Pairwise-Independent Contention Resolution

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Abstract. We study online contention resolution schemes (OCRSs) and prophet inequalities for non-product distributions. Specifically, when the active set is sampled according to a *pairwise-independent* (PI) distribution, we show a $(1-o_k(1))$-selectable OCRS for uniform matroids of rank k, and $\Omega(1)$-selectable OCRSs for laminar, graphic, cographic, transversal, and regular matroids. These imply prophet inequalities with the same ratios when the set of values is drawn according to a PI distribution. Our results complement recent work of Dughmi et al. [15] showing that no $\omega(1/k)$-selectable OCRS exists in the PI setting for general matroids of rank k.

Keywords: Online Algorithms, Prophet Inequalities, Contention Resolution.

1 Introduction

Consider the *prophet inequality* problem: a sequence of independent positive real-valued random variables $\mathbf{X} = \langle X_1, X_2, \dots, X_n \rangle$ are revealed one by one. Upon seeing X_i the algorithm must decide whether to select or discard the index i; these decisions are irrevocable. The goal is to choose some subset S of the indices $\{1, 2, \dots, n\}$ to maximize $\mathbb{E}[\sum_{i \in S} X_i]$, subject to the set S belonging to a well-behaved family $\mathcal{I} \subseteq 2^{[n]}$. The goal is to get a value close to $\mathbb{E}[\max_{S \in \mathcal{I}} \sum_{i \in S} X_i]$, the value that a clairvoyant "prophet" could obtain in expectation. This problem originally arose in optimal stopping theory, where the case of $\mathcal{I}$ being the set of all singletons was considered [25]: more recently, the search for good prophet inequalities has become a cornerstone of stochastic optimization and online decision making, with the focus being on generalizing to broad classes of downward-closed sets $\mathcal{I}$ [17, 24, 31], considering additional assumptions on the order in which these random variables are revealed [1, 2, 16, 18], obtaining optimal approximation guarantees [12, 26], and competing with nonlinear objectives [19, 32].

One important and interesting direction is to reduce the requirement of independence between the random variables: *what if the r.v.s are correlated*? The case of negative correlations is benign [29, 30], but general correlations present significant hurdles—even for the single-item case, it is impossible to get value much better than the trivial $\mathbb{E}[\max_i X_i]/n$ value obtained by random guessing [22]. As another example, the model with *linear correlations*, where $\mathbf{X} = A\mathbf{Y}$ for some independent random variables $\mathbf{Y} \in \mathbb{R}_+^d$ and known non-negative matrix $A \in \mathbb{R}_+^{d\times n}$, also poses difficulties in the single-item case [23].

Given these impossibility results, Caragiannis et al. [7] gave single-item prophet inequalities in the setting of *weak correlations*: specifically, for the setting of *pairwise-independent distributions.* As the name suggests, these are distributions that look like product distributions when projected on any two random variables. While pairwise-independent distributions have long been studied in algorithm design and complexity theory [27], these had not been studied previously in the setting of stochastic optimization. Caragiannis et al. [7] gave both algorithms and some limitations arising from just having pairwise-independence. They also consider related pricing and bipartite matching problems.

We ask the question: can we extend the prophet inequalities known for richer classes of constraint families $\mathcal{I}$ to the pairwise-independent case? In particular,

> *What classes of matroids admit good pairwise-independent prophet inequalities?*

Specifically, we investigate the analogous questions for *(online) contention resolution schemes (OCRS)* [17], another central concept in online decision making, and one closely related to prophet inequalities. In an OCRS, a random subset of a ground set is marked active. Elements are sequentially revealed to be active or inactive, and the OCRS must decide irrevocably on arrival whether to select each active element, subject to the constraint that the selected element set belongs to a constraint family $\mathcal{I}$. The goal is to ensure that each element, conditioned on being active, is picked with high probability. It is intuitive from the definitions (and formalized by Feldman et al. [17]) that good OCRSs imply good prophet inequalities (see also [26]).

1.1 Our Results

Our first result is for the k-uniform matroid where the algorithm can pick up to k items: we achieve a $(1 - o_k(1))$-factor of the expected optimal value.

Theorem 1 (Uniform Matroid PI Prophets). *There is an algorithm in the prophet model for k-uniform matroids that achives expected value at least $(1 - O(k^{-1/5}))$ of the expected optimal value.*

We prove this by giving a $(1 - O(k^{-1/5}))$-*selectable* online contention resolution scheme for k-unifrom matroids, even when the underlying generative process is only pairwise-independent. Feldman et al. [17] showed that selectable OCRSs immediately lead to prophet inequalities (against an almighty adversary) in the fully independent case, and we observe that their proofs translate to pairwise-independent distributions as well. Along the way we also show a $(1 - O(k^{-1/3}))$ (offline) CRS for the pairwise-independent k-uniform matroid case.

We then show $\Omega(1)$-selectable OCRSs for many common matroid classes, again via pairwise-independent OCRSs.

Theorem 2 (Other Matroids PI Prophets). *There exist $\Omega(1)$-selectable OCRSs for laminar (Section 3), graphic (Section 4), cographic (Appendix G), transversal (Appendix F), and regular (Section 5) matroids. These immediate imply $\Omega(1)$ prophet inequalities for these matroids against almighty adversaries.*

Finally, we consider the single-item case in greater detail in Appendix B. For this single-item case the current best result from Caragiannis et al. [7] uses a multiple-threshold algorithm to achieve a $(\sqrt{2}-1)$-prophet inequality; however, this bound is worse than the $1/2$-prophet inequality known for fully independent distributions. We show that no (non-adaptive) multiple-threshold algorithm (i.e., one that prescribes a sequence of thresholds τ_i up-front, and picks the first index i such that $X_i \geq \tau_i$) can beat $2(\sqrt{5}-2) \approx 0.472$, suggesting that if $1/2$ is at all possible it will require adaptive algorithms.

Theorem 3 (Upper Bound for Multiple Thresholds). *Any multiple-threshold algorithm for the single-item prophet inequality in the PI setting achieves a factor of at most* 0.472.

In Appendix B.3 we give a *single-sample* version of the single-item PI prophet inequality.

Theorem 4 (Single Sample Prophet Inequality). *There is an algorithm that draws a single sample from the underlying pairwise-independent distribution $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle \sim \mathcal{D}$ on $\mathbb{R}^n_+$, and then faced with a second sample $\langle X_1, \ldots, X_n \rangle \sim \mathcal{D}$ (independent from $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle$), picks a single item i from $X_1, \ldots, X_n$ with expected value at least $\Omega(1) \cdot \mathbb{E}_{\mathbf{X} \sim \mathcal{D}}[\max_i X_i]$.*

1.2 Related Work

In independent and concurrent work, Dughmi et al. [15] also study the pairwise-independent versions of prophet inequalities and (online) contention resolution schemes. This work can be considered complementary to ours: they show that for arbitrary linear matroids, nothing better than $O(1/r)$ factors can be achieved for pairwise-independent versions of OCRSs, and nothing better than $O(1/(\log r))$ factors can be achieved for pairwise-independent versions of matroid prophet inequalities. They also obtain $\Omega(1)$-selectable OCRSs for uniform, graphical, and bounded degree transversal matroids by observing that these have the α-partition property (see [5]), reducing to the single-item setting. Another motivation for our work is the famous matroid secretary problem, since the latter is known to be equivalent to the existence of good OCRSs for *arbitrary* distributions against a *random-order* adversary [14].

The original single-item prophet inequality for product distributions was proven by Krengel and Sucheston [25]; applications to computer science were given by Hajiaghayi et al. [21] and Chawla et al. [10]. There is a vast literature on variants and extensions of prophet inequalities, which we cannot survey here for lack of space. Contention resolution schemes were introduced by Chekuri et al. [11] in the context of constrained submodular function maximization, and these were generalized by Feldman et al. [17] to the online setting in order to give prophet inequalities for richer constraint families.

Limited-independence versions of prophet inequalities were studied from the early days e.g. by Hill and Kertz [22] and Rinott and Samuel-Cahn [30]. Many stochastic optimization problems have been studied recently in correlation-robust settings, e.g., by Bateni et al. [6], Chawla et al. [9], Immorlica et al. [23], and Makur et al. [28]; pairwise-independent prophet inequalities were introduced by Caragiannis et al. [7].

There is a line of work on single-sample prophet inequalities in the i.i.d. setting [3, 4, 8, 20, 33]. We are the first to study these for pairwise-independent distributions.

1.3 Preliminaries

We provide several essential definitions here, and a more complete preliminaries section in Appendix A. We assume the reader is familiar with the basics of matroid theory, and refer to Schrijver [35] for definitions. For a matroid $M = (E, \mathcal{I})$ the matroid polytope is defined to be $\mathcal{P}_M := \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x} \geq \mathbf{0},\ \mathbf{x}(S) \leq \operatorname{rank}(S)\ \forall S \subseteq E\}$. For polytope $\mathcal{P}$ and scalar $b \in \mathbb{R}$, define $b\mathcal{P} := \{b\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$.

We focus on pairwise independent versions of *contention resolution schemes* (CRSs), in both offline and online settings. Our setting entails a set $R \subseteq [n]$ drawn from a distribution $\mathcal{D}$ with marginal probabilities given by some $\mathbf{x} \in b\mathcal{P}_M$, and the goal is to select items $I \subseteq R$, $I \in \mathcal{I}$, such that $\Pr[i \in I \mid i \in R] \geq c$ for all i. An algorithm which does this is a (b, c)*-balanced* CRS.

For *online* contention resolution schemes (OCRSs) the items arrive one-at-a-time; a scheme must decide whether to include each arriving element into its independent set I or irrevocably reject it [17]. Generally the events $[i \in R]$ are taken to be independent, so that $\mathbf{x}$ determines $\mathcal{D}$. For a *pairwise-independent* (PI) OCRS these events are only pairwise independent under $\mathcal{D}$.

The Almighty Adversary The almighty adversary knows everything. It first sees the realization of $R \sim \mathcal{D}$, as well as all randomness the algorithm will use. It then adversarially orders R. To describe PI-OCRS's with guarantees against the almighty adversary, we adopt ideas from Feldman et al. [17].

Definition 1 ((b, c)-selectable PI-OCRS). *Let $\mathcal{P} \subseteq [0, 1]^n$ be some convex polytope. We call a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ a* (b, c)-selectable PI-OCRS *if it satisfies the following:*

1. *Algorithm π precommits to some feasible set family $\mathcal{F} \subseteq \mathcal{I}$, and then adds each arriving i to I only if the resulting set is in $\mathcal{F}$.*
2. *For any $\mathbf{x} \in b\mathcal{P}$, any distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$ and any $i \in [n]$, let $\mathcal{F}$ be the feasible set family defined by π. Let R be sampled according to $\mathcal{D}$, then*
$$\Pr_{R\sim\mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] \geq c. \tag{1.1}$$
Here the probability is over R and internal randomness of π in defining $\mathcal{F}$.

Notice the definition here is slightly different from [17], as we need to condition on the event $i \in R$. This is due to our limited independence over events $i \in R$. For the mutually independent case, one can prove that $\Pr_{R\sim\mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] = \Pr_{R\sim\mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F}]$, but this may not hold in the pairwise-independent case.

A (b, c)-selectable PI-OCRS implies a $(1, bc)$-selectable PI-OCRS (or for short bc-selectable PI-OCRS), and gives guarantees against an almighty adversary. For details see Appendix A.

The Offline Adversary and Prophet Inequalities The offline adversary does not know the randomness of π and must choose an arrival order for $[n]$ before $R \sim \mathcal{D}$ is sampled. PI-OCRSs that are (b, c)-*balanced* are effective against offline adversary, and since the offline adversary is weaker than the almighty adversary, a (b, c)-selectable PI-OCRS is always (b, c)-balanced. Once again, a (b, c)-balanced PI-OCRS may be converted to a $(1, bc)$-balanced PI-OCRS (or a bc-balanced PI-OCRS for short) via independent subsampling of R.

Feldman et al. [17] showed connections between OCRSs and prophet inequalities. In Appendix A we formally establish this connection in the pairwise-independent setting through the formulation of a *PI matroid prophet game*, and we demonstrate that balanced PI-OCRSs are enough to give prophet inequalities.

As an upshot, we show that our results imply matroid prophet inequalities for the pairwise-independent setting; for each class of matroids, any c-balanced PI-OCRS yields a c-competitive prophet inequality for values drawn from a pairwise-independent distribution. This generalizes the single-item PI prophet inequality of Caragiannis et al. [7] in the setting where the gambler knows the joint distribution as well as the marginals.

2 Uniform Matroids

Recall that the independent sets of a *uniform matroid* $M = (E, \mathcal{I})$ of rank k are all subsets of E of size at most k; hence our goal is to pick some set of size at most k. Identifying E with $[n]$, the corresponding matroid polytope is $\mathcal{P}_M := \{\mathbf{x} \in [0, 1]^n : \sum_{i=1}^n x_i \leq k\}$. Our main results for uniform matroids are the following, which imply Theorem 1.

Theorem 5 (Uniform Matroids). *For uniform matroids of rank k, there is*

(i) a $(1 - O(k^{-1/3}))$-balanced PI-CRS, and
(ii) a $(1 - O(k^{-1/5}))$-selectable PI-OCRS.

A simple greedy PI-OCRS follows by choosing the feasible set family $\mathcal{F} = \mathcal{I}$, i.e. selecting R as the resulting set if $|R| \le k$. However, conditioning on $i \in R$, pairwise independence only guarantees the marginals of the events $j \in R$ (and they might have arbitrary correlation), so we can only use Markov's inequality to bound $\Pr[|R| \le k \mid i \in R]$. Therefore this analysis only gives a $(b, 1-b)$-selectable PI-OCRS for k-uniform matroid. (For details see Appendix C.1.)

Hence instead of conditioning on some $i \in R$ and using Markov's inequality, we consider all of the items together, and use Chebyshev's inequality to give a bound for $\Pr[|R| \ge k, i \in R]$. The following lemma is key for both our PI-CRS and PI-OCRS.

Lemma 1. *Let $M = (E, \mathcal{I})$ be a k-uniform matroid, where E is identified as $[n]$. Given $\mathbf{x} \in (1-\delta)\mathcal{P}_M$ and a distribution $\mathcal{D}$ of subsets of E that is PI-consistent with $\mathbf{x}$, let $R \subseteq E$ be the random set sampled according to $\mathcal{D}$. Then*

$$\sum_{i=1}^{n} \Pr[|R| \ge k, i \in R] \le \frac{1-\delta^2}{\delta^2}.$$

Proof. The expression on the left can be written as

$$\begin{aligned}\sum_{i=1}^{n} \Pr[|R| \ge k, i \in R] &= \sum_{i=1}^{n} \sum_{t=k}^{n} \sum_{\substack{S: \\ |S|=t}} \mathbb{1}[i \in S] \Pr[R = S] = \sum_{t=k}^{n} \sum_{\substack{S: \\ |S|=t}} \Pr[R = S]|S| \\ &= \sum_{t=k}^{n} t \cdot \Pr[|R| = t] = k \Pr[|R| \ge k] + \sum_{t=k+1}^{n} \Pr[|R| \ge t].\end{aligned}$$

We now bound the two parts separately using Chebyshev's inequality. Let $X_i := \mathbb{1}[i \in R]$ be the indicator for i being active, and let $X = \sum_{i \in E} X_i$. Since the X_i are pairwise independent, $\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] \le \sum_i \mathbb{E}[X_i^2] = \sum_i \mathbb{E}[X_i] = \mathbb{E}[X]$. For the first part, we have

$$\begin{aligned}k \cdot \Pr[|R| \ge k] = k \cdot \Pr[X \ge k] &\le k \cdot \frac{\mathrm{Var}[X]}{(k - \mathbb{E}[X])^2} && \text{(Chebyshev's ineq.)} \\ &\le k \cdot \frac{1-\delta}{\delta^2 k} = \frac{1-\delta}{\delta^2}. && (2.1)\end{aligned}$$

For the second part,

$$\sum_{t=k+1}^{n} \Pr[|R| \ge t] = \sum_{t=k+1}^{n} \Pr[X \ge t] \le \sum_{t=k+1}^{n} \frac{\mathrm{Var}[X]}{(t - \mathbb{E}[X])^2} \quad \text{(Chebyshev's ineq.)}$$

$$\le \sum_{t=k+1}^{n} \frac{(1-\delta)k}{(t-(1-\delta)k)^2} \le (1-\delta)k \cdot \sum_{t\ge 1} \frac{1}{(\delta k + t)^2}$$
$$\le (1-\delta)k \cdot \frac{1}{\delta k} = \frac{1-\delta}{\delta}, \tag{2.2}$$

where we used the inequality

$$\sum_{j\ge 1} \frac{1}{(x+j)^2} \le \sum_{j\ge 1} \frac{1}{(x+j-1)(x+j)} = \sum_{j\ge 1} \left(\frac{1}{x+j-1} - \frac{1}{x+j}\right) = \frac{1}{x}.$$

Summing up the (2.1) and (2.2) finishes the proof. □

Using this lemma, we can bound $\min_i \Pr[|R| \ge k \mid i \in R]$ and obtain a $(1 - O(k^{-1/3}))$-balanced PI-CRS (in the same way that [11, Lemma 4.13] implies a $(b, 1-b)$-CRS in the i.i.d. setting). The details are deferred to Appendix C.1.

2.1 A $(1 - O(k^{-1/5}))$-Selectable PI-OCRS for Uniform Matroids

Our PI-CRS has to consider the elements in a specific order, and therefore it does not work in the online setting where the items come in adversarial order. The key idea for our PI-OCRS is to separate "good" items and "bad" items, and control each part separately. Let us assume R is sampled according to some distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$, and that $\mathbf{x}$ is on a face of $(1-\varepsilon)\mathcal{P}_M$, i.e.

$$\sum_{i=1}^{n} \Pr[i \in R] = (1-\varepsilon)k. \tag{2.3}$$

We will choose the value of ε later. For some other constants $r, b \in (0,1)$ define an item i to be *good* if $\Pr[|R| > \lfloor (1-r\varepsilon)k \rfloor \mid i \in R] \le b$. Let E_g denote the set of good items, and $E_b := E \setminus E_g$ the remaining *bad* items. Our algorithm keeps two buckets, one for the good items and one for the bad, such that

(i) the good bucket has a capacity of $\lfloor (1-r\varepsilon)k \rfloor$, and
(ii) the bad bucket has a capacity of $\lceil r\varepsilon k \rceil$.

When an item arrives, we put it into the corresponding bucket as long as that bucket is not yet full. Finally, we take the union of the items in the two buckets as the output of our OCRS. This algorithm is indeed a greedy PI-OCRS with the feasible set family

$$\mathcal{F} = \{I \in \mathcal{I} : |I \cap E_g| \le \lfloor (1-r\varepsilon)k \rfloor, |I \cap E_b| \le \lceil r\varepsilon k \rceil\}.$$

We show that for any item i, $\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \ge 1 - o(1)$. First, for a good item i, by definition

$$\Pr[I \cup \{i\} \in \mathcal{F} \ \ \forall I \in \mathcal{F},\ I \subseteq R \mid i \in R]$$
$$= 1 - \Pr[|R \cap E_g| > \lfloor (1-r\varepsilon)k \rfloor \mid i \in R]$$

$$\geq 1 - \Pr[|R| > \lfloor (1-r\varepsilon)k \rfloor \mid i \in R] \geq 1 - b,$$

since i is good. Next, for a bad item i, we can use Markov's inequality conditioning on $i \in R$:

$$\begin{aligned}
&\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \\
&= 1 - \Pr[|R \cap E_b| > \lceil r\varepsilon k \rceil \mid i \in R] \\
&\geq 1 - \frac{\sum_{j \in E_b} \Pr[j \in R \mid i \in R]}{r\varepsilon k} = 1 - \frac{\sum_{j \in E_b} \Pr[j \in R]}{r\varepsilon k}, \qquad (2.4)
\end{aligned}$$

where we use Markov's inequality, and the last step uses pairwise independence of events $i \in R$. We now need to bound $\sum_{j \in E_b} \Pr[j \in R]$. If we define ε' as $1 - \varepsilon' = \frac{1-\varepsilon}{1-r\varepsilon}$, then we have

$$\begin{aligned}
\sum_{j \in E_b} \Pr[j \in R] &= \sum_{j \in E_b} \frac{\Pr[|R| \geq \lfloor (1-r\varepsilon)k \rfloor, j \in R]}{\Pr[|R| \geq \lfloor (1-r\varepsilon)k \rfloor \mid j \in R]} \\
&\leq \sum_{j \in E_b} \frac{\Pr[|R| \geq \lfloor (1-r\varepsilon)k \rfloor, j \in R]}{b} \qquad \text{(since } j \text{ is bad)} \\
&\overset{(\star)}{\leq} \frac{(1-(\varepsilon')^2)/(\varepsilon')^2}{b} \leq \frac{1}{(1-r)^2\varepsilon^2 b},
\end{aligned}$$

where $(\star)$ uses Lemma 1. Substituting back into (2.4),

$$\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \geq 1 - ((1-r)^2 r\varepsilon^3 bk)^{-1}.$$

To balance the good and bad items, we set $b = ((1-r)^2 r\varepsilon^3 bk)^{-1} = ((1-r)^2 r\varepsilon^3 k)^{-1/2}$. If we set $r = 1/3$, then we have an $(1-\varepsilon, 1-(\frac{4}{27}\varepsilon^3 k)^{-1/2})$-selectable PI-OCRS. Finally, if we set $\varepsilon = k^{-1/5}$, then by Claim 5 we have a $(1-O(k^{-1/5}))$-selectable PI-OCRS.

3 Laminar Matroids

In this section we give an $\Omega(1)$-selectable PI-OCRS for laminar matroids. A laminar matroid is defined by a laminar family $\mathcal{A}$ of subsets of E, and a capacity function $c : \mathcal{A} \to \mathbb{Z}$; a set $S \subseteq E$ is independent if $|S \cap A| \leq c(A)$ for all $A \in \mathcal{A}$.

The outline of the algorithm is as follows: we construct a new capacity function c' by rounding down $c(A)$ to powers of two; satisfying these more stringent constraints loses only a factor of two. Then we run greedy PI-OCRSs for uniform matroids from Section 2.1 *independently* for each capacity constraint $c'(A)$, $A \in \mathcal{A}$. Finally, we output the intersection of these feasible sets. For our analysis, we apply a union bound on probability of an item being discarded by some greedy PI-OCRS; this is a geometric series by our choice of c'.

As the first step, we define $c'(A)$ to be the largest power of 2 smaller than $c(A)$, for each $A \in \mathcal{A}$. (For simplicity we assume that $E \in \mathcal{A}$.) Moreover, if sets $A, B \in \mathcal{A}$ with $A \subseteq B$ and $c'(A) \geq c'(B)$, then we can discard A from the collection. In conclusion, the final constraints satisfy:

1. The new laminar family is $\mathcal{A}' \subseteq \mathcal{A}$.
2. For any $A \in \mathcal{A}'$, $c'(A)$ is power of 2, and $c(A)/2 < c'(A) \leq c(A)$.
3. (Strict Monotonicity) For any $A, B \in \mathcal{A}'$ with $A \subsetneq B$, we have $c'(A) < c'(B)$.

Let M' denote the laminar matroid defined by the new set of constraints. We can check that any c-selectable PI-OCRS for M' is a $(1/2, c)$-selectable PI-OCRS for M. Hence, it suffices to give a $\Omega(1)$-selectable PI-OCRS for M'.

Now we run greedy OCRSs for uniform matroids to get a $(1/25, 1/2.661)$-selectable PI-OCRS: for a set A with capacity $c'(A)$, from Section 2 we have both a $(1-b, b)$-selectable PI-OCRS and a $(1-b, 1-(\frac{4}{27}b^3c'(A))^{-1/2})$-selectable PI-OCRS: the former is better for small capacities, whereas the latter is better for larger capacities. Setting a threshold of $t = 13$ and choosing $b = 24/25$, we use the former when $c'(A) < 2^t$, else we use the latter. Now a union bound over the the various sets containing an element gives us the result: the crucial fact is that we get a contribution of $t(1-b)$ from the first smallest scales and a geometric sum giving $O(2^{-t/2}b^{-3/2})$ from the larger ones. The details appear in Appendix D.

4 Graphic Matroids

Recall that graphic matroids correspond to forests (acyclic subgraphs) of a given (multi)graph. For these matroids we show the following.

Theorem 6. *For $b \in (0, 1/2)$, there is a $(b, 1-2b)$-selectable PI-OCRS scheme for graphic matroids.*

Let $M = (E, \mathcal{I})$ be a graphic matroid defined on (multi)graph $G = (V, E)$. Let $\mathcal{D}$ be any distribution over 2^E that is PI-consistent with some $\mathbf{x} \in b\mathcal{P}_M$, and R sampled according to $\mathcal{D}$. We follow the construction of OCRS of Feldman et al. [17]. Our goal is to construct a chain of sets: $\varnothing = E_l \subsetneq E_{l-1} \subsetneq \cdots \subsetneq E_0 = E$ where for any $i \in \{0 \cdots l-1\}$ and any $e \in E_i \setminus E_{i+1}$,

$$\Pr[e \in \underset{M/E_{i+1}}{\operatorname{span}} (((R \cap (E_i \setminus E_{i+1})) \setminus e) \mid e \in R] \leq 2b. \tag{4.1}$$

If we have this chain, then we can define the feasible set for our greedy PI-OCRS as $\mathcal{F} = \{I \subseteq E : \forall i, I \cap (E_i \setminus E_{i+1}) \in \mathcal{I}(M/E_{i+1})\}$. By definition of contraction, $\mathcal{F} \subseteq \mathcal{I}(M)$. It remains to check selectability. For an edge e in $E_i \setminus E_{i+1}$, we have

$$\begin{aligned} &\Pr[I \cup \{e\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid e \in R] \\ &= \Pr[e \notin \underset{M/E_{i+1}}{\operatorname{span}} (((R \cap (E_i \setminus E_{i+1})) \setminus e) \mid e \in R] \geq 1 - 2b. \quad \text{(Equation (4.1))} \end{aligned}$$

Therefore this is a $(b, 1-2b)$-selectable PI-OCRS. All is left is to show how to construct such a chain. We use the following recursive procedure:

1. Initialize $E_0 = E, i = 0$.
2. Set $S = \varnothing$.

3. While there exists $e \in E_i \setminus S$ such that $\Pr[e \in \mathrm{span}_{M/S}((R \cap (E_i \setminus S)) \setminus e) \mid e \in R] > 2b$, add e into S.
4. $i \leftarrow i+1$, set $E_i = S$.
5. If $E_i \neq \varnothing$, goto step 2; otherwise set $l = i$ and terminate.

Inequality (4.1) is automatically satisfied by this procedure. It remains to show that the process always terminates, i.e. that step 3 always leaves at least one element unidentified, and hence $E_{i+1} \subsetneq E_i$. We start with the following claim.

Claim 1. *If $u_0 \in V$ satisfies $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b$, then in the above procedure generating E_1 from E, we have that for all $e \in \mathcal{E}(u_0)$, $e \notin S$.*

Proof. We prove our claim using induction. For any edge $e \in \mathcal{E}(u_0) \cap R$, $e \in \mathrm{span}(R \setminus \{e\})$ implies the existence of a circuit $C \subseteq R$ containing e. By the definition of circuits, C must contain some edge $e' \in \mathcal{E}(u_0) \setminus \{e\}$. By the pairwise independence of events $e \in R$, we have

$$\begin{aligned}\Pr[e \in \mathrm{span}(R \setminus \{e\}) \mid e \in R] &\leq \Pr[\exists e' \in \mathcal{E}(u_0) \setminus \{e\}, e' \in R \mid e \in R] \\ &\leq \sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b.\end{aligned}$$

Therefore we do not add any $e \in \mathcal{E}(u_0)$ into S in the first iteration. Suppose no $e \in \mathcal{E}(u_0)$ has been added to S during the first i iterations, then before the $(i+1)^{th}$ iteration starts, u_0 has not been merged with any other vertex in the contracted graph G/S, so $\mathcal{E}(u_0)$ in G/S is the same as the original graph G. Thus $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b$ still holds for u_0 in G/S, and by the same argument as the first iteration, no $e \in \mathcal{E}(u_0)$ will be added to S in the $(i+1)^{th}$ iteration. $\square$

Since $\mathbf{x} \in b\mathcal{P}_M$, we have $\sum_{e \in E} x_e \leq b(n-1)$, which implies $\sum_{u \in V} \sum_{e \in \mathcal{E}(u)} x_e \leq 2b(n-1)$. By averaging, there exists a vertex $u_0 \in V$ such that $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b(n-1)/n \leq 2b$, and by Claim 1, $\mathcal{E}(u_0) \cap E_1 = \varnothing$. Assuming no isolated vertex in V, $E_1 \subsetneq E_0$. Similarly, for any i, since $M|_{E_i}$ is also a graphic matroid and $\mathbf{x}|_{E_i} \in b\mathcal{P}_{M|_{E_i}}$, the same argument holds for it. Therefore $E_{i+1} \subsetneq E_i$ always holds, which finishes our proof of termination for our construction.

5 Regular Matroids

We now give a $\Omega(1)$-competitive PI-OCRS for regular matroids. We use the regular matroid decomposition theorem of Seymour [36] and its modification by Dinitz and Kortsarz [13], which decomposes any regular matroid into 1-sums, 2-sums, and 3-sums of graphic matroids, cographic matroids, and a specific 10-element matroid R_{10}. (These matroids are called the *basic* matroids of the decomposition). We now define 1,2,3-sums, and argue that it suffices to run a PI-OCRS for each of the basic matroids and to output the union of their outputs.

Definition 2 (Binary Matroid Sums [13, 36]). *Given two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, the* matroid sum *$M$ defined on the ground set $E(M_1) \Delta E(M_2)$ is as follows. The set C is a cycle in M iff it can be written as $C_1 \Delta C_2$, where C_1 and C_2 are cycles of M_1 and M_2. respectively. Furthermore,*

1. *If $E_1 \cap E_2 = \varnothing$, then M is called 1-sum of M_1 and M_2.*
2. *If $|E_1 \cap E_2| = 1$, then we call M the 2-sum of M_1 and M_2.*
3. *If $|E_1 \cap E_2| = 3$, let $Z = E_1 \cap E_2$. If Z is a circuit of both M_1 and M_2, then we call M the 3-sum of M_1 and M_2.*

(The i-sum is denoted $M_1 \oplus_i M_2$.) Our definition differs from [13, 36] as we have dropped some conditions on the sizes of M_1 and M_2 that we do not need. A $\{1,2,3\}$-*decomposition* of a matroid $\widetilde{M}$ is a set of matroids $\mathcal{M}$ called the *basic matroids*, together with a rooted binary tree T in which $\widetilde{M}$ is the root and the leaves are the elements of $\mathcal{M}$. Every internal vertex in the tree is either the 1-, 2-, or 3-sum of its children. Seymour's decomposition theorem for regular matroids [36] says that every regular matroid $\widetilde{M}$ has a (poly-time computable) $\{1,2,3\}$-decomposition with all basic matroids being graphic, cographic or R_{10}.

The Dinitz-Kortsarz modification. Dinitz and Kortsarz [13] modified Seymour's decomposition to give an $O(1)$-competitive algorithm for the regular-matroid secretary problem, as follows. Given a $\{1,2,3\}$-decomposition T for binary matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, we define Z_M, the *sum-set* of a non-leaf vertex M in T, to be the intersection of the ground sets of its children (the sum-set is thus not in the ground set of M). A sum-set Z_M for internal vertex M is either the empty set (if M is the 1-sum of its children), a single element (for 2-sums), or three elements that form a circuit in its children (for 3-sums). A $\{1,2,3\}$-decomposition is *good* if for every sum-set Z_M of size 3 associated with internal vertex $M = M_1 \oplus_3 M_2$, the set Z_M is contained in the ground set of a single basic matroid below M_1, and in the ground set of a single basic matroid below M_2. For a given $\{1,2,3\}$-decomposition of a matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, define the *conflict graph* G_T to be the graph on $\mathcal{M}$ where basic matroids M_1 and M_2 share an edge if their ground sets intersect. [13] show that if T is a good $\{1,2,3\}$-decomposition of $\widetilde{M}$, then G_T is a forest. We can root each tree in such a forest arbitrarily, and define the parent $p(M)$ of each non-root matroid $M \in \mathcal{M}$. Let A_M be the sum-set for the edge between matroid M and its parent, i.e., $A_M = E(M) \cap E(p(M))$.

Theorem 7 (Theorem 3.8 of [13]). *There is a good $\{1,2,3\}$-decomposition T for any binary matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$ such that (a) each matroid $M \in \mathcal{M}$ has no circuits of size 2 consisting of an element of A_M and an element of $E(\widetilde{M})$, and (b) every basic matroid $M \in \mathcal{M}$ can be obtained from some $M' \in \widetilde{\mathcal{M}}$ by deleting elements and adding parallel elements.*

Dinitz and Kortsarz showed that a good $\{1,2,3\}$-decomposition for a matroid $\widetilde{M}$ can be used to construct independent sets for $\widetilde{M}$ as follows. Below, $\cdot|_S$ denotes restriction to the set S.

Lemma 2 (Lemma 4.4 of [13]). *Let T be a good $\{1,2,3\}$-decomposition for $\widetilde{M}$ with basic matroids $\mathcal{M}$. For each each $M \in \mathcal{M}$, let I_M be an independent set of $(M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$. Then $I = \bigcup_{M\in\mathcal{M}} I_M$ is independent in $\widetilde{M}$.*

Our Algorithm. Given the input matroid $\widetilde{M}$, our idea is to take a good decomposition T and run a PI-OCRS for $(M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$ for each vertex M in the conflict graph G_T. Then we need to glue the pieces together using Lemma 2. One technical obstacle is that the input to an OCRS is a feasible point in the matroid polytope, so to use the framework of [13] we need to convert it into a feasible solution to the polytopes of the (modified) basic matroids. Our insight is captured by the following lemma.

Lemma 3. *Let T be a good $\{1,2,3\}$-decomposition of regular matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, and let vector $\mathbf{x} \in \frac{1}{3}\mathcal{P}_{\widetilde{M}}$. Then for every basic matroid $M \in \mathcal{M}$, if $\widehat{M} := (M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$, then $\mathbf{x}|_{\widehat{M}} \in \mathcal{P}_{\widehat{M}}$.*

Proof. Fix a set $S \subseteq E(M)\cap E(\widetilde{M})$. We will show that $\operatorname{rank}_{\widehat{M}}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$, from which the claim follows.

Case 1: $A_M = \{z\}$. For any maximal independent set $I \subset S$ according to M, there always exists $a \in I$ such that $(I \cup \{z\}) \setminus \{a\}$ is independent in M, therefore $\operatorname{rank}_{\widehat{M}}(S) \geq \operatorname{rank}_{\widetilde{M}}(S) - 1$. Also since no element in S is parallel to z, for any non-empty S we have $\operatorname{rank}_{M/A_M}(S) \geq 1$, and we can conclude that $\operatorname{rank}_{M/A_M}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$.

Case 2: A_M is some 3-cycle $\{z_1, z_2, z_3\}$. For any maximal independent set $I \subset S$ according to M where $|I| \geq 3$, there always exists $a, b \in I$ such that $(I \cup \{z_1, z_2\}) \setminus \{a, b\}$ is independent in M. Therefore $\operatorname{rank}_{M_2/Z}(S) \geq \operatorname{rank}_M(S) - 2$. We claim that there does not exist e in $E(M) \setminus A_M$ such that $e \in \operatorname{span}(A_M)$.

Suppose for contradiction such an e exists. Then there is some circuit in $A_M \cup e$ containing e. Since there are no parallel elements, this circuit have size 3. Without loss of generality, assume this circuit is $C = \{z_1, z_2, e\}$. Since A_M is a circuit, by definition of binary matroids, the set $C \Delta A_M = \{z_3, e\}$ is a cycle, and thus e is parallel to z_3, a contradiction. Therefore for any non-empty S, we have that $\operatorname{rank}_{M/A_M}(S) \geq 1$, and we conclude that $\operatorname{rank}_{M/A_M}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$. □

With this we are able to conclude the main theorem of the section (the full proof of which we defer to Appendix H).

Theorem 8 (Regular Matroids). *There is a $(1/3, 1/12)$-selectable PI-OCRS for regular matroids.*

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A Preliminaries

We assume the reader is familiar with the basics of matroid theory, and refer to [35] for definitions. We will call elements of the ground set *parallel* if they form a circuit of size 2. The matroid polytope is defined to be $\mathcal{P}_M := \{\mathbf{x} \in \mathbb{R}^E \mid \mathbf{x} \geq \mathbf{0},\ \mathbf{x}(S) \leq \operatorname{rank}(S)\ \forall S \subseteq E\}$.

We focus on pairwise independent versions of *contention resolution schemes* (CRSs), both in the offline and online settings. For polytope $\mathcal{P}$ and scalar $b \in \mathbb{R}$, define $b\mathcal{P} := \{b\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$.

Definition 3 (PI-consistency). *Given a polytope $\mathcal{P} \subseteq [0,1]^n$ and a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{P}$, a probability distribution $\mathcal{D}$ over subsets of $[n]$ is* PI-consistent *with $\mathbf{x}$ if*

1. $\Pr_{R\sim\mathcal{D}}[i \in R] = x_i$*, for all* $i \in [n]$*, and*
2. $\Pr_{R\sim\mathcal{D}}[\{i,j\} \subseteq R] = x_i \cdot x_j$ *for all* $i, j \in [n], i \neq j$.

Claim 2. *Consider the distribution $\mathcal{D}'$ obtained by drawing $R \sim \mathcal{D}$ and then subsampling each element* independently *with probability $p \in [0,1]$. If $\mathcal{D}$ is PI-consistent for $\mathbf{x}$ then $\mathcal{D}'$ is PI-consistent for $b\mathbf{x} \in b\mathcal{P}$.*

Definition 4 ((b,c)-balanced PI-CRS). *Let $b, c \in [0,1]$. A* (b,c)-balanced pairwise independent contention resolution scheme *(or* (b,c)-balanced PI-CRS *for short) for a convex polytope $\mathcal{P}$ is a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ that given any distribution $\mathcal{D}$ over 2^E and any set $R \subseteq E$, returns a random set $\pi(R)$ satisfying the following properties:*

1. *(subset)* $\pi(R) \subseteq R$*,*
2. *(feasibility)* $\chi_{\pi(R)} \in \mathcal{P}$*, and*
3. *((b,c)-balanced) Let $\mathbf{x}$ be any vector in $b\mathcal{P}$, and $\mathcal{D}$ be any distribution PI-consistent with $\mathbf{x}$. Let R be sampled according to $\mathcal{D}$, then for any $i \in [n]$, $\Pr[i \in \pi(R) \mid i \in R] \geq c$, where the probability is over both the distribution $\mathcal{D}$ and any internal randomization of π.*

We use c-balanced PI-CRS *as an abbreviation for $(1,c)$-balanced PI-CRS.*

Claim 3. *Given a (b,c)-balanced PI-CRS π for a polytope $\mathcal{P}$, we can construct a bc-balanced PI-CRS π' for $\mathcal{P}$.*

Proof. The PI-CRS π' works as follows: on receiving $\mathbf{x} \in \mathcal{P}$ and $R \sim \mathcal{D}$ where $\mathcal{D}$ is PI-consistent with $\mathbf{x}$, flips a coin for each element $e \in R$ and discards e with probability $1 - b$. Let $R' \subseteq R$ denote the resulting set and $\mathcal{D}'$ the underlying distribution. The PI-CRS π' then returns $\pi(R')$.

By Claim 2, $\mathcal{D}'$ is PI-consistent with $b\mathbf{x}$. Let I be the set that $\pi(R')$ returns. Then $I \subseteq R' \subseteq R$, and $\chi_I \in \mathcal{P}$. Also $\Pr[i \in I \mid i \in R] = b \Pr[i \in I \mid i \in R'] \geq bc$. Therefore we conclude that π' is indeed a bc-balanced PI-CRS for $\mathcal{P}$. $\square$

An *online* contention resolution scheme is one where the n items are presented to the algorithm one-by-one in some order chosen by the adversary; the algorithm has to decide whether to accept an arriving item into the resulting independent set, or to irrevocably reject it [17]. The precise definition of PI-OCRS depends on the adversarial model. We define these precisely in Appendix A, and give just the high-level idea here.

The Almighty Adversary The almighty adversary knows everything about the game even before the game starts. More specifically, first the adversary sees the realization of $R \sim \mathcal{D}$, together with the outcome of all the random coins the algorithm uses. It then presents the items in R to the algorithm in the worst possible order. (Notice that since all the random coins of π is already tossed and known to the adversary, π is a deterministic algorithm from the perspective of the almighty adversary; therefore with enough computational power the almighty adversary can compute the worst order of items for π.) To describe a kind of PI-OCRS effective against the almighty adversary, we adopt ideas from [17].

Definition 5 (Greedy PI-OCRS). *Let $\mathcal{P} \subseteq [0,1]^n$ be some polytope. We call a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ a* greedy PI-OCRS *if:*

1. *Before the items arrive, π has access to $\mathcal{D}$, and (randomly) defines some down-closed set family $\mathcal{F} \subseteq \mathcal{I}$.*
2. *Let I be the set π selects, initially empty. When an item e arrives, it is added to I if and only if $I \cup \{e\} \in \mathcal{F}$.*
3. *Finally, π returns I.*

We say $\mathcal{F}$ is the feasible set family *defined by π, and for simplicity, we say the step 2 and 3 of this algorithm is a greedy PI-OCRS defined by $\mathcal{F}$.*

Here π need not list $\mathcal{F}$ explicitly. In fact, the ability to judge whether a given set S is in $\mathcal{F}$ is enough. Then we define the notion of selectable PI-OCRS, which is powerful even against almighty adversary:

Definition 6 ((b,c)-selectable PI-OCRS). *Let $\mathcal{P} \subseteq [0,1]^n$ be some convex polytope. We call a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ a (b,c)-*selectable PI-OCRS *if it satisfies the following:*

1. *Algorithm π is a* greedy *PI-OCRS.*
2. *For any $\mathbf{x} \in b\mathcal{P}$, any distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$ and any $i \in [n]$, let $\mathcal{F}$ be the feasible set family defined by π. Let R be sampled according to $\mathcal{D}$, then*
$$\Pr_{R \sim \mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] \geq c. \tag{A.1}$$
Here the probability is over R and internal randomness of π while defining $\mathcal{F}$.

We use c-selectable PI-OCRS as an abbreviation for $(1, c)$-selectable PI-OCRS.

Notice the definition here is slightly different from [17], as we need to condition on the event $i \in R$. This is due to our limited independence over events $i \in R$: For the mutually independent case, one can prove that $\Pr_{R\sim\mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] = \Pr_{R\sim\mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F}]$. But this may not hold in the pairwise independent case.

We now argue that any (b, c)-selectable PI-OCRS is powerful even against an almighty adversary.

Claim 4. *Let $\mathcal{P} \subseteq [0, 1]^n$ be a polytope, and π a (b, c)-selectable PI-OCRS. Let $\mathbf{x}$ be any vector in $b\mathcal{P}$, and $\mathcal{D}$ be any distribution PI-consistent with $\mathbf{x}$. Let R be sampled according to $\mathcal{D}$. Then for any item $i \in [n]$, $\Pr[i \in \pi(R) \mid i \in R] \geq c$, even when the order of the items is decided by the almighty adversary.*

Proof. The constraint (A.1) guarantees that no matter what set I we choose before i arrives, the event $I \cup \{i\} \in \mathcal{F}$ holds (and therefore i is selected when it is in R) with probability at least c. Since this argument holds regardless of the items appearing before i, we conclude that $\Pr[i \in \pi(R) \mid i \in R] \geq c$. □

Using the same idea of subsampling each item independently as Claim 2, we can also obtain a bc-selectable PI-OCRS from a (b, c)-selectable PI-OCRS.

Claim 5. *Given a (b, c)-selectable PI-OCRS π for some polytope $\mathcal{P}$, we can construct a bc-selectable PI-OCRS π' for $\mathcal{P}$.*

Proof. Let $\mathcal{D}$ be any distribution PI-consistent with some vector $\mathbf{x} \in \mathcal{P}_M$. First, π' defines a random set S, where for each item i, π' flips a random coin to include i in S with probability b. Let R be sampled according to $\mathcal{D}$, and let $\mathcal{D}'$ denote the distribution of $R \cap S$, then one can verify that $\mathcal{D}'$ is PI-consistent with $b\mathbf{x}$. Let $\mathcal{F}$ denote the feasible set family defined by π for $\mathcal{D}'$. Then π' defines $\mathcal{F}'$ as

$$\mathcal{F}' := \{I \in \mathcal{F} : I \subseteq S\}$$

Here $\mathcal{F}' \subseteq \mathcal{F} \subseteq \mathcal{I}$, and it remains to bound the selectability of the greedy PI-OCRS defined by $\mathcal{F}'$. Observe that for any $i \in S$, the following two propositions are equivalent:

1. $I \cup \{i\} \in \mathcal{F}$ holds for any $I \in \mathcal{F}, I \subseteq R \cap S$.
2. $I \cup \{i\} \in \mathcal{F}'$ holds for any $I \in \mathcal{F}', I \subseteq R$.

Then for any item i, we have

$$\begin{aligned}
&\Pr[I \cup \{i\} \in \mathcal{F}' \quad \forall I \in \mathcal{F}', I \subseteq R \mid i \in R] \\
&= \Pr[i \in S \mid i \in R] \Pr[I \cup \{i\} \in \mathcal{F}' \quad \forall I \in \mathcal{F}', I \subseteq R \mid i \in R \cap S] \\
&= b \Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \cap S \mid i \in R \cap S]
\end{aligned}$$

$$\geq bc. \qquad (R \cap S \sim \mathcal{D}', \ (b,c)\text{-selectability of } \pi)$$

This completes the proof. □

The Offline Adversary A much weaker type of adversary is the offline adversary, which knows nothing about the outcome of the random events, and has to decide the order of the items before they start to arrive. In other words, the adversary presents the worst, fixed order to the algorithm before $R \sim \mathcal{D}$ is drawn or the randomness of the algorithm is determined. Here we use (b,c)-*balanced* to denote PI-OCRS effective against offline adversary, compared to the (b,c)-*selectable* against almighty adversary.

Definition 7 ((b,c)-balanced PI-OCRS). *Let $\mathcal{P} \subseteq [0,1]^n$ be some convex polytope. We call a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ a (b,c)-balanced PI-OCRS for $\mathcal{P}$, if given any distribution $\mathcal{D}$ and any set $R \in E$, returns a set $\pi(R)$ satisfying the following properties:*

1. *(subset) $\pi(R) \subseteq R$,*
2. *(feasibility) $\chi_{\pi(R)} \in \mathcal{P}$, and*
3. *((b,c)-balanced) Let $\mathbf{x}$ be any vector in $b\mathcal{P}$, and $\mathcal{D}$ be any distribution PI-consistent with $\mathbf{x}$, and let R be sampled according to $\mathcal{D}$. Suppose the adversary has to choose the order of the items before R is sampled or the random coins of π are tossed, and when each item i is presented to π and is in R, π can make an irrevocable choice to select it or discard it. Then for any i,*

$$\Pr[i \in \pi(R) \mid i \in R] \geq c.$$

We use c-balanced PI-OCRS as an abbreviation for $(1,c)$-balanced PI-OCRS.

Using the same idea of subsampling each item independently as Claim 3, we can also obtain a bc-balanced PI-OCRS from a (b,c)-balanced PI-OCRS. The proof is similar to Claim 3 and therefore omitted.

Claim 6. *Given a (b,c)-balanced PI-OCRS π for some polytope $\mathcal{P}$, we can construct a bc-balanced PI-OCRS π' for $\mathcal{P}$.*

The property of *balancedness* is weaker than that of *selectability*: (b,c)-selectable PI-OCRS are also (b,c)-balanced, but the converse might not be true. However, a *balanced* PI-OCRS is enough to solve the prophet inequality problem, as we will see in the next section.

Prophet Inequality and OCRS Feldman et al. [17] showed connections between OCRSs and prophet inequalities; here we extend the relationship to the pairwise independent setting.

Definition 8 (Pairwise Independent Matroid Prophet Game). *Let $M = (E, \mathcal{I})$ be a matroid. For simplicity, we identify E as $[n]$. Let $W_1, \cdots, W_n$ be n pairwise independent non-negative random variables denoting the weight of the items, and let $\mathcal{D}_W$ be their joint distribution, which is known to the gambler. Now the n items are presented to the gambler one by one in some* fixed *order* unknown *to the gambler, and when each item arrives, the gambler has to make an irrevocable choice to select the item or not. The items selected must form an independent set of M. The gambler wants to maximize the sum of weights of the items he selects. We define the offline optimum* $\mathsf{OPT} := \mathbb{E}[\max_{I \in \mathcal{I}} \sum_{i \in I} W_i]$, *and let* ALG *denote the expected reward of the algorithm the gambler uses. We call an algorithm an α-approximation if* $\mathsf{ALG}/\mathsf{OPT} \geq \alpha$.

We claim that a good PI-OCRS gives a good approximation for the pairwise independent matroid prophet game:

Claim 7. *Let $M = (E, \mathcal{I})$ be a matroid. If there is a c-balanced PI-OCRS for $\mathcal{P}_M$, then there is a c-competitive algorithm for the pairwise independent matroid prophet game over M.*

Proof. For simplicity, we assume all the W_i are distributed continuously. (The extension to discrete distributions is standard, see e.g. [34, page 2].) Let $I_{\max}$ be an arbitrary optimal set, i.e., $I_{\max} := \operatorname{argmax}_{I \in \mathcal{I}} \sum_{i \in I} W_i$. Let $p_i := \Pr[i \in I_{\max}]$ denote the probability that element i is in the optimal set. Then we claim that $\mathbf{p}$ is in $\mathcal{P}$. This is because for any $S \subseteq E$,

$$\sum_{i \in S} p_i = \sum_{i \in S} \Pr[i \in I_{\max}] = \mathbb{E}[|I_{\max} \cap S|] \leq \operatorname{rank}(S).$$

After we obtain $\mathbf{p}$, we set a threshold τ_i for each $i \in E$ such that $\Pr[W_i \geq \tau_i] = p_i$. We then have the following lemma regarding the expected optimum value.

Lemma 4. *Let $M = (E, \mathcal{I})$ be a matroid, and W_i, p_i, τ_i be defined as above. Then*

$$\mathsf{OPT} = \mathbb{E}[\max_{I \in \mathcal{I}} \sum_{i \in I} W_i] \leq \sum_{i \in E} p_i \mathbb{E}[W_i \mid W_i \geq \tau_i].$$

Proof. First notice that $\mathsf{OPT} = \sum_{i \in E} p_i \mathbb{E}[W_i \mid i \in I_{\max}]$. Since $p_i = \Pr[i \in I_{\max}] = \Pr[W_i \geq \tau_i]$, we have

$$\begin{aligned} p_i\, \mathbb{E}[W_i | i \in I_{\max}] &= p_i \int_{t \geq 0} \Pr[W_i \geq t \mid i \in I_{\max}] \,\mathrm{d}t \\ &= \int_{t \geq 0} \Pr[W_i \geq t, i \in I_{\max}] \,\mathrm{d}t \\ &\leq \int_{t \geq 0} \Pr[W_i \geq t, W_i \geq \tau_i] \,\mathrm{d}t \end{aligned}$$

$$= p_i \int_{t \geq 0} \Pr[W_i \geq t \mid W_i \geq \tau_i] \, \mathrm{d}t$$
$$= p_i \, \mathbb{E}[W_i \mid W_i \geq \tau_i].$$

Summing over i finishes the proof. $\square$

Now we describe our algorithm of translating PI-OCRS into approximation of prophet games. Let π be a c-balanced PI-OCRS for M. Let the random set R denote the items with weights exceeding τ_i, i.e., $R := \{i \mid W_i \geq \tau_i\}$. Let $\mathcal{D}$ denote the distribution of R, and $\mathcal{D}$ is given to π before the items starts to arrive. When item i arrives, we add i to R and send it to π if $W_i \geq \tau_i$. Then we select i if and only if π selects i.

First, by our argument before, $\mathbf{p}$ is in $\mathcal{P}_M$. Then notice that since W_i are pairwise independent, the events $\{i \in R\}$, or $\{W_i \geq \tau_i\}$ are also pairwise independent. Therefore R is PI-consistent with $\mathbf{p}$. Then the definition of c-balanced PI-OCRS guarantees that for each i, $\Pr[i \in \pi(R) \mid i \in R] \geq c$. Since whether i is selected by π when it is in R is independent of W_i, we have $\mathbb{E}[W_i \mid i \in \pi(R)] = \mathbb{E}[W_i \mid i \in R] = \mathbb{E}[W_i \mid W_i \geq \tau_i]$. Then we can estimate the expected reward of our algorithm:

$$\begin{aligned}
\mathsf{ALG} &= \sum_{i \in E} \Pr[i \in \pi(R)] \cdot \mathbb{E}[W_i \mid i \in \pi(R)] = \sum_{i \in E} \Pr[i \in \pi(R)] \cdot \mathbb{E}[W_i \mid W_i \geq \tau_i] \\
&\geq \sum_{i \in E} c \Pr[i \in R] \cdot \mathbb{E}[W_i \mid W_i \geq \tau_i] && \text{(Definition 7)} \\
&= \sum_{i \in E} c \, p_i \cdot \mathbb{E}[W_i \mid W_i \geq \tau_i] \geq c \, \mathsf{OPT}. && \text{(Lemma 4)}
\end{aligned}$$

Therefore we conclude that our algorithm gives a c-approximation for the pairwise independent matroid prophet game. $\square$

B The Single Item Setting

B.1 A $(\sqrt{2}-1)$-Balanced PI-OCRS for Single Item

Caragiannis et al. [7] give a $(\sqrt{2}-1)$-prophet inequality for the single-item case; we show how the same ideas give a $(\sqrt{2}-1)$-balanced PI-OCRS as well, when the adversary is oblivious. (In Appendix C.1 we show a uniform threshold algorithm that works in the presence of almighty adversaries as well.) Indeed, when item i arrives and none items have been selected, we flip a random coin and select it with some probability q_i. For simplicity, let $X_i = \mathbb{1}[i \in R]$, and let Bernoulli random variable Y_i denote the event that the random coin flip is head up. Let I denote the final set we select, then for item i, we can estimate $\Pr[i \in I \mid i \in R]$:

$$\Pr[i \in I \mid i \in R] = \Pr[Y_i = 1 \mid X_i = 1] \Pr[X_j Y_j = 0 \ \forall j < i \mid X_i = 1]$$

$$
\begin{aligned}
&= q_i\left(1 - \Pr\left[\bigvee_{j<i} X_jY_j = 1 \mid X_i = 1\right]\right) \\
&\geq q_i\left(1 - \sum_{j<i} \Pr[X_jY_j = 1 \mid X_i = 1]\right) \\
&= q_i\left(1 - \sum_{j<i} \Pr[X_jY_j = 1]\right) \quad \text{(pair-wise independence of } X_i\text{)} \\
&= q_i\left(1 - \sum_{j<i} x_jq_j\right)
\end{aligned}
$$

Thus we want to choose q_i such that $c = \min_i q_i(1 - \sum_{j<i} x_jq_j)$ is as large as possible. A simple idea is to set $q_i = 1/2$ for all i, which guarantees that $c \geq 1/4$. In fact, we choose q_i such that $c \geq \sqrt{2} - 1$, and we can give an example showing that the $\sqrt{2} - 1$ here is asymptotically the best.

Lemma 5. *Let $\mathbf{x} \in \mathbb{R}_+^n$ be a real vector such that $\sum_i x_i \leq 1$. Then there exists $\mathbf{q}$ such that $q_i(1 - \sum_{j<i} x_jq_j) \geq \sqrt{2} - 1$ holds for all i. Moreover, q_i can be calculated online: we can calculate such q_i based only on $x_1, \cdots, x_i$.*

Proof. We set $q_i = f(\sum_{j<i} x_j)$ where $f(t) = 1/\sqrt{(3 + 2\sqrt{2}) - (2 + 2\sqrt{2})t}$ to show that $q_i(1 - \sum_{j<i} x_jq_j) \geq \sqrt{2} - 1$, we need the following claim:

Claim 8. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous non-decreasing function, and $\mathbf{x} \in R_+^n$ a non-negative vector such that $\sum_i x_i \leq 1$. Let S_i denote $\sum_{j=1}^{i} x_i$. Then for any i,*

$$\sum_{j<i} x_jf(S_{j-1}) \leq \int_0^{S_{i-1}} f(t)\,dt.$$

Proof. Since f is non-decreasing, we have

$$x_jf(S_{j-1}) \leq \int_{S_{j-1}}^{S_j} f(t)\mathrm{d}t.$$

Then summing over j proves the claim. □

Let S_i denote the prefix sum of $\mathbf{x}$, i.e.$S_i := \sum_{j=1}^{i} x_j$. Then we can bound $q_i(1 - \sum_{j<i} x_jq_j)$:

$$
\begin{aligned}
&q_i\left(1 - \sum_{j<i} x_jq_j\right) \\
&= f(S_{i-1})\left(1 - \sum_{j<i} x_jf(S_{j-1})\right) \geq f(S_{i-1})\left(1 - \int_0^{S_{i-1}} f(t)\mathrm{d}t\right)
\end{aligned}
$$

$$= \frac{1}{\sqrt{(3+2\sqrt{2})-(2+2\sqrt{2})S_{i-1}}} \cdot (\sqrt{2}-1)\sqrt{(3+2\sqrt{2})-(2+2\sqrt{2})S_{i-1}}$$
$$= \sqrt{2}-1. \qquad \square$$

Then we show that even when $\mathbf{x}$ is uniform ($\mathbf{x} = (1/n, \cdots, 1/n)$), $(\sqrt{2}-1)$ is asymptotically the best no matter what $\mathbf{q}$ we use.

Lemma 6. *For every $\mathbf{q} \in [0,1]^n$, we have*

$$\min_i q_i \left(1 - \frac{1}{n}\sum_{j<i} q_j\right) \le \sqrt{2} - 1 + O\left(\frac{1}{n}\right).$$

Proof. Let $r_i = q_i(1 - \frac{1}{n}\sum_{j<i} q_j)$. First, for any non-negative vector $\mathbf{p} \in \mathbb{R}_+^n$, we have

$$\min_i r_i \le \frac{\mathbf{r}\cdot\mathbf{p}}{\|\mathbf{p}\|_1}$$

In particular, the $\mathbf{p}$ we choose is $(1, \cdots, 1, \alpha n)$, i.e. first $(n-1)$ entries are 1, while the last entry is αn, where α is some constant whose value is to be determined. Then we compute $\mathbf{r}\cdot\mathbf{p}$:

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{p} &= \sum_{i=1}^{n-1} q_i \left(1 - \frac{1}{n}\sum_{j<i} q_j\right) + \alpha n q_n \left(1 - \frac{1}{n}\sum_{j<n} q_i\right) \\
&= (1-\alpha q_n)\sum_{i=1}^{n-1} q_i - \frac{1}{n}\sum_{1\le j<i\le n-1} q_i q_j + \alpha n q_n \\
&= (1-\alpha q_n)\sum_{i=1}^{n-1} q_i - \frac{1}{2n}\left(\sum_{i=1}^{n-1} q_i\right)^2 + \frac{1}{2n}\sum_{i=1}^{n-1} q_i^2 + \alpha n q_n \\
&\le \frac{n}{2}(1-\alpha q_n)^2 + \alpha n q_n + \frac{1}{2}.
\end{aligned}$$

Meanwhile, $\|\mathbf{p}\|_1 = n - 1 + \alpha n$. Therefore we have

$$\begin{aligned}
\min_i r_i &\le \frac{\mathbf{r}\cdot\mathbf{p}}{\|\mathbf{p}\|_1} \\
&\le \frac{\frac{1}{2}(1-\alpha q_n)^2 + \alpha q_n + \frac{1}{2n}}{\frac{n-1}{n} + \alpha} \\
&= \frac{1+\alpha^2 q_n^2}{2+2\alpha} + O\left(\frac{1}{n}\right) \\
&\le \frac{1+\alpha^2}{2+2\alpha} + O\left(\frac{1}{n}\right).
\end{aligned}$$

The expression $(1+\alpha^2)/(2+2\alpha)$ has a minimum value $\sqrt{2}-1$, achieved when $\alpha = \sqrt{2}-1$. Therefore we conclude that $\min_i r_i \le \sqrt{2} - 1 + O(\frac{1}{n})$. $\square$

However, the above analysis relies on the order of the items staying the same regardless of R, and therefore this PI-OCRS might not be be robust against the almighty adversary. In the next section we show a simple 1/4-selectable PI-OCRS for all uniform matroids, including the single item case.

B.2 An Upper Bound for Multiple-Threshold Algorithms

Here we show an example where no multiple-threshold algorithm can give an approximation ratio better than $2\sqrt{5}-4 \approx 0.472$. This result shows that we need new ideas to match the $1/2$-prophet inequality for the fully independent case.

The instance has $n+2$ items. Item 1 has deterministic value of t, where $0 < t < 1$ is some constant to be determined; item $2 \cdots n+1$ each has value 1 with probability $\frac{1}{n}$ and 0 otherwise; item $n+2$ has value rn with probability $\frac{1}{n}$ and 0 otherwise, where $r > 0$ is some constant to be determined. The joint distribution is constructed as follows:

- With probability $\frac{n+1}{2n}$, sample 2 items from $2 \cdots (n+2)$ and make them non-zero (i.e. for item $2 \cdots n+1$, set value as 1; for item $n+2$, set value as rn). The remaining items has value 0.
- With probability $\frac{n-1}{2n}$, items $2 \cdots n+2$ all have value zero.

It's easy to verify that this distribution is indeed pairwise independent, and the expected value of the largest item is

$$\mathrm{OPT} = \frac{n-1}{2n} \cdot t + \frac{1}{n} \cdot rn + \left(\frac{n+1}{2n} - \frac{1}{n}\right) \cdot 1 = r + \frac{1}{2}(1+t) + O\left(\frac{1}{n}\right)$$

Now for any multiple-threshold algorithm, its behaviour on this example can be described as:

- On seeing item i, if i is non-zero, toss an independent coin to take it with probability q_i; otherwise abandon this item. Here q_i is only determined by the ordering of items, and does not depend on values of item $1 \cdots i-1$.

Here we may as well assume $q_{n+2} = 1$. And we can calculate the expected reward for each item:

- For item 1, the expected reward is $q_i \cdot t$
- For item $n+2$, the expected reward is

$$\frac{1}{n} \cdot rn \cdot (1-q_1) \cdot \left(\sum_{i=2}^{n+1} \frac{1}{n}(1-q_i)\right) = r(1-q_1)\left(1 - \frac{1}{n}\sum_{i=2}^{n+1} q_i\right)$$

- For items $2 \cdots n+1$, the expected reward is

$$\sum_{i=2}^{n+1} \frac{1}{n} \cdot q_i \cdot (1-q_1) \cdot \left(\sum_{j=2}^{i-1} \frac{1}{n}(1-q_j) + \sum_{j=i+1}^{n+2} \frac{1}{n}\right)$$

$$
\begin{aligned}
&= (1-q_1)\left(\frac{1}{n}\sum_{i=2}^{n+1} q_i - \frac{1}{n^2}\sum_{2\le j<i\le n+1} q_i q_j\right)\\
&= (1-q_1)\left(\frac{1}{n}\sum_{i=2}^{n+1} q_i - \frac{1}{2n^2}\left(\sum_{i=2}^{n+1} q_i\right)^2 + \frac{1}{2n^2}\sum_{i=2}^{n+1} q_i^2\right)\\
&= (1-q_1)\left(\frac{1}{n}\sum_{i=2}^{n+1} q_i - \frac{1}{2n^2}\left(\sum_{i=2}^{n+1} q_i\right)^2\right) + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Therefore the total expected reward for the algorithm is

$$
\begin{aligned}
\mathrm{ALG} &= q_1 t + (1-q_1)\left(r + (1-r)\frac{\sum_{i=2}^{n+1} q_i}{n} - \frac{1}{2}\left(\frac{\sum_{i=2}^{n+1} q_i}{n}\right)^2\right) + O\left(\frac{1}{n}\right)\\
&\le q_1 t + (1-q_1)\left(r + \frac{1}{2}(1-r)^2\right) + O\left(\frac{1}{n}\right)\\
&= q_1 t + (1-q_1)\frac{1}{2}(1+r^2) + O\left(\frac{1}{n}\right)\\
&\le \max(t, {}^1\!/_2(1+r^2)) + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Putting these expressions together, the approximation ratio for our algorithm is at most

$$
\mathrm{APX} = \frac{\mathrm{ALG}}{\mathrm{OPT}} \le \frac{\max(t, \frac{1}{2}(1+r^2))}{r + \frac{1}{2}(t+1)} + O\left(\frac{1}{n}\right).
$$

The minimal value is achieved when $t = \frac{1}{2}(1+r^2)$, giving us

$$
\mathrm{APX} \le \frac{\frac{1}{2}(1+r^2)}{\frac{1}{4}r^2 + r + \frac{3}{4}} + O\left(\frac{1}{n}\right).
$$

The expression on the right is minimized when $r = \frac{1}{2}(\sqrt{5}-1)$, giving us a minimum value of $2\sqrt{5} - 4 + O(\frac{1}{n})$, as claimed.

B.3 A Single-Sample Prophet Algorithm

In this section we give an $O(1)$ competitive algorithm for the prophet problem under a pairwise independent distribution, in the restricted setting where the algorithm has only limited prior knowledge of the pairwise independent distribution $\mathcal{D}$ in the form of a single sample from $\mathcal{D}$.

Formally, the algorithm is given as input $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle \sim \mathcal{D}$. Subsequently, the values of a second draw $\langle X_1, \ldots, X_n \rangle \sim \mathcal{D}$ are revealed one by one in order. The algorithm's goal is to stop at the largest value. We show the following theorem.

Theorem 4 (Single Sample Prophet Inequality). *There is an algorithm that draws a single sample from the underlying pairwise-independent distribution* $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle \sim \mathcal{D}$ *on* $\mathbb{R}_+^n$*, and then faced with a second sample* $\langle X_1, \ldots, X_n \rangle \sim \mathcal{D}$ *(independent from* $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle$*), picks a single item* i *from* $X_1, \ldots, X_n$ *with expected value at least* $\Omega(1) \cdot \mathbb{E}_{\mathbf{X} \sim \mathcal{D}}[\max_i X_i]$*.*

Proof. Our algorithm sets a threshold $v^* = \max_i \widetilde{X}_i$ (which is a random variable), and then stops at the first $X_i \geq v^*$ if such an X_i exists.

To understand the performance of this algorithm, define the quantity $x(t) := \sum_i \mathbb{P}\left(\widehat{v}_i \geq t\right)$. The proof of [7, Theorem 2] shows that an algorithm using threshold t achieves competitive ratio $\min(1-t, t/(t+1))$. Letting f be the probability density function of $x(v^*)$, and defining

$$F(t) := \int_0^t f(r)\mathrm{d}r = \Pr[x(v^*) \leq t] = \Pr[v^* \geq x^{-1}(t)],$$

we can lower bound our competitive ratio as

$$\begin{aligned} \mathrm{APX} &\geq \int_0^1 f(t) \min\left(1-t, \frac{t}{1+t}\right) \mathrm{d}t \\ &= \int_0^{\frac{\sqrt{5}-1}{2}} f(t) \frac{t}{1+t} \mathrm{d}t + \int_{\frac{\sqrt{5}-1}{2}}^1 f(t)(1-t)\mathrm{d}t. \end{aligned}$$

We apply integration by parts to the first term to get

$$\begin{aligned} &\int_0^{\frac{\sqrt{5}-1}{2}} f(t) \frac{t}{1+t} \mathrm{d}t \\ &= F(t) \frac{t}{1+t}\Big|_0^{\frac{\sqrt{5}-1}{2}} - \int_0^{\frac{\sqrt{5}-1}{2}} f(t) \frac{1}{(1+t)^2} \mathrm{d}t \\ &= \frac{3-\sqrt{5}}{2} F\left(\frac{\sqrt{5}-1}{2}\right) - \int_0^{\frac{\sqrt{5}-1}{2}} F(t) \frac{1}{(1+t)^2} \mathrm{d}t, \end{aligned}$$

and to the second term which gives

$$\begin{aligned} &\int_{\frac{\sqrt{5}-1}{2}}^1 f(t)(1-t)\mathrm{d}t \\ &= F(t)(1-t)\Big|_{\frac{\sqrt{5}-1}{2}}^1 + \int_{\frac{\sqrt{5}-1}{2}}^1 F(t)\mathrm{d}t \\ &= -\frac{3-\sqrt{5}}{2} F\left(\frac{\sqrt{5}-1}{2}\right) + \int_{\frac{\sqrt{5}-1}{2}}^1 F(t)\mathrm{d}t. \end{aligned}$$

Since $F(t) = \Pr[v^* \geq x^{-1}(t)] = \Pr[\bigvee_{i=1}^n v_i \geq x^{-1}(t)]$, by [7, Lemma 1], we have $\frac{t}{1+t} \leq F(t) \leq t$. Thus

$$\begin{aligned}\mathrm{APX} &\geq \int_{\frac{\sqrt{5}-1}{2}}^{1} F(t)\mathrm{d}t - \int_0^{\frac{\sqrt{5}-1}{2}} F(t)\frac{1}{(1+t)^2}\mathrm{d}t \\ &\geq \int_{\frac{\sqrt{5}-1}{2}}^{1} \frac{t}{1+t}\mathrm{d}t - \int_0^{\frac{\sqrt{5}-1}{2}} \frac{t}{(1+t)^2}\mathrm{d}t \\ &= 3 - \sqrt{5} - \ln 2.\end{aligned}$$

Therefore this algorithm gives a competitive ratio of $(3-\sqrt{5}-\ln 2)$, or approximately 7%. □

C Details for Uniform Matroids

C.1 A $(b, 1-b)$-Selectable PI-OCRS for Uniform Matroid

Here we show a $(b, 1-b)$-selectable PI-OCRS for uniform matroids. Let $M = (E, \mathcal{I})$ be a k-uniform matroid, where E is identified as $[n]$. Let $\mathcal{D}$ be some distribution PI-consistent with $\mathbf{x} \in b\mathcal{P}_M$, and let R be sampled according to $\mathcal{D}$. Then we simply set the feasible set family $\mathcal{F} = \mathcal{I}$. We then bound the selectability of this PI-OCRS using Markov's inequality:

$$\begin{aligned}&\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \\ &= \Pr[|R| \setminus \{i\} \leq k-1 \mid i \in R] \\ &= 1 - \Pr[|R \setminus \{i\}| \geq k \mid i \in R] \\ &= 1 - \frac{\mathbb{E}[|R \setminus \{i\}| \mid i \in R]}{k} && \text{(Markov's inequality)} \\ &= 1 - \frac{\sum_{i' \in E \setminus \{i\}} \Pr[i' \in R \mid i \in R]}{k} \\ &= 1 - \frac{\sum_{i' \in E \setminus \{i\}} \Pr[i' \in R]}{k} && \text{(pairwise independence of events } i \in R) \\ &\geq 1 - b && (\mathbf{x}(E) \leq bk)\end{aligned}$$

Therefore $\mathcal{F}$ defines a $(b, 1-b)$-selectable PI-OCRS.

C.2 A $(1 - O(k^{-1/3}))$-Balanced PI-CRS for Uniform Matroid

First, assume that we are given any distribution $\mathcal{D}$ PI-consistent with some $\mathbf{x} \in (1-\varepsilon)\mathcal{P}_M$. Suppose that $\mathbf{x}(E) = (1-\delta)k$ for some $\delta \geq \varepsilon$. By Lemma 1 and an averaging argument, there exists i_n such that

$$\Pr[|R| \geq k \mid i_n \in R] = \frac{\Pr[|R| \geq k, i_n \in R]}{\Pr[i_n \in R]} \leq \frac{\frac{1-\delta^2}{\delta^2}}{(1-\delta)k} = \frac{1+\delta}{\delta^2 k} \leq \frac{1+\varepsilon}{\varepsilon^2 k}.$$

After we select i_n as above, observe that $M \setminus \{i_n\}$ is also a k-uniform matroid, and $\mathbf{x}(E \setminus \{i_n\}) \le (1-\varepsilon)k$. Therefore using the same argument as above, we can choose i_{n-1} such that $\Pr[|R \setminus \{i_n\}| \ge k \mid i_{n-1} \in R] \le \frac{1+\varepsilon}{\varepsilon^2 k}$. Similarly, we can choose i_{n-2} such that $\Pr[|R \setminus \{i_n, i_{n-1}\}| \ge k \mid i_{n-2} \in R] \le \frac{1+\varepsilon}{\varepsilon^2 k}$. We repeat this procedure, and obtain an ordering of items $\{i_j\}_{j=1}^n$ such that for any j,

$$\Pr[|R \cap \{i_{j'} : j' \le j\}| \ge k \mid i_j \in R] \le \frac{1+\varepsilon}{\varepsilon^2 k}.$$

Let I denote the set of items we select, initially empty. Now we consider the item one by one from i_1 to i_n and run a "greedy" procedure: we add $i_j \in R$ to I whenever $|I \cup \{i_j\}| \le k$, and finally return $\pi(R) = I$. Since $\Pr[|R \cap \{i'_j : j' \le j\}| \ge k \mid i_j \in R] \le \frac{1+\varepsilon}{\varepsilon^2 k}$, we have $\Pr[i \in \pi(R) \mid i \in R] \ge 1 - \frac{1+\varepsilon}{\varepsilon^2 k}$. In conclusion, the above procedure gives an $(1-\varepsilon, 1-(1+\varepsilon)/(\varepsilon^2 k))$-balanced PI-CRS. If we set $\varepsilon = k^{-1/3}$, by Claim 3 we have an $(1 - O(k^{-1/3}))$-balanced PI-CRS.

D Details for Laminar Matroids

Let b be some constant close to 1. Each constraint $(A, c'(A))$ is equivalent to a $c'(A)$-uniform matroid over A, therefore by Section 2, we have a $(1-b, b)$-selectable PI-OCRS as well as a $(1-b, 1-(\frac{4}{27}b^3 c'(A))^{-1/2})$-selectable PI-OCRS. We set some threshold t to use the first PI-OCRS when $c'(A) < 2^t$, and the second when $c'(A) \ge 2^t$. Let $\mathcal{F}_{A,c'(A)}$ denote the set family for the respective greedy PI-OCRS. We define the *extension* of $\mathcal{F}_{A,c'(A)}$ to E as $\mathcal{F}^E_{A,c'(A)} := \{I \subseteq E : I \cap A \in \mathcal{F}_{A,c'(A)}\}$. Finally, we set $\mathcal{F}_{\pi,\mathcal{D}} := \bigcap_{A \in \mathcal{A}'} \mathcal{F}^E_{A,c'(A)}$, and our greedy PI-OCRS is defined by $\mathcal{F}_{\pi,\mathcal{D}}$.

First, we claim that $\mathcal{F}_{\pi,\mathcal{D}} \subseteq \mathcal{I}'$, where $\mathcal{I}'$ denotes the independent sets of M'. This is true because all sets in $\mathcal{F}^E_{A,c'(A)}$ satisfies the constraint $(A, c'(A))$, and therefore $\mathcal{F}_{\pi,\mathcal{D}}$ satisfies all the constraints as the intersection of all $\mathcal{F}^E_{A,c'(A)}$. To bound the selectability of the greedy PI-OCRS defined by $\mathcal{F}_{\pi,\mathcal{D}}$, we first need the following observation:

$$\begin{aligned}&\Pr[I \cup \{i\} \in \mathcal{F}^E_{A,c'(A)} \quad \forall I \in \mathcal{F}^E_{A,c'(A)}, I \subseteq R \mid i \in R] \\ &= \begin{cases} 1 & , i \notin A \\ \Pr[I \cup \{i\} \in \mathcal{F}_{A,c'(A)} \quad \forall I \in \mathcal{F}_{A,c'(A)}, I \subseteq R \cap A \mid i \in R \cap A] & , i \in A \end{cases}\end{aligned} \tag{D.1}$$

Then we have

$$\begin{aligned}&\Pr[I \cup \{i\} \in \mathcal{F}_{\pi,\mathcal{D}} \quad \forall I \in \mathcal{F}_{\pi,\mathcal{D}}, I \subseteq R \mid i \in R] \\ &= \Pr\left[I \cup \{i\} \in \bigcap_{A \in \mathcal{A}'} \mathcal{F}^E_{A,c'(A)} \quad \forall I \in \bigcap_{A \in \mathcal{A}'} \mathcal{F}^E_{A,c'(A)}, I \subseteq R \mid i \in R\right] \\ &\ge 1 - \sum_{A \in \mathcal{A}'} \left(1 - \Pr\left[I \cup \{i\} \in \mathcal{F}^E_{A,c'(A)} \quad \forall I \in \bigcap_{B \in \mathcal{A}'} \mathcal{F}^E_{B,c'(B)}, I \subseteq R \mid i \in R\right]\right)\end{aligned}$$

$$\geq 1 - \sum_{A \in \mathcal{A}'} (1 - \Pr[I \cup \{i\} \in \mathcal{F}^E_{A,c'(A)} \quad \forall I \in \mathcal{F}^E_{A,c'(A)}, I \subseteq R \mid i \in R])$$

$$= 1 - \sum_{A \ni i} (1 - \Pr[I \cup \{i\} \in \mathcal{F}_{A,c'(A)} \quad \forall I \in \mathcal{F}_{A,c'(A)}, I \subseteq R \cap A \mid i \in R \cap A]) \qquad \text{(Equation (D.1))}$$

Since $\mathcal{A}'$ is a laminar set family, for $A, B \in \mathcal{A}', A \neq B, A \cap B \ni i$, either $A \subsetneq B$ or $B \subsetneq A$. Therefore by the strict monotonicity of c', $c'(A)$ for $A \ni i$ form a geometric series. Recall that we use the $(1-b, b)$-selectable PI-OCRS for $c'(A) < 2^t$, and $(1-b, 1-(\frac{4}{27}b^3c'(A))^{-1/2})$-selectable PI-OCRS for $c'(A) \geq 2^t$. Therefore we have

$$1 - \sum_{A \ni i} \left(1 - \Pr\left[I \cup \{i\} \in \mathcal{F}_{A,c'(A)} \quad \forall I \in \mathcal{F}_{A,c'(A)}, I \subseteq R \cap A \mid i \in R \cap A\right]\right)$$

$$\geq 1 - t\,(1-b) - \sum_{i \geq t} \left(\frac{4}{27}\, b^3\, 2^i\right)^{-1/2}$$

$$= 1 - t\,(1-b) - b^{-3/2}\, 2^{-t/2}\, \frac{3\sqrt{3}}{2-\sqrt{2}}.$$

Taking $t = 13$ and $b = 24/25$ gives a $(1/25, 1/2.661)$-selectable PI-OCRS. Therefore by Claim 5, we have a $1/67$-selectable PI-OCRS for M', which by the observation above gives a $(1/2, 1/67)$-selectable, and again by Claim 5, a $1/134$-selectable PI-OCRS for M.

E Details for Graphic Matroids

E.1 Limitations of our method

In Section 4, we showed that for any graphic matroid $M = (E, \mathcal{I})$ and R sampled according to any distribution $\mathcal{D}$ PI-consistent with some $\mathbf{x} \in b\mathcal{P}_M$, there exists some edge $e_0 \in E$ such that $\Pr[e_0 \in \mathrm{span}(R \setminus \{e_0\}) \mid e_0 \in R] \leq 2b$. Here we give an example showing that the $2b$ here is asymptotically tight. Consider the complete graph K_n where n is odd. Our construction of the distribution $\mathcal{D}$ is quite simple: with probability $\frac{n(n-1)}{(n+3)(n-2)}$, uniformly sample a cycle of length $\frac{n+3}{2}$ from K_n. Otherwise take no edge at all.

First, we want to verify that this is indeed a pairwise independent distribution. Since all edges are equivalent, we have

$$\forall e \in E, \Pr[e \in R] = \frac{n(n-1)}{(n+3)(n-2)} \cdot \frac{(n+3)/2}{n(n-1)/2} = \frac{1}{n-2}$$

For pairs of edges e_1 and e_2, we have to consider two cases.

1. e_1 and e_2 are "adjacent", i.e. they share one common vertex. Notice that a cycle of length $\frac{n+3}{2}$ contains $\frac{n+3}{2}$ "adjacent edge pairs". By taking expectation, we can see that

$$\sum_{e_i, e_j \text{ adjacent}} \Pr[e_i \in R \wedge e_j \in R] = \mathbb{E}[\# \text{ adjacent edge pairs in } R]$$
$$= \frac{n(n-1)}{(n+3)(n-2)} \cdot \frac{n+3}{2}$$

The number of adjacent edge pairs in K_n is $\binom{n}{3} \cdot 3$, and by symmetry, they all have same probability of appearing in R. Therefore we have

$$\Pr[e_1 \in R \wedge e_2 \in R] = \frac{n(n-1)}{2(n-2)} \cdot \frac{1}{3\binom{n}{3}} = \frac{1}{(n-2)^2}$$

2. e_1 and e_2 are "not adjacent", i.e., they share no common vertices. A cycle of length $\frac{n+3}{2}$ contains $\binom{(n+3)/2}{2} - \frac{n+3}{2} = \frac{(n+3)(n-3)}{8}$ "non-adjacent edge pairs". By taking expectation, we have

$$\sum_{e_i, e_j \text{ not adjacent}} \Pr[e_i \in R \wedge e_j \in R] = \mathbb{E}[\# \text{ non-adjacent edge pairs in } R]$$
$$= \frac{n(n-1)}{(n+3)(n-2)} \cdot \frac{(n+3)(n-3)}{8}$$

Similarly, the number of non-adjacent edge pairs in K_n is $\binom{n}{4} \cdot 3$, and by symmetry we have

$$\Pr[e_1 \in R \wedge e_2 \in R] = \frac{n(n-1)(n-3)}{8(n-2)} \cdot \frac{1}{3\binom{n}{4}} = \frac{1}{(n-2)^2}$$

So for both cases, the distribution is indeed pairwise independent. (Here we may as well calculate the probability of edge pairs directly, but this approach is easier.) If we set $x_i = \frac{2}{n}$ for each edge i, then $\mathbf{x}$ is in the matroid polytope, and we have $b = \frac{1}{n-2} / \frac{2}{n} = \frac{n}{2(n-2)}$. Notice that

$$\forall e \in E, \Pr[e \in \text{span}(R \setminus e) \mid e \in R] = 1 = \frac{2(n-2)}{n} b$$

because whenever $e \in R$, it is contained in a cycle. Therefore $2b$ is asymptotically tight.

F Transversal Matroids

In this section we show a $(b, 1-b)$-selectable PI-OCRS for transversal matroid. Recall that a transversal matroid matroids is represented by a bipartite graph

$G = (U, V, E)$, with vertex sets U, V and edge set $E \subseteq U \times V$. Let $M = (U, \mathcal{I})$ denote the transversal matroid, and let $\mathcal{D}$ be a distribution over U that is PI-consistent with some $\mathbf{x} \in b\mathcal{P}_M$. The OCRS works as follows: first, for each edge $(i, j) \in E$, we assign some probability $y_{i,j} \in [0, 1]$ to it, such that

$$\sum_{j \in \mathcal{N}(i)} y_{i,j} = 1 \qquad \forall i \tag{F.1}$$

$$\sum_{i \in \mathcal{N}(j)} x_i \, y_{i,j} \leq b \qquad \forall j, \tag{F.2}$$

These $y_{i,j}$ exist (and can be efficiently computed) by the following lemma:

Lemma 7. *Let $G = (U, V, E)$ denote a bipartite graph, and M the transversal matroid defined on U. Then for any $x \in b\mathcal{P}_M$, there exist corresponding $y_{i,j} \in [0, 1]$ satisfying (F.1) and (F.2), and such $y_{i,j}$ can be computed efficiently.*

For each left vertex $i \in U$, let $Y_i \in \mathcal{N}(i)$ be a random variable such that $\Pr[Y_i = j] = y_{i,j}$. Here $\{Y_i\}_i$ are mutually independent, and are independent from the events $\{i \in R\}_i$. Then the feasible sets $\mathcal{F}$ is defined as

$$\mathcal{F} = \{S \subseteq U : Y_i \neq Y_j \quad \forall i, j \in S, i \neq j\}$$

It's not hard to see that $\mathcal{F} \in \mathcal{I}$, because for any $S \in \mathcal{F}$, every $i \in S$ is matched to V by Y_i. It remains to bound the selectability of $\mathcal{F}$. For any $i \in U$, we have

$$\begin{aligned}
&\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \\
&= \sum_{j \in \mathcal{N}(i)} \Pr[Y_i = j] \Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R, Y_i = j] \\
&= \sum_{j \in \mathcal{N}(i)} y_{i,j} \Pr[Y_{i'} \neq j \quad \forall i' \in R \setminus \{i\} \mid i \in R] \\
&\geq \sum_{j \in \mathcal{N}(i)} y_{i,j} \left(1 - \sum_{i' \in \mathcal{N}(j) \setminus \{i\}} \Pr[Y_{i'} = j \wedge i' \in R \mid i \in R]\right) && \text{(union bound)} \\
&= \sum_{j \in \mathcal{N}(i)} y_{i,j} \left(1 - \sum_{i' \in \mathcal{N}(j) \setminus \{i\}} x_{i'} y_{i',j}\right) && \text{(pair-wise independence of } i \in R\text{)} \\
&\geq \sum_{j \in \mathcal{N}(i)} y_{i,j} (1 - b) && \text{(inequality (F.2))} \\
&= 1 - b. && \text{(equation (F.1))}
\end{aligned}$$

Therefore we conclude this algorithm is indeed a $(b, 1 - b)$-selectable PI-OCRS, and by Claim 5, setting $b = 1/2$ gives a $1/4$-selectable PI-OCRS.

Proof of Lemma 7. We give an algorithmic proof. First, construct a network flow model G' based on G by adding a source vertex s and a sink vertex t. For every

$i \in U$ add an arc from s to i with capacity x_i, and for every $j \in V$ add an arc from j to t with capacity 1. Then for each edge $(i,j) \in E$, add the arc (i,j) to G' with infinite capacity. Then compute a maximum $s \to t$ flow.

We claim that the maximum flow is equal to $\sum_{i \in U} x_i$. To see this, we argue that $\{s\} \times U$ is a minimum cut of the flow graph. Otherwise, let C be a minimum cut where $C \neq \{s\} \times U$. Let $C_U = C \cap (\{s\} \times U)$ and $C_V = C \cap (V \times \{t\})$, and let $\overline{C}_U = (\{s\} \times U) \setminus C_U$. Here $\{j \in N(i) : (s,i) \in \overline{C}_U\} \times \{t\} \subseteq C_V$ must hold, since otherwise C must contain infinite-capacity arcs. Then by the matroid polytope constraint, $\sum_{(s,i) \in \overline{C}_U} x_i \leq \operatorname{rank}(\overline{C}_U) \leq |C_V|$, meaning that the capacity of $\{s\} \times U = (C \setminus C_V) \cup \overline{C}_U$ is no larger than C. Therefore $\{s\} \times U$ is a minimum cut. And by the max-flow min-cut theorem, the maximum flow is equal to $\sum_{i \in U} x_i$.

Then finally set $y_{i,j} = f_{i,j}/x_i$, where $f_{i,j}$ is the flow on edge (i,j) in the maximum flow. The fact that f satisfies the capacity constraints on $s \times U$ and $V \times t$ implies (F.1) and (F.2), respectively. □

G Cographic Matroids

In this section we give a PI-OCRS for cographic matroids.

Theorem 9 (Cographic Matroids PI-OCRS). *There is a $1/12$-selectable PI-OCRS for cographic matroids.*

The idea is similar to [37]. We first transform the cographic matroid into a "low density" matroid by removing the parallel items, then combine the OCRS for low density matroids with a single-item OCRS to get an OCRS for cographic matroids. We first define the density of a matroid.

Definition 9. *Let $M = (E, \mathcal{I})$ be a loopless matroid (i.e. all circuits are of size ≥ 2). Then the* density *of $\mathcal{M}$ is defined as*

$$\gamma(M) := \max_{S \subseteq E, S \neq \varnothing} \frac{|S|}{\operatorname{rank}(S)}.$$

Just like the matroid secretary problem [37], we have a good PI-OCRS for low-density matroids.

Lemma 8. *There is a $1/\gamma(M)$-selectable PI-OCRS for matroids with density $\gamma(M)$.*

Proof. Let $\mathbf{y} \in \mathbb{R}^E$ be a vector with all its coordinates equal to $\frac{1}{\gamma(M)}$. Then for any $S \subset E$, by the definition of density, $\mathbf{y}(E) \leq \operatorname{rank}(E)$ holds. Therefore $\mathbf{y} \in P_M$, and hence we can write $\mathbf{y} = \sum_{I \in \mathcal{I}} p_I \chi_I$. The PI-OCRS π works as follows. Before items start arriving, π samples an independent set I_0 according to distribution $\{p_I\}_I$, and set $\mathcal{F} = 2^{I_0}$ for the greedy PI-OCRS. By definition $\mathcal{F} \subseteq \mathcal{I}$. For any item $i \in E$, we have

$$
\begin{aligned}
&\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \\
&= \Pr[I \cup \{i\} \subseteq I_0 \quad \forall I \in R \cap I_0 \mid i \in R] \\
&= \Pr[i \in I_0] \\
&= \sum_{I \ni i} p_I \\
&= \frac{1}{\gamma(M)}.
\end{aligned}
$$

Therefore this PI-OCRS is $1/\gamma(M)$-selectable. □

Let M be a loopless cographic matroid associated with graph G. M is not necessarily low-density: for example, if G is a cycle of length n, the density of M is n. Therefore to give a PI-OCRS for cographic matroid, first we show that if each vertex of G has degree at least 3, the density M is at most 3; Then we show how to reduce arbitrary graphs to a graph where degree of each vertex is at least 3.

Lemma 9. *Let $G = (V, E)$ be a graph such that $deg(v) \geq 3$ for all $v \in V$. Then the cographic matroid M associated with G has density $\gamma(M) \leq 3$.*

Proof. Since each vertex has degree at least 3, we have $|E| \geq \frac{3}{2}|V|$. Also since M^* (the duality of M) is graphic matroid on G, we have $\text{rank}(M^*) \leq |V| - 1$, and thus $\text{rank}(M) = |E| - \text{rank}(M^*) \geq |E| - |V| + 1$. Then we have

$$\frac{|E|}{\text{rank}(M)} \leq \frac{|E|}{|E| - |V| + 1} \leq \frac{|E|}{\frac{1}{3}|E|} = 3$$

Now for any $E' \subset E$, we want to give an upperbound for $\frac{|E'|}{\text{rank}(E')}$. Let $M' := M|_{E'}$, then the density of E' is equal to $\frac{|E'|}{\text{rank}(M')}$. Since deletion in cographic matroid is equivalent to contraction in graphic matroid, M' is associated with graph G' obtained by contracting $E \setminus E'$ in G. It's not hard to see that each vertex in G' also has degree at least 3. Therefore the above argument about density also holds for M', i.e. $\frac{|E'|}{\text{rank}(M')} \leq 3$. Therefore $\gamma(M) \leq 3$. □

Proof of Theorem 9. By Lemma 8 and Lemma 9, we already have a $1/3$-selectable PI-OCRS for cographic matroid, where each vertex of the underlying graph has degree at least 3. We now consider the case of general graphs. Given a cographic matroid $M = (E, \mathcal{I})$, we say two elements $a, b \in E$ are parallel if $a = b$ or $\{a, b\}$ is a circuit of size 2. We say two sets $A, B \subseteq E$ are parallel if there exists a bijection $\phi : A \to B$ such that for any $a \in A$, a is parallel to $\phi(a)$. Then one can verify that parallelism defines an equivalence relation over E, and thus E can be partitioned into disjoint union of equivalent classes: $E = \bigsqcup_{j=1}^{m} E_j$. For each equivalent class E_j, we select a representative e_j, and define $E' := \{e_j : j \in [m]\}$. We then define $M' = M|_{E'}$, then M' is also a cographic matroid, let $G' = (V', E')$ denote the underlying graph for M'. Assuming no isolated vertex in G', since M' contains no parallel elements, the degree of each vertex in G' is at least 3. Therefore by Lemma 8 and Lemma 9, we have a 1/3-selectable PI-OCRS for M'. Let $I_0 \subseteq E'$ denote the independent set selected in *Lemma* 8. Now for any $\mathbf{x} \in b\mathcal{P}_M$ and distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$, we set $\mathcal{F} = \{I \subseteq E : I \text{ is parallel to some } I' \subseteq I_0\}$. Then for any $e \in E_j$, we have

$$\begin{aligned}
&\Pr[I \cup \{e\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid e \in R] \\
&= \Pr[E_j \cap R = \{e\} \mid e \in R] \Pr[e_j \in I_0] \\
&\geq \frac{1}{3} \Pr\left[\bigwedge_{e' \in E_j \setminus \{e\}} e' \notin R \mid e \in R\right] && \text{(Lemma 8)} \\
&\geq \frac{1}{3}\left(1 - \sum_{e' \in E_j \setminus \{e\}} \Pr[e' \in R]\right) && \text{(union bound and pair-wise independence)} \\
&\geq \frac{1}{3}(1-b) && (\mathbf{x}(E_j) \leq b \operatorname{rank}(E_j) = b)
\end{aligned}$$

Therefore we have a $(b, \frac{1}{3}(1-b))$-selectable PI-OCRS. By Claim 5, setting $b = 1/2$ gives a $1/12$-selectable PI-OCRS for cographic matroid. □

H Details for Regular Matroids

First, we need to argue that for every M in the conflict graph, there exists a suitable PI-OCRS for $\widehat{M} := (M/A_M)|_{(E(M) \cap E(\widetilde{M}))}$. For a matroid M, let $S(M)$ denote the set of matroids that can be obtained from M by iteratively removing elements, adding parallel elements, and contracting either nothing, a single element, or a circuit of size 3.

Lemma 10. *If matroid M is either graphic, cographic, or R_{10}, then every matroid $M' \in S(M)$ admits a $1/12$-selectable PI-OCRS.*

Proof. The class of graphic (resp. cographic) matroids are closed under the operations of deletion, contraction, and addition of parallel elements. Hence if M is graphic or cographic, so are all matroids in $S(M)$.

Otherwise, $M = R_{10}$. It is known (see the proof of [13, Corollary 4.7]) that for any matroid $M' \in S(R_{10})$, either M' is graphic, cographic, or has a ground set can be partitioned into 10 parts such that deleting any one of the 10 parts makes M graphic.

If $M' \in S(M)$ is graphic or cographic, Theorems 6 and 9 imply a $1/12$-selectable OCRS. Else, deleting one of the 10 parts of M' uniformly at random and then running a graphic matroid OCRS yields a $(b, \frac{9}{10} \cdot (1-2b))$-selectable PI-OCRS, which can be tuned to give a $9/80$-selectable PI-OCRS. In conclusion, each matroid $M' \in S(M)$ admits a $\min(1/12, 9/80) = 1/12$-selectable PI-OCRS. $\square$

Finally, we have all we need to prove the main theorem of this section.

Theorem 8 (Regular Matroids). *There is a $(1/3, 1/12)$-selectable PI-OCRS for regular matroids.*

Proof. The input is a regular matroid $\widetilde{M}$ and any distribution $\mathcal{D}$ over $E(M)$ PI-consistent with some vector $\mathbf{x} \in \frac{1}{3}\mathcal{P}_{\widetilde{M}}$.

Our greedy PI-OCRS for regular matroid $\widetilde{M}$ proceeds as follows. First, compute a good $\{1,2,3\}$-decomposition T of $\widetilde{M}$ with basic matroid set $\mathcal{M}$, using Theorem 7. For each of the basic matroids $M \in \mathcal{M}$, recall that $\widehat{M} := (M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$. By Lemma 3, $\mathbf{x}|_{\widehat{M}} \in \mathcal{P}_{\widehat{M}}$. Since M is graphic/cographic/R_{10} and $\widehat{M} \in S(M)$, by Lemma 10 there is a $1/12$-selectable PI-OCRS for each $\widehat{M}$, and we denote the feasible sets of such greedy PI-OCRS as $\mathcal{F}_{\widehat{M}}$. Then we simply set $\mathcal{F}_{\widetilde{M}} = \{I \subseteq \widetilde{M} : I \cap E(\widehat{M}) \in \mathcal{F}_{\widehat{M}}, \quad \forall M \in \mathcal{M}\}$. By Lemma 2, $\mathcal{F}_{\widetilde{M}} \subseteq \mathcal{I}(\widetilde{M})$, i.e. $\mathcal{F}_{\widetilde{M}}$ is indeed feasible. Then we bound the selectability of the greedy PI-OCRS defined by $\mathcal{F}_{\widetilde{M}}$: for any item i in $\widehat{M}$, we have

$$\begin{aligned}
&\Pr[I \cup \{i\} \in \mathcal{F}_{\widetilde{M}} \quad \forall I \in \mathcal{F}_{\widetilde{M}}, I \subseteq R \mid i \in R] \\
&= \Pr[I \cup \{i\} \in \mathcal{F}_{\widehat{M}} \quad \forall I \in \mathcal{F}_{\widehat{M}}, I \subseteq R \cap \widehat{M} \mid i \in R \cap \widehat{M}] \\
&\geq \frac{1}{12}. && \text{(Lemma 3)}
\end{aligned}$$

We conclude that $\mathcal{F}_{\widetilde{M}}$ does indeed give a $(1/3, 1/12)$-selectable PI-OCRS for regular matroids. $\square$