Take A Relaxing Bath

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Contents

Constant	Description
k	Thermal conductivity of water
ρ	Density of water
ρ_a	Density of air
γ_1	Heat exchange coefficient between water and air
γ_2	Heat exchange coefficient between water and bather
γ_3	Heat exchange coefficient between water and bathtub
C_w	Specific heat capacity of water
C_a	Specific heat capacity of air
V	Total volume of water
V_a	Total volume of air above the water
v	Flow velocity of air
x_s	Humidity ratio in saturated air
x	Humidity ratio in the air
x_0	Initial humidity ratio in the air
A	Square of water surface
V_a	Volume of surrounding air
C_v	Evaporation heat of water

Table 1: Constant

1 Introduction

1.1 Restatement of the Problem

We are required to build a mathematical model to solve the problem of heating water in bathtub. Thus we have some subproblems:

- Build a basic model that demonstrates the change of the temperature in the bathtub in space and time without other interventions.
- Figure out the influence of parameters such as the shape of the bathtub or the motions made by the bather or so.
- Propose a strategy to keep the temperature as even as possible.

In the first step, we build the simplest model. The shape of the bathtub is a cuboid, and the bather will stay still. In the second step, the bather moves slowly and the shape of the bathtub changes. In the third step, we develop the conclusion of the strategy.

2 Terminology

Definitions of symbols employed in this paper are listed in Table 1 and Table 2.

3 Assumptions and Justifications

- **assumption 1** The surface temperature of bathtub and the bather keeps constant.

Variable	Description
t	time
u	Temperature of water
u	Temperature of water when t=0
u_1	Temperature of air
u_2	Temperature of bather's surface
u_3	Temperature of bathtub
u_h	Temperature of hot water from faucet
v_h	Velocity of hot water from faucet
g	Evaporation heat of water

Table 2: Variable

Since human-being is temperature-constant, and bathtub contacts the ground, we can assume that they have no heat exchange.

- **assumption 2** The volume of water flowing away is equivalent to that of hot water at any time.

  Although water has evaporation loss and other kinds of losses, it takes up a small proportion of the total volume. Thus we sensibly neglect them.

- **assumption 3** The hot water from the faucet has a constant flow velocity and temperature.

- **assumption 4** There is no convection in the water.

  Since the temperature of water in the bathtub is relatively uniform, the slight convection won't influence heat exchange.

- **assumption 5** Water, bathtub, bather and the surrounding air compose a closed heat system.

4 PDE Model for Heating Water

4.1 Model Overview

Partial differential equations are widely used in natural science, engineering and economical management. It is natural to use partial differential equation to solve this physical problem.

In our model, five factors account for heat transfer of water: heat exchange between water and air, heat exchange between water and the bather, heat exchange between water and the bathtub, heat energy of hot water from the faucet and heat energy of water flow outside. According to thermology, we deduce the equation of water temperature in the bathtub. The initial-boundary value conditions are deduced according to Fourier theorem, vaporization equation[1].

Since the equation is a relatively complex 3D partial differential equation, we use matlab to solve it. We build a geometry as the bathtub. By finite element method, the geometry is divided into several parts. The solution of each part is considered to be a relatively precise value solution.

4.2 Heat Equation

In this part, we use physical theorem to deduce the heat conduction equation in the bathtub.

Figure 1 shows thermal analysis of water in bathtub In the photo, W_1 is heat exchange between the bather and water, while W_2 is heat from water vaporization. W_3 stands for heat exchange between the bathtub and water, and W_4 is heat conductivity between water and air. W_5 represents heat exchange between the bathtub and air, and W_6 is heat increasement with hot in-water.

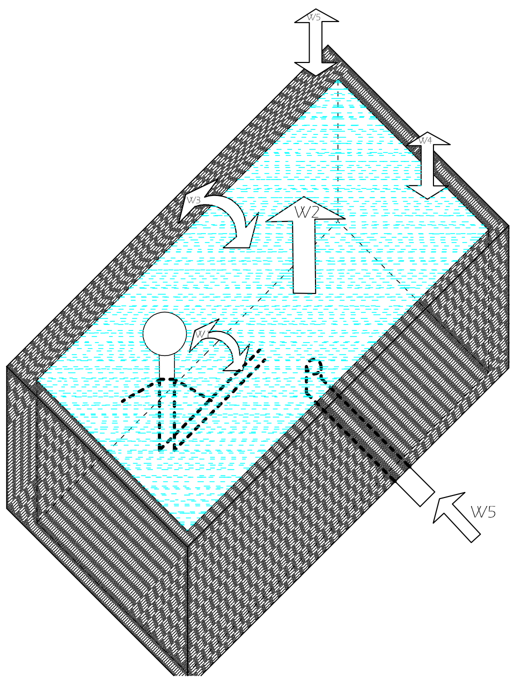

Figure 1: Thermal Analysis of Water

First, we assume that the heat change of all the water in a tiny time Δt from t_1 to $t_2 fv$ is ΔQ. As shown in Figure 1, heat of water in the bathtub exchanges with air, the bather and the bathtub. In addition, heat changes when the hot water flows in and the excess water flows out. Thus ΔQ is divided into five parts, i.e.

$$\Delta Q = \Delta Q_1 + \Delta Q_2 - \Delta Q_3 - \Delta Q_4 \tag{1}$$

in which ΔQ_1 is the heat transferred from air, bather and bathtub, ΔQ_2 is the heat of the hot water from the faucet in Δt, ΔQ_3 is the heat of the water flows away in Δt, ΔQ_4 is the heat loss because of vaporization.

Totally, internal energy of water changes when heat exchanges happen. Then the temperature changes according to the equation

$$Q = \iiint\limits_{\Omega} \rho C_w (u_f - u_i) dxdydz \tag{2}$$

in which Ω is the region of water, ρ represents the density of water, u_f stands for the final temperature when t=t_2, and u_i is the initial temperature when t=t_1.

Considering the right side of equation 1, Fourier theorem indicates that,

$$dQ = -k\frac{\partial u}{\partial n}dSdt \tag{3}$$

holds for the heat flows through surface with square of dS, as it shows in Figure 2. In the equation, k is thermal conductivity, $\frac{\partial u}{\partial n}$ is the derivative of u along the

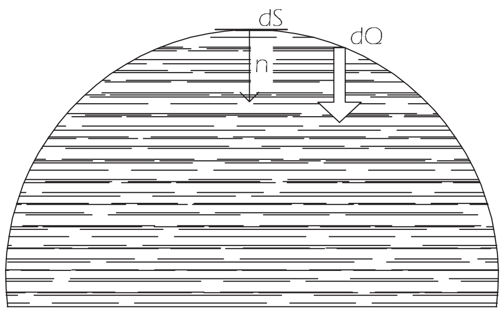

Figure 2: Thermal Analysis

normal line. Thus, after integration of dQ by time and space

$$\Delta Q_1 = \int_{t_1}^{t_2}[\int_{\Gamma} k\frac{\partial u}{\partial n}dS]dt \tag{4}$$

in which Γ is the boundary of Ω, the heat flows into the water by thermal conduction could be calculated.

ΔQ_2 is determined by the internal energy of hot water from the faucet in Δt. And ΔQ_2 can be calculated by $\Delta Q_2 = \Delta m C u_h$. Since the hot water is constant, $\Delta m = v\Delta t$.

It is obvious that the excess water is the same value of hot water in any time period. Thus ΔQ_3 can be calculated by $\Delta Q_3 = \Delta m C_w u$. Also, $\Delta m = v\Delta t$. Δv is the weight of hot water per second, Δm is the weight of hot water flows in during Δt.

According to thermal theorem, heat loss in vaporization is proportional to the weight of vapor water. $\Delta Q_4 = C_v \Delta m_v$. Δm_v is the weight of evaporation water in Δt, C_v is evaporation heat of water per kilogram.

Since water could be considered as manifold, according to Green formula, we deduce the formula

$$\begin{aligned}\int_{t_1}^{t_2}\iint_{\Gamma} k\frac{\partial u}{\partial n}dsdt &= \iiint_{\Omega} \rho C_w(u_f - u_i)dxdydz \\ &= \int_{t_1}^{t_2}\iiint_{\Omega}[\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial u}{\partial z})]dxdydx \\ &= \iiint_{\Omega} \rho C_w(\int_{t_1}^{t_2}\frac{\partial u}{\partial t}dt)dxdydz\end{aligned}$$

Combining the equations above, we conclude that

$$\rho C_w \frac{\partial u}{\partial t} = k \triangle u + \frac{C_w V_h (u_h - u)}{V} - \frac{C_v g}{V} \tag{5}$$

In this heat equation, u_h and v_h are two variables that control the strategy.

Now let's discuss g, evaporation rate per square. As Figure 3 shows, water evaporates into air and air humidity increases, until it is saturate. According to empirical formula[1], we know that

$$g = \frac{(25 + 19v)(x_s - x)}{3600} \tag{6}$$

in which v is the velocity of air, x_s is humidity ratio in saturated air, x is unsaturated humidity ratio in air. Considering the humidity ratio in the air, it indicates

$$x = x_0 + \int \frac{Ag}{\rho V_a} dt \tag{7}$$

in which x_0 is the initial humidity ratio in the air, ρ is the density of water, V_a is the total volume of surrounding air.

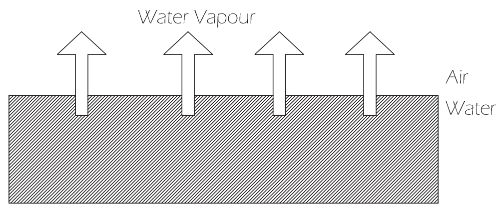

Figure 3: Evaporation

Combing and differentiating two equations, we get an ordinary partial equation

$$\frac{3600}{25 + 19v} \frac{dg}{dt} = \frac{A}{\rho V_a} \tag{8}$$

Solving the equation, we get the solution

$$g = \frac{(25 + 19v)(x_s - x_0)}{3600} e^{-\lambda t}, \lambda = \frac{A(25 + 19v)}{3600 \rho V_a} \tag{9}$$

4.3 Initial-Boundary Value Condition

Boundary value conditions are quite essential in partial differential equations. In our model, water is not a regular object. Thus boundary value condition is complex.

The boundary of water consists of four parts: water-air boundary, water-bather boundary, water-bathtub boundary, and the square for faucet. We denote them as Γ_1, Γ_2, Γ_3, Γ_4 respectively.

Since bather is temperature-constant, the boundary value condition of Γ_1 and Γ_3 could be considered as the Dirichlet value condition. That is

$$u = u_i, (x, y, z) \in \Gamma_i, i = 2.4 \tag{10}$$

In the equation, u_2 is the surface temperature of the bather. We assume it is constant. u_4 is the temperature of hot water.

The boundary value condition of Γ_1 and Γ_3 could be considered as the third boundary value condition approximately. That is

$$k\frac{\partial u}{\partial n} + \gamma_i u = \gamma_i u_i, (x, y, z) \in \Gamma_i \tag{11}$$

In the equation, i=1 or 3 ,u_i is the temperature of contact material, and γ_i is the heat exchange coefficient of Γ_i.

In the equation above, u_3 is temperature of bathtub, we assume it is unchangeable and as same as initial air temperature.

u_1, the temperature of air, is time-dependant because water vapor transfer heat to air. We can calculate u_1 by calculating the energy of water vapor evaporating in the air. Referencing the deduction of evaporation in the last section,

$$u_1 = u_0 + \frac{C_w \rho}{C_a \rho_a}(x_s - x_0)(1 - e^{-\lambda t}) \tag{12}$$

Consider the temperature of air, u_1. Heat exchange of air consists of two ways, heat conduction and evaporation.

In our model, we assume the temperature of water is equilibrium throughout the bathtub initially.

4.4 PDE Solution

Since our equations are complex and the region is quite irregular, it's impossible to get an analytical solution. Finite element method is normally adopted to deal with this kind of problem.

We use Partial Differential Equation Toolbox in matlab to solve our equations. Partial Differential Equation Toolbox makes use of finite element method to separate irregular geometry, calculate a relatively precise solution, and get solution of temperature corresponding to different u_h and v_h. By comparing temperature distributions, we choose proper parameters as the best strategy.

First, we build a 3D geometry model to simulate a proper-size bathtub, as it shown in Figure 4. Matlab divides the geometry into several elements.

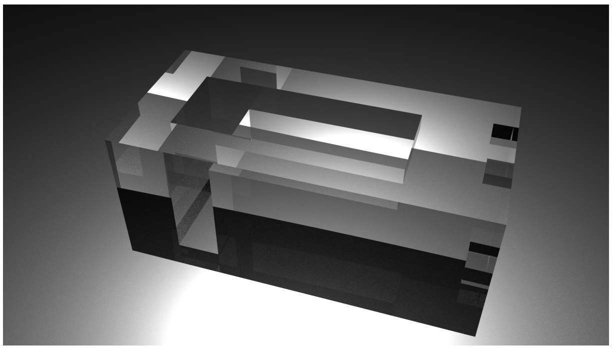

Figure 4: 3D geometry of water

Then we input the heat equation and initial-boundary value conditions. Boundary value conditions are different in three parts.

$v_h kg/s$	u_h °C
0.1	50
0.2	50
0.3	50
0.1	40
0.2	40
0.3	40
0.1	30
0.2	30
0.3	30

Table 3: Represenative Data Pairs

5 Results and Analysis

Our strategy focus on two variables, v_h and u_h. We test about fifty pairs of data to find the best strategy. And we list nine representative pairs of data in Table 3.

In each operation, u_2 is taken as 33.5 °C and the initial value condition is 35 °C. We set 120s as time period. The following graphs show the results we get.

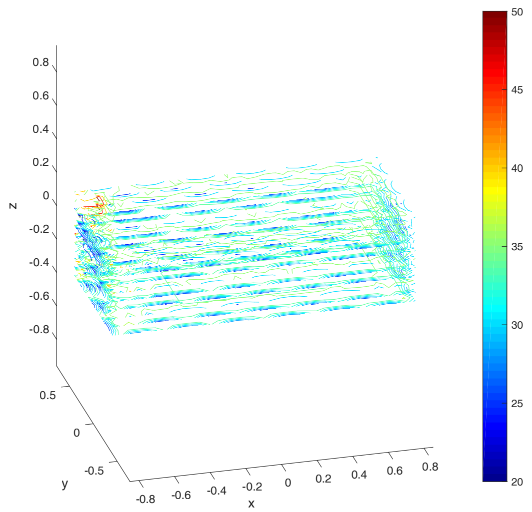

Figure 5: $u_h = 40$, $v_h = 0.1$, t=240

Figure 5, Figure 6, Figure 7 show the temperature of water at t=240, with $u_h = 40$ and $v_h = 0.1, 0.2 and 0.3$

Figure 8, Figure 9, Figure 10 show the temperature at t=600

It indicates that when u_h is constant, temperature increases when v_h increases, and the system becomes more stable as time goes on. It proves that our model is sensible.

Figure 11, Figure 6, Figure 12 show the temperature of water at t=240, with $v_h = 0.2$ and $u_h = 30, 40 and 50$

Figure 13, Figure 9, Figure 14 show the temperature at t=600

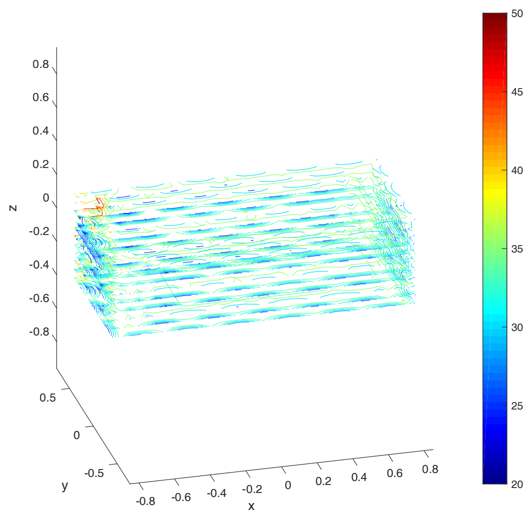

Figure 6: $u_h = 40$, $v_h = 0.2$, t=240

Shape	v_h	u_h
Square	0.2	40
Sphere	0.15	40
Cylinder	0.2	45

Table 4: Shape of Bathtub

It indicates that when v_h is constant, temperature increases when u_h increases, and the system becomes more stable as time goes on. It proves that our model is sensible.

Comparing the temperature distributions of each strategy, with least water waste, we choose $v_h = 0.2$, $u_h = 40$ as the best strategy to keep the temperature.

6 Sensitivity Analysis

6.1 Parameters of Bathtub

- We change the shape of the bathtub to sphere and cylinder to get the best strategy. The result shown in Table 4 that the bathtub with the shape of sphere is the most effective in keeping temperature.

- We change the volume of the bathtub without changing the proportion. The result in the following table shows that with the same v_h, u_h increases as volume increases; with the same u_h, v_h increases as volume increases.

6.2 Parameters of Bather

- Comparing the temperature graphs with different shapes of the bather, we find when the shape gets more regular, u_h, v_h decreases relatively.

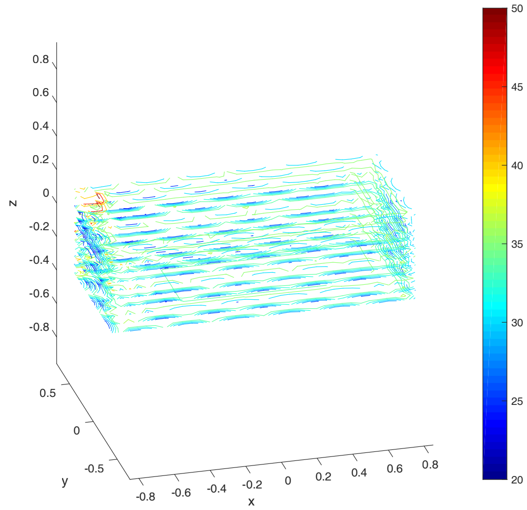

Figure 7: $u_h = 40$,$v_h = 0.3$,t=240

Surface Temperature	v_h	u_h
$u_2 = 33.5$	0.2	40
$u_2 = 33.5$	0.15	40
$u_2 = 33.5$	0.2	45

Table 5: Surface Temperature

- Change the volume of the bather. The result shows that, u_h, v_h for best strategy increases when the volume increases.
- Change the surface temperature of the bather to get the best strategy. The result shows no noticeable change.

6.3 Effects of Motion

When the bather in the bathtub makes motions, the heat exchange coefficient between the bather and water γ_2 increases. That means heat exchange between water and the bather increases. In another aspect, the air above flows, so that evaporation heat of water increases. Water loses more heat.

The effect is that more heat is called for to keep the temperature. So u_h increases for best strategy to keep the temperature.

6.4 Effects of Bubble Bath Additive

When bubble bath additive is added while initially filling the bathtub, bubbles stay on the surface and the evaporation of water to air is limited. So water loses less heat.

The effect is that less heat is needed to keep the temperature. So u_h for best strategy decreases to keep the temperature.

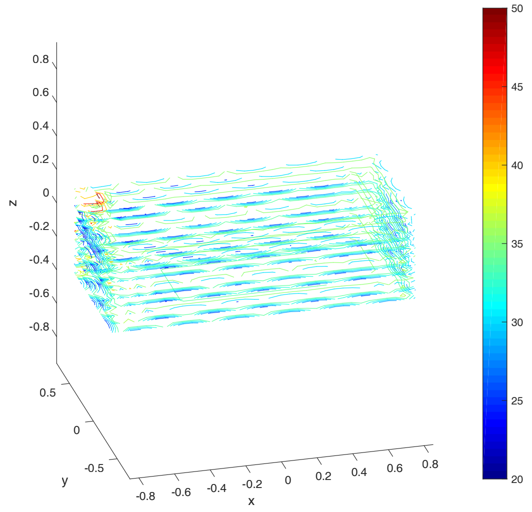

Figure 8: $u_h = 40$,$v_h = 0.1$,t=600

7 Strengths and Weaknesses

Strengths

- Our model makes good use of thermology knowledge so it can reflect the situation in the real world realistically.
- Our model builds a 3D geometry to simulate the research object so it can show the exact situation.
- Our model operates with different strategies and displays the result by 3D geometry graph, so they could be compared in a visible way.
- Our model proves the effect of time.

Weaknesses

- We assume that the water is still. In fact, it is better to take the streams flow and fluid dynamic into consideration. The actual situation is much more complicated.
- The model of the bather should be more realistic. In fact, model of human body includes hands, legs or other parts, and the surface of human skin isn't even.
- The condensation of water should be taken into consideration. Water that evaporates become water vapour, and some will condense into water on the bathtub.

References

[1] http://www.engineeringtoolbox.com

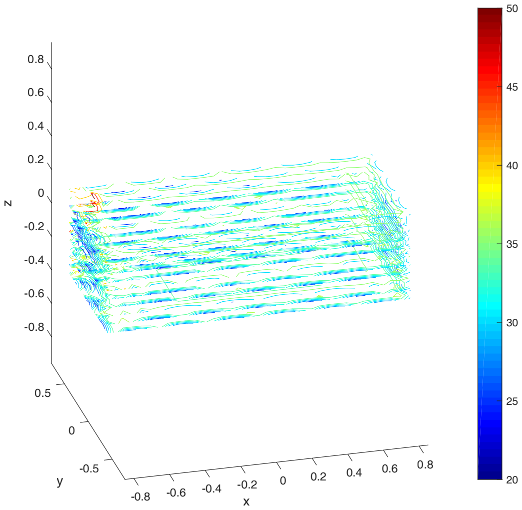

Figure 9: $u_h = 40$,$v_h = 0.2$,t=240

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8 Non-technical Explanation

Users,
If you'd like to use the bathtub, it's better to acknowledge that:

- Obviously, the water exchanges thermal energy directly with the air. That's why you can see water vapour floats above the water. The temperature of water decreases as a result of the heat exchange.

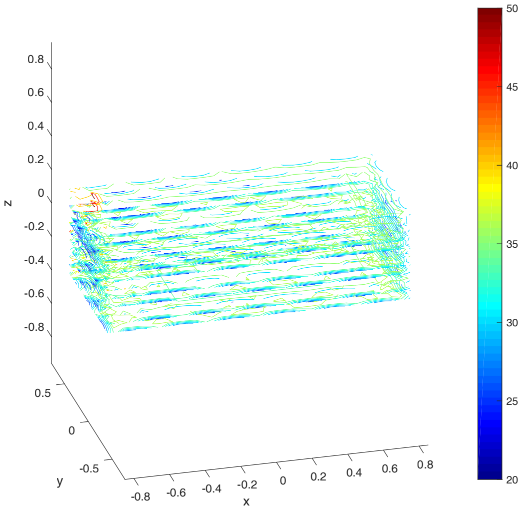

Figure 10: $u_h = 40$,$v_h = 0.3$,t=600

- On the other hand, the bathtub is cooler than the water in most occasions. It's reasonable that the bathtub absorbs heat from the water. However, the bathtub conducts heat easily. In this way the air's temperature rises. In other words, the temperature of water decreases faster.

From statement above, you may understand our bathtub can't stay in an constant temperature evenly. Consequently we have some suggestions for you, if you'd like to enjoy more during your bath :

- To keep the water warm and relax yourself, you may turn the faucet on. Water-in in high temperature loses it's energy quickly, and water-in in low temperature is unable to warm your surrounding. We recommend you set the speed of the water-in at 0.2kg/s, and temperature at 40 °C.

- Further more, your movement effects the water. It would be great if you stay still, but it doesn't count too much. For a better temperature-keeping situation, you may add some bubble bath additive to decrease the temperature loss.

We hope you enjoy your bath.

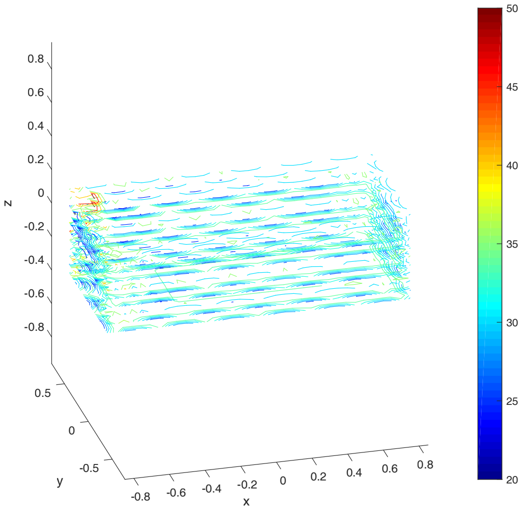

Figure 11: $u_h = 30$,$v_h = 0.2$,t=240

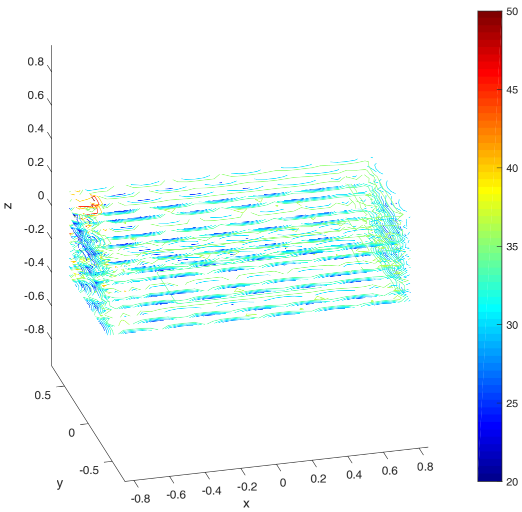

Figure 12: $u_h = 50$,$v_h = 0.2$,t=240

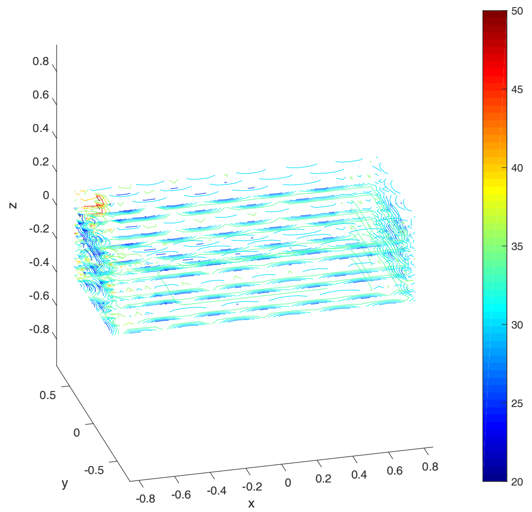

Figure 13: $u_h = 30$,$v_h = 0.2$,t=600

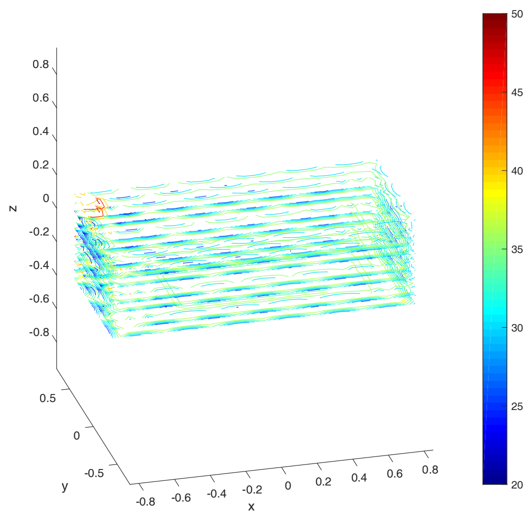

Figure 14: $u_h = 50$,$v_h = 0.2$,t=600