MODIFICATION OF DARCY'S LAW FOR THE FLOW OF WATER IN SOILS

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The classic equation of water movement set forth by Henry Darcy (1) in 1856 has occupied a unique place in the study of fluid flow through porous media. Darcy's law may be written as

$$v = Ki \qquad (1)$$

where v is the macroscopic flow velocity; i, the hydraulic gradient; and K, the hydraulic conductivity. Following Richards (14), K itself is written as

$$K = k\rho g/\eta \qquad (2)$$

where k is the intrinsic permeability; η, the fluid viscosity; ρ the fluid density (water density if i is in terms of water column); and g, the acceleration of gravity.

The simple proportionality between flow velocity and hydraulic gradient [equation (*1*)] is the condition of linearity which combines with the law of conservation of mass to yield the Laplace, heat-type, or diffusion-type equations, depending upon circumstances. Thus, the methods of mathematical physics are at once applicable, and the study of fluid flow through porous media is enhanced by mathematical techniques of considerable power and elegance. This is exemplified by the treatments of Slichter (16), Muskat (11), Polubarinova-Kochina (12), and Luthin (9). An extensive listing of the literature is provided by the recent book of Scheidegger (15).

Darcy himself recognized that his relationship was not valid for high fluid velocities. During the past forty years, much research effort has been concerned with the nature of this deviation, which occurs at large hydraulic gradients (11, 15). It seems well established that when the hydraulic gradient exceeds a critical value, the flow velocity is no longer proportional to the hydraulic gradient, but increases less rapidly than the gradient. This could be accounted for by saying that, since viscous forces no longer mask the inertia and turbulence forces, not all the driving force of the hydraulic gradient is used to overcome viscous resistance.

Compared with the extensive study conducted on deviations at large gradients, much less work has been directed toward the testing of Darcy's law for liquid flow at low gradients, even in soils, where such flow is common. Hubbert (5) simply indicated that there was no apparent reason to suspect failure at low gradients. Muskat (11) seemed not even to raise the question. Scheidegger (15), however, has recognized the possibility of deviations arising from a so-called "boundary effect," from ions in solution, and from non-Newtonian fluids.

The present paper considers the flow of water and electrolyte solutions through various porous media. A modified flow equation is proposed, based upon published data of several investigators.

OBSERVED DEPARTURES FROM DARCY'S LAW

As early as 1898, King (6) reported for low gradients in various porous media that the flow velocity was not proportional to the gradient; instead, the velocity increased more than proportionally with the gradient. This stood in contrast to the less-than-proportional increases at high gradients, of which King was well aware. Since his paper appears in the same volume as that of Slichter (16), it is interesting to speculate why it did not receive similar recognition. Perhaps the beauty of Slichter's mathematical extension was more compelling.

The flow media used by King included tube-confined brass-wire gauze discs, sand, sandstone, and ordinary glass capillaries. For all of these he reported greater-than-proportional increases for flow velocity versus gradient, but his evidence for the capillary tubes is perhaps the least convincing. Not only were the deviations small, but they essentially disappeared when evaporation from the outflow water droplets was prevented.

It is deemed instructive to present some of King's results in terms of hydraulic gradient,

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since he did not express them this way. For the brass-wire gauze, figure 1 shows that the zero-flow extrapolation of a straight line drawn through the points results in a gradient intercept of less than 0.002. For all practical purposes, Darcy's law is verified even though King's detailed calculations indicated a departure.

For the sand data in figure 1, a small but definite gradient intercept of about 0.2 is indicated; this becomes 16.5 for the Madison sandstone. Thus, even though the experimental points fall on a straight line, the line does not pass through the origin. Hence, a calculation of K by equation (*1*) at various gradients would show the hydraulic conductivity to increase with gradient. The greatest change would occur at small gradients.

In much of King's data there is evidence that, at constant gradient and temperature, his systems were subject to flow reduction with time. Perhaps, as reported recently (3), this might have been caused by bacterial activity. However, for the Madison sandstone of figure 1, which shows the largest gradient intercept, the check points taken at the end of the flow period appear to rule out any time-reduction effects.

During the past decade, more data similar to those of King have accumulated. Von Engelhardt and Tunn (17) conducted a particularly careful set of expriments in which no data were reported unless the initial results of a flow experiment could be duplicated at the end of the experiment. In Hansbo's (4) work, care also was taken to eliminate the effect of time changes in flow rate. Lutz and Kemper (10) obtained data for both successively increasing and decreasing gradients. Low (7) reported some results for high gradients. All these data will be analyzed in detail in the remainder of the paper.

ANALYSIS

Types of flow behavior

Some rheological considerations are demonstrated in A of figure 2 by the consistency curves for three materials. Newtonian flow is shown as a straight line passing through the origin. For non-Newtonian flow, the curve passes through the origin but is no longer straight. For Bingham flow, the curve is again linear but only for shear stresses in excess of the yield stress τ_0.

For a Bingham material, consider a capillary tube of length L, radius R, and in which the flowing material of viscosity η is subject to a pressure difference $\Delta P = P_2 - P_1$, where P_1 and P_2 are applied, respectively, at the inlet and outlet ends of the tube. The hydraulic gradient is $i = -\Delta P/\rho g L$, where ρ is the density and g is the acceleration of gravity. The relation between flow velocity v and gradient i is

$$v = \frac{R^2 \rho g i}{8\eta}\left[1 - \frac{4}{3}\frac{i_0}{i} + \frac{1}{3}\left(\frac{i_0}{i}\right)^4\right] \qquad (3)$$

and is called the Buckingham-Reiner equation (13). The quantity i_0 is a threshold gradient, and for the capillary tube is directly related to the yield stress by $i_0 = 2\tau_0/\rho g R$. The characteristic aspects of equation (*3*) are shown in B of figure 2. When $\tau_0 = 0$, then $i_0 = 0$, and equation (*3*) reduces to Poiseuille's law as it should.

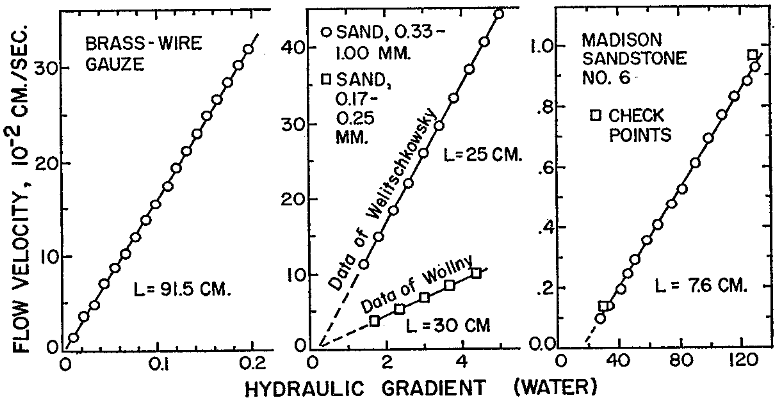

FIG. 1. Flow velocity versus hydraulic gradient for water in various porous media; data as reported by King (6). L denotes sample length. *Square symbols* for Madison sandstone are check points obtained at the end of the flow period.

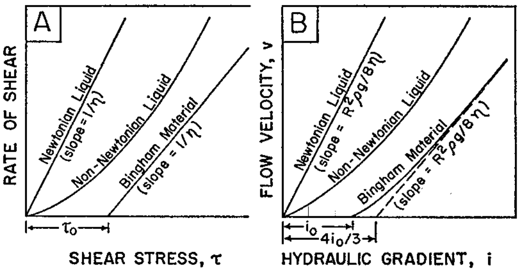

FIG. 2. (A) Shear rate versus shear stress for three flowing materials. (B) Flow velocity versus gradient for the same materials flowing in a capillary tube. R is the tube radius; ρ, the density; g, the acceleration of gravity; and η, viscosity.

For both Newtonian and non-Newtonian flow, there is no qualitative change in the type of curve as one passes from the shear rate-shear stress graph (A, fig. 2) to the flow-velocity–hydraulic-gradient graph (B, fig. 2). That is, for the Newtonian liquid, both relationships are straight lines through the origin, though of different slope. For the non-Newtonian liquid, both curves begin at the origin and then bend upward. For the Bingham material, however, τ_0 produces the threshold gradient i_0, such that $v = 0$ for $0 \leq i \leq i_0$. The relationship then curves upward as i exceeds i_0, but it eventually approaches the straight line of slope $R^2\rho g/8\eta$ and i-intercept $4i_0/3$.

Development of a modified flow equation

The assertion of Darcy's law is analogous to the graph of the Newtonian liquid (B, fig. 2). If the flow is not Newtonian, it is then either non-Newtonian or Bingham. Von Engelhardt and Tunn (17) and Low (8) have suggested the possibility of non-Newtonian behavior, but they did not propose a mathematical expression.

The straight lines for sand and sandstone in figure 1 are suggestive of the straight-line asymptote of the Bingham curve in B of figure 2. However, these lines (sand and sandstone, fig. 1) conceivably could bend through the origin if the necessary experimental points were available. Furthermore, the data of Von Engelhardt and Tunn (17), Lutz and Kemper (10), and Hansbo (4) indicate that such curvature does occur, which is suggestive of non-Newtonian flow (B of fig. 2). While recognizing that threshold gradients have been reported for ceramic and charcoal filters (2), the writer feels that adequate experimental proof of such gradients for sandstones and clays is, as yet, lacking. Hence, it is felt that the qualitative aspects of Hansbo's (4) approach are, at present, the most reasonable. These are shown in figure 3 by the solid-line curve which begins at the origin, bends upward, and then approaches a straight line which extrapolates to an i-intercept designated by I. This choice of curve combines the origin and curvature aspects of non-Newtonian flow with the straight-line aspect of Bingham flow. Further, for the gradient range in figure 3 it is assumed that deviations due to inertia and turbulence do not appear.

Hansbo suggested that the curved part of figure 3 be represented mathematically by a power function, and that this be replaced by a linear function when the curve has straightened out. Altogether, three independent parameters (or constants) are required. While the two mathematical forms are a complication, a worse drawback is that the three parameters cannot be evaluated unless data are available all the way from $i = 0$ out to and including an appreciable part of the linear portion of the velocity-gradient curve.

To circumvent these difficulties, the relationship was postulated as

$$dv/di = M(1 - e^{-i/I}) \qquad (4)$$

where M and I are constants. Integrating, and using $v = 0$ at $i = 0$ (for non-Newtonian flow), leads to

$$v = M[i - I(1 - e^{-i/I})] \qquad (5)$$

which then is the modified flow equation.

As i increases in equation (5), $e^{-i/I}$ approaches

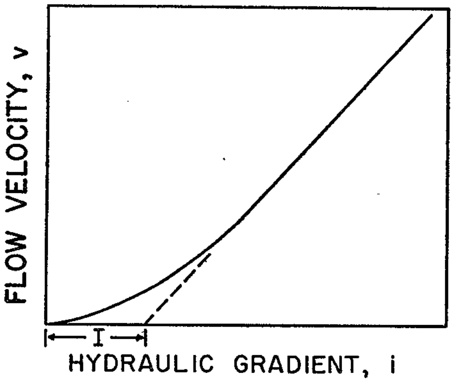

FIG. 3. General aspects of the suggested flow relationship.

zero, with the result

$$v = M(i - I) \qquad (6)$$

which is the equation of a straight line of slope M and i-intercept I. Thus, equation (5) is a two-parameter expression which possesses the behavior of figure 3. Note that I is not a threshold gradient. Instead it is an extrapolated intercept, and is a measure of non-Newtonian behavior.

Because of the similarity between M and K [compare equation (6) with equation (1)], it would seem reasonable to partition M in the manner of Richards (14) by writing

$$M = m\rho g/\eta \qquad (7)$$

where m is determined by particle geometry, and η is the normal bulk viscosity of the liquid. Perhaps the matter of nomenclature is premature, but to avoid confusion with Darcy's law, M could be called the hydraulic conductance, and m the permeance. The constant I could be called the non-Newtonian index. When $I = 0$, the behavior is Newtonian, and equation (5) reduces to Darcy's law, with $M = K$; also, $m = k$.

Fitting of constants

Define a variable u by

$$u = \int_0^i v di \qquad (8)$$

Combining this with equation (5) yields

$$u = Mi^2/2 - I[Mi - MI(1 - e^{-i/I})] \qquad (9)$$

where the quantity in square brackets is simply v of equation (5). Hence, with further rearrangement, equation (9) leads to

$$\frac{i^2}{v} = \frac{2}{M}\left(\frac{u}{v}\right) + \frac{2I}{M} \qquad (10)$$

To evaluate M and I in a given instance, v was plotted against i, and a smooth, "best-fitting" curve was drawn through the points by eye. Since u of equation (8) is simply the area under the curve for a given i, areas were determined both with a planimeter and by summation of narrow trapezoidal sections, with the latter method being the fastest. The resulting simultaneous values of u, v, and i permitted calculation and plotting of i^2/v versus u/v. For a straight line drawn through the points, equation (10) indicates the slope as $2/M$ and the intercept as $2\,I/M$. It was found the i^2/v and u/v were quite sensitive to relative experimental errors. Hence the straight line for equation (10) was generally drawn to give greater weight to experimental points derived from the larger gradients.

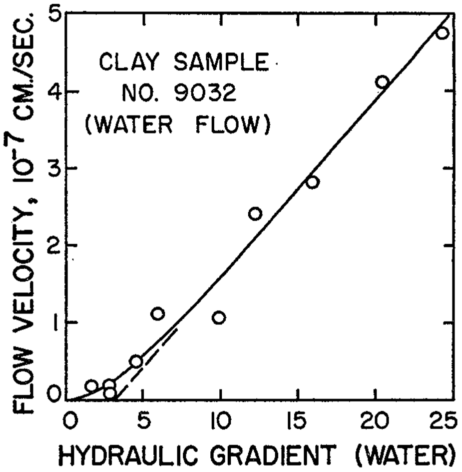

FIG. 4. Typical fit of the data of Hansbo (4), using equation (5).

The quality of curve-fitting is shown by representative examples. Figure 4 illustrates Hansbo's (4) data, figure 5 those of Von Engelhardt and Tunn (17), and figure 6 those of Lutz and Kemper (10). In each case the solid-line curve is drawn from equation (5), using M and I as fitted, while the experimental points are shown as circles. The dashed straight line is the portion of the linear asymptote shown as it intersects the i-axis. On the basis of figures 4, 5, and 6, the quality of fit is deemed adequate.

Behavior of flow parameters

For the sandstones of Von Engelhardt and Tunn, equation (7) was used to calculate m-values which are shown in table 1 along with I and some Darcy quantities. Notice that m always exceeds the liquid k_{max}, as would be expected since Darcy's law presumes a straight line through the origin for v versus i. However, m is always less than the air permeability; this is reasonable, since some particle rearrangement might be expected when the clay-bearing sandstone is wetted with water or salt solution. Finally, in every case m increases with salt concentration. This too is expected, since for higher salt concentrations the clay should be less subject to deflocculation and dispersion.

With one exception, I decreases as the salt

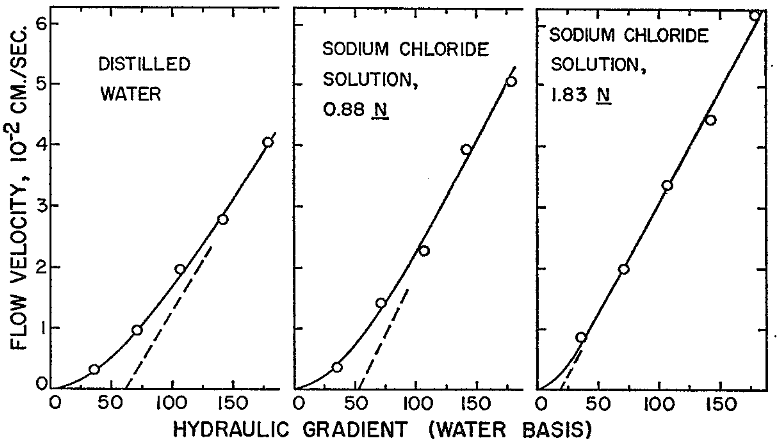

FIG. 5. Typical fit of the data of Von Engelhardt and Tunn (17) for sandstone sample No. 15L, using equation (*5*).

concentration increases. This implies that pure water exhibits the greatest degree of non-Newtonian behavior. Adding salt causes the behavior to be more in the direction of Newtonian. However, a substantial amount of non-Newtonian behavior still remains, since all values of I in table 1 are in excess of zero.

Results for the clays of Lutz and Kemper are shown in table 2. Air permeabilities were not available. In most cases m exceeds the liquid k_{max}. Again, m increases with increasing electrolyte concentration, except for hydrogen-saturated Utah bentonite where it remains constant. Again, I decreases with increasing electrolyte concentration, with the single exception of 0.05 N sodium chloride in sodium-saturated Bladen clay. As before, pure water exhibits the maximal non-Newtonian behavior, and in the sodium-saturated clays, a definite degree of this behavior remains even at the highest electrolyte concentrations. In contrast, however, the hydrogen-saturated clays do become essentially Newtonian as the hydrochloric acid concentration is increased.

For Low's (7) sodium bentonite, subjected to water flow at gradients ranging from 11000 to 18000, M and I were evaluated for a straight line fitted by least squares [equation (*6*)]. The resulting I-value of 327 is in reasonable relation to the I-values for bentonite in table 2. The m-value

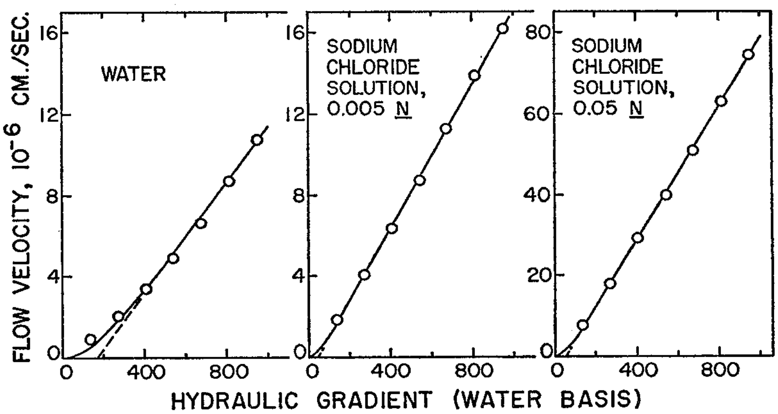

FIG. 6. Typical fit of the data of Lutz and Kemper (10) for sodium-saturated Bladen clay, using equation (*5*).

of $0.153 \times 10^{-5}\mu^2$ is also compatible, since Low's clay was likely packed more tightly than those of Lutz and Kemper.

Mathematical extension

By writing the derivative form of the gradient as $i_x = -\partial\varphi/\partial x$, we have for the flow velocity in the x-direction

$$v_x = -M[\partial\varphi/\partial x + I(1 - e^{\partial\varphi/I\partial x})] \quad (11)$$

where φ is the hydraulic head. The equation of continuity for steady state, incompressible flow in three dimensions is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad (12)$$

For an isotropic, homogeneous porous medium, v_y and v_z can be written as equation *(11)* with y or z replacing x. Doing this, and substituting the results into equation *(12)* yields

$$\frac{\partial^2\varphi}{\partial x^2}(1 - e^{\partial\varphi/I\partial x}) + \frac{\partial^2\varphi}{\partial y^2}(1 - e^{\partial\varphi/I\partial y}) + \frac{\partial^2\varphi}{\partial z^2}(1 - e^{\partial\varphi/I\partial y}) = 0 \quad (13)$$

TABLE 1

Flow parameters for the data of von Engelhardt and Tunn (17)

Sample No.	NaCl Conc., N	Darcy's Law k_{max}, Liquid* μ^2	Darcy's Law k_{air} μ^2	Modified Equation m μ^2	Modified Equation I
10L	0	0.36	0.66	0.49	47
13L	0	0.52	1.08	0.92	86
14L	0	0.16		0.19	29
	0.88	0.20	0.37	0.22	25
15L	0	0.22		0.32	62
	0.88	0.29		0.42	53
	1.83	0.42	0.57	0.46	19
16L	0	0.21		0.25	35
	0.88	0.19		0.27	68
	1.83	0.30	0.44	0.32	11
17L	0	0.18		0.26	54
	3.27	0.38	0.70	0.50	42

* Calculated for the largest gradient.

TABLE 2

Flow parameters for the data of Lutz and Kemper (10)

Sample	Electrolyte* Conc., N	k_{max} Liquid† $10^{-5}\mu^2$	Modified Equation m $10^{-5}\mu^2$	Modified Equation I
Na-sat.	0	7.5	13.5	506
Utah bentonite	0.005	18.0	30.6	436
	0.500	48.5	52.4	84
H-sat.	0	98.6	100.0	27.0
Utah bentonite	0.005	102.3	100.0	1.0
Na-sat.	0	106.1	115.8	81.3
halloysite	0.005	134.6	131.9	19.4
	0.500	359.0	353.0	19.0
H-sat.	0	97.1	98.6	26.6
halloysite	0.005	101.0	100.6	12.3
	0.050	126.0	126.0	0.0
Na-sat.	0	1.16	1.39	170.0
Bladen clay	0.005	1.74	1.83	45.6
	0.050	7.97	8.44	54.1
H-sat.	0	0.74	1.05	307.0
Wyoming bentonite	0.005	1.89	2.17	122.3
	0.050	27.07	29.70	91.7

* NaCl for Na-sat. clays, HCl for H-sat. clays.
† Calculated for the largest gradient.

Equation *(13)* is thus the governing partial differential equation which arises from the modified flow equation [equation *(5)*]. Remembering that the first derivatives of φ with respect to x, y, and z are inherently negative, equation *(13)* reduces to Laplace's equation when $I = 0$, as it should. However, the solution of equation *(13)* for various boundary conditions will be much more difficult than the corresponding solutions for Laplace's equation. For certain ranges of gradient, the series expansions for e might enable the solution of equation *(13)* with sufficient accuracy.

DISCUSSION

Let us consider some conditions within a porous medium which could give rise to the behavior specified by equations *(5)* and *(7)*. In non-Newtonian flow, the viscosity, taken as a point function, is no longer constant throughout the liquid region, but decreases with the shear rate. This is seen from A of figure 2 by taking the slope of the non-Newtonian curve to be the reciprocal of the variable point viscosity.

Now the reduction in viscosity with shear rate leads consequently to the more-than-proportional increase in the velocity-gradient curve (B of fig. 2). Hence, it would appear reasonable for a porous medium that the reduction of point viscosity with shear rate is giving rise to the early, upward-curving aspect of equation (*5*). If, for increasing gradients, the variable point viscosity is eventually reduced to a uniform and constant value, the linear portion of equation (*5*) could result. Its slope M could then involve the normal bulk viscosity η and the permeance m as expressed by equation (*7*).

With the limit of the variable point viscosity presumed to be the normal bulk viscosity, the concept just outlined provides for liquid domains in the porous medium where the point viscosity is higher than the normal bulk viscosity. Such a condition could be accounted for by a water structure as postulated by Low (8). This would allow for the reduction of the non-Newtonian index I with added electrolyte, since ions usually break down a water structure.

A gradient-dependent electroviscous effect might possibly be proposed to account for equation (*5*). It would seem necessary, however, to restrict this to water and relatively dilute electrolyte solutions, because in concentrated solutions the effect is usually negligible.

In any event, the modified flow equation [equation (*5*)] has been found to fit satisfactorily the data of four independent investigators, and its fitted parameters behave consistently. The equation is a compatible modification, in that Darcy's law is included as a special case. Finally, it provides two parameters to describe the flow of water and ionic solutions in the presence of clay; heretofore, all such description has of necessity been forced into the single Darcy parameter. Since most soils contain clay, the relevance for soil science is obvious.

Since equation (*5*) is empirical, it must be made the subject of much more testing before it can be taken as established. Further research should either verify the equation or establish a refined modification. This would then justify the necessary efforts to solve equation (*13*) or its resulting refinement.

The present study has been concerned entirely with liquid-saturated porous media. The same non-Newtonian effects, perhaps to an even greater degree, might well be expected in partially saturated porous media, since liquid movement must then take place in closer proximity to particle surfaces.

Because of the relationship of saturated water flow to soil drainage, the present study has clear relevance for plant growth as affected by conditions of excess soil water. Also, if non-Newtonian flow is manifest in partially saturated soils, then further relevance is provided on the question of water flow to plant roots in general.

SUMMARY

A modified equation is proposed for liquid flow in porous media containing clay, for which the flow is considered to be non-Newtonian. The equation has two parameters, and includes Darcy's law as a special case. The equation fits the published data of several independent investigators. The behavior of the fitted parameters is consistent, and capable of reasonable interpretation.

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