined graphically with an equilateral glass prism. Soil Sci. Soc. Amer. Proc. 27:108–110.
32. Topp, G. C., A. Klute, and D. B. Peters. Comparison of water content-pressure head data obtained by equilibrium, steady-state, and unsteady-state methods. Soil Sci. Soc. Amer. Proc. 31:312–314.
33. Vachaud, G. 1967. Determination of the hydraulic conductivity of unsaturated soils from an analysis of transient flow data. Water Resources Res. 3:697–705.
34. Von Engelhardt, W., and W. L. M. Tunn. 1955. The flow of fluids through sandstones (translated by P. A. Witherspoon from Heidelberger Beitrage zur Mineralogie und Petrographie 2:12–25. 1954). Illinois State Geol. Survey Circ. 194.
35. Watson, K. K. 1966. An instantaneous profile method for determining the hydraulic conductivity of unsaturated porous materials. Water Resources Res. 2:709–715.
36. Youngs, E. G. 1957. Moisture profiles during vertical infiltration. Soil Sci. 84:283–290.