Comprehensive List of Quant Questions

Increasing Uniform Chain II

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, Five Rings, HRT

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Let N be the first index n where $X_n \neq \max\{X_1, \dots, X_n\}$. Find $\mathbb{E}[N]$. The answer will be in the form $a + be$ for integers a and b. Note here that e is Euler's constant. Find $a + b$.

Hint:

Use the tail-sum formula for non-negative integer-valued random variables $\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}[N > n]$. What does the event $\{N > n\}$ mean in terms of the X_i random variables?

Explanation:

We can use the tail-sum formula for non-negative integer-valued random variables $\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}[N > n]$. Clearly $\mathbb{P}[N > 0] = \mathbb{P}[N > 1] = 1$, as we need at least 2 random variables to compare to get one that is not the max. Suppose that $n \geq 2$. Then the event $\{N > n\}$ means that $X_1 < X_2 < \cdots < X_n$, which occurs with probability $\frac{1}{n!}$, as this is one specific ordering of the indices among $n!$ possible orderings. Therefore, $\mathbb{P}[N > n] = \frac{1}{n!}$. In particular, this holds for $n = 0, 1$ as well, which is ideal. Summing up,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

The answer is therefore $0 + 1 = 1$.

Answer:

1

Reference:

https://www.quantguide.io/questions/increasing-uniform-chain-ii

Ten Consecutive Heads

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Two Sigma, Jane Street

Question:

On average, how many times must a fair coin be flipped to obtain 10 consecutive heads?

Hint:

Let f_k be the expected number flips of the coin needed to obtain 10 heads when you have $0 \leq k \leq 10$ heads consecutively obtained already. Use Law of Total Expectation to derive a recurrence for f_k and notice a pattern.

Explanation:

We are going to solve this more generally for n consecutive heads. Let f_k be the expected number flips of the coin needed to obtain n heads when you have $0 \leq k \leq n$ heads consecutively obtained already. Our boundary condition is that $f_n = 0$, as you already have all n heads obtained. We can see that $f_{n-1} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + f_0) = 1 + \frac{1}{2} f_0$, as with probability 1/2, you reach n heads, and with probability 1/2, you start over. Similarly, we have that

$$f_{n-2} = 1 + \frac{1}{2} f_{n-1} + \frac{1}{2} f_0 = 1 + \frac{1}{2}\left(1 + \frac{1}{2} f_0\right) + \frac{1}{2} f_0 = \frac{3}{2} + \frac{3}{4} f_0$$

By doing this again, we obtain

$$f_{n-3} = 1 + \frac{1}{2} f_{n-2} + \frac{1}{2} f_0 = 1 + \frac{1}{2}\left(\frac{3}{2} + \frac{3}{4} f_0\right) + \frac{1}{2} f_0 = \frac{7}{4} + \frac{7}{8} f_0$$

A pattern now arises. By recursively applying this process, we can see that

$$f_{n-k} = \left(2 - \frac{1}{2^{k-1}}\right) + \left(1 - \frac{1}{2^k}\right) f_0$$

By plugging in $k = n$, we obtain

$$f_0 = f_0 \left(1 - \frac{1}{2^n}\right) + 2 - \frac{1}{2^{n-1}} \iff \frac{1}{2^n} f_0 = 2 - \frac{1}{2^{n-1}} \iff f_0 = 2^{n+1} - 2$$

In particular, for $n = 10$, our answer is $2^{11} - 2 = 2046$.

Answer:

2046

Reference:

https://www.quantguide.io/questions/ten-consecutive-heads

Correlation Ranges

Category: Probability
Difficulty: Medium
Companies: Citadel, Jane Street

Question:

Suppose that X, Y, and Z are three random variables. We know that $\text{Corr}(X, Y) = \frac{5}{13}$ and $\text{Corr}(Y, Z) = \frac{12}{13}$. The range of possible values for $\text{Corr}(X, Z)$ is an interval in the form $[0, b]$, where b is a fraction in fully reduced form. Find b.

Hint:

The key idea here is that the correlation matrix for any collection of random variables must be positive semi-definite. A condition to show that a matrix is positive definite is that each of the sub-matrices of order $1 \times 1, 2 \times 2, \ldots, n \times n$, where n is a size of the matrix, originating from the top left corner have positive determinant. The case where you have determinant of 0 can be individually analyzed.

Explanation:

The key idea here is that the correlation matrix for any collection of random variables must be positive semi-definite. Denoting $\text{Corr}(X, Z) = \rho$, the correlation matrix for (X, Y, Z), say C, is a 3×3 matrix

$$\begin{bmatrix} 1 & \frac{5}{13} & \rho \\ \frac{5}{13} & 1 & \frac{12}{13} \\ \rho & \frac{12}{13} & 1 \end{bmatrix}$$

A condition to show that a matrix is positive definite is that each of the sub-matrices of order $1 \times 1, 2 \times 2, \ldots, n \times n$, where n is a size of the matrix, originating from the top left corner have non-negative determinant. In this case, it is easy to see that the top left 2×2 matrix has determinant $C_{11}C_{22} - C_{12}C_{21} = \frac{144}{169} > 0$, so we only need to check that the entire matrix C has non-negative determinant.

Evaluating this determinant as a function of ρ, we obtain that it is $\frac{120}{169}\rho - \rho^2$. Setting this equal to 0, we obtain that $\rho = 0, \frac{120}{169}$. As this polynomial had a negative leading coefficient for ρ^2, it follows that the parabola (i.e. determinant) must have been positive between the two roots, so $\rho^* = \frac{120}{169}$ is the largest value where this determinant is non-negative.

Answer:

120169

Reference:

https://www.quantguide.io/questions/correlation-ranges

First Ace

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, Jane Street, HRT, Optiver

Question:

On average, how many cards in a normal deck of 52 playing cards do you need to flip over to observe your first ace?

Hint:

How can you use the linearity of expectation to distribute expectation across a sum of random variables, regardless of independence?

Explanation:

The four aces are dispersed throughout the deck and cut the 48 remaining cards into 5 distinct sections, each of some random length $X_i \in [0, 48]$ where $1 \leq i \leq 5$; that is, $\sum_{i=1}^{5} X_i = 48$. Furthermore, by symmetry, $E[X_1] = E[X_2] = \cdots = E[X_5]$, as in the absence of any additional information, none of the sections is expected to be any larger or smaller than any other. Thus, by linearity of expectation:

$$E\left[\sum_{i=1}^{5} X_i\right] = 5 \times E[X_1] = 48 \Rightarrow E[X_1] = \frac{48}{5}$$

We have found that the expected number of cards in the first section is $\frac{48}{5}$, so we will observe the first ace on the next card. Hence, we expect to flip $\frac{48}{5} + 1 = 10.6$ cards to observe the first ace.

Answer:

10.6

Reference:

https://www.quantguide.io/questions/first-ace

Replacement Orbs

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG

Question:

An urn containing 2 red and 1 blue orb is in front of you. At each step, you will take out an orb uniformly at random and replace it with a blue orb, regardless of the color selected. Find the expected amount of draws needed until the urn only contains blue orbs.

Hint:

Let X_1 represent the number of draws needed to go from 1 blue to 2 blue orbs in the box. Similarly, let X_2 be the amount of draws needed to go from 2 blue to 3 blue orbs in the box. Then $T = X_1 + X_2$ gives us the total amount of draws needed to go from our current state to 3 blue orbs.

Explanation:

Let X_1 represent the number of draws needed to go from 1 blue to 2 blue orbs in the box. Similarly, let X_2 be the amount of draws needed to go from 2 blue to 3 blue orbs in the box. Then $T = X_1 + X_2$ gives us the total amount of draws needed to go from our current state to 3 blue orbs. By linearity of expectation, $\mathbb{E}[T] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$. Note that $X_1 \sim \text{Geom}(2/3)$, as there is a $\frac{2}{3}$ probability on each draw that the orb will be red (and hence replaced by a blue after). Similarly, $X_2 \sim \text{Geom}(1/3)$. The means of these two are $\frac{3}{2}$ and 3, respectively, so the answer is $\mathbb{E}[T] = \frac{3}{2} + 3 = \frac{9}{2}$.

Answer:

92

Reference:

https://www.quantguide.io/questions/replacement-orbs

Cylindrical Intersection

Category: Pure Math
Difficulty: Medium
Companies: Citadel

Question:

Two cylinders of radius 1 intersect at right angles to one another. Furthermore, their central axes also intersect. Find the volume of the enclosed region.

Hint:

We can model this as two cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. Suppose that we fix some y. What is the point set (x, z) of interest? What is the shape of it?

Explanation:

We can model this as two cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. Suppose that we fix some y. We then want to find the set of points (x, z), such that $x^2 \leq 1 - y^2$ and $z^2 \leq 1 - y^2$. Namely, this just because $-\sqrt{1-y^2} \leq x, z \leq \sqrt{1-y^2}$. This point set is just a square of side length $2\sqrt{1-y^2}$, which has area $4(1-y^2)$. Then, we can just integrate the cross sections from $y = -1$ to 1, as those are the bounds of y in our intersected region. Therefore, the answer is

$$\int_{-1}^{1} 4(1-y^2)dy = \frac{16}{3}$$

Answer:

163

Reference:

https://www.quantguide.io/questions/cylindrical-intersection

Comparing Flips II

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, SIG, Jane Street

Question:

You and your friend are playing a game with a fair coin, tossing it and writing down the outcomes. You win if HTH appears before HHT, else your friend wins. What is the probability that your friend wins?

Hint:

Both sequences contain the subsequence HT, so imagine writing down an infinite number of toss outcomes and scanning for HT subsequences. When will HTH occur before HHT?

Explanation:

Both sequences contain the subsequence HT, so imagine writing down an infinite number of toss outcomes and scanning for HT subsequences. In order for HTH to occur first, an H must not precede HT (else HHT occurs) and an H must succeed it (to create HTH). Hence, the probability HTH wins is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ given there is an HT. In order for HHT to occur first, an H must precede HT, which happens with probability $\frac{1}{2}$ given there is an HT. Note that these are the only two outcomes that can occur and thus define our sample space $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. The probability that your friend wins is $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$ and the probability that you win is $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$. In other words, HHT is twice as likely to occur before HTH.

Answer:

23

Reference:

https://www.quantguide.io/questions/comparing-flips-ii

Basic Die Game VI

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, SIG, Drw, Jane Street, HRT, Optiver

Question:

Alice rolls a fair 6−sided die with the values $1-6$ on the sides. She sees that value showing up and then is allowed to decide whether or not she wants to roll again. Each re-roll costs $1. Whenever she decides to stop, Alice receives a payout equal to the upface of the last die she rolled. Note that there is no limit on how many times Alice can re-roll. Assuming optimal play by Alice, what is her expected payout on this game?

Hint:

Let v_k be the value of the game if you accept a roll of k or more and re-roll otherwise. Our optimal strategy will be in this form because we can view each round of the game as an independent trial. Therefore, our strategy should be the same between trials.

Explanation:

Let v_k be the value of the game if you accept a roll of k or more and re-roll otherwise. Our optimal strategy will be in this form because we can view each round of the game as an independent trial. Therefore, our strategy should be the same between trials. Then

$$
\begin{aligned}
v_1 &= (1/6)(1+2+3+4+5+6) && \Rightarrow \quad v_1 = 7/2 \\
v_2 &= (1/6)\,(v_2-1) + (1/6)(2+3+4+5+6) && \Rightarrow \quad v_2 = 19/5 \\
v_3 &= (2/6)\,(v_3-1) + (1/6)(3+4+5+6) && \Rightarrow \quad v_3 = 4 \\
v_4 &= (3/6)\,(v_4-1) + (1/6)(4+5+6) && \Rightarrow \quad v_4 = 4 \\
v_5 &= (4/6)\,(v_5-1) + (1/6)(5+6) && \Rightarrow \quad v_5 = 7/2 \\
v_6 &= (5/6)\,(v_6-1) + (1/6)(6) && \Rightarrow \quad v_6 = 1
\end{aligned}
$$

All of these are derived from the fact that if we obtain a value at least our minimum stopping value, we just stop immediately. Otherwise, we just go again and it is the same game but we lose 1 in payout from the cost of the roll. Therefore, this means that our optimal EV is 4.

Answer:

4

Reference:

https://www.quantguide.io/questions/basic-die-game-vi

Spacious Uniform Values II

Category: Probability
Difficulty: Hard
Companies: Citadel

Question:

You sample 101 uniformly random numbers in the interval $(0,1)$. Find the expected length of the shortest distance between any two selected points.

Hint:

Recall the identity for non-negative real-valued continuous random variables that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x]dx$. Apply this to the minimum distance random variable.

Explanation:

In Spacious Uniform Values I, we found that the probability that no two points are within a distance x of one another is $(1-100x)^{101}$. In other words, the probability that the minimum distance between two values is at least x is that. Now, recall the identity for non-negative real-valued continuous random variables that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x]dx$. We apply this to M, the minimum spacing between the points. We have that $\mathbb{E}[M] = \int_0^\infty \mathbb{P}[M \geq x]dx$. The maximum value of the minimum possible spacing between the points is $\frac{1}{100}$. If no point was within a distance of something larger than $\frac{1}{100}$, the total length of the interval formed would be larger than 1, contradicting that these are only supported on $(0,1)$. Therefore, our integral upper bound is $\frac{1}{100}$. This yields

$$\mathbb{E}[M] = \int_0^{\frac{1}{100}} (1-100x)^{101}\, dx$$

Let $u = 1-100x$ so that $du = -100dx$. It is simple to verify the bounds of our integral are now 0 and 1. This yields the new integral $\frac{1}{100}\int_0^1 u^{101}du$, which is evaluated to be $\frac{1}{10200}$.

Answer:

110200

Reference:

https://www.quantguide.io/questions/spacious-uniform-values-ii

2 Below I

Category: Probability
Difficulty: Medium
Companies: Citadel, Jane Street

Question:

You and friend play a game where you both select an integer $1-100$. The winner receives \$1 from the loser. The winner is the player who selects the strictly higher number. If there is a tie, then nothing happens. However, a player can also win by selecting a value exactly 2 below the larger integer. For example, if you select 80 and your friend selects 82, you are the winner in this case. Assume both you and your friend play optimally. The optimal strategy here is a mixed strategy, where you select a random value X from some appropriately determined distribution. Find $\text{Var}(X)$.

Hint:

Suppose you were planning to select a value n. Note that $n+3$ is strictly better than n, as $n+3$ will beat all integers n would beat, as well as the integer $n+2$. Therefore, whenever you can select n, you should select $n+3$.

Explanation:

Suppose you were planning to select a value n. Note that $n+3$ is strictly better than n, as $n+3$ will beat all integers n would beat, as well as the integer $n+2$. Therefore, whenever you can select n, you should select $n+3$. This means that your strategy should be to select $98, 99$, or 100 with some probabilities.

By the symmetry of the game, your friend should also select those values with the same probabilities. In particular, if you select $98-100$ with equal probability, no matter what probabilities your friend selects $98, 99$, and 100 with, the expected payout for each player is 0 by symmetry. This is because we see that 98 is beat by 99, 99 is beat by 100, and 100 is beat by 98, so each of the three outcomes is dominated by one other outcome.

Furthermore, if you select a non-uniform distribution on $98, 99, 100$ for your values, there exists a strategy your friend can select that yields positive expected payout for them. The probabilities of selecting each of the values for you would be p_1, p_2, and $1-p_1-p_2$. For a numerical demonstration, say $p_1 = 1/5$ and $p_2 = 1/2$. Then your friend should always select the value that maximizes their probability of winning. In this case, they should select 100, as the expected payout would be

$$(-1) \cdot \frac{1}{5} + (1) \cdot \frac{1}{2} + (0) \cdot \frac{3}{10} > 0$$

Namely, if x is the value assigned to probability $\max\{p_1, p_2, 1-p_1-p_2\}$, then your friend should always select the value that beats x. Therefore, to eliminate this opportunity, you should select a uniform distribution among $98-100$.

This means $X \sim \text{DiscreteUnif}(\{98, 99, 100\})$, so $\mathbb{E}[X] = 99$ and $\text{Var}(X) = \dfrac{(98-99)^2 + (99-99)^2 + (100-99)^2}{3} = \dfrac{2}{3}$.

Answer:

23

Reference:

https://www.quantguide.io/questions/2-below-i

Machine Variance

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

A pizza shop consists of 25 independent workers that make pizzas. The standard deviation of the number of pizzas produced by each worker in a day is 60. Find the standard deviation of the total number of pizzas produced by the shop per day.

Hint:

Let $X_1, \ldots, X_{25}$ be the number of pizzas produced by each of the workers per day. Then $T = X_1 + \cdots + X_{25}$ gives the total number of pizzas produced per day by the shop.

Explanation:

Let $X_1, \ldots, X_{25}$ be the number of pizzas produced by each of the workers per day. Then $T = X_1 + \cdots + X_{25}$ gives the total number of pizzas produced per day by the shop. We have that $\text{Var}(T) = 25 \cdot \text{Var}(X_1)$ because of the fact that each of the people are independent and have the same variance. Thus, $\sigma_T = \sqrt{\text{Var}(T)} = 5 \cdot \sigma_{X_1} = 5 \cdot 60 = 300$.

Answer:

300

Reference:

https://www.quantguide.io/questions/machine-variance

Throwing Darts I

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG

Question:

A dart lands uniformly randomly on a dartboard composed of three concentric circles with radii of 1ft, 2ft, and 3ft. Assuming the dart lands on the dartboard, what is the probability that it lands in the central ring, between the 1ft and 2ft circles?

Hint:

The probability that the dart lands in the central ring is the ratio of the area of the central ring to the area of the entire board, since the dart lands uniformly at random on the board.

Explanation:

The probability that the dart lands in the central ring is the ratio of the area of the central ring to the area of the entire board, since the dart lands uniformly at random on the board. The area of the entire board is 9π and the area of the central ring is $4\pi - 1\pi$. Hence, the probability that the dart lands in the central ring is:

$$\frac{4\pi - \pi}{9\pi} = \frac{1}{3}$$

Answer:

13

Reference:

https://www.quantguide.io/questions/throwing-darts-i

10 Die Sum

Category: Probability
Difficulty: Medium
Companies: Citadel, Jane Street, Old Mission

Question:

Find the probability of obtaining a sum of 10 when rolling three fair 6−sided dice.

Hint:

The possibilities of values (not ordered in any way currently) that result in a sum of 10 are

$$(1, 6, 3), (1, 5, 4), (2, 6, 2), (2, 5, 3), (2, 4, 4), (3, 4, 3)$$

Explanation:

The most efficient approach here would be generating functions. However, we are going to solve via casework instead. The possibilities of values (not ordered in any way currently) that result in a sum of 10 are

$$(1, 6, 3), (1, 5, 4), (2, 6, 2), (2, 5, 3), (2, 4, 4), (3, 4, 3)$$

Three of this combinations have all distinct values. Namely, these are $(1, 6, 3)$, $(1, 5, 4)$, and $(2, 5, 3)$. For these sets of outcomes, there are $3! = 6$ ways they can be assigned to the three dice. These yields $3 \cdot 6 = 18$ possibilities.

The other three outcomes have two distinct values. For these, there are only 3 ways they can each be assigned to the dice, as you only select the die that shows the value that differs from the other two. Then, the other two dice are fixed. These yields $3 \cdot 3 = 9$ possibilities.

Combining all of this, this yields $18 + 9 = 27$ possibilities that result in a sum of 10. We divide by $6^3 = 216$ to get the probability, yielding a final answer of $\frac{1}{8}$.

Answer:

18

Reference:

https://www.quantguide.io/questions/10-die-sum

Modified Even Coins

Category: Probability
Difficulty: Easy
Companies: Citadel, Akuna, SIG, Jane Street

Question:

n coins are laid out in front of you. One of the coins is fair, while the other $n-1$ have probability $0 < \lambda < 1$ of showing heads. If all n coins are flipped, find the probability of an even amount of heads.

Hint:

Try to condition on the outcome of the fair coin. Note that in order to observe an even number of heads, either the fair coin shows heads and the other $n-1$ coins show an odd number of heads, or the fair coin shows tails and the other $n-1$ coins show an even number of heads.

Explanation:

To obtain an even amount of heads, we have two options. If the fair coin shows H, then we must show an odd amount of heads on the other $n-1$ coins to obtain an even total. Alternatively, if the fair coin shows T, then we must show an even amount of heads on the other $n-1$ coins to obtain. If p is the probability of an even amount of heads on the other $n-1$ non-fair coins, then by Law of Total Probability, our probability of interest is $\frac{1}{2} \cdot p + \frac{1}{2} \cdot (1-p) = \frac{1}{2}$. This is because the complement of an even amount of heads is an odd amount, and these probabilities are weighted equally in the computation, so they cancel out.

Answer:

12

Reference:

https://www.quantguide.io/questions/modified-even-coins

Spacious Uniform Values I

Category: Probability
Difficulty: Hard
Companies: Citadel

Question:

You sample 101 uniformly random numbers in the interval $(0, 1)$. Find the probability that no two of the values selected are within a distance of $\frac{1}{1000}$ of one another. The answer should be in the fully reduced form $\left(\frac{a}{b}\right)^c$. Find $a + b + c$.

Hint:

Imagine having a deck of cards of size m and marking N of them. Find the probability that the cards are all at least a distance d apart. Consider this intuition in the limiting case.

Explanation:

Consider the following discrete scenario: Imagine having a deck of cards of size m and marking N of them. Find the probability that the marked cards are all at least a distance d apart. If you deal out the cards one-by-one from the top of the deck, each time you obtain a marked card, deal out d cards from the bottom of the deck and put them on top of your marked card you just dealt. All of the marked cards are at least a distance d apart precisely when none of the marked cards were in the bottom $(N - 1)d$ cards. If they were, then they would be within d distance of the previous marked card dealt out, as they would lie in the region that we put between two marked cards, which is supposed to have none if the marked cards are to truly be at least d cards apart.

Let $m \to \infty$ in the above example and the intuition for this problem holds. Instead of throwing out d cards, we are going to throw out an interval of length d at each point selected. Thus, we want to find the probability that none of the "marked values" (points) are within the interval of total length $(N - 1)d$ that was thrown away. This probability is $1 - (N - 1)d$ for each random variable. By independence and there being N such random variables (as we have N such points here), the probability is $(1 - (N - 1)d)^N$. In this case, $N = 101$ and $d = \frac{1}{1000}$, so this evaluates to $\left(\frac{9}{10}\right)^{101}$. By the form of our solution, we want $9 + 10 + 101 = 120$, which is the answer.

Answer:

120

Reference:

https://www.quantguide.io/questions/spacious-uniform-values-i

Better In Red V

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG

Question:

The surfaces of a $3 \times 3 \times 3$ cube (initially white) are painted red. The large cube is then cut up into 27 $1 \times 1 \times 1$ small cubes. One of the small cubes is selected uniformly at random and is rolled twice. The face appearing is red both times. What is the probability the cube selected was a corner cube?

Hint:

Given that we have new information entering this problem (both rolls of the selected mini cube yielded a red face shown), use Bayes Theorem to update the probability that the chosen cube was a corner.

Explanation:

Given that we have new information entering this problem (both rolls of the selected mini cube yielded a red face shown), we need to update the probability that the chosen cube was a corner. Thus, we have to use Bayes Theorem. Let C be the event of picking a corner cube and R_2 be the event that the chosen cube shows a red face twice when rolled twice. Then we have:

$$\mathbb{P}[C \mid R_2] = \frac{\mathbb{P}[R_2 \mid C] \cdot \mathbb{P}[C]}{\mathbb{P}[R_2]}$$

The probability we roll a red face on a corner cube is $\frac{3}{6}$. Thus the probability we roll a red face twice on a corner cube is $(\frac{3}{6})^2 = \frac{1}{4}$. Thus $\mathbb{P}[R_2 \mid C] = \frac{1}{4}$. Since there are 8 total corner cubes out of a total of 27, $\mathbb{P}[C] = \frac{8}{27}$.Finally, the probability we roll a red face twice is $\frac{1}{4} \cdot \frac{8}{27}$ from corner pieces, $(\frac{1}{3})^2 \cdot \frac{12}{27}$ from edge pieces (two red faces), and $(\frac{1}{6})^2 \cdot \frac{6}{27}$ from center side pieces (one red face). Putting this all together we get:

$$\mathbb{P}[C \mid R_2] = \frac{\mathbb{P}[R_2 \mid C] \cdot \mathbb{P}[C]}{\mathbb{P}[R_2]} = \frac{\frac{1}{4} \cdot \frac{8}{27}}{\frac{1}{4} \cdot \frac{8}{27} + \frac{1}{9} \cdot \frac{12}{27} + \frac{1}{36} \cdot \frac{6}{27}} = \frac{4}{7}$$

Answer:

47

Reference:

https://www.quantguide.io/questions/better-in-red-v

Extra Coin

Category: Probability
Difficulty: Easy
Companies: Citadel, De Shaw, Akuna, SIG, Jane Street, HRT

Question:

Alice has $n+1$ fair coins and Bob has n fair coins. What is the probability that Alice will flip more heads than Bob if both flip all of their coins?

Hint:

This can be solved with symmetry. How can you compare the number of heads in Alice's first n coins and Bob's n coins? What does this tell you about Alice's $n+1$th coin?

Explanation:

This can be solved with symmetry. Let us compare the number of heads in Alice's first n coins and Bob's n coins. There are three possible outcomes. Let E_1 be the event that Alice has strictly more heads than Bob; E_2 be the event that Alice and Bob have the same number of heads; E_3 be the event that Alice has fewer heads than Bob. By symmetry, $P(E_1) = P(E_3) = x$ and $P(E_2) = y$. Since $\sum_{\omega \in \Omega} P(\omega) = 1$, $2x + y = 1$. For E_1, Alice will always have more heads than Bob, regardless of Alice's final coin result. Furthermore, for E_3, Alice will never have more heads than Bob, regardless of Alice's final coin result. Thus, only event E_2 depends on Alice's $n+1$th coin flip, in which Alice will win half the time (with a result of heads). Thus, Alice's total probability of winning is: $x + 0.5y = x + (1 - 2x) = \frac{1}{2}$.

Answer:

0.5

Reference:

https://www.quantguide.io/questions/extra-coin

Threepeat Dice

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

You have two urns presented to you. One of them has the values $1 - 8$ written on 8 different pieces of paper. The other one only has the values 1 and 2 on 4 pieces of paper each. You select one urn uniformly at random and then select a piece of paper from it uniformly at random. You see that the paper selected has the value 2 on it. You then replace the paper in the urn you selected from. If you select from the same urn 40 more times with replacement between trials, find the expected number of times 2 would appear in these 40 draws.

Hint:

Condition on which urn is selected and adjust your probabilities of selecting each urn based on the first turn.

Explanation:

Since there are 4 pieces of paper labelled 2 in the second urn and only 1 piece labelled 2 in the first, the probability we selected the second urn is $\frac{4}{5}$. This is directly by Bayes' rule. Given we selected this urn, we would expect to see the value 2 appear $\frac{1}{2} \cdot 40 = 20$ times, as there is probability $\frac{1}{2}$ per selection it appears in this urn. If we selected the other urn, which occurs with probability $\frac{1}{5}$, then we would expect to see the value 2 to appear $\frac{1}{8} \cdot 40 = 5$ times. Therefore, by the Law of Total Expectation, the expected number of times we see the value 2 is

$$\frac{4}{5} \cdot 20 + \frac{1}{5} \cdot 5 = 17$$

Answer:

17

Reference:

https://www.quantguide.io/questions/threepeat-dice

Managerial Oversight

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

Two quant firms have client inflow that is well-modeled by independent Poisson processes. The respective intensity parameters are 6 and 10. Clients give outstanding reviews about the service with respective probabilities $\frac{1}{6}$ and $\frac{1}{5}$ for the two firms, independent of one another. The clients that give outstanding reviews are sent a thank you card. Find the expected time between clients that receive thank you cards.

Hint:

Use Poisson thinning to note that the inflow of clients sent thank you cards from either firm is also a Poisson process. What are the parameters for each? How can you relate that to the rate of clients being sent thank you cards?

Explanation:

We first use Poisson thinning to note that the inflow of clients sent a thank you card from each firm is also a Poisson process. For the two respective firms, the rates are $6 \cdot \frac{1}{6} = 1$ and $10 \cdot \frac{1}{5} = 2$. Therefore, the time between clients being sent thank you cards for each firm follow IID Exp(1) and Exp(2) distributions, respectively. The time between any clients being sent thank you cards is $\min\{X_1, X_2\}$, where $X_i \sim \text{Exp}(i)$, as whichever firm has a shorter interarrival time of clients being sent thank you cards will represent the time between clients (from either firm) being sent cards.

Then, we can note that for any $t > 0$, from the known facts about the exponential random variable,

$$\mathbb{P}[\min\{X_1, X_2\} > t] = \mathbb{P}[X_1 > t]\mathbb{P}[X_2 > t] = e^{-t} \cdot e^{-2t} = e^{-3t}$$

Therefore, we have that $\min\{X_1, X_2\} \sim \text{Exp}(3)$. In particular, this means the mean of this is $\frac{1}{3}$, which is our answer.

Answer:

13

Reference:

https://www.quantguide.io/questions/managerial-oversight

Fixed Point Limit I

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

Let F_n be the number of fixed points of a random function $f : S_n \to S_n$, where $S_n = \{1, 2, \ldots, n\}$. Find $\lim_{n\to\infty} \mathbb{P}[F_n = 5]$. The answer is in the form $\frac{c}{e}$, where e is Euler's constant and c is a constant. Find c.

Hint:

The events that each of the values is a fixed point are independent, as we can choose a function f from the set of all functions

Explanation:

The events that each of the values is a fixed point are independent, as we can choose a function f from the set of all functions, so as there are n possible values in S_n and each has probability $1/n$ of being fixed (exactly one of the n values would make it fixed), we have that $F_n \sim \text{Binom}(n, 1/n)$. By Poisson Limit Theorem, we have that $F_n \implies \text{Poisson}(1)$, so $\mathbb{P}[F_n = 5] = \frac{1}{5!}e^{-5}$, so our answer is $1/120$.

Answer:

1120

Reference:

https://www.quantguide.io/questions/fixed-point-limit-i

Standing Table

Category: Probability
Difficulty: Hard
Companies: Citadel, De Shaw, Akuna, Jane Street

Question:

We make a table from a circular disk and three legs. We attach the three legs to the circumference of the circular disk. What is the probability that the table stands up?

Hint:

What can we say about the center of mass of the table in relation to the legs for the table to stand up?

Explanation:

Lets fix the first leg at a random spot on the circumference. The second leg is expected to be $\frac{1}{4}^{\text{th}}$ of the circumference away from the initial leg (draw out the circle, you will see that the average spot for the second leg is half way between where the first leg is and the opposite point of the first leg). Finally, for the table to stand up, the center of the table (center of mass) needs to be within the triangle that is formed by connecting the three legs. This leaves us with $\frac{1}{4}^{\text{th}}$ of the circumference to place the final leg which allows for the table to stand. Thus the answer is $\frac{1}{4}$.

Answer:

14

Reference:

https://www.quantguide.io/questions/standing-table

Straddle Output

Category: Finance
Difficulty: Medium
Companies: Citadel, Drw

Question:

We have a straddle with strike $K = 0$. The underlying asset price is $S \sim N(0, 1)$. What is the value of v, the expected value of the straddle? v^2 can be written in the form $\frac{a}{\pi}$ for a rational number a. Find a.

Hint:

The expected value of the straddle is $\mathbb{E}[|S - K|] = \mathbb{E}[|S|]$, as $K = 0$. This reduces to a pure probability problem.

Explanation:

The expected value of the straddle is $\mathbb{E}[|S - K|] = \mathbb{E}[|S|]$, as $K = 0$. We are going to generalize this calculation for when $S \sim N(0, \sigma^2)$.

$$
\begin{aligned}
E[|S|] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} |x| \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \sqrt{\frac{2}{\pi\sigma^2}} \left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\Bigg|_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}\sigma,
\end{aligned}
$$

From line 1 to line 2, we use symmetry of the integrand about 0. Then, from line 2 to line 3, we apply a $u-$substitution to obtain that indefinite integral. In particular, $\sigma = 1$ here, so our answer is $v = \sqrt{\frac{2}{\pi}}$, meaning that $v^2 = \frac{2}{\pi}$, so $a = 2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/straddle-output

2d Paths I

Category: Brainteasers
Difficulty: Medium
Companies: Citadel, Five Rings

Question:

You are playing a 2D game where your character is trapped within a 6×6 grid. Your character starts at $(0, 0)$ and can only move up and right. There is a power-up located at $(2, 3)$. How many possible paths can your character take to get to $(6, 6)$ such that it can collect the power-up?

Hint:

The character has two sub-paths: from $(0, 0)$ to $(2, 3)$ and from $(2, 3)$ to $(6, 6)$. Try to solve these separately.

Explanation:

The character has two sub-paths: from $(0, 0)$ to $(2, 3)$ and from $(2, 3)$ to $(6, 6)$. The total number of paths from $(0, 0)$ to $(2, 3)$ is $\binom{5}{2}$. The total number of paths from $(2, 3)$ to $(6, 6)$ is $\binom{7}{4}$. Thus, the total number of paths to $(6, 6)$ such that your character can pass through $(2, 3)$ is:

$$\binom{5}{2} \times \binom{7}{4} = 10 \times 35 = 350$$

Answer:

350

Reference:

https://www.quantguide.io/questions/2d-paths-i

Balanced Beans II

Category: Brainteasers
Difficulty: Hard
Companies: Citadel, De Shaw, SIG, Optiver

Question:

There are 12 beans; one weighs slightly heavier or lighter than the others. What is the minimum number of times a balance scale must be used to guarantee the determination of the abnormal bean?

Hint:

Start the same way we do for ”Balanced Beans”.

Explanation:

The first step is similar to the other ”Balanced Beans” problem and its to split the beans into 3 groups of four. Put one group on one side and another group on the other side, leaving one group unweighted. The best case scenario is if the scale is balanced on both sides which means the abnormal bean is in the unweighted group. In this case, you can take two beans from that group and weigh them on either side. If the scale is balanced, you know the abnormal bean is one of the other two beans in the 3^{rd} group. Thus you can pick one of these two beans and replace either side with that bean. If the scale continues to stay balanced, you know the last bean (never touched the scale) is the abnormal one. Otherwise, its the new bean you put on the scale. Its a similar approach if the scales are not balanced after you selected two beans from the 3^{rd} group. You replace one side with a bean you know is normal and if it continues to remain unbalanced, the bean that stayed on the scale is abnormal. Otherwise its the bean you replaced with a normal bean.Now lets cover the more difficult situation where the two groups of four aren't balanced. The key realization with this problem is to move around subsets of each group, specifically subsets of three beans. Lets call the two groups of four on the scale Group 1 (left side) and Group 2 (right side) while the unweighted group is Group 3. We take three beans from Group 1 and move them to Group 2, take three beans from Group 2 and move them to Group 3, and three beans from Group 3 and move them to Group 1. Now the best case scenario is the balance doesn't change its orientation (if the left side was lighter before the moving of groups of three, it continues to stay lighter). This is the best case scenario because we know the abnormal bean is either the bean that didn't move from from Group 1 or the bean that didn't move from Group 2. We can easily deduce which one is the abnormal one by weighing one of these beans against a known normal bean and deduce if the bean weighed is the normal or abnormal bean. Lets say the scale does change orientation after the 2^{nd} weighing. If the scale flips orientations (still unbalanced but not the same side being lighter than the other), then we know the abnormal bean was in the group of three beans we moved from Group 1 to Group 2. We would also know if this bean is lighter or heavier than the others. If the left side (Group 1) weighed more than the right (Group 2), and it flipped to the right weighing more, then we know the abnormal bean weighs more than the others. Same thing if its lighter but we'd notice the left side going from being lighter to heavier. Weigh any two beans from this group of three that's currently in Group 2 to deduce the abnormal bean. Lets go back to the case where the scales were originally unbalanced but after the 2nd weighing, the becomes balanced. This means that the abnormal bean is in the group of three we moved from Group 2 to Group 3. We would also know if this bean is heavier or lighter than the others as we could remember if the right side (Group 2) was up or down after the second weighing. Now weigh any two beans from this group of three to deduce the abnormal bean. The total number of 3 weightings needed to guarantee you know which bean is the abnormal one.

Answer:

3

Reference:

https://www.quantguide.io/questions/balanced-beans-ii

Probability Of Unfair Coin II

Category: Probability
Difficulty: Easy
Companies: Citadel, Akuna, SIG, Drw, Five Rings, Two Sigma, Jane Street, HRT

Question:

Flip 98 fair coins, one double-headed coin, and 1 double-tailed coin and observe the first coin tossed. A coin is selected uniformly at random and you see it shows heads. What is the probability that this coin is the double-headed coin?

Hint:

Let D be the event that the double-headed coin is selected and H be the event that the coin showed heads. You want $\mathbb{P}[D \mid H]$. Can you use Bayes' Rule to compute this?

Explanation:

Let D be the event that we select the double-headed coin and H be the event that the coin showed heads. We want $\mathbb{P}[D \mid H]$. By Bayes' Rule, we have that $\mathbb{P}[D \mid H] = \frac{\mathbb{P}[H \mid D]\mathbb{P}[D]}{\mathbb{P}[H]}$. For the denominator, we should condition on the ways a head could be obtained. There are three types of coins: the fair coins, the double-headed coin, and the double-tailed coin. Let F be the event of selecting a fair coin and T be the event of selecting the double-tailed coin. We have that $\mathbb{P}[H \mid T] = 0$ since this coin has no heads on it. We know $\mathbb{P}[H \mid F] = \frac{1}{2}$ since it is fair and $\mathbb{P}[F] = \frac{98}{100}$ since 98 of the coins are fair. Lastly, $\mathbb{P}[H \mid D] = 1$ since both sides are heads and $\mathbb{P}[D] = \frac{1}{100}$ as there is only 1 of them. Combining all of this,

$$\mathbb{P}[D \mid H] = \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{98}{100} \cdot \frac{1}{2} + 0} = \frac{1}{50}$$

Answer:

150

Reference:

https://www.quantguide.io/questions/probability-of-unfair-coin-ii

2d Paths II

Category: Brainteasers
Difficulty: Medium
Companies: Citadel, Five Rings

Question:

You are playing a 2D game where your character is trapped within a 6×6 grid. Your character starts at $(0, 0)$ and can only move up and right. There is a power-up located at $(2, 3)$. How many possible paths can your character take to get to $(6, 6)$ such that it can collect the power-up?

Hint:

The character has two sub-paths: from $(0, 0)$ to $(2, 3)$ and from $(2, 3)$ to $(6, 6)$. Try to solve these separately.

Explanation:

The character has two sub-paths: from $(0, 0)$ to $(2, 3)$ and from $(2, 3)$ to $(6, 6)$. The total number of paths from $(0, 0)$ to $(2, 3)$ is $\binom{5}{2}$. The total number of paths from $(2, 3)$ to $(6, 6)$ is $\binom{7}{4}$. Thus, the total number of paths to $(6, 6)$ such that your character can pass through $(2, 3)$ is:

$$\binom{5}{2} \times \binom{7}{4} = 10 \times 35 = 350$$

Answer:

350

Reference:

https://www.quantguide.io/questions/2d-paths-ii

Stacked Derivative

Category: Pure Math
Difficulty: Medium
Companies: Citadel, Akuna, Five Rings

Question:

Find the derivative of $f(x) = (\ln(x))^x$ at $x = e$.

Hint:

The trick here is to use logarithmic differentiation.

Explanation:

The trick here is to use logarithmic differentiation. In other words, let $y = f(x) = (\ln(x))^x$. Then $\ln(y) = x \ln(\ln(x))$. Using the product and chain rules, we have that

$$\frac{y'}{y} = \ln(\ln(x)) + x \cdot \frac{1}{x \ln(x)} \iff y' = y\left[\ln(\ln(x)) + \frac{1}{\ln(x)}\right] = (\ln(x))^x \left[\ln(\ln(x)) + \frac{1}{\ln(x)}\right]$$

Plugging in $x = e$, the above expression evaluates to 1.

Answer:

1

Reference:

https://www.quantguide.io/questions/stacked-derivative

Collecting Toys I

Category: Probability
Difficulty: Medium
Companies: Citadel, Drw, Jane Street, Optiver

Question:

Every box of cereal contains one toy from a group of five distinct toys, each of which is mutually independent from the others and is equally likely to be within a given box. On average, how many boxes of cereal will you need to open in order to collect all five toys?

Hint:

Choosing a distinct toy on each new box follows a geometric distribution. What is the expected value of a geometric distribution?

Explanation:

Choosing a distinct toy on each new box follows a geometric distribution. For the first distinct toy, we know that the first box we open will be a new toy. For the second distinct toy, there is a $\frac{4}{5}$ probability of receiving a new toy; thus, it should take $\frac{5}{4}$ boxes on average to receive a new toy. This logic follows for the third ($\frac{5}{3}$), fourth ($\frac{5}{2}$), and fifth ($\frac{5}{1}$) toy. Hence, the total number of boxes we opened on average to collect all five toys is:

$$1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1} = \frac{137}{12}$$

Answer:

13712

Reference:

https://www.quantguide.io/questions/collecting-toys-i

Soccer Practice

Category: Probability
Difficulty: Hard
Companies: Citadel

Question:

You're practicing soccer by taking 100 penalty kicks. Assume that you have made the first goal but missed the second. For each of the following kicks, the probability that you score is the fraction of goals you've made thus far. For example, if you made 17 goals out of the first 30 attempts, then the probability that you make the 31st goal is $\frac{17}{30}$. After 100 attempts, including the first two, what is the probability that you score exactly 66 penalty kicks?

Hint:

Break down the problem. What is the probability of making exactly 1 successful kick? What is the probability of making exactly 2 successful kicks? What is the probability of making exactly n successful kicks?

Explanation:

Among the 98 remaining penalty kicks, you must score exactly 65, which occurs $\binom{98}{65}$ ways. Let us now find the probability of each one of these outcomes occurring. The denominator of the probability must be 99!, which is the product of the total number of kicks at each time step ($\frac{1}{2} \times \frac{x}{3} \times \frac{y}{4} \times ... = \frac{z}{99!}$). Similarly, the numerator must be $65! \times 33!$, since there must be exactly 65 successful kicks and exactly 33 failed kicks. Thus, the probability of any such arrangement of is $\frac{65!33!}{99!}$, and the probability that you score exactly 66 penalty kicks is:

$$\binom{98}{65} \times \frac{65! \times 33!}{99!} = \frac{1}{99}$$

Answer:

199

Reference:

https://www.quantguide.io/questions/soccer-practice

Basic Dice Game I

Category: Probability
Difficulty: Easy
Companies: Citadel, Akuna, SIG, Drw, Jane Street, HRT, Optiver

Question:

A casino offers you a game with a six-sided die where you are paid the value of the roll. The casino lets you roll the first time. If you are happy with your roll, you can cash out. Else, you can choose to re-roll and cash out on this second value. What is the fair value of this game?

Hint:

What is the expected value of the final roll and how does this affect your choice to re-roll?

Explanation:

The fair value of the second roll is $\frac{1+2+3+4+5+6}{6} = 3.5$. Thus, you should only opt to re-roll if your first roll was below this, since you know that the second roll will do better on average. In other words, you will not re-roll if you obtain a 4, 5, or 6, which happens $\frac{1}{2}$ of the time with an expectation of 5. Hence, the fair value of this game is:

$$\frac{1}{2} \times 5 + \frac{1}{2} \times 3.5 = 4.25$$

Answer:

4.25

Reference:

https://www.quantguide.io/questions/basic-dice-game-i

Throwing Darts II

Category: Probability
Difficulty: Easy
Companies: Citadel, Drw

Question:

You throw 3 darts that land uniformly randomly on a circular dartboard composed of three concentric circles with radii 1ft, 2ft, and 3ft. Assuming the dart lands on the dartboard, what is the probability that at least one of the darts land in the inner ring (within the 1ft circle)?

Hint:

What is the probability that none of the three darts land within the inner ring? How can you use complementary probability to find the answer?

Explanation:

We first want to calculate the probability of one dart landing in the inner ring. Because the dart lands uniformly at random on the board, we can calculate the probability as the ratio of the area of the inner ring to the area of the entire board. The area of the entire board is 9π and the area of the inner ring is 1π. Thus, the probability that the dart lands in the inner ring is:

$$\frac{\pi}{9\pi} = \frac{1}{9}$$

We now want to calculate the probability that at least one of our three darts will land in this inner circle. To do this, we will calculate the probability of not getting any darts in the inner ring, which is:

$$\left(\frac{8}{9}\right)^3 = \frac{512}{729}$$

Finally, we use the complement rule, $1 - P(A^c) = P(A)$ to come to the answer:

$$1 - \frac{512}{729} = \frac{217}{729}$$

Answer:

217729

Reference:

https://www.quantguide.io/questions/throwing-darts-ii

Absolutely Brownian

Category: Pure Math
Difficulty: Medium
Companies: Citadel

Question:

Let W_t be a standard Brownian Motion and let $T_4 = \inf\{t > 0 : |W_t| > 4\}$. Find $\text{Var}(T_4)$.

Hint:

Consider $f(t, w) = w^4 - 6w^2t + 3t^2$ and show that this defines a martingale.

Explanation:

We are going to solve this for general $T_a = \inf\{t > 0 : |W_t| > a\}$ for any $a > 0$. It is a well-known fact that $Z_t = f(t, W_t)$ defines a martingale, where $f(t, w) = w^4 - 6w^2t + 3t^2$. By using the Optional Stopping Theorem, we have that

$$\mathbb{E}[Z_{T_4}] = \mathbb{E}[W_{T_a}^4 - 6W_{T_a}^2 T_a + 3T_a^2] = \mathbb{E}[Z_0] = 0$$

Since $W_{T_a} = \pm a$ with equal probability, we know that $W_{T_a}^4 = a^4$ with probability 1. Similarly, $W_{T_a}^2 = a^2$ with probability 1. Therefore, by linearity, we have that $a^4 - 6a^2\mathbb{E}[T_a] + 3\mathbb{E}[T_a^2] = 0$. From a more basic martingale argument on $W_t^2 - t$, one can easily prove that $\mathbb{E}[T_a] = a^2$. Therefore, we have that $\mathbb{E}[T_a^2] = \frac{5}{3}a^4$ by rearrangement.

To find the variance, we just use our classic relationship $\text{Var}(T_a) = \mathbb{E}[T_a^2] - (\mathbb{E}[T_a])^2 = \frac{5}{3}a^4 - a^4 = \frac{2}{3}a^4$. In our case, $a = 4$, so the answer is $\frac{512}{3}$.

Answer:

5123

Reference:

https://www.quantguide.io/questions/absolutely-brownian

Probability Discussion

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, SIG, Jane Street

Question:

Gabe and a student have decided to meet up to discuss probability. The student is very prompt and will show up at a uniformly random time between 4 and 5 PM. Gabe is a tad late, so he will show up at a uniformly random time between $4:30$ and 5 PM. For whichever person gets there first, they will wait up to 10 minutes, and if the other person has not shown up, they will leave. Find the probability that the meeting occurs.

Hint:

Let G and S be the number of minutes after 4 PM that Gabe and the student show up, respectively. Then we have that $G \sim \text{Unif}(30, 60)$ and $S \sim \text{Unif}(0, 60)$. What is the event that the meeting occurs in terms of G and S?

Explanation:

First, let's model this mathematically. Let G and S be the number of minutes after 4 PM that Gabe and the student show up, respectively. Then we have that $G \sim \text{Unif}(30, 60)$ and $S \sim \text{Unif}(0, 60)$. The event that the meeting occurs is the just the event that $|G - S| \leq 10$, as they must differ by up to 10 minutes.

We should draw out the region in the plane to see what we are working with. If you put G on the x-axis and S on the y-axis, we have a tall rectangle. When drawing the two lines corresponding to $|G - S| \leq 10$, you will notice that it is easier to calculate the probability of the complement, as those are easier regions to work with. You will see that there is one trapezoid and one triangle. It is easy enough to use some basic algebra to note that the slope is 1, so the line will go as far vertically as it does horizontally. You will find that the base of the trapezoid is 30, and the two heights are 20 and 50, so its area is 1050. The two sides of the triangle are 20 each, so the area is 200. Thus, the probability of the complement is $\frac{25}{36}$. Thus, the probability in the question is $\frac{11}{36}$

Answer:

1136

Reference:

https://www.quantguide.io/questions/probability-discussion

Balanced Beans

Category: Brainteasers
Difficulty: Hard
Companies: Citadel, De Shaw, SIG, Optiver

Question:

There are 12 beans; one weighs slightly heavier or lighter than the others. What is the minimum number of times a balance scale must be used to guarantee the determination of the abnormal bean?

Hint:

Start the same way we do for ”Balanced Beans”.

Explanation:

The first step is similar to the other ”Balanced Beans” problem and its to split the beans into 3 groups of four. Put one group on one side and another group on the other side, leaving one group unweighted. The best case scenario is if the scale is balanced on both sides which means the abnormal bean is in the unweighted group. In this case, you can take two beans from that group and weigh them on either side. If the scale is balanced, you know the abnormal bean is one of the other two beans in the 3^{rd} group. Thus you can pick one of these two beans and replace either side with that bean. If the scale continues to stay balanced, you know the last bean (never touched the scale) is the abnormal one. Otherwise, its the new bean you put on the scale. Its a similar approach if the scales are not balanced after you selected two beans from the 3^{rd} group. You replace one side with a bean you know is normal and if it continues to remain unbalanced, the bean that stayed on the scale is abnormal. Otherwise its the bean you replaced with a normal bean.Now lets cover the more difficult situation where the two groups of four aren't balanced. The key realization with this problem is to move around subsets of each group, specifically subsets of three beans. Lets call the two groups of four on the scale Group 1 (left side) and Group 2 (right side) while the unweighted group is Group 3. We take three beans from Group 1 and move them to Group 2, take three beans from Group 2 and move them to Group 3, and three beans from Group 3 and move them to Group 1. Now the best case scenario is the balance doesn't change its orientation (if the left side was lighter before the moving of groups of three, it continues to stay lighter). This is the best case scenario because we know the abnormal bean is either the bean that didn't move from from Group 1 or the bean that didn't move from Group 2. We can easily deduce which one is the abnormal one by weighing one of these beans against a known normal bean and deduce if the bean weighed is the normal or abnormal bean. Lets say the scale does change orientation after the 2^{nd} weighing. If the scale flips orientations (still unbalanced but not the same side being lighter than the other), then we know the abnormal bean was in the group of three beans we moved from Group 1 to Group 2. We would also know if this bean is lighter or heavier than the others. If the left side (Group 1) weighed more than the right (Group 2), and it flipped to the right weighing more, then we know the abnormal bean weighs more than the others. Same thing if its lighter but we'd notice the left side going from being lighter to heavier. Weigh any two beans from this group of three that's currently in Group 2 to deduce the abnormal bean. Lets go back to the case where the scales were originally unbalanced but after the 2nd weighing, the becomes balanced. This means that the abnormal bean is in the group of three we moved from Group 2 to Group 3. We would also know if this bean is heavier or lighter than the others as we could remember if the right side (Group 2) was up or down after the second weighing. Now weigh any two beans from this group of three to deduce the abnormal bean. The total number of 3 weightings needed to guarantee you know which bean is the abnormal one.

Answer:

3

Reference:

https://www.quantguide.io/questions/balanced-beans

Dinner At Dorsia

Category: Probability
Difficulty: Medium
Companies: Citadel, Akuna, SIG, Jane Street

Question:

Two quants are planning for dinner at Dorsia. Assume that each independently arrives at some uniformly random time between 8:00pm and 9:00pm, for which they stay for exactly 10 minutes before leaving. What is the probability that they will meet each other and stay for dinner?

Hint:

Note that their arrival times are uniformly distributed. Geometry can be useful for this type of problem. How can you define the sample space and constraint of this problem?

Explanation:

Let quant x arrive X minutes after 8:00pm and quant y arrive Y minutes after 8:00pm. The two quants meet if and only if $|Y - X| \leq 10$. The area covered by this constraint in the sample space $X, Y \in [0, 60]$ is equal to $60^2 - 50^2 = 1100$. Since both X and Y are uniformly distributed, the probability that the two quants meet is $\frac{1100}{3600} = \frac{11}{36}$.

Answer:

1136

Reference:

https://www.quantguide.io/questions/dinner-at-dorsia

Carded Pair

Category: Probability
Difficulty: Hard
Companies: Citadel, Five Rings

Question:

Cards are dealt one-by-one from a standard deck of 52 cards without replacement. The card dealing ends when either 2 kings appear OR at least 1 king and 1 ace appear, whichever comes first. Find the expected number of cards that are dealt in the game.

Hint:

Use a "first ace" approach and condition on whether or not the first card between king and ace to appear is king vs. ace. What needs to happen afterwards in each scenario?

Explanation:

To observe either 2 kings or 1 king and 1 ace, we need to condition on which of king or ace appears first in the deck. If king appears first, then we need to find the number of cards until we observe either a king or an ace. If ace appears first, then we need to find the number of cards until we observe a king. For the first king/ace to be drawn, it is equally likely to be either of them. From here on out, K is king and A is ace.

First, we need to find the expected number of cards until the first K or A appears. We can use the First Ace methodology to use the dividers as K or A, yielding 8 dividers. These 8 dividers divide the other $52-8=44$ cards into 9 equally-sized regions in expectation, so there are $44/9$ cards on average before the first K or A. Then, we need to add in 1 card for actually picking that K or A.

Now, with probability $1/2$, we selected K. In this case, we need to find the expected number of cards until either a K or A appears in the remaining deck. There are now 7 dividers (4 A and 3 K) and $52-44/9-1-7=352/9$ cards left in the deck on average. Since there are 7 dividers, there are now 8 equally-sized regions in expectation, so the expected number of cards before observing the next K or A is $352/72$. Then, we must add in 1 for selecting the K or A, yielding a conditional expectation of $424/72$ in this case.

With probability $1/2$, we selected A. In this case, we need to find the expected number of cards until a K appears. There are now 4 dividers (corresponding to the 4 K) and $52-44/9-1-4=379/9$ cards left in the deck on average. The 4 dividers split up our remaining deck into 5 equally-sized regions in expectation, so there are $379/45$ cards before the first K on average. Then, we need to add in 1 more back for selecting the K, so our conditional expectation here is $424/45$.

Putting it all together, our final answer is

$$\frac{44}{9}+1+\frac{1}{2}\cdot\frac{424}{45}+\frac{1}{2}\cdot\frac{424}{72}=\frac{1219}{90}\approx 13.54$$

Answer:

121990

Reference:

https://www.quantguide.io/questions/carded-pair

Probability Of Unfair Coin I

Category: Probability
Difficulty: Easy
Companies: Citadel, Akuna, SIG, Drw, Five Rings, Two Sigma, Jane Street, HRT

Question:

Flip 98 fair coins, one double-headed coin, and 1 double-tailed coin and observe the first coin tossed. A coin is selected uniformly at random and you see it shows heads. What is the probability that this coin is the double-headed coin?

Hint:

Let D be the event that the double-headed coin is selected and H be the event that the coin showed heads. You want $\mathbb{P}[D \mid H]$. Can you use Bayes' Rule to compute this?

Explanation:

Let D be the event that we select the double-headed coin and H be the event that the coin showed heads. We want $\mathbb{P}[D \mid H]$. By Bayes' Rule, we have that $\mathbb{P}[D \mid H] = \dfrac{\mathbb{P}[H \mid D]\mathbb{P}[D]}{\mathbb{P}[H]}$. For the denominator, we should condition on the ways a head could be obtained. There are three types of coins: the fair coins, the double-headed coin, and the double-tailed coin. Let F be the event of selecting a fair coin and T be the event of selecting the double-tailed coin. We have that $\mathbb{P}[H \mid T] = 0$ since this coin has no heads on it. We know $\mathbb{P}[H \mid F] = \dfrac{1}{2}$ since it is fair and $\mathbb{P}[F] = \dfrac{98}{100}$ since 98 of the coins are fair. Lastly, $\mathbb{P}[H \mid D] = 1$ since both sides are heads and $\mathbb{P}[D] = \dfrac{1}{100}$ as there is only 1 of them. Combining all of this,

$$\mathbb{P}[D \mid H] = \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{98}{100} \cdot \frac{1}{2} + 0} = \frac{1}{50}$$

Answer:

150

Reference:

https://www.quantguide.io/questions/probability-of-unfair-coin-i

Spherical Coodinates

Category: Probability
Difficulty: Hard
Companies: Citadel, Drw, Five Rings

Question:

A random point $(X_1, X_2, \ldots, X_{10})$ is uniformly at random selected from the $10-$ball that has radius 12 centered at the origin. Find the variance of X_1, the first coordinate.

Hint:

Let R be the random radius (distance from the origin) of this point. We know that $R^2 = X_1^2 + \cdots + X_{10}^2$.

Explanation:

We know that our point is uniformly at random selected from this sphere. Let R be the random radius (distance from the origin) of this point. We know that $R^2 = X_1^2 + \cdots + X_{10}^2$. Since this sphere is clearly symmetric about the origin and all of the coordinates share the same marginal distributions, they are exchangeable and hence has the same expectation. Therefore, $\mathbb{E}[R^2] = 10\mathbb{E}[X_1^2]$.

We know that $\text{Var}(X_1) = \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2$. However, as our sphere is again symmetric about the origin, the mean of X_1 is 0, so we just need to find the second moment of X_1 and we are done. This boils down to, by the above, finding the expected squared distance from the center.

Let's compute $\mathbb{P}[R \leq r]$. The volume of a $n-$dimensional sphere of radius r is a constant $C_n \cdot r^n$. Therefore, as we select uniformly at random from this sphere, the probability of the event $\{R \leq r\}$ just means that our point belongs to the sub-sphere of radius r from the big sphere of radius 12. The probability of this is $\frac{C_{10}r^{10}}{C_{10} \cdot 12^{10}} = \frac{r^{10}}{12^{10}}$. Therefore, the probability density of R (found by taking the derivative with respect to r of the CDF) is $f_R(r) = \frac{10r^9}{12^{10}} I_{(0,12)}(r)$. The indicator comes from the fact that the radius must be somewhere between 0 and 12. Obtaining $\mathbb{E}[R^2]$ is now straightforward using LOTUS. We have that $\mathbb{E}[R^2] = \frac{1}{12^{10}} \int_0^{12} 10r^{11} dr = \frac{10}{12^{11}} 12^{12} = 120$.

Plugging this back into our original equation, we have that $120 = 10\mathbb{E}[X_1^2]$, meaning $\text{Var}(X_1) = \mathbb{E}[X_1^2] = 12$.

Answer:

12

Reference:

https://www.quantguide.io/questions/spherical-coodinates

Exponential Normal

Category: Probability
Difficulty: Easy
Companies: Citadel, Akuna

Question:

Suppose that $\ln(X) \sim N(0, 1)$. Find $\ln(\mathbb{E}[X])$.

Hint:

Let $Z \sim N(0, 1)$. Then we can say that $X = e^Z$. We want to find $\mathbb{E}[X] = \mathbb{E}[e^Z] = M_Z(1)$, where $M_Z(\theta)$ is the MGF of Z.

Explanation:

Let $Z \sim N(0, 1)$. Then we can say that $X = e^Z$. We want to find $\mathbb{E}[X] = \mathbb{E}[e^Z] = M_Z(1)$, where $M_Z(\theta)$ is the MGF of Z. The MGF of $Z \sim N(0, 1)$ is $M_Z(\theta) = e^{\frac{1}{2}\theta^2}$. Therefore, $M_Z(1) = e^{\frac{1}{2}}$, so our answer is $\ln(e^{\frac{1}{2}}) = \dfrac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/exponential-normal

Friendly Competition

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG, Old Mission, Optiver

Question:

Players 1 and 2 respectively start with \$1 and \$2. They flip a coin with probability $\frac{2}{3}$ of heads on each toss. Player 1 receives \$1 from Player 2 if the coin shows heads. Otherwise, Player 2 receives \$1 from Player 1. Find the probability Player 2 goes broke first i.e. Player 1 wins.

Hint:

Let p be the probability player 1 wins. Consider what happens under the two coin flip sequences TT, TH, HT, HH.

Explanation:

Let p be the probability player 1 wins. If the first flip is tails, then player 1 loses automatically. If we go HT, then we are back where we started and player 1 has probability p of winning. This occurs with probability $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ by independence of the coins. If we go HH, then player 1 wins. This occurs with probability $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$ Therefore, by Law of Total Probability, $p = \frac{2}{9}p + \frac{4}{9}$. Solving this for p yields $p = \frac{4}{7}$.

Answer:

47

Reference:

https://www.quantguide.io/questions/friendly-competition

Safe Cracking

Category: Brainteasers
Difficulty: Hard
Companies: Citadel

Question:

An electronic safe has a three digit passcode. You are given three constraints regarding the code. Firstly, the code is not an odd number. Secondly, the code does not contain the number six. Lastly, one of the digits appears more than once. How many possible three digit entries satisfy these three requirements?

Hint:

Consider building the initial space of possibilities using the first two constraints, before casing on the third constraint.

Explanation:

Let us look at the first two constraints while thinking about the possibilities of each of the three digits. The units digit must be an even number that is not six, and thus has four possibilities. The tens and hundreds place both have nine possibilities. Now onto the third constraint to determine the total number of codes that satisfy all three. In order for one of the digits to appear, there are four cases to consider: first two digits are the same and the third is different; the first and third digits are the same and the second is different, the second and third digits are the same and the first is different; all three digits are the same. We will consider each one separately.

Case 1: The first two digits are either the same odd number or the same even number. If they are both odd, then the third digit can be any of its four choices. If they are both even, then the third digit can be any of the three choices such that it is a different even number. The number of total possibilities is $5 \times 4 + 4 \times 3 = 32$.

Case 2: The first and third digits can be any one of four choices, limited by the options that the third digit has. The second digit can be any one of the eight choices such that it is a different number. The number of total possibilities is $4 \times 8 = 32$.

Case 3: Note that this case is symmetrical to the previous case. The second and third digits can be any one of four choices, limited by the options that the third digit has. The first digit can be any one of the eight choices such that it is a different number. The number of total possibilities is $4 \times 8 = 32$.

Case 4: The three digits can be any one of four choices, limited by the options that the third digit has. The number of total possibilities is 4.

In conclusion, the total number of three digit entries satisfy these three requirements is $32 + 32 + 32 + 4 = 100$.

Answer:

100

Reference:

https://www.quantguide.io/questions/safe-cracking

Car Crash

Category: Probability
Difficulty: Hard
Companies: Citadel, De Shaw, Optiver

Question:

N employees are driving their individual cars on their way to work at QuantGuide. All cars are reasonably spaced apart, and they all travel at distinct speeds which are randomly assigned. When a faster car catches up to a slower car, it assumes the slower car's speed. After a very long period of time has passed, all N cars have settled into K clusters traveling at distinct speeds. What is the expected value of K when $N = 10$?

Hint:

Let us define an indicator variable Z_i as follows:

$$Z_i = \begin{cases} 1 & \text{if there are no slower cars in front of } X_i \\ 0 & \text{otherwise} \end{cases}$$

Then, the total number of clusters K is simply $K = \sum_{i=1}^{N} Z_i$.

Explanation:

We can solve this question with linearity of expectation. Let us define an indicator variable Z_i as follows:

$$Z_i = \begin{cases} 1 & \text{if there are no slower cars in front of } X_i \\ 0 & \text{otherwise} \end{cases}$$

Then, the total number of clusters K is simply $K = \sum_{i=1}^{N} Z_i$. By linearity of expectation, it follows that $\mathbb{E}[K] = \sum_{i=1}^{N} \mathbb{E}[Z_i]$.

Consider Z_N, the indicator variable corresponding to the car at the front of the pack. No matter what, $Z_N = 1$ since there are no other cars in front of it. Next, let's consider Z_i. In this case, there are $N - i$ cars in front of car X_i. Including car X_i, there are $N - i + 1$ total cars. The probability that X_i is the slowest car among the total $N - i + 1$ cars is simply $\frac{1}{N-i+1}$. Hence, $\mathbb{E}[Z_i] = \frac{1}{N-i+1}$. Plugging this in, we find

$$\mathbb{E}[K] = \sum_{i=1}^{N} \mathbb{E}[Z_i]$$

$$= \sum_{i=1}^{N} \frac{1}{N-i+1}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

When $N = 10$, we compute that

$$\mathbb{E}[K] = \sum_{i=1}^{10} \frac{1}{i}$$

$$= \frac{7381}{2520}$$

Answer:

73812520

Reference:

https://www.quantguide.io/questions/car-crash

Defeating Dragons

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG

Question:

A 3-headed dragon is attacking your village! You, the mightiest of knights, are tasked with taking down this dragon with your trusty sword. The dragon is defeated if you chop off all of its heads. However, this dragon uses magic and has the ability to grow heads over time! If you attack the dragon, one of three events can happen. You either chop of 2 of its heads, you chop of 1 head (which will ALWAYS grow back), or you miss entirely and the dragon grows 1 head. The probability you chop of 2 heads and miss the dragon are the same (assuming the dragon has 2 or more heads). You know that if the dragon has 5 or more heads, the dragon is too strong for you to take care of (in which case you and the rest of the villagers will perish). What is the probability you defeat the dragon?

Hint:

How can we model the different states of this scenario?

Explanation:

Since there are changing states to this situation, we need to employ a Markov Chain to answer this question. Each state is differentiated by the amount of heads the dragon has. Let P_x be the probability you defeat the dragon when the dragon is at x heads. We already know that $P_0 = 1$ and $P_5 = 0$, thus we need to make equations for every other state in between. Also, its important to notice you are either going to a state with two less, the same amount, or one more head. Since the probability of going down two heads is the same as one more head, the probability of each scenario happening is $\frac{1}{2}$ as if you repeat the same state, you'll still have to either go down two heads or up one head again. Also, when the dragon is at one head, you will either stay at the same number or increase the number of heads by one, so $P_1 = P_2$. Every other state equation is as follows:

$$P_2 = \frac{1}{2} \cdot P_0 + \frac{1}{2} \cdot P_3$$

$$P_3 = \frac{1}{2} \cdot P_1 + \frac{1}{2} \cdot P_4$$

$$P_4 = \frac{1}{2} \cdot P_2 + \frac{1}{3} \cdot P_5$$

Solving all these equations, we obtain the answer of $P_3 = \frac{3}{5}$.

Answer:

35

Reference:

https://www.quantguide.io/questions/defeating-dragons

Plane Boarding

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

100 people are in line waiting to board a plane. Let us define the ith passenger in line as having a ticket for plane seat i. Intoxicated, the first passenger in line picks a random seat on the plane to sit on with equal probability. The other 99 passengers are sober, and will sit in their assigned seat unless taken, in which they will sit in a random free seat. You are the last person in line. What is the probability that you end up in your assigned seat?

Hint:

This can be solved with symmetry- try considering only your seat and the drunk passenger's seat.

Explanation:

This can be solved with symmetry. Consider your seat and the drunk passenger's seat. Either your seat is taken before the drunk passenger's seat, or the drunk passenger's seat is taken before your seat. Because there is nothing special about your seat or the drunk passenger's seat, each passenger that has to make a random choice of seats will have an equal probability of filling either your seat or the drunk passenger's seat, assuming both are vacant. Hence, the probability is $\frac{1}{2}$.

Answer:

0.5

Reference:

https://www.quantguide.io/questions/plane-boarding

Circular Delete

Category: Brainteasers
Difficulty: Medium
Companies: Citadel

Question:

The numbers $1, 2, \ldots, 2000$ are put on the circumference of a circle in clockwise increasing order. You start at the integer 1 and delete it. Then you, move clockwise to 2 and keep it. Afterwards, you are going to repeat in this fashion of alternating between deleting integers and keeping integers repeatedly until you reach the last integer of a given rotation. Afterwards, you start again by deleting the first integer in the new cycle, keeping the second, etc. Find the last integer to be deleted from the circle.

Hint:

At the first step, you are keeping all integers n with $n \equiv 0 \bmod 2$. In the second step, what do you see happening?

Explanation:

At the first step, you are keeping all integers n with $n \equiv 0 \bmod 2$. Afterwards, you are keeping all integers with $n \equiv 0 \bmod 4$. More generally, at step k, you are keeping all integers $n \equiv 0 \bmod 2^k$. Therefore, we just need to find the largest k such that $2^k \leq 2000$. 2^k will thus be our last integer. This is $k = 10$, which corresponds to $2^{10} = 1024$.

Answer:

1024

Reference:

https://www.quantguide.io/questions/circular-delete

Tigers Love Sheep

Category: Brainteasers
Difficulty: Easy
Companies: Citadel

Question:

Tigers on this magical grass island are rational and prioritize survival first and eating sheep over grass second. Assume that at each time step, only one tiger can eat one sheep, and that tiger itself will become a sheep after it eats the sheep. One hundred tigers and one sheep are on the magic grass island. How many sheep are left?

Hint:

Begin with a simplified version of the problem. In the 1-tiger case, the tiger will eat the sheep since it does not need to worry about being eaten after.

Explanation:

In the 1-tiger case, the tiger will eat the sheep since it does not need to worry about being eaten after. In the 2-tiger case, either tiger knows that if they eat the sheep, then they will be eaten by the other tiger once they become a sheep. Thus, neither tiger will eat the sheep. In the 3-tiger case, the sheep will be eaten since each tiger will realize that once it eats the sheep and turns into a sheep, there will be 2 tigers left, and neither tiger will eat in the 2-tiger case. In the 4-tiger case, each tiger understands that if it eats the sheep, then it will be eaten after turning into a sheep, since they know that the sheep is eaten in the 3-tiger case. Following this logic, we can naturally show that if the number of tigers is even, the sheep will not be eaten. If the number is odd, the sheep will be eaten.

Answer:

1

Reference:

https://www.quantguide.io/questions/tigers-love-sheep

Big Or Small Deck

Category: Brainteasers
Difficulty: Medium
Companies: Citadel

Question:

A standard deck is split up into two subdecks of 39 and 13 cards. You and a friend may choose one of the decks to draw cards from the top of one-by-one. The goal is to minimize the expected number of cards needed to be drawn to obtain your first ace. Whoever needs to draw fewer cards to see their first ace wins the round. You get to select first. Which deck do you select? Answer 13 if you should select the deck of 13 to draw from, 39 if you should select the deck of 39 to draw from, and -1 if it doesn't matter. Note that if a given deck has no aces, we consider the other deck faster and that we disregard ties i.e. neither person wins.

Hint:

Think about which deck has larger probability of having no cards.

Explanation:

The idea here is that the deck of size 13 has a significantly larger probability of having no aces in it. Conditional on there being an ace in the deck of 13, the deck of 13 is faster on average. However, the probability of no ace is so much larger in the 13 than it is in the 39 that the deck of 39 is the better selection.

Answer:

39

Reference:

https://www.quantguide.io/questions/big-or-small-deck

Big Bubble I

Category: Pure Math
Difficulty: Easy
Companies: Citadel

Question:

A spherical bubble has a radius increasing at a rate of $\frac{9}{4\pi}$ inches per second. At the moment when the volume of the bubble is increasing at 36 cubic inches per second, what rate is the surface area increasing at (in square inches per second)?

Hint:

We know that $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$ represent the volume and the surface area of the bubble as a function of radius. Take the derivative of the volume function to solve for the radius first.

Explanation:

We know that $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$ represent the volume and the surface area of the bubble as a function of radius. We know that $r' = \frac{9}{4\pi}$ is constant. Taking the derivatives of each, we see that $V' = 4\pi r^2 r'$ and $S' = 8\pi r r'$. We know at this moment that $V' = 36$, so plugging these in and solving yields

$$36 = 4\pi r^2 \cdot \frac{9}{4\pi} \iff r = 2$$

Afterwards, we have that

$$S' = 8\pi(2) \cdot \frac{9}{4\pi} = 36$$

Answer:

36

Reference:

https://www.quantguide.io/questions/big-bubble-i

100 Lights

Category: Brainteasers
Difficulty: Medium
Companies: Citadel, SIG, Jane Street

Question:

There are 100 light bulbs in a room, each initially off and with its switch by its side. The first person enters and flips every switch. The second person enters and flips every other switch. The third person enters and flips every third switch. This process continues until the 100-th person flips every 100-th switch, or the last switch. How many lights are on?

Hint:

Come up with some examples where the light remains on. What causes the light to remain on at the end of the process?

Explanation:

The i-th bulb is switched by person j if j is a factor of i. Thus, the i-th bulb will remain on if it has an odd number of factors. We call a number that has an odd number of factors a perfect square, and there are 10 perfect squares from 1 to 100.

Answer:

10

Reference:

https://www.quantguide.io/questions/100-lights

2030 Die Split III

Category: Probability
Difficulty: Hard
Companies: Citadel, Drw, Jane Street

Question:

Alice and Bob have fair 30−sided and 20−sided dice, respectively. The goal for each player is to have the largest value on their die. Alice and Bob both roll their dice. However, Bob has the option to re-roll his die in the event that he is unhappy with the outcome. He can't see Alice's die beforehand. Bob then keeps the value of the new die roll. In the event of a tie, Bob is the winner. Assuming optimal play by Bob, find the probability Alice is the winner.

Hint:

Let W be the event Alice wins. First, we want to find $\mathbb{P}[W]$. Let A and B be value of Alice's roll and Bob's first roll, respectively. The key here is to condition on whether or not $A \leq 20$. Bob is going to roll again whenever $\mathbb{P}[W \mid B = b] > \mathbb{P}[W]$.

Explanation:

Let W be the event Alice wins. First, we want to find $\mathbb{P}[W]$. Let A and B be value of Alice's roll and Bob's first roll, respectively. The key here is to condition on whether or not $A \leq 20$. Namely, this means

$$\mathbb{P}[W] = \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20] + \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20]$$

We can quickly see that $\mathbb{P}[A \leq 20] = \frac{2}{3}$, as this accounts for 20 of 30 equally-likely values. If $A > 20$, then clearly Alice wins regardless of what Bob obtains, so $\mathbb{P}[W \mid A > 20] = 1$.

If $A \leq 20$, then we are really rolling 2 independent 20−sided fair dice. Namely, to compute $\mathbb{P}[W \mid A \leq 20]$, we see that the probability of a tie is $\frac{1}{20}$, as there are 20 values they can agree upon out of $20^2 = 400$ outcomes. Therefore, the probability the two rolls are different is $\frac{19}{20}$. Because of the fact that the two dice are symmetric, $\mathbb{P}[A > B] = \frac{1}{2} \cdot \frac{19}{20} = \frac{19}{40}$, as when the two rolls are not the same, it is equally-likely whether $A > B$ or $A < B$.

Combining these all together, we see that $\mathbb{P}[W] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{19}{40} = \frac{39}{60}$. Now, Bob is going to roll again whenever $\mathbb{P}[W \mid B = b] > \mathbb{P}[W] = \frac{39}{60}$. This is because he wants to minimize Alice's chance of winning. It is easy to see that $\mathbb{P}[W \mid B = b] = 1 - \frac{b}{30}$, as Alice just needs to roll a value of at least $b + 1$.

We just need to find the maximum value of b, say b^*, such that $\mathbb{P}[W \mid B = b] = 1 - \frac{b}{30} > \frac{39}{60}$.

Solving this inequality, we get that $b \leq 10.5$, so $b^* = 10$ is the maximum value that Bob rolls again. Let W_1 be the event that Alice wins in this new game where Bob can roll again. We have that

$$\mathbb{P}[W_1] = \mathbb{P}[W] + \frac{10}{20}\left(\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]\right)$$

We get this because of the fact that Bob rolls again with probability $\frac{1}{2}$, which is when he rolls at most 10. Then, the probability Alice wins is going to change by $\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]$.

To calculate $\mathbb{P}[W \mid B \leq 10]$, this is simply just average all the cases where $B = b$ for $1 \leq b \leq 10$, as when $B \leq 10$, the value is uniformly distributed on the first 10 positive integers. Therefore,

$$\mathbb{P}[W \mid B \leq 10] = \frac{1}{10}\sum_{b=1}^{10} 1 - \frac{b}{30} = \frac{49}{60}$$

Therefore, we get that

$$\mathbb{P}[W] = \frac{39}{60} + \frac{1}{2}\left(\frac{39}{60} - \frac{49}{60}\right) = \frac{17}{30}$$

Answer:

1730

Reference:

https://www.quantguide.io/questions/2030-die-split-iii

Better In Red III

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Jane Street, HRT

Question:

A $3 \times 3 \times 3$ cube is composed of 27 $1 \times 1 \times 1$ cubes that are white by default. All the surfaces of the $3 \times 3 \times 3$ cube are painted red and then the cube is disassembled such that all orientations of all cubes are equally likely. You select a random small cube and notice that all 5 sides that are visible to you are white. What is the probability the last face not visible to you is red?

Hint:

Consider Bayes' Rule and the orientation of the cube.

Explanation:

Let R be the event that the last side not visible to you is red and W be the event that the other sides visible are not red. We want $\mathbb{P}[R \mid W] = \dfrac{\mathbb{P}[R \cap W]}{\mathbb{P}[W]}$. For $\mathbb{P}[R \cap W]$, we need the last side to be red and have that side be facing away from us. Since the orientation is random, the probability the red side is facing away from us is $\dfrac{1}{6}$. In addition, 6 of the 27 cubes have exactly one red side, so $\mathbb{P}[R \cap W] = \dfrac{1}{27}$. Then, in the denominator, we can have 5 white sides from choosing a cube that would satisfy the numerator event or just choosing the center cube. The center cube is white on all sides so there is no need to orient it. Therefore, $\mathbb{P}[W] = \dfrac{1}{27} + \dfrac{1}{27} = \dfrac{2}{27}$. Substituting in, we get $\mathbb{P}[R \mid W] = \dfrac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/better-in-red-iii

Uniform Movement

Category: Probability
Difficulty: Easy
Companies: Citadel, Drw

Question:

Let $X, Y \sim \text{Unif}(3,4)$ IID. Find $\mathbb{E}[|X-Y|]$.

Hint:

Note that $X = 3 + U_1$ and $Y = 3 + U_2$, where $U_1, U_2 \sim \text{Unif}(0,1)$ IID. Therefore, we can really write this as

$$\mathbb{E}[|(3+U_1) - (3+U_2)|] = \mathbb{E}[|U_1 - U_2|]$$

Explanation:

Note that $X = 3 + U_1$ and $Y = 3 + U_2$, where $U_1, U_2 \sim \text{Unif}(0,1)$ IID. Therefore, we can really write this as

$$\mathbb{E}[|(3+U_1) - (3+U_2)|] = \mathbb{E}[|U_1 - U_2|]$$

The trick here is that $|U_1 - U_2|$ is the length of the middle segment of two order statistics of the $\text{Unif}(0,1)$ random variable. This is because of the absolute value here denoting that we look at the length of the middle segment between the two order statistics. In expectation, each of the intervals $(0, \min\{U_1, U_2\}), (\min\{U_1, U_2\}, \max\{U_1, U_2\})$, and $(\max\{U_1, U_2\}), 1)$ should be the same length. Therefore, as their lengths sum to 1, each should be $\frac{1}{3}$ in length. In particular, this means that $\mathbb{E}[|X-Y|] = \mathbb{E}[|U_1 - U_2|] = \frac{1}{3}$.

Answer:

13

Reference:

https://www.quantguide.io/questions/uniform-movement

Sequence Terminator

Category: Probability
Difficulty: Hard
Companies: Citadel, SIG, Five Rings, Two Sigma

Question:

A fair 6−sided die is rolled repetitively, forming a sequence of values, under the following rules: If any even value is rolled, add it to the current sequence. If a 3 or 5 is rolled, discard the entire sequence and don't add the 3 or 5 to the start of the new sequence. If a 1 is rolled, add the 1 to the current sequence and end the game. Find the expected length of the sequence at the end of the game.

For example if the sequence of rolls is 34265241, we would reset to an empty sequence at the first roll, reset to an empty sequence at the 5, and our final sequence would be 241, which is of length 3.

Hint:

An incorrect solution goes as follows: We will have no 3s or 5s appearing before the first 1, as they reset the sequence and are not added. Therefore, our sample space is really $1, 2, 4$, and 6. On average, the 1 will appear after 4 throws of the die, so the answer is 4.

This above solution is incorrect because it more implicitly assumes that we just
”ignore
” 3s and 5s that appear. Therefore, this is not the correct assumption we want to make. Instead, we can reframe this problem as follows: At some point, the last 3 or 5 will appear before the first 1. As the die is memoryless (i.e. is not affected by what has already appeared), given that the first 1 appears before both the first 3 and first 5, what is the expected number of rolls needed to obtain the 1?

Explanation:

An incorrect solution goes as follows: We will have no 3s or 5s appearing before the first 1, as they reset the sequence and are not added. Therefore, our sample space is really $1, 2, 4$, and 6. On average, the 1 will appear after 4 throws of the die, so the answer is 4.

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Let's let N represent the number of rolls needed to see a 1, and let B be the event that 1 occurs before both 3 and 5. By symmetry, $\mathbb{P}[B] = 1/3$, as each is equally likely to appear first. We now condition on the value of the first roll. Let R_i represent the event that we roll value i on the first roll. This would imply by Law of Total Expectation that

$$\mathbb{E}[N \mid B] = \sum_{i=1}^{6} \mathbb{E}[N \mid R_i, B]\mathbb{P}[R_i \mid B]$$

For $i = 2, 4, 6$, we have that $\mathbb{P}[R_i \mid B] = \dfrac{P[B \mid R_i]\mathbb{P}[R_i]}{\mathbb{P}[B]}$. Since the event that we roll $i = 2, 4, 6$ on the first roll doesn't affect the probability of appearance of 1 before 3 and 5, we have that $\mathbb{P}[B \mid R_i] = \mathbb{P}[B]$. Clearly $\mathbb{P}[R_i] = 1/6$ based on the fact the die is fair, so $\mathbb{P}[R_i \mid B] = 1/6$ for $i = 2, 4, 6$.

We have that $\mathbb{P}[R_3 \mid B] = \mathbb{P}[R_5 \mid B] = 0$, as there is no chance that our first roll can be 3 or 5 if 1 comes before them. Thus, since the conditional probability measure is a probability measure, we have that $\mathbb{P}[R_1 \mid B] = 1 - 3 \cdot 1/6 = 1/2$.

If we roll a 1 on the first roll, then our sequence is complete and of length 1, so $\mathbb{E}[N \mid B, R_1] = 1$. Otherwise, if $i = 2, 4, 6$, then $\mathbb{E}[N \mid B, R_i] = 1 + \mathbb{E}[N \mid B]$, as we used up one roll and need to continue forward. Therefore, the equation we need to solve is

$$\mathbb{E}[N \mid B] = \frac{1}{2}\left(1 + \mathbb{E}[N \mid B]\right) + \frac{1}{2}$$

This is easily solved to yield $\mathbb{E}[N \mid B] = 2$.

You can instead view this problem as saying that whenever we roll a $1, 3$, or 5, we make it a 1 so that the 1 occurs before the 3 and 5. Therefore, the probability of rolling an odd integer on each trial is $1/2$, meaning that $N \mid B \sim \text{Geom}(1/2)$, which has mean 2. This agrees with the answer above.

Answer:

2

Reference:

https://www.quantguide.io/questions/sequence-terminator

Cube Colorer

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG, Jane Street

Question:

Dyann paints the outer faces of a $5 \times 5 \times 5$ cube green and then cuts this large cube up into 125 $1 \times 1 \times 1$ cubes. One of the cubes is then uniformly at random selected and rolled. Find the probability that this cube shows a green face up.

Hint:

There are $6 \cdot 5^2 = 150$ painted sides, as each side has $5^2 = 25$ faces painted.

Explanation:

We are going to solve this for a $k \times k \times k$ cube. In particular, there are $6k^2$ painted sides, as each side has k^2 faces painted. There are $6k^3$ total faces in the cube, as each of the k^3 cubes has 6 faces. Therefore, the probability that we obtained a painted side when we roll the selected $1 \times 1 \times 1$ cube is $\frac{6k^2}{6k^3} = \frac{1}{k}$. In this case, $k = 5$, so our answer is $\frac{1}{5}$.

Answer:

15

Reference:

https://www.quantguide.io/questions/cube-colorer

Show The Face

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Five Rings, Two Sigma

Question:

Roll a fair standard 6−sided die until a 6 appears. Given that the first 6 occurs before the first 5, find the expected number of times the die was rolled.

Hint:

An incorrect solution goes as follows: Since we want the first 6 to appear before the first 5, we can eliminate 5 from our sample space and look at the reduced sample space conditional on no roll being a 5. There are 5 remaining values in this sample space that are equally probable, so the number of rolls needed to see a 6 conditional on this information is Geom(1/5), which has mean 5. Therefore, the answer is 5.

Consider finding the conditional distribution of N given this information.

Explanation:

An incorrect solution goes as follows: Since we want the first 6 to appear before the first 5, we can eliminate 5 from our sample space and look at the reduced sample space conditional on no roll being a 5. There are 5 remaining values in this sample space that are equally probable, so the number of rolls needed to see a 6 conditional on this information is Geom(1/5), which has mean 5. Therefore, the answer is 5

This above solution is incorrect because it more implicitly assumes that we just
"ignore
" 5s that appear. Therefore, this is not the correct assumption we want to make.

Let's let N represent the number of rolls needed to see a 6, and let B be the event that 6 occurs before 5. By symmetry, $\mathbb{P}[B] = 1/2$. We now condition on the value of the first roll. Let R_i represent the event that we roll value i on the first roll. This would imply by Law of Total Expectation that

$$\mathbb{E}[N \mid B] = \sum_{i=1}^{6} \mathbb{E}[N \mid R_i, B]\mathbb{P}[R_i \mid B]$$

For $i = 1, 2, 3, 4$, we have that $\mathbb{P}[R_i \mid B] = \frac{P[B \mid R_i]\mathbb{P}[R_i]}{\mathbb{P}[B]}$. Since the event that we roll $i = 1, 2, 3, 4$ on the first roll doesn't affect the probability of appearance of 6 before 5, we have that $\mathbb{P}[B \mid R_i] = \mathbb{P}[B]$. Clearly $\mathbb{P}[R_i] = 1/6$ based on the fact the die is fair, so $\mathbb{P}[R_i \mid B] = 1/6$ for $i = 1, 2, 3, 4$.

We have that $\mathbb{P}[R_5 \mid B] = 0$, as there is no chance that our first roll can be 5 is 6 comes before 5. Thus, since the conditional probability measure is a probability measure, we have that $\mathbb{P}[R_6 \mid B] = 1 - 4 \cdot 1/6 = 1/3$.

If we roll a 6 on the first roll, then we get it in 1 roll, so $\mathbb{E}[N \mid B, R_6] = 1$. Otherwise, if $i = 1, 2, 3, 4$, then $\mathbb{E}[N \mid B, R_i] = 1 + \mathbb{E}[N \mid B]$, as we used up one roll and need to continue forward. Therefore, the equation we need to solve is

$$\mathbb{E}[N \mid B] = \frac{2}{3}\left(1 + \mathbb{E}[N \mid B]\right) + \frac{1}{3}$$

This is easily solved to yield $\mathbb{E}[N \mid B] = 3$.

You can instead view this problem as saying that whenever we roll a 5 or 6, we make it a 6 so that the 6 occurs before the 5. Therefore, the probability of rolling a 5 or 6 on each trial is 1/3, meaning that $N \mid B \sim \text{Geom}(1/3)$, which has mean 3. This agrees with the answer above.

Answer:

3

Reference:

https://www.quantguide.io/questions/show-the-face

Fixed Point Limit II

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

Let F_n be the number of fixed points of a random permutation $f : S_n \to S_n$, where $S_n = \{1, 2, \ldots, n\}$. Find $\lim_{n\to\infty} \mathbb{P}[F_n = 5]$. The answer is in the form $\frac{c}{e}$, where e is Euler's constant and c is a constant. Find c.

Hint:

Think about Poisson Limit Theorem and the approximate distribution of F_n for large n.

Explanation:

For large n, the distribution of $F_n \approx \text{Binom}(n, 1/n)$ because each of the n values has probability $1/n$ of being mapped to itself. However, each value being a fixed point are not independent events, as knowledge of one being fixed increases the probability of another being fixed. However, for large n, the change is so small that it is nearly independent. One can make this argument rigorous to show that the (formal) conditions of the Poisson Limit Theorem apply. Since Poisson Limit Theorem applies, we can conclude that as $n \to \infty$, $F_n \implies \text{Poisson}(1)$ (this means convergence in distribution). Therefore, $\mathbb{P}[F_n = 5]$ will converge to $\frac{1}{5!}e^{-5}$, so our answer is $\frac{1}{120}$.

Answer:

1120

Reference:

https://www.quantguide.io/questions/fixed-point-limit-ii

Fair Bounty I

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

You're on a game show! The host tells you that \$200 is located behind one of 7 doors. The other 6 doors have nothing behind them. You have no idea which door the money is behind. You keep selecting randomly until you find the money. You do not select any doors that you have selected prior. If each door opening costs $\$x$ and the game is fair, what is x?

Hint:

As the money is behind a random door, the number of selections needed to find the money is discrete uniform on $\{1, 2, \ldots, 7\}$.

Explanation:

As the money is behind a random door, the number of selections needed to find the money is discrete uniform on $\{1, 2, \ldots, 7\}$. Note that it is not geometric since the success probability changes each time. The expected number of selections needed to find the money is therefore 4. For the game to be fair, this means each draw must cost $x = \frac{200}{4} = 50$.

Answer:

50

Reference:

https://www.quantguide.io/questions/fair-bounty-i

Absolute Normal Difference

Category: Probability
Difficulty: Medium
Companies: Citadel, Drw

Question:

Suppose that $X \sim N(0,1)$ and $Y \sim N(0,4)$ are independent. Compute $\mathbb{E}[|Y-X|]$. The answer will be in the form $\left(\frac{K}{\pi}\right)^b$ for rational b and K. Find bK.

Hint:

Since X and Y are independent, we have that $Y-X \sim N(0,5)$. Therefore, if $W = Y-X \sim N(0,5)$, we want to compute $\mathbb{E}[|W|]$. Note that $W = \sqrt{5}Z$, where $Z \sim N(0,1)$, so we can really just compute $\sqrt{5}\mathbb{E}[|Z|]$.

Explanation:

Since X and Y are independent, we have that $Y-X \sim N(0,5)$. Therefore, if $W = Y-X \sim N(0,5)$, we want to compute $\mathbb{E}[|W|]$. Note that $W = \sqrt{5}Z$, where $Z \sim N(0,1)$, so we can really just compute $\sqrt{5}\mathbb{E}[|Z|]$. We can compute this via LOTUS for a general $Z \sim N(0,\sigma^2)$, as the calculation is the same:

$$
\begin{aligned}
E[|Z|] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}|x| \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \sqrt{\frac{2}{\pi\sigma^2}} \left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\Bigg|_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}\sigma,
\end{aligned}
$$

From line 1 to line 2, we use symmetry of the integrand about 0. Then, from line 2 to line 3, we apply a $u-$substitution to obtain that indefinite integral. In particular, $\sigma = \sqrt{5}$ here, as $\sigma^2 = 5$, so our answer is $\sqrt{\frac{10}{\pi}}$, meaning that $bK = 0.5 \cdot 10 = 5$.

Answer:

5

Reference:

https://www.quantguide.io/questions/absolute-normal-difference

Meek Mill

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG

Question:

Meek Mill is performing at Spring Fair. The administration does not know where he is, and they estimate there is an 80% chance he is in Philadelphia and 20% chance he is in Baltimore. If he is in Baltimore, there is a 80% chance he will perform. If he is in Philadelphia, there is a 10% chance he will perform. Find the probability that if Meek Mill performs, he came from Philadelphia.

Hint:

Let P be the event Meek Mill performs, L be the event he is in Philadelphia, and B be the event he is in Baltimore. You want $\mathbb{P}[L \mid P]$.

Explanation:

Let P be the event Meek Mill performs, L be the event he is in Philadelphia, and B be the event he is in Baltimore. We want

$$\mathbb{P}[L \mid P] = \frac{\mathbb{P}[PL]}{\mathbb{P}[L]} = \frac{\mathbb{P}[P \mid L]\mathbb{P}[L]}{\mathbb{P}[P \mid L]\mathbb{P}[L] + \mathbb{P}[P \mid B]\mathbb{P}[B]} = \frac{0.8 \cdot 0.1}{0.8 \cdot 0.1 + 0.2 \cdot 0.8} = \frac{1}{3}$$

All of the quantities are derived from the question itself by interpreting them in our language of events.

Answer:

13

Reference:

https://www.quantguide.io/questions/meek-mill

Close Couples

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

n married couples (n husbands and n wives) sit at a round table of $2n$ seats at random. The seating scheme is such that the seats must alternate between men and women. Find the number of couples that will sit together $\mathbb{E}[N]$ on average. Assume $n > 2$.

Hint:

Let I_i be the indicator that couple i sits together. How can you use the Linearity of Expectation to solve this problem?

Explanation:

Let I_i be the indicator that couple i sits together. Then we have that $T = \sum_{i=1}^{n} I_i$ counts the number of couples that sit together total. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[I_i]$. We have that $\mathbb{E}[I_i]$ is the probability couple i sits together. Fix the husband within couple i at an arbitrary seat in the table. The two people on either side of him must be women, as there is an alternating pattern at the table. Therefore, of the n spots that this man's wife can sit at, 2 are adjacent to him, and all arrangements are equally likely. Therefore, the probability of this is just $\frac{2}{n}$. Therefore, we have that $\mathbb{E}[I_i] = \frac{2}{n}$, so $\mathbb{E}[T] = \sum_{i=1}^{n} \frac{2}{n} = 2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/close-couples

Fair Bounty II

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

You're on a game show! The host tells you that \$200 is located behind 2 of 7 doors. The other 5 doors have nothing behind them. You have no idea which doors the money is behind. You keep selecting randomly until you find the first door with money behind it. You do not select any doors that you have selected prior. If each door opening costs \$$x$ and the game is fair, what is x?

Hint:

Use a "First Ace" approach, where the aces here are the doors with the money.

Explanation:

Thinking of the two doors with money as aces, we want to find the expected number of doors that need to be opened, say b, until we find the first door with money. Our answer would then be $\frac{200}{b}$, as the expected cost to find the first door with money is $b \cdot \frac{200}{b} = 200$.

With the doors having money fixed, there are 5 other empty doors. The two doors with money divide up our total selection into 3 equally-sized regions in expectation. Therefore, there are, on average, $\frac{5}{3}$ doors before the first door with money. We must add one in for the selection of the door with actual money. We get $b = \frac{8}{3}$, so

$$x = 200 \cdot \frac{3}{8} = 75$$

Answer:

75

Reference:

https://www.quantguide.io/questions/fair-bounty-ii

Longest Rope

Category: Statistics
Difficulty: Hard
Companies: Citadel, Five Rings, Two Sigma, Jane Street

Question:

A rope that is one meter long is divided into three segments by two random points. What is the expected length of the longest segment?

Hint:

Graph shading is a powerful technique that will be useful for this problem. Start by writing your constraints. Because each segment is equally likely to be the longest, the expected length of the longest piece doesn't depend on which piece we choose, so you are solving for $E[X|X \text{ is the longest}]$.

Explanation:

Let $X \in [0,1]$ be the location of the first cut and $Y \in [0,1]$ be the location of the second cut such that $X < Y$. Because each segment is equally likely to be the longest, the expected length of the longest piece doesn't depend on which piece we choose, so we can solve for $E[X|X \text{ is the longest}]$. This, with our earlier constraint of $X < Y$, gives us two additional constraints:

$$X > Y - X \Rightarrow Y < 2X$$

$$X > 1 - Y \Rightarrow Y > 1 - X$$

These two constraints derive from the fact that X is given to be greater than the other two segments, $Y - X$ and $1 - Y$. Graphing the constraints within our sample space, we see that the area satisfying our inequalities is the triangle A with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$, and $(\frac{1}{3}, \frac{2}{3})$, and triangle B with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$, and $(1, 1)$. We wish to find the probability of an outcome occurring in either areas and the expected value of X within each area.

$$\text{Area of A: } \frac{1}{2} \times \frac{1}{2} \times (\frac{1}{2} - \frac{1}{3}) = \frac{1}{24}$$

$$\text{Area of B: } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = \frac{3}{24} = 3 \times \text{Area of A}$$

$$\text{Expected value of X within A: } \frac{1}{2} - \frac{1}{3}(\frac{1}{2} - \frac{1}{3}) = \frac{4}{9}$$

$$\text{Expected value of X within B: } \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}$$

The expected value of X within each triangle is calculated using the property of centroids. A centroid, or the center of area for a triangle, is $\frac{1}{3}$ of the way from the base to the opposite vertex, a property that is most easily seen by integration. Finally, we can solve for the expected value of X given X is the longest piece:

$$E[X|X \text{ is the longest}] = \frac{1}{4} \times \frac{4}{9} + \frac{3}{4} \times \frac{2}{3} = \frac{11}{18}$$

Answer:

1118

Reference:

https://www.quantguide.io/questions/longest-rope

Colorless Sides

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Jane Street, HRT

Question:

A 333 cube that is colored red on the outside is cut into 27 111 smaller cubes. A random cube is selected from these 27 and you see five sides without any color. What is the probability that the cube has one colored side?

Hint:

Since we are updating probabilities given new information, try using Bayes Theorem.

Explanation:

Given that we have new information entering this problem (that we can see 5 sides are not painted), we need to update the probability that the chosen cube has a colored side. Thus, we have to use Bayes theorem. Let S be the event of the cube having exactly one colored side and B being the event that the five sides of the cube shown are blank. Then we have

$$\mathbb{P}[S \mid B] = \frac{\mathbb{P}[B \mid S] \cdot \mathbb{P}[S]}{\mathbb{P}[B]}$$

There's a $\frac{1}{6}$ chance that a one colored side cube has the painted side not showing. Thus $\mathbb{P}[B \mid S] = \frac{1}{6}$. And since there's 6 cubes that have only one painted face out of 27. Thus $\mathbb{P}[S] = \frac{6}{27} = \frac{2}{9}$. Finally, the probability we have a cube that shows five unpainted sides is $\frac{1}{6} \cdot \frac{2}{9}$ from the cubes with one painted side and $1 \cdot \frac{1}{27}$ from the center cube with no painted sides. Combining everything together we get

$$\mathbb{P}[S \mid B] = \frac{\mathbb{P}[B \mid S] \cdot \mathbb{P}[S]}{\mathbb{P}[B]} = \frac{\frac{1}{6} \cdot \frac{2}{9}}{\frac{1}{6} \cdot \frac{2}{9} + 1 \cdot \frac{1}{27}} = \frac{1}{2}$$

Answer:

12

Reference:

https://www.quantguide.io/questions/colorless-sides

Three Consecutive Heads

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Two Sigma, Jane Street

Question:

On average, how many fair coin tosses will it take to observe three consecutive heads?

Hint:

This can be done in a myriad of ways: Markov chains, recursive equations, Law of Total Probability, binary trees, and others.

Explanation:

Let x be the expected number of coin tosses it takes to observe three consecutive heads. On the first toss, there is a $\frac{1}{2}$ probability that you observe a tail and essentially start over with 0 consecutive heads, except that your expected number of coin tosses is $x+1$ since you wasted a toss observing a tail. There is a $\frac{1}{4}$ probability of receiving HT and essentially starting over with 0 consecutive heads, except that your expected number of coin tosses is $x+2$ since you wasted two tosses observing HT. There is a $\frac{1}{8}$ probability of receiving HHT and essentially starting over with 0 consecutive heads, except that your expected number of coin tosses is $x+3$ since you wasted three tosses observing tails HHT. Finally, there is a $\frac{1}{8}$ probability of receiving HHH and requiring three tosses. Thus, we can write this as:

$$x = \frac{1}{2}(x+1) + \frac{1}{4}(x+2) + \frac{1}{8}(x+3) + \frac{1}{8} \times 3x = 14$$

Answer:

14

Reference:

https://www.quantguide.io/questions/three-consecutive-heads

Tennis Gambling

Category: Probability
Difficulty: Medium
Companies: Citadel, De Shaw, SIG

Question:

The rules of tennis are the following: The scores go $0 - 15 - 30 - 40$ to start. If one player has 40 points and the other has less than 40 and the player with 40 points wins the next serve, this player wins the game. If both players have 40 points, the player who wins the next serve gets advantage. If the player with advantage wins again, they win the game. Otherwise, if the advantaged player loses, the game goes back to having no advantages (40-40). Imagine Rob and Bob are in a tennis match that starts out $30-30$. Rob has a probability 0.6 of winning each serve, independent between serves. Find the probability Rob wins the game.

Hint:

What needs to happen for a winner to occur?

Explanation:

The clever idea is to see that the winning player needs to obtain two consecutive wins. The probability that Rob obtains two consecutive wins is $0.6^2 = 0.36$. The probability that Bob obtains two consecutive wins is $0.4^2 = 0.16$. The probability that in the next two serves they each receive one win is $2 \cdot 0.6 \cdot 0.4 = 0.48$. If they each obtain one win in the next two serves, then the scenario is exactly the same as being tied and needing 2 consecutive wins on the next serves to win the game. Therefore, to get a winner, we just need one of the two players to obtain two consecutive wins. Accordingly, we can condition on the event that someone has obtained two consecutive wins. Given this, the probability the winner is Rob is $\dfrac{0.6^2}{0.6^2 + 0.4^2} = \dfrac{9}{13}$.

Answer:

913

Reference:

https://www.quantguide.io/questions/tennis-gambling

Normal Conditions

Category: Probability
Difficulty: Medium
Companies: Citadel, De Shaw, Two Sigma, Jane Street

Question:

Suppose that $X_1 \sim N(0, 9)$ and $X_2 \sim N(0, 16)$ are independent. Compute $\mathbb{E}[X_1 \mid X_1 + X_2 = 5]$.

Hint:

Compute the conditional distribution of $X_1 \mid X_1 + X_2 = 5$ and note that if $X_1 = x$, $X_2 = 5 - x$.

Explanation:

Let ϕ_1 and ϕ_2 be the PDFs of X_1 and X_2, respectively. We know that as X_1 and X_2 are independent, $X_1 + X_2 \sim N(0, 25)$, as we just add the variances. We now compute the conditional distribution of $X_1 \mid X_1 + X_2 = 5$. By the conditional PDF formula, we know that

$$f_{X_1 \mid X_1 + X_2 = 5}(x) = \frac{f_{X_1, X_1 + X_2}(x, 5)}{f_{X_1 + X_2}(5)}$$

The numerator here can be greatly simplified by noting that if $X_1 = x$, then $X_2 = 5 - x$ so that we get a sum of 5. Therefore, $f_{X_1, X_1 + X_2}(x, 5) = \phi_1(x)\phi_2(5 - x)$.

Plugging all of the PDFs in,

$$f_{X_1 \mid X_1 + X_2 = 5}(x) = \frac{\frac{1}{3\sqrt{2\pi}} e^{-\frac{x^2}{18}} \cdot \frac{1}{4\sqrt{2\pi}} e^{-\frac{(5-x)^2}{32}}}{\frac{1}{5\sqrt{2\pi}} e^{-\frac{25}{50}}}$$

After doing a good amount of simplification, you will note that $X_1 \mid X_1 + X_2 = 5 \sim N\left(\frac{9}{5}, \frac{144}{25}\right)$. In particular, this means the mean is $\frac{9}{5}$, and we are done. You can also reason this answer intuitively as you would expect the contribution from each normal to be in proportion with the variance, so we are really saying that $\frac{9}{9+16} \cdot 5$ comes from the normal with variance 9.

Answer:

95

Reference:

https://www.quantguide.io/questions/normal-conditions

Coin Flipper Ruin

Category: Probability
Difficulty: Easy
Companies: Citadel, Five Rings

Question:

Vishnu and Jason have \$1000 and \$500 in their respective bank accounts. They flip a fair coin repeatedly. If heads, Vishnu gives Jason \$1. Else, Jason gives Vishnu \$1. What is the probability that Jason runs out of money first?

Hint:

What is the expected value of Jason's account after n tosses?

Explanation:

Either Jason or Vishnu will run out of money first. Let p denote the probability that Jason runs out of money first. Naturally, $1 - p$ is Vishu's probability of running out of money first, leaving Jason with 1500. After n coin tosses, Jason should expect to neither lose nor gain any money, since the coin is fair; in other words, Jason should expect to have \$500 in his account after n tosses. By the end of the game—let's say, after the n_{end}-th toss—the expected value of Jason's account is $p \cdot 0 + (1-p) \cdot 1500$. This value must be equal to 500, as discussed previously. Solving for p, we find

$$p \cdot 0 + (1 - p) \cdot 1500 = 500$$

$$1500p = 1000$$

$$p = \frac{2}{3}$$

Answer:

23

Reference:

https://www.quantguide.io/questions/coin-flipper-ruin

Expected Returns

Category: Probability
Difficulty: Hard
Companies: Citadel

Question:

A frog performs a simple symmetric random walk on the integers, starting at position 5 and hopping 1 unit up or down at each step with equal probability. Find the expected of number of times that the frog lands on position 1000 before landing on position 0.

Hint:

If a random walk starts at integer position a and $0 < a < b$, b being an integer, then the probability that the random walk hits b before 0 is $\frac{a}{b}$

Explanation:

We are going to use one important fact about simple symmetric random walks here repeatedly: If a random walk starts at integer position a and $0 < a < b$, b being an integer, then the probability that the random walk hits b before 0 is $\frac{a}{b}$. This can be proved using martingale theory.

With this fact equipped, if N is the number of visits to 1000 before 0, we are going to calculate $\mathbb{P}_5[N = k]$, where $\mathbb{P}_a$ denotes that we are calculating starting at position a. First, we can see by the above fact that $\mathbb{P}_5[N = 0] = \frac{199}{200}$. This is because the probability we reach 1000 before 0 is $\frac{5}{1000} = \frac{1}{200}$, so the probability we reach 0 before 1000 is the complement of this.

Now, to calculate $\mathbb{P}_5[N = 1]$, we must reach 1000 and then go back to 0. The probability of reaching 1000 before 0 is $\frac{1}{200}$. Then, once we are at position 1000, we must return down to 0 and not visit back. This means that we must go down from 1000 to 999 the moment we hit position 1000, occurring with probability $\frac{1}{2}$. Then, afterwards, starting from position 999, we must hit 0 before 1000. This is just $\frac{1}{1000}$ by the same complementation idea above, so our total probability is $\frac{1}{200} \cdot \frac{1}{2} \cdot \frac{1}{1000} = \frac{1}{200} \cdot \frac{1}{2000}$.

To calculate $\mathbb{P}_5[N = 2]$, we must reach 1000, return to 1000 one time, and then go back to 0. The probability of reaching 1000 before 0 starting from position 5 is $\frac{1}{200}$. Then, to return to 1000 once, we condition on the first step once we are at 1000. If we go up, which occurs with probability $\frac{1}{2}$, then we will reach 1000 before going to 0 with probability 1. If we go down, which occurs with probability $\frac{1}{2}$, we will return to 1000 from 999 with probability $\frac{999}{1000}$. Therefore, the total probability of returning to 1000 once is $\frac{\frac{999}{1000} + 1}{2} = \frac{1999}{2000}$. After we return once to 1000, from there, we need to go back to 0 before returning again. This probability is $\frac{1}{2000}$ from before. Therefore,

$\mathbb{P}_5[N=1] = \frac{1}{200} \cdot \frac{1999}{2000} \cdot \frac{1}{2000}$.

With all of this equipped, we will now calculate $\mathbb{P}_5[N=k]$. This means that we first need to hit 1000 once, which occurs with probability $\frac{1}{200}$. From there, we must return to 1000 exactly $k-1$ times before hitting 0. This occurs with probability $\left(\frac{1999}{2000}\right)^{k-1}$, as it was $\frac{1999}{2000}$ for returning exactly once, so every time we hit 1000, by the Strong Markov Property, we can view the future independent of the past, and we do this $k-1$ times. Lastly, after this, we need to hit 0 before 1000, which occurs with probability $\frac{1}{2000}$. Therefore, for integers $k > 0$,

$$\mathbb{P}_5[N=k] = \frac{1}{200} \cdot \left(\frac{1999}{2000}\right)^{k-1} \cdot \frac{1}{2000}$$

In other words, we can say that $N = 0$ with probability $\frac{199}{2000}$ and $N \sim \text{Geom}\left(\frac{1}{2000}\right)$ with probability $\frac{1}{200}$.

Therefore

$$\mathbb{E}_5[N] = \mathbb{E}_5[N \mid N=0]\mathbb{P}_5[N=0] + \mathbb{E}_5[N \mid N>0]\mathbb{P}_5[N>0]$$

The first term is clearly just 0 due to the expected value part. $\mathbb{E}_5[N \mid N > 0] = 2000$ as that is the mean of a $\text{Geom}\left(\frac{1}{2000}\right)$ random variable. Lastly, we know the last part is just $\frac{1}{200}$ from our prior calculations, so $\mathbb{E}_5[N] = 10$.

Answer:

10

Reference:

https://www.quantguide.io/questions/expected-returns

Fish Slice

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

Noelle is preparing a tuna fish of length L in anticipation of dinner. She selects 2 cutting points uniformly at random and independently along the length of the fish so that the cutting points are on opposite sides of the midpoint of the fish. Find the probability that the distance between the two points Noelle selects is greater than $\frac{L}{3}$.

Hint:

Let X be the random variable representing the location of the first cutting point and Y be the random variable representing the location of the second cutting point. Restrict to the region where X is to the left and Y is to the right. What are the distributions of X and Y?

Explanation:

Let X be the random variable representing the location of the first cutting point and Y be the random variable representing the location of the second cutting point. Since X and Y are equally likely to be on either side of the midpoint due to symmetry, we are going to consider the case of X being left of the midpoint and Y to the right. Since X is to the left, $X \sim \text{Unif}[0, \frac{L}{2}]$. Similarly, $Y \sim \text{Unif}[\frac{L}{2}, L]$. We want to find $\mathbb{P}[|X - Y| \geq \frac{L}{3}]$. Since the joint PDF is uniform over the indicator region, we can just take ratios of areas to find the probability. If you are to sketch out the region of interest and then the region in which $|X - Y| \geq \dfrac{L}{3}$, you will receive a region whose complement is a right triangle whose sides are of length $L/3$. Therefore, the ratio of the areas is $\frac{\frac{L^2}{18}}{\frac{L^2}{4}} = \dfrac{2}{9}$, so the probability of the complement is 2/9, meaning our probability of interest is 7/9.

Answer:

79

Reference:

https://www.quantguide.io/questions/fish-slice

2 Lead

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG

Question:

Alice and Bob are playing basketball. Both start from 0 points. Each game, Alice has a 30% chance of winning, independent between games. In each game, the winner gains 1 point and the loser loses 1 point. The first person to 2 points wins the set. Find the probability Alice wins the set.

Hint:

We can view this is a standard Gambler's Ruin problem that is non-symmetric.

Explanation:

We can view this is a standard Gambler's Ruin problem that is non-symmetric. In this case, $p = 0.3, q = 1 - p = 0.7$, and $n = 2$. The formula for the probability that Alice wins is commonly known to be

$$\frac{1 - \left(\frac{q}{p}\right)^n}{1 - \left(\frac{q}{p}\right)^{2n}}$$

We know that $\frac{q}{p} = \frac{7}{3}$, so plugging this all in, we have that our answer is

$$\frac{1 - \frac{49}{9}}{1 - \frac{2401}{81}} = \frac{81 - 441}{81 - 2401} = \frac{9}{58}$$

Another way to see this without using the machinery above is to note that we just need to see a sequence of two consecutive wins from a player to win, and this occurs with probability $\left(\frac{3}{10}\right)^2 = \frac{9}{100}$ for Alice and $\left(\frac{7}{10}\right)^2 = \frac{49}{100}$ for Bob. As these are disjoint events in each set of two turns, the probability it occurs for Alice first is $\frac{9/100}{9/100 + 49/100} = \frac{9}{58}$.

Answer:

958

Reference:

https://www.quantguide.io/questions/2-lead

View All Sides

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG, Drw, Two Sigma, Jane Street, Old Mission, HRT

Question:

On average, how many times do you need to roll a standard fair 6−sided die to observe all of the sides?

Hint:

Let T_i be the number of rolls needed to observe the ith distinct side given that $(i-1)$ distinct sides have already appeared. Then $T = T_1 + \cdots + T_6$ gives the total number of rolls needed to observe all of the sides of the die.

Explanation:

Let T_i be the number of rolls needed to observe the ith distinct side given that $(i-1)$ distinct sides have already appeared. Then $T = T_1 + \cdots + T_6$ gives the total number of rolls needed to observe all of the sides of the die. We know that $T_1 = 1$, as the first roll always yields a side not seen yet. We also have that $T_2 \sim \text{Geom}(5/6)$, as 5 of the 6 remaining sides are not observed yet. More generally, $T_i \sim \text{Geom}\left(\frac{7-i}{6}\right)$, as $7-i$ of the sides have not been observed yet. Thus, $\mathbb{E}[T_i] = \frac{6}{7-i}$. By linearity of expectation, we get that

$$\mathbb{E}[T] = \sum_{i=1}^{6} \frac{6}{7-i} = 6\sum_{i=1}^{6} \frac{1}{i} = 14.7$$

The equality between sums occurs by re-indexing the sum

Answer:

14.7

Reference:

https://www.quantguide.io/questions/view-all-sides

Queen First

Category: Probability
Difficulty: Easy
Companies: Citadel, SIG

Question:

Hannah shuffles a standard deck of cards. She wins $100 if the last queen appears before the last king. Her friend Alissa looks at the card order and tells Hannah that there are exactly two queens before the first king. Given this additional information, how much should Hannah expect to earn?

Hint:

We only need to consider possible orderings of queens and kings between 8 arbitrary spots.

Explanation:

Suppose that we are given that all the queens and all the kings appear in the $x_1, x_2, \ldots, x_8$-th spots, where $1 \leq x_1 < x_2 < \cdots < x_8 \leq 52$. Each possible ordering of the queens and kings among the 8 spots occurs with the same probability. Hence, we only need to consider possible orderings of queens and kings between 8 arbitrary spots.

Since we are given that the first 2 cards are queens, and the third card is a king, our problem becomes: how many different orderings of the remaining 5 cards (3 kings and 2 queens) end with a king. This probability is simply $\frac{3}{5}$, since there is a $\frac{3}{5}$ chance that the last card is a king. Putting it all together, we find that Hannah's total expected earnings is $\frac{3}{5} \cdot 100 = 60$.

Answer:

60

Reference:

https://www.quantguide.io/questions/queen-first

Archery Accuracy

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

An archer is shooting at a dartboard. If she has a $\frac{3}{4}$ chance of hitting any given shot, find the probability that she hits at least one of her next three shots.

Hint:

The complementary event is that she misses all three of her shots.

Explanation:

The complementary event is that she misses all three of her shots. This occurs with probability $\frac{1}{4}$ per shot, so the probability she misses all of the next three shots is $\frac{1}{4^3} = \frac{1}{64}$. Thus, the probability she makes at least one of the next three shots is $1 - \frac{1}{64} = \frac{63}{64}$.

Answer:

6364

Reference:

https://www.quantguide.io/questions/archery-accuracy

Sum Exceedance IV

Category: Probability
Difficulty: Hard
Companies: Citadel, Jane Street, HRT

Question:

A fair $5-$sided die with the values $1-5$ on the sides is rolled repeatedly until the sum of all upfaces is at least 5. Find the expected number of times the die needs to be rolled.

Hint:

Let's denote the expected number of rolls needed to obtain a sum of at least n starting from a sum of k by E_k. Clearly we have that $E_n = 0$, as we already have a sum of n. Further, we have that $E_{n-1} = 1$, as no matter what is obtained, we have a sum of at least n. This is the more general version, and you can plug in $n = 5$ to get the answer here.

Explanation:

We are going to solve the generalized version for $n-$sided die with a sum of at least n. Let's denote the expected number of rolls needed to obtain a sum of at least n starting from a sum of k by E_k. Clearly we have that $E_n = 0$, as we already have a sum of n. Further, we have that $E_{n-1} = 1$, as no matter what is obtained, we have a sum of at least n.

Then, we have that $E_{n-2} = 1 + \frac{1}{n}E_{n-1} = 1 + \frac{1}{n}$, as with probability $\frac{1}{n}$, we obtain the value 1 and we have a total sum of $n-1$. Similarly, $E_{n-3} = 1 + \frac{1}{n}E_{n-1} + \frac{1}{n}E_{n-2} = 1 + \frac{2}{n} + \frac{1}{n^2}$. By continuing with this pattern, one can prove by induction that

$$E_{n-k} = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}}{n^j}$$

Therefore, by the Binomial Theorem, $E_0 = E_{n-n} = \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{n}\right)^j = \left(1 + \frac{1}{n}\right)^{n-1}$ We applied the Binomial Theorem with $x = \frac{1}{n}$ and $y = 1$. Therefore, for $n = 5$, our answer is $\frac{6^4}{5^4} = \frac{1296}{625}$.

Answer:

1296625

Reference:

https://www.quantguide.io/questions/sum-exceedance-iv

Forming A Triangle

Category: Probability
Difficulty: Hard
Companies: Citadel, De Shaw, Jane Street

Question:

A stick of unit length is randomly broken into three pieces. Assuming each break follows a uniform distribution along the stick, what is the probability that the three segments can form a triangle?

Hint:

Try to graph the sample space in 2D space. Let $x \in [0, 1]$ be the first break and $y \in [0, 1]$ be the second break. What points in space do and do not work? How can you represent this algebraically?

Explanation:

Let x be the first break and y be the second break such that $x < y$. Thus, the lengths of the three segments are x, $y - x$, and $1 - y$. As you recall from geometry, in order for three side lengths to form a triangle, each side length must be less than the sum of the other two side lengths. We can rewrite this as:

$$x < (y - x) + (1 - y) \Rightarrow x < \frac{1}{2}$$

$$y - x < x + (1 - y) \Rightarrow y < x + \frac{1}{2}$$

$$1 - y < x + (y - x) \Rightarrow y > \frac{1}{2}$$

These constraints (including $x < y$) cover $\frac{1}{8}$ of the sample space of $x, y \in [0, 1]$, which can be seen visually. Note that the $x < y$ constraint only accounts for half of the possibilities since x is equally likely to be greater than or less than y. The final answer is $2 \times \frac{1}{8} = \frac{1}{4}$.

Answer:

14

Reference:

https://www.quantguide.io/questions/forming-a-triangle

Points On A Circle II

Category: Probability
Difficulty: Medium
Companies: Citadel, De Shaw, SIG, Jane Street

Question:

n points are selected randomly at uniform around a circle. What is the probability that all n points are on the same semicircle for $n = 100$? The answer is in the form

$$\frac{a}{b^c}$$

for integers $a, b, c > 0$ with b minimal. Find $a + b + c$.

Hint:

Suppose we already have the n points selected on the circle. Choose one of those points, let's call it A. We draw a line through A and O, the center of the circle; this diameter forms two semicircles. To prevent overcounting, only consider the semicircle that start from A in the counterclockwise direction.

Explanation:

Suppose we already have the n points selected on the circle. Choose one of those points, let's call it A. We draw a line through A and O, the center of the circle; this diameter forms two semicircles. To prevent overcounting, we will only consider the semicircle that start from A in the counterclockwise direction. Then, each of the $n-1$ remaining points has a $\frac{1}{2}$ chance of being within that semicircle. Repeating this for all n points, we find that the solution is simply $\frac{n}{2^{n-1}}$. Plugging in $n = 100$, we find our answer to be $\frac{100}{2^{99}}$. Therefore, the answer to our question is $100 + 2 + 99 = 201$.

Answer:

201

Reference:

https://www.quantguide.io/questions/points-on-a-circle-ii

Nonuniform Fix

Category: Probability
Difficulty: Medium
Companies: Citadel, Drw, HRT

Question:

Let $T, X_1, X_2, \cdots \sim \text{Beta}(12, 8)$ IID. Then, let $N = \min\{n \in \mathbb{N} : X_n > T\}$. Find $\mathbb{E}[N]$. If this is not finite, enter -1.

Hint:

Condition on $T = t$ and use Law of Total Expectation.

Explanation:

We can generalize this to any non-negative continuous distribution $F(x)$ with PDF $f(x)$. Suppose $T = t$. Then on any trial, there is a probability $1 - F(t)$ of $X_i > t$. Therefore, $N \mid T = t \sim \text{Geom}(1 - F(t))$. Using the Law of Total Expectation, we get

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid T]] = \mathbb{E}\left[\frac{1}{1 - F(T)}\right] = \int_0^\infty \frac{f(t)}{1 - F(t)} dt = -\ln(1 - F(t))\Big|_0^\infty = \infty$$

This means the answer is -1.

Answer:

1

Reference:

https://www.quantguide.io/questions/nonuniform-fix

Better In Red IV

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Jane Street, HRT

Question:

The surfaces of a $3 \times 3 \times 3$ cube (initially white) are painted red and then is cut up into 27 $1 \times 1 \times 1$ small cubes. One of the cubes is selected uniformly at random and is rolled. The face appearing is red. What is the probability the cube selected was a corner cube?

Hint:

We are conditioning our sample space to the cubes with at least one red side. How many total red sides are there?

Explanation:

We are conditioning our sample space to the cubes with at least one red side. There are 54 total sides that are red on the cube, as each face of the larger cube has area 9. There are 8 corner cubes that each have 3 red painted faces. These contribute 24 of the 54 total red faces. Therefore, the answer is just $\frac{24}{54} = \frac{4}{9}$.

Answer:

49

Reference:

https://www.quantguide.io/questions/better-in-red-iv

Uniform Fix

Category: Probability
Difficulty: Easy
Companies: Citadel, Drw, HRT

Question:

Let $U \sim \text{Unif}(0,1)$ and $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Define $N = \min\{n \in \mathbb{N} : X_n > U\}$. Find $\mathbb{E}[N]$. Enter -1 if the answer is infinite.

Hint:

The key here is to condition on U. What is the conditional distribution of $N \mid U = u$?

Explanation:

The key here is to condition on U. This is because of the fact that $N \mid U = u \sim \text{Geom}(1-u)$, as each trial has probability $1-u$ of being larger than u. Therefore, $\mathbb{E}[N \mid U] = \dfrac{1}{1-U}$ by known formulas about geometric random variables. Using the Law of Total Expectation and LOTUS, we have that

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid U]] = \mathbb{E}\left[\frac{1}{1-U}\right] = \int_0^1 \frac{du}{1-u} = \infty$$

Therefore, you should enter in -1.

Answer:

1

Reference:

https://www.quantguide.io/questions/uniform-fix

First And Last Heads

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

A fair coin is flipped 5 times. Find the probability that the number of heads in the first 3 flips is equal to the number of heads in the last 2 flips.

Hint:

Condition based on the number of heads in both segments.

Explanation:

We condition based on the number of heads in both segments. The only way to get 0 heads in both segments is $TTTTT$, so that yields 1 outcome. To get one head in both segments, there are 3 initial sequences in the first 3 flips that have 1 head (namely, HTT, THT, and TTH) and 2 sequences in the last two flips that have 1 head (HT and TH). This yields $3 \cdot 2 = 6$ outcomes here. Lastly, to get two heads in each segment, there are 3 initial sequences in the first 3 flips that have 2 heads (namely, HHT, HTH, and THH). There is only one sequence with 2 heads in the last two flips. Therefore, this yields $3 \cdot 1 = 3$ sequences. Adding these all up, we get $1 + 6 + 3 = 10$ sequences where this is satisfied. There are $2^5 = 32$ total sequences of 5 flips, so our answer is

$$\frac{10}{32} = \frac{5}{16}$$

Answer:

516

Reference:

https://www.quantguide.io/questions/first-and-last-heads

Corner Meet

Category: Probability
Difficulty: Medium
Companies: Citadel

Question:

You have a 5×5 checkerboard. You randomly place a checker in the center 3×3 grid. The checkers can move in two equally random ways: up to the left, and up to the right. If the checker is on the boundary, then it can only make valid moves. What is the probability that the checker ends up in the top-left or top-right corner?

Hint:

Look at the probability of reaching a corner from each of the center squares in the 3×3 grid. Consider the top-left inner square, for example. We see that we have probability $\frac{1}{2}$ of immediately reaching the left corner and probability $\frac{1}{2}$ of reaching the top-center, in which case there's no chance of reaching a corner.

Explanation:

Let's look at the probability of reaching a corner from each of the center 3 x 3 grid. We can see that there are 5 squares that have some possibility of reaching the corner. We can use law of total probability. First, we have a equal chance of selecting each of the 9 inner squares. From here, we can look at the probability of reaching a corner from each square. This can be done in a recursive manner, working from top to bottom.

First, let's look at the top-left square in the inner 3 x 3 grid. We see that we have probability $\frac{1}{2}$ of immediately reaching the left corner and probability $\frac{1}{2}$ of reaching the top-center, in which case there's no chance of reaching a corner. Hence, we have overall probability $\frac{1}{2}$. By symmetry, the top-right square has the same probability. We can see that the top-center square of the 3 x 3 grid has zero probability of reaching a corner. We can iterate downwards, and see that we can calculate probabilities of the lower squares using the probabilities that we just calculated.

This gives us

$$\frac{1}{9}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)=\frac{5}{18}$$

Answer:

518

Reference:

https://www.quantguide.io/questions/corner-meet

Repetitious Game II

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG

Question:

Audrey repeatedly draws cards from a standard 52-card deck with replacement. Compute the expected number of draws for Audrey to get at least one card from each suit.

Hint:

Let X_1 denote the number of draws it takes for Audrey to draw a card from a unique suit. Let X_2 denote the number of draws it takes for Audrey to draw a card from a second unique suit after a card from the first unique suit has been draw. We'll define X_3 and X_4 in a similar way for the third and fourth unique suit draws, respectively. Then, the number of total draws it takes for Audrey to get at least one card from each suit is $Z = X_1 + X_2 + X_3 + X_4$.

Explanation:

Let X_1 denote the number of draws it takes for Audrey to draw a card from a unique suit. Let X_2 denote the number of draws it takes for Audrey to draw a card from a second unique suit after a card from the first unique suit has been draw. We'll define X_3 and X_4 in a similar way for the third and fourth unique suit draws, respectively. Then, the number of total draws it takes for Audrey to get at least one card from each suit is $Z = X_1 + X_2 + X_3 + X_4$. By the Linearity of Expectation,

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + X_2 + X_3 + X_4]$$

$$= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4].$$

Consider X_1. $\mathbb{P}(\mathbb{X}_{\nVdash} = \nVdash) = 1$, since no matter what card is drawn first, it will be from a unique suit. So, $\mathbb{E}[X_1] = 1$

Next, consider X_2. Since one suit has already been accounted for, three suits remain. The probability that a drawn card belongs to one of the three unaccounted for suits is $\frac{3}{4}$. Hence,

$$X_2 \sim \text{Geo}\left(\frac{3}{4}\right)$$

$$\Rightarrow \mathbb{E}[X_2] = \frac{4}{3}.$$

Following this reasoning for X_3 and X_4, we find

$$X_3 \sim \text{Geo}\left(\frac{2}{4}\right)$$

$$\Rightarrow \mathbb{E}[X_3] = \frac{4}{2}, \quad \text{and}$$

$$X_4 \sim \text{Geo}\left(\frac{1}{4}\right)$$

$$\Rightarrow \mathbb{E}[X_4] = \frac{4}{1}.$$

Substituting, we conclude

$$\mathbb{E}[Z] = 1 + \frac{4}{3} + \frac{4}{2} + \frac{4}{1}$$

$$= \frac{25}{3}.$$

Answer:

253

Reference:

https://www.quantguide.io/questions/repetitious-game-ii

Basic Dice Game II

Category: Probability
Difficulty: Medium
Companies: Citadel, SIG, Drw, Jane Street, HRT

Question:

A casino offers you a game with a six-sided die where you are paid the value of the roll. The casino lets you roll the first time. If you are happy with your roll, you can cash out. Else, you can choose to roll a second time. If you are happy with your roll, you can cash out on this second value. Else, you can choose to roll for your third and final time and cash out on this third value. What is the fair value of this game?

Hint:

From the expected value of the last roll, you know the expected value of the second roll. Based on the expected value of the second roll, you can understand when you want to re-roll and find your expected value at the start of the game.

Explanation:

The fair value of the last roll is 3.5, and thus you should only opt to roll a third time if your second roll is less than 3.5. Your expected value for the second roll is then $\frac{3}{6} \times 5 + \frac{3}{6} \times 3.5 = 4.25$. Similarly, you should only opt to roll your second time if the first roll is less than 4.25. Your expected value for the first roll is thus $\frac{2}{6} \times 5.5 + \frac{4}{6} \times 4.25 \approx 4.67$.

Answer:

143

Reference:

https://www.quantguide.io/questions/basic-dice-game-ii

Redblue Die Match

Category: Probability
Difficulty: Easy
Companies: Citadel

Question:

You have three fair $6-$sided dice in front of you. One of the dice is red, while the other two are blue. Find the probability exactly one of the blue dice matches the red die in value.

Hint:

Let the value of the red die be arbitrary. There are 2 ways to pick the blue die that matches it in value.

Explanation:

Let the value of the red die be arbitrary. There are 2 ways to pick the blue die that matches it in value. The probability of a match is $\frac{1}{6}$. The probability the other die does not match the red die is $\frac{5}{6}$, so the total probability of exactly one match is $2 \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{18}$.

Answer:

518

Reference:

https://www.quantguide.io/questions/redblue-die-match

Determination II

Category: Statistics
Difficulty: Hard
Companies: De Shaw, Two Sigma

Question:

Given three data sets X_1, X_2, and Y , we run two linear regressions to obtain $y \sim \alpha_1 + \beta_1 x_1$ and $y \sim \alpha_2 + \beta_2 x_2$. The R^2 value for both regressions is 0.05. Find the lowest upper bound on R^2 value of the regression $y \sim \alpha + \beta' x_1 + \beta'' x_2$.

Hint:

Consider the dataset Y that is sampled perfectly from a parabola far from the origin.

Explanation:

Consider the dataset Y that is sampled perfectly from a parabola far from the origin. Let X_2 be the dataset where each value of X_1 is squared. If Y is far enough from the origin, $y \sim \alpha_1 + \beta_1 x_1$ has low R^2 since it is linear, while Y is quadratic. Furthermore, $y \sim \alpha_2 + \beta_2 x_2 = \alpha_2 + \beta_2 x_1^2$ is also low R^2 because although it is a parabola, it is not able to be shifted around horzontally to match Y. Therefore, individually, these linear regressions have low R^2. However, the model $y \sim \alpha + \beta' x_1 + \beta'' x_2$ has $R^2 = 1$, as you can now shift the parabola around in the plane to perfectly match the non-noisy dataset Y.

Answer:

1

Reference:

https://www.quantguide.io/questions/determination-ii

Diagonal Eigenvalue

Category: Pure Math
Difficulty: Medium
Companies: De Shaw, Akuna

Question:

Consider the matrix 30×30 matrix A with $A_{ii} = 30$ for $1 \leq i \leq 30$ and $A_{ij} = 1$ for $i \neq j$. Let $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ be the vector of distinct eigenvalues $\lambda_1 \neq \lambda_2$ of A and $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ be the vector of geometric multiplicities corresponding to λ_1 and λ_2, respectively. Find $||\lambda + g||^2$

Hint:

A fact from Linear Algebra states that if M is $n \times n$ with eigenvalues (repeated according to multiplicity) $\{\lambda_i\}_{i=1}^n$, then the eigenvalues of $M + kI_n$ is $\{\lambda_i + k\}_{i=1}^n$. Note that we can write $A = 1_{30} + 29I_{30}$, where 1_{30} is the 30×30 matrix of all ones.

Explanation:

A fact from Linear Algebra states that if M is $n \times n$ with eigenvalues (repeated according to multiplicity) $\{\lambda_i\}_{i=1}^n$, then the eigenvalues of $M + kI_n$ is $\{\lambda_i + k\}_{i=1}^n$. Note that we can write $A = 1_{30} + 29I_{30}$, where 1_{30} is the 30×30 matrix of all ones.

To find the eigenvalues of 1_{30}, we can simply note that every row is dependent on the first, so the geometric multiplicity of the eigenvalue 0 is 29. Since the sum of the eigenvalues is the trace, we can see that the algebraic multiplicity is also 29 and that the last eigenvalue must be 30, as $\text{tr}(1_{30}) = 30$. Therefore, the eigenvalues of A are 29 with geometric multiplicity 29 and $30 + 29 = 59$ with geometric multiplicity 1.

This means that $\lambda + g = \begin{bmatrix} 58 \\ 60 \end{bmatrix}$, so $||\lambda + g||^2 = 58^2 + 60^2 = 6964$

Answer:

6964

Reference:

https://www.quantguide.io/questions/diagonal-eigenvalue

Car Question I

Category: Probability
Difficulty: Hard
Companies: De Shaw

Question:

Suppose that we have 2 cars parked in a line occupying spaces 1 and 2 of a parking lot. Spaces 3 and 4 are initially empty. Every minute, a car is considered eligible to move forward one space if a) the space in front of them is empty prior to any potential moves and b) label of the space they are on is less than 4. Suppose that every minute, if a car is eligible to move, it moves forward one space with probability 1/2. What is the expected amount of minutes before the cars occupy spaces 3 to 4?

Note: The space in front must be empty before any moves. For example, on the first turn, only the car in space 2 is eligible to move.

Hint:

Let E_{ij} represent the expected number of minutes before the cars are in spaces 3 and 4 when the cars are in spaces $i < j$. Clearly $E_{34} = 0$, as we already have reached our goal. We want E_{12}. The number of turns needed for a car to move forward to the next spot is $N \sim \text{Geom}(1/2)$. The first step is that we need the car in space 2 to move to space 3. This takes 2 minutes on average, so $E_{12} = 2 + E_{13}$. Use the result of First Flip to say $\mathbb{E}[X \mid X < Y] = \mathbb{E}[X \mid X = Y] = \frac{4}{3}$, where $X, Y \sim \text{Geom}(1/2)$ IID.

Explanation:

Let E_{ij} represent the expected number of minutes before the cars are in spaces 3 and 4 when the cars are in spaces $i < j$. Clearly $E_{34} = 0$, as we already have reached our goal. We want E_{12}. The number of turns needed for a car to move forward to the next spot is $N \sim \text{Geom}(1/2)$. The first step is that we need the car in space 2 to move to space 3. This takes 2 minutes on average, so $E_{12} = 2 + E_{13}$.

Now, from this state, there are three possibilities that occur with equal probability: the car in space 1 moves before the car in space 3, the car in space 3 moves before the car in space 1, or the cars both move at the same time. We will use Law of Total Expectation to condition on which of the cars moves first from this state. You can prove these are equally probable by computing $\mathbb{P}[X = Y]$, where $X, Y \sim \text{Geom}(1/2)$ IID and using the exchangeability of the random variables. Furthermore, we use the result of First Flip to say $\mathbb{E}[X \mid X < Y] = \mathbb{E}[X \mid X = Y] = \frac{4}{3}$. Taking these for granted, let's compute the conditional expectation in each case.

Space 1 Moves Before Space 3: In this case, we reach a state where the cars are in spaces 2 and 3. By the first fact above, it takes an average of $\frac{4}{3}$ minutes for the first car to move conditioned on this. Then, the car in space 3 needs to move to space 4, taking an average of 2 minutes. Lastly, the car in space 2 needs to move to space 3, taking an average of 2 minutes as well. The expected time in this case to reach the end is $\frac{4}{3} + 2 + 2 = \frac{16}{3}$.

Space 3 Moves Before Space 2: In this case, we reach a state where the cars are in spaces 1 and 4. By the first fact above, it takes an average of $\frac{4}{3}$ minutes for the car in space 3 to move conditioned on this. Then, the car in space 1 needs to move to space 3, which means it must move from space 1 to 2 and then 2 to 3. This takes an average of 2 minutes per space, meaning the expected time to reach the end in this case is $\frac{4}{3} + 4 = \frac{16}{3}$.

Both Move Simultaneously: In this case, we reach a state where the cars are in spaces 2 and 4. It again takes an average of $\frac{4}{3}$ minutes for the cars to move conditioned on this. Then, all that needs to occur is a movement of the car in space 2 to space 3, which takes 2 minutes on average. Therefore, the expected time to reach the end in this case is $\frac{4}{3} + 2 = \frac{10}{3}$.

By the Law of Total Expectation, $E_{13} = \frac{1}{3}\left(\frac{16}{3} + \frac{16}{3} + \frac{10}{3}\right) = \frac{14}{3}$. Therefore, by our initial equation $E_{12} = 2 + E_{13} = \frac{20}{3}$.

Answer:

203

Reference:

https://www.quantguide.io/questions/car-question-i

60 Heads

Category: Probability
Difficulty: Easy
Companies: De Shaw, Akuna, SIG, Optiver

Question:

A fair coin is tossed 100 times. Using the Central Limit Theorem without continuity correction, estimate the probability that at least 60 heads are observed. Round to 5 digits after the decimal point.

Hint:

The number of heads that appear is $X \sim \text{Binom}(100, 1/2)$. What is the mean and variance of this?

Explanation:

The number of heads that appear is $X \sim \text{Binom}(100, 1/2)$. The mean and variance of X are, respectively, $100 \cdot 1/2 = 50$ and $100 \cdot 1/2 \cdot (1 - 1/2) = 25$. Therefore, as X is the sum of 100 IID Bernoulli(1/2) random variables, we can approximate X well by $M \sim N(50, 25)$ via Central Limit Theorem. Thus, we want to calculate

$$\mathbb{P}[M \geq 60] = \mathbb{P}\left[\frac{M - 50}{5} \geq \frac{60 - 50}{5}\right] = \mathbb{P}\left[\frac{M - 50}{5} \geq 2\right]$$

The LHS of the last term is now standard normal, so our approximation is $1 - \Phi(2) = \Phi(-2) \approx 0.02275$

Answer:

0.02275

Reference:

https://www.quantguide.io/questions/60-heads

Sum Exceedance II

Category: Probability
Difficulty: Hard
Companies: De Shaw, Akuna, HRT

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID and $N_2 = \min\{n : X_1 + \cdots + X_n > 2\}$. Find $\mathbb{E}[N_2]$. Your answer will be in the form $ae^2 + be$ for integers a and b. e here is Euler's constant. Find $a + b$.

Hint:

Write a delayed differential equation that $f(x) = \mathbb{E}[N_x]$ should satisfy on $(1,2)$. You can obtain this by using Law of Total Expectation on X_1 and differentiating the result. What is a reasonable initial condition for $f(1)$? You may also want to recall $f(x) = e^x$ on $(0,1)$.

Explanation:

We are going to use the result of Sum Exceedance I that states that for any $0 < x < 1$, if $f(x) = \mathbb{E}[N_x]$, with $N_x = \min\{n : X_1 + \cdots + X_n > x\}$, $f(x) = e^x$. We want to find $f(2)$. We should condition on X_1, as that will tell us how much further of a sum we need to exceed. Namely, $f(2) = \mathbb{E}[N_2] = \mathbb{E}[\mathbb{E}[N_2 \mid X_1]]$. Now, given X_1, we have $2 - X_1$ left to exceed starting from the next turn and we used 1 turn, so $\mathbb{E}[N_2 \mid X_1] = 1 + N_{2-X_1}$. Therefore, $\mathbb{E}[N_2] = 1 + \mathbb{E}[N_{2-X_1}]$ by plugging back into the above. In other words, $f(2) = 1 + f(2 - X_1)$. Now, we need to condition on $X_1 = x_1$ and integrate. Namely, this means that $f(2) = \int_0^1 (1 + f(2 - x_1))\, dx_1 = 1 + \int_0^1 f(2 - x_1) dx_1$ By the u-substitution $u = 2 - x_1$, we get that $f(2) = 1 + \int_1^2 f(u) du$.

More generally, for $1 < x < 2$, by replacing 2 with x, note, that the bounds of the integral would now become $x - 1$ to x in this case, so

$$f(x) = 1 + \int_{x-1}^{x} f(u) du$$

Taking the derivative, we have that $f'(x) = f(x) - f(x-1)$. However, as $1 < x < 2$, $0 < x - 1 < 1$, in which we know f in the interval $(0,1)$. Therefore, using the result stated at the beginning, $f'(x) = f(x) - e^{x-1}$, as $f(x) = e^x$ in $(0,1)$. Rearranging this differential equation, we have $f'(x) - f(x) = -e^{x-1}$. Our initial condition is that $f(1) = e$, as this makes it continuous with the part of f on $(0,1)$.

This here is a first order linear differential equation that can be solved by integrating factors. Namely, the integrating factor here is $\mu(t) = e^{\int (-1) dx} = e^{-x}$. Multiplying by this on both sides yields $e^{-x} f'(x) - e^{-x} f(x) = -e^{-1}$. The LHS is just $(e^{-x} f(x))'$. Therefore, integrating both sides, $e^{-x} f(x) = \int -e^{-1} dx = -xe^{-1} + C$. Multiplying by e^x on both sides, $f(x) = -xe^{x-1} + Ce^x$. We know that $f(1) = e$, so $e = -1 + Ce$, so $C = 1 + e^{-1}$.

Therefore, $f(x) = -xe^{x-1} + \left(1 + e^{-1}\right) e^x$. Plugging in $x = 2$ yields $f(2) = e^2 - e$. Thus, $a = 1$ and $b = -1$, so $a + b = 0$.

Answer:

0

Reference:

https://www.quantguide.io/questions/sum-exceedance-ii

Greedy Pirates

Category: Brainteasers
Difficulty: Medium
Companies: De Shaw, SIG

Question:

There are five pirates with distinct seniority. Each is rational in that they prioritize staying alive first and getting as much gold as possible second. The five pirates agree on the following method to divide 100 gold coins. The most senior pirate will propose a distribution of the coins. All pirates five pirates vote on whether or not to pass the proposition. If at least 50

Hint:

Reduce this down to the $2-$pirate case to start and work your way up.

Explanation:

In the 2-pirate case, the most senior pirate, denoted pirate 2, will distribute all of the gold to himself since he will always get at least 50

Answer:

98

Reference:

https://www.quantguide.io/questions/greedy-pirates

Simulation Scheme I

Category: Probability
Difficulty: Easy
Companies: De Shaw, Akuna, Optiver

Question:

Suppose we want to simulate an event with probability $\frac{1}{3}$. This is possible via the following scheme: Flip a fair coin twice. If the outcome is HH, we say the event happened. If the outcome is HT or TH, we say that the event did not happen. If the outcome is TT, we re-run the 2 flips experiment under the same rules. Find the expected amount of coin flips needed to simulate this event.

Hint:

Condition on whether or not the simulation is finished within the first 2 flips.

Explanation:

Let N be the number of coin flips needed to simulate the event. With probability $\frac{3}{4}$, we determine whether or not the event happened on the first trial. If we obtain TT, which occurs with probability $\frac{1}{4}$, then we restart our trial process, but the number of flips we have done increases by 2. Therefore, with probability $\frac{3}{4}$, it takes 2 flips to simulate, and with probability $\frac{1}{4}$, it takes $2 + \mathbb{E}[N]$ flips to simulate, as we have performed 2 flips and we end up exactly where we started with the next trial. Thus, we have that $\mathbb{E}[N] = \frac{3}{4} \cdot 2 + \frac{1}{4}(2 + \mathbb{E}[N])$. Solving for $\mathbb{E}[N]$ yields $\mathbb{E}[N] = \frac{8}{3}$

Answer:

83

Reference:

https://www.quantguide.io/questions/simulation-scheme-i

Circular Hop

Category: Probability
Difficulty: Medium
Companies: De Shaw, Jump Trading

Question:

You are in a circle with 100 points labelled $0 - 99$ clockwise. You start at 1. At each turn, you move one unit left or right with equal probability. What is the expected time to hit 0?

Hint:

Use the fact that for a (non-circular) symmetric random walk starting at 0 with boundaries a and $-b$, for $a, b > 0$ being integers, the expected time to hit a boundary is ab.

Explanation:

We will use the fact that for a (non-circular) symmetric random walk starting at 0 with boundaries a and $-b$, for $a, b > 0$ being integers, the expected time to hit a boundary is ab. This is a fairly well-known fact and is a good exercise to prove if you have never done so before. Imagine we flattened out the circle into a line. In this case, to hit 0 from 1, we either need to move 1 unit up or 99 units down (this would get us to 0 by hitting spot 99 and circling back to 0). The answer is immediate from this fact. Namely, if we treat 1 as our starting point i.e. 0, we can treat the boundaries as 1 and -99, so our answer is just 99 by the lemma above.

Answer:

99

Reference:

https://www.quantguide.io/questions/circular-hop

Determination I

Category: Statistics
Difficulty: Hard
Companies: De Shaw, Two Sigma

Question:

Given three data sets X_1, X_2, and Y , we run two linear regressions to obtain $y \sim \alpha_1 + \beta_1 x_1$ and $y \sim \alpha_2 + \beta_2 x_2$. The R^2 value for both regressions is 0.05. Find the lowest upper bound on R^2 value of the regression $y \sim \alpha + \beta' x_1 + \beta'' x_2$.

Hint:

Consider the dataset Y that is sampled perfectly from a parabola far from the origin.

Explanation:

Consider the dataset Y that is sampled perfectly from a parabola far from the origin. Let X_2 be the dataset where each value of X_1 is squared. If Y is far enough from the origin, $y \sim \alpha_1 + \beta_1 x_1$ has low R^2 since it is linear, while Y is quadratic. Furthermore, $y \sim \alpha_2 + \beta_2 x_2 = \alpha_2 + \beta_2 x_1^2$ is also low R^2 because although it is a parabola, it is not able to be shifted around horzontally to match Y. Therefore, individually, these linear regressions have low R^2. However, the model $y \sim \alpha + \beta' x_1 + \beta'' x_2$ has $R^2 = 1$, as you can now shift the parabola around in the plane to perfectly match the non-noisy dataset Y.

Answer:

1

Reference:

https://www.quantguide.io/questions/determination-i

Likely Targets

Category: Probability
Difficulty: Hard
Companies: De Shaw

Question:

Two linear targets, say A and B, of radius $\varepsilon << 1$ are placed on an infinitely long line. The targets are centered at $x_A = -1$ and $x_B = 3$. In other words, target A covers the interval $[1-\varepsilon, 1+\varepsilon]$, and similarly with target B. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Hint:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would just be the $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. Since the two targets are disjoint, our goal is to maximize the sum of the probability densities.

Explanation:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. A similar statement holds for B. Since the two targets are disjoint, our goal is to maximize the sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$f(\mu) = \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(-1-\mu)^2}{8}} + e^{-\frac{(3-\mu)^2}{8}}\right)$$

The two interior terms are just the density of a $N(\mu, 4)$ distribution at -1 and 3. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}}\left[\frac{-1-\mu}{4}e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{4}e^{-\frac{(3-\mu)^2}{8}}\right] = 0$$

In particular, note that if $\mu = 1$, then the two terms inside are the same but of opposite sign by symmetry, so $\mu = 1$ is a critical point. One can verify that this is indeed a maxima, so $\mu = 1$ is our answer. This intuitively makes sense, as the symmetry of the normal distribution should imply that the maxima would be symmetric in both points for our target.

Answer:

1

Reference:

https://www.quantguide.io/questions/likely-targets

Janes Children

Category: Probability
Difficulty: Easy
Companies: De Shaw, SIG, Jane Street

Question:

There is a conference for parents with at least one daughter. Jane, a mother with two children, is invited. What is the probability that Jane has two daughters?

Hint:

What is the sample space conditioned on the information you were given about Jane's children?

Explanation:

The sample space of two children given at least one child is a daughter is $\Omega = \{(b, g), (g, b), (g, g)\}$, where (b, g) means that the older child is a boy and the younger child is a girl. Thus, the probability that Jane has two daughters is $\frac{1}{3}$.

Answer:

13

Reference:

https://www.quantguide.io/questions/janes-children

Equicorrelated

Category: Probability
Difficulty: Medium
Companies: De Shaw, Jane Street

Question:

7 random variables $X_1, \ldots, X_7$ are all identically distributed with mean 0 and variance 1. However, they also all have the same pairwise correlation, say ρ. Find the minimum possible value of ρ.

Hint:

Consider $\text{Var}(\overline{X})$, where $\overline{X} = \dfrac{X_1 + \cdots + X_7}{7}$.

Explanation:

We are going to consider $\text{Var}(\overline{X})$ here, where $\overline{X} = \dfrac{X_1 + \cdots + X_7}{7}$. By plugging this in and using properties of variance, we see that

$$\text{Var}(\overline{X}) = \frac{1}{49}\text{Var}(X_1 + \cdots + X_7) = \frac{1}{49}\left[\sum_{i=1}^{7} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)\right]$$

Since we know the mean and variance of each of the random variables, we know that the first sum is just 7. Similarly, we know that $\text{Cov}(X_i, X_j) = \rho(1)(1) = \rho$ and that there are $7 \cdot 6 = 42$ terms in that sum. Therefore, the second sum is just 42ρ. Thus,

$$\text{Var}(\overline{X}) = \frac{7 + 42\rho}{49}$$

Our condition is that $\text{Var}(\overline{X}) \geq 0$, as the variance of any random variable must be non-negative. Thus, we can disregard the denominator and find that

$$7 + 42\rho \geq 0 \iff \rho \geq -\frac{1}{6}$$

Therefore, our answer is $-\dfrac{1}{6}$.

Answer:

16

Reference:

https://www.quantguide.io/questions/equicorrelated

Sum Exceedance I

Category: Probability
Difficulty: Hard
Companies: De Shaw, Akuna, HRT

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID and $N_2 = \min\{n : X_1 + \cdots + X_n > 2\}$. Find $\mathbb{E}[N_2]$. Your answer will be in the form $ae^2 + be$ for integers a and b. e here is Euler's constant. Find $a + b$.

Hint:

Write a delayed differential equation that $f(x) = \mathbb{E}[N_x]$ should satisfy on $(1,2)$. You can obtain this by using Law of Total Expectation on X_1 and differentiating the result. What is a reasonable initial condition for $f(1)$? You may also want to recall $f(x) = e^x$ on $(0,1)$.

Explanation:

We are going to use the result of Sum Exceedance I that states that for any $0 < x < 1$, if $f(x) = \mathbb{E}[N_x]$, with $N_x = \min\{n : X_1 + \cdots + X_n > x\}$, $f(x) = e^x$. We want to find $f(2)$. We should condition on X_1, as that will tell us how much further of a sum we need to exceed. Namely, $f(2) = \mathbb{E}[N_2] = \mathbb{E}[\mathbb{E}[N_2 \mid X_1]]$. Now, given X_1, we have $2 - X_1$ left to exceed starting from the next turn and we used 1 turn, so $\mathbb{E}[N_2 \mid X_1] = 1 + N_{2-X_1}$. Therefore, $\mathbb{E}[N_2] = 1 + \mathbb{E}[N_{2-X_1}]$ by plugging back into the above. In other words, $f(2) = 1 + f(2 - X_1)$. Now, we need to condition on $X_1 = x_1$ and integrate. Namely, this means that $f(2) = \int_0^1 (1 + f(2 - x_1))\, dx_1 = 1 + \int_0^1 f(2 - x_1) dx_1$ By the u-substitution $u = 2 - x_1$, we get that $f(2) = 1 + \int_1^2 f(u) du$.

More generally, for $1 < x < 2$, by replacing 2 with x, note, that the bounds of the integral would now become $x - 1$ to x in this case, so

$$f(x) = 1 + \int_{x-1}^{x} f(u) du$$

Taking the derivative, we have that $f'(x) = f(x) - f(x-1)$. However, as $1 < x < 2$, $0 < x - 1 < 1$, in which we know f in the interval $(0,1)$. Therefore, using the result stated at the beginning, $f'(x) = f(x) - e^{x-1}$, as $f(x) = e^x$ in $(0,1)$. Rearranging this differential equation, we have $f'(x) - f(x) = -e^{x-1}$. Our initial condition is that $f(1) = e$, as this makes it continuous with the part of f on $(0,1)$.

This here is a first order linear differential equation that can be solved by integrating factors. Namely, the integrating factor here is $\mu(t) = e^{\int(-1)dx} = e^{-x}$. Multiplying by this on both sides yields $e^{-x}f'(x) - e^{-x}f(x) = -e^{-1}$. The LHS is just $(e^{-x}f(x))'$. Therefore, integrating both sides, $e^{-x}f(x) = \int -e^{-1} dx = -xe^{-1} + C$. Multiplying by e^x on both sides, $f(x) = -xe^{x-1} + Ce^x$. We know that $f(1) = e$, so $e = -1 + Ce$, so $C = 1 + e^{-1}$.

Therefore, $f(x) = -xe^{x-1} + \left(1 + e^{-1}\right) e^x$. Plugging in $x = 2$ yields $f(2) = e^2 - e$. Thus, $a = 1$ and $b = -1$, so $a + b = 0$.

Answer:

0

Reference:

https://www.quantguide.io/questions/sum-exceedance-i

Likely Targets III

Category: Probability
Difficulty: Hard
Companies: De Shaw

Question:

Three linear targets, say A, B, and C, of respective radii $\varepsilon, 2\varepsilon$, and 3ε, where $\varepsilon << 1$, are placed on an infinitely long line. The targets are centered at $x_A = -1, x_B = 3$, and $x_C = 5$. In other words, target A covers the interval $[1-\varepsilon, 1+\varepsilon]$, target B covers the interval $[3-2\varepsilon, 3+2\varepsilon]$, and target C covers the interval $[5-3\varepsilon, 5+3\varepsilon]$. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Hint:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. How do we account for difference in radii? How do we account for the 3 targets instead of 2?

Explanation:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. However, since B is also of small radius 2ε, the probability we hit target B is approximately $2f(x_B)\varepsilon$. Likewise, the probability we hit target C is approximately $3f(x_C)\varepsilon$. Since the targets are disjoint, our goal is to maximize the weighted sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$f(\mu) = \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(-1-\mu)^2}{8}} + 2e^{-\frac{(3-\mu)^2}{8}} + 3e^{-\frac{(5-\mu)^2}{8}}\right)$$

The interior terms are just the density of a $N(\mu, 4)$ distribution at $-1, 3$, and 5, respectively. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}}\left[\frac{-1-\mu}{4}e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{2}e^{-\frac{(3-\mu)^2}{8}} + 3\cdot\frac{5-\mu}{4}e^{-\frac{(5-\mu)^2}{8}}\right] = 0$$

Using a computer system, $f'(\mu) = 0$ for $\mu \approx 4.211$. One can verify that this is indeed a maxima, so 4.21 is our answer. This intuitively makes sense, as the increased size of the region around 3 and 5 means that we should assign more density there. Furthermore, the region around 5 is slightly larger than the region around 3, so the center should be placed closer to 5 than 3.

Answer:

4.21

Reference:

https://www.quantguide.io/questions/likely-targets-iii

Lock And Key Pair

Category: Brainteasers
Difficulty: Easy
Companies: De Shaw

Question:

There are 5 keys that unlock exactly one of 5 locks. Assuming an optimal strategy, what is the maximum number of times you need to try the locks to identify which key unlocks each lock?

Hint:

Try 4 of the keys on the first lock.

Explanation:

Try 4 of the keys on the first lock. If any work, then we can stop early. However, the worst case scenario is that the one we don't test is the one that works, in which we would attempt all 4 keys. Similarly, of the 4 remaining unassigned keys, pick 3 of them to test. Repeat this with 3 locks and 2 locks left to yield 2 and 1 attempts, respectively. Therefore, we need to try the locks a total of $4 + 3 + 2 + 1 = 10$ times.

Answer:

10

Reference:

https://www.quantguide.io/questions/lock-and-key-pair

Baby Boy

Category: Probability
Difficulty: Easy
Companies: De Shaw, SIG

Question:

A hospital nursery has 3 baby boys and an unknown number of baby girls. A mother just gave birth to a child of unknown gender and it is added to the nursery. The doctor then picks up a random child in the nursery and it is a baby boy. Assuming that births of boys and girls are equally likely, find the probability that the mother just gave birth to a boy.

Hint:

Assume there are g girls in the nursery. How many total children would there be? Let C represent the event that the doctor chose a boy and B represent the event that the mother just had a boy. What is the event you are looking for the probability of and apply Bayes' Rule.

Explanation:

Let g be the number of girls in the nursery. Let C represent the event that the doctor chose a boy and B represent the event that the mother just had a boy. Then there are $4+g$ total children in the nursery. We want $\mathbb{P}[B \mid C]$. By Bayes' Theorem, this is $\dfrac{\mathbb{P}[C \mid B]\mathbb{P}[B]}{\mathbb{P}[C]}$. We can choose a boy with the mother just birthing a boy vs. the mother just birthing a girl. Thus, $\mathbb{P}[C] = \mathbb{P}[C \mid B]\mathbb{P}[B] + \mathbb{P}[C \mid B^c]\mathbb{P}[B^c]$.

$\mathbb{P}[C \mid B] = \dfrac{4}{g+4}$, as the new child being a boy means there are 4 boys and g girls. $\mathbb{P}[C \mid B^c] = \dfrac{3}{g+4}$, as the new child being a girl means there are 3 boys and $g+1$ girls. Plugging all of these in, we have that

$$\mathbb{P}[B \mid C] = \frac{\frac{4}{g+4} \cdot \frac{1}{2}}{\frac{4}{g+4} \cdot \frac{1}{2} + \frac{3}{g+4} \cdot \frac{1}{2}} = \frac{4}{7}$$

Answer:

47

Reference:

https://www.quantguide.io/questions/baby-boy

Likely Target II

Category: Probability
Difficulty: Hard
Companies: De Shaw

Question:

Two linear targets, say A and B, of respective radii ε and 2ε, where $\varepsilon << 1$, are placed on an infinitely long line. The targets are centered at $x_A = -1$ and $x_B = 3$. In other words, target A covers the interval $[1-\varepsilon, 1+\varepsilon]$, while target B covers the interval $[3-2\varepsilon, 3+2\varepsilon]$. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Hint:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. How do you account for the difference in radii?

Explanation:

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. However, since B is also of small radius 2ε, the probability we hit target B is approximately $2f(x_B)\varepsilon$. Since the two targets are disjoint, our goal is to maximize the weighted sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$f(\mu) = \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(-1-\mu)^2}{8}} + 2e^{-\frac{(3-\mu)^2}{8}}\right)$$

The two interior terms are just the density of a $N(\mu, 4)$ distribution at -1 and 3. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}}\left[\frac{-1-\mu}{4}e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{2}e^{-\frac{(3-\mu)^2}{8}}\right] = 0$$

Using a computer system, $f'(\mu) = 0$ for $\mu \approx 2.649$. One can verify that this is indeed a maxima, so 2.65 is our answer. This intuitively makes sense, as the increased size of the region around 3 means that we should assign more density there. Thus, our answer should be closer to 3 than it is to -1.

Answer:

2.65

Reference:

https://www.quantguide.io/questions/likely-target-ii

Rubiks Cube Stickers

Category: Brainteasers
Difficulty: Easy
Companies: Akuna

Question:

A $4 \times 4 \times 4$ Rubik's Cube is composed of $1 \times 1 \times 1$ mini-cubes. The Rubik's Cube has stickers on its outer layer to denote the colors of the cube. How many mini-cubes have at least one sticker on them?

Hint:

Instead of computing how many mini-cubes have stickers on them, an easier approach may be to calculate how many mini-cubes are unseen and thus do not have stickers on them.

Explanation:

There are a total of 4^3 mini-cubes that compose the Rubik's Cube. Of these, the core 2^3 mini-cubes that make up the inside of the cube and that are not showing do not have stickers on it. Thus, the total number of mini-cubes that do have stickers on it is:

$$4^3 - 2^3 = 56$$

Answer:

56

Reference:

https://www.quantguide.io/questions/rubiks-cube-stickers

Petersburg Paradox

Category: Probability
Difficulty: Easy
Companies: Akuna

Question:

Suppose you are offered to play a game where you flip a fair coin until you obtain a heads for the first time. If the first heads occurs on the nth flip, you are paid out $\$2^n$. What is the fair value of this game? If your answer is infinite, enter -1.

Hint:

The number of flips needed to see the first heads is $N \sim \text{Geom}(1/2)$. We are looking for $\mathbb{E}[2^N]$, which is our expected payout.

Explanation:

The number of flips needed to see the first heads is $N \sim \text{Geom}(1/2)$. We are looking for $\mathbb{E}[2^N]$, which is our expected payout. Namely,

$$\mathbb{E}[2^N] = \sum_{n=1}^{\infty} 2^n \mathbb{P}[N = n] = \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} 1 = \infty$$

Therefore, the answer is infinite, meaning the input is -1.

Answer:

1

Reference:

https://www.quantguide.io/questions/petersburg-paradox

Hand Meet

Category: Brainteasers
Difficulty: Easy
Companies: Akuna, Jane Street

Question:

It is presently 12 : 00 PM. The minute and hour hands meet there. After how many hours will the minute and hour hands meet again? Enter a fraction if it is not an integer amount of hours.

Hint:

Let h be the number of hours. The minutes hand will move $360h$ degrees. In addition, we know that the hours hand moves $30h$ degrees. How do you account for rotation?

Explanation:

Let h be the number of hours. The minutes hand will move $360h$ degrees. In addition, we know that the hours hand moves $30h$ degrees. However, as we need to account for rotation, the first time they meet, the minutes hand will have travelled an additional 360 degrees beyond the hours hand, as it completes a rotation. Therefore, we get the equation $360h = 30h + 360$, which yields $h = \frac{12}{11}$.

Answer:

1211

Reference:

https://www.quantguide.io/questions/hand-meet

Coloring Components II

Category: Probability
Difficulty: Hard
Companies: Akuna

Question:

Consider a line of 20 adjacent colorless squares. Color in each individual square black or white with equal probability, independent of all other squares. A connected 5-component is a group of 5 consecutive black squares. Note that overlapping components are not counted. For example, WBBBBBWWB and WBBBBBBWB have 1 connected 5-component, but WBBBBBBBBBBW has 2 connected 5-components. Find the expected number of connected 5-components in our line. The answer can be written as a simplified fraction of the form $\frac{p}{q}$. Find $p+q$.

Hint:

Try computing the expected number of connected 5-components if we counted overlaps. Then consider subtracting out the subsequent term.

Explanation:

If we were to count overlaps, we could simply set up an indicator variable over every window of 5 squares and applying Linearity of Expectation. This is simply 16 windows of probability 1/32 each, for an expected value of 1/2.

To account for overlaps, let's analyze how many times we want to count consecutive chains of black squares of certain lengths. For example, if we have exactly 6 black squares in a row, we want to count this as 1 connected 5-component, but if there are exactly 10 black squares in a row, we want to count this as 2. In general, for exactly n black squares in a row, we want to count this as $\lfloor n/5 \rfloor$ connected 5-components. This can be achieved by repeating our scheme for counting overlaps for multiples of 5 but subtracting the expected number of results we would get from considering overlapping chains of length 1 greater. Denoting C_n as the expected number of chains of n consecutive black squares while counting overlaps, our answer is

$$C_5 - C_6 + C_{10} - C_{11} + C_{15} - C_{16} + C_{20}$$

$$= \left(16 \cdot \frac{1}{2^5}\right) - \left(15 \cdot \frac{1}{2^6}\right) + \left(11 \cdot \frac{1}{2^{10}}\right) - \left(10 \cdot \frac{1}{2^{11}}\right) + \left(6 \cdot \frac{1}{2^{15}}\right) - \left(5 \cdot \frac{1}{2^{16}}\right) + \left(1 \cdot \frac{1}{2^{20}}\right) = \frac{284785}{1048576}.$$

We get the term $C_k = \frac{21-k}{2^k}$ from the fact that there are $21-k$ spots for a length k chain to start at and the probability of it starting at any given spot is $\frac{1}{2^k}$. Our answer is $284785+1048576 = 1333361$.

Answer:

1333361

Reference:

https://www.quantguide.io/questions/coloring-components-ii

Solo Or Pair

Category: Brainteasers
Difficulty: Medium
Companies: Akuna, Jane Street

Question:

You are given the option between two games involving fair standard 6−sided dice. Which one gives you a higher probability of winning? Answer 1 for the first game, 2 for the second game, or 3 if both give you equal probability of winning.

Game 1: You are given 4 rolls of a single die and you must roll 6 at least once.
Game 2: You are given 24 rolls of a pair of dice and you must roll 66 at least once.

Hint:

The probability of winning in game 1 is $1 - \left(\frac{5}{6}\right)^4$ by complementation, as to lose, you must roll each of the other 5 values per roll. The probability of winning in game 2 is $1 - \left(\frac{35}{36}\right)^{24}$, as to lose, you must roll any of the other 35 outcomes of the two dice per turn. Try to reason out by approximation which is larger.

Explanation:

The probability of winning in game 1 is $1 - \left(\frac{5}{6}\right)^4$ by complementation, as to lose, you must roll each of the other 5 values per roll. The probability of winning in game 2 is $1 - \left(\frac{35}{36}\right)^{24}$, as to lose, you must roll any of the other 35 outcomes of the two dice per turn. We need to determine which of these is larger, which is equivalent to determine which of $\left(\frac{5}{6}\right)^4$ and $\left(\frac{35}{36}\right)^{24}$ is smaller. We know that $\frac{35^2}{36^2} = \frac{1225}{1296} \approx \frac{1225}{1300} = \frac{49}{52}$. Then, squaring this yields $\frac{2401}{2704} \approx \frac{8}{9}$. Therefore, $\left(\frac{35}{36}\right)^8 \approx \frac{64}{81} \approx \frac{4}{5}$. Now, we just need to compare $\left(\frac{5}{6}\right)^4$ to $\frac{4^3}{5^3} = \frac{64}{125}$. However, this is easy to see, as $\frac{5^4}{6^4} = \frac{625}{1296} < \frac{1}{2} < \frac{64}{125}$. Therefore, we can conclude, at least approximately, that $\left(\frac{5}{6}\right)^4$ is smaller, so Game 1 gives you better odds. When plugging into a calculator, this is indeed confirmed.

Answer:

1

Reference:

https://www.quantguide.io/questions/solo-or-pair

Confused Ant II

Category: Probability
Difficulty: Medium
Companies: Akuna, Jane Street

Question:

An ant walks the corner of a 3D cube and moves to one of the three adjacent vertices with equal probability at each step. Find the expected number of steps needed for the ant to return to the vertex it started at.

Hint:

Think about the different states of this phenomenon.

Explanation:

This seems like a very complicated problem on the surface but you can actually use Markov Chains to solve this problem. There is a lot of symmetry in cubes to help us decrease the number of states involved in this question. Let's let your starting state be E_{00}. No matter which way the ant goes, by symmetry it'll basically always be the same position. That position being 1 side length away from the starting vertex. Lets denote the expected number of moves to get back to the initial vertex as E_1. Thus $E_{00} = E_1 + 1$. Let E_2 be the expected number of moves from being in the state of 2 side lengths away from the starting vertex. Finally, let E_3 be the expected moves from being at the opposite vertex of the starting vertex. The equations for all these states are:

$$E_{00} = E_1 + 1E_1 = \frac{2}{3}E_2 + \frac{1}{3}E_0 + 1E_2 = \frac{2}{3}E_1 + \frac{1}{3}E_3 + 1E_3 = E_2 + 1$$

We know that $E_0 = 0$. Solving all of these equations, we get $E_{00} = 8$. Thus it takes 8 steps for the ant to start at one vertex of the cube and return to it.

Answer:

8

Reference:

https://www.quantguide.io/questions/confused-ant-ii

Clock Angle

Category: Brainteasers
Difficulty: Easy
Companies: Akuna, Drw, Jane Street, Old Mission, Optiver

Question:

Find the angle (in degrees) between the minutes and hours hands at $9:45$ PM on a standard clock.

Hint:

At 9:45 PM, where is the minutes hand located on the clock? How long into an hour is 45 minutes?

Explanation:

At 9:45 PM, we know the minutes hand is at the 9. The hours hand is $\frac{3}{4}$ of the way between the 9 and the 10, as $\frac{3}{4}$ of an hour has elapsed. Therefore, the angle between them is just $\frac{3}{4}$ of the angular distance from 9 to 10 on the clock. Namely, as there are 12 equally spaced numerical values and the total degrees of a circle is 360, there are 30 degrees between each numerical value. Therefore, our answer is $\frac{3}{4} \cdot 30 = 22.5$.

Answer:

22.5

Reference:

https://www.quantguide.io/questions/clock-angle

Lily Pads II

Category: Brainteasers
Difficulty: Medium
Companies: Akuna, SIG

Question:

Lily pads double in area each day, and each lily pad is one square foot in area. How many days pass until a 20480-square foot pond that initially has 10 lily pads is covered in lily pads? Assume the region each lily pad covers is disjoint from the others.

Hint:

How much area must each lily pad cover in order for the entirety of the pond to be covered?

Explanation:

Each lily pad must cover $\frac{20480}{10}$ square feet to cover the entirety of the pond. Since the size of each pad is 2^N after N days of growth, then:

$$2^N = \frac{20480}{10} \Rightarrow N = 11$$

Answer:

11

Reference:

https://www.quantguide.io/questions/lily-pads-ii

Normal Activities

Category: Statistics
Difficulty: Easy
Companies: Akuna, Five Rings, HRT

Question:

Suppose that X and Y are independent standard normal random variables. Find $\mathbb{P}[Y > 4X]$.

Hint:

What is the distribution of $Y - 4X$?

Explanation:

As X and Y are mean 0 normal random variables, $Y-4X$ is also a mean 0 normal random variable. Therefore, $\mathbb{P}[Y > 4X] = \mathbb{P}[Y - 4X > 0]$ is asking the probability that a normal random variable is greater than its mean. This is $\frac{1}{2}$ by the symmetry of the normal distribution about its mean.

Answer:

12

Reference:

https://www.quantguide.io/questions/normal-activities

Observing Cars

Category: Probability
Difficulty: Easy
Companies: Akuna, SIG, Jane Street, Optiver

Question:

You are standing at a bus stop watching cars go by. The probability of observing at least one accident in an hour interval is $\frac{3}{4}$. What is the probability of observing at least one accident within thirty minutes? Assume that the probability of observing an accident at any moment within an hour interval is uniform.

Hint:

The hour interval can be broken down into two disjoint thirty-minute interval. How can you use complementary events to further define the problem?

Explanation:

The hour interval can be broken down into two disjoint thirty-minute interval such that the probability p of observing any accident is uniform. The probability that we do not observe an accident within a thirty-minute interval is $1 - p$. Furthermore, the probability that we do not observe any accidents within two independent thirty-minute intervals (or an hour) can be written as:

$$(1-p)^2 = 1 - \frac{3}{4} \Rightarrow p = 1/2$$

Answer:

12

Reference:

https://www.quantguide.io/questions/observing-cars

Egg Drop II

Category: Brainteasers
Difficulty: Hard
Companies: Akuna, Akuna, Optiver, Optiver

Question:

You are holding three identical eggs in a really large building with n stories. If an egg is dropped at an elevation under story X, then the egg will survive; else, the egg breaks. It is known that the egg will break at some floor number $1-n$. What is the maximum value of n such that at most 9 drops are required to determine X in the worst-case scenario?

Hint:

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, make larger jumps at lower floors in the tower so that you can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, you could test $\dfrac{d(d+1)}{2}$ floors completely. The key here is to note that when you drop one egg and it breaks, the problem is reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what you do in the two egg case), make jumps that decrease by the maximum increment of the two egg case.

Explanation:

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, we make larger jumps at lower floors in the tower so that we can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, we could test $\dfrac{d(d+1)}{2}$ floors completely. The key here is to note that when we drop one egg and it breaks, we really get reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what we do in the two egg case), we are going to make jumps that decrease by the maximum increment of the two egg case. This is so that we are sufficiently able to search everything in a given range in the two egg case afterwards.

Putting this into math, we want to find a first cutoff point such that if the egg breaks at floor X on the first drop, we can examine the floors below it in $9-1=8$ trials. We can examine up to $\dfrac{8 \cdot 9}{2} = 36$ floors in 8 trials by Egg Drop I, so we should drop our first egg at floor 37. If it doesn't break, then we need to increment such that if it breaks on our next trial, we can examine the remaining space between 37 and the next floor in $9-2=7$ drops. We can explore up to $\dfrac{8 \cdot 7}{2} = 28$ floors with 7 drops, so we should place our next egg at floor 66 such that we can explore floors $38-65$ (28 floors) in the remaining 7 drops.

Continuing this pattern, when we have k drops left, we are going to increment by $\dfrac{k(k+1)}{2}+1$ floors. This means that the next egg after would be placed at floor $66+21+1=88$, the egg after at $88+15+1=104$, and so on. The remaining floors to drop eggs at would be $115, 122, 126$, and

128. If it breaks at 128 but not at 126, we try 127 to verify the answer. If it doesn't break at 128, we try 129 instead and see if it breaks. If it breaks, we found our answer. If it doesn't break, then if $n = 130$, we would know that the egg breaks at floor 130, as it must break on one of the floors in our building. We can't explore any higher than 130 due to the fact that if it breaks at some point above 130, we would have to search between 129 and the breaking point, which takes strictly more than one egg drop. Therefore, $n = 130$ is our answer.

Answer:

130

Reference:

https://www.quantguide.io/questions/egg-drop-ii

Lily Pads I

Category: Brainteasers
Difficulty: Easy
Companies: Akuna, SIG

Question:

A single lily pad sits in an empty pond. Everyday, the lily pad doubles in area until the whole pond is covered- it is known that it would take 10 days for the single lily pad to cover the entire pond. Imagine instead starting with 8 lily pads on the first day. How many days will it take for the surface of the pond to be covered?

Hint:

Each lily pad grows at the game rate, so the 8 lily pads can be thought of as a single lily pad that is three days old.

Explanation:

The 8 lily pads can be thought of as a single lily pad that is three days old, since $2^3 = 8$. Thus, it will take $10 - 3 = 7$ days for the surface of the pond to be covered.

Answer:

7

Reference:

https://www.quantguide.io/questions/lily-pads-i

Short Wood

Category: Probability
Difficulty: Hard
Companies: Akuna, SIG, Two Sigma

Question:

A lumberjack cuts a piece of wood of 1 meter in length at 2 uniformly randomly selected locations along the length of the wood. Find the probability the shortest piece of wood is at most 5 centimeters.

Hint:

Let X and Y be the two distances from the LHS where the wood is cut (in meters). Both of these are IID $\text{Unif}(0, 1)$ random variables. Since X and Y are IID, they are exchangeable. Thus, for simplicity, assume $Y > X$. Multiply by 2 to account for when $Y < X$. Write your probability statement in terms of X and Y.

Explanation:

Let X and Y be the two distances from the LHS where the wood is cut (in meters). Both of these are IID $\text{Unif}(0, 1)$ random variables. Since X and Y are IID, they are exchangeable. Thus, for simplicity, assume $Y > X$. We will multiply by 2 at the end to account for when $X > Y$. The lengths of the three pieces would then by given by $X, Y - X$, and $1 - Y$. Thus, we want $\mathbb{P}[\min(X, Y - X, 1 - Y) < 0.05] = 1 - \mathbb{P}[\min(X, Y - X, 1 - Y) > 0.05]$.

By properties of the minimum, this is the same as $1 - \mathbb{P}[X > 0.05, Y > X + 0.05, Y < 0.95]$. The last two terms are obtained by rearranging the inequalities. Drawing the region where all three of those conditions are true out in the plane yields a triangular region. Since the slope is one, the two legs are of the same length. Note that $X > 0.05$ and $Y < 0.95$. Thus, we have that the endpoint of where we leave our region of interest is $X = 0.9$, as then $Y = 0.95$.

Thus, the length of each leg is $\frac{17}{20}$, so the area of the triangle is just $\frac{1}{2}\left(\frac{17}{20}\right)^2 = \frac{289}{800}$. Thus, we have that since X and Y are exchangeable, we multiply the above probability by 2, so we get $\frac{289}{400}$, which means the probability is $\frac{111}{400}$ by complementation.

Answer:

111400

Reference:

https://www.quantguide.io/questions/short-wood

Central Containment

Category: Probability
Difficulty: Hard
Companies: Akuna, Jane Street

Question:

What is the probability that three random points on a unit circle would form a triangle that includes the center of the unit circle?

Hint:

Fix the first point arbitrarily. The second point can also lie anywhere, but notice that to form a triangle which contains the center, the third point must lie on the portion which has equivalent size as the length of the arc between the first two points, reflected over the center.

Explanation:

Fix the first point arbitrarily. The second point can also lie anywhere, but notice that to form a triangle which contains the center, the third point must lie on the portion which has equivalent size as the length of the arc between the first two points, reflected over the center. Thus, think about the position of the second point. The size of that portion could be anywhere between 0 and π. On average, it is $\frac{\pi}{2}$, and hence the answer is

$$\frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$$

Answer:

14

Reference:

https://www.quantguide.io/questions/central-containment

Overlapping Segments

Category: Probability
Difficulty: Medium
Companies: Akuna

Question:

Suppose that $X_1, X_2, X_3, X_4 \sim \text{Unif}(0,1)$ IID. Draw two line segments: One between X_1 and X_2 and one between X_3 and X_4. Find the probability that there is overlap between the two line segments.

Hint:

We only care about which endpoint is paired with the lowest of the four endpoints. Why?

Explanation:

There is a sneaky trick here. We only care about which endpoint is paired with the lowest of the four endpoints (in terms of value). If it is the second smallest, then we get no overlap. Otherwise, we have overlap. As it is equally likely for the remaining three random variables to be ordered in any way with respect to the last three spots, the probability is just $\frac{2}{3}$.

Answer:

23

Reference:

https://www.quantguide.io/questions/overlapping-segments

Covariance Review II

Category: Probability
Difficulty: Easy
Companies: Akuna

Question:

You are given the following information:

$$\mathbb{E}[X_1] = -3$$

$$\mathbb{E}[X_2] = 7$$

$$\text{Var}[X_1] = 24$$

$$\text{Var}[X_2] = 6$$

Compute the maximum possible value for $\text{Cov}(X_1, X_2)$.

Hint:

You are given the following information:

$$\mathbb{E}[X_1] = -3$$

$$\mathbb{E}[X_2] = 7$$

$$\text{Var}[X_1] = 24$$

$$\text{Var}[X_1] = 6$$

Compute the maximum possible value for $\text{Cov}(X_1, X_2)$.

Explanation:

The correlation coefficient ρ is defined as

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}, \text{ where}$$

$$\sigma_1^2 = \text{Var}(X_1) \text{ and}$$

$$\sigma_2^2 = \text{Var}(X_2).$$

The correlation coefficient must satisfy $-1 \leq \rho \leq 1$.

$$-1 \leq \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \leq 1$$

$$-1 \leq \frac{\text{Cov}(X_1, X_2)}{12} \leq 1$$

$$-12 \leq \text{Cov}(X_1, X_2) \leq 12$$

The maximum possible value for $\text{Cov}(X_1, X_2)$ is 12.

Answer:

12

Reference:

https://www.quantguide.io/questions/covariance-review-ii

Oil Transport

Category: Brainteasers
Difficulty: Medium
Companies: Akuna, Optiver

Question:

An oil tanker is tasked with the goal of transporting 3000 gallons of oil from Port A to Port B which are 1000 miles apart. However, the oil tanker loses 1 gallon of oil for every mile it travels from spillage (at a constant rate) and can carry a max of 1000 gallons at any time. The tanker can also dump any oil that it is carrying at any number of storage ports on the way from Port A to Port B and pick it up later. Assuming an optimal travel plan where you decide where to place any number of storage ports and how to carry the oil, how many gallons can you transport form Port A to Port B (round to the nearest gallon)?

Hint:

Where should be put our storage ports to maximize our fuel efficiency?

Explanation:

The key realization in this question is how to place the storage ports to maximize the amount of oil you are carrying per oil lost. This realization leads to the idea of having multiples of 1000 gallons at any storage port. If you don't, you aren't maximizing the efficiency metric above. Where do we place the first storage port to hold 2000 gallons of oil? Well, we need to leave $\frac{2000}{3}$ gallons every time we arrive there from Port A. This means that the distance should be $1000 - \frac{2000}{3} = \frac{1000}{3}$ miles away from Port A. If we only have this storage port, we will bring $1000 - \frac{2000}{3} = \frac{1000}{3}$ gallons twice to Port B $= \frac{2000}{3} \approx 667$ gallons. But we can do better by having another storage port that is a multiple of 1000 gallons, that being the port that will have 1000 gallons. From the first storage port, we can go 500 miles and drop off 500 gallons to the second storage port twice. This means this second storage port should be located 500 miles away from the first storage port $= 500 + \frac{1000}{3} = \frac{2500}{3}$ miles away from Port A. Now we have 1000 gallons of oil that is $1000 - \frac{2500}{3} = \frac{500}{3}$ miles away from Port B. We can carry these 1000 gallons all at once and lose $\frac{500}{3}$ gallons of oil which leaves us with $1000 - \frac{500}{3} = \frac{2500}{3} \approx 833$ gallons of oil at Port B.

Answer:

833

Reference:

https://www.quantguide.io/questions/oil-transport

The Picking Hat

Category: Probability
Difficulty: Hard
Companies: Akuna, Jane Street

Question:

A hat contains the numbers 1-100. The rules to the game are as follows. Each round, AJ draws a number out of the hat, writes it down, and puts the number back in the hat. The last number written down is the number of dollars awarded to AJ. AJ may play as many rounds as he would like, but each round costs \$1. Assuming optimal play, what is the fair value of this game?

Hint:

Let's say AJ gets unlucky and draws a 5 on his first turn. Then, AJ should of course draw again, despite the \$1 penalty. But, if AJ gets lucky and draws a 70 on his first turn? An 80? Define a loss function where to determine an optimal threshold x, where, if AJ's draw is at least x, then he will not play another round.

Explanation:

We want to determine an optimal threshold x, where, if AJ's draw is at least x, then he will not play another round. Notice that, if AJ draws again, without taking into consideration the \$1 penalty, then AJ's expected payoff is the same as the previous round. So, we can write the following expression, where A denotes the total winnings.

$$\mathbb{E}[A] = -1 + \sum_{i=x}^{100} \frac{i}{100} + \sum_{i=1}^{x-1} \frac{\mathbb{E}[A]}{100}$$

$$= -1 + \frac{(100 - x + 1)(100 + x)}{200} + \frac{x-1}{100}\mathbb{E}[A]$$

Let's isolate $\mathbb{E}[A]$.

$$\mathbb{E}[A] = \frac{(101 - x)(100 + x) - 200}{2(101 - x)}$$

We now wish to determine

$$x_{\text{optimal}} = \arg\max_x \left\{ \frac{(101 - x)(100 + x) - 200}{2(101 - x)} \right\}.$$

Let's take the derivative with respect to x.

$$\frac{d}{dx}\mathbb{E}[A] = \frac{2(101-x)(-2x+1)+2((101-x)(100+x)-200)}{4(101-x)^2}$$

$$= \frac{10001-202x+x^2}{4(101-x)^2}$$

Setting $\frac{d}{dx}\mathbb{E}[A] = 0$ and solving for x, we find

$$10001 - 202x_{\text{optimal}} + x^2_{\text{optimal}} = 0$$

$$x_{\text{optimal}} = \frac{202 \pm \sqrt{800}}{2}$$

$$= 101 \pm 10\sqrt{2}$$

Since $x \leq 100$, $x_{\text{optimal}} = 101 - 10\sqrt{2} \approx 86.9$. Our optimal integer x is then either 86 or 87. Plugging both into $\mathbb{E}[A]$, we find $\mathbb{E}[A]$ is maximized when $x = 87$ and has value $\frac{1209}{14}$.

Answer:

120914

Reference:

https://www.quantguide.io/questions/the-picking-hat

Captive Marbles

Category: Probability
Difficulty: Easy
Companies: Akuna

Question:

You are on death row presently. The judge offers you the following game: You have 50 white balls, 50 black balls, and 2 empty urns in front of you. You can distribute all 100 balls between the 2 urns in any way you please. Then, you are blindfolded and must pick one of the urns to select a ball from. If you pick a white ball, you are free. However, you are executed on the spot if you draw a black ball. Assuming optimal strategy, what is your chance of survival?

Hint:

If the bowls are not perfectly balanced, then we know that one of the bowls must have a larger than 0.5 probability of you drawing a white ball while the other has less than a 0.5 probability. Taking this to the extreme, the best case is that one of the bowls has probability 1 of you surviving and the other has probability 0.5 of you surviving, which yields a 75% chance of survival.

Explanation:

If the bowls are not perfectly balanced, then we know that one of the bowls must have a larger than 0.5 probability of you drawing a white ball while the other has less than a 0.5 probability. Taking this to the extreme, the best case is that one of the bowls has probability 1 of you surviving and the other has probability 0.5 of you surviving, which yields a 75% chance of survival. The closest we can get to this is by putting 1 white marble in one of the urns and then the other 99 marbles in the other urn. This gives you probability 1 of survival in one urn and $\frac{49}{99}$ probability in the other. This is as close as we can get to the theoretical optimum, so the probability of survival here is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{49}{99} = \frac{74}{99}$$

Answer:

7499

Reference:

https://www.quantguide.io/questions/captive-marbles

Game Show III

Category: Probability
Difficulty: Medium
Companies: Akuna

Question:

You're on a game show and are given the choice of 10 doors to choose from. Behind one door is \$1000 and behind the other nine are goats. You pick one of the doors at random and announce your choice to the game show host. The game show host, knowing which prize is behind each door, opens three doors that you did not choose and shows goats behind all of them after hearing your initial choice. He offers you the opportunity to either keep your original door or switch to the other closed door. What is the expected value of this game in dollars? Assume goats are worth \$0.

Hint:

Which strategy should you go into this game with?

Explanation:

If you have not done the problem "Game Show I", please do so before this question. From "Game Show I", we already know the best choice is to always switch doors. Now the problem is to calculate the EV of the game. $\frac{9}{10}^{\text{ths}}$ of the time, you pick a door with a goat initially. Given the strategy of always switching after the three doors with goats behind them are revealed, we are left with 6 doors to switch to, one of which has the cash prize. Thus the probability we switch to the cash prize door is $\frac{9}{10} \cdot \frac{1}{6} = 0.15$. The other $\frac{1}{10}^{\text{th}}$ of the time, we initially pick the cash prize door and switch to a losing door.
Thus the EV of this game is $0.15 \cdot \$1000 = \$150$.

Answer:

150

Reference:

https://www.quantguide.io/questions/game-show-iii

Confident Double Heads

Category: Probability
Difficulty: Easy
Companies: Akuna, SIG, Drw, Five Rings

Question:

A bag has 19 fair coins and one double-headed coin. We pick a coin uniformly at random from the bag. What is the minimum number of times this coin would need to appear heads in a row to be at least 95% certain that the coin we have is the double-headed coin?

Hint:

Use Bayes' Rule to get the probability that given we observe n consecutive heads, our coin is the double-headed coin.

Explanation:

Let p_n be the probability that given we observe n consecutive heads, our coin is the double-headed coin. Let H_n be the event that n consecutive heads are observed and D be the event that we selected the double-headed coin. We want $p_n = \mathbb{P}[D \mid H_n] = \frac{\mathbb{P}[D \cap H_n]}{\mathbb{P}[H_n]}$ by the definition of conditional probability. On the numerator, we can use the definition of conditional probability again to receive $\mathbb{P}[D \mid H_n] = \mathbb{P}[H_n \mid D]\mathbb{P}[D]$. The probability of selecting the double-headed coin is $\frac{1}{20}$. Given we select the double-headed coin, we will flip any amount of heads consecutively with probability 1, so $\mathbb{P}[H_n \mid D] = 1$.

On the denominator, we need to condition on whether or not we receive the double-headed coin. Thus,

$$\mathbb{P}[H_n] = \mathbb{P}[H_n \mid D]\mathbb{P}[D] + \mathbb{P}[H_n \mid D^c]\mathbb{P}[D^c]$$

We already calculated the first term in the numerator. For the second term, we know $\mathbb{P}[D^c] = 1 - \mathbb{P}[D] = \frac{19}{20}$. Then, $\mathbb{P}[H_n \mid D^c] = \frac{1}{2^n}$, as we flip a heads on a fair coin with probability $\frac{1}{2}$ in each turn. Therefore, $\mathbb{P}[H_n] = \frac{1}{20} + \frac{19}{20 \cdot 2^n}$. Therefore, $p_n = \frac{\frac{1}{20}}{\frac{1}{20} + \frac{19}{20 \cdot 2^n}} = \frac{1}{1 + \frac{19}{2^n}}$. Our goal is to find the smallest n such that $p_n > \frac{19}{20}$.

This means that $\frac{1}{1 + \frac{19}{2^n}} > \frac{19}{20}$. Multiplying by the denominator on both sides yields $1 > \frac{19}{20} + \frac{361}{20 \cdot 2^n}$. Multiplying by $20 \cdot 2^n$ on both sides yields $2^n > 361$, which occurs for $n \geq 9$. This means 9 times is the minimum number we need.

Answer:

9

Reference:

https://www.quantguide.io/questions/confident-double-heads

9 Appearance

Category: Brainteasers
Difficulty: Easy
Companies: Akuna

Question:

How many times does the digit 9 occur when counting 1 to 1000? Note that the number 919, for example, would count as 2 9s.

Hint:

The first thing is to count the number of times 9 appears in $1 - 99$. Can you use this to account for the different cases of the starting digit?

Explanation:

The first thing is to count the number of times 9 appears in $1 - 99$. Then, we can multiply this by 10, as there are 10 starting digits $0 - 9$ that can go in front of this. A starting digit of 0 here really means that the number is less than 100. Afterwards, we can add in 100 9s afterwards for the 900s to account for the starting digit of 9.

9 occurs in each of $9, 19, \ldots, 89$ once, yielding 9 9s. Afterwards, $90 - 99$ has 11 9s, as 99 counts as 2. Therefore, in $1 - 99$, the digit 9 occurs 20 times. Thus, the answer is $10 \cdot 20 + 100 = 300$.

Answer:

300

Reference:

https://www.quantguide.io/questions/9-appearance

Egg Drop

Category: Brainteasers
Difficulty: Hard
Companies: Akuna, Optiver

Question:

You are holding three identical eggs in a really large building with n stories. If an egg is dropped at an elevation under story X, then the egg will survive; else, the egg breaks. It is known that the egg will break at some floor number $1 - n$. What is the maximum value of n such that at most 9 drops are required to determine X in the worst-case scenario?

Hint:

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, make larger jumps at lower floors in the tower so that you can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, you could test $\frac{d(d+1)}{2}$ floors completely. The key here is to note that when you drop one egg and it breaks, the problem is reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what you do in the two egg case), make jumps that decrease by the maximum increment of the two egg case.

Explanation:

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, we make larger jumps at lower floors in the tower so that we can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, we could test $\frac{d(d+1)}{2}$ floors completely. The key here is to note that when we drop one egg and it breaks, we really get reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what we do in the two egg case), we are going to make jumps that decrease by the maximum increment of the two egg case. This is so that we are sufficiently able to search everything in a given range in the two egg case afterwards.

Putting this into math, we want to find a first cutoff point such that if the egg breaks at floor X on the first drop, we can examine the floors below it in $9 - 1 = 8$ trials. We can examine up to $\frac{8 \cdot 9}{2} = 36$ floors in 8 trials by Egg Drop I, so we should drop our first egg at floor 37. If it doesn't break, then we need to increment such that if it breaks on our next trial, we can examine the remaining space between 37 and the next floor in $9 - 2 = 7$ drops. We can explore up to $\frac{8 \cdot 7}{2} = 28$ floors with 7 drops, so we should place our next egg at floor 66 such that we can explore floors $38 - 65$ (28 floors) in the remaining 7 drops.

Continuing this pattern, when we have k drops left, we are going to increment by $\frac{k(k+1)}{2} + 1$ floors. This means that the next egg after would be placed at floor $66 + 21 + 1 = 88$, the egg after at $88 + 15 + 1 = 104$, and so on. The remaining floors to drop eggs at would be $115, 122, 126$, and

128. If it breaks at 128 but not at 126, we try 127 to verify the answer. If it doesn't break at 128, we try 129 instead and see if it breaks. If it breaks, we found our answer. If it doesn't break, then if $n = 130$, we would know that the egg breaks at floor 130, as it must break on one of the floors in our building. We can't explore any higher than 130 due to the fact that if it breaks at some point above 130, we would have to search between 129 and the breaking point, which takes strictly more than one egg drop. Therefore, $n = 130$ is our answer.

Answer:

130

Reference:

https://www.quantguide.io/questions/egg-drop

Numerical Triangle

Category: Brainteasers
Difficulty: Hard
Companies: Akuna

Question:

The integers 9 integers $1-9$ are put on the sides and vertices of an equilateral triangle. Any integer put on a vertex of the triangle will count for both sides intersecting at that vertex. If all three vertices have an integer fixed on them and the sum of the integers on each side (including the vertices) is 17, find number of distinct triangles that can be formed. Any triangle that can be formed as a rotation of another triangle is not considered distinct. Order of the integers on the sides is irrelevant as well.

Hint:

The sum of all the sides, including the vertices, must be 51. The sum of the integers $1-9$ is 45. Therefore, a sum of 6 must be on the three vertices so that they are double-counted and the sum turns out to be 51.

Explanation:

The sum of all the sides, including the vertices, must be 51. The sum of the integers $1-9$ is 45. Therefore, a sum of 6 must be on the three vertices so that they are double-counted and the sum turns out to be 51. The only way to do this is fix $1, 2,$ and 3 on the vertices, as these are the only 3 integers summing to 6.

Let A be the side with vertices 1 and 2, B be the side with vertices 1 and 3, and C be the side with vertices 2 and 3. We have the integers $4, 5, 6, 7, 8,$ and 9 left. We have to assign sums of $14, 13,$ and 12, respectively, to sides $A, B,$ and C.

We can get a sum of 14 by either using $6/8$ or $5/9$. This gives us two options for side A. Once we pick what is assigned to A, our remaining sides are fixed. Namely, if we assign $6/8$ to A. Then we must assign $4/9$ to side B and $5/7$ to side C. Similarly, if we assign $5/9$ to side A, then we must assign $6/7$ to side B and $4/8$ to side C. Therefore, there are only 2 arrangements of the triangle's values that are distinct.

All that is left is reflections of the triangle to evaluate distinctness. There are 3 reflection symmetries of an equilateral triangle corresponding to axes of symmetry from the three vertices. Therefore, we have 3 reflections that create distinct triangles. This means we have a total of $2 \cdot 3 = 6$ distinct answers.

Answer:

6

Reference:

https://www.quantguide.io/questions/numerical-triangle

Positively Normal

Category: Probability
Difficulty: Medium
Companies: Akuna, HRT

Question:

Suppose that X and Y are two IID standard normal random variables. Compute $\mathbb{P}\left[Y > \sqrt{3}X \mid Y > 0\right]$.

Hint:

The key idea here is that (X, Y) is radially symmetric about the origin. In other words, the density at each point of the circumference of the circle of radius $r > 0$ centered at the origin is constant.

Explanation:

The key idea here is that (X, Y) is radially symmetric about the origin. In other words, the density at each point of the circumference of the circle of radius $r > 0$ centered at the origin is constant. One can see this from the fact that the $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \dfrac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$, so the density at a point (x, y) depends on it's distance from the origin. This problem boils down to finding the angle swept our by our region of interest compared to the total region.

The region $\{Y > 0\}$ in the plane is the upper-half plane, which spans π radians. The region $Y > \sqrt{3}X$ covers all of the 2nd quadrant and also $\dfrac{\pi}{6}$ radians in the 1st quadrant, as the line $y = \sqrt{3}x$ makes an angle $\dfrac{\pi}{3}$ with the positive $x-$axis. Our region of interest is $\dfrac{\pi}{6} + \dfrac{\pi}{2} = \dfrac{2\pi}{3}$ radians of the total π radians, so our answer is just

$$\frac{\frac{2\pi}{3}}{\pi} = \frac{2}{3}$$

Answer:

23

Reference:

https://www.quantguide.io/questions/positively-normal

Confused Ant

Category: Probability
Difficulty: Medium
Companies: Akuna, Jane Street

Question:

An ant is standing on one vertex of a cube and can only walk along the edges. The ant is confused and moves randomly along the edges at random. How many edges, on average, will the ant travel to reach the opposite vertex of the cube?

Hint:

Define the problem as calculating the expecting hitting time within a Markov chain. What are your states and what are the transition probabilities? As an additional hint, there should not be eight states.

Explanation:

There are a total of eight vertices: one is the starting vertex denoted v_0, three are the vertices adjacent to the starting vertex denoted v_1, three are the vertices adjacent to the ending vertex denoted v_2, and one is the ending vertex denoted v_3. Let $E[v_i]$ be the expected number of edges travelled to arrive at v_3 from v_i. This problem has now been set up as calculating the expecting hitting time of a state within a Markov chain:

$$E[V_0] = 1 + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1]$$

$$E[V_1] = 1 + \frac{1}{3} \times E[V_0] + \frac{1}{3} \times E[V_2] + \frac{1}{3} \times E[V_2]$$

$$E[V_2] = 1 + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_3]$$

$$E[V_3] = 0$$

Substituting in $E[V_3]$ and solving, we see that $E[V_0] = 10$.

Answer:

10

Reference:

https://www.quantguide.io/questions/confused-ant

Zero Appearance

Category: Brainteasers
Difficulty: Easy
Companies: Akuna

Question:

How many positive integers at most 10000 have the number 0 somewhere?

Hint:

Perform complementary counting.

Explanation:

We should perform complementary counting. There are 10000 positive integers at most 10000. There are 9 single-digit integers that don't have a 0. There are 9^2 two-digit integers without 0. There are 9^3 three-digit integers that don't have 0. Lastly, there are 9^4 four-digit integers that don't have 0. Therefore, the answer is

$$10000 - (9 + 9^2 + 9^3 + 9^4) = 2620$$

Answer:

2620

Reference:

https://www.quantguide.io/questions/zero-appearance

Clock Angle II

Category: Brainteasers
Difficulty: Easy
Companies: Akuna, Drw, Jane Street, Old Mission, Optiver

Question:

What is the angle between the hands of a clock at 3:15 PM (in degrees)?

Hint:

Its not 0!

Explanation:

We know that both hands will be very close to each other but should note that the hour hand will be slightly further clockwise than the minute hand because its been 15 minutes into the 3^{rd} hour. In fact, it is precisely one-fourth of the way between the 3 and 4. Since there are 12 total space between numbers of the clock, each space has to occupy $30°$. Thus, $\frac{1}{4} \cdot 30° = 7.5°$.

Answer:

7.5

Reference:

https://www.quantguide.io/questions/clock-angle-ii

1 Head Up

Category: Probability
Difficulty: Easy
Companies: Akuna

Question:

3 fair coins are flipped, and you are told that at least one of the coins showed heads. Find the probability exactly one coin showed heads.

Hint:

Of the $2^3 = 8$ possible outcomes, all but TTT have at least one head. How many have exactly one head?

Explanation:

Of the $2^3 = 8$ possible outcomes, all but TTT have at least one head, so our sample space consists of $2^3 - 1 = 7$ equally-likely outcomes. Of these outcomes, 3 of them have exactly one head. Namely, these are HTT, THT, and TTH. Therefore, our probability is $\frac{3}{7}$.

Answer:

37

Reference:

https://www.quantguide.io/questions/1-head-up

Atm Option Pricing

Category: Finance
Difficulty: Medium
Companies: Akuna

Question:

You have an at-the-money call option with maturity $T = 1000$ years of a stock S with initial value 190. What is the price of this option at time-0 assuming Black-Scholes dynamics? Do not use a calculator.

Hint:

Relate this to the case where an option is deep-in-the-money. What does it behave like?

Explanation:

Think about this result intuitively. If $T = 1000$, then the option essentially acts like the underlying itself. This result can also be verified with Black-Scholes, taking $K = S_0$ and $t \to \infty$.

Answer:

190

Reference:

https://www.quantguide.io/questions/atm-option-pricing

Birthday Twins II

Category: Probability
Difficulty: Medium
Companies: Jump Trading, Five Rings, Two Sigma

Question:

Assuming 365 days in a year, how many people do we need in a class to make the probability that at least two people have the same birthday more than $\frac{1}{2}$?

Hint:

In order for the probability of at least two people having the same birthday to be greater than $\frac{1}{2}$, then the probability of no one having the same birthday must be less than $\frac{1}{2}$.

Explanation:

In order for the probability of at least two people having the same birthday to be greater than $\frac{1}{2}$, then the probability of no one having the same birthday must be less than $\frac{1}{2}$. Let there be n people in the class. The probability that there are no birthday collisions is $\frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-n+1}{365} < \frac{1}{2}$. The smallest such n is 23, which can be found via calculator.

Answer:

23

Reference:

https://www.quantguide.io/questions/birthday-twins-ii

Bubbly Sort

Category: Probability
Difficulty: Medium
Companies: Jump Trading

Question:

Suppose that you run one iteration of Bubble Sort on some random permutation of $(1, 2, \ldots, 100)$. Find the probability that the list is now sorted in order. The probability is in the form

$$\frac{a^b}{c!}$$

for integers a, b, and c. Find $a + bc$.

Hint:

There are 99 comparisons to be made in Bubble Sort. Namely, between positions i and $i + 1$ for $1 \leq i \leq 99$. At each iteration, you decide to either swap or skip the pair.

Explanation:

There are 99 comparisons to be made in Bubble Sort. Namely, between positions i and $i + 1$ for $1 \leq i \leq 99$. At each iteration, you decide to either swap or skip the pair. Each list of swaps or skips generates a unique original permutation that would be sorted after one iteration of the algorithm. For example, with $n = 3$, the sorted list would be $(1, 2, 3)$. Consider the two example cases below:

Case 1 - Swap Swap: We want to work backwards here, so first we unswap positions 2 and 3, yielding $(1, 3, 2)$. Afterwards, we unswap positions 1 and 2, yielding $(3, 1, 2)$.

Case 2 - Swap Skip: We skip switching positions 2 and 3. However, we swap positions 1 and 2, yielding $(2, 1, 3)$.

Generally, each of the $2^{100-1} = 2^{99}$ sequences of swap and skip will lead to a unique initial configuration that yields a sorted list after 1 iteration. Therefore, the solution is $\frac{2^{99}}{100!}$, meaning the answer to our question is $2 + 99 \cdot 100 = 9902$.

Answer:

9902

Reference:

https://www.quantguide.io/questions/bubbly-sort

Red Card Deal

Category: Probability
Difficulty: Medium
Companies: Jump Trading

Question:

A dealer presents to you the following game: The dealer is going to deal cards from the top of a standard deck one card at a time and turn them over on a table for you to see. You can tell the dealer to stop dealing at any time. If the next card is red, you win. Assuming optimal play, what is the highest probability you can achieve of being correct?

Hint:

Initially, it seems as if you can gain a small advantage in this game by waiting for the first moment when the red cards in the remaining deck outnumber the black. However, this may never happen. The question is now if that benefit outweighs the risk.

Explanation:

Initially, it seems as if you can gain a small advantage in this game by waiting for the first moment when the red cards in the remaining deck outnumber the black. However, this may never happen. The question is now if that benefit outweighs the risk. The answer is no, and we can show this now.

Suppose you elect to use some strategy, say S, to this game. Now, apply S to the game where instead of the next card being red, the last card in the deck is red. The probability of you winning is the same in each game, as each of the remaining unturned cards has the same probability of being red. However, the probability of winning in the second game, regardless of your strategy, is $1/2$, as we always ask for the last card to be red. Therefore, it doesn't matter what strategy you use for this game, as it will always be probability $1/2$ of winning.

Answer:

12

Reference:

https://www.quantguide.io/questions/red-card-deal

100 Factorial Digits

Category: Brainteasers
Difficulty: Medium
Companies: Jump Trading, SIG, Drw, Five Rings, Five Rings

Question:

How many digits are in 100!?

Hint:

What function can we use to get the number of digits in a number? Think about 100, where there are 3 digits.

Explanation:

First, we can see that $\log_{10}(100) = 2$, so to get the number of digits, we just have to add 1. Thus, we want to find $\log_{10}(100!) = \log(1) + \log(2) + \cdots + \log(100)$ and then add 1. We can approximate this as the integral of $\log_{10}(x)$ on $[1, 100]$.

Integrating $\int_1^{100} \log(x)\ dx \approx 157$. Adding 1, we get 158.

Answer:

158

Reference:

https://www.quantguide.io/questions/100-factorial-digits

Factorial Zeros

Category: Brainteasers
Difficulty: Easy
Companies: Jump Trading, SIG, Drw

Question:

How many zeros does the expansion of 100! have?

Hint:

We get a zero when we have a power of 10. As there are more integers divisible by 2 than by 5 in the first 100 integers, we really just need to count the exponent of 5 in the prime factorization of 100!.

Explanation:

We get a zero when we have a power of 10. As there are more integers divisible by 2 than by 5 in the first 100 integers, we really just need to count the exponent of 5 in the prime factorization of 100!. We get one power of 5 from each of the integers $5, 10, 15, \ldots, 100$, yielding an exponent of 20. However, we also need to account for the terms that are divisible by 5 multiple times. In this case, those would be divisible by 25, so these have an extra power of 5 that has not been stripped yet, so the integers $25, 50, 75$, and 100 have an extra power of 5 to strip, yielding 5^{24} in our prime factorization. Therefore, our answer is 24.

Answer:

24

Reference:

https://www.quantguide.io/questions/factorial-zeros

Splitwise

Category: Brainteasers
Difficulty: Easy
Companies: Jump Trading

Question:

You and seven friends go out for lunch. The subtotal is 182.30. You all decide to add a generous 60

Hint:

Let X be the subtotal. Then, the post-tip total is $1.6X$. What is your split?

Explanation:

Let X be the subtotal. Then, the post-tip total is $1.6X$, and your split is $\frac{1.6X}{8} = .2X$. Hence, your split is:

$$2 \times \frac{182.30}{10} = 36.46$$

Answer:

36.46

Reference:

https://www.quantguide.io/questions/splitwise

Coin On Chess Board

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

A chess board consists of 2 inch by 2 inch squares. We toss a coin (diameter of 1 inch) that lands somewhere on the board randomly. What is the probability that the coin is completely within one of the 2 inch by 2 inch squares (not on more than 1 square)?

Hint:

Instead of analyzing the whole chess board, think about the coin only landing inside one square.

Explanation:

Because of the layout and symmetry of squares, we only need to consider if the coin fits in a 2 inch by 2 inch square. If it doesn't, we know that it overlaps onto a neighboring square and thus doesn't fit the criteria. In order to find the probability that a coin is fully within a square, we can calculate the area that the center of the coin can land in and divide it by the total area of a square. Given the radius of the coin is 0.5 inches, it can't be any closer than 0.5 inches to any edge of the square. This gives a total area that the center can land in of $(2-0.5-0.5)^2 = 1$ square inch. The total area of the square (and thus total area that the center of the coin can land in) is $2^2 = 4$ square inches. This gives us the answer of $\frac{1}{4} = 0.25$.

Answer:

0.25

Reference:

https://www.quantguide.io/questions/coin-on-chess-board

Counting Digits

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Drw, Five Rings

Question:

How many digits are in 99 to the 99th power?

Hint:

$(1-\frac{1}{n})^n$ approaches $\frac{1}{e}$ as n approaches infinity. How can you manipulate 99^{99} to achieve this form?

Explanation:

There are 198 digits in 99^{99}.

$$99^{99} = 99^{99} \times (\frac{100}{100})^{99} = 100^{99} \times (\frac{99}{100})^{99} = 10^{198} \times (1 - \frac{1}{100})^{99} \approx 10^{198} \times (1 - \frac{1}{100})^{100} \approx 10^{198} \times \frac{1}{e}$$

Thus, 99^{99} has 198 digits.

Answer:

198

Reference:

https://www.quantguide.io/questions/counting-digits

Even Before Odd

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Suppose you are continually rolling a standard fair $6-$sided die. Find the probability that all of the odd values show up before the first even value.

Hint:

We need three things to happen: The first odd must show up before the first even, the second odd must show up before the first even, and the final odd must show up before the first even. This is really a chain of conditional probabilities, as the second odd showing up must occur given the first odd has already shown up.

Explanation:

We need three things to happen: The first odd must show up before the first even, the second odd must show up before the first even, and the final odd must show up before the first even. This is really a chain of conditional probabilities, as the second odd showing up must occur given the first odd has already shown up.

The probability the first odd shows up before the first even is $\frac{1}{2}$, as the first roll is either odd or even with equal probability, so the first roll must appear odd. Afterwards, given the first odd has appeared, we now have reduced our sample space to 5 outcomes of interest. If we roll the odd that has already appeared, it is as if we ignore it. Therefore, given one of the other 5 outcomes occurs, the probability it is odd is $\frac{2}{5}$, as two of the remaining 5 outcomes are odd integers. By the same logic, once the second odd has occurred, if we see either of those two odds again, we ignore. Conditional on one of the other 4 outcomes appearing, the probability that it is the final odd is $\frac{1}{4}$, as 1 of the last 4 outcomes is odd.

This implies that the probability of this event is $\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$.

Alternatively, consider all $6! = 720$ ways to arrange the orders of the appearance of the 6 values. We must have $1, 3, 5$ appear in the first 3 spots (in any order), while $2, 4, 6$ must appear in the last three spots in any order. There are $3! \cdot 3! = 6 \cdot 6 = 36$ ways to arrange the values satisfying the condition, so the probability of this occurring is $\frac{36}{720} = \frac{1}{20}$.

Answer:

120

Reference:

https://www.quantguide.io/questions/even-before-odd

3 Heads 3 Tails II

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Anna and Brenda are playing a game. They repeatedly toss a coin. Anna wins if 3 heads appear in a row. Brenda wins if 3 tails appear in a row. What is the expected number of coin tosses for a winner to be determined?

Hint:

Build a Markov Chain with 4 states (besides the absorption state)

Explanation:

After building a transition graph, we find can produce the following set of equations for the expcted time to absorption:

$$\mu_H = 1 + \frac{1}{2}\mu_T + \frac{1}{2}\mu_{HH}$$

$$\mu_T = 1 + \frac{1}{2}\mu_H + \frac{1}{2}\mu_{TT}$$

$$\mu_{TT} = 1 + \frac{1}{2}\mu_H$$

$$\mu_{HH} = 1 + \frac{1}{2}\mu_T$$

Solving, we find $\mu_T = 6$, meaning that beginning with state T, we expect 6 more flips before 3 in a row is reached. By symmetry, $\mu_H = 6$. By law of total expectation, we expect 6 more flips before 3 in a row is reached after the first coin has been flipped. Adding the first coin flip, our final answer is 7.

Answer:

7

Reference:

https://www.quantguide.io/questions/3-heads-3-tails-ii

Car Crashes

Category: Probability
Difficulty: Easy
Companies: SIG, Five Rings, Jane Street, Optiver

Question:

On a given busy intersection, the probability of at least one car crash in a 1 hour time interval is $\frac{8}{9}$. Assuming that car crashes occur independently of one another and at a constant rate throughout time, find the probability of at least one car crash in a 30 minute interval.

Hint:

Break up the hour into two 30-minute interval. If there is no car crash in an hour, there is no car crash in each of the independent 30-minute intervals.

Explanation:

Let the probability in question be p. We know that the probability there is no car crash in the hour interval is $\frac{1}{9}$ by complementation. This is the same as saying that there is no car crash in each of the intervals consisting of the first and last 30 minutes of the hour. By the question, we can say that the number of car crashes in disjoint intervals are independent. This is because they occur at a constant rate throughout time and the arrivals are independent. The probability of no car crash in each of those two intervals individually is $1-p$, so the probability of no car crash in both intervals is $(1-p)^2$. Thus, $(1-p)^2 = \frac{1}{9}$, so $p = \frac{2}{3}$.

Answer:

23

Reference:

https://www.quantguide.io/questions/car-crashes

Row Your Boat

Category: Brainteasers
Difficulty: Medium
Companies: SIG

Question:

A group of friends travels in a canoe. They travel upstream from their campsite for three hours. After a while, they want to go back home. However, they travel five hours downstream and end up 32 miles downstream below their initial campsite. The next morning, they travel back up the river to their campsite and arrive at 7:00 PM. If the river flows downstream at a constant rate of 2 miles per hour and the friends row their canoe at a constant speed, what time did they leave to go back home the next morning? Enter your answer in military time. For example, if the time is 4:30 AM, then enter 430 as your answer. If the answer is 4:30 PM, enter 1630 as your answer.

Hint:

Let x be the constant rowing speed in miles per hour. When travelling upstream, they move at $x-2$ miles per hour because the stream opposes them. How about travelling downstream?

Explanation:

The first order of business to find the speed at which the friends travelled. Let x be this constant speed in miles per hour. When travelling upstream, they move at $x-2$ miles per hour because the stream opposes them. When travelling downstream, the move at $x+2$ miles per hour. Therefore, we have that $3(x-2)-5(x+2)=-32$, as they travel upstream for 3 hours and downstream for 5 hours and end up 26 miles below their initial spot. Solving for x here yields $x=8$. Therefore, as they travel upstream going back to the campsite the next morning, they will be travelling at 6 miles per hour going back up. Therefore, it takes the friends $\frac{32}{6}=\frac{16}{3}$ hours to get back to their campsite. This is 5 hours and 20 minutes. This means they must have started at 1:40 PM. Therefore, our answer is 1340.

Answer:

1340

Reference:

https://www.quantguide.io/questions/row-your-boat

Windless Mile

Category: Brainteasers
Difficulty: Easy
Companies: SIG, Jane Street

Question:

A man riding on a bike can travel 1 mile in 3 minutes if the wind blows in the direction he is moving and in 4 minutes if it is against him. How fast can the man travel 1 mile if there is no wind?

Hint:

Thinking about this as a rate, let x be the (in miles/min) that the wind blows and r be the rate (in miles/min) that the man bikes at. Set up the equations in the question in terms of r and x.

Explanation:

Thinking about this as a rate, let x be the (in miles/min) that the wind blows and r be the rate (in miles/min) that the man bikes at. We know that $r + x = 1/3$ and $r - x = 1/4$ from the question above. Therefore, we get that $r = 7/24$ by solving this system, meaning it would take him 24/7 minutes to bike a mile windless.

Answer:

247

Reference:

https://www.quantguide.io/questions/windless-mile

3 Heads 3 Tails I

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Anna and Brenda are playing a game. They repeatedly toss a coin. Anna wins if 3 heads appear. Brenda wins if 3 tails appear. The heads and tails do not need to be consecutive. What is the expected number of coin tosses for a winner to be determined?

Hint:

The probability that Anna wins is $\frac{1}{2}$ by symmetry. Further, the expected total number of tosses for Anna to win is equal to the expected total number of tosses for Brenda to win by symmetry. By the law of total expectation, our desired answer is simply double the expected total number of tosses for Anna to win.

Explanation:

The probability that Anna wins is $\frac{1}{2}$ by symmetry. Further, the expected total number of tosses for Anna to win is equal to the expected total number of tosses for Brenda to win by symmetry. By the law of total expectation, our desired answer is simply double the expected total number of tosses for Anna to win. Anna wins in 3 total tosses with probability $\frac{1}{8}$. Anna wins in 4 total tosses with probability $\frac{3}{16}$ (there are 3 arrangements of 3 heads and 1 tail such that the last flip is heads). Anna wins in 5 total tosses with probability $\frac{6}{32}$ (there are 6 arrangements of 3 heads and 2 tails such that the last flip is tails). Putting it all together, we find the expected number of tosses for Anna to win is

$$\frac{1}{8} \cdot 3 + \frac{3}{16} \cdot 4 + \frac{6}{32} \cdot 5 = \frac{66}{32}$$

Multiplying by 2 gives us $\frac{33}{8}$.

Answer:

338

Reference:

https://www.quantguide.io/questions/3-heads-3-tails-i

Stone Ripple

Category: Pure Math
Difficulty: Easy
Companies: SIG

Question:

A stone is thrown into a pond and sends out a circular ripple whose radius increases at a rate of 2 feet per second. After 10 seconds, how fast is the area of the ripple increasing (in square feet per second)? The answer is in the form $k\pi$ for an integer k. Find k.

Hint:

We know that $A(r) = \pi r^2$ as a function of r and that $r' = 2$. After 10 seconds, the radius of the circle is $2 \cdot 10 = 20$ feet.

Explanation:

We know that $A(r) = \pi r^2$ as a function of r and that $r' = 2$. After 10 seconds, the radius of the circle is $2 \cdot 10 = 20$ feet. Lastly, we find that $A' = 2\pi r r'$, so plugging in our values, we get that the rate of increase is $2\pi(20)(2) = 80\pi$. Therefore, $k = 80$.

Answer:

80

Reference:

https://www.quantguide.io/questions/stone-ripple

112 Appearance

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Suppose you roll a fair 6−sided die until you either obtain a 2 for the first time or 2 consecutive 1s. Find the expected number of rolls you perform.

Hint:

Let μ represent the mean number of rolled needed. Apply law of total expectation here to compute μ based on the first roll.

Explanation:

Let μ represent the mean number of rolled needed. We are going to apply law of total expectation here to compute μ based on our first roll. If our first roll is 2, we are done. If our first roll is 1, then we only need one more 1 until we are done. We will call the expected number of rolls to finish starting with a 1 in our sequence μ_1. Otherwise, we just are at our current state again. Together, these imply that

$$\mu = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot (1 + \mu_1) + \frac{2}{3} \cdot (1 + \mu)$$

To compute μ_1, note that rolling either a 1 or 2 will make the game finish in 1 turn. Otherwise, we are back to the "nothing" state. Therefore,

$$\mu_1 = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (1 + \mu) = 1 + \frac{2}{3}\mu$$

Substituting this into the first equation, we get that

$$\mu = 1 + \frac{1}{6} \cdot \left(1 + \frac{2}{3}\mu\right) + \frac{2}{3}\mu$$

Rearranging this yields $\frac{2}{9}\mu = \frac{7}{6}$, so $\mu = \frac{21}{4}$.

Answer:

214

Reference:

https://www.quantguide.io/questions/112-appearance

Always Profit II

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

QuantGuide stock will either double its value or go down 50% by tomorrow. You can also bet whether the stock goes up or down with your friend at 1 : 1 payout. Let A and B be the lowest round bet (can only bet whole dollars) amount you long/short the stock and bet on the stock going up/down with your friend respectively. Assume that if you are betting, you want to make the same amount profit no matter the outcome of the stock's movement tomorrow (bank balance will be the same no matter the outcome). How much do you profit from betting A and B? If you can't guarantee profit, enter 0.

Hint:

Make equations for profits including returned bets for all scenarios.

Explanation:

Something to notice about this situation is that being long on the stock is better than betting the stock goes up against your friend because in the case the stock goes down, you'll still retain 50% of your investment in the stock. Thus we should look to go long on the stock and bet the stock goes down against your friend. The profit (including returned bets) equations are as follows:

Stock goes up: $A - B$

Stock goes down: $B - 0.5A$

Making these two equations equal, we get $A = 4B/3$. Thus A and B are equal to \$4 and \$3 respectively which allows us to always profit \$1.

Answer:

1

Reference:

https://www.quantguide.io/questions/always-profit-ii

7 Multiple

Category: Probability
Difficulty: Easy
Companies: SIG, SIG

Question:

You and your friend play a game in which you take turns rolling a fair $6-$sided die and keep a running tally of the sum of the upfaces obtained in each roll. The winner of the game is the person who most recently rolled the die when the running sum first becomes a multiple of 7. You get to decide whether to go first or second. Under rational strategy from you, what is your probability of winning?

Hint:

Do you have a chance of winning on the first roll if you go first? What if you go second?

Explanation:

Intuitively, it is clear that you should select to go second. This is because you have no chance of winning on the first roll, but every roll after the first roll, there is a 1/6 probability that you win the game, as exactly 1 of the 6 possible values on the will make the sum a multiple of 7.

Let's compute the probability that you, the second player to roll, wins. Call this probability p. By conditioning on your first roll, there is a 5/6 probability you do not roll a value resulting in a sum divisible by 7 on the first turn. To come back to you, you would need your friend to also not roll a value resulting in a sum divisible by 7, which occurs with probability 5/6 too. In this case, your probability of winning when it comes back to you is p. Alternatively, you do roll the value that results in a sum divisible by 7, occurring with probability 1/6. Therefore, we have the equation

$$p = \frac{5}{6} \cdot \frac{5}{6}p + \frac{1}{6} \iff p = \frac{6}{11}$$

Answer:

611

Reference:

https://www.quantguide.io/questions/7-multiple

Socks And Shelves

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

There are two shelves, the top and bottom. The top shelf contains red socks with probability 0.4 and black socks with probability 0.6. The bottom shelf contains red socks with probability 0.7 and black socks with probability 0.3. I pick a shelf randomly. From that self, I picked 2 red socks. What is the probability that those 2 red socks come from the top shelf? Assume you have so many socks that taking from either shelf does not change the probability of future socks that are drawn.

Hint:

What methods can we use to solve conditional probability questions?

Explanation:

This is a classic conditional probability question. A sure fire way to do this problem is to use Bayes Theorem. Applying Bayes for

$$\mathbb{P}[\text{Top Shelf}|RR] =$$

$$\frac{\mathbb{P}[RR|\text{Top Shelf}] \cdot \mathbb{P}[\text{Top Shelf}]}{\mathbb{P}[RR]} = \frac{0.4 \cdot 0.4 \cdot 0.5}{0.4 \cdot 0.4 \cdot 0.5 + 0.7 \cdot 0.7 \cdot 0.5} = \frac{16}{65}$$

Answer:

1665

Reference:

https://www.quantguide.io/questions/socks-and-shelves

Paired Pumpkins II

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Jane Street

Question:

Dracula has 5 pumpkins. When he pairs any two pumpkins, they weigh $21, 22, \ldots, 30$ pounds. What's the sum of the weights of all the pumpkins?

Hint:

Is there anything you can do with all the paired weights to directly solve for the sum of all the pumpkins?

Explanation:

Let a, b, c, d, and e be the weight of each pumpkin from lightest to heaviest. If we sum all the pairwise weights, we know we will have 4 sets of a, b, c, d, and e. Thus

$$4 \cdot (a + b + c + d + e) = 21 + \cdots + 30 = 255$$

or

$$a + b + c + d + e = \frac{255}{4}$$

You could also create equations for all the pairs to find the weights of each pumpkin but the question only asks for the sum of the weights and the above method is a clever way of reaching the answer.

Answer:

2554

Reference:

https://www.quantguide.io/questions/paired-pumpkins-ii

Prime Janitors

Category: Brainteasers
Difficulty: Hard
Companies: SIG

Question:

Janitors at a large trading company come to work to see 100 open doors. These doors are labeled 1 to 100. Throughout the day, janitors open/close doors based off of their number. The first janitor closes every door that's a multiple of 2. Then the next janitor closes every door that's a multiple of 3 unless its currently closed, in which case he opens it back up. This will continue with by having the kth janitor open/close all door numbers that are a multiple of the kth positive prime integer. How many doors are open at the end of the day?

Hint:

For a door to be closed at the end of the day, this means that it must have been touched an odd amount of times. What does this say about the number of distinct prime factors?

Explanation:

Looking at this scenario a bit longer, you'll see that the opening/closing of doors are dependent on their factors. For example, door 2 starts opened and is closed by the first janitor. Door 3 is going to be closed by the second janitor. However, door 6 is closed by the first janitor, but reopened by the second janitor. In general, door k will be closed at the end of the day if it has an odd number of distinct prime factors. This is all of the integers that are prime integers themselves, powers of prime integers (both have 1 factor), or integers that have 3 distinct prime factors.

The primes that are at most 100 are: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$ (25 integers)

The powers of primes that are at most 100 are: $4, 8, 16, 32, 64, 9, 27, 81, 25, 49$ (10 integers)

Integers that are at most 100 with 3 prime factors are:

$$2 \cdot 3 \cdot 5 = 30$$

$$2 \cdot 3 \cdot 7 = 42$$

$$2 \cdot 3 \cdot 11 = 66$$

$$2 \cdot 3 \cdot 13 = 78$$

$$2 \cdot 5 \cdot 7 = 70$$

$$2^2 \cdot 3 \cdot 5 = 60$$

$$2^2 \cdot 3 \cdot 7 = 84$$

$$2 \cdot 3^2 \cdot 5 = 90$$

There are 8 integers above. This means that there are $25 + 10 + 8 = 43$ closed doors, meaning that $100 - 43 = 57$ are still open.

Answer:

57

Reference:

https://www.quantguide.io/questions/prime-janitors

Better In Red I

Category: Probability
Difficulty: Medium
Companies: SIG, Five Rings, Jane Street, HRT

Question:

A $3 \times 3 \times 3$ cube is composed of 27 $1 \times 1 \times 1$ cubes that are white by default. All the surfaces of the $3 \times 3 \times 3$ cube are painted red and then the cube is disassembled such that all orientations of all cubes are equally likely. You select a random small cube and notice that all 5 sides that are visible to you are white. What is the probability the last face not visible to you is red?

Hint:

Consider Bayes' Rule and the orientation of the cube.

Explanation:

Let R be the event that the last side not visible to you is red and W be the event that the other sides visible are not red. We want $\mathbb{P}[R \mid W] = \dfrac{\mathbb{P}[R \cap W]}{\mathbb{P}[W]}$. For $\mathbb{P}[R \cap W]$, we need the last side to be red and have that side be facing away from us. Since the orientation is random, the probability the red side is facing away from us is $\dfrac{1}{6}$. In addition, 6 of the 27 cubes have exactly one red side, so $\mathbb{P}[R \cap W] = \dfrac{1}{27}$. Then, in the denominator, we can have 5 white sides from choosing a cube that would satisfy the numerator event or just choosing the center cube. The center cube is white on all sides so there is no need to orient it. Therefore, $\mathbb{P}[W] = \dfrac{1}{27} + \dfrac{1}{27} = \dfrac{2}{27}$. Substituting in, we get $\mathbb{P}[R \mid W] = \dfrac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/better-in-red-i

Resell Painting

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street

Question:

You are currently bidding for a painting. You know that the value of the painting is between \$0 and \$100,000 uniformly. If your bid is greater than the value of the painting, you win and sell it to an art museum at a price of 1.5 times the value. What's your bid to max your profit? If you can not profit, bid \$0.

Hint:

Do you lose out on anything if your bid doesn't win the painting?

Explanation:

If your bid is lower than the value of the painting, you won't lose anything, so no harm no foul. However, let's say that you do win the painting. This means that you bid higher than the actual value. Let x be your bid. Given you won the bid, we know that the the value is at most x. Therefore, the conditional distribution of the value of the painting is $\text{Unif}(0, x)$. On average, the value of the painting will be $\frac{x}{2}$. This means we will be able to sell it, on average, for $\frac{1.5 \cdot x}{2} = \frac{3}{4}x$. However, this is less than x, the amount we paid for the painting. So we shouldn't bid on this painting, thus the answer is \$0.

Answer:

0

Reference:

https://www.quantguide.io/questions/resell-painting

Three Repeat II

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street, Old Mission, Optiver

Question:

A fair coin is flipped 6 times. Find the probability of obtaining exactly 3 consecutive heads somewhere in the 6 flips.

Hint:

What are the possible forms of such a sequence? Do you notice any symmetry in the sequences?

Explanation:

We have 4 forms that the sequence can be in, which are $HHH---$, $-HHH--$, $--HHH-$, and $---HHH$. Note that the first and second are respectively reflection of the fourth and third sequences, so we can just do the first two sequences and then multiply by 2. For the first sequence, the 4th flip must be tails, but the other two can be either parity, so this yields 4 possibilities. For the second outcome, the first and second dashes must be T, but the last can be either, so this yields 2 outcomes. Therefore, we have a total of $2(4+2) = 12$ sample points that have a single run of three heads, so the probability is $\frac{12}{64} = \frac{3}{16}$.

Answer:

316

Reference:

https://www.quantguide.io/questions/three-repeat-ii

Prime Pair

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street

Question:

You roll two fair $6-$sided dice. Each of the two dice have the first 6 prime numbers on the sides. Find the probability that the sum of the two upfaces is also prime?

Hint:

The numbers on the sides would be $2, 3, 5, 7, 11,$ and 13. Therefore, the sum of the two upfaces has to be between 4 and 26, inclusive. What must one of the dice show for the sum to be prime?

Explanation:

The numbers on the sides would be $2, 3, 5, 7, 11,$ and 13. Therefore, the sum of the two upfaces has to be between 4 and 26, inclusive. The prime integers in this interval are $5, 7, 11, 13, 17, 19,$ and 23. We now need to determine how each prime can be obtained from these dice. One thing to note is that 2 must be one of the rolls, as all other values on the die are primes larger than 2, which must be odd. Therefore, the outcomes are just primes p such that $p - 2$ is also a prime and $p - 2 \leq 13$. The values of p where this holds true is $p = 5, 7,$ and 13. Each of these have two permutations of the die outcomes that yield that sum. Therefore, 6 such outcomes of the $6^2 = 36$ yield a prime sum, so our answer is

$$\frac{6}{36} = \frac{1}{6}$$

Answer:

16

Reference:

https://www.quantguide.io/questions/prime-pair

Optimal Card Pick

Category: Brainteasers
Difficulty: Medium
Companies: SIG

Question:

You are given a standard 52 card deck. You may select any 5 cards from the deck to be your hand as long as they form a full house (3 cards of one rank and 2 cards or another). Then, your opponent receives 5 random cards from the remaining 47. Let Jack, Queen, King, and Ace have the values $11 - 14$. The value of a full house is the sum of all the values of the cards that comprise it. For example, 22299 has value 24. Find the sum of the values of all optimal full houses. We define optimal here as maximizing the probability of winning with respect to standard poker rules.

Hint:

$AAAKK$ is not optimal because the kings do not block many straight flushes, which are the main things that can defeat a full house here.

Explanation:

Among all full houses, selecting AAA confirms that no other full house will defeat our hand. Therefore, we now want to select our paired rank as to minimize the probability of losing. In this case, we want to minimize the probability of four of a kind or a straight flush. The probability of a four of a kind is the same regardless of which two kicker cards we select, so this doesn't give us any information. However, we do want to minimize the probability of a straight flush. This means we want to select the rank that is involved in the most number of straights.

There are fewer straight that involve Aces since we already selected 3 Aces. Therefore, any rank possessing a straight involving Aces is eliminated. We see that $2 - 5$ and $10 - K$ are all eliminated with this logic. The remaining ranks would be $6 - 9$, and one can check that there are equal amounts of straights involving each of these values and not Aces. Since no other full house can defeat our hand, we are indifferent to each of these 4 hands. This means we would pick any of $AAA66, AAA77, AAA88$, or $AAA99$ to be our hand. The value of each hand is, respectively, $54, 56, 58$, and 60, so the sum of these values is 228.

Answer:

228

Reference:

https://www.quantguide.io/questions/optimal-card-pick

Dog Days

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Every day is either good (G) or bad (B). If today was G, tomorrow is also G with probability $\frac{3}{5}$. If today was B, tomorrow is also B with probability $\frac{7}{10}$. You woke up to find that today is a bad day. Find the expected number of days until another bad day.

Hint:

How can you use the Law of Total Expectation to condition on the outcome of outcome of the first day?

Explanation:

Let T be the number of days until another bad day and X_1 be the outcome of the first day since today. Then $\mathbb{E}_B[T] = \mathbb{E}_B[T \mid B]\mathbb{P}_B[B] + \mathbb{E}_B[T \mid G]\mathbb{P}_B[G]$. The subscript B is to denote that we start on a bad day. We have that $\mathbb{P}_B[B] = \frac{7}{10}$, so $\mathbb{P}_B[G] = \frac{3}{10}$. Then, $\mathbb{E}_B[T \mid G] = 1 + \mathbb{E}_G[T]$, as we take one step and then find the expected number of days starting from a good day until a bad day. Then, $\mathbb{E}_B[T \mid B] = 1$, as it takes just 1 day to get another bad day if the first day after is bad. Therefore,

$$\mathbb{E}_B[T] = (1 + \mathbb{E}_G[T]) \cdot \frac{3}{10} + 1 \cdot \frac{7}{10} = 1 + \frac{3}{10} \cdot \mathbb{E}_G[T]$$

Note that the distribution of the number of days until a bad day starting from a good day is $\text{Geom}\left(\frac{2}{5}\right)$, as there is a $\frac{2}{5}$ probability each day of going to a bad day, so $\mathbb{E}_G[T] = \frac{5}{2}$. This implies that $\mathbb{E}_B[T] = 1 + \frac{3}{10} \cdot \frac{5}{2} = \frac{7}{4}$.

Answer:

74

Reference:

https://www.quantguide.io/questions/dog-days

Say 50

Category: Brainteasers
Difficulty: Medium
Companies: SIG

Question:

You and your friend are playing a game where you each pick an integer from 1 to 10 inclusive and add that to a running total. The person that states 50 is the winner. You get to go first, what number do you say first?

Hint:

Work backwards from 50.

Explanation:

Notice here that if you say 39, its functionally checkmate because you can always compliment whatever your opponent adds to make 50. Lets say they say 1 and bring the total to 40. Then you can say 10 and win. No matter what your opponent says, you can always compliment them to add 11 between both of your turns. Now we can work backwards. If we want to say 39, saying 28 is again, functionally checkmate because you can compliment whatever your friend says to make 39. Go back another 11 and we get 17. One more time and we get 6. Thus you should say the number 6 first and then compliment your friend to keep adding 11 to this 6 till it becomes 50.

Answer:

6

Reference:

https://www.quantguide.io/questions/say-50

Rolls In A Row

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

On average, how many rolls of a fair 6-sided die must be rolled on average to get two 6s in a row?

Hint:

Think about the different states up to getting two 6s in a row.

Explanation:

We can use Markov Chains for this problem. Let E_0 be the expectation state of having no 6s in a row. Let E_1 be the expectation state of having one 6 in a row (so 6 is the most recent roll). Finally, let E_2 be the expectation state of having two 6s in a row (our goal). Then our equations become:

$$E_0 = \frac{1}{6}(E_1 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_1 + \frac{5}{6}E_0 + 1$$

$$E_1 = \frac{1}{6}(E_2 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_2 + \frac{5}{6}E_0 + 1$$

$$E_2 = 0$$

Solving these equations, we get $E_0 = 42$. Thus it takes 42 rolls on average to get two 6s in a row.

Answer:

42

Reference:

https://www.quantguide.io/questions/rolls-in-a-row

Fish Capture

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

You are at a pond. One day, you decide to catch 50 fish and tag them. The next day, you capture 40 fish and 10 of them have tags. What is your best guess at the number of fish in the pond?

Hint:

$\frac{1}{4}$ of the captured fish were tagged.

Explanation:

$\frac{1}{4}$ of the captured fish were tagged. Therefore, this implies that you tagged roughly $\frac{1}{4}$ of the fish population in the pond. This means there are roughly $50 \cdot 4 = 200$ fish in the pond.

Answer:

200

Reference:

https://www.quantguide.io/questions/fish-capture

Rps Galore

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

Jack and Jane play Rock (R) Paper (P) Scissors (S) 10 times. It is known that each round had a winner. Additionally, it is known that Jack played R, P, and S, respectively, $3, 1$, and 6 times, while Jane played R, P, and S, respectively, $2, 4$, and 4 times. How many wins did Jack have? If it is not possible to know, answer "-1".

Hint:

Note that if Jack plays S, Jane must have played either R or P to get a distinct winner.

Explanation:

Note that if Jack plays S, Jane must have played either R or P to get a distinct winner. Therefore, from the 6 times Jack played S, they would have 4 wins from the times that Jane played P and 2 losses from the two R. From the last 4 games, Jane played S 4 times, so they would 3 losses to the three R from Jack and 1 win from the time that they played P. Therefore, Jack has 7 total wins.

Answer:

7

Reference:

https://www.quantguide.io/questions/rps-galore

Card Turner

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Suppose you have 20 cards with the values $1-10$ each appearing twice in a deck. You draw 2 cards at a time uniformly at random from the 20 cards. If they match in value, you remove them from the deck. Otherwise, they are put back into the deck. The game finishes once there are no more cards to draw. Each drawing of two cards is a turn. Find the expected number of turns needed to finish the game.

Hint:

Let T be the total duration of the game. Then $T = T_1 + T_2 + \cdots + T_{10}$, where T_i is the amount of draws needed to go from $i-1$ pairs drawn to i pairs drawn.

Explanation:

Let T be the total duration of the game. We are going to generalize this to n pairs of cards. Then $T = T_1 + T_2 + \cdots + T_n$, where T_i is the amount of draws needed to go from $i-1$ pairs drawn to i pairs drawn. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[T_i]$. We need to find $\mathbb{E}[T_i]$ now. If $i-1$ pairs are already drawn, then there are $2n - 2(i-1) = 2(n-i+1)$ cards left in the deck. For each drawing, fix the first card. The probability the second card matches that first card is $\frac{1}{2(n-i+1)-1}$, as there are $2(n-i+1)-1$ cards left after the first of the pair is drawn, and only 1 of the cards is the same value as first card. Therefore, as this is independent between trials, $T_i \sim \text{Geom}\left(\frac{1}{2(n-i+1)-1}\right)$, meaning $\mathbb{E}[T_i] = 2(n-i+1)-1$. Therefore,

$$\mathbb{E}[T] = \sum_{i=1}^{n} 2(n-i+1) - 1 = \sum_{i=1}^{n} (2i-1) = n^2$$

In particular, $n = 10$ here, so the answer is 100.

Answer:

100

Reference:

https://www.quantguide.io/questions/card-turner

Maximal Variance

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Let X be any random variable only defined on $[-1, 1]$. Find the maximum possible value of $\text{Var}(X)$.

Hint:

For the variance to be a maximum, we need to optimize in two ways. Namely, we have to maximize the spread from the mean as well as the randomness about that mean. Think about discrete vs. continuous.

Explanation:

For the variance to be a maximum, we need to optimize in two ways. Namely, we have to maximize the spread from the mean as well as the randomness about that mean. To maximize the distance of all points away from the mean, we should have a discrete random variable. This is because a continuous random variable has density throughout an interval, which makes it more compact about the mean. We can't have just one value in our discrete random variable, as then the variance would be 0. Therefore, with 2 values, we should aim to separate them from the mean the furthest. This can be done by taking the values -1 and 1, as those are the endpoints of our allowed interval. Then, to maximize how "random" our distribution is, we should set $p = \frac{1}{2}$. This would maximize something called the entropy. Note that with $p = \frac{1}{2}$, the mean is 0.

Since the mean is 0, all that needs to be computed is $\mathbb{E}[X^2]$, where X is as above. however, X^2 is just 1 with probability 1, so $\mathbb{E}[X^2] = 1$. Thus, $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1$.

Answer:

1

Reference:

https://www.quantguide.io/questions/maximal-variance

Six Card Sum

Category: Probability
Difficulty: Medium
Companies: SIG, Five Rings, Jane Street

Question:

Jamie is told there are 3 aces and 3 jacks in a pile. Each turn, a card is drawn without replacement; Jamie earns $1 if he guesses the drawn card correctly. Jamie plays 6 turns under the optimal strategy. How much money should Jamie expect to earn?

Hint:

Take a "dynamic programming" approach in the sense that you compute the expected payout based on what happened before.

Explanation:

Let $X_1, X_2, \ldots, X_6$ denote the amount of money won on turn $1, 2, \ldots, 6$, respectively. By linearity of expectation, the total amount of money won is simply $\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_6]$. Clearly, $\mathbb{E}[X_1] = \frac{1}{2}$, so let's begin our discussion with the second turn.

After Jamie makes a guess from round 1, regardless of the guess and the true card, Jamie will have information for the correct 3 card-2 card split of the remaining 5 cards. Under the optimal strategy, Jamie should guess the value of the card that appears 3 times in the remaining 5 cards. Hence, $\mathbb{E}[X_2] = \frac{3}{5}$.

The expected value of round 3 depends on the previous two guesses; specifically, there is either a 3-1 split or a 2-2 split. Once again, Jamie should know the true split and guess accordingly. It is not difficult to conclude that the probability of a 3-1 split is $\frac{2}{5}$, since there are 20 total orderings of 3 aces and 3 jacks, and $\frac{4}{1} \cdot 2$ of those orderings begin with ace-ace or jack=jack. That means that there is a $\frac{3}{5}$ chance that there is a 2-2 split. By the law of total expectation, we have $\mathbb{E}[X_3] = \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}$.

The expected value of round 4 can be computed similarly. There is either a 2-1 split or a 3-0 split. The 3-0 split occurs with probability $\frac{1}{10}$, while the 2-1 split occurs with probability $\frac{9}{10}$. By linearity of expectation, we find $\mathbb{E}[X_4] = \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \frac{2}{3} = \frac{7}{10}$.

The expected value of round 5 can be conditioned on a 2-0 split which occurs with probability $\frac{2}{5}$ or a 1-1 split which occurs with probability $\frac{3}{5}$. By linearity of expectation, we find $\mathbb{E}[X_5] = \frac{2}{5} \cdot 1 + \frac{3}{5} \cdot \dfrac{1}{2} = \frac{7}{10}$.

By round 6, there is only one card remaining, so $\mathbb{E}[X_6] = 1$. Putting it all together, our expected value is $\frac{1}{2} + \frac{3}{5} + \frac{3}{5} + \frac{7}{10} + \frac{7}{10} + 1 = \frac{5+6+6+7+7+10}{10} = \frac{41}{10}$.

Answer:

4.1

Reference:

https://www.quantguide.io/questions/six-card-sum

Same Heads

Category: Probability
Difficulty: Easy
Companies: SIG, Optiver

Question:

2 people flip a fair coin 4 times each and record their flips. What is the probability that the two people flipped the same number of heads?

Hint:

Find the probability of just one person getting each number of heads.

Explanation:

To solve this question, we first need to find the probability that each player gets each number of heads. We can use the binomial formula for this.

$$\mathbb{P}[0 \text{ heads}] = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$\mathbb{P}(1 \text{ head}) = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$\mathbb{P}(2 \text{ heads}) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

$$\mathbb{P}(3 \text{ heads}) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$\mathbb{P}(4 \text{ heads}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Since the probability of rolling a certain amount of heads is independent for the two people, we can just square all these probabilities to find the probability that two people get the same number of heads.

$$\mathbb{P}(0 \text{ heads for both}) = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

$$\mathbb{P}(1 \text{ head for both}) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\mathbb{P}(2 \text{ heads for both}) = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

$$\mathbb{P}(3 \text{ heads for both}) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\mathbb{P}(4 \text{ heads for both}) = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

Summing up these probabilities, we get

$$\mathbb{P}(\text{same \# of heads}) = 2 \cdot \frac{1}{256} + 2 \cdot \frac{1}{16} + \frac{9}{64} = \frac{35}{128}$$

Answer:

35128

Reference:

https://www.quantguide.io/questions/same-heads

Taylor Sum

Category: Pure Math
Difficulty: Medium
Companies: SIG

Question:

Evaluate $f^{(7)}(0)$, where $f(x) = (1 + x + 2x^2 + 3x^3)e^{x^2}$.

Hint:

Write $e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$. Plug this in and expand out the sums. Then, you need to sum up $c_7 \cdot 7!$ in each sum, where c_7 is the coefficient of x^7.

Explanation:

Using Taylor Series, we can write $e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$. Therefore,

$$f(x) = (1 + x + 2x^2 + 3x^3) \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + 2x^2 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + 3x^3 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

Multiplying in the powers of x, we have that

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} + 2\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k!} + 3\sum_{k=0}^{\infty} \frac{x^{2k+3}}{k!}$$

To find the 7th derivative at 0, we need to sum up $c_7 \cdot 7!$ in each sum, where c_7 is the coefficient of x^7. However, we quickly see that the first and third sums do not have any such term, as all of their terms have even powers. Therefore, those can immediately be eliminated. Looking at the second sum, we see that $c_7 = \frac{1}{3!} = \frac{1}{6}$, as we get an exponent of 7 with $k = 3$. Therefore, we get a contribution of $\frac{7!}{3!}$ from that term. From the last sum, we see that $k = 2$ yields an exponent of 7, so the coefficient there is $\frac{3}{2!} = \frac{3}{2}$, so that sum adds a total of $\frac{3}{2} \cdot 7!$. Adding these two terms up, we get a sum of 8400, which we can conclude to be $f^{(7)}(0)$.

Answer:

8400

Reference:

https://www.quantguide.io/questions/taylor-sum

Poisoned Kegs

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Five Rings, Two Sigma, Jane Street, Old Mission

Question:

A king has 10 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants only agree to participate in the wine tasting for 1 month. What is the maximum value of n such that the king is guaranteed to have at least a 10% chance to determine which keg among the n is poisoned?

Hint:

Use a similar idea to Poisoned Kegs II, but now the number of kegs needs to only be narrowed down to at most 10 possibilities.

Explanation:

Using the same idea as Poisoned Kegs II, we can let each of the 10 servants represent a binary indicator for the first 2^{10} kegs. Thus, in the first 2^{10} kegs, we will know exactly which keg is poisoned. However, now we just need to narrow it down to 10 kegsper possible death sequence. The easiest way to do this is to have each of the 2^{10} subsets of servants test 10 different kegs. Then, if some subset of servants die, the king will know that one of those 10 kegs that the subset of servants died to has the poison. Therefore, the king can test 10 times as many kegs if he only wants at least a 10% chance of identifying it. Therefore, the king can test up to $n = 2^{10} \cdot 10 = 10240$ kegs

Answer:

10240

Reference:

https://www.quantguide.io/questions/poisoned-kegs

Dice Roll Intuition

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Do not use a pen for the question. We are playing a game; we roll a fair, 6-sided die repeatedly. We stop playing the game when we get two specific numbers in a row; the payoff is equal to the total number of rolls of the die. For our two specific numbers, should we have a preference for (case 1) 4 then 5, or (case 2) 4 then 4, or (case 3) would it not matter? Respond with the number of the case.

Hint:

Think about which one takes more time to appear on average.

Explanation:

Consider the transition graph for both cases, where we consider three states: (A) no specific numbers, (B) the first specific number, and (C) 2 specific numbers in a row. There is a slight difference in the two transition graphs; namely, there is a $\frac{2}{3}$ probability of returning to state A when we work with case 1, whereas there is a
$\frac{5}{6}$ probability of returning to state A when considering case 2. Since we want to maximize the number of rolls it takes for us to finish the game, we would prefer case 2.

Answer:

2

Reference:

https://www.quantguide.io/questions/dice-roll-intuition

Leg Count

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

A group of chickens (2 legs), cows (4 legs), and spiders (8 legs) are hanging out. There are a total of 440 legs between all of them. There are twice as many chickens as cows and twice as many spiders as chickens. How many spiders are there?

Hint:

Let x be the number of cows. Then $2x$ is the number of chickens and $2(2x) = 4x$ is the number of spiders.

Explanation:

Let x be the number of cows. Then $2x$ is the number of chickens and $2(2x) = 4x$ is the number of spiders. Therefore, with x cows, there are $4x + 2(2x) + 8(4x) = 40x$ legs. Therefore, $40x = 440$, so $x = 11$. The number of spiders then is $4 \cdot 11 = 44$.

Answer:

44

Reference:

https://www.quantguide.io/questions/leg-count

Pick Your Urn Wisely

Category: Probability
Difficulty: Medium
Companies: SIG, Drw, Jane Street

Question:

You have 2 indistinguishable urns in front of you. One of the urns has 4 \$1 chips and 6 \$10 chips. The other urn has 3 \$1 chips and 7 \$10 chips. You reach into one urn at random and select a \$1 chip. You are given the opportunity to pick another chip at random either from same urn as the first chip or at random from the other urn. Your payout is the value of the second chip you select. Under optimal gameplay, what is your expected payout?

Hint:

Calculate the expected payout if you switch urns versus if you stay with your current urn. For notational purposes, let urn 1 be the urn that started with 4 \$1 chips and urn 2 be the one that started with 3. Define the random variable X as the urn that you selected your first chip from. We can say that $\mathbb{P}[X = 1] = \frac{4}{3}\mathbb{P}[X = 2]$ as urn 1 started with 4 \$1 chips. Therefore, as those 2 values must sum to 1, $\mathbb{P}[X = 1] = \frac{4}{7}$ and $\mathbb{P}[X = 2] = \frac{3}{7}$.

Explanation:

Lets calculate the expected payout if you switch urns versus if you stay with your current urn. For notational purposes, let urn 1 be the urn that started with 4 \$1 chips and urn 2 be the one that started with 3. Define the random variable X as the urn that you selected your first chip from. We can say that $\mathbb{P}[X = 1] = \frac{4}{3}\mathbb{P}[X = 2]$ as urn 1 started with 4 \$1 chips. Therefore, as those 2 values must sum to 1, $\mathbb{P}[X = 1] = \frac{4}{7}$ and $\mathbb{P}[X = 2] = \frac{3}{7}$.

Let P_0 be the payout if we stay with the same urn. If we don't switch, then $\mathbb{E}[P_0] = \mathbb{E}[P_0 \mid X = 1]\mathbb{P}[X = 1] + \mathbb{E}[P_0 \mid X = 2]\mathbb{P}[X = 2]$. If $X = 1$, then after drawing one \$1 chip, there are 6 \$10 and 3 \$1 chips left, so $\mathbb{E}[P_0 \mid X = 1] = \frac{63}{9} = 7$. Similarly, there would be 7 \$10 and 2 \$1 chips after the first draw if $X = 2$, so $\mathbb{E}[P_0 \mid X = 2] = \frac{72}{9} = 8$. Plugging these values in yields

$$\mathbb{E}[P_0] = 7 \cdot \frac{4}{7} + 8 \cdot \frac{3}{7} = \frac{52}{7}$$

Let P_1 be your profit if you switch. Similarly, we have $\mathbb{E}[P_1] = \mathbb{E}[P_1 \mid X = 1]\mathbb{P}[X = 1] + \mathbb{E}[P_1 \mid X = 2]\mathbb{P}[X = 2]$. If $X = 1$, then by switching, we pick from an untouched urn 2, so $\mathbb{E}[P_1 \mid X = 1] = \frac{73}{10} = 7.3$. Similarly, if $X = 2$, then we pick from an untouched urn 1, so $\mathbb{E}[P_1 \mid X = 2] = \frac{64}{10} = 6.4$. Therefore,

$$\mathbb{E}[P_1] = 6.4 \cdot \frac{3}{7} + 7.3 \cdot \frac{4}{7} = \frac{242}{35} < \frac{245}{35} = 7$$

Therefore, you should not switch, and your expected payout is $\frac{52}{7}$.

Answer:

527

Reference:

https://www.quantguide.io/questions/pick-your-urn-wisely

Builders

Category: Brainteasers
Difficulty: Medium
Companies: SIG

Question:

Alice and Bob can each build a pipe on their own in 120 and 100 minutes, respectively. Instead, they work together on building a pipe. After 40 minutes of building, Charlie assists them in building the pipe. Together, they finish building the pipe 10 minutes after Charlie joins. In how many minutes could Charlie build the pipe on his own?

Hint:

Consider the rate at which each person builds and suppose a scenario where 600 units must be built to fully create the pipe. It follows then that Alice builds at 5 units per minute and Bob builds at 6 units per minute (indeed, their respective completion times are 120 and 100 minutes). How does Charlie affect this process?

Explanation:

Suppose 600 total units need to be built to fully create the pipe. It follows then that Alice builds at 5 units per minute and Bob builds at 6 units per minute (indeed, their respective completion times are 120 and 100 minutes). After 40 minutes, Alice and Bob together would have completed $40 \cdot (5 + 6) = 440$ of the 600 units. After Charlie joins, they complete the remaining 160 units in 10 minutes, meaning that they were building at a rate of 16 units per minute with all three working. As they were at 11 per minute before Charlie joined, this means Charlie builds at 5 units per minute, the same rate as Alice, so Charlie also would take 120 minutes to build the pipe on his own.

Answer:

120

Reference:

https://www.quantguide.io/questions/builders

First Pair

Category: Probability
Difficulty: Easy
Companies: SIG, Old Mission

Question:

Find the probability that the top two cards of a well-shuffled standard deck form a pair (two cards of the same rank).

Hint:

The first card dealt out will determine the rank of the pair.

Explanation:

The first card dealt out will determine the rank of the pair. It can be any card as all cards have the same number of duplicates of the rank. Therefore, let the first card be arbitrary. There are 3 cards of that rank remaining of 51 cards in the deck, so the probability of a pair is $\frac{3}{51} = \frac{1}{17}$.

Answer:

117

Reference:

https://www.quantguide.io/questions/first-pair

Dollar Draw

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

A bin consists of 4 \$10 bills, 3 \$20 bills, and 1 \$100 bill. You select 7 bills out of this bin uniformly at random without replacement. Find the expected value of the sum of all the bills you select.

Hint:

The average value of each bill selected is $\dfrac{100 + 3 \cdot 20 + 4 \cdot 10}{8} = 25$.

Explanation:

The average value of each bill selected is $\dfrac{100 + 3 \cdot 20 + 4 \cdot 10}{8} = 25$. Since draws are exchangeable, the expected payout of each draw is the exact same, so the expected total of the bills we select is $25 \cdot 7 = 175$.

Alternatively, we can compute this by considering the bill we don't select, and then subtract this result from 200. With probability $\frac{1}{2}$, we leave a \$10 bill. With probability $\frac{3}{8}$, we leave a \$20 bill. With probability $\frac{1}{8}$, we leave the \$100 bill. Therefore, the expected value of the bill we leave is $\frac{1}{2} \cdot 10 + \frac{3}{8} \cdot 20 + \frac{1}{8} \cdot 100 = 25$. This means the expected sum of the bill we do select is $200 - 25 = 175$.

Answer:

175

Reference:

https://www.quantguide.io/questions/dollar-draw

Sequence Probability

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

What is the probability that you flip a fair coin 4 times and you get the sequence $HTHT$ in that order?

Hint:

What's the probability of any sequence of 4 flips?

Explanation:

Every sequence of n coin flips has a $\left(\frac{1}{2}\right)^n$ chance of happening (assuming a fair coin). This is since there are 2^n possible sequences of length n. For 4 flips, this comes out to $\frac{1}{16}$.

Answer:

116

Reference:

https://www.quantguide.io/questions/sequence-probability

Doubly 5 II

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Jenny has a fair 6−sided die with numbers $1-6$ on the sides. Jenny continually rolls the die and keeps track of the outcomes in the order they appear. Jenny rolls until she sees two 5s (not necessarily consecutive) OR both 4 and 6. Find the probability Jenny stops rolling due to seeing two 5s.

Hint:

We are going to compute the complement of seeing 4 and 6 before two 5s. The trick here is that no other rolls matter besides $4-6$. Therefore, conditioned on being in $4-6$, each of the three values appears with probability 1/3. We only need to consider the orderings of the rolls $4, 6, 5$, and 5. What are the cases?

Explanation:

We are going to compute the complement of seeing 4 and 6 before two 5s. The trick here is that no other rolls matter besides $4-6$. Therefore, conditioned on being in $4-6$, each of the three values appears with probability 1/3. We only need to consider the orderings of the rolls $4, 6, 5$, and 5. There are three cases where we get 4 and 6 before two 5s.

Case 1 - 5 on First Roll: In this case, we need the next two rolls to be 4/6 and then the other of 4/6 that wasn't first. The probability of this case is $\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{9}$, as there is a 1/3 chance 5 is first, then 2/3 chance of rolling 4/6 before another 5, and then 1/2 chance of rolling the other of 4/6 before a 5.

Case 2 - 4/6 Comes First and Second: In this case, the probability is just $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, as it is the same as Case 1 after we roll the 5 in Case 1.

Case 3 - 4/6 Comes First and 5 Comes Second: In this case, the probability is $\frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{6}$. There is a 2/3 probability 4/6 comes first, then a 1/2 probability that 5 comes before the other of 4/6, and 1/2 probability that the other of 4/6 comes before the other 5.

Adding these up, we obtain a probability of $\frac{11}{18}$ that both 4 and 6 appear before two 5s. Therefore, our answer is $1 - \frac{11}{18} = \frac{7}{18}$. This makes intuitive sense, as there are different permutations of 4 and 6 that they can appear in, whereas two 5s only have one permutation.

Answer:

718

Reference:

https://www.quantguide.io/questions/doubly-5-ii

Leaked Ip

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Someone leaked IP from a quant firm! A witness claims that the IP was leaked by a trader. Suppose that witnesses are correct 2/3 of the time. At this particular firm, 2/3 of the employees are traders, while the other 1/3 are researchers. Taking into account the witness statement, what is the probability that the person who leaked IP was a trader?

Hint:

Be careful about how to interpret the statements in the question. Let T be the event the culprit is a trader and S be the event that the witness says that the person leaking IP is a trader. Formulate the items in the question in terms of T and S.

Explanation:

Let T be the event the culprit is a trader and S be the event that the witness says that the person leaking IP is a trader. We want $\mathbb{P}[T \mid S]$. We know that $\mathbb{P}[T] = 2/3$ from the question. Also from the question, we can conclude $\mathbb{P}[S \mid T] = 2/3$, as the witness is correct 2/3 of the time, so given the culprit was a trader, 2/3 of the time, the witness would also say it is a trader. This also means $\mathbb{P}[S \mid T^c] = 1/3$, as the witness is incorrect 1/3 of the time. Therefore, we have that

$$\mathbb{P}[T \mid S] = \frac{\mathbb{P}[S \mid T]\mathbb{P}[T]}{\mathbb{P}[S \mid T]\mathbb{P}[T] + \mathbb{P}[S \mid T^c]\mathbb{P}[T^c]} = \frac{2/3 \cdot 2/3}{2/3 \cdot 2/3 + 1/3 \cdot 1/3} = \frac{4}{5}$$

Answer:

45

Reference:

https://www.quantguide.io/questions/leaked-ip

Kiddie Pool

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

You are trying to fill up an inflatable kiddie pool. One hose can fill it up in 20 minutes. Another hose can fill it up in 10 minutes. If the pool is full, it takes 15 minutes to drain out the water after the drain is opened. How long does it take in minutes to fill the pool with both hoses and the drain being open?

Hint:

Write the rates for all these variables.

Explanation:

This is a rates problem so let's answer it as such. For every minute, the first hose fills up $\frac{1}{20}^{\text{th}}$ of the pool, the second hose fills up $\frac{1}{10}^{\text{th}}$, and the drain removes $\frac{1}{15}^{\text{th}}$ of the pool. Thus the combined rate is $\frac{1}{20} + \frac{1}{10} - \frac{1}{15} = \frac{1}{12}$. With the two hoses and the drain open, we fill up $\frac{1}{12}^{\text{th}}$ of the pool every minute. Thus it takes 12 minutes to completely fill up the pool.

Answer:

12

Reference:

https://www.quantguide.io/questions/kiddie-pool

Missing Product

Category: Brainteasers
Difficulty: Medium
Companies: SIG

Question:

You have 4 positive real numbers a, b, c, and d. When you multiply each pair of the numbers together, you get some element of the 6 element set $\{9, 10, 12, 27, 30, x\}$. Find x.

Hint:

Note that since a, b, c, and d don't necessarily have to be integers, prime factorization matching may not solve the question entirely. One thing to quickly notice is that the integers 9 and 10, as well as 27 and 30 are in our set. Therefore, this implies two of the values, say a and b, satisfy, $a = \frac{9}{10}b$.

Explanation:

Note that since a, b, c, and d don't necessarily have to be integers, prime factorization matching may not solve the question entirely. One thing to quickly notice is that the integers 9 and 10, as well as 27 and 30 are in our set. Therefore, this implies two of the values, say a and b, satisfy, $a = \frac{9}{10}b$. One may now take the guess that $a = 9$ and $b = 10$. However, this doesn't work, as one of the other integers would have to be $c = 1$, and there would be too many missing values remaining. The next natural guess would be $a = \frac{9}{2}$ and $b = 5$, as you can obtain the values 9 and 10 from setting another integer $c = 2$. Running with this, we see that $ac = 9$ and $bc = 10$. To find d, the values left that we haven't matched yet are $12, 27$, and 30. However, we see that $12 = 2 \cdot 6, 27 = \frac{9}{2} \cdot 6$, and $30 = 5 \cdot 6$, which implies $d = 6$. The last value should be $ab = \frac{9}{2} \cdot 5 = \frac{45}{2}$

Answer:

452

Reference:

https://www.quantguide.io/questions/missing-product

Oil Profits I

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

A drilling company must decide whether or not to drill oil at some site. The company can drill the site for a price of $\$350,000$ paid up front. Afterwards, the company will learn if there is actually oil at this site. If there is oil, the company will generate $\$1,400,000$ in revenue. If there is no oil, there will be no future profits. Let p be the probability there is oil at this site. What is the minimum value of p such that the company should drill if they act rationally?

Hint:

What are the two outcomes, and what is the profit associated to each?

Explanation:

In the event that there is oil (which occurs with probability p), the company makes $\$1,050,000$ in profit. If there is no oil, which occurs with probability $1-p$, the company loses $\$350,000$. Therefore, we need to find the smallest p such that

$$1050000p - 350000(1-p) \geq 0$$

The smallest such p can be solved to be $1/4$.

Answer:

14

Reference:

https://www.quantguide.io/questions/oil-profits-i

Option Dice II

Category: Finance
Difficulty: Easy
Companies: SIG, Old Mission, Optiver

Question:

Pretend you have a simple options chain on the expected value of the product of two dice rolls. What would the put option at 4 be priced at in this market?

Hint:

When does our contract make money and how much?

Explanation:

First, we need to see how many combinations of dice rolls land us in profit. Listed in increasing order they are $(1,1),(1,2),(2,1),(1,3),(3,1)$. With these combinations, a roll of $(1,1)$ yields a profit of 3 units $(4-1=3)$ with a $\frac{1}{36}$ chance in doing so, a roll of $(1,2)$ or $(2,1)$ yields a profit of 2 units $(4-2=2)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(1,3)$ or $(3,1)$ yields a profit of 1 units $(4-3=1)$ with a $\frac{2}{36}$ chance in doing so.Putting this all together we have $\frac{1}{36}\cdot 3+\frac{2}{36}\cdot 2+\frac{2}{36}\cdot 1=\frac{9}{36}$, showing our contract should be priced at 1/4.

Answer:

14

Reference:

https://www.quantguide.io/questions/option-dice-ii

Soccer Bets

Category: Probability
Difficulty: Easy
Companies: SIG, Optiver

Question:

You are betting on the outcome of a soccer match. Team Quant has 2 : 1 odds (you get $3 back if you win, $2 profit) and Team Guide has 3 : 1 odds. You can also bet that both teams will tie which has 10 : 1 odds. Let A, B, and C be the lowest round bet (can only bet whole dollars) amount you bet on Team Quant, Team Guide, and a tie respectively. Assume that if you are betting, you want to make the same amount no matter the outcome of the game. How much do you profit? If you shouldn't bet on this game, enter 0.

Hint:

Think about the payouts for each situational outcome.

Explanation:

The best way to tackle this question is to look at the outcomes and make equations for the net amount given each case. For example, if Team Quant wins, your net value is $2A - B - C$, as you receive $2A$ profit, and then lose the B you bet on Team Guide and the C you bet on a tie. Repeating this logic, you obtain the following equations:

$$\text{If Team Quant wins: } 2A - B - C$$

$$\text{If Team Guide wins: } 3B - A - C$$

$$\text{If tie: } 10C - A - B$$

Since we want our profit to be the same irrespective of the outcome, we need to make all these equations equal to each other. Solving these systems of equations, we get $A = 44$, $B = 33$, $C = 12$ as the smallest even bets you can make. To double check, when we plug these values into the three equations we made, we seem to profit 43 no matter the scenario. Thus, the answer is 43.

Answer:

43

Reference:

https://www.quantguide.io/questions/soccer-bets

Ferry Stops

Category: Brainteasers
Difficulty: Easy
Companies: SIG, Jane Street

Question:

There are an unknown amount of people on a ferry. After the first stop, $\frac{3}{4}$ of them get off and 7 people get on. This happens again at 2 more stops. After this process, what is the minimum amount of possible people that could be aboard the ferry?

Hint:

Let be the amount of people we started with. After the first round, there are $\frac{1}{4}x + 7$ people on the ferry. Iterate this and find the smallest x so that the result is an integer

Explanation:

Let be the amount of people we started with. After the first round, there are $\frac{1}{4}x + 7$ people on the ferry. After the second stop, there are

$$\frac{1}{4}\left(\frac{1}{4}x + 7\right) + 7 = \frac{1}{16}x + \frac{35}{4}$$

After the last stop, there are

$$\frac{1}{4}\left(\frac{1}{16}x + \frac{35}{4}\right) + 7 = \frac{1}{64}x + \frac{147}{16} = \frac{x + 588}{64}$$

people on the ferry. We must find the smallest integer x such that $x + 588$ is divisible by 64. Note that $10 \cdot 64 = 640$, which is the smallest integer larger than 588 that is divisible by 64. Thus, there are $640 - 588 = 52$ people on the ferry at the start, meaning there are 10 people on at the end.

Answer:

10

Reference:

https://www.quantguide.io/questions/ferry-stops

Investment Arbitrage

Category: Finance
Difficulty: Easy
Companies: SIG

Question:

You are in a market where you can invest in a stock or bet at a casino in $100 increments. Call these increments units. Note that you may not purchase fractions of a unit. If you invest in the stock, then it goes up or down. If it goes up in value, you gain an additional 100 dollars per unit. If it goes down, you lose 50 dollars per unit. Additionally, you can bet on whether the stock will go up or down at the casino, where you either win 100 per unit if you're right or lose all 100 if not. Devise a strategy where you will always make a profit. Find the guaranteed profit if you hold a minimal amount of total units.

Hint:

We can hedge our position in investing in the stock by betting on it doing down at the casino. Set up a system of equations based on what actually happens.

Explanation:

We can hedge our position in investing in the stock by betting on it doing down at the casino. Let p be the profit you make. Suppose you purchase s stocks and purchase b units in the casino to bet on the stock going down. Then $p = 100s - 100b$ if the stock goes up, as you make 100 per stock if it goes up and lose 100 in the casino since you bet on it going down. If it goes down, then $p = 100b - 50s$. Equating these, we see that $100s - 100b = 100b - 50s$, which means $s = \dfrac{4b}{3}$. Therefore, to ensure both are integers, we can let $b = 3$ and $a = 4$, which is the minimal position with integer units, yields a profit of 100 guaranteed.

Answer:

100

Reference:

https://www.quantguide.io/questions/investment-arbitrage

Comparing Flips I

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Audrey repeatedly flips a coin until she sees a pattern of HHT or HTT. What is the probability that HHT occurs before HTT?

Hint:

Because both HHT and HTT begin with heads, we can ignore any pattern with leading tails. Instead, consider the four outcomes of flipping the coin three times. For each case, what is the probability that HHT or HTT will occur before the other?

Explanation:

Both HHT and HTT begin with heads, so we can ignore any pattern with leading tails. Consider the case where our sequence begins with heads. Given that the first flip is heads, each of the following four cases occurs with probability $\frac{1}{4}$: HHH, HHT, HTH, HTT. In the case of HHH, then Audrey will see the pattern HHT first with probability 1, since it is impossible for her to roll two tails without first seeing HHT. In the case of HHT, HHT appears first. In the case of HTT, HTT appears first. And finally, in the case of HTH, we are returned to our original state of our sequence of interest beginning with heads. Since HHT, HTT, and HHH occur with equal probability, and since HHT and HHH result in HHT appearing first, the probability that HHT appears first is $\frac{2}{3}$.

Answer:

23

Reference:

https://www.quantguide.io/questions/comparing-flips-i

Cheater

Category: Probability
Difficulty: Easy
Companies: SIG, Optiver

Question:

As a good test taker, on a multiple choice exam with 5 options per question, Gabe either knows the answer to a question beforehand or chooses an answer completely at random. If he knows the answer beforehand, he selects the correct answer. If the probability that Gabe knows the answer to any given question is 0.6, find the probability that an answer that was correct was one for which he knew the answer.

Hint:

Let K be the event that Gabe knew the answer to the question, C be the event he gets a question right, and G be the event he guessed on a question.

Explanation:

Let K be the event that Gabe knew the answer to the question, C be the event he gets a question right, and G be the event he guessed on a question. We know that $\mathbb{P}[K] = 0.6$, $\mathbb{P}[G] = \mathbb{P}[K^c] = 1 - \mathbb{P}[K] = 0.4$, $\mathbb{P}[C \mid G] = 0.2$, as he selects completely at random, and it is reasonable to assume that $\mathbb{P}[C \mid K] = 1$, as if he knows the answer, he will get it correct. We want to find $\mathbb{P}[K \mid C]$. By definition of conditional probability, this is $\mathbb{P}[K \mid C] = \dfrac{\mathbb{P}[KC]}{\mathbb{P}[C]}$. On the top, we apply conditional probability again to write this as

$$\mathbb{P}[KC] = \mathbb{P}[C \mid K]\mathbb{P}[K]$$

and on the bottom, we apply the Law of Total Probability to get

$$\mathbb{P}[C] = \mathbb{P}[C \mid K]\mathbb{P}[K] + \mathbb{P}[C \mid G]\mathbb{P}[G]$$

Thus, we know that $\mathbb{P}[K \mid C] = \dfrac{\mathbb{P}[C \mid K]\mathbb{P}[K]}{\mathbb{P}[C \mid K]\mathbb{P}[K] + \mathbb{P}[C \mid G]\mathbb{P}[G]} = \dfrac{1 \cdot 0.6}{1 \cdot 0.6 + 0.2 \cdot 0.4} = \dfrac{15}{17}$

Answer:

1517

Reference:

https://www.quantguide.io/questions/cheater

Signed Correlation

Category: Statistics
Difficulty: Easy
Companies: SIG

Question:

Let X and Y be two random variables with $\text{Corr}(X, Y) = -0.4$. If possible, find $\text{Corr}(3X+1, 1-Y)$. If it is not possible, enter 100.

Hint:

Correlation is a measure of linear association between X and Y. It is unaffected by scaling and shifting.

Explanation:

Correlation is a measure of linear association between X and Y that is unaffected by scaling and shifting. Thus, we have that $\text{Corr}(aX+b, cY+d) = \text{sign}(ac)\text{Corr}(X, Y)$. In this case, $\text{sign}(ac) = -1$, so our answer is $-1 \cdot -0.4 = 0.4$.

Answer:

0.4

Reference:

https://www.quantguide.io/questions/signed-correlation

Primes And Composites

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

There is a game where an integer is picked uniformly between 2 and 10 inclusive. You win $\$n$ if the integer is prime and lose $\$\frac{n}{2}$ if its composite (where n is the integer picked). How much would you pay to play this game? If you don't wish to play, enter 0.

Hint:

Write down all the outcomes and if they are prime or composite.

Explanation:

Let's split up all the outcomes into prime and composite.

$$\text{Primes: } 2, 3, 5, 7 \text{Composites: } 4, 6, 8, 9, 10$$

The EV of this game comes out to $(2 + 3 + 5 + 7 - 2 - 3 - 4 - 4.5 - 5)/9 = -\frac{1}{6}$.
Since the EV is negative, the answer is 0.

Answer:

0

Reference:

https://www.quantguide.io/questions/primes-and-composites

Optimal Bidders I

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Carter has come into contact with a bounty of gold. He takes it to an auction shop. The auction shop says that each person that bids will place a bid that is uniformly distributed between \$500 and \$1000. They also state that they can recruit people to bid for a price of \$5 per person. Assuming Carter selects the optimal number of people to bid on his gold, what is his expected payout?

Hint:

Write the expected payout as a function of n, the number of people that they recruit. Try to maximize it with respect to n.

Explanation:

It is known that given $X_1, \dots, X_n \sim \text{Unif}(0,1)$ IID and $M_n = \max\{X_1, \dots, X_n\}$, $\mathbb{E}[M_n] = \dfrac{n}{n+1} = 1 - \dfrac{1}{n+1}$. This also extends now to Unif(500, 1000), as this is just a scaling and shifting of a Unif(0, 1). Namely, if $X \sim \text{Unif}(0,1)$, $500X + 500 \sim \text{Unif}(500, 1000)$. Therefore, if M_n' is the maximum of n IID Unif(500, 1000) random variables and M_n is as above, $\mathbb{E}[M_n'] = 500\mathbb{E}[M_n] + 500 = 1000 - \dfrac{500}{n+1}$. The cost of obtaining n people is $5n$. Therefore, our profit, which we can denote as P_n, is $P_n = M_n' - 5n$, so $\mathbb{E}[P_n] = \mathbb{E}[M_n'] - 5n = 1000 - \dfrac{500}{n+1} - 5n$. We want to find n that maximizes this. Calling this function $f(n)$, we can treat n as continuous and find the value of n that maximizes it by taking the derivative and setting it equal to 0.

This means $f'(n) = \dfrac{500}{(n+1)^2} - 5 = 0$. Rearranging yields $(n+1)^2 = 100$, so as n must be positive, $n = 9$ bidders gives us the maximal expected profit. Plugging this in, $\mathbb{E}[P_9] = 905$, which is our solution.

Answer:

905

Reference:

https://www.quantguide.io/questions/optimal-bidders-i

Options Dice III

Category: Finance
Difficulty: Medium
Companies: SIG, Old Mission, Optiver

Question:

Pretend you have a simple options chain on the expected value of the product of two dice rolls. Make a 2 unit wide market on the call at 19.Enter your answer as the sum of the bid and ask. For example, if your market was 5 @ 7, you would enter 12.

Hint:

Calculate the theoretical value of the call option, then make a market around that.

Explanation:

First we need to calculate the true value of this call option. To do this, we first need to see how many combinations of dice rolls land us in profit. Listed in increasing order they are $(4,5)$, $(5,4)$, $(4,6)$, $(6,4)$, $(5,5)$, $(5,6)$, $(6,5)$, and $(6,6)$. With these combinations, a roll of $(4,5)$ or $(5,4)$ yields a profit of 1 units $(20-19=2)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(5,5)$ yields a profit of 6 units $(25-19=6)$ with a $\frac{1}{36}$ chance in doing so, a roll of $(5,6)$ or $(6,5)$ yields a profit of 11 units $(30-19=11)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(6,6)$ yields a profit of 17 units $(36-19=17)$ with a $\frac{1}{36}$ chance in doing so. Lastly, $(4,6)$ and $(6,4)$ yield profit of 5 units $(24-19=5)$ with probability $\frac{2}{36}$.Putting this all together we have $\frac{2}{36}\cdot 1+\frac{2}{36}\cdot 5+\frac{1}{36}\cdot 6+\frac{2}{36}\cdot 11+\frac{1}{36}\cdot 17 = 19/12$, showing our contract should be priced at 19/12.To make a $2-$wide market around this value, we buy 1 unit cheaper than the theoretical value of 19/12 and sell 1 unit more expensive than 19/12, which comes out to a market of $\frac{7}{12}$ @ $\frac{31}{12}$, and adding them together (the same as multiplying our theo by 2) yields our final answer of $\frac{19}{6}$.

Answer:

196

Reference:

https://www.quantguide.io/questions/options-dice-iii

5 Pairwise Sum

Category: Brainteasers
Difficulty: Hard
Companies: SIG, Five Rings

Question:

Let $S = \{a, b, c, d, e\}$ be a set of 5 (possibly repeated) real numbers. The pair sums of elements in S are:

$$5, 11, 11, 13, 13, 14, 16, 19, 22, 22$$

Find $a^2 + b^2 + c^2 + d^2 + e^2$.

Hint:

Impose an ordering with $a \leq b \leq c \leq d \leq e$. What do you know about 4 and 22 now in terms of this ordering you imposed?

Explanation:

We can impose an ordering with $a \leq b \leq c \leq d \leq e$. We then know that $a + b = 5$ and $d + e = 22$, as the smallest two integers must add up to the smallest sum and the two largest integers must add up to the largest sum. Additionally, we have that every single of the a, b, c, d, and e appears in 4 of the 10 pairwise sums, as each integer is added up to every other integer but itself. Therefore, if we sum up all the values in the list, the result is $4 \cdot (a + b + c + d + e)$. Adding these integers up yields $5 + 11 + 11 + 13 + 13 + 14 + 16 + 19 + 22 + 22 = 146$, so $4 \cdot (a + b + c + d + e) = 146$. However, we know $a + b = 5$ and $d + e = 22$, so we can get that

$$4 \cdot (5 + c + 22) = 146 \iff c = 9.5$$

The trick now is to note that by the ordering we have, $a + c$ must be the second smallest sum possible after $a + b$, so we have that $a + c = 11$, meaning $a = 1.5$. Since we know $a + b = 5$, we get that $b = 3.5$. By similar logic, we know that the second largest sum after $d + e$ is $c + e$, so we know that $22 = c + e$, meaning that $e = 12.5$. Lastly, since $d + e = 22$ as well, we know $d = c = 9.5$. Putting it all together, we obtained the solution

$$(a, b, c, d, e) = (1.5, 3.5, 9.5, 9.5, 12.5)$$

Therefore, $a^2 + b^2 + c^2 + d^2 + e^2 = \dfrac{3^2 + 7^2 + 19^2 + 19^2 + 25^2}{2^2} = \dfrac{1405}{4}$

Answer:

14054

Reference:

https://www.quantguide.io/questions/5-pairwise-sum

Poisoned Kegs II

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Five Rings, Two Sigma, Jane Street, Old Mission

Question:

A king has 10 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants only agree to participate in the wine tasting for 1 month. What is the maximum value of n such that the king is guaranteed to have at least a 10% chance to determine which keg among the n is poisoned?

Hint:

Use a similar idea to Poisoned Kegs II, but now the number of kegs needs to only be narrowed down to at most 10 possibilities.

Explanation:

Using the same idea as Poisoned Kegs II, we can let each of the 10 servants represent a binary indicator for the first 2^{10} kegs. Thus, in the first 2^{10} kegs, we will know exactly which keg is poisoned. However, now we just need to narrow it down to 10 kegsper possible death sequence. The easiest way to do this is to have each of the 2^{10} subsets of servants test 10 different kegs. Then, if some subset of servants die, the king will know that one of those 10 kegs that the subset of servants died to has the poison. Therefore, the king can test 10 times as many kegs if he only wants at least a 10% chance of identifying it. Therefore, the king can test up to $n = 2^{10} \cdot 10 = 10240$ kegs

Answer:

10240

Reference:

https://www.quantguide.io/questions/poisoned-kegs-ii

The Last Roll

Category: Brainteasers
Difficulty: Easy
Companies: SIG, Optiver

Question:

A fair 6−sided die is continually rolled until the sum of all the numbers is at least 1000. What value on the die was most likely to show up on the last roll?

Hint:

What current sums can reach 1000 if the last die roll is i, $1 \le i \le 6$?

Explanation:

This is because 6 can reach 1000 from a current sum of anything between $994 - 999$. Conversely, a 1 only works if the current sum is 999. Having a smaller value on the last roll reduces the possibilities of the previous sums before exceeding 1000.

Answer:

6

Reference:

https://www.quantguide.io/questions/the-last-roll

Stranded At Sea

Category: Probability
Difficulty: Easy
Companies: SIG, Five Rings, Jane Street, Optiver

Question:

You are boating through the Mediterranean Sea when your motor breaks down. You remember learning from your divemaster that the probability of seeing at least one boat within a given hour is 73.7856

Hint:

Let p be the probability that you do not see a boat in 10 minutes- you are solving for $1 - p$. How can you use complementary probability to define the probability that you do not see a boat within an hour?

Explanation:

Let p be the probability that you do not see a boat in 10 minutes. Then the probability of not seeing a boat within one hour is:

$$p^6 = 1 - .737856 \Rightarrow p = 0.8$$

The probability that you will see a boat within the next 10 minutes is:

$$1 - p = 0.2$$

Answer:

0.2

Reference:

https://www.quantguide.io/questions/stranded-at-sea

Always Profit I

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

QuantGuide stock will either double its value or go down 50% by tomorrow. You can also bet whether the stock goes up or down with your friend at 1 : 1 payout. Let A and B be the lowest round bet (can only bet whole dollars) amount you long/short the stock and bet on the stock going up/down with your friend respectively. Assume that if you are betting, you want to make the same amount profit no matter the outcome of the stock's movement tomorrow (bank balance will be the same no matter the outcome). How much do you profit from betting A and B? If you can't guarantee profit, enter 0.

Hint:

Make equations for profits including returned bets for all scenarios.

Explanation:

Something to notice about this situation is that being long on the stock is better than betting the stock goes up against your friend because in the case the stock goes down, you'll still retain 50% of your investment in the stock. Thus we should look to go long on the stock and bet the stock goes down against your friend. The profit (including returned bets) equations are as follows:

Stock goes up: $A - B$

Stock goes down: $B - 0.5A$

Making these two equations equal, we get $A = 4B/3$. Thus A and B are equal to \$4 and \$3 respectively which allows us to always profit \$1.

Answer:

1

Reference:

https://www.quantguide.io/questions/always-profit-i

Option Dice I

Category: Finance
Difficulty: Easy
Companies: SIG, Old Mission, Optiver

Question:

Pretend you have a simple options chain on the expected value of the product of two dice rolls. What would the put option at 4 be priced at in this market?

Hint:

When does our contract make money and how much?

Explanation:

First, we need to see how many combinations of dice rolls land us in profit. Listed in increasing order they are $(1,1),(1,2),(2,1),(1,3),(3,1)$. With these combinations, a roll of $(1,1)$ yields a profit of 3 units $(4-1=3)$ with a $\frac{1}{36}$ chance in doing so, a roll of $(1,2)$ or $(2,1)$ yields a profit of 2 units $(4-2=2)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(1,3)$ or $(3,1)$ yields a profit of 1 units $(4-3=1)$ with a $\frac{2}{36}$ chance in doing so.Putting this all together we have $\frac{1}{36}\cdot 3+\frac{2}{36}\cdot 2+\frac{2}{36}\cdot 1=\frac{9}{36}$, showing our contract should be priced at 1/4.

Answer:

14

Reference:

https://www.quantguide.io/questions/option-dice-i

Lawn Teamwork

Category: Brainteasers
Difficulty: Easy
Companies: SIG, Jane Street

Question:

There are two lawnmowers: Mr. Rabbit can mow a lawn in 1 hour, Mr. Turtle needs 1 hour and 15 minutes. If they work together, how long does it take to mow one lawn (in minutes)?

Hint:

rate $\cdot$ time $=$ work

Explanation:

Given the information we have, we can solve for the rate at which both men mow a lawn per minute. Mr. Rabbit mows $\frac{1}{60}$ of a lawn per minute. Mr. Turtle mows $\frac{1}{75}$ of a lawn per minute. We can use rate $\cdot$ time $=$ work done to solve for how long it takes. Let x be the number of minutes it takes for both of them to do one lawn. $(\frac{1}{60}+\frac{1}{75})\cdot x = 1$ lawn. Solving for x, we get $\frac{100}{3}$ minutes.

Answer:

1003

Reference:

https://www.quantguide.io/questions/lawn-teamwork

Golden Pancakes

Category: Probability
Difficulty: Easy
Companies: SIG, HRT

Question:

You have three pancakes in front of you. One is toasted on both sides, one is toasted on one side, and one is not toasted at all. You hungrily choose a pancake at random and notice the top is toasted. What is the probability that the other side of your pancake is also toasted?

Hint:

Look at each side individually. How many sides are toasted? Of these sides, how many share a pancake with another toasted side?

Explanation:

Denote the sides of the six pancakes s_i where $1 \leq i \leq 6$. Without loss of generality, let s_1, s_2, and s_3 be the toasted sides and s_1 and s_2 be on the same pancake. Out of the six possible sides we could have observed, we are given to have observed one of the three toasted sides, s_1, s_2, or s_3. Of these three sides, two (s_1 and s_2) belong to a pancake that is toasted on both sides. Thus, the probability that the other side of the pancake is also toasted is $\frac{2}{3}$.

Answer:

23

Reference:

https://www.quantguide.io/questions/golden-pancakes

Pulling Cards In Order

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

You have a 100 card deck in front of you containing the numbers from 1 to 100. You randomly pick one card after the other 4 times (without replacement). What is the probability that the cards you picked out are in ascending order?

Hint:

Imagine you pull out the four cards at once instead of one at a time. What are the odds they are in order in your hand?

Explanation:

Instead of thinking about the situation as picking out cards one at a time, think about picking 4 cards out of the 100 all at once. Since all the numbers are unique, there is only 1 way the numbers can show up in ascending order out of 4! orderings. Thus the answer is $\frac{1}{4!} = \frac{1}{24}$.

Answer:

124

Reference:

https://www.quantguide.io/questions/pulling-cards-in-order

Optimal Bidders II

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Carter has come into contact with a bounty of gold. He takes it to an auction shop. The auction shop says that each person that bids will place a bid that is uniformly distributed between \$500 and \$1000. They also state that they can recruit people to bid for a price of \$10 per person. However, the auction shop also states that Carter will only receive the payment of the second highest bid. Assuming Carter selects the optimal number of people to bid on his gold, what is his expected payout?

Hint:

Write the expected payout as a function of n, the number of people that they recruit. Try to maximize it with respect to n.

Explanation:

It is known that given $X_1, \dots, X_n \sim \text{Unif}(0,1)$ IID and $S_{i,n}$ is the ith smallest, $1 \leq i \leq n$, among $X_1, \dots, X_n$, $\mathbb{E}[S_{i,n}] = \dfrac{i}{n+1}$. This also extends now to $\text{Unif}(500, 1000)$, as this is just a scaling and shifting of a $\text{Unif}(0,1)$. Namely, if $X \sim \text{Unif}(0,1)$, $500X + 500 \sim \text{Unif}(500, 1000)$. Therefore, if $S'_{i,n}$ is the ith smallest among n IID $\text{Unif}(500, 1000)$ random variables and $S_{i,n}$ is as above, $\mathbb{E}[S'_{i,n}] = 500\mathbb{E}[S_{i,n}] + 500 = 500 + \dfrac{500i}{n+1}$. In this case, $i = n-1$, as we are paid out the second largest bid, so the expected payout with n people is $500 + 500\dfrac{n-1}{n+1} = 1000\left(1 - \dfrac{1}{n+1}\right)$. The cost of obtaining n people is $10n$. Therefore, our profit, which we can denote as P_n, is $P_n = S'_{i,n} - 10n$, so $\mathbb{E}[P_n] = \mathbb{E}[S'_{i,n}] - 10n = 1000\left(1 - \dfrac{1}{n+1}\right) - 10n$. We want to find n that maximizes this. Calling this function $f(n)$, we can treat n as continuous and find the value of n that maximizes it by taking the derivative and setting it equal to 0.

This means $f'(n) = \dfrac{1000}{(n+1)^2} - 10 = 0$. Rearranging yields $(n+1)^2 = 100$, so as n must be positive, $n = 9$ bidders gives us the maximal expected profit. Plugging this in, $\mathbb{E}[P_9] = 810$, which is our solution.

Answer:

810

Reference:

https://www.quantguide.io/questions/optimal-bidders-ii

Poisoned Kegs III

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Five Rings, Two Sigma, Jane Street, Old Mission

Question:

A king has 10 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants only agree to participate in the wine tasting for 1 month. What is the maximum value of n such that the king is guaranteed to have at least a 10% chance to determine which keg among the n is poisoned?

Hint:

Use a similar idea to Poisoned Kegs II, but now the number of kegs needs to only be narrowed down to at most 10 possibilities.

Explanation:

Using the same idea as Poisoned Kegs II, we can let each of the 10 servants represent a binary indicator for the first 2^{10} kegs. Thus, in the first 2^{10} kegs, we will know exactly which keg is poisoned. However, now we just need to narrow it down to 10 kegsper possible death sequence. The easiest way to do this is to have each of the 2^{10} subsets of servants test 10 different kegs. Then, if some subset of servants die, the king will know that one of those 10 kegs that the subset of servants died to has the poison. Therefore, the king can test 10 times as many kegs if he only wants at least a 10% chance of identifying it. Therefore, the king can test up to $n = 2^{10} \cdot 10 = 10240$ kegs

Answer:

10240

Reference:

https://www.quantguide.io/questions/poisoned-kegs-iii

Whole Lotta Dice

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

We have 100 fair $100-$sided dice in front of us and we toss them all at the same time. What is the probability that the resulting sum of the faces is divisible by 100?

Hint:

Don't be fooled by the number of dice! There is a simpler solution to this question than counting outcomes.

Explanation:

No matter what the sum of any 99 of the dice, there is only one face on the 100th die that will yield a sum that's divisible by 100. For example, say the first 99 rolls add up to 3075. Only rolling a 25 on the last die will make the sum divisible by 100. Thus the answer is $\frac{1}{100}$.

Answer:

1100

Reference:

https://www.quantguide.io/questions/whole-lotta-dice

Maximize Head Ratio II

Category: Probability
Difficulty: Hard
Companies: SIG

Question:

Say that you are flipping a fair coin where you can stop flipping whenever you want. Your goal is to maximize the ratio between the number of heads you get with the total number of flips. The ratios are in the form $a : 1$. What is the expectation of a when you follow the strategy of stopping when you have more heads than tails? Round your answer to the nearest thousandths.

Hint:

When trying to find the probability of outcomes, use Dyck words. The Taylor Series for $\arcsin(x) = \sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)}\binom{2n}{n}x^{2n+1}$

Explanation:

The strategy here is close to optimal but needs simulations to fully reach optimality. However, this strategy makes sense intuitively because if the current ratio is $0.5 : 1$, we should always try and get another head as we can always get back to a ratio of $0.5 : 1$.

To put this strategy in a mathematical context, the optimal ratio will be $\frac{n+1}{2n+1}$ where n is a number from 0 to infinity. Thus we should always end on an odd roll. To find the expectation, we need to find a weighted sum of this series, the weight being the probability we end on $2n + 1$. When we stop on $2n+1$ and $n \neq 0$, we know the last toss needs to be heads and for $k = 1, 2, ..., 2n$, the first k tosses can't include more heads than tails (we would've stopped the flips earlier in that case).

To count such sequences, we can use Dyck words (length $2n$) and the n-th Catalan number can be found by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Since each of the Catalan numbers have probability of $\left(\frac{1}{2}\right)^{2n}$ and the last flip is heads with probability of $\frac{1}{2}$, the expectation of the ratio becomes $\sum_{n=0}^{\infty} p_n \cdot \frac{n+1}{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} \cdot \binom{2n}{n}$. Using the Taylor series for $\arcsin(x)$, this summation simplifies to

$$\frac{1}{2} \cdot \arcsin(1) = \frac{\pi}{4} \approx 0.785$$

Answer:

0.785

Reference:

https://www.quantguide.io/questions/maximize-head-ratio-ii

Maximize Head Ratio I

Category: Probability
Difficulty: Hard
Companies: SIG

Question:

Say that you are flipping a fair coin where you can stop flipping whenever you want. Your goal is to maximize the ratio between the number of heads you get with the total number of flips. The ratios are in the form $a:1$. What is the expectation of a when you follow the strategy of stopping when you have more heads than tails? Round your answer to the nearest thousandths.

Hint:

When trying to find the probability of outcomes, use Dyck words. The Taylor Series for $\arcsin(x) = \sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} \binom{2n}{n} x^{2n+1}$

Explanation:

The strategy here is close to optimal but needs simulations to fully reach optimality. However, this strategy makes sense intuitively because if the current ratio is $0.5:1$, we should always try and get another head as we can always get back to a ratio of $0.5:1$.

To put this strategy in a mathematical context, the optimal ratio will be $\frac{n+1}{2n+1}$ where n is a number from 0 to infinity. Thus we should always end on an odd roll. To find the expectation, we need to find a weighted sum of this series, the weight being the probability we end on $2n+1$. When we stop on $2n+1$ and $n \neq 0$, we know the last toss needs to be heads and for $k = 1, 2, ..., 2n$, the first k tosses can't include more heads than tails (we would've stopped the flips earlier in that case).

To count such sequences, we can use Dyck words (length $2n$) and the n-th Catalan number can be found by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Since each of the Catalan numbers have probability of $\left(\frac{1}{2}\right)^{2n}$ and the last flip is heads with probability of $\frac{1}{2}$, the expectation of the ratio becomes $\sum_{n=0}^{\infty} p_n \cdot \frac{n+1}{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} \cdot \binom{2n}{n}$. Using the Taylor series for $\arcsin(x)$, this summation simplifies to

$$\frac{1}{2} \cdot \arcsin(1) = \frac{\pi}{4} \approx 0.785$$

Answer:

0.785

Reference:

https://www.quantguide.io/questions/maximize-head-ratio-i

Picky Primes

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street

Question:

4 distinct integers are drawn from the set of the first 16 positive prime integers. Find the probability that the sum is even.

Hint:

All primes besides 2 are odd.

Explanation:

All primes besides 2 are odd, so we get a sum that is odd precisely when we select integers that are all not 2. In other words, we select our 4 integers from the other 15 primes. There are $\binom{15}{4}$ ways to pick the 4 from the other 15 and $\binom{16}{4}$ total ways to pick 4 primes from the 16. Therefore, our probability is

$$\frac{\binom{15}{4}}{\binom{16}{4}} = \frac{3}{4}$$

Answer:

34

Reference:

https://www.quantguide.io/questions/picky-primes

Bet Size I

Category: Finance
Difficulty: Easy
Companies: SIG, Old Mission

Question:

Say you are betting on a game where you win 75% where you get paid 1 : 1 on it. Assuming optimal strategy, what percentage of your bankroll should you bet on this game?

Hint:

Use Kelly Criterion.

Explanation:

Use Kelly Criterion formula. The formula follows $\frac{p(b+1)-1}{b}$ where p is the probability of winning the bet and b is the profit ratio. In this case, p is 0.75 and b is 1. Thus you should bet $\frac{0.75(1+1)-1}{1} = 0.5$. Thus you should bet 50% of your bankroll on this bet.

Answer:

0.5

Reference:

https://www.quantguide.io/questions/bet-size-i

Ranged Max

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Let $X_1, X_2, X_3, X_4 \sim \text{Unif}(0, 4)$ IID. Find the probability that the maximum of these 4 random variables is in the interval $(2, 3]$.

Hint:

Let $M = \max\{X_1, X_2, X_3, X_4\}$. Can you derive the CDF of M easily?

Explanation:

Let $M = \max\{X_1, X_2, X_3, X_4\}$. We want $\mathbb{P}[2 < M \leq 3] = \mathbb{P}[M \leq 3] - \mathbb{P}[M \leq 2]$. The CDF of M is easy to derive. $\{M \leq x\}$ means that $\{X_1, X_2, X_3, X_4 \leq x\}$. By independence,

$$\mathbb{P}[M \leq x] = \mathbb{P}[X_1 \leq x]\mathbb{P}[X_2 \leq x]\mathbb{P}[X_3 \leq x]\mathbb{P}[X_4 \leq x] = (\mathbb{P}[X_1 \leq x])^4 = \frac{x^4}{4^4}$$

Therefore, $\mathbb{P}[2 < M \leq 3] = \dfrac{3^4 - 2^4}{4^4} = \dfrac{65}{256}$

Answer:

65256

Reference:

https://www.quantguide.io/questions/ranged-max

Circular Birthdays

Category: Probability
Difficulty: Easy
Companies: SIG, Drw, Jane Street, Optiver

Question:

7 people sit around a circular table uniformly at random. All of them have a distinct age. Find the probability that they sit down at the table in age order. Note that the ages can be increasing in either the clockwise or counter-clockwise directions.

Hint:

There are $(7-1)! = 6! = 720$ distinct ways for the people to sit at the table with no restrictions. How many arrangements are there of the people in age order?

Explanation:

There are $(7-1)! = 6! = 720$ distinct ways for the people to sit at the table with no restrictions. Only two of these seating arrangements have the people in age order: Namely, they are just when they increase CCW or CW. Therefore, our probability is

$$\frac{2}{720} = \frac{1}{360}$$

Answer:

1360

Reference:

https://www.quantguide.io/questions/circular-birthdays

Big Bubble II

Category: Pure Math
Difficulty: Easy
Companies: SIG

Question:

A spherical bubble has a radius increasing at a rate of $\frac{9}{4\pi}$ inches per second. At the moment when the volume of the bubble is increasing at 36 cubic inches per second, what rate is the surface area increasing at (in square inches per second)?

Hint:

We know that $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$ represent the volume and the surface area of the bubble as a function of radius. Take the derivative of the volume function to solve for the radius first.

Explanation:

We know that $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$ represent the volume and the surface area of the bubble as a function of radius. We know that $r' = \frac{9}{4\pi}$ is constant. Taking the derivatives of each, we see that $V' = 4\pi r^2 r'$ and $S' = 8\pi r r'$. We know at this moment that $V' = 36$, so plugging these in and solving yields

$$36 = 4\pi r^2 \cdot \frac{9}{4\pi} \iff r = 2$$

Afterwards, we have that

$$S' = 8\pi(2) \cdot \frac{9}{4\pi} = 36$$

Answer:

36

Reference:

https://www.quantguide.io/questions/big-bubble-ii

Football Bets

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Connor and Calvin bet \$10 each on opposing outcomes of a football game. During the match, Connor raises the bet to \$20. If Calvin doesn't match Connor's bet, he automatically loses what he has bet thus far. Let p be the probability that Calvin's team will win the game when Connor raises. Find the smallest value of p so that Calvin should accept Connor's bet.

Hint:

What is Calvin's expected value if he doesn't accept the bet? What if he does accept it?

Explanation:

We know that if Calvin doesn't accept the bet, his expected value is -10, as he loses his initial bet. If Calvin accepts the bet, then with probability p he wins 20 and with probability $1-p$ he loses 20. Therefore, we need to find the smallest p such that $20p - 20(1-p) \geq -10$. Solving this equation for p yields $p \geq \frac{1}{4}$, so $p = \frac{1}{4}$ is the smallest value where he should accept.

Answer:

14

Reference:

https://www.quantguide.io/questions/football-bets

Estimating Pi

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

You decide to estimate π by generating N independent samples from the unit square $[0,1] \times [0,1]$. Let I_i represent the indicator random variable of the event that the ith sample, $1 \leq i \leq N$, lands inside the region enclosed by $x^2 + y^2 = 1$ in the first quadrant. Using

$$T_N = \frac{I_1 + \cdots + I_N}{N}$$

in your estimate for π, what is variance of your estimate for π? Note that T_N is not necessarily an estimator for π itself. The answer is in the form

$$\frac{a\pi^2 + b\pi}{N}$$

for integers a and b. Find $a + b$. Note that this variance is somewhat useless given that you need π to calculate your variance.

Hint:

Note that $T_N \to \mathbb{E}[I_1] = \dfrac{\pi}{4}$ with probability 1 by the Strong Law of Large Numbers. $\mathbb{E}[I_1] = \dfrac{\pi}{4}$ because of the fact that the circle occupies a proportion of $\dfrac{\pi}{4}$ of the total area of the square. Our estimate for π should be $4T_N$. This is the quantity we want to find the variance of.

Explanation:

Note that $T_N \to \mathbb{E}[I_1] = \dfrac{\pi}{4}$ with probability 1 by the Strong Law of Large Numbers. $\mathbb{E}[I_1] = \dfrac{\pi}{4}$ because of the fact that the circle occupies a proportion of $\dfrac{\pi}{4}$ of the total area of the square. Our estimate for π should be $4T_N$. This is the quantity we want to find the variance of.

By properties of variance and the fact that each I_i is IID, we have that

$$\text{Var}(4T_N) = 16\text{Var}(T_N) = 16\text{Var}\left(\frac{I_1 + \cdots + I_N}{N}\right) = \frac{16}{N^2} \cdot N\text{Var}(I_1) = \frac{16\text{Var}(I_1)}{N}$$

We can use the standard formula to compute $\text{Var}(I_1)$. This is just $\text{Var}(I_1) = \mathbb{E}[I_1^2] - (\mathbb{E}[I_1])^2$. However, note that $I_1^2 = I_1$, as I_1 only takes the values 0 and 1. Therefore, $\text{Var}(I_1) = \dfrac{\pi}{4} - \dfrac{\pi^2}{16} = \dfrac{4\pi - \pi^2}{16}$. Accordingly, our final answer is just

$$\text{Var}(4T_N) = \frac{-\pi^2 + 4\pi}{N}$$

This means $a = -1$ and $b = 4$, so $a + b = 3$.

Answer:

3

Reference:

https://www.quantguide.io/questions/estimating-pi

Poisoned Kegs IV

Category: Brainteasers
Difficulty: Hard
Companies: SIG, Five Rings, Two Sigma, Jane Street, Old Mission

Question:

A king has 5 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants agree to participate in the wine tasting for 3 months. What is the maximum value of n such that the king is guaranteed to determine which keg among the n is poisoned?

Hint:

Think about a base 4 expansion.

Explanation:

We are going to use a similar idea to Poisoned Kegs II. We used binary expansion in Poisoned Kegs II since we only had 1 month, so each servant could either drink or not drink from a given keg. Now, we can assign some servants to drink at the first month, second month, and third month. With 5 servants, we convert each keg label $i = a_{0i} + a_{1i} \cdot 4^1 + a_{2i} \cdot 4^2 + a_{3i} \cdot 4^3 + a_{4i} \cdot 4^4$, where each $a_{ki} = 0, 1, 2, 3$ for $0 \leq k \leq 4$. If $a_{ki} = 0$, we don't have servant k drink from keg i at all. If $a_{ki} = r$ for $r > 0$, servant k drinks from keg i in month r.

When a servant dies, we now record which servant died and what hour they died in. From that, we can reconstruct the keg that was poisoned. For example, if our sequence is 10321, the poisoned keg was $1 + 2 \cdot 4^1 + 3 \cdot 4^2 + 0 \cdot 4^3 + 1 \cdot 4^4 = 313$. We have now created a bijection between base 4 expansions of length 5 and the keg labels, so we can test up to $n = 4^5 = 1024$ kegs.

Answer:

1024

Reference:

https://www.quantguide.io/questions/poisoned-kegs-iv

Ronaldos House

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Cristiano Ronaldo lives in Portugal in a house numbered 122577. A hurricane hits Portugal, and two random numbers on Ronaldo's house fall off. After the hurricane passes, he sees the numbers on the ground, but has amnesia, so he doesn't remember what his house number is. Therefore, he places the two numbers up in the two empty spots uniformly at random. Find the probability Ronaldo creates the number 122577 again.

Hint:

The key here is to see that if either 22 or 77 falls off, then no matter how Ronaldo places up the numbers, he will get back the correct house number.

Explanation:

The key here is to see that if either 22 or 77 falls off, then no matter how Ronaldo places up the numbers, he will get back the correct house number. If any other pair falls off, there is a 1/2 chance of obtaining the correct house number. Therefore, let S be the event that the two numbers that fall off are the same i.e. either 22 or 77 and C be the event that Ronaldo puts up the numbers in the correct order again. By Law of Total Probability,

$$\mathbb{P}[C] = \mathbb{P}[C \mid S]\mathbb{P}[S] + \mathbb{P}[C \mid S^c]\mathbb{P}[S^c]$$

There are $\binom{6}{2} = 15$ equally likely pairs of numbers to fall off, of which 2 of them are 22 or 77, so $\mathbb{P}[S] = \frac{2}{15}$, meaning $\mathbb{P}[S^c] = \frac{13}{15}$. If the two numbers that fall off are the same, Ronaldo will be correct with probability 1, so $\mathbb{P}[S \mid C] = 1$. If the numbers are not the same, he will be correct with probability $1/2$, as one of the two arrangements is the true house number, so $\mathbb{P}[C \mid S^c] = 1/2$. Adding all of these up, we get

$$\mathbb{P}[C] = 1 \cdot \frac{2}{15} + \frac{1}{2} \cdot \frac{13}{15} = \frac{17}{30}$$

Answer:

1730

Reference:

https://www.quantguide.io/questions/ronaldos-house

Median Uniform

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Let $X_1, X_2, X_3 \sim \text{Unif}(0,3)$ IID. Find the probability that $X_{(2)}$, the median of the three random variables, is between 1 and 2.

Hint:

Split up the interval $(0,3)$ into three subintervals $(0,1], (1,2]$, and $(2,3)$. Each X_i has equal chance of landing in each of the three subintervals. Label the random variables $1-3$, based on which interval they end up in. Therefore, we want the probability the median is in 2.

Explanation:

We can approach this with an intuitive perspective, as the random variables here are uniformly distributed. Split up the interval $(0,3)$ into three subintervals $(0,1], (1,2]$, and $(2,3)$. Each X_i has equal chance of landing in each of the three subintervals. Label the random variables $1-3$, based on which interval they end up in. Therefore, we want the probability the median is in 2. Let's consider the different permutations that allow that to happen:

$$122, 123, 223, 222$$

These are the different ways the values of the intervals can be assigned to the random variables. All that remains is to count the permutations. There are 3 ways to assign the values 122 to the three random variables (select the random variable that receives 1 in 3 ways, the other 2 receive 2). Similarly, there are $3! = 6$ ways for 123, 3 ways for 223, and 1 way for 222. Adding these all up, there are $3+6+3+1 = 13$ permutations of the values. Then, there are $3^3 = 27$ equally-likely ways to assign the values to the random variables in general. Therefore, the answer is $\frac{13}{27}$.

Answer:

1327

Reference:

https://www.quantguide.io/questions/median-uniform

Coin Duel

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Bill and Bob are simultaneously flipping coins with probability $1/3$ of heads per flip. Find the probability that Bill obtains his first heads in strictly less flips than Bob.

Hint:

Find the probability that Bill and Bob obtain their first heads at the same type and use an exchangeability argument.

Explanation:

Let X_1 and X_2 represent the number of flips needed for Bill and Bob, respectively, to obtain their first heads. We are going to generalize this to probability p of heads per flip. We immediately know that $X_1, X_2 \sim \text{Geom}(p)$ by the fact we are looking for a waiting time until first heads. One might guess that the answer to this is $\frac{1}{2}$, but the issue is that there is positive probability of equality, so we first need to subtract that out, and then we can use exchangeability to claim that $\mathbb{P}[X_1 > X_2] = \mathbb{P}[X_1 < X_2]$.

First, let's find $\mathbb{P}[X_1 = X_2]$. This is simple enough to find, as X_1 and X_2 are independent, so

$$\mathbb{P}[X_1 = X_2] = \sum_{n=1}^{\infty} \mathbb{P}[X_1 = X_2 \mid X_2 = n]\mathbb{P}[X_2 = n] = \sum_{n=1}^{\infty} \mathbb{P}[X_1 = n]\mathbb{P}[X_2 = n]$$

as we want the probability $X_1 = X_2$ and we know that $X_2 = n$. The previous work is just by the Law of Total Probability applied by conditioning on X_2. We have that

$$\mathbb{P}[X_1 = n] = \mathbb{P}[X_2 = n] = p(1-p)^{n-1}$$

by the fact that X_1 and X_2 are IID. Thus, previous sum becomes

$$\sum_{n=1}^{\infty} p^2(1-p)^{2n-2} = p^2 \sum_{n=1}^{\infty} \left[(1-p)^2\right]^{n-1}$$

by rearranging slightly. We do this because we want to get an $n-1$ in the exponent of the interior since the lower index of the sum is $n = 1$ and we would ideally by able to apply the geometric sum formula. The sum inside is now geometric with $r = (1-p)^2 < 1$, so the above expression becomes $\frac{p^2}{1-(1-p)^2} = \frac{p^2}{p(2-p)} = \frac{p}{2-p}$. This is $\mathbb{P}[X_1 = X_2]$, so

$$\mathbb{P}[X_1 \neq X_2] = 1 - \mathbb{P}[X_1 = X_2] = 1 - \frac{p}{2-p} = \frac{2(1-p)}{2-p}$$

. This is $\mathbb{P}[X_1 \neq X_2]$, and this is partitioned into two events, namely that $\mathbb{P}[X_1 \neq X_2] = \mathbb{P}[X_1 > X_2] + \mathbb{P}[X_1 < X_2]$. This is because if they are not equal, then one of them must be larger than the other. Since X_1 and X_2 are IID, they are exchangeable, so we have that the should have equally likely probabilities of being larger or smaller than one another. Thus, the two probabilities on the RHS are equal. Substituting in our found value on the LHS for $\mathbb{P}[X_1 \neq X_2]$, we have that $\frac{2(1-p)}{2-p} = 2\mathbb{P}[X_1 < X_2]$, meaning that $\mathbb{P}[X_1 < X_2] = \frac{1-p}{2-p}$. Since there is positive probability of equality between X_1 and X_2, it makes sense that the probability slightly less than $\frac{1}{2}$.

Alternatively, we can give an argument by a one-step Law of Total Probability approach. Let x be the probability in question. We can condition on the 4 outcomes of the first flip. If Bob flips a heads on the first flip, which occurs with probability p, Bill can't obtain a heads before him, so this conditional probability is 0. If Bob flips a tails, then if Bill flips a heads, he wins immediately. This occurs with probability $p(1-p)$. If Bill flips a tails as well, then the probability is just the same as at the start, which is x. Therefore, we obtain the equation

$$x = p(1-p) + (1-p)^2 x \iff (2p - p^2)x = p(1-p) \iff x = \frac{1-p}{2-p}$$

In this case, $p = 1/3$, so our answer is $\frac{2}{5}$.

Answer:

25

Reference:

https://www.quantguide.io/questions/coin-duel

Unknown Baby

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

There are 5 babies in a room: 2 boys and 3 girls. one baby with unknown sex is added. One baby is random selected and it is a boy. What's the probability that the added baby is a boy? You may assume that each gender is equally likely at birth.

Hint:

Let B be the event that the baby added was a boy. Let S be the event that the selected baby was a boy. We want $\mathbb{P}[B \mid S] = \dfrac{\mathbb{P}[S \mid B]\mathbb{P}[B]}{\mathbb{P}[S]}$.

Explanation:

Let B be the event that the baby added was a boy. Let S be the event that the selected baby was a boy. We want $\mathbb{P}[B \mid S] = \dfrac{\mathbb{P}[S \mid B]\mathbb{P}[B]}{\mathbb{P}[S]}$. We know that $\mathbb{P}[B] = \dfrac{1}{2}$, as this is with no other information. We further know that if the added baby was a boy, then $\mathbb{P}[S \mid B] = \dfrac{1}{2}$, as there are equal amounts of boys and girls. On the denominator, we condition on B and B^c. Namely,

$$\mathbb{P}[S] = \mathbb{P}[S \mid B]\mathbb{P}[B] + \mathbb{P}[S \mid B^c]\mathbb{P}[B^c]$$

The first term is the same as the numerator. We know that $\mathbb{P}[B^c] = \dfrac{1}{2}$ as well. B^c is the event a girl was added, so in this case, $\mathbb{P}[S \mid B^c] = \dfrac{1}{3}$, as there are only 2 boys among 6 total. Therefore, our total probability is

$$\mathbb{P}[B \mid S] = \frac{1/2 \cdot 1/2}{1/2 \cdot 1/2 + 1/2 \cdot 1/3} = \frac{3}{5}$$

Answer:

35

Reference:

https://www.quantguide.io/questions/unknown-baby

Wire Connection

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

10 wires each of length 1 mile lay parallel to each other. Electrician A can only see one side of the wires, while Electrician B can only see the other side of the wires. They randomly connect pairs of distinct wire ends until all ends are accounted for. What is the probability that the connections form a closed circuit with total length 10 miles?

Hint:

There are 10 ends on the left side of the wires and 10 ends on the right side of the wires. Without loss of generality, we connect the right end of the first wire to the right end of the second wire. In order to retain the possibility of forming a 10 mile loop, there are 8 possible left wire ends with which the left end of the second wire can join with, out of a possible 9.

Explanation:

There are 10 ends on the left side of the wires and 10 ends on the right side of the wires. Without loss of generality, we connect the right end of the first wire to the right end of the second wire. In order to retain the possibility of forming a 10 mile loop, there are 8 possible left wire ends with which the left end of the second wire can join with, out of a possible 9. Without loss of generality, we connect the left end of the second wire to the left end of the third wire. The right end of the third wire can then be joined with any other remaining wire's right end. Back to the left end. There are 7 remaining ends on the left side. Of these, 6 are viable connections. Repeating this process, we get a final answer of

$$\frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{128}{315}$$

Answer:

128315

Reference:

https://www.quantguide.io/questions/wire-connection

Circular Partition

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

Suppose X_1, X_2, X_3, and X_4 are all independent random points selected uniformly from the perimeter of the unit circle. Draw a chord between X_1 and X_2. Draw another chord between X_3 and X_4. Find the expected number of disjoint regions the circle is partitioned into.

Hint:

Fix X_1 as the "smallest" point in the sense of smallest angle CCW from the origin. Consider the different orderings of X_2, X_3, and X_4.

Explanation:

Fix X_1 as the "smallest" point in the sense of smallest angle CCW from the origin. This doesn't matter since all of them are independent and uniform on the perimeter. If X_2 is the smallest (in terms of CCW angle from $(1, 0)$) among X_2, X_3, and X_4, then we get 3 distinct regions, as the chords do not intersect. If X_2 is the median of the remaining 3, then we get 4 regions, as there is an intersection. If X_2 is the largest among the remaining 3, there is no intersection of chords, so we have 3 regions. As all orderings occur with equal probability, there are 3 regions with probability $\frac{2}{3}$ and 4 regions with probability $\frac{1}{3}$, so the expected number is

$$3 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{10}{3}$$

Answer:

103

Reference:

https://www.quantguide.io/questions/circular-partition

Rolls In A Row II

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

How many rolls of a fair 6-sided die must be rolled on average to get 5 and 6 in a row in that order?

Hint:

Make Markov state equations.

Explanation:

We can use Markov Chains for this problem. Let E_0 be the expectation state of not having the starting 5. Let E_1 be the expectation state of having the 5). Finally, let E_2 be the expectation state of having 5 and 6 in a row (our goal). Then our equations become:

$$E_0 = \frac{1}{6}(E_1 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_1 + \frac{5}{6}E_0 + 1$$

$$E_1 = \frac{1}{6}(E_2 + 1) + \frac{1}{6}(E_1 + 1) + \frac{2}{3}(E_0 + 1) = \frac{1}{6}E_2 + \frac{5}{6}E_0 + 1$$

$$E_2 = 0$$

Solving these equations, we get $E_0 = 36$. Thus it takes 36 rolls on average to roll a 5 and 6 in a row.

Answer:

36

Reference:

https://www.quantguide.io/questions/rolls-in-a-row-ii

Random Tic Tac Toe

Category: Probability
Difficulty: Medium
Companies: SIG

Question:

We toss 3 pebbles at a tic-tac-toe board at the same time. Each of those pebbles will land in one of the nine squares of the board with uniform probability. What is the probability that the pebbles will land in such a way that they form a tic-tac-toe (all pebbles in a row either vertically, horizontally, or diagonally)?

Hint:

Each pebble doesn't have to be in a unique square!

Explanation:

There are 8 ways to forma a tic-tac-toe (3 vertically, 3 horizontally, 2 diagonally). Including the permutations, we multiply by 3!. Thus the total ways is $8 * 3! = 48$. Now we have to see how many different ways the pebbles can land on the board.

The first scenario is that they all land in distinct squares. Since there are 9 total squares and we pick 3 to be where the pebbles land, the total combinations are $\binom{9}{3} = 84$. Each of these combinations has 3! permutations in their order. Thus $84 * 3! = 504$.

Now we have to calculate the number of ways the pebbles could land in two distinct squares. Similar to the three square scenario, we choose two squares from the nine. Including permutations, the total number of arrangements with two squares is $\binom{9}{2} * 3! = 216$.

Finally, all three pebbles can land in the same square which adds an extra 9 ways. Thus in total there are $504 + 216 + 9 = 729$ ways. Thus the probability is

$$\frac{48}{729} = \frac{16}{243}$$

Alternatively, a faster way to do this is to consider that there are $9 * 9 * 9 = 729$ total possibilities how each pebble lands. Since there are 8 winning patterns and each has 3! permutations, there are $\frac{8 * 3!}{729} = \frac{48}{729} = \frac{16}{243}$.

Answer:

16243

Reference:

https://www.quantguide.io/questions/random-tic-tac-toe

Horse Racing

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Jane Street

Question:

There are 25 horses, each of which runs at a distinct and constant speed. The track has five lanes, and thus can race at most five horses at once. By simply racing different sets of horses and observing the placing of the horses within those races without access to a stopwatch, what is the minimum number of races needed to uniquely identify the three fastest horses?

Hint:

To find the three fastest horses, all horses need to be tested. A natural first step will then be to divide the horses into five racing groups labeled 1-5, 6-10, 11-15, 16-20, and 21-25.

Explanation:

To find the three fastest horses, all horses need to be tested. A natural first step will then be to divide the horses into five groups labeled 1-5, 6-10, 11-15, 16-20, and 21-25. After these five races, we have the order of the horses within each group. Without loss of generality, assume the order within each group follows the order of the numbers (e.g., 6 is the fastest and 10 is the slowest within the 6-10 group). Therefore, the five fastest horses are 1, 6, 11, 16, and 21. We can eliminate the last two horses from each group since they cannot possibly be within the top three. We will then race the top five horses to establish the fastest horse. Again, assume the order of this group follows the order of the numbers (e.g., 1 is the fastest and 21 is the slowest). This tells us that from the first group 1-5, only 2 and 3 are in the running for second and third; from the second ground 6-10, only 6 and 7 are in the running for second and third; and from the third group 11-15, only 11 is in the running for second or third. This is because if the fastest horse within a group ranks second or third, then only one or no other horses within that group can be in the top three, respectively. Hence, the last group will race 2, 3, 6, 7, and 11, for a total of 7 races.

Answer:

7

Reference:

https://www.quantguide.io/questions/horse-racing

Contracts And Options IV

Category: Probability
Difficulty: Easy
Companies: SIG, Old Mission

Question:

An energy company is drilling for oil. There is a 30% chance the well will be successful and a 70% chance it will fail. If they find oil, the stock is worth \$100. If they fail its worth \$20. You own a right (but not an obligation) to buy the stock for \$50 the day after they find out if the well is successful. What is the fair value of this option?

Hint:

We only exercise our option if the well is successful.

Explanation:

We only exercise our option if the well is successful. We save \$50 in this case and it occurs with probability 0.3. Therefore, the answer is $50 \cdot 0.3 = 15$.

Answer:

15

Reference:

https://www.quantguide.io/questions/contracts-and-options-iv

Odd Coin Flips

Category: Probability
Difficulty: Medium
Companies: SIG, Jane Street

Question:

You flip 100 fair coins simultaneously. What is the probability that you observe an odd number of heads?

Hint:

Look closely at the last coin flipped.

Explanation:

Imagine flipping 99 coins first. The last coin always decides whether we observe an odd or even number of heads. Thus, the probability of observing an odd number of heads is $\frac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/odd-coin-flips

Win By 2

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

Person A has a 60% chance to win a round of a game against Person B. To win the entire game, one of the players must win N more rounds than the other person and they keep playing till someone wins. What is the probability Person A wins the entire game as N approaches infinity?

Hint:

Consider this thought experiment. Would you rather play a 1v1 basketball game against LeBron to 3 points or to 21 points? What are your chances of winning when you play to a very large N?

Explanation:

Lets start out with the case where $N = 1$. The probability of Person A winning is $\frac{0.6}{0.6+0.4} = 0.6$. Lets see what happens when $N = 2$. The probability that Person A wins two games in a row is $0.6 \cdot 0.6 = 0.36$. The probability that Person B wins two games in a row is $0.4 \cdot 0.4 = 0.16$. This problem becomes a random walk where there is a 0.36 chance of going $+1$ and 0.16 chance of going -1. The other 0.48 chance our walk doesn't increase or decrease. We end the walk when we get to $+2$ or -2. The probability Person A wins two games in a row before Person B is

$$\frac{0.36}{0.36 + 0.16} = \frac{9}{13}$$

As we keep increasing N, we see the general case becomes

$$\frac{0.6^N}{0.6^N + 0.4^N} = \frac{1}{1 + \left(\frac{2}{3}\right)^N}$$

As N tends to infinity, the probability Person A wins approaches 1. Thus the answer is 1.

Answer:

1

Reference:

https://www.quantguide.io/questions/win-by-2

Complementary Dice

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Two players are playing a game by rolling a single standard die. Player 1 rolls first. If the top surface of the die shows 1 or 2, then he wins. If not, then Player 2 will roll the die. If the top surface of the die shows 3, 4, 5, or 6, then he wins. Else, the game is continues until there is a winner. Let p_1 and p_2 represent the probabilities that the two players win, respectively. Find $\max\{p_1, p_2\}$.

Hint:

Compute p_1 as in the question. With probability 1/3, he rolls a winning value on the first roll.

Explanation:

Let's compute p_1 as in the question. With probability 1/3, he rolls a winning value on the first roll. Otherwise, with probability 2/3, he does not roll a winning value, and then player 2 needs to not roll a winning value on their first roll, which occurs with probability 1/3. In that case, the probability player 1 wins is p_1 again. Therefore,

$$p_1 = \frac{1}{3} + \frac{2}{9}p_1 \iff p_1 = \frac{3}{7}$$

This means that $p_2 = 1 - p_1 = \frac{4}{7}$, so the answer is $\frac{4}{7}$.

Answer:

47

Reference:

https://www.quantguide.io/questions/complementary-dice

Better In Red II

Category: Probability
Difficulty: Medium
Companies: SIG, Five Rings, Jane Street, HRT

Question:

A $3 \times 3 \times 3$ cube is composed of 27 $1 \times 1 \times 1$ cubes that are white by default. All the surfaces of the $3 \times 3 \times 3$ cube are painted red and then the cube is disassembled such that all orientations of all cubes are equally likely. You select a random small cube and notice that all 5 sides that are visible to you are white. What is the probability the last face not visible to you is red?

Hint:

Consider Bayes' Rule and the orientation of the cube.

Explanation:

Let R be the event that the last side not visible to you is red and W be the event that the other sides visible are not red. We want $\mathbb{P}[R \mid W] = \frac{\mathbb{P}[R \cap W]}{\mathbb{P}[W]}$. For $\mathbb{P}[R \cap W]$, we need the last side to be red and have that side be facing away from us. Since the orientation is random, the probability the red side is facing away from us is $\frac{1}{6}$. In addition, 6 of the 27 cubes have exactly one red side, so $\mathbb{P}[R \cap W] = \frac{1}{27}$. Then, in the denominator, we can have 5 white sides from choosing a cube that would satisfy the numerator event or just choosing the center cube. The center cube is white on all sides so there is no need to orient it. Therefore, $\mathbb{P}[W] = \frac{1}{27} + \frac{1}{27} = \frac{2}{27}$. Substituting in, we get $\mathbb{P}[R \mid W] = \frac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/better-in-red-ii

Paired Pumpkins

Category: Brainteasers
Difficulty: Medium
Companies: SIG, Jane Street

Question:

Dracula has 5 pumpkins. When he pairs any two pumpkins, they weigh $21, 22, \ldots, 30$ pounds. What's the sum of the weights of all the pumpkins?

Hint:

Is there anything you can do with all the paired weights to directly solve for the sum of all the pumpkins?

Explanation:

Let a, b, c, d, and e be the weight of each pumpkin from lightest to heaviest. If we sum all the pairwise weights, we know we will have 4 sets of a, b, c, d, and e. Thus

$$4 \cdot (a + b + c + d + e) = 21 + \cdots + 30 = 255$$

or

$$a + b + c + d + e = \frac{255}{4}$$

You could also create equations for all the pairs to find the weights of each pumpkin but the question only asks for the sum of the weights and the above method is a clever way of reaching the answer.

Answer:

2554

Reference:

https://www.quantguide.io/questions/paired-pumpkins

Card Up

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

There are 83 cards in a deck. Each is labeled with a distinct value $1 - 83$. A dealer randomly shuffles them up and deals the top 5 cards of the deck are dealt. If the top 5 cards are in strictly ascending or descending order, your payout is $\$x$; otherwise, you receive nothing. If the cost is \$1 to play this case, what must x be for the game to be fair?

Hint:

Consider how many permutations of the first 5 cards there are.

Explanation:

The number 83 here is irrelevant, as we only care about the top 5 cards drawn. There are 5! permutations of the values of the first 5 cards, as they are all distinct. Two of these 5! permutations are cases where all the cards are in strictly ascending or descending order, so the probability of this event is $\frac{2}{5!} = \frac{1}{60}$. Therefore, the expected value on the game is $(x-1) \cdot \frac{1}{60} - \frac{59}{60}$, which implies that $x = 60$ for the game to be fair.

Answer:

60

Reference:

https://www.quantguide.io/questions/card-up

A Low Median

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

Suppose we draw 3 numbers from a Unif$[0, 1]$ distribution independently. We then sort the numbers in ascending order. What is the probability that the median is less than $2/3$?

Hint:

Break up into cases of exactly one larger and none larger.

Explanation:

If we want the median to be less than $\frac{2}{3}$ then it must be that we can have at most one number be larger than $\frac{2}{3}$. Therefore, we have that

$$P[\text{exactly one number larger}] = \binom{3}{1}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}$$

and we also have

$$P[\text{exactly zero numbers larger}] = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Adding these together, we have that the probability that at most one number is larger than $\frac{2}{3}$ is

$$\frac{4}{9} + \frac{8}{27} = \frac{20}{27}$$

Answer:

2027

Reference:

https://www.quantguide.io/questions/a-low-median

Rough Day

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street

Question:

A bagel factory has a conveyor belt that delivers bagels continuously. Every bagel is either good (G) or bad (B). If a given bagel was G, the next bagel is also G with probability $\frac{2}{5}$. If a given bagel was B, the next bagel is also B with probability $\frac{3}{5}$. You woke up to find that the first bagel is a bad one. Find the expected number of bagels that pass until another bad bagel.

Hint:

How can you use the Law of Total Expectation to condition on the outcome of outcome of the first bagel since our present one?

Explanation:

Let T be the number of bagels until another bad one and X_1 be the outcome of the first bagel since our present one. Then

$$\mathbb{E}_B[T] = \mathbb{E}_B[T \mid B]\mathbb{P}_B[B] + \mathbb{E}_B[T \mid G]\mathbb{P}_B[G]$$

The subscript B is to denote that we start on a bad bagel. We have that $\mathbb{P}_B[B] = \frac{3}{5}$, so $\mathbb{P}_B[G] = \frac{2}{5}$. Then, $\mathbb{E}_B[T \mid G] = 1 + \mathbb{E}_G[T]$, as we take one step and then find the expected number of bagels starting from a good bagel until a bad bagel. Then, $\mathbb{E}_B[T \mid B] = 1$, as it takes just 1 bagel to get another bad bagel if the first bagel after is bad. Therefore,

$$\mathbb{E}_B[T] = (1 + \mathbb{E}_G[T]) \cdot \frac{2}{5} + 1 \cdot \frac{3}{5} = 1 + \frac{2}{5} \cdot \mathbb{E}_G[T]$$

Note that the distribution of the number of bagels until a bad bagel starting from a good bagel is Geom $\left(\frac{3}{5}\right)$, as there is a $\frac{3}{5}$ probability each bagel of going to a bad bagel, so $\mathbb{E}_G[T] = \frac{5}{3}$. This implies that $\mathbb{E}_B[T] = 1 + \frac{2}{5} \cdot \frac{5}{3} = \frac{5}{3}$.

Answer:

53

Reference:

https://www.quantguide.io/questions/rough-day

Three Repeat

Category: Probability
Difficulty: Easy
Companies: SIG, Jane Street, Old Mission

Question:

A fair coin is flipped 6 times. Find the probability of obtaining exactly 3 consecutive heads somewhere in the 6 flips.

Hint:

What are the possible forms of such a sequence? Do you notice any symmetry in the sequences?

Explanation:

We have 4 forms that the sequence can be in, which are $HHH---$, $-HHH--$, $--HHH-$, and $---HHH$. Note that the first and second are respectively reflection of the fourth and third sequences, so we can just do the first two sequences and then multiply by 2. For the first sequence, the 4th flip must be tails, but the other two can be either parity, so this yields 4 possibilities. For the second outcome, the first and second dashes must be T, but the last can be either, so this yields 2 outcomes. Therefore, we have a total of $2(4+2) = 12$ sample points that have a single run of three heads, so the probability is $\frac{12}{64} = \frac{3}{16}$.

Answer:

316

Reference:

https://www.quantguide.io/questions/three-repeat

Dice Labels

Category: Probability
Difficulty: Medium
Companies: SIG, Drw

Question:

How many distinct ways can you label a 6 sided dice if you wipe off all the numbers? Arrangements that can be formed by rotating the die around are not considered distinct.

Hint:

Fix one side of the die and then find the possibilities for the opposite side. Then count the orderings for the last 4 faces.

Explanation:

Lets start with placing the number 1 on any side and let's consider it the top. Then there's 5 different numbers that can be on the opposite side of the dice. Finally, with the 4 remaining numbers, there's 4! different orderings of the numbers but this counts the same orderings but starting on different faces so we divide by 4. Think about it this way, lets say we have 1 on the top and 6 on the bottom. The last 4 numbers are 2, 3, 4, and 5. There are 4! orderings of these numbers. But let's specifically consider the order $2, 3, 4, 5$. This order is the exact same as $5, 2, 3, 4$ and $4, 5, 2, 3$ and $3, 4, 5, 2$. As you can see, any order will have 4 different orderings and placements, so we divide by 4. Putting everything together, $5 * 4!/4 = 5 * 3! = 30$.

Answer:

30

Reference:

https://www.quantguide.io/questions/dice-labels

Prepped Offer

Category: Probability
Difficulty: Easy
Companies: SIG

Question:

45% of students don't use QuantGuide, 25% are heavy prep users and 30% are light prep users. If the heavy prep users are twice as likely to get an offer as light prep users, and light prep users are twice as likely to get an offer as those that don't use the platform, then what is the probability that if someone got an offer that they were a heavy prep user of QuantGuide?

Hint:

We are looking to update the probability of something given new information. What theorem can help us out here?

Explanation:

Since we can see this is a conditional probability question as they give you information to try and find an associated value, let's use Bayes to solve this. We are solving for $\mathbb{P}[\text{Heavy User}|\text{Offer}]$. Bayes says that this value is equivalent to

$$\mathbb{P}[\text{Offer}|\text{Heavy User}] \cdot \mathbb{P}[\text{Heavy User}]/\mathbb{P}[\text{Offer}]$$

In the question, we aren't given any hard value on the probability of a person getting an offer given their usage of QuantGuide, only relational information. However, we can still use this information. Lets let p be the probability a non-user gets an offer. Then the probability that a light user gets an offer is $2p$ and the probability a heavy user gets an offer is $4p$. Thus, our Bayes equation becomes

$$4p \cdot 0.25/(4p \cdot 0.25 + 2p \cdot 0.3 + p \cdot 0.45) = \frac{20}{41}$$

Answer:

2041

Reference:

https://www.quantguide.io/questions/prepped-offer

Up Or Down

Category: Brainteasers
Difficulty: Easy
Companies: SIG

Question:

If you have an asset that goes up or down 20% each day from its price the day prior with equal probability, what's the probability that after 10 days the asset's price is unchanged?

Hint:

If there was a price rise on k of the days, then for the price to be unchanged, we would need $(1.2)^k(0.8)^{10-k} = 1$

Explanation:

If there was a price rise on k of the days, then for the price to be unchanged, we would need $(1.2)^k(0.8)^{10-k} = 1$, as the price went down the other $10 - k$ days. Equivalently, this means that $(1.2)^k = (1.25)^{10-k}$, which does not hold for an integer k. Therefore, it is impossible for this to occur.

Answer:

0

Reference:

https://www.quantguide.io/questions/up-or-down

Bridge Crossing

Category: Brainteasers
Difficulty: Easy
Companies: SIG, Jane Street

Question:

Alice, Bob, Charlie, and Daniel need to get across a river. The only way to cross the river is by an old wooden bridge, which holds at most two people at a time. Because it is dark, they can't cross the bridge without a lantern, of which they only have one. Each pair can only walk at the speed of the slower person. They need to get all of them across to the other side as quickly as possible. Alice takes 10 minutes to cross; Bob takes 5 minutes; Charlie takes 2 minutes; Daniel takes 1 minute. What is the minimum time to get all of them across to the other side?

Hint:

The key point is to realize that the person returning with the lantern does not have to be from the most recent pairing.

Explanation:

The key point is to realize that the person returning with the lantern does not have to be from the most recent pairing. Another point to realize is that Alice and Bob should go together, and not in the first crossing- otherwise, one of them has to go back, which will take too long. Therefore, Charlie and Daniel should go across first, which takes two minutes. Then, send Daniel back with the lantern, which takes one minute. Alice and Bob go across, which takes ten minutes. Charlie returns with the lantern, which takes two minutes. Finally, Charlie and Daniel cross again, which takes two minutes, for a total of 17 minutes.

Answer:

17

Reference:

https://www.quantguide.io/questions/bridge-crossing

Cheese Lover III

Category: Probability
Difficulty: Easy
Companies: SIG, Drw

Question:

Jon loves cheese. He decides to make 100 blocks of cheese. The distribution of the weight (in grams) of each block he makes follows IID $\text{Exp}\left(\frac{1}{250}\right)$ distribution. Let W_i denote the weight of the ith block of cheese, and T_{100} represent the total weight of the 100 blocks of cheese. Using Central Limit Theorem, what is an approximation for $\mathbb{P}[T_{100} > 26000]$? The answer is in the form $\Phi(a)$ for some real a. Φ here is the CDF of the standard normal distribution. Find a.

Hint:

First standardize T_{100} by subtracting the mean and dividing by standard deviation.

Explanation:

We subtract the expectation of $T_{100} = 25000$ from both sides, then divide by $\text{Var}(T_{100}) = 100(250)^2$ (by recognition of using linearity of variance), implying that $\sigma_{T_{100}} = \sqrt{100(250)^2} = 250\sqrt{100} = 2500$. Thus. we get that

$$\mathbb{P}[T_{100} > 26000] = 1-\mathbb{P}[T \leq 26000] = 1-\mathbb{P}[T_{100}-25000 \leq 26000-25000] = 1-\mathbb{P}\left[\frac{T_{100}-25000}{2500} \leq \frac{1000}{2500}\right]$$

We approximate the LHS using a standard normal random variable by the CLT, so we are using the approximation $1 - \mathbb{P}\left[Z \leq \frac{2}{5}\right]$, where $Z \sim N(0,1)$. This last quantity is $1 - \Phi(\frac{2}{5}) = \Phi(-\frac{2}{5})$. Therefore, $a = -2/5$.

Answer:

25

Reference:

https://www.quantguide.io/questions/cheese-lover-iii

Poker Hands I

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

A poker hand consists of five cards from a standard deck of 52 cards. If p is the probability that you have a four-of-a-kind (four of the five cards have the same rank), find the reciprocal of p.

Hint:

How many ways can you pick 4 cards of the same rank?

Explanation:

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain a four-of-a-kind, we can break the problem down into the four cards that have the same value and the fifth card. The value that is shared amongst four of them can be any of the 13 faces. The fifth card can be any of the 48 remaining cards left in the deck. Thus, the probability that you have a four-of-a-kind is:

$$\frac{13 \times 48}{\binom{52}{5}} = \frac{1}{4165}$$

Therefore, the answer to the question is 4165.

Answer:

4165

Reference:

https://www.quantguide.io/questions/poker-hands-i

Matrix Exponential

Category: Pure Math
Difficulty: Medium
Companies: Drw

Question:

Find trace(e^A) to 3 decimal points, where A is defined as

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

The answer will be in the form $e^a + e^b$ for integers a and b. Find ab.

Hint:

Recall that e^A can be rewritten as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Explanation:

Recall that e^A can be rewritten as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

In addition, note the following:

$$\text{trace}(e^A) = \text{trace}\left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k\right)$$

$$= \sum_{k=0}^{\infty} \text{trace}\left(\frac{1}{k!} A^k\right)$$

$$= \text{trace}\,(\;$$

$$\begin{bmatrix} \sum_{k=0}^{\infty} \frac{3^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{6^k}{k!} \end{bmatrix}$$

$$= e^3 + e^6$$

Therefore, our answer is $3 \cdot 6 = 18$

Answer:

18

Reference:

https://www.quantguide.io/questions/matrix-exponential

Absolute Difference

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

Let $x_0 = 0$ and let $x_1, \ldots, x_{10}$ satisfy that $|x_i - x_{i-1}| = 1$ for $1 \leq i \leq 10$ and $x_{10} = 4$. How many such sequences are there satisfying these conditions?

Hint:

Consider a walk where you move up 1 for heads and move down 1 for tails.

Explanation:

Let's put this problem in a different light. Consider a walk where you move up 1 for heads and move down 1 for tails. The condition states that in the 10 flips, we have 4 more heads than tails, so this means that there are 7 movements up and 3 movements down. In other words, each valid sequence corresponds to some ordering of $UUUUUUUDDD$. There are $\binom{10}{3} = 120$ such arrangements, so this is our answer.

Answer:

120

Reference:

https://www.quantguide.io/questions/absolute-difference

3 Larger Die

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

You have two fair 10−sided dice that are colored green and yellow with values $1 - 10$ on each side. You roll both dice simultaneously. If they show the same value, then the game is over. If the two dice are not equal, but the yellow die shows a value at least 3 larger than the green die, you receive \$1 and can roll both dice again. In all other cases, you receive nothing and can roll again. The game only ends once the two dice show the same value. What is your expected payout on this game?

Hint:

The yellow die is at least 3 larger than the green die with probability $\frac{7}{25}$.

Explanation:

Let e be the expected payout. The yellow die is at least 3 larger than the green die with probability $\frac{7}{25}$. You can verify this by conditioning on the value of the green die and noting the number of outcomes that are satisfactory if the green die shows k is $8 - k$. Summing those up, you get 28 of 100 equally-likely outcomes. The dice show the same value with probability $\frac{1}{10}$. Therefore, they show different values with probability $\frac{31}{50}$. Applying the Law of Total Expectation on the outcome of the first roll,

$$e = \frac{7}{25} \cdot (e + 1) + \frac{31}{50}e$$

If we roll a value at least 3 larger on the yellow than the green, then it is as if our game restarts with our money being 1 instead of 0. Solving for e, we get that $\frac{1}{10}e = \frac{7}{25}$, so $e = \frac{14}{5}$.

Answer:

145

Reference:

https://www.quantguide.io/questions/3-larger-die

Pass The Ball

Category: Probability
Difficulty: Medium
Companies: Drw

Question:

You and 4 other people are sitting in a circle. You are given a ball to start the game. Every second of this game, the person with the ball has three choices they can make. They can either pass the ball to the left, pass the ball to the right, or keep the ball (all with equal probability). This game goes on till someone keeps the ball. What is the probability that you are the person to end the game and keep the ball?

Hint:

Consider the states of this problem.

Explanation:

We can use Markov states to solve this question. You'll notice there is some symmetry in this problem that will allow us to solve this question with less states. The symmetry is in the placement of the other people. For example, the two people immediately next to you can be treated as the same state functionally (since they both behave the exact same), and the two people furthest from you are also functionally the same (again, since they both behave the exact same). Let P_1 be the probability you end with the ball. Let P_2 be the probability that someone immediately next to you has the ball and you are the one to end the game. Finally, let P_3 be the probability that someone furthest from you has the ball and you are the one to end the game. The equations for the states are as follows:

$$P_1 = \frac{1}{3} + \frac{2}{3}P_2$$

$$P_2 = \frac{1}{3}P_1 + \frac{1}{3}P_3$$

$$P_3 = \frac{1}{3}P_2 + \frac{1}{3}P_3$$

Note that in the last equation, if one of the two people not immediately next to you has the ball, one of the passes will be to someone that is also not next to you, which is why there is only a $\frac{1}{3}P_2$ term. Solving these equations, we see that $P_1 = \frac{5}{11}$ which is our final answer.

Answer:

511

Reference:

https://www.quantguide.io/questions/pass-the-ball

6040 Split

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

You have a fair $40-$sided and fair $60-$sided die in front of you. If you roll both, find the probability the $60-$sided die shows a strictly larger number than the $40-$sided die.

Hint:

Condition on the value of the $60-$sided die being > 40 or between 1 and 40.

Explanation:

The trick here is to condition on whether or not the $60-$sided die is larger than 40 or in the range $1, 2, \ldots, 40$. In the case where it is strictly larger than 40, the $60-$sided die wins with probability 1. This event occurs with probability $\frac{1}{3}$. In the event that it is in the range $1, 2, \ldots, 40$, then it is like we are rolling two identical dice.

Therefore, we just need to find the probability that the $60-$sided die would be the one that is strictly larger. Conditioning on the fact that we are in the range $1, 2, \ldots, 40$, each of the two dice is equally likely to be strictly larger than the other. Therefore, we just need to subtract out the probability that they are equal and then divide by 2. The probability they are equal is $\frac{1}{40}$. This is because if we fix the first die arbitrarily, there is 1 of 40 equally likely values the second die that is equal to the first. Therefore, there is a $\frac{39}{40}$ chance they are not equal. If they are not equal, it is equally likely for either of the two to be larger, so the probability that the larger one is the $60-$sided die is $\frac{39}{80}$. This means the total probability is

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{39}{80} = \frac{79}{120}$$

Answer:

79120

Reference:

https://www.quantguide.io/questions/6040-split

Leap Frog

Category: Probability
Difficulty: Medium
Companies: Drw, Five Rings

Question:

A frog starting at position 0 is going to jump on the integers. At each step, the frog will choose to jump forward 1 or 2 steps with equal probability. Let p_k be the probability that the frog hits position $k > 0$. Find the largest value of p_k.

Hint:

If we want to reach position k, what are the possible positions we can reach it from? From those positions, how many steps need to be taken to reach position k? You may consider deriving a second-order recurrence relation for p_k.

Explanation:

We want to maximize p_k. We are going to explicitly solve for p_k. We know that the frog jumps 1 or 2 steps at each time. Therefore, for position k, we can only reach it from positions $k-1$ or $k-2$. Namely, by Law of Total Probability, $p_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k-2}$. The $\frac{1}{2}$ represents the probability of selecting 1 or 2 steps when at position $k-1$ and $k-2$, respectively. This is a homogeneous second-order recurrence relation with constant coefficients. We can see this by rearranging to get $-p_k + \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k-2} = 0$. The characteristic polynomial is $-r^2 + \frac{1}{2}r + \frac{1}{2} = 0$. By multiplying both sides by -2, this is equivalent to $2r^2 - r - 1 = 0$. By using the quadratic equation, the roots are $r = 1, -\frac{1}{2}$. Therefore, our solution is in the form $p_k = c_0 + c_1\left(-\frac{1}{2}\right)^k$. We need some initial conditions on this. Namely, it is very easy to calculate p_0 and p_1. We have that $p_0 = 1$ as we start at 0. Additionally, $p_1 = \frac{1}{2}$ since we must take 1 step starting from 0 to reach it. Otherwise, we do not hit it. We get the system of equations $c_0 + c_1 = 1$ and $c_0 - \frac{1}{2}c_1 = \frac{1}{2}$. It is simple enough to verify that $c_0 = \frac{2}{3}$ and $c_1 = \frac{1}{3}$ solve this system.

Plugging this in $p_k = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^k$. Note that $\left(-\frac{1}{2}\right)^k$ is positive for even k and negative for odd k. Therefore, to maximize this probability, we want to maximize how positive $\left(-\frac{1}{2}\right)^k$ is for $k > 0$. Namely, we just want k to be as small as possible and even, as evenness makes it positive and k being the smallest even one makes the term as large as possible over the positive integers. The smallest positive integer is $k = 2$, so $k = 2$ will maximize p_k. Namely, $p_2 = \frac{3}{4}$, and this is our solution.

Answer:

34

Reference:

https://www.quantguide.io/questions/leap-frog

Statistical Test Review VIII

Category: Statistics
Difficulty: Easy
Companies: Drw

Question:

Consider a random sample of size 25 from the distribution $\mathcal{N}(160, 900)$. To the nearest ten thousandth, what is the probability that the sample mean is 165 or greater?

Hint:

Consider a random sample of size 25 from the distribution $\mathcal{N}(160, 900)$. To the nearest ten thousandth, what is the probability that the sample mean is 165 or greater?

Explanation:

Let $\hat{\mu}$ denote the sample mean. Our test statistic is defined as follows:

$$z = \frac{\hat{\mu} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{\sqrt{25}(165 - 160)}{30}$$

$$= \frac{5}{6}$$

The desired probability is simply $\mathbb{P}\left(Z \geq \frac{5}{6}\right) = 1 - \mathbb{P}\left(Z \leq \frac{5}{6}\right)$ for $Z \sim \mathcal{N}(0, 1)$. With a calculator, we find this value to be 0.2023.

Answer:

0.2023

Reference:

https://www.quantguide.io/questions/statistical-test-review-viii

New Question

Category: Probability
Difficulty: Medium
Companies: Drw

Question:

You flip a fair coin till you get two heads in a row. What is the probability that you stop on the 10^{th} toss?

Hint:

Think about possible sub-sequences of a valid 10 flip sequence.

Explanation:

We know that our 9^{th} and 10^{th} toss have to be heads. This means that flips 1 through 8 can't have consecutive heads. Through deduction, we can start our sequence with either T or HT.

Let F_n be the number of valid sequences from the first flip to the nth flip. We need to ensure our sequence does not end with heads, as we are placing HH right after. If the sequence ended with heads, there would be two consecutive heads before our desired length. We know F_1 is 1, as only T does not end with H. $F_2 = F_1 + 1 = 2$, as we added the one for the HT outcome. $F_3 = F_2 + F_1 = 3$ because we can either add T to all of the sequences in F_2 or HT to all of the sequences in F_1 to make F_3. We continue this recurrence and see that this sequence is the Fibonacci sequence.

$F_4 = F_3 + F_2 = 5$, $F_5 = 5 + 3 = 8, F_6 = 8 + 5 = 13, F_7 = 13 + 8 = 21$. Finally $F_8 = 21 + 13 = 34$. Thus there are 34 possible ways for the first 8 tosses. All of these possibilities gets HH added onto the end to make 34 possible ways to make 10 toss sequences that fits our criteria. Each of these sequences has a $\left(\frac{1}{2}\right)^{10}$ probability of occurrence, thus the answer is

$$34 \cdot \left(\frac{1}{2}\right)^{10} = \frac{17}{512}$$

Answer:

17512

Reference:

https://www.quantguide.io/questions/new-question

Checkmate

Category: Probability
Difficulty: Medium
Companies: Drw

Question:

Andy is playing chess. In order to receive a prize, he must win at least two consecutive games out of three. Andy may either play Michael, then Aaron, then Michael (option 1), or he may play Aaron, then Michael, then Aaron (option 2). Aaron is better at chess than Michael. Which option, 1 or 2, should Andy pick in order to maximize his chances at receiving a prize?

Hint:

Let m denote the probability that Andy wins a game against Michael, and let a denote the probability that Andy wins a game against Aaron. Since Aaron is better than Michael, we assume $a < m \Rightarrow m - a > 0$. In order to receive a prize, Andy can either win the first two games, or he can lose the first game and win the next two games. How can you set up equations to compare $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1})$ and $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2})$?

Explanation:

Let m denote the probability that Andy wins a game against Michael, and let a denote the probability that Andy wins a game against Aaron. Since Aaron is better than Michael, we assume $a < m \Rightarrow m - a > 0$. In order to receive a prize, Andy can either win the first two games, or he can lose the first game and with the next two games. For option 1, the probability that Andy receives a prize can be written as

$$\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1}) = m \cdot a + (1 - m) \cdot a \cdot m$$

And for option 2, we have

$$\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2}) = a \cdot m + (1 - a) \cdot m \cdot a$$

Note the following:

$$\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2}) - \mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1})$$

$$= [(1 - a) - (1 - m)]\, am$$

$$= (m - a)am > 0$$

Therefore, we conclude that $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2})$ is greater than $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1})$. Andy should pick option 2.

Answer:

2

Reference:

https://www.quantguide.io/questions/checkmate

Writing To Recruiters

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

You have N letters for recruiters at N firms. You forgot which recruiter works for which firm so you randomly assign each letter to a firm (only one letter going to each firm). What is the expected number of letters going to their correct firm?

Hint:

Try using indicator variables.

Explanation:

This is a classic indicator variable problem. Let E_k be the indicator variable for each k letter where it is 1 if the letter goes to the right firm and 0 if it doesn't. Then $T = \sum_{i=1}^{N} E_i$ gives the total number of letters that go to the correct firm. By linearity of expectation and the fact that the random variables are exchangeable, $\mathbb{E}[T] = N\mathbb{E}[E_1]$. The probability for each of these letters to go to their correct firm is $\frac{1}{N}$. Thus,

$$\mathbb{E}[T] = N \cdot \frac{1}{N} = 1$$

You can double check this is the case for small values N, such as 2 and 3.

Answer:

1

Reference:

https://www.quantguide.io/questions/writing-to-recruiters

Counting Up

Category: Brainteasers
Difficulty: Medium
Companies: Drw

Question:

Jay and Kay play a game where Jay calls out a number between 1 and 8 to start. Afterwards, Kay gets to pick a number that is between 1 and 8 larger than the number Jay keeps. They repeatedly select in this fashion. The first player to call out the number 200 wins. There is a value that Jay can choose to start with so that if he plays optimally, he will always win. What is this value? If this is not possible, answer "-1".

Hint:

Note that $200 = 9 \cdot 22 + 2$. How can you use modular arithmetic to find Jay's optimal starting value?

Explanation:

When Jay guesses 2, Kay can only pick the values 3 to 10. Regardless of what Kay says, Jay should then select 11. In this pattern, Jay will select all values in the form $9k + 2$ for k an integer. The largest value in this form less than 200 is 191. Once Jay says 191, Kay must pick a value 192 to 199. Regardless of what Kay says, the value will be within 8 of 200, so Jay can say 200 on the next turn. The reason to think of this is because 200 is also in the form $9k + 2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/counting-up

Basic Dice Game III

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

A casino offers you a game with a six-sided die where you are paid the value of the roll. If you roll a $1, 2$, or 3, the game stops and you are paid out. Else, you add the value to your total sum and get to continue rolling. What is the fair value of this game?

Hint:

Your total payoff is dependent on the outcome of the first dice. What are the possible outcomes, their associated probabilities, and how can you utilize the Law of Total Expectation?

Explanation:

Your payoff is dependent on the first roll, so we can use the Law of Total Expectation. Let x be the fair value of this game. There is a $\frac{1}{2}$ probability that you roll a 1, 2, or 3 (on average 2). The other $\frac{1}{2}$ of the time, you roll a 4, 5, or 6 (on average 5), add this to your total, and essentially restart the game with an expected value of x. Thus, we can write:

$$x = \frac{1}{2} \times (2) + \frac{1}{2} \times (x + 5)$$

$$x = 7$$

Answer:

7

Reference:

https://www.quantguide.io/questions/basic-dice-game-iii

Full Solutions

Category: Pure Math
Difficulty: Easy
Companies: Drw, Five Rings, Jane Street

Question:

Find all pairs of integers (x, y) such that $3x + 7y = 10000$. All of the solutions can be written in the form $x = a + 7n$ and $y = b - 3n$, where n is any arbitrary integer and a, b are integers such that b is a minimal positive integer. Find ab.

Hint:

Since 3 and 7 are relatively prime, there must exist a solution to this equation. Note that $3(-2) + 7(1) = 1$.

Explanation:

Since 3 and 7 are clearly relatively prime, there must exist a solution to this equation. We note that $3(-2) + 7(1) = 1$, so if we multiply by 10000 on both sides, $3(-20000) + 7(10000) = 10000$, which means $(x, y) = (-20000, 10000)$ is a solution to this equation. By the latter hint, we can write all solutions to this equation in the form

$$(-20000 + 7n, 10000 - 3n)$$

We now need to reduce this equation so that the integer in front of $-3n$ is as small as possible. We note that $9999 = 3 \cdot 3333$, so we note that if $m = n - 3333$, we can rewrite the form of solutions as

$$(-20000 + 7(m + 3333), 10000 - 3(m + 3333)) = (3331 + 7m, 1 - 3m)$$

This is the general form where b is as small as possible but possible. In particular $ab = 3331$.

Answer:

3331

Reference:

https://www.quantguide.io/questions/full-solutions

Basic Dice Game IV

Category: Probability
Difficulty: Medium
Companies: Drw, HRT

Question:

You roll a fair die. Then, you get to roll until you obtain a value that differs from your first roll. You get paid out the value equal to the sum of all of your rolls (including the first one). What is the fair value of this game?

Hint:

Condition on the value of the first roll. Afterwards, once you have conditioned on the value of the first roll, condition on the value of the second roll to see if it matches the first roll.

Explanation:

We know that the value we need to not roll depends on the value we roll first. This implores us to use Law of Total Expectation and condition on the first value that we roll. Namely, if X_1 is our first roll and T is our total, $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X_1]] = \sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]\mathbb{P}[X_1 = i]$.

We know $\mathbb{P}[X_1 = i] = \frac{1}{6}$ for all $1 \leq i \leq 6$, so $\mathbb{E}[T] = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]$

To compute $\mathbb{E}[T \mid X_1 = i]$, we first know that we start with i in the bank from our first roll. Therefore, $\mathbb{E}[T \mid X_1 = i] = i + \mathbb{E}[T' \mid X_1 = i]$, where T' is the money that is made after the first roll.

We know that with probability $\frac{1}{6}$, we roll the value i again, and our bank is now i larger than before and we are at the same game as before. Otherwise, with probability $\frac{5}{6}$, we cash out next round, as we don't roll the value i. If we are given that we don't roll i on the second die, then our expected value would be $\frac{21-i}{5}$, as the sum of the other faces is $21 - i$ and each of those 5 would be equally likely if it is not the value i. Therefore, by Law of Total Expectation,

$$\mathbb{E}[T' \mid X_1 = i] = \frac{1}{6}\left(i + \mathbb{E}[T' \mid X_1 = i]\right) + \frac{5}{6} \cdot \frac{21-i}{5}$$

Solving this, we see that i cancels and we obtain that $\mathbb{E}[T' \mid X_1 = i] = \frac{21}{5}$ for each i. Adding back the first roll, $\mathbb{E}[T \mid X_1 = i] = i + \frac{21}{5}$.

Therefore, as $\mathbb{E}[T]$ is just the average of $\mathbb{E}[T \mid X_1 = i]$ for each i, so $\mathbb{E}[T] = \frac{1}{6}\sum_{i=1}^{6}\left(\frac{21}{5} + i\right) = \frac{21}{5} + \frac{7}{2} = \frac{77}{10}$.

Answer:

7.7

Reference:

https://www.quantguide.io/questions/basic-dice-game-iv

Visible Number Of Heads

Category: Probability
Difficulty: Easy
Companies: Drw, HRT

Question:

Suppose 20 people whose heights follow some unknown continuous distribution are arranged in a single-file line. We then stand at the front of the line and observe that we can see someone's head if they are taller than everyone that comes before. Let X be the number of visible heads. Compute $\mathbb{E}[X]$? Round to the nearest tenths.

Hint:

Use an indicator random variable for the event that the $i-$th person is the tallest among the first i people.

Explanation:

We decompose $X = X_1 + X_2 + ... + X_{20}$ where we define:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ is taller than everyone before him,} \\ 0 & \text{otherwise.} \end{cases}$$

We then claim that $E[X_i] = \frac{1}{i}$, since among the first i people, they are all equally likely to be the tallest, so the probability that the $i-$th person is tallest is $\frac{1}{i}$. Then, it follows from Linearity of Expectation that

$$\mathbb{E}[X] = \sum_{i=1}^{20} \frac{1}{i} \approx 3.6$$

For large n (number of people), a good approximation is $\log(n)$.

Answer:

3.6

Reference:

https://www.quantguide.io/questions/visible-number-of-heads

Squared Matrix

Category: Pure Math
Difficulty: Easy
Companies: Drw

Question:

Suppose that $A = \begin{bmatrix} 1 & x & -1 \\ 2 & -2 & y \\ 0 & 3 & 0 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 9 & -7 & 11 \\ -2 & 21 & -8 \\ 6 & -6 & 9 \end{bmatrix}$. Find xy.

Hint:

Multiply certain rows and columns together to get nice and simple equations for x and y. Utilize the 0s.

Explanation:

By multiplying the first row of A by the first column of A, we get that $1 + 2x = 9$ by equating entries, meaning $x = 4$. Afterwards, by multiplying the second row of A by the by the second column of A, we get that $2x + 4 + 3y = 21$. Since $x = 4$ from before, we have that $3y = 9$ after rearrangement, yielding $y = 3$. Therefore, $xy = 12$.

Answer:

12

Reference:

https://www.quantguide.io/questions/squared-matrix

Swapping X And Y

Category: Statistics
Difficulty: Easy
Companies: Drw

Question:

Consider a simple univariate linear regression. We have the following X and Y values.

$$(5, 5), (4, 0), (6, 10)$$

We regress Y onto X and obtain some β. Now, we regress X onto Y. Without doing any calculation, how does β change? Enter 1 for increase, 0 for stay the same, and -1 for decrease.

Hint:

Consider the formula to calculate β in a simple linear regression, regressing Y onto X

Explanation:

Consider the formula to calculate β in a simple linear regression, regressing Y onto X: $\beta = \frac{\text{Cov}(X,Y)}{Var(X)}$. The denominator corresponds to the variance of the independent variable. If we swap Y and X, the denominator becomes $\text{Var}(Y)$. From looking at the data, we can see that the variance of Y is larger than the variance of X. So, the denominator will be bigger in the case of the second regression, decreasing the value of β.

Answer:

1

Reference:

https://www.quantguide.io/questions/swapping-x-and-y

2030 Split II

Category: Probability
Difficulty: Medium
Companies: Drw, Jane Street

Question:

Alice and Bob have fair 30−sided and 20−sided dice, respectively. The goal is to obtain the largest possible value. Alice rolls her die one time. However, Bob can roll his die twice and keep the maximum of the two values. In the event of a tie, Bob is the winner. Find the probability Alice is the winner.

Hint:

Let W be the event Alice wins. Then we can use Law of Total Probability to condition on whether or not Alice's roll is larger than 20. Let A be the value Alice rolls. Furthermore, let B_1 and B_2 be the values that Bob rolls on the first and second roll. We have that

$$\mathbb{P}[W] = \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20] + \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20]$$

Explanation:

Let W be the event Alice wins. Then we can use Law of Total Probability to condition on whether or not Alice's roll is larger than 20. Let A be the value Alice rolls. Furthermore, let B_1 and B_2 be the values that Bob rolls on the first and second roll. We have that

$$\mathbb{P}[W] = \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20] + \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20]$$

It is clear to see that $\mathbb{P}[A > 20] = \frac{1}{3}$, as this accounts for 10 of 30 values. Furthermore, $\mathbb{P}[W \mid A > 20] = 1$, as this is strictly larger than any value Bob can roll.

For the other case, we compute

$$\mathbb{P}[W \mid A \leq 20] = \sum_{a=1}^{20} \mathbb{P}[\max\{B_1, B_2\} < a]\mathbb{P}[A = a \mid A \leq 20] = \sum_{a=1}^{20} \mathbb{P}[B_1 < a]^2 \mathbb{P}[A = a \mid A \leq 20]$$

This is the event of interest as we want the Alice to have a strictly higher value than Bob. We simplify the maximum statement by noting that B_1 and B_2 are IID and that $\max\{B_1, B_2\} < a$ is equivalent to saying that $B_1 < a, B_2 < a$.

We know $\mathbb{P}[A = a \mid A \leq 20] = \frac{1}{20}$, as we know $A \leq 20$, so there are 20 equally-likely values. Furthermore, we have that $\mathbb{P}[B_1 < a] = \frac{a-1}{20}$, as we want B_1 to take some value between 1 and $a - 1$, inclusive of both. Therefore,

$$\mathbb{P}[W \mid A \leq 20] = \frac{1}{20} \sum_{a=1}^{20} \left(\frac{a-1}{20}\right)^2 = \frac{247}{800}$$

Combining these, we have that

$$\mathbb{P}[W] = \frac{1}{3} + \frac{2}{3} \cdot \frac{247}{800} = \frac{647}{1200}$$

Answer:

6471200

Reference:

https://www.quantguide.io/questions/2030-split-ii

Marble Runs

Category: Probability
Difficulty: Hard
Companies: Drw, HRT

Question:

You repeatedly draw marbles from a bag containing 50 red and 50 blue marbles until there are no more marbles left, recording the order of red and blue marbles drawn. You then count the number of "runs," where a run is defined as any number of consecutive marbles of the same color. For example, $RBBRRRBRR$ contains 5 runs. What is the expected number of runs that you observe?

Hint:

The number of runs is equal to the number of starts of runs. How can you use indicator variables and the linearity of expectation to break down the problem into the expected number of starts of runs per marble?

Explanation:

Let us define random variables $X_1, \ldots, X_{100}$ such that

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th draw is the beginning of a new run} \\ 0 & \text{otherwise} \end{cases}$$

In the case of $RBBRRRBRR$, for example, we would have $X_1 = 1, X_2 = 1, X_4 = 1, X_7 = 1, X_8 = 1$. Note that, for a sequence of 100 red and blue marbles drawn in any order, it is always the case that $X_1 = 1$. Let Z represent the total number of runs.

$$Z = \sum_{i=1}^{100} X_i$$

By the linearity of expectation,

$$\mathbb{E}\left[Z\right] = \mathbb{E}\left[\sum_{i=1}^{100} X_i\right]$$

$$= \sum_{i=1}^{100} \mathbb{E}\left[X_i\right]$$

$$= 1 + \sum_{i=2}^{100} \mathbb{E}\left[X_i\right]$$

Now let's consider $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1)$ for $i \geq 2$.

$$\mathbb{P}(X_i = 1) = \mathbb{P}(((i-1)\text{-th draw } R, i\text{-th draw } B) \cup ((i-1)\text{-th draw } B, i\text{-th draw } R))$$

$$= 2 \cdot \mathbb{P}((i-1)\text{-th draw } R, i\text{-th draw } B)$$

$$= 2 \cdot \mathbb{P}(i\text{-th draw } B \mid (i-1)\text{-th draw } R) \cdot \mathbb{P}((i-1)\text{-th draw } R)$$

The probability that the $(i-1)$-th draw is red is simply $\frac{50}{100}$. The probability that the i-th draw is blue given that the previous draw is red is simply $\frac{50}{99}$. Plugging this in, we find

$$\mathbb{P}(X_i = 1) = 2 \cdot \frac{50}{100} \cdot \frac{50}{99}$$

$$= \frac{50}{99}$$

$$= \mathbb{E}[X_i]$$

Therefore,

$$\mathbb{E}\left[Z\right] = 1 + \sum_{i=2}^{100} \frac{50}{99}$$

$$= 51$$

Answer:

51

Reference:

https://www.quantguide.io/questions/marble-runs

30 Side Difference

Category: Probability
Difficulty: Easy
Companies: Drw, Jane Street

Question:

Let X and Y be the upfaces of two independent rolls of fair $30-$sided dice with values $1-30$ on the sides. Find $\mathbb{E}[|X-Y|]$.

Hint:

Let $Z = |X-Y|$. Compute the PMF of Z directly. If the absolute difference between the rolls is exactly k, then what form are in the outcomes in?

Explanation:

Let $Z = |X-Y|$. We are going to compute the PMF of Z directly. We know that there are a total of $30^2 = 900$ possible outcomes for the rolls. Now, for a fixed k, we want to find the number of outcomes satisfying $\{Z = k\}$.

If the absolute difference between the rolls is exactly k, then the outcomes are in the form $(i, i+k)$ for some appropriate i. What is the range of i we can have? Namely, we know that $i \geq 1$, as that is the minimal possible value of the die. The upper bound is $i = 30-k$, as that would correspond to the pair $(30-k, 30)$. We can permute the values to either of the 2 dice, so there are $2(30-k)$ outcomes corresponding to $\{Z = k\}$ Therefore,

$$\mathbb{E}[Z] = \sum_{k=0}^{29} k \cdot \frac{2(30-k)}{900} = \frac{1}{450}\sum_{k=1}^{29} 30k - k^2 = \frac{1}{450}\left(\frac{30(29)(30)}{2} - \frac{29(30)(59)}{6}\right) = \frac{899}{90}$$

Answer:

89990

Reference:

https://www.quantguide.io/questions/30-side-difference

Soccer Jerseys

Category: Probability
Difficulty: Easy
Companies: Drw

Question:

Liverpool FC sells player jerseys at random prices. The price of a Mohamed Salah jersey is normally distributed with mean \$80 and standard deviation \$15. The price of a Sadio Mane jersey is normally distributed with mean \$70 and standard deviation \$20, independent of Salah jerseys. Gabe wants to be 97.5% sure that he can purchase both jerseys. How much money (in dollars) should Gabe bring with him? For the purposes of this question, take $\Phi^{-1}(0.95) = 1.65$ and $\Phi^{-1}(0.975) = 1.96$

Hint:

We want to find a value c so that $\mathbb{P}[S + M \leq c] = 0.975$. $M \sim N(70, 20^2)$ and $S \sim N(80, 15^2)$ represent the prices of Mane and Salah jerseys, respectively. What is the distribution of the sum of independent normal random variables?

Explanation:

We want to find a value c so that $\mathbb{P}[S + M \leq c] = 0.975$. $M \sim N(70, 20^2)$ and $S \sim N(80, 15^2)$ represent the prices of Mane and Salah jerseys, respectively. Since these are independent, $M + S \sim N(150, 25^2)$ by adding the parameters. Then, standardizing,

$$\mathbb{P}[S + M \leq c] = \mathbb{P}\left[\frac{S + M - 150}{25} \leq \frac{c - 150}{25}\right] = 0.975$$

The LHS is now standard normal, so the LHS is $\Phi\left(\frac{c-150}{25}\right)$. Therefore, we need to find c such that $\Phi\left(\frac{c-150}{25}\right) = 0.975$. Using the inverse CDF, we get that $\frac{c-150}{25} \leq \Phi^{-1}(0.975) = 1.96$. Rearranging, $c = 150 + 25 \cdot 1.96 = 199$.

Answer:

199

Reference:

https://www.quantguide.io/questions/soccer-jerseys

Exponent Reverse

Category: Brainteasers
Difficulty: Medium
Companies: Drw

Question:

Is π^e or e^π larger? Answer 1 and 2 for π^e and e^π, respectively.

Hint:

Comparing e^π and π^e is equivalent to comparing $\pi \ln(e) = \pi$ and $e \ln(\pi)$.

Explanation:

Comparing e^π and π^e is equivalent to comparing $\pi \ln(e) = \pi$ and $e \ln(\pi)$. This is since $\ln(x)$ is a strictly increasing function. Consider **?** as an unknown relationship between the two quantities. Therefore,

$$\pi \ln(e) \mathbf{?}\, e \ln(\pi) \iff \frac{\ln(e)}{e} \mathbf{?} \frac{\ln(\pi)}{\pi}$$

Consider the function $f(x) = \dfrac{\ln(x)}{x}$. By the quotient rule, $f'(x) = \dfrac{1 - \ln(x)}{x^2}$. We have that $f'(x) > 0$ for all $0 < x < e$ and $f'(x) < 0$ for $x > e$. Therefore, $x = e$ is the maximum of this function. Therefore, we can confirm that $\dfrac{\ln(e)}{e} > \dfrac{\ln(\pi)}{\pi}$, so $e^\pi > \pi^e$. Our answer is 2.

Answer:

2

Reference:

https://www.quantguide.io/questions/exponent-reverse

Rabbit Hop IV

Category: Probability
Difficulty: Medium
Companies: Drw, Five Rings

Question:

A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of steps strictly larger than 1 at each movement. In particular, this means that the rabbit can't go up a one stair staircase. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Hint:

Let h_n be the number of distinct paths to the top of a n stair staircase. Condition on the size of the first jump.

Explanation:

Let h_n be the number of distinct paths to the top of a n stair staircase. We can condition on the size of the first jump. The size of the first jump can be anywhere from 2 to n, inclusive. If the rabbit hops k stairs on the first step, it must make a unique path through the other $n-k$ stairs. Hence, you could view the problem now as stair k being the floor and the top stair being $n-k$. This means that

$$h_n = h_{n-2} + h_{n-3} + h_{n-4} + \cdots + h_1 + h_0$$

However, let's look at the tail term $h_{n-3} + h_{n-4} + \cdots + h_1 + h_0$. This is equivalent to the problem where we start on stair 1 (instead of the ground) and need to jump up strictly more than one stair at each jump. This is because stair 3 would now be the first available stair to jump on, and the rabbit can jump to any other stair at least 3. Thus, the tail term there is just h_{n-1}, as starting on stair 1, the rabbit needs a path through the other $n-1$ stairs to the top. Our recurrence relation is now

$$h_n = h_{n-1} + h_{n-2}$$

Our initial conditions are that $h_1 = 0$ and $h_2 = 1$, which can just be counted directly. We note now that this is just the Fibonacci sequence shifted by 1 index up, as $F_0 = 0$ and $F_1 = 1$, so we can conclude that $h_n = F_{n-1}$. In particular, this means that $h_{10} = F_9 = 34$.

Answer:

34

Reference:

https://www.quantguide.io/questions/rabbit-hop-iv

2030 Die Split I

Category: Probability
Difficulty: Hard
Companies: Drw, Jane Street

Question:

Alice and Bob have fair 30−sided and 20−sided dice, respectively. The goal for each player is to have the largest value on their die. Alice and Bob both roll their dice. However, Bob has the option to re-roll his die in the event that he is unhappy with the outcome. He can't see Alice's die beforehand. Bob then keeps the value of the new die roll. In the event of a tie, Bob is the winner. Assuming optimal play by Bob, find the probability Alice is the winner.

Hint:

Let W be the event Alice wins. First, we want to find $\mathbb{P}[W]$. Let A and B be value of Alice's roll and Bob's first roll, respectively. The key here is to condition on whether or not $A \leq 20$. Bob is going to roll again whenever $\mathbb{P}[W \mid B = b] > \mathbb{P}[W]$.

Explanation:

Let W be the event Alice wins. First, we want to find $\mathbb{P}[W]$. Let A and B be value of Alice's roll and Bob's first roll, respectively. The key here is to condition on whether or not $A \leq 20$. Namely, this means

$$\mathbb{P}[W] = \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20] + \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20]$$

We can quickly see that $\mathbb{P}[A \leq 20] = \dfrac{2}{3}$, as this accounts for 20 of 30 equally-likely values. If $A > 20$, then clearly Alice wins regardless of what Bob obtains, so $\mathbb{P}[W \mid A > 20] = 1$.

If $A \leq 20$, then we are really rolling 2 independent 20−sided fair dice. Namely, to compute $\mathbb{P}[W \mid A \leq 20]$, we see that the probability of a tie is $\dfrac{1}{20}$, as there are 20 values they can agree upon out of $20^2 = 400$ outcomes. Therefore, the probability the two rolls are different is $\dfrac{19}{20}$. Because of the fact that the two dice are symmetric, $\mathbb{P}[A > B] = \dfrac{1}{2} \cdot \dfrac{19}{20} = \dfrac{19}{40}$, as when the two rolls are not the same, it is equally-likely whether $A > B$ or $A < B$.

Combining these all together, we see that $\mathbb{P}[W] = \dfrac{1}{3} \cdot 1 + \dfrac{2}{3} \cdot \dfrac{19}{40} = \dfrac{39}{60}$. Now, Bob is going to roll again whenever $\mathbb{P}[W \mid B = b] > \mathbb{P}[W] = \dfrac{39}{60}$. This is because he wants to minimize Alice's chance of winning. It is easy to see that $\mathbb{P}[W \mid B = b] = 1 - \dfrac{b}{30}$, as Alice just needs to roll a value of at least $b + 1$.

We just need to find the maximum value of b, say b^*, such that $\mathbb{P}[W \mid B = b] = 1 - \dfrac{b}{30} > \dfrac{39}{60}$.

Solving this inequality, we get that $b \leq 10.5$, so $b^* = 10$ is the maximum value that Bob rolls again. Let W_1 be the event that Alice wins in this new game where Bob can roll again. We have that

$$\mathbb{P}[W_1] = \mathbb{P}[W] + \frac{10}{20}\left(\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]\right)$$

We get this because of the fact that Bob rolls again with probability $\frac{1}{2}$, which is when he rolls at most 10. Then, the probability Alice wins is going to change by $\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]$.

To calculate $\mathbb{P}[W \mid B \leq 10]$, this is simply just average all the cases where $B = b$ for $1 \leq b \leq 10$, as when $B \leq 10$, the value is uniformly distributed on the first 10 positive integers. Therefore,

$$\mathbb{P}[W \mid B \leq 10] = \frac{1}{10}\sum_{b=1}^{10} 1 - \frac{b}{30} = \frac{49}{60}$$

Therefore, we get that

$$\mathbb{P}[W] = \frac{39}{60} + \frac{1}{2}\left(\frac{39}{60} - \frac{49}{60}\right) = \frac{17}{30}$$

Answer:

1730

Reference:

https://www.quantguide.io/questions/2030-die-split-i

Chess Tournament I

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

A chess tournament has 128 players, each with a distinct rating. Assume that the player with the higher rating always wins against a lower rated opponent and that the winner proceeds to the subsequent round. What is the probability that the highest rated and second-highest rated players will meet in the final?

Hint:

Consider visualizing this problem as two sets of 64 players that halves at each round until there is a finalist.

Explanation:

The highest rated player (P_1) will always make it to the final since no other player can knock him or her out. The second-highest rated player (P_2) makes it to the final if and only if he or she does not play P_1 beforehand, in which the P_2 will be knocked out before the final. In the first round, the 128 players are divided into two subgroups of 64. P_2 will make it to the final if he or she is in a different subgroup as P_1. More concretely, P_2 has 127 possible match-ups, 64 of which are not in the same subgroup as P_1. Thus, the probability that P_1 and P_2 meet in the final is

$$\frac{64}{127}$$

Answer:

64127

Reference:

https://www.quantguide.io/questions/chess-tournament-i

Coefficient Swap

Category: Statistics
Difficulty: Medium
Companies: Five Rings

Question:

Suppose that we have two datasets X and Y with $\text{Var}(X) = 10$ and $\text{Var}(Y) = 20$. We perform the linear regression $y \sim \alpha_x + \beta_x x$ and obtain $\beta_x = 1$. Suppose now that we perform the regression $x \sim \alpha_y + \beta_y y$. Find β_y. If the value can't be determined, enter -100.

Hint:

Let ρ be the Pearson Correlation Coefficient of X and Y. Then $\beta_x = r\dfrac{\sigma_y}{\sigma_x}$ and $\beta_y = r\dfrac{\sigma_x}{\sigma_y}$.

Explanation:

Let r be the Pearson Correlation Coefficient of X and Y. Then $\beta_x = r\dfrac{\sigma_y}{\sigma_x}$ and $\beta_y = r\dfrac{\sigma_x}{\sigma_y}$. Therefore,

$$\frac{\beta_y}{\beta_x} = \frac{\sigma_x^2}{\sigma_y^2} \iff \beta_y = \beta_x\frac{\sigma_x^2}{\sigma_y^2} = 1 \cdot \frac{10}{20} = \frac{1}{2}$$

Answer:

12

Reference:

https://www.quantguide.io/questions/coefficient-swap

Conditional Head Starter

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

A fair coin is flipped n times. It turns out that no heads appear consecutively in the sequence. Given this information, in terms of n, find the probability that the first flip was a heads. You should get a function $p(n)$. Evaluate $p(10)$.

Hint:

Condition on the sequence starting with T or HT and derive a recurrence relation.

Explanation:

Let $A(n)$ be the number of sequences of length n with no consecutive heads. Then $A(1) = 2$ and $A(2) = 3$ (only HH doesn't satisfy this). For a sequence of length n to satisfy our constraint, it either starts with T or HT. It can't start with HH, as that would violate the condition.

There are $A(n-1)$ sequences starting with T that satisfy this condition, as the first flip is T and the other $n-1$ spots must not contain any consecutive heads. There are $A(n-2)$ sequences starting with HT by the same logic with with $n-2$ remaining spots. Thus, $A(n) = A(n-1) + A(n-2)$ with $A(1) = 2$ and $A(2) = 3$. This is just the Fibonacci sequence shifted. In particularly, $A(n) = F_{n+2}$, where F_k is the kth Fibonacci number.

Given this information, all sequences that satisfy the condition are equally likely. We know that there are $A(n-2)$ sequence starting with HT (and hence H) out of the $A(n)$ total. Therefore, we get that

$$p(n) = \frac{A(n-2)}{A(n)} = \frac{F_n}{F_{n+2}}$$

In particular, $p(10) = \frac{F_{10}}{F_{12}} = \frac{55}{144}$

Answer:

55144

Reference:

https://www.quantguide.io/questions/conditional-head-starter

Compound Interest I

Category: Brainteasers
Difficulty: Easy
Companies: Five Rings

Question:

You start with \$100 in your bank account today. You invest in a stock that yields 1% interest that is compounded daily. To the nearest dollar, how much will you have in your bank account after 100 days?

Hint:

The answer will be $100(1 + 0.01)^{100}$ by simple interest formula. How do you approximate it?

Explanation:

The answer will be $100(1+0.01)^{100}$ by simple interest formula. Estimating that requires some skill. Noting the fact that $(1 + 1/n)^n \approx e$ for large n, we know it is about $100e \approx 271$. However, we must adjust down a little bit, so one may guess it is \$270. 270 is indeed correct, though it is on the boundary between 270 and 271.

Answer:

270

Reference:

https://www.quantguide.io/questions/compound-interest-i

Longest Rope II

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose you have a rope that is 1 meter in length. You cut the rope uniformly at random along the length. Find the variance of the length of the longer piece (in meters).

Hint:

Let X be the distance from the LHS that the rope is cut. Then we know that $X \sim \text{Unif}(0, 1)$ and $L = \max\{X, 1 - X\}$ is the length of the longer piece, as the division at point X yields two pieces of length X and $1 - X$. Try to find the distribution of L.

Explanation:

Let X be the distance from the LHS that the rope is cut. Then we know that $X \sim \text{Unif}(0, 1)$ and $L = \max\{X, 1 - X\}$ is the length of the longer piece, as the division at point X yields two pieces of length X and $1 - X$. We know that $L \geq 0.5$, as at least one of the two pieces must be longer than $1/2$ in length. However, we see that the event $\{L = c\}$ corresponds to $X = c$ or $X = 1 - c$, so we get that $L \sim \text{Unif}(0.5, 1)$, meaning that $\text{Var}(L) = \frac{(0.5)^2}{12} = \frac{1}{48}$.

Answer:

148

Reference:

https://www.quantguide.io/questions/longest-rope-ii

Rabbit Hop II

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of stairs at each move, but it must make an even amount of hops to get to the top. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Hint:

Can you match up paths to some subsets of $\{1, 2, \ldots, 9\}$? What is the restriction?

Explanation:

Using the same idea as Rabbit Hop I, we are looking for subsets of $\{1, 2, \ldots, 9\}$ that have odd size. This is because we have to select an odd amount of integers $1 - 9$ such that when we move to step 10, the total number of moves is even. In other words, we need to find the number of odd-sized subsets of a set of size 9.

We can match up odd and even subsets by matching a subset $A \subseteq \{1, 2, \ldots, 9\}$ with odd cardinality to A^c, which would have even cardinality. This is since if $|A| = k$, where k is odd, $9 - k$ must be even, so we have a one-to-one matching of odd and even subsets, so there must be equal amounts of odd and even subsets. Thus, there are $\frac{1}{2} \cdot 2^9 = 256$ such paths.

Answer:

256

Reference:

https://www.quantguide.io/questions/rabbit-hop-ii

Increasing Uniform Chain

Category: Probability
Difficulty: Hard
Companies: Five Rings

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Let N be the first index n where $X_n \neq \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[X_{N-1}]$ i.e. the largest value among the first N values selected. The answer will be in the form $a + be$ for integers a and b. Note here that e is Euler's constant. Find $a + b$.

Hint:

Use Law of Total Probability to condition on the next value.

Explanation:

Let $f(x)$ be the expected largest number given that the current largest is x. We want $f(0)$, as we are selecting uniform values on $(0,1)$, so our initial largest value is 0. We don't choose a starting value lower than 0 since we can't generate values lower than 0.

With probability x, the next value is smaller than x, and our largest value will be x. Otherwise, if our next value is $y > x$, y is uniformly distributed on $(x, 1)$, so the expected maximum in that case is $f(y)$. We integrate over all $x < y < 1$ since we are applying Law of Total Probability. Written as an equation, this is

$$f(x) = x^2 + \int_x^1 f(y)dy$$

Taking the derivative to convert into a differential equation, we get that $f'(x) = 2x - f(x)$. Equivalently, $f'(x) + f(x) = 2x$. An initial condition for this differential equation is $f(1) = 1$, as we can't go any higher than 1.

This is a linear first order differential equation, so we can use the method of integrating factors here. In particular, the integrating factor is $\mu(x) = e^{\int 1dx} = e^x$. Multiplying this on both sides, $(e^x f(x))' = 2xe^x$, meaning $e^x f(x) = \displaystyle\int 2xe^x dx = 2e^x(x-1) + C$. This means that $f(x) = 2x - 2 + Ce^{-x}$. Using our initial condition $f(1) = 1$, this yields $C = e$, so $f(x) = 2(x-1) + e^{1-x}$. In particular, $f(0) = e - 2$, so our answer is $1 - 2 = -1$.

For a sanity check, we can note that the answer to this question and the answer to Decreasing Uniform Chain sum to 1. This makes perfect sense. If $X_i \sim \text{Unif}(0,1)$, then $1 - X_i \sim \text{Unif}(0,1)$, so there is a symmetry between the two problems.

Answer:

1

Reference:

https://www.quantguide.io/questions/increasing-uniform-chain

Conditional Uniform

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Suppose $X, Y \sim \text{Unif}(0, 1)$ IID. Compute $\mathbb{P}[X - Y > 1/2 \mid X + Y < 1]$.

Hint:

Draw the region out in the plane and take ratios of areas.

Explanation:

Drawing this out in the plane, the region that is bound by the axes and $x + y = 1$ is a triangle with side lengths 1. The region of interest for our event is the region that is below $y = x - 1/2$. Shading in this region, we see that it is also a triangle. The intersection point is $(3/4, 1/4)$, so the length of each side of the triangle of our region of interest is $\frac{\sqrt{2}}{4}$. Since X and Y are IID $\text{Unif}(0, 1)$, their conditional joint distribution is uniform over the bigger triangle bound by $x + y = 1$ and the axes. Thus, we can take ratios of areas, and determine that the area of the big triangle is $1/2$, while the area of our smaller region of interest is $\frac{1}{16}$, meaning our probability is $\frac{1}{8}$.

Answer:

18

Reference:

https://www.quantguide.io/questions/conditional-uniform

Crossing Pairs

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose 6 points are chosen uniformly at random along the perimeter of the circle. The 6 points are then divided into three pairs, and a chord is drawn between every pair of points. What is the probability none of the chords intersect?

Hint:

Imagine we pick the pairs sequentially, and say we pick a random point of the six. Which of the remaining 5 points can we pair with the chosen one? Condition on the selected pair.

Explanation:

Suppose the pairs of points are picked sequentially, then for our first point pick any point along the circle. As for the possible partner of the first point, we can either choose to pick its left or right neighbor, or the point across from it. Any other choice will result in one point trapped inside the formed chord forcing an intersection.
First, the probability that the first point is matched with a left/right neighbor is $\frac{2}{5}$. Applying the same logic now to the remaining four points, we get that the probability these pairs do not intersect is $\frac{2}{3}$.
In the other $\frac{1}{5}$ chance that the first point is paired with the point directly across, we need the remaining four points not to intersect the chord. The only way for this to happen is if the pairs on each side of the chord are paired with each other. This has probability of $\frac{1}{3}$.
Putting these together, we find that the probability of no intersection is $\frac{2}{5}(\frac{2}{3}) + \frac{1}{5}(\frac{1}{3}) = \frac{1}{3}$.

Answer:

13

Reference:

https://www.quantguide.io/questions/crossing-pairs

Second Moment

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Given that $\text{Var}(X) = 25$ and $\mathbb{E}[X] = 3$, find $\mathbb{E}[X^2]$.

Hint:

Use the relation $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

Explanation:

Using the relation $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, we have $25 = \mathbb{E}[X^2] - 3^2$, so $\mathbb{E}[X^2] = 34$.

Answer:

34

Reference:

https://www.quantguide.io/questions/second-moment

Deja Vu

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Find the expected number of distinct faces observed when rolling a fair $6-$sided die before rolling a face previously observed.

Hint:

$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$. What does the event $\{X \geq k\}$ mean here for an appropriately selected random variable X?

Explanation:

We solve this using the classic trick $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$ for non-negative integer-valued random variables X. Let X be the number of distinct faces showing up. We now need to find $\mathbb{P}[X \geq k]$ for $k = 1, 2, \ldots, 6$. We know for $k \geq 7$ this is 0 since there are only 6 sides on the die.

$\mathbb{P}[X \geq 1] = 1$, as you have to roll at least one distinct side to be able to repeat it. Now, for $\mathbb{P}[X \geq 2]$, we need to roll any value besides our first face before the first face again. This is determined within one roll, and the probability of this is $\frac{5}{6}$, as there are 5 values that haven't been selected yet. For $\mathbb{P}[X \geq 3]$, we need to obtain 2 distinct faces, which occurs with probability $\frac{5}{6}$, and then we need to roll another distinct face on the next turn, which occurs with probability $\frac{2}{3}$. Therefore, $\mathbb{P}[X \geq 3] = \frac{5}{6} \cdot \frac{2}{3} = \frac{5}{9}$.

Continuing this pattern, $\mathbb{P}[X \geq 4] = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$, $\mathbb{P}[X \geq 5] = \frac{5}{18} \cdot \frac{1}{3} = \frac{5}{54}$, and $\mathbb{P}[X \geq 6] = \frac{5}{54} \cdot \frac{1}{6} = \frac{5}{324}$. Therefore, $\mathbb{E}[X] = \sum_{k=1}^{6} \mathbb{P}[X \geq k] = 1 + \frac{5}{6} + \cdots + \frac{5}{324} = \frac{899}{324}$.

Answer:

899324

Reference:

https://www.quantguide.io/questions/deja-vu

Plane Partition

Category: Probability
Difficulty: Medium
Companies: Five Rings, Jane Street

Question:

Find the maximum number of disjoint regions the plane can be divided into by 10 non-parallel lines. For example, one line splits the plane into 2 regions.

Hint:

Let L_n be the maximum number of regions with n lines. Note that $L_1 = 2$ by the above and $L_2 = 4$, as we want to draw two non-parallel lines. The key observation is here that the $(n+1)$st line that is added should intersect each of the first n lines, and this yields $n+1$ distinct regions.

Explanation:

Let L_n be the maximum number of regions with n lines. Note that $L_1 = 2$ by the above and $L_2 = 4$, as we want to draw two non-parallel lines. The key observation is here that the $(n+1)$st line that is added should intersect each of the first n lines, and this yields $n+1$ distinct regions. We can see this in the small cases of $n = 1, 2$, and 3. More rigorously, the $(n+1)$st line splits k of the old regions exactly when it intersects the existing lines in $k-1$ places. As two lines can only intersect in at most one point, then this implies $L_{n+1} \leq L_n + (n+1)$, as we get $k \leq n+1$ given that the new line can intersect the old lines in at most n places.

However, it is also possible to place the line so that it is non-parallel to every other existing line and it doesn't intersect at any existing intersection points. Therefore, $L_{n+1} \geq L_n + (n+1)$ by the same logic. This means we derived the recurrence $L_{n+1} = L_n + (n+1)$, which simply has the solution

$$L_n = \frac{n(n+1)}{2} + 1$$

Plugging in $n = 10$, we see that our answer is 56.

Answer:

56

Reference:

https://www.quantguide.io/questions/plane-partition

Perfect Correlation II

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose that X and Y are perfectly correlated i.e. $\rho(X, Y) = 1$ with $\sigma_Y < \sigma_X$. Find the correlation of $X + Y$ and $X - Y$.

Hint:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Compute $\text{Cov}(X + Y, X - Y)$ and their respective variances using the common formulae.

Explanation:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\text{Cov}(X + Y, X - Y) = \text{Cov}(X, X) - \text{Cov}(Y, Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\text{Var}(X + Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\sigma_{X+Y} = \sigma_X + \sigma_Y$. By a similar argument, we get $\text{Var}(X - Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\text{Var}(X - Y) = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance.

Therefore, we have that $\rho(X + Y, X - Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = 1$. We get the 1 from the condition that $\sigma_X > \sigma_Y$, so the numerator is positive.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_X > \sigma_Y$, it must be the case that $|a| < 1$. Therefore, $X + Y = X(1 + a) + b$ and $X - Y = X(1 - a) + b$. As $|a| < 1$, both of the constants $1 + a$ and $1 - a$ are positive. Therefore, as these are both linear transformation of X with the same signs, they must have correlation 1.

Answer:

Reference:

https://www.quantguide.io/questions/perfect-correlation-ii

Rebuy Gambler

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Alice and Bob start with \$15 and \$10, respectively. They bet \$1 at a time on opposite outcomes of a fair coin. If Alice bankrupts, she has the ability to buy in again with \$15, while Bob has the stack he won. If she bankrupts again, she loses the game. Find the probability Bob loses the game.

Hint:

The easier quantity to compute is the probability that Alice loses the game, which is the complement here. For Alice to lose the game, she first needs to go bankrupt in the initially, and then bankrupt a second time.

Explanation:

The easier quantity to compute is the probability that Alice loses the game, which is the complement here. For Alice to lose the game, she first needs to go bankrupt in the initially, and then bankrupt a second time. Using the Gambler's Ruin paradigm for equal-odds rounds, the probability she initially goes bankrupt is $\frac{10}{25} = \frac{2}{5}$. Afterwards, Bob would have \$25 and Alice would have \$15 upon rebuying. The probability Alice goes bankrupt here is $\frac{25}{40} = \frac{5}{8}$. Combining these, the probability Alice loses is

$$\frac{2}{5} \cdot \frac{5}{8} = \frac{1}{4}$$

Therefore, Bob loses with probability $\frac{3}{4}$. This makes sense, as this is the same result as if Alice had \$30 bankroll initially and Bob had \$10 initially. Alice essentially does have \$30, as she can rebuy for \$15 midway.

Answer:

34

Reference:

https://www.quantguide.io/questions/rebuy-gambler

Ranged Stars And Bars

Category: Probability
Difficulty: Hard
Companies: Five Rings

Question:

Find the number of non-negative integer solutions to $6 \le x_1 + \cdots + x_5 \le 10$.

Hint:

We know by stars and bars that for a fixed k, the number of non-negative integer solutions to $x_1 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1}$. Use the hockey stick identity $\displaystyle\sum_{i=k-1}^{r-1} \binom{i}{k-1} = \binom{r}{k}$, $1 \le k < r$.

Explanation:

We are going to solve this problem more generally. Namely, we want to find the number of non-negative integer solutions to

$$n + 1 \le x_1 + \cdots + x_n \le 2n$$

We know by stars and bars that for a fixed k, the number of non-negative integer solutions to $x_1 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1}$. Therefore, the number of non-negative integer solutions to $x_1 + \cdots + x_n \le 2n$ is just the sum of the number of non-negative integer solutions to $x_1 + \cdots + x_n = k$ for $0 \le k \le 2n$, so this yields

$$\binom{n-1}{n-1} + \cdots + \binom{2n-1}{n-1} + \binom{2n}{n-1} + \cdots + \binom{3n-1}{n-1}$$

Using the hockey stick identity $\displaystyle\sum_{i=k-1}^{r-1} \binom{i}{k-1} = \binom{r}{k}$, $1 \le k < r$, with $k = n$ and $r = 3n$, we get that the above is just $\binom{3n}{n}$. We do similar to find that the number of non-negative integer solutions to $x_1 + \cdots + x_n \le n$ is

$$\binom{n-1}{n-1} + \cdots + \binom{2n-1}{n-1} = \binom{2n}{n}$$

Therefore, the number of non-negative integer solutions to $n + 1 \le x_1 + \cdots + x_n \le 2n$ is just the difference of the above values, as $x_1 + \cdots + x_n \le 2n$ includes solutions that are also $\le n$. Therefore, the number solutions to this is

$$\binom{2n}{n-1} + \cdots + \binom{3n-1}{n-1} = \binom{3n}{n} - \binom{2n}{n}$$

Plugging in $n = 5$ yields that our answer is $\binom{15}{5} - \binom{10}{5} = 2751$

Answer:

2751

Reference:

https://www.quantguide.io/questions/ranged-stars-and-bars

Product Over Sum

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Roll two fair standard $6-$sided dice. Find the probability that the sum of the upfaces is at least the product of the upfaces.

Hint:

Let X and Y be the outcomes of the two rolls. Then we want to find when $X + Y \geq XY$.

Explanation:

Let X and Y be the outcomes of the two rolls. Then we want to find when $X + Y \geq XY$. Solving for Y, we see that $Y \leq \dfrac{X}{X-1}$. If $X = 1$, then this holds for any Y. If $X = 2$, then this holds for $Y = 1, 2$. However, for $X > 2$, this only holds when $Y = 1$ by plugging into the inequality. Therefore, the condition is just equivalent to when is there at least one 1 rolled or we get the outcome $(2, 2)$. The probability of this is $\dfrac{12}{36} = \dfrac{1}{3}$ by drawing out the table or using Inclusion-Exclusion and adding in the case of $(2, 2)$.

Answer:

13

Reference:

https://www.quantguide.io/questions/product-over-sum

Ping Pong Tournament II

Category: Probability
Difficulty: Medium
Companies: Five Rings, Two Sigma

Question:

50 people are competing in a ping pong tournament where there is only one ping pong table. The competitors are numbered 1 through 50. Suppose that if two competitors meet, the one with the larger number wins. Two competitors are chosen at random, and the loser is removed from the tournament. The winner moves on to the next round, where their opponent is chosen at random. This process is repeated until one person is left (a total 49 rounds will be played). Compute the probability that competitor 49 is still in the tournament after the first 10 rounds have been played.

Hint:

In order for competitor 49 to survive the first 10 rounds, either (1) competitor 49 does not appear until the 11th round, or (2) competitor 49 plays their first round within the first 10 rounds, and competitor 50 does not play within the first 10 rounds.

Explanation:

In order for competitor 49 to survive the first 10 rounds, either (1) competitor 49 does not appear until the 11th round, or (2) competitor 49 plays their first round within the first 10 rounds, and competitor 50 does not play within the first 10 rounds.

Consider case 1. Note that there are a total of 39 rounds between rounds 11 and 49, inclusive. If we want competitor 49 to play their first round after the 10th round, then competitor 49 must be at position 12 or later (39 possible positions) in a random ordering of the 50 competitors. The probability of case 1 occuring is then $\frac{39}{50}$.

Next, consider case 2. In this case, in a random ordering of 50 competitors, competitor 49 must be among the first 11 competitors, and competitor 50 must be among the last 39 competitors. This occurs with probability $\frac{\binom{11}{1}\binom{39}{1}48!}{50!} = \frac{429}{2450}$. By countable additivity, we find the answer to be $\frac{39}{50} + \frac{429}{2450} = \frac{234}{245}$.

Answer:

234245

Reference:

https://www.quantguide.io/questions/ping-pong-tournament-ii

Sum Standard Deviation

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Let X and Y be random variables with standard deviations 1 and 3, respectively. The standard deviation of $X + Y$ is 4. Find $\text{Corr}(X, Y)$.

Hint:

Use the variance of sum formula $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

Explanation:

Using the variance of sum formula, we know $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)+2\text{Cov}(X, Y)$. Plugging in our given values, $4^2 = 1^2 + 3^2 + 2\rho(3)(1)$, so $6\rho = 6$, meaning that $\rho = 1$.

Answer:

1

Reference:

https://www.quantguide.io/questions/sum-standard-deviation

Complex Circle

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose that Z is a uniformly random point selected from the boundary of the unit circle in the complex plane $|z| = 1$. Compute $\mathbb{E}[Z^2]$.

Hint:

Write $Z = X + Yi$, where X and Y are real-valued and satisfy $X^2 + Y^2 = 1$. On the real unit circle, X and Y are uncorrelated and identically distributed.

Explanation:

We can write $Z = X + Yi$, where X and Y are real-valued and satisfy $X^2 + Y^2 = 1$. We know that on the real unit circle, X and Y are uncorrelated and identically distributed. Thus, we have that

$$\mathbb{E}[Z^2] = \mathbb{E}[(X + Yi)^2] = \mathbb{E}[X^2 - Y^2] + 2i\mathbb{E}[XY] = 0$$

The first term vanishes by the identical distribution of X and Y, while the second term vanishes by X and Y being uncorrelated and mean 0.

The easiest solution is to note that the boundary of the unit circle maps to itself under $f(z) = z^2$, as every point on it can be written as $e^{i\theta}$ for some θ, so $Z^2 = e^{2i\theta}$, which has image just the boundary of the unit circle. Thus, $\mathbb{E}[Z^2] = \mathbb{E}[Z] = 0$, as the circle is symmetric in both components.

Answer:

0

Reference:

https://www.quantguide.io/questions/complex-circle

Perfect Correlation III

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose that X and Y are perfectly negatively correlated i.e. $\rho(X, Y) = -1$ with $\sigma_Y > \sigma_X$. Find the correlation of $X + Y$ and $X - Y$.

Hint:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Compute $\text{Cov}(X + Y, X - Y)$ and their respective variances using the common formulae.

Explanation:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\text{Cov}(X + Y, X - Y) = \text{Cov}(X, X) - \text{Cov}(Y, Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\text{Var}(X + Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\sigma_{X+Y} = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance. By a similar argument, we get $\text{Var}(X - Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\text{Var}(X - Y) = \sigma_X + \sigma_Y$.

Therefore, we have that $\rho(X + Y, X - Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = -1$. We get the -1 from the condition that $\sigma_Y > \sigma_X$, so the numerator is negative.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_Y > \sigma_X$, it must be the case that $|a| > 1$. Therefore, $X+Y = X(1+a)+b$ and $X - Y = X(1 - a) + b$. As $|a| > 1$, one of the constants between $1 + a$ and $1 - a$ is positive and one is negative. Therefore, as these are both linear transformation of X with opposing signs, they must have correlation -1. Note that this argument does not depend at all on whether or not the correlation of X and Y is -1 or 1.

Answer:

1

Reference:

https://www.quantguide.io/questions/perfect-correlation-iii

Discrete Walker

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Consider a random walk on a discrete line of 11 points (0 through 10). Supposed you have equal probability of stepping up or down at each movement. If you reach either 10 or 0 you must stop. If you start at point 6, what is the probability that you arrive at 10 before you arrive at 0?

Hint:

This is just a standard symmetric random walk problem.

Explanation:

This is just a standard symmetric random walk problem. The probability that 10 is hit before 0 starting at 6 is just $\frac{6}{10}$ by considering the ratio of the distance from the lower boundary (0) to our starting position (6) to the entire length of our region (10).

Answer:

35

Reference:

https://www.quantguide.io/questions/discrete-walker

Heads And Tails II

Category: Probability
Difficulty: Hard
Companies: Five Rings

Question:

Aaron flips a fair coin until he receives an equal amount of heads and tails. For example, HHTHTT would correspond to 6 flips. What is the reciprocal of the expected number of flips Aaron does?

Hint:

Consider the absolute difference between the number of heads and tails.

Explanation:

Our first flip will either turn up to be H or T. Therefore, the difference (in absolute value) between the number of H and T will be 1 after 1 flip. Let H_n and T_n, respectively, be the number of heads and tails that occur in the first n flips. If N represents our stopping time, then $N = \min\{n > 0 : H_n = T_n\}$.

The key here is that we consider $S_n = |H_n - T_n|$, the absolute difference between the number of heads and tails in the first n flips. We can see that S_n is a symmetric random walk on the non-negative integers with a reflecting boundary at 0. This is because we either move left or right 1 with equal probability at each step when $S_n > 0$. These correspond to either the difference between heads and tails increasing or decreasing by 1 at each step. Furthermore, the fact that this is a reflecting random walk is irrelevant to this question. This is because we want to stop at the first time where we return to $S_n = 0$, when $H_n = T_n$ and don't care what happens after that. Therefore, our question simplifies to finding the expected return time to 0 in a symmetric random walk on the integers starting from 1. Mathematically, we want $\mathbb{E}[N]$, where $N = \min\{n > 0 : S_n = 0\}$ and we start at $S_0 = 0$.

There are some further simplifications that can be made. Since the random walk is shift-invariant, this is the expected time starting from 0 to hit -1 in our random walk. Since our random walk is symmetric, the expected time to hit -1 is the same as the expected time to hit 1. If R_n is a symmetric random walk on the integers (recall that we no longer care that it is on the positive integers), then we want $\mathbb{E}[N_1]$, with $N_1 = \min\{n : R_n = 1\}$.

Suppose that $\mathbb{E}[N_1] < \infty$. As we can represent $R_n = \displaystyle\sum_{i=1}^{n} X_i$, with each X_i being ± 1 with equal probability IID (this is the sum of left and right steps), by Wald's Identity, $\mathbb{E}[R_{N_1}] = \mathbb{E}[N_1]\mathbb{E}[X_1]$. We know that the LHS is 1 with probability 1 by the definition of N_1. By a direct calculation, we know $\mathbb{E}[X_1] = 0$. Therefore, if $\mathbb{E}[N_1]$ were finite, the LHS would be 1 and the RHS 0, leading to a contradiction. Thus, this expectation must be infinite and the answer is 0.

Answer:

0

Reference:

https://www.quantguide.io/questions/heads-and-tails-ii

Binary Zeroes

Category: Brainteasers
Difficulty: Medium
Companies: Five Rings

Question:

In the binary expansion of 142!, how many trailing zeroes are there?

Hint:

The number of trailing zeroes is the highest power of 2 dividing 142!.

Explanation:

In base 10, we know that the number of trailing zeros for n is the highest power of 10 that divides n. Similarly, in binary, we want the highest power of 2 dividing 142!. We get 71 powers of 2 from all the even terms. Then, there are floor $\left(\frac{142}{2^2}\right) = 35$ additional powers of 2 coming from terms divisible by 4. We keep doing this to obtain that there are

$$\sum_{k=1}^{\infty} \text{floor}\left(\frac{n}{2^k}\right)$$

trailing zeroes in binary for $n!$. Note that this sum is finite since eventually $2^k > n$, so the floor will become 0. With $n = 142$, we get

$$\sum_{k=1}^{\infty} \text{floor}\left(\frac{142}{2^k}\right) = 71 + 35 + 17 + 8 + 4 + 2 + 1 = 138$$

Answer:

138

Reference:

https://www.quantguide.io/questions/binary-zeroes

Perfect Correlation I

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

Suppose that X and Y are perfectly negatively correlated i.e. $\rho(X,Y) = -1$ with $\sigma_Y > \sigma_X$. Find the correlation of $X+Y$ and $X-Y$.

Hint:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Compute $\mathrm{Cov}(X+Y, X-Y)$ and their respective variances using the common formulae.

Explanation:

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X,X) - \mathrm{Cov}(Y,Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\mathrm{Var}(X+Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\sigma_{X+Y} = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance. By a similar argument, we get $\mathrm{Var}(X-Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\mathrm{Var}(X-Y) = \sigma_X + \sigma_Y$.

Therefore, we have that $\rho(X+Y, X-Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = -1$. We get the -1 from the condition that $\sigma_Y > \sigma_X$, so the numerator is negative.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_Y > \sigma_X$, it must be the case that $|a| > 1$. Therefore, $X+Y = X(1+a)+b$ and $X - Y = X(1-a) + b$. As $|a| > 1$, one of the constants between $1+a$ and $1-a$ is positive and one is negative. Therefore, as these are both linear transformation of X with opposing signs, they must have correlation -1. Note that this argument does not depend at all on whether or not the correlation of X and Y is -1 or 1.

Answer:

1

Reference:

https://www.quantguide.io/questions/perfect-correlation-i

Largest Inscribed Circle

Category: Pure Math
Difficulty: Medium
Companies: Five Rings

Question:

What is the area of the largest circle that can be inscribed inside a triangle with side lengths of 16, 16, and 24? The answer can be written in the form $\frac{p\pi}{q}$, where $\frac{p}{q}$ is a simplified fraction. Find $p+q$.

Hint:

Consider the area of the triangle. Can you relate this to the quantity that we want?

Explanation:

Recall from euclidean geometry that the area of a triangle can be expressed as the product of its inradius and its semiperimeter. Following routine application of the pythagorean theorem, we find that the altitude of the triangle from the base of length 24 is $4\sqrt{7}$, so the area is $48\sqrt{7}$. The semiperimeter, being half the perimeter, is 28, so we can compute the inradius as $\frac{12\sqrt{7}}{7}$. The area of the incircle is $\frac{144\pi}{7}$, so our answer is $144 + 7 = 151$.

Answer:

151

Reference:

https://www.quantguide.io/questions/largest-inscribed-circle

Very Very Normal

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Let $X, Y, Z \sim N(12, 12)$ IID. Compute $\mathbb{P}[X - Y > Z]$. The answer is in the form $\Phi(a)$ for some a. Find a.

Hint:

Write this as $\mathbb{P}[X - Y - Z > 0]$. What is the distribution of $X - Y - Z$?

Explanation:

We can write this as $\mathbb{P}[X - Y - Z > 0]$. $W = X - Y - Z \sim N(-12, 36)$ by the properties of independent normal distributions. Therefore, we can write as $\mathbb{P}[W > 0]$, and we can standardize this to obtain $\mathbb{P}\left[\frac{W+12}{6} > \frac{12}{6}\right]$. The LHS is now standard normal, so the answer is $\Phi(-2)$ by symmetry of the normal distribution. This means $a = -2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/very-very-normal

Local Maxima

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

14 pieces of paper labelled $1 - 14$ are placed in a line at random. We spot i is a local maxima if the paper at the ith position is strictly larger than all of it's adjacent papers. Find the expected number of local maxima in the sequence. For example, with 6 numbers, 513246 has 3 local maxima at the first, third, and last spots.

Hint:

14 pieces of paper labelled 1-14 are placed in a line at random. Find the expected number of local maxima in the sequence. For example, with 6 numbers, 513246 has 2 local maxima at the first and third spots.

Explanation:

We will want to model whether or not the ith spot is a local maxima or not. Being a local maxima means that the numbers to the left and right of the spot are strictly smaller than our current spot. For the endpoints, this means that it is just larger than the value to the right/left (accordingly for the endpoint). Thus, if I_i is the indicator that spot i is a local maxima, $T = I_1 + \cdots + I_{14}$ gives the total number of local maxima.

By linearity of expectation $\mathbb{E}[T] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_{14}]$. Recall that the expectation of an indicator random variable is the probability of the event it indicates. For indicators $2 - 13$, they are surrounded by a spot on each side, so the probability that the maximum value of the papers in spots $i-1, i, i+1$ is the one at spot i is $\frac{1}{3}$ by symmetry. This holds for all $2 \leq i \leq 13$. For the endpoints, there is just one value they need to be larger than, so the probability is $\frac{1}{2}$ for spots 1 and 14. Therefore, $\mathbb{E}[T] = \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 12 = 5$.

Answer:

5

Reference:

https://www.quantguide.io/questions/local-maxima

Rabbit Hop V

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any odd amount of stairs at each movement. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Hint:

Let o_n be the number of distinct paths that the rabbit can take to go up a staircase of height n. Condition on the size of the first jump.

Explanation:

Let o_n be the number of distinct paths that the rabbit can take to go up a staircase of height n. We can condition on the size of the first jump. We can jump either $1, 3, 5, \dots, \alpha$, where $\alpha = n$ if n is odd and $n-1$ if n is even. In other words, α is the largest odd integer at most n. By conditioning, we have that

$$o_n = o_1 + o_3 + \cdots + o_\alpha$$

Let's take a closer look at the tail term $o_3 + \cdots + o_\alpha$. If the rabbit started on the second stair, this is exactly the sum that would be obtained, as the rabbit must can up $1, 3, 5, \dots, \alpha - 1$ stairs from stair 2. These correspond to the exact terms in the tail sum when taking the ground floor to be stair 2. This means that

$$o_2 = o_3 + \cdots + o_\alpha$$

so we can update our recurrence relation to be

$$o_n = o_{n-1} + o_{n-2}$$

Our initial conditions are $o_1 = o_2 = 1$, which can be directly counted quite easily. These are exactly the first two terms and the recurrence of the Fibonacci sequence, so $o_n = F_n$. In particular, $o_{10} = F_{10} = 55$.

Answer:

55

Reference:

https://www.quantguide.io/questions/rabbit-hop-v

Rabbit Hop I

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of stairs at each move, but it must make an even amount of hops to get to the top. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Hint:

Can you match up paths to some subsets of $\{1, 2, \dots, 9\}$? What is the restriction?

Explanation:

Using the same idea as Rabbit Hop I, we are looking for subsets of $\{1, 2, \dots, 9\}$ that have odd size. This is because we have to select an odd amount of integers $1 - 9$ such that when we move to step 10, the total number of moves is even. In other words, we need to find the number of odd-sized subsets of a set of size 9.

We can match up odd and even subsets by matching a subset $A \subseteq \{1, 2, \dots, 9\}$ with odd cardinality to A^c, which would have even cardinality. This is since if $|A| = k$, where k is odd, $9 - k$ must be even, so we have a one-to-one matching of odd and even subsets, so there must be equal amounts of odd and even subsets. Thus, there are $\frac{1}{2} \cdot 2^9 = 256$ such paths.

Answer:

256

Reference:

https://www.quantguide.io/questions/rabbit-hop-i

Correlation Variance

Category: Statistics
Difficulty: Easy
Companies: Five Rings

Question:

Suppose we have that $\sigma_X = 4$, $\sigma_Y = 7$, and $\sigma_{X+Y} = 11$. What is $\rho(X, Y)$?

Hint:

Use the variance of a sum formula

Explanation:

We are going to generalize this to when $\sigma_X = a, \sigma_Y = b$, and $\sigma_{X+Y} = a + b$. We know that

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \implies \text{Cov}(X, Y) = \frac{1}{2}((a + b)^2 - a^2 - b^2) = ab$$

After, we know that

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{ab}{ab} = 1$$

Answer:

1

Reference:

https://www.quantguide.io/questions/correlation—variance

Decreasing Uniform Chain

Category: Probability
Difficulty: Hard
Companies: Five Rings

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Let N be the first index n where $X_n \neq \min\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[X_{N-1}]$ i.e. the smallest value among the first N values selected. The answer will be in the form $a + be$ for integers a and b. Note here that e is Euler's constant. Find $a + b$.

Hint:

Use Law of Total Expectation to condition on the next value.

Explanation:

Let $f(x)$ be the expected smallest number given that the current smallest is x. We want $f(1)$, as we are selecting uniform values on $(0,1)$, so our initial value is 1. We don't choose a starting value higher than 1 since we can't generate values larger than 1.

With probability $1-x$, the next value is larger than x, and our smallest value will be x. Otherwise, if our next value is $y < x$, y is uniformly distributed on $(0,x)$, so the expected minimum in that case is $f(y)$. We integrate over all $0 < y < x$ since we are applying Law of Total Probability. Written as an equation, this is

$$f(x) = x(1-x) + \int_0^x f(y)dy$$

Taking the derivative to convert into a differential equation, we get that $f'(x) = f(x) + (1-2x)$. Equivalently, $f'(x) - f(x) = 1 - 2x$. An initial condition for this differential equation is $f(0) = 0$, as we can't go any lower than 0.

This is a linear first order differential equation, so we can use the method of integrating factors here. In particular, the integrating factor is $\mu(x) = e^{\int -1dx} = e^{-x}$. Multiplying this on both sides, $(e^{-x}f(x))' = (1-2x)e^{-x}$, meaning $e^{-x}f(x) = \int e^{-x} - 2xe^{-x}dx = (2x+1)e^{-x} + C$. This means that $f(x) = 1 + 2x + Ce^x$. Using our initial condition $f(0) = 0$, this immediately yields $C = -1$, so $f(x) = 1 + 2x - e^x$. In particular, $f(1) = 3 - e$, so our answer is $3 - 1 = 2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/decreasing-uniform-chain

Wedding Handshakes

Category: Brainteasers
Difficulty: Easy
Companies: Five Rings

Question:

At a wedding of 100 people, each person shakes hands with every other person once. How many handshakes occur in total?

Hint:

Start with a smaller scenario or think of a mathematical operation that models this situation well.

Explanation:

There are a couple ways to do this question. The fastest method is to realize that $\binom{100}{2}$ perfectly models this situation as we see how many different pairs of people can be chosen from the 100. This results in the answer of 4950. The other method is to start with a smaller scenario. Start with 1 person. Every time another person joins the wedding, they have to shake hands with the current number of guests. This means the total number of handshakes is the integer sums from 1 to 99. There are 49 sets of 100 within this sum and an extra 50 which results in a total of 4950.

Answer:

4950

Reference:

https://www.quantguide.io/questions/wedding-handshakes

Compound Interest II

Category: Brainteasers
Difficulty: Easy
Companies: Five Rings

Question:

You start with \$100 in your bank account today. You invest in a stock that yields 1% interest that is compounded daily. To the nearest dollar, how much will you have in your bank account after 15 days?

Hint:

Use the Taylor approximation $e^x \approx 1 + x$

Explanation:

Using the Taylor approximation $e^x \approx 1 + x$, we can see that $100(1.01)^{15} \approx 100(1 + 0.15) = 115$. However, we must adjust up a little bit for the interest that accrues over the 15 days i.e. it is not exactly \$1 per day. Therefore, a natural guess is \$116, which is correct. One can see it will not go to \$117 by the fact that the absolute maximum that could be attained is $1.15 \cdot 15 = 17.25$, which is if you constantly receive 1.15 interest a day. However, this is a large overestimate, so 116 is a better guess.

Answer:

116

Reference:

https://www.quantguide.io/questions/compound-interest-ii

Betting Orbs

Category: Probability
Difficulty: Medium
Companies: Five Rings, Jane Street

Question:

You are presented with a bag of 4 orbs. You know that 2 are blue and 2 are red. You start drawing them without replacement from the bag one-by-one. Before each draw, you are given the opportunity to guess the color of the orb that is about to be drawn. If you get it correct, you earn \$1. Under an optimal strategy, what is your expected profit of this game?

Hint:

Make the best prediction on the color based on the colors that have already been observed.

Explanation:

Let $I_1, \ldots, I_4$ be the indicator of the event that the ith orb's color is guessed correctly. Then $T = I_1+I_2+I_3+I_4$ gives the total profit of the game. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{4} \mathbb{P}[A_i]$, where A_i is the event that the ith orb's color is guessed correctly. We want to find a strategy that maximizes $\mathbb{P}[A_i]$.

On the first draw, we have absolutely no information about what color will come out. Therefore, regardless of which color we select, our probability of getting it correct is $\mathbb{P}[A_1] = \frac{1}{2}$. Afterwards, once we have observed the color of the first orb, note that the probability that the second orb differs in color from the first orb is $\frac{2}{3}$. This also means the probability they are the same color is $\frac{1}{3}$. Therefore, we should guess the opposite color of the orb to be drawn on the second draw. This means $\mathbb{P}[A_2] = \frac{2}{3}$. For the third draw, we really need to think about what has come out thus far. If the first two draws are the same color, which occurs with probability $\frac{1}{3}$, then with probability 1 you know the color of the last two orbs to be drawn, so you should guess that. If they differ, which occurs with probability $\frac{2}{3}$, then you are back at random for the third draw, so you get it correct with probability $\frac{1}{2}$. Therefore, by applying Law of Total Probability via conditioning on the matching or differing colors of the orbs, $\mathbb{P}[A_3] = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 = \frac{2}{3}$. Regardless of what is drawn on the third turn, you know the last color, so $\mathbb{P}[A_4] = 1$. Therefore, under this strategy, $\mathbb{E}[T] = \frac{1}{2} + \frac{2}{3} + \frac{2}{3} + 1 = \frac{17}{6}$. Note that under just guessing randomly, your expected value would be 2, so this is a significant improvement.

Answer:

176

Reference:

https://www.quantguide.io/questions/betting-orbs

Couple Handshakes

Category: Brainteasers
Difficulty: Easy
Companies: Five Rings

Question:

A room of four couples greet each other by shaking hands. If each person shakes the hand of every other person besides their partner, how many handshakes occur?

Hint:

How many pairs of people can be formed? How many pairs of people are couples?

Explanation:

Each handshake consists of 2 people. The order of the people in the handshake is irrelevant, so there are $\binom{8}{2} = 28$ ways to pick pairs of people to shake hands. However, we know that no couple shakes hands with their partner. Therefore, we must remove one handshake for each couple representing the one that is with their partner. As there are 4 couples, we remove 4 handshakes, and our total number of handshakes is $28 - 4 = 24$.

Answer:

24

Reference:

https://www.quantguide.io/questions/couple-handshakes

Rabbit Hop III

Category: Probability
Difficulty: Medium
Companies: Five Rings

Question:

A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up 1 or 2 stairs at each step. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Hint:

Let p_n be the number of distinct paths to the top of a staircase of length n. Condition on the last jump.

Explanation:

Let p_n be the number of distinct paths to the top of a staircase of length n. We can condition on the last jump. If the last jump is a movement by 1 stair, we need to count the number of ways to get to stair $n-1$. There are p_{n-1} ways to do this by definition. Similarly, if the last movement is movement by 2 stairs, we need to count the number of paths to get to stair $n-2$. There are p_{n-2} ways to do this by definition. Since these two cases are disjoint and exhaustive, we have that $p_n = p_{n-1} + p_{n-2}$. The initial conditions are $p_1 = 1$ and $p_2 = 2$, which can just be enumerated directly. We can see that $p_n = F_{n+1}$, the $(n+1)$st Fibonacci number. In particular, $p_{10} = F_{11} = 89$.

Answer:

89

Reference:

https://www.quantguide.io/questions/rabbit-hop-iii

Ping Pong Tournament I

Category: Probability
Difficulty: Medium
Companies: Five Rings, Two Sigma

Question:

50 people are competing in a ping pong tournament where there is only one ping pong table. The competitors are numbered 1 through 50. Suppose that if two competitors meet, the one with the larger number wins. Two competitors are chosen at random, and the loser is removed from the tournament. The winner moves on to the next round, where their opponent is chosen at random. This process is repeated until one person is left (a total 49 rounds will be played). Compute the probability that competitor 49 is still in the tournament after the first 10 rounds have been played.

Hint:

In order for competitor 49 to survive the first 10 rounds, either (1) competitor 49 does not appear until the 11th round, or (2) competitor 49 plays their first round within the first 10 rounds, and competitor 50 does not play within the first 10 rounds.

Explanation:

In order for competitor 49 to survive the first 10 rounds, either (1) competitor 49 does not appear until the 11th round, or (2) competitor 49 plays their first round within the first 10 rounds, and competitor 50 does not play within the first 10 rounds.

Consider case 1. Note that there are a total of 39 rounds between rounds 11 and 49, inclusive. If we want competitor 49 to play their first round after the 10th round, then competitor 49 must be at position 12 or later (39 possible positions) in a random ordering of the 50 competitors. The probability of case 1 occuring is then $\frac{39}{50}$.

Next, consider case 2. In this case, in a random ordering of 50 competitors, competitor 49 must be among the first 11 competitors, and competitor 50 must be among the last 39 competitors. This occurs with probability $\frac{\binom{11}{1}\binom{39}{1}48!}{50!} = \frac{429}{2450}$. By countable additivity, we find the answer to be $\frac{39}{50} + \frac{429}{2450} = \frac{234}{245}$.

Answer:

234245

Reference:

https://www.quantguide.io/questions/ping-pong-tournament-i

Radioactive Decay

Category: Probability
Difficulty: Easy
Companies: Five Rings

Question:

Plutonium decays at an average rate of 1 time per second. Find the probability an atom of Plutonium doesn't decay in next 3 seconds. The answer is in the form e^a for some a. Find a.

Hint:

What is the distribution of time between decays?

Explanation:

The average rate of decay is 1 per second, so the time between decays is $T \sim \text{Exp}(1)$. We want $\mathbb{P}[T > 3]$. This is just $\displaystyle\int_3^\infty e^{-x}dx = e^{-3}$, so $a = -3$.

Answer:

3

Reference:

https://www.quantguide.io/questions/radioactive-decay

1 Glove Off

Category: Probability
Difficulty: Hard
Companies: Five Rings

Question:

You have 5 pairs of gloves that each have a distinct number $1-5$. The 10 gloves are randomly paired up. Find the probability that the gloves are paired up such that the values of any pair differ by at most 1.

Hint:

Note that there are $\frac{(2n)!}{2^n \cdot n!}$ ways to pair up the $2n$ gloves, as there are $(2n)!$ total arrangements, $n!$ ways to re-label the pairs, and then 2 ways to switch around the order in each pair. Let g_n be the number of arrangements that satisfy our condition. Derive a recurrence relation for g_n based on conditioning with one of the gloves labelled n.

Explanation:

We are going to solve the more general case with n pairs of gloves labelled $1-n$. Note that there are $\frac{(2n)!}{2^n \cdot n!}$ ways to pair up the $2n$ gloves, as there are $(2n)!$ total arrangements, $n!$ ways to re-label the pairs, and then 2 ways to switch around the order in each pair. Let g_n be the number of arrangements that satisfy our condition. Consider the two gloves labelled n, say n_1 and n_2. Decide the partner for n_1 first. We either have that n_2 is paired with n_1, in which case, we go back to the same problem but with $n-1$ pairs of gloves instead of n. Otherwise, n_1 is paired with one of the gloves labelled $n-1$, of which there are 2 ways to pick that glove. Afterwards, we know that n_2 is paired with the other glove labelled $n-1$, and that becomes fixed. Then, this goes back to the same problem but with $n-2$ pairs of gloves instead. Therefore, we get the recurrence relation

$$g_n = g_{n-1} + 2g_{n-2}$$

Remember that the 2 in front of g_{n-2} represents the fact that we have 2 options of the glove labelled $n-1$ to match with n_1 in that sub-case. We now need some initial conditions. Note that $g_1 = 1$, as there is clearly only one pair. Furthermore, we have $g_2 = 3$, as we can pick the partner for any one of the gloves in 3 ways, and that fixes the other pair immediately. The characteristic equation of this recurrence relation is $r^2 - r - 2 = 0$, of which the solutions are $r = 2, -1$. Therefore, $g_n = c_0 \cdot 2^n + c_1 \cdot (-1)^n$. Plugging in the initial conditions yields that $1 = 2c_0 - c_1$ and $3 = 4c_0 + c_1$. Solving these yields that $c_0 = \frac{2}{3}$ and $c_1 = \frac{1}{3}$. Therefore,

$$g_n = \frac{2^{n+1} + (-1)^n}{3}$$

Therefore, the probability of this event occurring with n pairs is given by

$$p_n = \frac{g_n}{\frac{(2n)!}{2^n \cdot n!}} = \frac{(2^{n+1} + (-1)^n) \cdot 2^n \cdot n!}{3(2n)!}$$

Substituting in $n = 5$, we get that $p_5 = \frac{1}{45}$.

Answer:

145

Reference:

https://www.quantguide.io/questions/1-glove-off

Double Data Trouble II

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{\beta}_{OLS}$. Recall that if X is the data and σ^2 is the variance of the IID normal errors, then $\text{Var}\left(\hat{\beta}_{OLS}\right) = \sigma^2(X^TX)^{-1}$. If you were to run linear regression again on the dataset where you duplicate each point in your original dataset and obtain new coefficients $\hat{\beta}'_{OLS}$, find the constant c such that $\text{Var}\left(\hat{\beta}'_{OLS}\right) = c\text{Var}\left(\hat{\beta}_{OLS}\right)$. If no such constant exists, enter -1.

Hint:

When we duplicate the data, we are just replacing the matrix X with the matrix $X' = \begin{bmatrix} X \\ \\ X \end{bmatrix}$, so $X'^TX' = 2X^TX$.

Explanation:

When we duplicate the data, we are just replacing the matrix X with the matrix $X' = \begin{bmatrix} X \\ \\ X \end{bmatrix}$, so $X'^TX' = 2X^TX$. Therefore, $\sigma^2(2X^TX)^{-1} = \frac{1}{2} \cdot \sigma^2(X^TX)^{-1}$, which proves our statement.

Answer:

12

Reference:

https://www.quantguide.io/questions/double-data-trouble-ii

Double Data Trouble IV

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain an R^2 value of 0.48. If you were to run linear regression again on the dataset where you double all of the values of each point in the dataset (for example, if $(2, 5)$ were in the original dataset, it would now be $(4, 10)$), what would R^2 be? If it can't be determined, enter -1.

Hint:

Recall that $R^2 = 1 - \frac{SSE}{SST}$. How does SSE change, and how does SST change?

Explanation:

Recall that $R^2 = 1 - \frac{SSE}{SST}$. With each of the data points doubled, the new SSE is 4 times as large as it was before. This is because we are summing squared errors and we have doubled each of the points. Therefore, the errors (in magnitude) are doubled. Once we square these, the SSE would then be 4 times as large. For this same reason, SST also is quadrupled. Accordingly, the ratio $\frac{SSE}{SST}$ is unchanged, meaning that R^2 is unchanged from before.

Answer:

0.48

Reference:

https://www.quantguide.io/questions/double-data-trouble-iv

Adult Concert

Category: Brainteasers
Difficulty: Medium
Companies: Two Sigma, Jane Street

Question:

$\frac{5}{12}$ of concert attendees are adults. A bus carrying 50 people arrives at the concert. Now, $\frac{11}{25}$ of concert attendees are adults. What is the minimum number of adults who could have been at the concert before the bus arrived?

Hint:

Let x denote the total number of people at the concert before the bus arrives. $x \equiv 0 \mod 12$ from the question. What do you know about $x + 50$?

Explanation:

Let x denote the total number of people at the concert before the bus arrives. $x \equiv 0 \mod 12$. Then, $x + 50 \equiv 0 \mod 25$. Since $50 \equiv 0 \mod 25$, we can rewrite the previous expression as $x \equiv 0 \mod 25$. The minimum possible value of x satisfying these two conditions is $x = 300$. $x \cdot \frac{5}{12} = 125$.

Answer:

125

Reference:

https://www.quantguide.io/questions/adult-concert

Deriving Logprice Dynamics

Category: Pure Math
Difficulty: Medium
Companies: Two Sigma

Question:

Let's say S_t follows the dynamics:

$$dS_t = 3S_t dt + 4S_t dW_t$$

where W_t is a standard Brownian motion.

Find the dynamics of $X_t = \ln(S_t)$, where we can write the dynamics as:

$$dX_t = adt + bdW_t$$

Find $a^2 + b^2$

Hint:

Use Ito's Lemma

Explanation:

We can use Ito's Lemma on $f(x) = \ln(x)$. We have $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$.

Ito's Lemma says that $df = f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2$. Plugging in the dynamics from above into dS_t, the derivatives, and using the fact that $(dS_t)^2 = 16S_t^2 dt$, we get the following.

$$df = \frac{1}{S_t}(3S_t dt + 4S_t dW_t) + \frac{1}{2}\frac{-1}{S_t^2}16S_t^2 dt$$

$$= 3dt + 4dW_t - 8dt$$

$$= -5dt + 4dW_t$$

So, we have $a = -5$ and $b = 4$. Plugging the values in, we get $(-5)^2 + 4^2 = 41$.

Answer:

41

Reference:

https://www.quantguide.io/questions/deriving-logprice-dynamics

The Big Three

Category: Probability
Difficulty: Easy
Companies: Two Sigma, Jane Street

Question:

Suppose we roll 5 standard fair dice and sum the upfaces of the largest 3 values showing. Find the probability that the sum is 18.

Hint:

What needs to happen to get a sum of 18 from three standard dice?

Explanation:

To obtain a sum of 18 from three dice, we need to have at least 3 sixes among the 5 dice. Therefore, we really are just asking for the probability of $3, 4$, or 5 sixes among 5 die rolls. The probability of 5 sixes is just $\frac{1}{6^5}$, as it is just $\frac{1}{6}$ probability for each die. The probability of 4 sixes is $5 \cdot \frac{5}{6} \cdot \frac{1}{6^4}$, as we have 5 ways to select the die that isn't a 6, a $\frac{5}{6}$ probability that the select die is not a 6, and then $\frac{1}{6}$ probability for each of the other 4 dice to be a 6. Lastly, the probability that we have exactly 3 sixes is $\binom{5}{2} \cdot \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6^3}$. This is because there are $\binom{5}{2} = 10$ ways to pick the two dice that aren't a 6, $\frac{5}{6}$ probability for each of them to not be a 6, and $\frac{1}{6}$ probability for each of the remaining three dice to be a 6. Adding all of these up yields

$$\frac{1}{6^5} + \frac{25}{6^5} + \frac{250}{6^5} = \frac{276}{6^5} = \frac{23}{648}$$

Answer:

23648

Reference:

https://www.quantguide.io/questions/the-big-three

Double Data Trouble III

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Let t_0 be the $t-$statistic obtained when testing the null hypothesis $H_0 : \beta_1 = 0$ against $H_A : \beta_1 \neq 0$. Suppose that we duplicate our data and then run linear regression again on this new data set. Let $\hat{y}_i = \hat{\beta}'_0 + \hat{\beta}'_1 x_i$ be the new coefficients that are obtained. If t_1 is the corresponding $t-$statistic obtained when testing the null hypothesis $H_1 : \beta'_1 = 0$ against $H_A : \beta'_1 \neq 0$, find $\frac{t_1^2}{t_0^2}$. If it can't be determined, enter -1.

Hint:

From Double Data Trouble II, we know that the variance of our estimate is reduced by $\frac{1}{2}$, so the standard error is reduced by $\frac{1}{\sqrt{2}}$.

Explanation:

From Double Data Trouble II, we know that the variance of our estimate is reduced by $\frac{1}{2}$, so the standard error is reduced by $\frac{1}{\sqrt{2}}$. As $t_0 = \frac{\hat{\beta}_1}{s\left\{\hat{\beta}_1\right\}}$, we know that the estimate $\hat{\beta}'_1$ is the same from Double Data Trouble I, and that the standard error in the denominator is $\frac{1}{\sqrt{2}}$ as large for $\hat{\beta}'_1$, we get that $t_1 = \sqrt{2} t_0$, meaning that $t_1^2 = 2t_0^2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/double-data-trouble-iii

Double Data Trouble VI

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Let t_0 be the $t-$statistic obtained when testing the null hypothesis $H_0 : \beta_1 = 0$ against $H_A : \beta_1 \neq 0$. Suppose that we double the values of each of our data points and then run linear regression again on this new data set. Let $\hat{y}_i = \hat{\beta}_0' + \hat{\beta}_1' x_i$ be the new coefficients that are obtained. If t_1 is the corresponding $t-$statistic obtained when testing the null hypothesis $H_1 : \beta_1' = 0$ against $H_A : \beta_1' \neq 0$, find $\frac{t_1^2}{t_0^2}$. If it cannot be determined, enter -1.

Hint:

From Double Data Trouble V, we know that the variance of our estimate is reduced by $\frac{1}{4}$, so the standard error is reduced by $\frac{1}{2}$.

Explanation:

From Double Data Trouble V, we know that the variance of our estimate is the same, so the standard error is also the same. As

$$t_0 = \frac{\hat{\beta}_1}{s\left\{\hat{\beta}_1\right\}}$$

we know that the estimate $\hat{\beta}_1'$ is the same from Double Data Trouble I, and that the standard error in the denominator is same for $\hat{\beta}_1'$. Therefore, we get that $t_1 = t_0$, meaning that $t_1^2 = t_0^2$.

Answer:

1

Reference:

https://www.quantguide.io/questions/double-data-trouble-vi

Double Data Trouble V

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Let t_0 be the $t-$statistic obtained when testing the null hypothesis $H_0 : \beta_1 = 0$ against $H_A : \beta_1 \neq 0$. Suppose that we double the values of each of our data points and then run linear regression again on this new data set. Let $\hat{y}_i = \hat{\beta}_0' + \hat{\beta}_1' x_i$ be the new coefficients that are obtained. If t_1 is the corresponding $t-$statistic obtained when testing the null hypothesis $H_1 : \beta_1' = 0$ against $H_A : \beta_1' \neq 0$, find $\frac{t_1^2}{t_0^2}$. If it cannot be determined, enter -1.

Hint:

From Double Data Trouble V, we know that the variance of our estimate is reduced by $\frac{1}{4}$, so the standard error is reduced by $\frac{1}{2}$.

Explanation:

From Double Data Trouble V, we know that the variance of our estimate is the same, so the standard error is also the same. As

$$t_0 = \frac{\hat{\beta}_1}{s\left\{\hat{\beta}_1\right\}}$$

we know that the estimate $\hat{\beta}_1'$ is the same from Double Data Trouble I, and that the standard error in the denominator is same for $\hat{\beta}_1'$. Therefore, we get that $t_1 = t_0$, meaning that $t_1^2 = t_0^2$.

Answer:

1

Reference:

https://www.quantguide.io/questions/double-data-trouble-v

Double Data Trouble I

Category: Statistics
Difficulty: Easy
Companies: Two Sigma

Question:

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{\beta}_{OLS}$. Recall that if X is the data and σ^2 is the variance of the IID normal errors, then $\text{Var}\left(\hat{\beta}_{OLS}\right) = \sigma^2(X^TX)^{-1}$. If you were to run linear regression again on the dataset where you duplicate each point in your original dataset and obtain new coefficients $\hat{\beta}'_{OLS}$, find the constant c such that $\text{Var}\left(\hat{\beta}'_{OLS}\right) = c\text{Var}\left(\hat{\beta}_{OLS}\right)$. If no such constant exists, enter -1.

Hint:

When we duplicate the data, we are just replacing the matrix X with the matrix $X' = \begin{bmatrix} X \\ \\ X \end{bmatrix}$, so $X'^TX' = 2X^TX$.

Explanation:

When we duplicate the data, we are just replacing the matrix X with the matrix $X' = \begin{bmatrix} X \\ \\ X \end{bmatrix}$, so $X'^TX' = 2X^TX$. Therefore, $\sigma^2(2X^TX)^{-1} = \frac{1}{2} \cdot \sigma^2(X^TX)^{-1}$, which proves our statement.

Answer:

12

Reference:

https://www.quantguide.io/questions/double-data-trouble-i

Rsquared Range

Category: Statistics
Difficulty: Medium
Companies: Two Sigma

Question:

Using OLS, we regress y onto X_1 and find that the model has an R^2 of 0.15. We also regress y onto X_2 but this time the model has an R^2 of 0.2. Let $[\min, \max]$ denote the lower and upper-bound of the R^2 of a model which regresses y onto X_1, X_2. Express your answer as $\frac{\max}{\min}$.

Hint:

Can adding a variable to a regression ever decrease R^2?

Explanation:

We start with a basic fact of OLS that an additional predictive variable can never decrease R^2 (as OLS can always set the coefficient to a predictor to be 0!)
Therefore, our lower bound is the max of individual R^2 values meaning $\min = 0.2$.
As an upper bound, we could have that $y = X_1 + X_2$ meaning $\max = 1.0$. Note that this does not restrict our freedom of picking X_1 and X_2 so that their individual R^2 are 0.15 and 0.2.

Answer:

5

Reference:

https://www.quantguide.io/questions/rsquared-range

Shopping Habits

Category: Statistics
Difficulty: Hard
Companies: Two Sigma, Jane Street

Question:

A researcher is investigating the temporal effect of shopping habits at the mall. She independently and randomly selects 20 weekend and 20 weekday shoppers and finds that weekend shoppers spend \$78 on average with a standard deviation of \$22, while weekday shoppers spend \$67 on average with a standard deviation of \$20. The researcher would like to test if there is sufficient evidence to claim that there is a significant difference in the average amount spent by weekend and weekday shoppers. What is the attained significance level? Round to 3 significant figures. Assume simple random sampling, variance homogeneity, and that spending is approximately normally distributed.

Hint:

Because neither sample size is sufficiently large, you should use a t test with $20 + 20 - 2 = 38$ degrees of freedom.

Explanation:

We are testing the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_a : \mu_1 \neq \mu_2$. Because neither sample size is sufficiently large, we must use a t test with 38 degrees of freedom:

$$t = \frac{\bar{x_1} - \bar{x_2}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \text{ where } S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

Solving for S_p:

$$S_p = \sqrt{\frac{(20 - 1) \times 22^2 + (20 - 1) \times 20^2}{20 + 20 - 2}} \approx 21.024$$

Solving for t:

$$t = \frac{78 - 67}{21.024\sqrt{\frac{1}{20} + \frac{1}{20}}} \approx 1.655$$

Because this is a two-tailed test, the attained significance level is double the p-value of a one-tailed test. Thus,

$$p\text{-value} = 2 \times P(t \geq 1.655) = 2 \times .05308 \approx 0.106$$

Answer:

0.106

Reference:

https://www.quantguide.io/questions/shopping-habits

Place Or Take

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

You are playing a one-player game with two opaque boxes. At each turn, you can choose to either "place" or "take". "Place" places \$1 from a third party into one box randomly. "Take" empties out one box randomly and that money is yours. This game consists of 100 turns where you must either place or take. Assuming optimal play, what is the expected payoff of this game? Note that you do not know how much money you have taken until the end of the game.

Hint:

There are two aspects to this game that should be noted: the ordering of the places and takes, and the number of places and takes. We can tackle the former aspect first. Intuitively, no places should come after a take. Greedily so, you might as well put a take after all places such that you have a higher expected value of that take (considering a single take, every place you put after the take is \$1 less that could be taken); thus, all takes should be stacked at the end of the 100 turns. This can also be proved with induction, and this will be left as an exercise to the reader. The next aspect is regarding the number of places and takes.

Explanation:

There are two aspects to this game that should be noted: the ordering of the places and takes, and the number of places and takes. We can tackle the former aspect first. Intuitively, no places should come after a take. Greedily so, you might as well put a take after all places such that you have a higher expected value of that take (considering a single take, every place you put after the take is \$1 less that could be taken); thus, all takes should be stacked at the end of the 100 turns. This can also be proved with induction, and this will be left as an exercise to the reader. The next aspect is regarding the number of places and takes. If there are p places, then there must be $100 - p$ takes, and we know that the p places must occur before the $100 - p$ takes. In order to find the value of p, we can calculate at what position does adding an additional place action not improve our expected payoff. Consider the p-th action where $1 < p < 100$ and where this is the last place before we begin taking, or the next action $(p + 1)$ will be the last pace before we begin taking. What is the expected value of the game if action p is the last place? There are p dollars in aggregate and we have $100 - p$ takes. Our expected payoff will be $p \times \sum_{i=1}^{100-p} \frac{1}{2^i}$, which can be seen from the linearity of expectation- at every take, we will take half of what is remaining on average. What is the expected value of the game if action $p + 1$ is the last place? There are $p + 1$ dollars in aggregate and we have $100 - (p + 1)$ takes. Thus, our expected payoff is $(p + 1) \times \sum_{i=1}^{100-p-1} \frac{1}{2^i}$. We are looking for the largest value of p such that the payoff where the p-th place is the last place is greater the payoff where the incremental $p + 1$-th place is the last place. Thus:

$$p \sum_{i=1}^{100-p} \frac{1}{2^i} > (p + 1) \sum_{i=1}^{100-p-1} \frac{1}{2^i}$$

$$p\sum_{i=1}^{100-p}\frac{1}{2^i} > p\sum_{i=1}^{99-p}\frac{1}{2^i} + \sum_{i=1}^{99-p}\frac{1}{2^i}$$

$$\frac{p}{2^{100-p}} > \sum_{i=1}^{99-p}\frac{1}{2^i}$$

$\sum_{i=1}^{99-p}\frac{1}{2^i}$ approaches 1, and thus $p > 2^{100-p} \Rightarrow p > 93$. With 94 places, the expected payoff is $94\sum_{i=1}^{6}\frac{1}{2^i} \approx 92.5$. To check this is the maximum payoff, we can look at the 93 and 95 placing cases. With 93 places, the expected payoff is $93\sum_{i=1}^{7}\frac{1}{2^i} \approx 92.3$. With 95 places, the expected payoff is $95\sum_{i=1}^{5}\frac{1}{2^i} \approx 92.0$. Both surrounding values are less than our expected payoff, and hence the optimal number of times we should place is 94 with an expected payoff of $\frac{2961}{32} \approx 92.5$.

Answer:

296132

Reference:

https://www.quantguide.io/questions/place-or-take

River Length

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

You are in a river of unknown length and velocity. You know that if you are stationary on the river, then it takes you 6 seconds for you to reach the other end. If you swim down the river at a rate of 3 feet per second, then it takes you 4 seconds to reach the other end. Find the length of the river (in feet).

Hint:

Let v be the velocity of the river and l be the length. Then we know that $l = 6v$ from the first part.

Explanation:

Let v be the velocity of the river and l be the length. Then we know that $l = 6v$ from the first part. Furthermore, we have that $l = 4(v + 3)$ from the part where we swim. By equating these, we have that $6v = 4(v + 3) \iff 2v = 12 \iff v = 6$. Therefore, we have that

$$l = 6 \cdot 6 = 36$$

Answer:

36

Reference:

https://www.quantguide.io/questions/river-length

Unlucky Seven II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

You are given a fair 6−sided die and you roll it. You can either choose to keep your roll and receive the observed value in dollars. Alternatively, you are allowed to roll again, but if the sum of your two rolls is at least 7, you pay the value equal to your first roll. If the sum of the two rolls is less than 7, you receive the sum of the two observed values in dollars. Assuming optimal play, what is your expected payout?

Hint:

Condition on the value of the cube and find the expected payout upon re-roll in each case.

Explanation:

Condition on the value of the first roll. If we roll a 1, we either lose \$1 with probability $\frac{1}{6}$ (if we roll a 6 on the second roll). Otherwise, we make a total of $2, 3, \ldots, 6$, each with probability $\frac{1}{6}$. Therefore our expected payout if we roll again is $\frac{1}{6} \cdot (-1) + \frac{2+3+4+5+6}{6} = \frac{19}{6} > 1$. This implies we should roll again if we receive a 1.

If we roll a 2, then with probability $\frac{1}{3}$ we lose \$2 (if we roll a 5 or 6). Otherwise, we receive $3, 4, 5,$ or 6 with equal probability $\frac{1}{6}$. Therefore, the expected value upon rolling again is $\frac{1}{3} \cdot (-2) + \frac{3+4+5+6}{6} = \frac{7}{3} > 2$. This implies we should roll. If we roll a 3, then with probability $\frac{1}{2}$ we lose \$3 (rolling 4 or more). Otherwise, we earn $4, 5,$ or 6 each with probability $\frac{1}{6}$. Therefore, our expected profit upon rolling again is $\frac{1}{2} \cdot (-3) + \frac{4+5+6}{6} = 1 < 3$. Therefore, for any value 3 or more, we should not roll again. This implies that our expected value on this game with this strategy is $\frac{\frac{19}{6} + \frac{14}{6} + 3 + 4 + 5 + 6}{6} = \frac{47}{12}$.

Answer:

4712

Reference:

https://www.quantguide.io/questions/unlucky-seven-ii

Prime First

Category: Brainteasers
Difficulty: Hard
Companies: Jane Street

Question:

Consider the set of integers $S = \{2, 3, \ldots, 30\}$. Alice and Bob play a game where they take turns selecting integers from S. The first player to select an integer that shares a common factor with a previously picked integer loses. Alice has the ability to determine if she wants to select first or second. Assume both players play optimally. Let $p = 1$ if Alice should choose to go first and $p = 2$ if she should go second. There are 6 optimal selections $v_1, \ldots, v_5$ for Alice's first turn. Find $100p + \dfrac{v_1 + v_2 + v_3 + v_4 + v_5 + v_6}{6}$.

Hint:

Consider all the primes in S : $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. There are 10 such integers. If you eliminate 2 prime factors on the first turn, what happens?

Explanation:

Consider all the primes in S : $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. There are 10 such integers. The key idea to notice here is that for whoever selects the integer 6, that eliminates 2 and 3. Therefore, no other pairwise products of primes among the remaining prime integers are at most 30. Therefore, there would be 8 primes left, and whoever is selecting first in this new game loses, as they can only eliminate one prime factor a turn. However, this "new game" is really just the second turn our game after we eliminate 2 and 3. Therefore, Alice should go first, select the value 6 (alternatively, $12, 18$, or 24 also work, as they have the same primes in the factorization) so that she can eliminate both 2 and 3, and then Bob and Alice alternate selecting among the remaining prime integers. Then, Bob would be forced to pick some non-prime integer, which will have a prime factor among those already eliminated, so Bob will lose. Therefore, Alice should go first, and the answer is 1.

However, there is also another sneaky starting integer, which is 5. This is because regardless of what Bob selects on his first turn after Alice selects 5, Alice can force Bob to eliminate one prime factor at a time after her next turn, so this also works. Since 5 works, $5^2 = 25$ also works. Therefore,

$$100(1) + \frac{6 + 12 + 18 + 24 + 5 + 25}{6} = 115$$

Answer:

115

Reference:

https://www.quantguide.io/questions/prime-first

Sharpe Marbles

Category: Statistics
Difficulty: Medium
Companies: Jane Street

Question:

Siblings Alice and Bob play a game with marbles. Each player has one red and one blue marble and shows one marble to the other uniformly at random. If both show blue, Alice wins \$1. If both show red, Alice wins \$3. Else, Bob wins \$2. Note that the winnings come from their mother, not the other player. Let A_r and B_r define the ratio between the expected return and variance of Alice's and Bob's payoffs, respectively. What is $B_r - A_r$?

Hint:

Calculate the expected payoffs and payoff variances of Alice and Bob separately before calculating A_r and B_r. Remember to weigh the payoffs and deviations according to their probabilities of occurring .

Explanation:

Siblings Alice and Bob play a game with marbles. Each player has one red and one blue marble and shows one marble to the other uniformly at random. If both show blue, Alice wins \$1. If both show red, Alice wins \$3. Else, Bob wins \$2. Note that the winnings come from their mother, not the other player. Let A_r and B_r define the ratio between the expected return and variance of Alice's and Bob's payoffs, respectively. What is $B_r - A_r$?

We start by solving for the expected payoffs and their variances:

$$E[A] = \frac{1}{4}(1) + \frac{1}{4}(3) + \frac{1}{2}(0) = 1$$

$$E[B] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$

$$V[A] = \frac{1}{4}(1-1)^2 + \frac{1}{4}(3-1)^2 + \frac{1}{2}(0-1)^2 = \frac{3}{2}$$

$$V[B] = \frac{1}{2}(2-1)^2 + \frac{1}{2}(0-1)^2 = 1$$

Now, we can calculate A_r and B_r:

$$A_r = 1 \div \frac{3}{2} = \frac{2}{3}$$

$$B_r = 1 \div 1 = 1$$

$$B_r - A_r = \frac{1}{3}$$

This conclusion shows how expected return is not the only metric that should be used when calculating the practical payoff of an investment. Even though the two players have the same expected return, Alice is taking on additional risk for her position. Ceteris paribus, Bob's position is favored over Alice's because he achieves higher expected return per unit of positional risk he takes on.

Answer:

13

Reference:

https://www.quantguide.io/questions/sharpe-marbles

Prime Subset

Category: Brainteasers
Difficulty: Hard
Companies: Jane Street

Question:

Consider all subsets of $S = \{1, 2, 3, \ldots, 30\}$ so that each pair of numbers in that subset are coprime. Find the subset $\Omega \subseteq S$ satisfying the previous condition whose elements have the largest sum. What is the sum of all the elements in Ω?

Hint:

To ensure all of the elements are coprime, the easiest way to make a candidate set would be the collection of all prime numbers at most 30. However, we also want to include 1, as 1 is coprime to every positive integer. Therefore, a candidate set would be

$$\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$$

This set has sum 130. Can you do better by combining and substituting different values?

Explanation:

To ensure all of the elements are coprime, the easiest way to make a candidate set would be the collection of all prime numbers at most 30. However, we also want to include 1, as 1 is coprime to every positive integer. Therefore, a candidate set would be

$$\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$$

This set has sum 130. However, we can do better than this by noting that we could replace any prime integers in the set by a combination of their factors that would have a larger sum. For example, you may remove 3 and 5 to replace it with 15, increasing the sum to 137. The question then becomes which integers can we do this for and not violate our condition. Let's start with taking out the 2. This is natural since 2 is our smallest integer besides 1 in our subset. The closest we can get to 30 with elements currently in the subset is 28, as $2^2 \cdot 7 = 28$. 29 is already in our subset, and 30 can't be added in presently, as 3 and 5 divide 30. Therefore, we should replace 2 and 7 with 28, increasing our sum by 19 to 149. Our remaining set is

$$\{1, 3, 5, 11, 13, 17, 19, 23, 28, 29\}$$

Afterwards, we note that $3^3 = 27$, and 27 is still coprime to everything our in our set. Similarly, $5^2 = 25$ would also be coprime to everything else, so we make these substitutions as well. This yields our final set of

$$\Omega = \{1, 11, 13, 17, 19, 23, 25, 27, 28, 29\}$$

The elements of this set sum to 193.

Answer:

193

Reference:

https://www.quantguide.io/questions/prime-subset

Prime Sum

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

Two distinct prime integers between 1 and 20, inclusive, are selected uniformly at random. Find the probability their sum is even.

Hint:

The prime integers at most 20 are $2, 3, 5, 7, 11, 13, 17$, and 19, which yields 8 total integers.

Explanation:

The prime integers at most 20 are $2, 3, 5, 7, 11, 13, 17$, and 19, which yields 8 total integers. There are $8 \cdot 7 = 56$ ways to pick 2 integers ordered for the sum. Our sum is even exactly when we don't select 2. Therefore, there are $7 \cdot 6 = 42$ ways to pick two prime integers that aren't 2. Therefore, our probability is

$$\frac{42}{56} = \frac{3}{4}$$

Answer:

34

Reference:

https://www.quantguide.io/questions/prime-sum

Russian Roulette IV

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two consecutively placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between not spinning and spinning the barrel?

Hint:

Look at the two cases separately. What is the probability of surviving if you do not spin the barrel conditioned on your friend surviving?

Explanation:

If you spin the barrel, the probability that you survive is $\frac{4}{6}$. If you don't spin the barrel, the probability that you survive is $\frac{3}{4}$. To understand this case, let us define the chambers 1-6 and further define chambers 5 and 6 to hold the two consecutive bullets, without loss of generality. Because our friend survives, we know that he shot either chambers 1, 2, 3, or 4. Of these 4 equal possibilities, only one of these leads to our loss (if he shot the fourth chamber, then the following is a bullet); hence, the probability of survival in this case is $\frac{3}{4}$ and the difference between these probabilities is:

$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Answer:

112

Reference:

https://www.quantguide.io/questions/russian-roulette-iv

First Right

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You and your opponent flip a coin. If the first player gets heads, the second player pays him \$30. If he flips tails, the coin goes to the second player. If the second player flips heads, he wins \$30 from the first player. If the second player also gets tails, the process repeats. What is the maximum amount (in dollars) you would pay for the right to go first?

Hint:

Let x be the expected value of the game if you go first. With probability $\frac{1}{2}$, you will receive \$30. What are the other cases?

Explanation:

Let x be the expected value of the game if you go first. With probability $\frac{1}{2}$, you will receive \$30. With probability $\frac{1}{4}$ (you land tails and your opponent lands heads), you will lose \$30. Then, with probability $\frac{1}{4}$, the game resets and your expected gain is x. This yields the equation

$$x = 0.5(30) + 0.25(-30) + 0.25(x)$$

When solved, this yields $x = 10$, so the right is worth at most \$10.

Answer:

10

Reference:

https://www.quantguide.io/questions/first-right

Half Cycle

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

Let C_n count the number of cycles in a random permutation of $\{1, 2, \ldots, 2n\}$ that are larger than n in length. Compute $\lim_{n\to\infty} \mathbb{E}[C_n]$. The answer is in the form $\ln(q)$ for some rational number q. Find q.

Hint:

First, find the expected number of $k-$cycles in the permutation for $k > n$. Then, sum those up from $n+1$ to $2n$ to get $\mathbb{E}[C_n]$. Use linearity of expectation to compute this part. You will end up with a sum that may be able to be re-written to look like a Riemann Integral.

Explanation:

First, we need to find the expected number of $k-$cycles in our permutation for $k > n$. Then, we can sum those up from $n+1$ to $2n$ to get $\mathbb{E}[C_n]$.

First, let's fix some $n+1 \leq k \leq 2n$. Let X_i be the indicator of whether or not the value i belongs to a cycle of length k. Then we have that

$$T = \frac{\sum_{i=1}^{2n} X_i}{k}$$

gives the total number of $k-$cycles, as we need to remove rotations (k such rotations per $k-$cycle). Using linearity of expectation, we have that

$$\mathbb{E}[T] = \frac{1}{k}\sum_{i=1}^{2n} \mathbb{E}[X_i] = \frac{2n}{k}\mathbb{E}[X_1]$$

$\mathbb{E}[X_1]$ is just the probability 1 belongs to a $k-$cycle. There are $(2n)!$ permutations of $\{1, 2, \ldots, 2n\}$. There are $\binom{2n-1}{k-1}$ ways to pick $k-1$ more elements to belong to the $k-$cycle. Now that we have selected the elements, there are $(k-1)!$ ways to arrange those elements. Then, there are $(2n-k)!$ ways to arrange the other $2n-k$ elements. This yields that our probability that 1 belongs to a cycle is

$$\frac{\binom{2n-1}{k-1}(k-1)!(2n-k)!}{(2n)!} = \frac{1}{2n}$$

Therefore, $\mathbb{E}[T] = \frac{1}{k}$ after substituting in.

The above tells us that the expected number of $k-$cycles for $1 \leq k \leq 2n$ is $\frac{1}{k}$. Therefore, $\mathbb{E}[C_n]$ is just the sum of the terms above from $n+1$ to $2n$. Namely,

$$\mathbb{E}[C_n] = \sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{\frac{k}{n}} \cdot \frac{1}{n}$$

This is starting to look like a Riemann Integral. Let $x = \frac{k}{n}$ so that $dx = \frac{1}{n}$ (this is a discrete sum currently, so dx is just change between terms). The lower bound of our integral is $x_L = \frac{n+1}{n} \to 1$, while the upper bound is $x_U = \frac{2n}{n} = 2$. As $n \to \infty$, this sum converges to the Riemann Integral

$$\int_1^2 \frac{1}{x} dx = \ln(2)$$

Therefore, $q = 2$.

Answer:

2

Reference:

https://www.quantguide.io/questions/half-cycle

Penny Stack

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street

Question:

You are given 100 pennies. You may arrange these pennies into however many stacks you want and put as many pennies into each stack as you would like. You are paid out the product of the number of pennies among all of the stacks. For example, if you make three stacks of sizes $10, 20$, and 70, your payout is $10 \cdot 20 \cdot 70 = 14000$. The optimal arrangement of the pennies will give you a payout of $a \cdot b^c$, where a and b are relatively prime. Find abc.

Hint:

The stacks should be of equal size roughly. Try starting with say 10 stacks of 10.

Explanation:

We want to make the product as large as possible. Therefore, if we view the number of coins in each pile as a side length, we want to maximize our value as much as possible. A cube maximizes the volume here, which is the product of the side lengths, so we want to get to a cube or as close to it as possible. Let's start with 10 stacks of 10, as $100 = 10 \cdot 10$. If we divide each stack of 10 into two stacks of 5, then we get a larger product, as each stack of 10 now contributes $5 \cdot 5 = 25$ to our product instead of 10. Therefore, we now have 5 stacks of 20. However, as $5 = 3 + 2$ and $3 \cdot 2 = 6 > 5$, we should divide them up again. Now, we are going to try as many stacks of 3 as possible. This would yields 33 stacks of 3 and a lone penny. However, this is not optimal, as removing one of our stacks of 3 and putting it with the lone penny yields $4 \cdot 3^{32} > 3 \cdot 3^{32} = 3^{33}$. As we can't divide 4 into anything more optimal, our solution is $4 \cdot 3^{32}$, so the answer to the answer is $4 \cdot 3 \cdot 32 = 384$.

Answer:

384

Reference:

https://www.quantguide.io/questions/penny-stack

Perfect Seating I

Category: Probability
Difficulty: Easy
Companies: Jane Street, Optiver

Question:

Seven people with distinct ages randomly sit down at a circular table with seven seats. What is the probability that the people sit themselves in increasing order of age, irrespective of direction?

Hint:

If you start from the youngest person, how many arrangements of the other 6 people at the circular table are increasing?

Explanation:

Seat the youngest person at the table in any spot. There are 6! ways to arrange the remaining 6 people that have not been seated at the table. Of those arrangements, exactly 2 of them are in increasing order of age. Namely, these occur when they are increasing in age clockwise or counter-clockwise. Therefore, 2 of these 6! equally likely permutations are in increasing order of age, so our result is $\frac{2}{6!} = \frac{1}{360}$.

Answer:

1360

Reference:

https://www.quantguide.io/questions/perfect-seating-i

Counting Nash Equillibria

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

A treasure chest is up for auction and there are 10 participants participating in the auction. The value of the treasure chest is determined as follows: behind a curtain 20 fair coins are independently flipped and for each head that shows, \$1 is added to the chest (participants cannot see behind the curtian but are aware of the way the chest is valued).

Each person chooses a non-negative integer number of dollars to submit as a sealed bid. The chest goes to a uniformly random person who was among the highest bidders, in return for the amount bid and everyone else has their bid returned. Count how many pure Nash equilibria there are for this game. (You may assume everyone's utility for money is linear in the range \$0 to \$10.)
Note: A pure Nash equilibrium means the strategy of each player must choose a single bid value with probability 1 (e.g. a player's strategy cannot be to bid 3 with probability 0.4 and bid 4 with probability 0.6).

Hint:

Count the number of Nash Equilibria for which exactly k people bid \$10.

Explanation:

Let's first simplify the valuation of the chest, since we are interested in Nash equilibria we are interested in a set of strategies such that a **single** person cannot increase their *expected value* from the chest when everyone uses the Nash strategy. Therefore, we can simply think of the chest containing \$10 since this is the expected number of heads. Now we begin with the observation that if one person submits a \$10 bid, then all 10 players at best will have a 0 EV strategy. It is not hard to see the person bidding \$10 will be 0 EV since the chest is \$10 in expectation. As for the remaining 9 people, if any of them bid less than \$10, they don't win or lose anything since the chest goes to the \$10 bidder and their bid is returned. If they bid exactly \$10 then again this is 0 EV, and if they bid any more than \$10 their EV is negative so at best the remaining 9 people can do a 0 EV strategy.However, it might be in the interest of a person bidding \$10 to decrease their bid. Consider a strategy where all 9 people will bid \$9, what should the last person do to maximize their EV? Bidding \$10 here is 0 EV as discussed before, but bidding \$9 is actually positive EV since there is a $\frac{1}{10}$ probability their \$9 bid wins the chest worth \$10 for netting them a \$1 profit. This motivates our first statement.**Claim:** All people bidding \$9 is a Nash Equilibrium. **Proof:** The expectation of this strategy for a given person is \$0.1 as shown above. A given person cannot exceed this in expectation since bidding any higher will result in a zero or negative expectation and bidding any lower will result in zero expectation since they will never win the auction due to the other people bidding \$9.
Now, we begin to generalize. Let's consider counting the number of Nash Equilibrium by the number of people who are bidding \$10. If there are only 0 people bidding \$10, then the only Nash Equilibrium that can exist is when all bid \$9, if there was someone who was not bidding \$9 then they could increase their expectation by bidding \$9.

If there is only 1 person bidding \$10, then no such Nash equilibrium can exist since this person can increase their expectation by bidding \$9.

If there are only 2 people bidding \$10, then we claim this is a Nash equilibrium because everyone's strategy has an expected value of 0 and no one can gain by deviating. In fact, this logic holds as long as there are *at least* 2 people bidding \$10. Therefore, to count the total number of strategies let us reframe the problem. Let's denote a strategy as a 10-d tuple so $S = \{b_1, b_2, ..., b_{10}\}$ where b_i denotes the $i-$th person's bid meaning $b_i \in \{\$0, \$1, ..., \$10\}$. To count the number of Nash Equilibria which correspond to at least 2 people bidding \$10 we count the complement. That is, the number of Nash Equilibria with at least 2 people bidding ten $= \underbrace{11^{10}}_{\text{all strategies}} - \underbrace{10^{10}}_{\text{no ten bids}} - \underbrace{\binom{10}{1}10^9}_{\text{one ten bids}} = 5937424601.$

Lastly, we must add one in the case of all bids being \$9 giving us our final answer of 5937424602.

Answer:

5937424602

Reference:

https://www.quantguide.io/questions/counting-nash-equillibria

Power Grid

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

A 3×3 grid of light bulbs is formed. Then, each light bulb is powered on with probability $\frac{1}{2}$. Find the probability that no two adjacent (grid cells that share a common side) light bulbs are powered on.

Hint:

First, break into cases based on the middle light bulb being on or off. Then, further consider sub-cases with the middle being off.

Explanation:

Consider the light bulbs in a matrix arrangement as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

There are clearly $2^9 = 512$ equally-likely outcomes of the light bulbs being activated or not. Now, we need to count the outcomes that satisfy our event.

The key here is to consider light bulb 5. If it is powered on, then none of $2, 4, 6,$ and 8 can be powered on. However, $1, 3, 7,$ and 9 are free to be on or off. Therefore, there are $2^4 = 16$ outcomes in this case. The other case is more complicated.

Suppose that light bulb 5 is off now. We now have 4 sub-cases to consider corresponding to the different combinations of when light bulbs 4 and 6 are powered or not.

Case 1 - Both On: In this case, $1, 3, 7,$ and 9 all must be off. However, 2 and 8 are free to be on or off, so there are $2^2 = 4$ outcomes in this case.

Case 2 - One On: To simplify, we will consider when 4 is on and 6 is off. Then, we can just multiply by 2 to account for the other case. Since 4 is on, 1 and 7 must be off. For 2 and 3, either exactly one is on or neither is on. This accounts for 3 cases. This holds similarly for light bulbs 8 and 9, so there are $2 \cdot 3 \cdot 3 = 18$ combinations in this case.

Case 3 - Both Off: If both 4 and 6 are off, then light bulbs $1 - 3$ can either be all off (1 case), have exactly one on (3 cases), or have 1 and 3 on with 2 off (1 case). This yields 5 total cases. This similarly holds for lightbulbs $7-9$, so there are $5^2 = 25$ combinations for this case.

Adding all of these up, we get that there are $16 + 4 + 18 + 25 = 63$ total combinations that are favorable, so our answer is $\frac{63}{512}$.

Answer:

63512

Reference:

https://www.quantguide.io/questions/power-grid

Russian Roulette I

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two consecutively placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between not spinning and spinning the barrel?

Hint:

Look at the two cases separately. What is the probability of surviving if you do not spin the barrel conditioned on your friend surviving?

Explanation:

If you spin the barrel, the probability that you survive is $\frac{4}{6}$. If you don't spin the barrel, the probability that you survive is $\frac{3}{4}$. To understand this case, let us define the chambers 1-6 and further define chambers 5 and 6 to hold the two consecutive bullets, without loss of generality. Because our friend survives, we know that he shot either chambers 1, 2, 3, or 4. Of these 4 equal possibilities, only one of these leads to our loss (if he shot the fourth chamber, then the following is a bullet); hence, the probability of survival in this case is $\frac{3}{4}$ and the difference between these probabilities is:

$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Answer:

112

Reference:

https://www.quantguide.io/questions/russian-roulette-i

Big Smalls

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Alice and Bob play the following game: Alice generates a uniformly random integer between 1 and 100, inclusive of both. Bob can pick any number that he wants between 1 and 100, inclusive of both. Whoever has the larger number must pays out the amount of the smaller number to the person with the smaller number. For example, if Alice selects 70 and Bob selects 50, Bob will receive 50 from Alice. If they select the same number, nobody pays out anything. Assuming optimal play by Bob, what is his expected payout?

Hint:

Let Z_x be Bob's gain if he selects the value x. Furthermore, let Y be the randomly generated number. We can ignore the case where $Y = x$, as the payout there is 0 for both people. By Law of Total Expectation, we have

$$\mathbb{E}[Z_x] = \mathbb{E}[Z_x \mid x > Y]\mathbb{P}[x > Y] + \mathbb{E}[Z_x \mid x < Y]\mathbb{P}[x < Y]$$

Maximize this function.

Explanation:

Let Z_x be Bob's gain if he selects the value x. Furthermore, let Y be the randomly generated number. We can ignore the case where $Y = x$, as the payout there is 0 for both people. By Law of Total Expectation, we have

$$\mathbb{E}[Z_x] = \mathbb{E}[Z_x \mid x > Y]\mathbb{P}[x > Y] + \mathbb{E}[Z_x \mid x < Y]\mathbb{P}[x < Y]$$

We can see that $\mathbb{P}[x > Y] = (x-1)/100$ and $\mathbb{P}[x < Y] = (100-x)/100$, as there are $x-1$ values in $\{1, \ldots, x-1\}$ and $100-x$ numbers in $\{x+1, \ldots, 100\}$

To compute $\mathbb{E}[Z_x \mid x > Y]$, we can see that if $Y < x$, it is uniform on $\{1, \ldots, x-1\}$, so the expected amount Bob pays is $x/2$.
To compute $\mathbb{E}[Z_x \mid x < Y]$, we just simply note that Bob receives his number as the payout, which is x. Combining all of the above,

$$\mathbb{E}[Z_x] = \left(\frac{x-1}{100}\right)\left(-\frac{x}{2}\right) + \left(\frac{100-x}{100}\right)(x)$$

$$= \frac{201x - 3x^2}{200}.$$

The derivative of this function if $\frac{1}{200}(201 - 6x)$, so this derivative equals 0 precisely when $x^* = \frac{201}{6} = \frac{67}{2}$. Since the parabola is symmetric about the axis of symmetry, which is at the $x-$value of the maximum (33.5 in this case), we conclude that guessing 33 and 34 give equal payouts. Namely, $\mathbb{E}[Z_{33}] = \mathbb{E}[Z_{34}] = \frac{1683}{100}$

Answer:

16.83

Reference:

https://www.quantguide.io/questions/big-smalls

Bacterial Survival II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

There is one bacterium on a cell plate. After every minute, the bacterium may die, stay the same, split into two, or split into three with equal probabilities. Assuming all bacteria behave the same and independent of other bacteria, what is the probability that the bacterial population will die out? The answer is in the form $\sqrt{a} - b$ for integers a and b. Find $a + b$.

Hint:

How can you derive the probability conditioned on what happens to the first bacterium at the next time step?

Explanation:

Let x be the probability that the bacterial population will die out. There are 4 total possibilities for the first bacterium: either it dies, stays the same, splits into two, or splits into three. In the first case, the bacterial population dies with probability 1. In the second case, the bacterial population dies with probability x, since the bacterial population dies when the bacterium that stays the same dies, which happens with probability x. In the third case, the bacterial population dies with probability x^2, as both bacteria that are formed must die in order for the population to die. In the fourth and final case, the bacterial population dies with probability x^3, as all three bacteria that are formed must die in order for the population to die. Thus, we can write

$$x = \frac{1}{4} \times 1 + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3 = \sqrt{2} - 1 \approx 0.41$$

Therefore, our answer is $2 + 1 = 3$.

Answer:

3

Reference:

https://www.quantguide.io/questions/bacterial-survival-ii

Fishy Bear

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

A bear wants to catch 3 fish from a river. When the bear has caught 3 fish, it will leave. The bear catches each fish with probability $\frac{1}{2}$. Find the probability the 5th fish is caught.

Hint:

Let C represent the event that the 5th fish is caught and F represent the event that the bear is fishing on the 5th turn. By the Law of Total Probability,

$$\mathbb{P}[C] = \mathbb{P}[C \mid F]\mathbb{P}[F] + \mathbb{P}[C \mid F^c]\mathbb{P}[F^c]$$

$\mathbb{P}[F]$ is just the probability that the bear has caught at most 2 fish in the first 4 turns.

Explanation:

Let C represent the event that the 5th fish is caught and F represent the event that the bear is fishing on the 5th turn. By the Law of Total Probability,

$$\mathbb{P}[C] = \mathbb{P}[C \mid F]\mathbb{P}[F] + \mathbb{P}[C \mid F^c]\mathbb{P}[F^c]$$

$\mathbb{P}[F]$ is just the probability that the bear has caught at most 2 fish in the first 4 turns. This probability is just $\frac{\binom{4}{0} + \binom{4}{1} + \binom{4}{2}}{2^4} = \frac{11}{16}$ Therefore, $\mathbb{P}[F^c] = 1 - \frac{11}{16} = \frac{5}{16}$.
Then, we know that $\mathbb{P}[C \mid F] = \frac{1}{2}$ by the question. Additionally, $\mathbb{P}[C \mid F^c] = 0$, as if the bear isn't fishing at the 5th turn, it can't be caught. Therefore, $\mathbb{P}[C] = \frac{11}{16} \cdot \frac{1}{2} + 0 = \frac{11}{32}$.

Answer:

1132

Reference:

https://www.quantguide.io/questions/fishy-bear

Expected Chord Length

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

Two points are uniformly at random selected from the circumference of the unit circle. Find the expected length of the chord (line segment) between the two points. The answer is in the form $\frac{a}{\pi}$ for some rational number a. Find a.

Hint:

Let θ and ϕ be IID Unif$(0, 2\pi)$. Picking the two angles is equivalent to picking the two points uniform on the circumference. The measure of the angle between θ and ϕ is $\theta - \phi$. To get the length of the line segment connecting the two points, we can take a slight shortcut. The length here only depends on the distance between our points we select in terms of angle. Use this to fix one of the points at $(1, 0)$.

Explanation:

Let θ and ϕ be IID Unif$(0, 2\pi)$. Picking the two angles is equivalent to picking the two points uniform on the circumference. The measure of the angle between θ and ϕ is $\theta - \phi$. To get the length of the line segment connecting the two points, we can take a slight shortcut. The length here only depends on the distance between our points we select in terms of angle. In other words, regardless of where we end up having our points located, we can just rotate the circle so that one of them is located at $(1, 0)$. Therefore, we can arbitrarily fix $\theta_1 = 0$. Thus, the length between $\theta_1 = 0$ and the point at $\theta_2 = \theta$ is $\sqrt{(1-\cos(\theta))^2 + \sin^2(\theta)} = \sqrt{2}\sqrt{1-\cos(\theta)}$. This means that finding the expected length of the chord is equivalent to finding $\sqrt{2}\mathbb{E}[\sqrt{1-\cos(\theta)}]$, where $\theta \sim$ Unif$(0, 2\pi)$.

Now, $\sqrt{2}\mathbb{E}[\sqrt{1-\cos\theta}] = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$. Now, the best approach is to conjugate the interior by multiplying and dividing by $\sqrt{1+\cos\theta}$ so that we get $\sqrt{1-\cos^2\theta} = |\sin\theta|$. Doing this, we get $\frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\frac{|\sin\theta|}{\sqrt{1+\cos\theta}}d\theta$.

Note that the region on $(0, \pi)$ and $(\pi, 2\pi)$, our integrand is symmetric, so we can just evaluate over one interval and double it. This means our new integral is $\frac{\sqrt{2}}{\pi}\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$. Let $u = 1 + \cos\theta$. Then $du = -\sin\theta d\theta$. Our bounds would respectively become 2 and 0, but the negative from the du flips them back. Therefore, our new integral is $\frac{\sqrt{2}}{\pi}\int_0^2 u^{-\frac{1}{2}}du$. Evaluating this, we get $\frac{2\sqrt{2}}{\pi}\cdot\sqrt{u}\Big|_0^2 = \frac{4}{\pi}$.

Answer:

4

Reference:

https://www.quantguide.io/questions/expected-chord-length

Least Multiple Of 15

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

What is the least positive multiple of 15 whose digits consist of 1's and 0's only?

Hint:

In order for a number to be a multiple of 15, it must be perfectly divisible by both 3 and 5. How can you use 1's and 0's to build a number that is perfectly divisible by both?

Explanation:

In order for a number to be a multiple of 15, it must be perfectly divisible by both 3 and 5. In order for the number to be divisible by 5, it must end in either a 0 or 5- thus, our number must end in a 0. In order for the number to be divisible by 3, the sum of its digits must be a multiple of 3- thus, we will add three 1's digits to the number. Hence, the smallest positive multiple of 15 whose digits consist of 1's and 0's only is 1110.

Answer:

1110

Reference:

https://www.quantguide.io/questions/least-multiple-of-15

Russian Roulette III

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two randomly placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. The two of you take turns pulling the trigger until the person who fires the gun loses. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between spinning and not spinning the barrel?

Hint:

Because your friend survived, if you don't spin, you lose a safe chamber. How can you use this to calculate the probability of survival given you don't spin?

Explanation:

If you don't spin the barrel, the probability that you survive is $\frac{3}{5}$, since one of the safe chambers has already been pulled by your friend. If you do spin the barrel, the probability that you survive is $\frac{4}{6}$, since any of the 6 chambers is possible, 2 of which are lethal. Thus, the difference in probability is:

$$\frac{4}{6} - \frac{3}{5} = \frac{1}{15}$$

Answer:

115

Reference:

https://www.quantguide.io/questions/russian-roulette-iii

Combinatorial Electrician

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

An electrician sets up 12 light bulbs on a circle. Each pair of light bulbs has a wire connecting them. Each individual wire is active with probability $\frac{1}{2}$, independent of all other wires. A collection of $2 \leq k \leq 12$ light bulbs is said to form a complete $k-$circuit if between any 2 light bulbs in the circuit, we have an active wire. Find the expected number of $4-$circuits in our electrician's arrangement of light bulbs.

Hint:

How many $4-$circuits are there? Set up an indicator random variable for each one. How many wires need to be active in a $4-$circuit to be complete?

Explanation:

There are $\binom{12}{4} = 495$ ways to pick a subset of 4 light bulbs to be in our circuit. Therefore, label the circuits $1 - 495$ and let $X_1, \dots, X_{495}$ be the indicators that circuit i is a complete $4-$circuit. Then $T = X_1 + \cdots + X_{495}$ gives the total number of $k-$circuits. By the exchangeability of the circuits (no circuit is more likely to be complete than any other) and linearity of expectation, $\mathbb{E}[T] = 495\mathbb{E}[X_1]$. $\mathbb{E}[X_1]$ is just the probability that circuit 1 is $4-$complete. We know that between any 2 pairs of light bulbs, there must be an active wire, and there are 4 light bulbs here, so there are $\binom{4}{2} = 6$ active wires for this circuit to be complete. The probability of all 6 being active is $\frac{1}{2^6} = \frac{1}{64}$. Therefore, $\mathbb{E}[T] = \frac{495}{64}$.

Answer:

49564

Reference:

https://www.quantguide.io/questions/combinatorial-electrician

Matching Socks I

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

Your drawer contains 2 red socks, 4 yellow socks, and 5 blue socks. In a rush, you randomly grab socks from the drawer. What is the minimum number of socks you need to grab to guarantee a pair of socks of the same color?

Hint:

Pigeon Hole Principle

Explanation:

By Pigeon Hole Principle, you will need to have 4 socks to guarantee that at least two have the same color. This is because in the worst-case scenario, the first 3 socks you grab are each of a different color- the fourth sock must pair one of the first 3 socks.

Answer:

4

Reference:

https://www.quantguide.io/questions/matching-socks-i

Random Triangle

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

Three points are selected uniformly at random on the circumference of the unit circle and are labelled points A, B, and C. Find the expected perimeter of the triangle ABC. The answer is in the form $\frac{a}{\pi}$ for a rational number a. Find a.

Hint:

Three points are selected uniformly at random on the perimeter of the unit circle and are labelled points A, B, and C. Find the expected perimeter of the triangle ABC. The answer is in the form $\frac{a}{\pi}$ for a rational number a. Find a.

Explanation:

The triangle is formed by drawing chords between each pair of points. As the points are IID, the three side lengths are exchangeable. This means that the expected perimeter is just obtained by tripling the expected length of one of the sides of the triangle. In other words, if P is the perimeter and S_{AB} is the length of the chord between points A and B, $\mathbb{E}[P] = 3\mathbb{E}[S_{AB}]$. This question now boils down to finding the expected length of a chord drawn in a circle.

Let θ and ϕ be IID $\text{Unif}(0, 2\pi)$. Picking the two angles is equivalent to picking the two points uniform on the circumference. The measure of the angle between θ and ϕ is $\theta - \phi$. To get the length of the line segment connecting the two points, we can take a slight shortcut. The length here only depends on the distance between our points we select in terms of angle. In other words, regardless of where we end up having our points located, we can just rotate the circle so that one of them is located at $(1, 0)$. Therefore, we can arbitrarily fix $\theta_1 = 0$. Thus, the length between $\theta_1 = 0$ and the point at $\theta_2 = \theta$ is $\sqrt{(1-\cos(\theta))^2 + \sin^2(\theta)} = \sqrt{2}\sqrt{1-\cos(\theta)}$. This means that finding the expected length of the chord is equivalent to finding $\sqrt{2}\mathbb{E}[\sqrt{1-\cos(\theta)}]$, where $\theta \sim \text{Unif}(0, 2\pi)$.

Now, $\sqrt{2}\mathbb{E}[\sqrt{1-\cos\theta}] = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \sqrt{1-\cos\theta}\,d\theta$. Now, the best approach is to conjugate the interior by multiplying and dividing by $\sqrt{1+\cos\theta}$ so that we get $\sqrt{1-\cos^2\theta} = |\sin\theta|$. Doing this, we get $\frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \frac{|\sin\theta|}{\sqrt{1+\cos\theta}}d\theta$.

Note that the region on $(0, \pi)$ and $(\pi, 2\pi)$, our integrand is symmetric, so we can just evaluate over one interval and double it. This means our new integral is $\frac{\sqrt{2}}{\pi}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$. Let $u = 1 + \cos\theta$. Then $du = -\sin\theta d\theta$. Our bounds would respectively become 2 and 0, but the negative from the du flips them back. Therefore, our new integral is $\frac{\sqrt{2}}{\pi}\int_0^2 u^{-\frac{1}{2}}du$. Evaluating this, we get $\frac{2\sqrt{2}}{\pi} \cdot \sqrt{u}\Big|_0^2 = \frac{4}{\pi}$.

Thus, the expected perimeter of the triangle is $3 \times \frac{4}{\pi} = \frac{12}{\pi}$ and $a = 12$.

Answer:

12

Reference:

https://www.quantguide.io/questions/random-triangle

Coin Pair IV

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of coin flips needed until you are unable to better your position.

Hint:

Let T be the total number of coin flips needed and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin.

Explanation:

Let T be the total number of coin flips needed and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin. To set our notation, let e_i represent the additional number of flips needed to stop our process once we have i heads.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. This would mean $e_4 = 0$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 4$. This would mean $e_3 = 0$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. We have that $\mathbb{E}[T \mid X = 2] = 4 + e_2$, as it takes 4 flips at the beginning and then need e_2 additional flips to stop. We have that

$$e_2 = \frac{1}{4} \cdot (2 + e_4) + \frac{1}{2} \cdot (2 + e_3) + \frac{1}{4} \cdot (2 + e_2)$$

This stems from the fact that we flip 2 coins, and we get $0, 1$, or 2 heads with those prescribed probabilities fries those two coins. Solving for e_2 here by plugging in $e_3 = e_4 = 0$ is $e_2 = \frac{8}{3}$, so $\mathbb{E}[T \mid X = 2] = 4 + \frac{8}{3} = \frac{20}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. We know that $\mathbb{E}[T \mid X = 1] = 4 + e_1$ by the same logic as above. By doing the same conditioning argument, we have that

$$e_1 = \frac{1}{4} \cdot (2 + e_3) + \frac{1}{2} \cdot (2 + e_2) + \frac{1}{4} \cdot (2 + e_1)$$

Plugging in $e_3 = 0$ and $e_2 = \frac{8}{3}$ yields that $e_1 = \frac{40}{9}$. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{40}{9} + 4 = \frac{76}{9}$.

Lastly, if $X = 0$, we have that $\mathbb{E}[T \mid X = 0] = 4 + e_0$. By doing the same conditioning argument once again, we get that

$$e_0 = \frac{1}{4} \cdot (2 + e_2) + \frac{1}{2} \cdot (2 + e_1) + \frac{1}{4} \cdot (2 + e_0)$$

Solving for e_0 by plugging in $e_2 = \frac{8}{3}$ and $e_1 = \frac{40}{9}$ yields $e_0 = \frac{176}{27}$. We can now compute $\mathbb{E}[T \mid X = 0] = \frac{176}{27} + 4 = \frac{284}{27}$.

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 4 + \frac{6}{16} \cdot \frac{20}{3} + \frac{4}{16} \cdot \frac{76}{9} + \frac{1}{16} \cdot \frac{284}{27} = \frac{176}{27}$$

Answer:

17627

Reference:

https://www.quantguide.io/questions/coin-pair-iv

Stamp Sum

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

You have infinitely many stamps of values 5 and 21. Find the largest value that is unable to be expressed as a combination of these stamps.

Hint:

Note that $21, 42, 63$, and 84 are the smallest values that can be expressed as change that are equivalent to $1, 2, 3$, and 4 modulo 5, respectively.

Explanation:

Note that $21, 42, 63$, and 84 are the smallest values that can be expressed as change that are equivalent to $1, 2, 3$, and 4 modulo 5, respectively. Afterwards, if we add some amount of 5 value stamps to these, we are able to obtain stamps of any number equivalent to those in modulo 5. Therefore, the largest value we can't create would is something 4 modulo 5, as that is the highest starting point. Namely, the largest integer that can't be created here is 79, as anything that is 4 modulo 5 and at least 84 can be created, so 79 is our answer.

Answer:

79

Reference:

https://www.quantguide.io/questions/stamp-sum

Bus Wait I

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

You are waiting for a bus to get to QuantGuide on time! The bus runs on a fixed schedule of appearing every 10 minutes. However, the driver, independently between appearances, may want to refill on gas. The driver refills on gas with 10% probability per trial, independently between trials. If the driver fills up on gas, 1 hour is added to his travel time. If you arrive at a uniformly random time throughout the day, what is the expected time until the next bus appears (in minutes)?

Hint:

Use Law of Total Expectation to condition on whether or not the bus stops for gas.

Explanation:

Let G be the event of refilling on gas and T be the time it takes for the bus to arrive. We are going to condition on showing up on a turn when the bus refills for gas vs. when it doesn't. On average, 1 in every 10 trips will have a gas refill. In this case, the trip length is 70 minutes.

Therefore, in 9 of every 10 trips, the bus doesn't refill. Therefore, on average, in every 160 minutes, 70 of those will be when the bus is on a refill trip. Therefore, the probability of you appearing during a refill trip is $\frac{7}{16}$. The expected wait of that trip would be $\frac{1}{2} \cdot 70 = 35$. Then, with probability $\frac{9}{16}$ the trip is regular and the expected wait is 5 minutes as per the previous part of this question. Therefore, by Law of Total Expectation, the total wait time is

$$35 \cdot \frac{7}{16} + 5 \cdot \frac{9}{16} = \frac{145}{8}$$

Answer:

1458

Reference:

https://www.quantguide.io/questions/bus-wait-i

Discard Game

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

A casino offers a card game using a normal deck of 52 cards. The rule is that you turn over two cards each time. For each pair, if both cards are black, they go to the dealer's pile; if both are red, they go to your pile; else, they are discarded. The process is repeated 26 times until all cards are exhausted. If you have more cards in your pile, you win $10. Else, including ties, you get nothing. What is the fair value of this game?

Hint:

Look at this problem through the lens of symmetry. Each discarded pair has one black card and one red card. What does this tell you about the number of black and red cards remaining?

Explanation:

No matter how the cards are permuted, you and the dealer will always have the same number of cards in your respective piles because each pair of discarded cards has one black and one red card, so an equal number of red and black cards are always discarded. In other words, there are an equal number of red and black cards that are not discarded and in your piles. Thus, the number of red cards left for you and the number of black cards left for the dealer are always the same and the casino always wins. You should pay $0 to play this game.

Answer:

0

Reference:

https://www.quantguide.io/questions/discard-game

Basic Die Game V

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Alice rolls a fair 6−sided die with the values $1-6$ on the sides. She sees that value showing up and then is allowed to decide whether or not she wants to roll again. Each re-roll costs \$1. Whenever she decides to stop, Alice receives a payout equal to the upface of the last die she rolled. Note that there is no limit on how many times Alice can re-roll. Assuming optimal play by Alice, what is her expected payout on this game?

Hint:

Let v_k be the value of the game if you accept a roll of k or more and re-roll otherwise. Our optimal strategy will be in this form because we can view each round of the game as an independent trial. Therefore, our strategy should be the same between trials.

Explanation:

Let v_k be the value of the game if you accept a roll of k or more and re-roll otherwise. Our optimal strategy will be in this form because we can view each round of the game as an independent trial. Therefore, our strategy should be the same between trials. Then

$$\begin{aligned}
v_1 &= (1/6)(1+2+3+4+5+6) && \Rightarrow \quad v_1 = 7/2 \\
v_2 &= (1/6)\,(v_2-1)+(1/6)(2+3+4+5+6) && \Rightarrow \quad v_2 = 19/5 \\
v_3 &= (2/6)\,(v_3-1)+(1/6)(3+4+5+6) && \Rightarrow \quad v_3 = 4 \\
v_4 &= (3/6)\,(v_4-1)+(1/6)(4+5+6) && \Rightarrow \quad v_4 = 4 \\
v_5 &= (4/6)\,(v_5-1)+(1/6)(5+6) && \Rightarrow \quad v_5 = 7/2 \\
v_6 &= (5/6)\,(v_6-1)+(1/6)(6) && \Rightarrow \quad v_6 = 1
\end{aligned}$$

All of these are derived from the fact that if we obtain a value at least our minimum stopping value, we just stop immediately. Otherwise, we just go again and it is the same game but we lose 1 in payout from the cost of the roll. Therefore, this means that our optimal EV is 4.

Answer:

4

Reference:

https://www.quantguide.io/questions/basic-die-game-v

Infected Dinner I

Category: Brainteasers
Difficulty: Hard
Companies: Jane Street

Question:

There are 1000 people having dinner at a grand hall. One of them is known to be sick, while the other 999 are healthy. Each minute, each person talks to one other person in the room at random. However, as everyone is social, nobody talks to people they have previously talked to. In each pair, if one is sick and one is healthy, the healthy person is infected and becomes sick. Once a person becomes sick, they are assumed to be sick for the rest of the dinner. Find the maximum amount of time (in minutes) until every person in the hall becomes sick.

Hint:

Consider grouping the 1000 people into 25 groups of 40 people. In 25 rounds of 39 minutes each, every person in a given bubble talks to every other person in that bubble. Additionally, each of the other bubbles are paired up. Suppose the original sick person is in Bubble 25 and the last healthy person is in Bubble 24. In round k, we can have all the people in bubble b talk to all of the people in bubble $24 - b + k \bmod 25$, where 0 maps to 25. What happens with this?

Explanation:

Consider grouping the 1000 people into 25 groups of 40 people. In 25 rounds of 39 minutes each, every person in a given bubble talks to every other person in that bubble. Additionally, each of the other bubbles are paired up.

Suppose the original sick person is in Bubble 25 and the last healthy person is in Bubble 24. In round k, we can have all the people in bubble b talk to all of the people in bubble $24-b+k \bmod 25$, where 0 maps to 25. Therefore, we see that after k rounds, the sick people belong to people in Bubble 25 and Bubbles $1, 2, \ldots, k-1$. This is since at round 1, Bubble 25 talks with itself. Afterwards, it talks with Bubbles $1, 2, \ldots, k-1$. Therefore, Bubble 24 will be completely untouched until after the first 24 rounds, which is 936 minutes. Then, in the 937th minute, everyone in the 24th bubble talks to everyone in the 25th bubble, meaning all are infected in the first minute. Therefore, after 937 minutes, everyone is infected.

Answer:

937

Reference:

https://www.quantguide.io/questions/infected-dinner-i

Bus Wait II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

You are waiting for a bus to get to QuantGuide on time! The bus runs on a fixed schedule of appearing every 10 minutes. However, the driver, independently between appearances, may want to refill on gas. The driver refills on gas with 10% probability per trial, independently between trials. If the driver fills up on gas, 1 hour is added to his travel time. If you arrive at a uniformly random time throughout the day, what is the expected time until the next bus appears (in minutes)?

Hint:

Use Law of Total Expectation to condition on whether or not the bus stops for gas.

Explanation:

Let G be the event of refilling on gas and T be the time it takes for the bus to arrive. We are going to condition on showing up on a turn when the bus refills for gas vs. when it doesn't. On average, 1 in every 10 trips will have a gas refill. In this case, the trip length is 70 minutes.

Therefore, in 9 of every 10 trips, the bus doesn't refill. Therefore, on average, in every 160 minutes, 70 of those will be when the bus is on a refill trip. Therefore, the probability of you appearing during a refill trip is $\frac{7}{16}$. The expected wait of that trip would be $\frac{1}{2} \cdot 70 = 35$. Then, with probability $\frac{9}{16}$ the trip is regular and the expected wait is 5 minutes as per the previous part of this question. Therefore, by Law of Total Expectation, the total wait time is

$$35 \cdot \frac{7}{16} + 5 \cdot \frac{9}{16} = \frac{145}{8}$$

Answer:

1458

Reference:

https://www.quantguide.io/questions/bus-wait-ii

Digit Multiplication II

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street

Question:

Let A be the set of all integers whose digits multiply to 96. Furthermore, let x and y be the minimal and maximal elements of A, respectively. What is $y - x$? Note that no digit can be 1, so that A is finite.

Hint:

What is the prime factorization of 96?

Explanation:

The prime factorization of 96 is $3 \cdot 32 = 3 \cdot 2^5$. Therefore, we need to have 5 twos and a three in our number accounted for. For the largest possible value, we should arrange the values in descending order left to right, as this would give the largest weight to digits of largest value. Therefore, our largest value is 322222. For the smallest value, note that $2^3 = 8$ and $2 \cdot 3 = 6$. Therefore, we condense down 4 of the twos and the three into an 8 and 6. We can't condense more, as otherwise those digits will be larger than 10, so the smallest value that can be made from 8, 2, and 6 is clearly 268. Thus, our answer is $322222 - 268 = 321954$.

Answer:

321954

Reference:

https://www.quantguide.io/questions/digit-multiplication-ii

Double Dice Payoff

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You roll two fair dice. If you get double 6s, you receive \$100. If you get a 6 and a non-6, you lose \$$x$. If you get anything else, you reroll both dice until you get double 6s or a 6 and non-6. What is the maximum value of x where the game still has non-negative expected value?

Hint:

How does the probability you roll again related to the other outcomes?

Explanation:

First we need to find the probability of every outcome. Double 6s occur with probability $(\frac{1}{6})^2 = \frac{1}{36}$. We can count the number of 6 and non-6 combos there are and you'll see there are 10 (1-5 on the first die and 6 on the other with two different orderings). This gives that event a probability of $\frac{10}{36}$. Since the other event is just a recurrence of these two events, we can normalize the probabilities of the two events we care about. We do this by finding the probability of our target event and divide it with the probability of the non-recurrent events. So the probability of a double 6s in this case can be updated to $\frac{\frac{1}{36}}{(\frac{1}{36}+\frac{10}{36})} = \frac{1}{11}$. This gives the probability the game ends in a 6 and non 6 of $\frac{10}{11}$. Finally, we can solve for x. The profit equation here is $100 \cdot \frac{1}{11} - x \cdot \frac{10}{11}$ which has to be greater than or equal to 0. Thus $\frac{100}{11} \geq \frac{10x}{11}$. The maximum value of x is thus \$10.

Answer:

10

Reference:

https://www.quantguide.io/questions/double-dice-payoff

Unlucky Seven I

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

You are given a fair 6−sided die and you roll it. You can either choose to keep your roll and receive the observed value in dollars. Alternatively, you are allowed to roll again, but if the sum of your two rolls is at least 7, you pay the value equal to your first roll. If the sum of the two rolls is less than 7, you receive the sum of the two observed values in dollars. Assuming optimal play, what is your expected payout?

Hint:

Condition on the value of the cube and find the expected payout upon re-roll in each case.

Explanation:

Condition on the value of the first roll. If we roll a 1, we either lose \$1 with probability $\frac{1}{6}$ (if we roll a 6 on the second roll). Otherwise, we make a total of $2, 3, \ldots, 6$, each with probability $\frac{1}{6}$. Therefore our expected payout if we roll again is $\frac{1}{6} \cdot (-1) + \frac{2+3+4+5+6}{6} = \frac{19}{6} > 1$. This implies we should roll again if we receive a 1.

If we roll a 2, then with probability $\frac{1}{3}$ we lose \$2 (if we roll a 5 or 6). Otherwise, we receive $3, 4, 5$, or 6 with equal probability $\frac{1}{6}$. Therefore, the expected value upon rolling again is $\frac{1}{3} \cdot (-2) + \frac{3+4+5+6}{6} = \frac{7}{3} > 2$. This implies we should roll. If we roll a 3, then with probability $\frac{1}{2}$ we lose \$3 (rolling 4 or more). Otherwise, we earn $4, 5$, or 6 each with probability $\frac{1}{6}$. Therefore, our expected profit upon rolling again is $\frac{1}{2} \cdot (-3) + \frac{4+5+6}{6} = 1 < 3$. Therefore, for any value 3 or more, we should not roll again. This implies that our expected value on this game with this strategy is $\frac{\frac{19}{6} + \frac{14}{6} + 3 + 4 + 5 + 6}{6} = \frac{47}{12}$.

Answer:

4712

Reference:

https://www.quantguide.io/questions/unlucky-seven-i

True Statement

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street

Question:

On a sheet of paper, you have 100 statements written down. The first statement says, "at most 0 of these 100 statements are true." The second says, "at most 1 of these 100 statements are true." More generally, the nth says, "at most $n-1$ of these 100 statements are true. How many of the statements are true?

Hint:

We know that the first statement must be false, as if it was true, it would contradict itself that none of the statements are true. The trick here is to notice that for some threshold r, all the statements saying "at least r of the statements are true" must be true for everything at least that r, and false for everything below it.

Explanation:

We know that the first statement must be false, as if it was true, it would contradict itself that none of the statements are true. Therefore, let's consider the second statement. If it was false, then this implies that at least 2 statements are true. However, the statement after says that at most 2 statements are true. If that were to be true, then that claims exactly 2 statements would be true, of which one is itself. However, every statement that says at most $k > 2$ statements are true would also be true, causing a logical contradiction. Therefore, the third statement must be false.

The trick here is to notice that for some threshold r, all the statements saying "at least r of the statements are true" must be true for everything at least that r, and false for everything below it. For such an r, that statement being true implies that both at most and at least r statements are true, meaning exactly r statements are true. Namely, there must be $100 - r$ statements that are false and r statements that are true. To find that threshold r, we just equate the above, yielding $r = 50$ statements must be true. We can verify this by seeing that the 50 statements "at most $50, 51, \ldots, 99$ statements are true" must be true and other 50 are false.

Answer:

50

Reference:

https://www.quantguide.io/questions/true-statement

Party Groups II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

There are 50 guests at a party and they are making groups for a game. To do this they each write their name on a piece of paper and put it into a hat. One by one, each guest picks a name from the hat. Each guest will be a part of a group with the guest they pulled the name out of from the hat. If a guest pulls out their own name, they are in a group all by themselves. If Guest A pulls out Guest B's name, Guest B pulls out Guest C's name, and Guest C pulls out Guest A's name, they are all a part of the same closed group and no one else will be able to join them. What is the average size of a group? Round your answer to the nearest tenth.

Hint:

You discovered the average number of groups in Part I of the question. What is the intuitive answer, and how can you show it is correct?

Explanation:

An intuitive answer is that we discovered in Party Guests I that there are $\sum_{k=1}^{50} \frac{1}{k}$ groups on average, and since there are 50 people total, the average size of a given group is $\frac{50}{\sum_{k=1}^{50} \frac{1}{k}} \approx 11.1$. This is indeed correct. We show this using the "cycles" interpretation from the previous part of the question.

There are $n!/k$ $k-$cycles among the permutations, while there are $n! \cdot H_n$ total cycles in all of the permutations (adding up the cycles of each length), where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Therefore, the probability that a given cycle is a $k-$cycle is $\frac{n!/k}{n!H_n} = \frac{1}{kH_n}$. Therefore, the expected length of a cycle would be

$$\sum_{k=1}^{n} k \cdot \frac{1}{kH_n} = \frac{n}{H_n}$$

Answer:

11.1

Reference:

https://www.quantguide.io/questions/party-groups-ii

Planets Aligned

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

There is a solar system with three planets orbiting around the sun. One of them has a translation period of 40 years, another one of 60 years and another one of 90 years. Today, the three planets are aligned with the sun. When is the next time, in years, the three planets will be aligned with the sun?

Hint:

The planets can be aligned either on the same or opposite side of the sun. Cut all the periods in half.

Explanation:

The planets can be aligned either on the same or opposite side of the sun. Therefore, the periods are really $20, 30$, and 45. We want the least common multiple of these, which is simply 180.

Answer:

180

Reference:

https://www.quantguide.io/questions/planets-aligned

Coin Pair III

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Hint:

Four fair coins appear in front of you. You all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Explanation:

Let T be the total number of heads and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 3$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. Conditioned on the fact that we don't obtain TT, 2 of the 3 equally-likely outcomes result in us having 3 heads, while 1 of the 3 results in having 4 heads, so $\mathbb{E}[T \mid X = 2] = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 4 = \frac{10}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. With probability $\frac{2}{3}$ we end up with 2 heads, but we can iterate the process again from there on the remaining two tails and reach an expected $\frac{10}{3}$ heads. With probability $\frac{1}{3}$, we reach 3 heads and we're done. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{2}{3} \cdot \frac{10}{3} + \frac{1}{3} \cdot 3 = \frac{29}{9}$.

Lastly, if $X = 0$, with probability $\frac{2}{3}$, we end up with 1 head, so we can iterate the process again to have an expected number of heads of $\frac{29}{9}$. With probability $\frac{1}{3}$, we end up with 2 heads, and we iterate the process again to get an expected number of heads of $\frac{10}{3}$. Therefore,

$$\mathbb{E}[T \mid X = 0] = \frac{2}{3} \cdot \frac{29}{9} + \frac{1}{3} \cdot \frac{10}{3} = \frac{88}{27}$$

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 3 + \frac{6}{16} \cdot \frac{10}{3} + \frac{4}{16} \cdot \frac{29}{9} + \frac{1}{16} \cdot \frac{88}{27} = \frac{88}{27}$$

Answer:

8827

Reference:

https://www.quantguide.io/questions/coin-pair-iii

Infected Dinner II

Category: Brainteasers
Difficulty: Hard
Companies: Jane Street

Question:

There are 1000 people having dinner at a grand hall. One of them is known to be sick, while the other 999 are healthy. Each minute, each person talks to one other person in the room at random. However, as everyone is social, nobody talks to people they have previously talked to. In each pair, if one is sick and one is healthy, the healthy person is infected and becomes sick. Once a person becomes sick, they are assumed to be sick for the rest of the dinner. Find the maximum amount of time (in minutes) until every person in the hall becomes sick.

Hint:

Consider grouping the 1000 people into 25 groups of 40 people. In 25 rounds of 39 minutes each, every person in a given bubble talks to every other person in that bubble. Additionally, each of the other bubbles are paired up. Suppose the original sick person is in Bubble 25 and the last healthy person is in Bubble 24. In round k, we can have all the people in bubble b talk to all of the people in bubble $24 - b + k \bmod 25$, where 0 maps to 25. What happens with this?

Explanation:

Consider grouping the 1000 people into 25 groups of 40 people. In 25 rounds of 39 minutes each, every person in a given bubble talks to every other person in that bubble. Additionally, each of the other bubbles are paired up.

Suppose the original sick person is in Bubble 25 and the last healthy person is in Bubble 24. In round k, we can have all the people in bubble b talk to all of the people in bubble $24-b+k \bmod 25$, where 0 maps to 25. Therefore, we see that after k rounds, the sick people belong to people in Bubble 25 and Bubbles $1, 2, \ldots, k-1$. This is since at round 1, Bubble 25 talks with itself. Afterwards, it talks with Bubbles $1, 2, \ldots, k-1$. Therefore, Bubble 24 will be completely untouched until after the first 24 rounds, which is 936 minutes. Then, in the 937th minute, everyone in the 24th bubble talks to everyone in the 25th bubble, meaning all are infected in the first minute. Therefore, after 937 minutes, everyone is infected.

Answer:

937

Reference:

https://www.quantguide.io/questions/infected-dinner-ii

Dot Removal

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

A fair 6−sided die has the values $1 - 6$ on the sides. The values of one of the sides (selected uniformly at random) is decreased by 1. Find the probability an even value is rolled on this die.

Hint:

It is equally likely to select an even or odd value to decrease the value by 1 on. Subtracting one turns the side into one of opposite parity.

Explanation:

It is equally likely to select an even or odd value to decrease the value by 1 on. Subtracting one turns the side into one of opposite parity. If we select an even value to decrease, with probability $\frac{2}{6} = \frac{1}{3}$ we roll an even value. If we select an odd value to decrease, with probability $\frac{4}{6} = \frac{2}{3}$ we roll an even value. Therefore, the total probability is

$$\frac{\frac{1}{3} + \frac{2}{3}}{2} = \frac{1}{2}$$

Answer:

12

Reference:

https://www.quantguide.io/questions/dot-removal

Russian Roulette II

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two randomly placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. The two of you take turns pulling the trigger until the person who fires the gun loses. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between spinning and not spinning the barrel?

Hint:

Because your friend survived, if you don't spin, you lose a safe chamber. How can you use this to calculate the probability of survival given you don't spin?

Explanation:

If you don't spin the barrel, the probability that you survive is $\frac{3}{5}$, since one of the safe chambers has already been pulled by your friend. If you do spin the barrel, the probability that you survive is $\frac{4}{6}$, since any of the 6 chambers is possible, 2 of which are lethal. Thus, the difference in probability is:

$$\frac{4}{6} - \frac{3}{5} = \frac{1}{15}$$

Answer:

115

Reference:

https://www.quantguide.io/questions/russian-roulette-ii

No Rock

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

You play rock, paper, scissors with an opponent, but your opponent cannot play rock. Every time you win, you receive \$1, every time you lose, you lose \$1, and every time you draw, nothing happens. Assuming optimal play, what is your expected profit per round?

Hint:

Note that there is no point in playing paper, as paper doesn't defeat scissors nor paper. Therefore, suppose that you play rock with probability a and scissors with probability $1-a$. Furthermore, suppose that your opponent plays paper with probability b and scissors with probability $1-b$. Write a function $P(a,b)$ that represents your payout as a function of a and b. Find the equilibrium.

Explanation:

Note that there is no point in playing paper, as paper doesn't defeat scissors nor paper. Therefore, suppose that you play rock with probability a and scissors with probability $1-a$. Furthermore, suppose that your opponent plays paper with probability b and scissors with probability $1-b$. If $P(a,b)$ represents your expected profit on this game with a and b fixed as above, we have that

$$P(a,b) = ab(-1) + a(1-b)(1) + (1-a)b(1) + (1-a)(1-b)(0) = a + b - 3ab$$

Keeping b fixed, taking the partial derivative, we see that $\frac{\partial P}{\partial a} = 1 - 3b$. Similarly, taking the partial derivative in b yields that $\frac{\partial P}{\partial b} = 1 - 3a$. Setting these both equal to 0 yields that there is an equilibrium at $a = b = \frac{1}{3}$. Therefore, your expected profit per round is

$$P(1/3, 1/3) = \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

Answer:

13

Reference:

https://www.quantguide.io/questions/no-rock

Digit Multiplication I

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street

Question:

Let A be the set of all integers whose digits multiply to 96. Furthermore, let x and y be the minimal and maximal elements of A, respectively. What is $y - x$? Note that no digit can be 1, so that A is finite.

Hint:

What is the prime factorization of 96?

Explanation:

The prime factorization of 96 is $3 \cdot 32 = 3 \cdot 2^5$. Therefore, we need to have 5 twos and a three in our number accounted for. For the largest possible value, we should arrange the values in descending order left to right, as this would give the largest weight to digits of largest value. Therefore, our largest value is 322222. For the smallest value, note that $2^3 = 8$ and $2 \cdot 3 = 6$. Therefore, we condense down 4 of the twos and the three into an 8 and 6. We can't condense more, as otherwise those digits will be larger than 10, so the smallest value that can be made from 8, 2, and 6 is clearly 268. Thus, our answer is $322222 - 268 = 321954$.

Answer:

321954

Reference:

https://www.quantguide.io/questions/digit-multiplication-i

Coin Pair V

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you select any pair of coins to reconsider. One of the two coins can be turned over, while the other must be flipped again. If you already have 3 heads, you still must perform both actions. You iterate this process until you end up with all 4 coins being heads. A movement of a coin is when you either turn over or flip it. Find the expected number of coin movements needed to obtain 4 heads.

Hint:

Four fair coins appear in front of you. You all four at once and observe the outcomes of the coins. After seeing the outcomes, you select any pair of coins to reconsider. One of the two coins can be turned over, while the other must be flipped again. You iterate this process until you end up with all 4 coins being heads. A movement of a coin is when you either turn over or flip it. Find the expected number of coin movements needed to obtain 4 heads.

Explanation:

Let T be the total number of coin flips needed and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin. To set our notation, let e_i represent the additional number of flips needed to stop our process once we have i heads.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. This would mean $e_4 = 0$.

If $X = 3$, then we would be consider one heads and one tails. We should turn over the tails so that we have a guaranteed head i.e. we end up with either 3 or 4 heads after this process. Then, with equal probability, we would get a heads or tails on the remaining coin, so $e_3 = 2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$. Putting in $e_4 = 0$ yields that $e_3 = 4$. This means $\mathbb{E}[T \mid X = 3] = 8$.

If $X = 2$, then we are considering 2 tails, meaning we turn one over and flip the other. This is exactly the same scenario as above, as we are guaranteed to have 3 or 4 heads after with the same probabilities, so $e_2 = 4$ as well. This means $\mathbb{E}[T \mid X = 2]$
$= 8$.

If $X = 1$, then we consider 2 tails. We turn one over, and then we flip the other. We will end with either 2 or 3 heads after this iteration with equal probability. Therefore, $e_1 = 2 + \frac{1}{2}e_2 + \frac{1}{2}e_3$. As $e_2 = e_3 = 4$, $e_1 = 6$. This means $\mathbb{E}[T \mid X = 1] = 10$.

Lastly, if $X = 0$, we just consider 2 tails one again. We turn one over, and then we flip the other. We will end with either 1 or 2 heads after this iteration with equal probability, so $e_0 = 2 + \frac{1}{2}e_1 + \frac{1}{2}e_2$. Plugging in $e_1 = 6$ and $e_2 = 4$ yields $e_0 = 7$, so $\mathbb{E}[T \mid X = 0] = 11$.

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 8 + \frac{6}{16} \cdot 8 + \frac{4}{16} \cdot 10 + \frac{1}{16} \cdot 11 = \frac{135}{16}$$

Answer:

13516

Reference:

https://www.quantguide.io/questions/coin-pair-v

Matching Socks II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

6 distinct pairs of socks are in a drawer. You're in a hurry to pack for a trip, so you draw out 8 socks uniformly at random from the drawer. Find the expected number of pairs of socks that remain in the drawer.

Hint:

Set up indicators representing the events that each pair of socks is in the drawer after drawing out 8 socks and apply linearity of expectation.

Explanation:

Since we want to count the expected number of pairs that remain in the drawer, it makes sense to use indicators and linearity of expected here to count. Label the pairs of socks $1-6$ and let I_i be the indicator that pair i is still in the drawer after drawing out the pairs of socks. Then $T = I_1 + \cdots + I_6$ gives the total number of pairs of socks remaining in the drawer. By linearity of expectation and the exchangeability of the sock pairs, $\mathbb{E}[T] = 6\mathbb{E}[I_1]$.

$\mathbb{E}[I_1]$ is just the probability of the event that I_1 indicates. Namely, this is the probability that pair 1 is still in the drawer after drawing out the socks. To not pick any of pair 1 socks, we must pick 8 socks from the other 10, so there are $\binom{10}{8}$ ways to do this. There are $\binom{12}{8}$ ways to pick 8 socks in general, so the probability (and hence $\mathbb{E}[I_1]$) is $\frac{\binom{10}{8}}{\binom{12}{8}} = \frac{1}{11}$. Therefore, $\mathbb{E}[T] = \frac{6}{11}$.

Answer:

611

Reference:

https://www.quantguide.io/questions/matching-socks-ii

Die Rank

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Let $X_1, \ldots, X_5$ be the upfaces of 5 independent rolls of a fair 6−sided die with values $1-6$ on each side. Rank each of the following from largest to smallest in terms of value: a) $\mathbb{E}[X_1 X_2]$; b) $\mathbb{E}[X_1^2]$; c) $\mathbb{E}[X_{(3)}^2]$, where $X_{(3)}$ is the median of the 5 die rolls.

Let $a = 1, b = 2$, and $c = 3$. Answer with the integer corresponding to the concatenation of the order from largest to smallest. For example, if you believe $c > b > a$, answer with 321.

Hint:

Compute the first two values explicitly and then think about the distribution of the values of the median.

Explanation:

We can calculate $\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2] = (3.5)^2 = 12.25$ by direct calculation via independence of the rolls. Furthermore, we can calculate $\mathbb{E}[X_1^2] = \dfrac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} \approx 15.17$, so we know that $b > a$. We now need to find where c lies relative to these. It's worth just thinking about this intuitively, as the calculations are quite difficult.

By inspection, we note that the distribution here is skewed towards the middle (we would need 3 6s or 3 1s to get either extreme). Note that in the sum for $\mathbb{E}[X_1^2]$, we gain much more through the squared 6 than we lose in the squared 1. This ranks it below $\mathbb{E}[X_1^2]$.

To compare to $\mathbb{E}[X_1 X_2]$, we note if either one of the dice is low, it reduces our product greatly. This reduction occurs with very high probability, whereas the median is more likely to reach at least a moderate value (say 3 or 4), so the median is skewed more right. Therefore, we get that $b > c > a$, corresponding to the answer 231.

Answer:

231

Reference:

https://www.quantguide.io/questions/die-rank

Coin Pair II

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Hint:

Four fair coins appear in front of you. You all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Explanation:

Let T be the total number of heads and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 3$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. Conditioned on the fact that we don't obtain TT, 2 of the 3 equally-likely outcomes result in us having 3 heads, while 1 of the 3 results in having 4 heads, so $\mathbb{E}[T \mid X = 2] = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 4 = \frac{10}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. With probability $\frac{2}{3}$ we end up with 2 heads, but we can iterate the process again from there on the remaining two tails and reach an expected $\frac{10}{3}$ heads. With probability $\frac{1}{3}$, we reach 3 heads and we're done. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{2}{3} \cdot \frac{10}{3} + \frac{1}{3} \cdot 3 = \frac{29}{9}$.

Lastly, if $X = 0$, with probability $\frac{2}{3}$, we end up with 1 head, so we can iterate the process again to have an expected number of heads of $\frac{29}{9}$. With probability $\frac{1}{3}$, we end up with 2 heads, and we iterate the process again to get an expected number of heads of $\frac{10}{3}$. Therefore,

$$\mathbb{E}[T \mid X = 0] = \frac{2}{3} \cdot \frac{29}{9} + \frac{1}{3} \cdot \frac{10}{3} = \frac{88}{27}$$

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 3 + \frac{6}{16} \cdot \frac{10}{3} + \frac{4}{16} \cdot \frac{29}{9} + \frac{1}{16} \cdot \frac{88}{27} = \frac{88}{27}$$

Answer:

8827

Reference:

https://www.quantguide.io/questions/coin-pair-ii

Card Diff

Category: Brainteasers
Difficulty: Hard
Companies: Jane Street

Question:

You have all the clubs from a standard deck (13 cards) and you can choose 2 from the deck and get paid the product of their values. All face cards are considered to have value 0 and Ace is considered to have value 1. You can pay \$1 to reveal the difference of any two cards you choose. You can exercise this "difference" option between any two cards as many times as you would like. However, in the end, you must select two cards be to the ones you take the product of for your payout. Under a rational strategy, what is the maximum guaranteed profit you can achieve?

Hint:

Considering fixing your first card and differencing with the other 12 cards. Can you optimize in any way?

Explanation:

The claim here is that one can always locate the places of the 9 and 10 within 11 card draws. Therefore, the payout would be $9 \cdot 10 - 11 = 79$.

Suppose you pick one card. You start paying \$1 to reveal the difference between that card and each other card of the remaining 12. We can split up into two cases: The first card you selected does have value 0 or does not have value 0.

If the first card is a 0, then at some point in the first 11 differences, you will either two differences of 0 (in which you completely identify your card as a 0) and a difference of 9 or you obtain a difference of 10 (or both!). You would only need to spend \$11 at worst, as if one of the two scenarios doesn't occur in the first 11 differences, we know that the 9 or 10 (whichever difference wasn't obtained) is the last card whose value you didn't difference yet. This means that in the worst case scenario you only need to ask for the difference between your card and 11 more cards out of the 12 total cards, which yields \$11 payment.

If the value of the first card is not 0, we know that there are 3 cards with value 0 still remaining in the deck. You can identify which card you picked as soon as you obtain 2 face cards, as their values are 0 and you would be told the same number twice. Therefore, in this case, you also need to pay only \$11 at most in order to be able to have full knowledge of where the 9 and the 10 are located.

Answer:

79

Reference:

https://www.quantguide.io/questions/card-diff

Party Groups

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

There are 50 guests at a party and they are making groups for a game. To do this they each write their name on a piece of paper and put it into a hat. One by one, each guest picks a name from the hat. Each guest will be a part of a group with the guest they pulled the name out of from the hat. If a guest pulls out their own name, they are in a group all by themselves. If Guest A pulls out Guest B's name, Guest B pulls out Guest C's name, and Guest C pulls out Guest A's name, they are all a part of the same closed group and no one else will be able to join them. How many groups will there be on average? Round your answer to the nearest tenth.

Hint:

Think of this scenario as each guest being a cable extender there the input end is their hand picking a name out of the hat and the output end is the piece of paper with their name on it.

Explanation:

You can also think of this scenario as each guest acting like a cable extender. Each cable has an "input" side and an "output" side. When the first guest picks a name and they pick their own name, its like connecting the two ends of the same wire together. Otherwise you are connected two wires together and making one larger wire with one input end and one output end.
A more algebraically-inclined person may want to think of this scenario as the expected number of cycles of a random $50-$permutation.Whichever way you want to think about it, the math stays the same. When the first guest picks a name, there is a $\frac{1}{50}$ chance that the number of groups does not change and a $\frac{49}{50}$ chance that the number of groups effectively decrease by one (because two groups join together). When the next guest picks a number, the given scenario is very similar whether the first guest formed their own group or connected with another guest. Both scenarios have 49 groups or "wires". The only difference is that there's one closed group if the first guest picks their own name versus no closed groups if they pick another guest's name. If we keep continuing this for every guest, we add a closed group with probability $\frac{1}{k}$ where k are the number of names still left in the hat. Thus the answer is

$$\sum_{n=1}^{50} \frac{1}{n} \approx 4.5$$

Answer:

4.5

Reference:

https://www.quantguide.io/questions/party-groups

Take And Roll I

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

You are given a fair 20−sided die and 100 actions in a game. The die starts with upface 1. The two options you can perform are to roll and to take. Performing a roll re-rolls the current upface of the die. Performing a take allows you to cash out the current upface of the die. Note that the game does not end when you perform a take. However, you must roll the die again before doing another take. Your strategy is to accept any number that is at least some threshold n. This n must be decided in advance and is fixed for the entire game. Assuming rational play in selecting n, find your expected payout.

Hint:

The optimal strategy will once again to be accept any number that it at least some threshold n. To get a baseline to compare to, suppose we just roll and take in an alternating fashion. We will be able to perform this 50 times (as each is one action) and the expected value per roll is 10.5, so our expected payout would be $50 \cdot 10.5 = 525$ with this strategy. Write the expected payoff as a function of n.

Explanation:

To get a baseline to compare to, suppose we just roll and take in an alternating fashion. We will be able to perform this 50 times (as each is one action) and the expected value per roll is 10.5, so our expected payout would be $50 \cdot 10.5 = 525$ with this strategy.

Now, let's write the expected payoff as a function of n. If we accept any value at least n, then the expected value we roll given we accept is $\frac{20+n}{2}$. There are $21 - n$ values that are at least value n, so the probability on each roll that we obtain a value at least n is $\frac{21-n}{20}$. As this probability is constant between rolls, the expected number of terms of obtain a value at least n is $\frac{20}{21-n}$. However, we now must claim it after we obtain a roll satisfying this threshold, so the expected number of turns needed to roll and claim the money is $\frac{20}{21-n} + 1$. Therefore, on average, we are able to roll and claim the money $\frac{100}{\frac{20}{21-n}+1}$ times in the game, as it takes us that many turns on average to roll and claim and we have 100 total turns. Lastly, this implies our expected payout is $f(n) = \frac{100}{\frac{20}{21-n}+1} \cdot \frac{20+n}{2}$, as we multiply the expected number of times we are paid by the expected payout per time.

To find the n maximizing this, one can treat f as continuous and use the derivative of it to find the optimal n. The details of taking the derivative messy and not enlightening, so the steps are excluded. However, after using the basic rules and simplifying,

$$f'(n) = \frac{50}{(n-41)^2} \cdot \left(n^2 - 82n + 461\right) = 0$$

The roots of the polynomial are $n_{1,2} = 41 \pm 2\sqrt{305}$. The root adding $2\sqrt{305}$ is larger than 20, so $n^* = 41 - 2\sqrt{305}$ must be the maximizer. As $17 < \sqrt{305} < 18$ and our optimal n must be an integer, we can test $n = 5, 6, 7$ to see which gives us the largest expected payout.

Plugging all three of these in reveals $n = 6$ maximizes $f(n)$ with payout $\dfrac{3900}{7}$.

Answer:

39007

Reference:

https://www.quantguide.io/questions/take-and-roll-i

Coin Pair I

Category: Probability
Difficulty: Medium
Companies: Jane Street

Question:

Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Hint:

Four fair coins appear in front of you. You all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Explanation:

Let T be the total number of heads and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 3$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. Conditioned on the fact that we don't obtain TT, 2 of the 3 equally-likely outcomes result in us having 3 heads, while 1 of the 3 results in having 4 heads, so $\mathbb{E}[T \mid X = 2] = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 4 = \frac{10}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. With probability $\frac{2}{3}$ we end up with 2 heads, but we can iterate the process again from there on the remaining two tails and reach an expected $\frac{10}{3}$ heads. With probability $\frac{1}{3}$, we reach 3 heads and we're done. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{2}{3} \cdot \frac{10}{3} + \frac{1}{3} \cdot 3 = \frac{29}{9}$.

Lastly, if $X = 0$, with probability $\frac{2}{3}$, we end up with 1 head, so we can iterate the process again to have an expected number of heads of $\frac{29}{9}$. With probability $\frac{1}{3}$, we end up with 2 heads, and we iterate the process again to get an expected number of heads of $\frac{10}{3}$. Therefore,

$$\mathbb{E}[T \mid X = 0] = \frac{2}{3} \cdot \frac{29}{9} + \frac{1}{3} \cdot \frac{10}{3} = \frac{88}{27}$$

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 3 + \frac{6}{16} \cdot \frac{10}{3} + \frac{4}{16} \cdot \frac{29}{9} + \frac{1}{16} \cdot \frac{88}{27} = \frac{88}{27}$$

Answer:

8827

Reference:

https://www.quantguide.io/questions/coin-pair-i

Take And Roll II

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

You are given a fair 20−sided die and 100 actions in a game. The die starts with upface 1. The two options you can perform are to roll and to take. Performing a roll re-rolls the current upface of the die. Performing a take allows you to cash out the current upface of the die. Note that the game does not end when you perform a take. However, you must roll the die again before doing another take. Your strategy is to accept any number that is at least some threshold n. This n must be decided in advance and is fixed for the entire game. Assuming rational play in selecting n, find your expected payout.

Hint:

The optimal strategy will once again to be accept any number that it at least some threshold n. To get a baseline to compare to, suppose we just roll and take in an alternating fashion. We will be able to perform this 50 times (as each is one action) and the expected value per roll is 10.5, so our expected payout would be $50 \cdot 10.5 = 525$ with this strategy. Write the expected payoff as a function of n.

Explanation:

To get a baseline to compare to, suppose we just roll and take in an alternating fashion. We will be able to perform this 50 times (as each is one action) and the expected value per roll is 10.5, so our expected payout would be $50 \cdot 10.5 = 525$ with this strategy.

Now, let's write the expected payoff as a function of n. If we accept any value at least n, then the expected value we roll given we accept is $\dfrac{20+n}{2}$. There are $21-n$ values that are at least value n, so the probability on each roll that we obtain a value at least n is $\dfrac{21-n}{20}$. As this probability is constant between rolls, the expected number of terms of obtain a value at least n is $\dfrac{20}{21-n}$. However, we now must claim it after we obtain a roll satisfying this threshold, so the expected number of turns needed to roll and claim the money is $\dfrac{20}{21-n}+1$. Therefore, on average, we are able to roll and claim the money $\dfrac{100}{\frac{20}{21-n}+1}$ times in the game, as it takes us that many turns on average to roll and claim and we have 100 total turns. Lastly, this implies our expected payout is $f(n) = \dfrac{100}{\frac{20}{21-n}+1} \cdot \dfrac{20+n}{2}$, as we multiply the expected number of times we are paid by the expected payout per time.

To find the n maximizing this, one can treat f as continuous and use the derivative of it to find the optimal n. The details of taking the derivative messy and not enlightening, so the steps are excluded. However, after using the basic rules and simplifying,

$$f'(n) = \frac{50}{(n-41)^2} \cdot \left(n^2 - 82n + 461\right) = 0$$

The roots of the polynomial are $n_{1,2} = 41 \pm 2\sqrt{305}$. The root adding $2\sqrt{305}$ is larger than 20, so $n^* = 41 - 2\sqrt{305}$ must be the maximizer. As $17 < \sqrt{305} < 18$ and our optimal n must be an integer, we can test $n = 5, 6, 7$ to see which gives us the largest expected payout.

Plugging all three of these in reveals $n = 6$ maximizes $f(n)$ with payout $\frac{3900}{7}$.

Answer:

39007

Reference:

https://www.quantguide.io/questions/take-and-roll-ii

Same Month

Category: Brainteasers
Difficulty: Easy
Companies: Jane Street

Question:

What is the minimum amount of people needed in a group to guarantee at least two of them have a birthday in the same month?

Hint:

There are 12 months in a year.

Explanation:

Given that there are 12 months, if we have 13 people, by the Pigeonhole Principle, two must share a birthday in the same month.

Answer:

13

Reference:

https://www.quantguide.io/questions/same-month

Ramen Bowl

Category: Probability
Difficulty: Hard
Companies: Jane Street, HRT

Question:

There are 100 noodles in your bowl of ramen. You take the ends of two noodles uniformly at random and connect the two, putting the connected noodle back into the bowl and continuing until there are no ends left to connect. On average, how many circles will you create? Round to the nearest whole number.

Hint:

How many circles do you create when there is only one noodle? Two noodles? n noodles?

Explanation:

Let $f(n)$ be the number of circles creates from n starting noodles- we are looking for $E[f(100)]$. When $n = 1$, it is clear that $E[f(n)] = 1$ since we connect the two ends of the only noodle. For $n = 2$, there are a total of $\binom{4}{2} = 6$ possible combinations to connect the first two ends. Of these 6 combinations, 4 will yield a single noodle ($f(1)$) and 2 will yield a noodle and a circle ($1 + f(1)$)). Hence,

$$E[f(2)] = \frac{4}{6} \times E[f(1)] + \frac{2}{6} \times (1 + E[f(1)]) = 1 + \frac{1}{3}$$

For $n = 3$, there are a total of $\binom{6}{2} = 15$ possible combinations to connect the first two ends. Of these 15 combinations, 12 will yield will yield two noodles ($f(2)$) and 3 will yield two noodles and a circle ($1 + f(2)$)). Hence,

$$E[f(3)] = \frac{12}{15} \times E[f(2)] + \frac{3}{15} \times (1 + E[f(2)]) = 1 + \frac{1}{3} + \frac{1}{5}$$

The pattern is now noticeable: $E[f(n)] = 1 + \frac{1}{3} + ... + \frac{1}{2n-1}$, and thus $E[f(100)] \approx 3$.

Answer:

3

Reference:

https://www.quantguide.io/questions/ramen-bowl

Face Value

Category: Probability
Difficulty: Easy
Companies: Jane Street

Question:

Ten cards with values $1 - 10$ are face down in front of you. You select one card at random and look at it. You can either choose the payout of \$3.50 or the face value of the card. What is the fair value of this game?

Hint:

Use Law of Total Expectation to condition on the outcome of the card.

Explanation:

With probability $\frac{3}{10}$, the card is worth less than the payout, so you should choose the payout of 3.5. Otherwise, you should take the face value of the card. Therefore, the expected value is

$$\frac{3}{10} \cdot \frac{7}{2} + \frac{7}{10} \cdot \frac{4+5+6+7+8+9+10}{7} = \$5.95$$

.

Answer:

5.95

Reference:

https://www.quantguide.io/questions/face-value

Distinct Date I

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street, Old Mission

Question:

Find the next date where all of the digits, when expressed in the form $MM/DD/YYYY$ are distinct. For example, 01/23/4567 would be a valid date. Express your answer in the form $MMDDYYYY$.

Hint:

We want to minimize the year, as that will be the biggest influence on how soon it is. Note that the first M is either 0 or 1. If the first M is 0, then the first day must be 1 or 2 OR the day is 31 (30 doesn't work since we already have a 0. If the first M is 1, then either the second M is a 0 OR the second M is a 2 and the day contains a 0 somewhere.

Explanation:

We want to minimize the year, as that will be the biggest influence on how soon it is. Note that the first M is either 0 or 1. If the first M is 0, then the first day must be 1 or 2 OR the day is 31 (30 doesn't work since we already have a 0. If the first M is 1, then either the second M is a 0 OR the second M is a 2 and the day contains a 0 somewhere. Therefore, these both imply that the 0 and either the 1 or 2 must be used in the $MMDD$ portion. Since we ideally don't want to skip 1000 years, we should put the 2 in the year portion, so thus far, we have that our date is in the form

$$0M/1D/2YYY$$

After this, it's just an objective to minimize the rest of the digits. Namely, $YYY = 345$, as that is the smallest number that can be made with the remaining digits. Then, we want to minimize the month, so $M = 6$. Then, lastly, $D = 7$, as that is the smallest remaining number. Our answer is therefore

$$06/17/2345$$

Answer:

06172345

Reference:

https://www.quantguide.io/questions/distinct-date-i

Distinct Date II

Category: Brainteasers
Difficulty: Medium
Companies: Jane Street, Old Mission

Question:

Find the most recent date where all of the digits, when expressed in the form $MM/DD/YYYY$ are distinct. For example, 01/23/4567 would be a valid date. Express your answer in the form $MMDDYYYY$.

Hint:

Our goal now is the maximize the year to make it as close as possible to our present date. The biggest thing to first note is that the year must start with a 1. Why?

Explanation:

Our goal now is the maximize the year to make it as close as possible to our present date. The biggest thing to first note is that the year must start with a 1. Suppose that it started with a 2. We know that the first M is either a 0 or 1. If it is a 0, then the day either starts with a 1 or is 31. Note that the other M can't be a 1 in this case. Otherwise, that would eliminate the day possibilities. If the month starts with a 1, then the day either contains 0 or is 30. However, since the 0 is taken already in the $MMDD$ part, regardless of what we start with, we can't get a year that is in the past.

Therefore, the year must start with a 1. To maximize the year, we can just fix the rest as the three largest integers. Namely, 987, so our year is 1987. Since the year starts with 1, the month must start with 0. Then, we choose the largest month integer remaining, which is 6. Then, we must pick the largest possible day among the integers remaining. We can see that 25 is the largest possible day, as anything starting with 3 is not possible. Therefore, our date is

06/25/1987

Answer:

06251987

Reference:

https://www.quantguide.io/questions/distinct-date-ii

Optimal Marbles I

Category: Probability
Difficulty: Hard
Companies: Jane Street

Question:

Two players, say A and B, play the following game: Both players have 100 marbles and may put anywhere between 1 and 100 marbles in the box each. This decision is not revealed to the other player. Then, they draw 2 marbles with replacement between trials. If the marble belongs to A, then assuming that A put a marbles in the box, A is paid $100 - a$ monetary units from a third party. Similarly if the marble belongs to B, then assuming B but b marbles in the box, B is paid $100 - b$ monetary units from a third party. Assume both players play optimally. Find the expected total payout of player A.

Hint:

If the strategy is optimal, then if this game were to be repeated many times, they would not change their strategy. Therefore, the optimal strategy for the game where there are 2 consecutive marble draws is the same as the optimal strategy for the game with one draw. Then, we just multiply the expected profit by 2 to represent the two draws. Furthermore, as this game is symmetric for the two players, their optimal strategy will be the same. Set up a function $A(a, b)$ the represents the expected payout if A puts in a marbles and b puts in b marbles. Find the equilibrium.

Explanation:

We can first make some simplifications to the game. Firstly, if the strategy is optimal, then if this game were to be repeated many times, they would not change their strategy. Therefore, the optimal strategy for the game where there are 2 consecutive marble draws is the same as the optimal strategy for the game with one draw. Then, we just multiply the expected profit by 2 to represent the two draws. Furthermore, as this game is symmetric for the two players, their optimal strategy will be the same. This point will be important later.

Let $A(a, b)$ be the expected profit that A obtains with player A putting in a balls and B putting in b balls. Namely, for the one draw game,

$$A(a, b) = \frac{a}{a+b} \cdot (100 - a)$$

As player a draws his ball with probability $\frac{a}{a+b}$ and $100 - a$ is the payout. Let's fix b and find the a that is the best response to this b. In other words, given b, what a optimizes $A(a, b)$? To do this, we take the partial derivative of $A(a, b)$ in a and treat a as continuous for now. We will then account for discreteness at the end.

This yields that

$$\frac{\partial}{\partial a} A(a, b) = -\frac{a^2 + 2ba - 100b}{(a+b)^2} = 0 \iff a^2 + 2ba - 100b = 0$$

Solving the above with the quadratic equation yields that $a^* = \dfrac{-2b \pm \sqrt{4b^2 + 400b}}{2}$. However, the $-$ root results in a negative value, so $a^* = \sqrt{b^2 + 100b} - b$ is the best response for player A if player B puts b marbles in. Similarly, as this game is symmetric, the optimal response for player B if player A puts a marbles in is $b^* = \sqrt{a^2 - 100a} - a$.

To find the optimal strategy for each player, this means that we need to find the combo (a^*, b^*) such that neither of the players can do better by adjusting their strategy. We already are aware from before that $a^* = b^*$ by the symmetry of the game. Therefore, to solve for this, we just substitute in b as $b^* = a^*$ in the first equation. This yields we can say that

$$a^* = \sqrt{(a^*)^2 + 100a^*} - a^* \iff 4(a^*)^2 = (a^*)^2 + 100a^* \iff a^*(3a^* - 100) = 0 \iff a^* = 0, \frac{100}{3}$$

As 0 is not possible, we conclude that $a^* = b^* = \dfrac{100}{3}$ is the optimal strategy. However, this is not actually possible, as our marbles must be an integer value. Therefore, we should test $(33, 33)$ and $(34, 34)$ to see if they are Nash equilibria.

For $(33, 33)$, the expected payout for one draw for each player is $\dfrac{67}{2}$. One can check that by varying a and keeping b fixed at 33, player A can't do any better. Therefore, $(33, 33)$ is a Nash equilibrium. For $(34, 34)$, the expected payout is 33. However, one can also verify that the expected payout for $(33, 34)$ is also 33. However, this can't be an equilibrium, as b should change to 33 marbles to yield higher expected payout. Thus, while $(34, 34)$ is also a Nash equilibrium, $(33, 33)$ is preferrable because of the higher expected payout. This means that the optimal strategy is for both players to place 33 marbles and have a total expected payout of $\dfrac{67}{2} \cdot 2 = 67$.

Answer:

67

Reference:

https://www.quantguide.io/questions/optimal-marbles-i

4 Die Sum

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

Calculate the probability that when we roll 4 fair $6-$sided dice, the sum of their upfaces is 20.

Hint:

Count the different permutations that sum of 20. Alternatively, condition on the sum of the first 2 dice.

Explanation:

Clearly there are $6^4 = 1296$ total outcomes of the 4 dice rolls. We need to divide this up into all of the permutations that sum to 20, as well as count all of their arrangements. The permutations that sum to 20 are

$$6, 6, 6, 2; 6, 6, 5, 3; 6, 6, 4, 4; 6, 5, 5, 4; 5, 5, 5, 5$$

by brute force. Note that we now need to count all of the arrangements for each combination, as we created our sample space so that it include all 4-tuples. For $6, 6, 6, 2$, there are 4 permutations corresponding to where the 2 is put. For $6, 6, 5, 3$, there are 12 permutations corresponding to just dividing 4! (total permutations) by 2! for the two sixes. By the same idea, $6, 6, 4, 4$ has 6 permutations, $6, 5, 5, 4$ has 12 permutations, and $5, 5, 5, 5$ has 1 permutation. Adding all of these up, we get 35 possible permutations that add to 20. Therefore, the probability is $\frac{35}{1296}$.

Answer:

351296

Reference:

https://www.quantguide.io/questions/4-die-sum

Word Shift II

Category: Probability
Difficulty: Medium
Companies: Old Mission

Question:

Find the number of anagrams of $BOOLAHUBBOO$ that have at least 2 B's before the first O.

Hint:

There are 3 B and 4 O in the word above. There are $\binom{11}{7}$ ways to pick the locations of these 7 letters in our anagram. How many anagrams of $BBBOOOO$ have at least 2 B's before the first O?

Explanation:

There are 3 B and 4 O in the word above. There are $\binom{11}{7}$ ways to pick the locations of these 7 letters in our anagram. However, there are 5 anagrams of $BBBOOOO$ that have at least 2 of the B's before all of the O's. Namely, you can move one B around among the 4 O's, yielding 5 total letters to place. You just select one of those spots to be B and those yield the 5 anagrams. Afterwards, the other 4 letters $LAHU$ can be arranged in the remaining 4 blanks completely at will in 4! ways. Therefore, the answer is

$$5 \cdot 4! \cdot \binom{11}{7} = 5 \cdot \frac{11!}{7!} = 39600$$

Answer:

39600

Reference:

https://www.quantguide.io/questions/word-shift-ii

Colorful Bracelet

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

How many unique bracelet configurations can be made with 3 red and 3 blue beads? Configurations that can be made as rotations of other configurations are considered indistinguishable. Do not consider symmetry over flipping the bracelet. For example,

$$RRGRGG \neq RRGGRG$$

Hint:

Consider cases based on the number of distinct regions that the red beads form.

Explanation:

There are three cases to consider here. The first is that all three red beads are together. In this case, we get exactly 1 distinguishable bracelet, as the others are just rotations. The second case is that we have 2 distinct regions i.e. 2 beads are together separated from the last one. This means that blue beads are at both ends of the 2 consecutive red bead region. Therefore, you get 2 distinct bracelets by ordering the last red and blue bead. The last case is that all of the red bead are separated from one another, in which case we also get 1 bead that alternates in color. Adding these up yields 4 bracelets.

Answer:

4

Reference:

https://www.quantguide.io/questions/colorful-bracelet

Unfriendly

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

John has 8 friends. He will invite 5 to his party. However, two of his friends have beef and they refuse to attend together. How many possible combinations of guests are possible given this constraint?

Hint:

The easiest way to compute this is to calculate the total number of ways to invite people and then remove those that invite both.

Explanation:

The easiest way to compute this is to calculate the total number of ways to invite people and then remove those that invite both. There are $\binom{8}{5} = 56$ total ways to select 5 people to invite. To count the number of ways with both of the friends that refuse to attend together, fix both of them among the 5 selected. We then need to select 3 more from the remaining 6, so this yields $\binom{6}{3} = 20$ combinations. Therefore, we have $56 - 20 = 36$ ways to invite the people.

Answer:

36

Reference:

https://www.quantguide.io/questions/unfriendly

Jellybean Jar I

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

A pack contains 6 red and 10 blue jellybeans. A child wants to eat 4 jellybeans and grabs into the pack to select 4 jellybeans one-by-one uniformly at random without replacement. Find the probability that the first two jellybeans are red and the last two are blue.

Hint:

To get two red jellybeans at the start, the child has $6 \cdot 5 = 30$ options of red jellybean. How many options do they have for the blue jellybeans?

Explanation:

To get two red jellybeans at the start, the child has $6 \cdot 5 = 30$ options of red jellybean. Then, for the two blues, the child has $10 \cdot 9 = 90$ options. Thus, there are 2700 ways for the child to obtain this sequence. The total number of ways to draw 4 jellybeans without replacement is $16 \cdot 15 \cdot 14 \cdot 13$. Therefore, our probability is $\frac{2700}{16 \cdot 15 \cdot 14 \cdot 13} = \frac{45}{728}$.

Answer:

45728

Reference:

https://www.quantguide.io/questions/jellybean-jar-i

Word Shift I

Category: Probability
Difficulty: Medium
Companies: Old Mission

Question:

Find the number of anagrams of $BOOLAHUBBOO$ that have all of the B's before the first O.

Hint:

There are 3 B and 4 O in the word above. There are $\binom{11}{7}$ ways to pick the locations of these 7 letters in our anagram.

Explanation:

There are 3 B and 4 O in the word above. There are $\binom{11}{7}$ ways to pick the locations of these 7 letters in our anagram. However, there is only one anagram of $BBBOOOO$ that has all of the B's before all of the O's. Namely, it is the one listed there. Afterwards, the other 4 letters $LAHU$ can be arranged in the remaining 4 blanks completely at will in 4! ways. Therefore, the answer is

$$4! \cdot \binom{11}{7} = \frac{11!}{7!} = 7920$$

Answer:

7920

Reference:

https://www.quantguide.io/questions/word-shift-i

English And History Majors

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

At a local university, 30% of students major in English, 45% major in History, and 15% double major in both. What proportion of students major in neither English nor History?

Hint:

We are able to obtain the proportion of students majoring in just English by subtracting out the intersection proportion from the English proportion. Alternatively, use inclusion-exclusion.

Explanation:

We are able to obtain the proportion of students majoring in just English by subtracting out the intersection proportion from the English proportion. This means 15% of students major in only English. Similarly, 30% of students major in just History. Adding all the single major students and the double major students, we get 60% of students major in either English or History. Therefore, 40% do not major in either of the subjects.

Answer:

40

Reference:

https://www.quantguide.io/questions/english-and-history-majors

Conditional 8 Sum

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

Two fair $6-$sided dice were rolled and the sum of the upfaces is 8. Find the probability that one of the dice showed 6.

Hint:

List out all the ways to obtain a sum of 8. How many have a 6?

Explanation:

There are 5 outcomes of the dice that show a sum of 8. These are $(6,2),(5,3),(4,4),(3,5)$, and $(2,6)$. Of those, two of them are $(6,2)$ and $(2,6)$. Therefore, our answer is simply $\frac{2}{5}$.

Answer:

25

Reference:

https://www.quantguide.io/questions/conditional-8-sum

Pizza Munching

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

Garrett loves eating pizza, but enjoys savoring the slices too. The time it takes for him to eat a pizza slice is uniformly distributed between 1 and 5 minutes. If he has been eating for 2 minutes and still isn't done, find the probability that he finishes the slice in the next minute.

Hint:

Let T be the amount of time that Garrett takes to eat his slice of pizza. We know that $T \sim \text{Unif}(1, 5)$ from the question. We know he has spent at least 2 minutes eating his slice, and we want the probability he finishes in the next minute. What specific probability are we interested in?

Explanation:

Let T be the amount of time that Garrett takes to eat his slice of pizza. We know that $T \sim \text{Unif}(1, 5)$ from the question. We know he has spent at least 2 minutes eating his slice, and we want the probability he finishes in the next minute, so we want $\mathbb{P}[T \leq 3 \mid T \geq 2]$. By the definition of conditional probability, this is

$$\frac{\mathbb{P}[2 \leq T \leq 3]}{\mathbb{P}[T \geq 2]} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Answer:

13

Reference:

https://www.quantguide.io/questions/pizza-munching

Same Side

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

What is the probability of flipping the same side of a coin 4 times a row?

Hint:

The first flip will dictate what side the rest of the flips must be.

Explanation:

The first flip will dictate what side the rest of the flips must be. Therefore, this flip is arbitrary. Now, once flipped, the other three flips match that first result each with probability $\frac{1}{2}$, so our answer is $\frac{1}{2^3} = \frac{1}{8}$.

Answer:

18

Reference:

https://www.quantguide.io/questions/same-side

Paying Bail

Category: Probability
Difficulty: Medium
Companies: Old Mission

Question:

You have been arrested and need to post bail. You have a bag of $100 bills of which one half are real and the other half are fake. You need to pay $500 for bail. Every time you give the jail a fake bill, the funds you have given are given back to you and you need to start again. How many bills will you have to pull out of the bag on average? Assume the probability of drawing each type of bill is constant per trial.

Hint:

Can you translate the states of this problem into equations?

Explanation:

This is a Markov Chain question and you can tell by the existence of states in this question (state of given $0, $100, $200, $300, $400, and $500). Let E_0 be the number of bills you have to give on average given the state of having given the jailor $0 or starting over after giving a fake bill. Then E_i for i in $\{100, 200, 300, 400, 500\}$ be the states of having given the jailor $\$i$ (all real) in a row (again, the average number of bills needed to be given). The equations become:

$$E_0 = \frac{1}{2}E_1 + \frac{1}{2}E_0 + 1$$

$$E_1 = \frac{1}{2}E_2 + \frac{1}{2}E_0 + 1$$

$$E_2 = \frac{1}{2}E_3 + \frac{1}{2}E_0 + 1$$

$$E_3 = \frac{1}{2}E_4 + \frac{1}{2}E_0 + 1$$

$$E_4 = \frac{1}{2}E_5 + \frac{1}{2}E_0 + 1$$

$$E_5 = 0$$

When you solve for E_0 (the state of expectation we are trying to find), you get 62 bills needing to be pulled out on average.

Answer:

62

Reference:

https://www.quantguide.io/questions/paying-bail

Dangerous Doubles

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

Two fair coins are flipped at once. You receive \$2 if exactly one heads that appears, but you lose \$7 if you flip two heads. What is your expected profit/loss on this game if you play once?

Hint:

Two fair coins are flipped at once. You receive \$2 for each heads that appears but you lose \$7 if you flip two heads. What is your expected profit/loss on this game if you play once?

Explanation:

The four possible outcomes of this game are TT, HT, TH, and HH. TT gives you \$0, as there are no heads. HT and TH both give you \$2, as there is exactly one head. HH makes you pay \$7 by the information in the question. As all four outcomes are equally likely, our expected profit/loss on the game is $\frac{0+2+2-7}{4} = -\frac{3}{4}$.

Answer:

34

Reference:

https://www.quantguide.io/questions/dangerous-doubles

Clarences Bread

Category: Probability
Difficulty: Hard
Companies: Old Mission

Question:

Clarence is getting this bread. Literally. Clarence has 7 small loaves and 3 large loaves of bread in a bag. He draws in his bag and pulls out a loaf of bread uniformly at random, one-by-one. If it is a small loaf, Clarence eats it. If it is a large loaf, Clarence cuts it into two small loaves. He eats one one of them and puts the other one back in the bag. Find the expected number of small loaves that Clarence eats until the bag contains only small loaves left, inclusive of the small loaf he eats as a result of the last large loaf he draws.

Hint:

Use the "First Ace" approach three separate times on the first appearances of the next large loaf after previous ones have been eaten.

Explanation:

Here, we are looking for the expected number of draws needed until Clarence grabs all 3 large loaves of bread from the bag. Since we are doing this drawing without replacement, this implies we should implement some type of "First Ace" approach. However, the issue here is that when Clarence draws a large loaf, he replaces it by another small loaf. Therefore, this isn't exactly like the paradigm above, as when you draw an Ace, it is not replaced in any way. Our new strategy is to thus consider the expected number of draws needed to see each consecutive large loaf.

First, let's find the number of loaves Clarence eats until (and including) his first selection of a large loaf. There are 3 large loaves and 7 small loaves. Therefore, we can treat the large loaves as our "dividers" and the 7 small as filler between them. The loaves split up our drawing process into 4 equally-sized components (in expectation).

Therefore, we expect $\frac{7}{4}$ small loaves before the first large loaf. Afterwards, Clarence eats one of the two small loaves that results from the large loaf and then replaces the other in the bag. Therefore, he has eaten a total of $\frac{7}{4} + 1 = \frac{11}{4}$ loaves thus far. Additionally, there are $\frac{7}{4} - 1 = \frac{3}{4}$ fewer small loaves in the bag on average.

Now, we apply the same paradigm again but on $7 - \frac{3}{4} = \frac{25}{4}$ small loaves and 2 large loaves. We get 3 equally-sized regions in expectation from the two large loaves, so there should be $\frac{25}{12}$ small loaves coming before the next large loaf. Therefore, Clarence eats $\frac{25}{12} + 1 = \frac{37}{12}$ small loaves on average between the first and second large loaf selection. This brings our total thus far to $\frac{11}{4} + \frac{37}{12} = \frac{35}{6}$.

With one large loaf left and an (on average) $\frac{25}{4} - \frac{13}{12} = \frac{31}{6}$ small loaves left, we would expect half of those small loaves to come before the large loaf. Therefore, we expect Clarence to eat $\frac{31}{12} + 1 = \frac{43}{12}$ more large loaves until completion.

This yields that Clarence would need to eat an expected number of $\frac{35}{6} + \frac{43}{12} = \frac{113}{12}$ small loaves total.

Answer:

11312

Reference:

https://www.quantguide.io/questions/clarences-bread

Cat Dog Line

Category: Brainteasers
Difficulty: Medium
Companies: Old Mission

Question:

You're walking down an infinite street, and you notice that six out of every seven cats on a sidewalk are followed by a dog, while one out of every four dogs is followed by a cat. What proportion of animals on the sidewalk are dogs?

Hint:

Let c and d, respectively, be the proportion of cats and dogs on the sidewalk. Note that of all the cats, $\frac{6}{7}$ are followed by a dog. Iterate this type of logic.

Explanation:

Let c and d, respectively, be the proportion of cats and dogs on the sidewalk. We know $c + d = 1$. Let's find d in terms of c. Note that of all the cats, $\frac{6}{7}$ are followed by a dog. Then, of those dogs, $\frac{3}{4}$ of them are followed by another dog. Of those dogs, $\frac{3}{4}$ are followed by another dog, etc. Therefore, we can write

$$d = \frac{6}{7}c + \frac{6}{7} \cdot \frac{3}{4}c + \frac{6}{7} \cdot \left(\frac{3}{4}\right)^2 c + \cdots = \frac{6c}{7} \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{24c}{7}$$

Substituting this into our initial equation, $\frac{31c}{7} = 1$, so $c = \frac{7}{31}$. This means $d = \frac{24}{31}$.

Alternatively, you can create a Markov chain representing this scenario. Namely, if state 1 is dog and state 2 is cat, the Markov chain is

$$\begin{bmatrix} 3/4 & 1/4 \\ 6/7 & 1/7 \end{bmatrix}$$

The steady state of this Markov chain yields the same answer as above.

Answer:

2431

Reference:

https://www.quantguide.io/questions/cat-dog-line

Quintuple Value

Category: Probability
Difficulty: Easy
Companies: Old Mission, Optiver

Question:

35 cards are in a deck numbered $1 - 35$. One card is dealt uniformly at random and you are paid out 5 times the value on the card. Find the expected payout if you draw 2 cards from this deck.

Hint:

Let X_1 and X_2 be the value of the cards we select. Then our payout is $5(X_1 + X_2)$ by the question. We have that $X_1, X_2 \sim \text{DiscreteUnif}(\{1, 2, \ldots, 35\})$.

Explanation:

Let X_1 and X_2 be the value of the cards we select. Then our payout is $5(X_1 + X_2)$ by the question. We have that $X_1, X_2 \sim \text{DiscreteUnif}(\{1, 2, \ldots, 35\})$. We are looking for $\mathbb{E}[5(X_1+X_2)]$. By linearity, this is $5\mathbb{E}[X_1] + 5\mathbb{E}[X_2]$. As the two draws are exchangeable, their means are equal, so this is really just $10\mathbb{E}[X_1] = 10 \cdot 18 = 180$.

Answer:

180

Reference:

https://www.quantguide.io/questions/quintuple-value

Jellybean Jar II

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

A child has a pack of 6 red and 10 blue jellybeans. The child wants to eat 4 jellybeans, so they grab 4 jellybeans one-by-one uniformly at random without replacement from the pack. Find the probability that the first two and last two jellybeans agree in colors (not necessarily in order). For example, $RBBR$ and $RRRR$ are both valid, but $RRBR$ is not.

Hint:

There are three possibilities for the colors appearing in the first two spots. Namely, they are RR, BR, and BB.

Explanation:

There are three possibilities for the colors appearing in the first two spots. Namely, they are RR, BR, and BB. The first and last case respectively occur when we obtain $RRRR$ and $BBBB$. The probabilities of each of these are $\frac{6 \cdot 5 \cdot 4 \cdot 3}{16 \cdot 15 \cdot 14 \cdot 13}$ and $\frac{10 \cdot 9 \cdot 8 \cdot 7}{16 \cdot 15 \cdot 14 \cdot 13}$. The second case occurs when we have one blue and one red in each of the first and last two spots. As we can arrange the blue and red as RB or BR in each of the first and last two spots, we can multiply the number of ways to get some fixed sequence of one red and one blue in each of the first and last two by 4. For simplicity, let's say the fixed sequence is $RBRB$. Then we have $4 \cdot \frac{10 \cdot 6 \cdot 9 \cdot 5}{16 \cdot 15 \cdot 14 \cdot 13}$ total ways to obtain this case.

Adding up our three cases, our total probability is

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 + 10 \cdot 9 \cdot 8 \cdot 7 + 4 \cdot 10 \cdot 6 \cdot 9 \cdot 5}{16 \cdot 15 \cdot 14 \cdot 13} = \frac{135}{364}$$

Answer:

135364

Reference:

https://www.quantguide.io/questions/jellybean-jar-ii

Big Mac

Category: Brainteasers
Difficulty: Easy
Companies: Old Mission

Question:

In the year 2000, Big Macs sold for 89 cents each. Let x be the number of Big Macs you can buy with a \$20 bill and y be the change (in cents) you have left over. Find xy. Ignore tax or other expenses.

Hint:

Convert to cents first and then find a simple multiple that works.

Explanation:

Converting to cents, we know that $89 \cdot 20 = (89 \cdot 2) \cdot 10 = 1780$, so $1780 + 89 \cdot 2 = 1958 = 22 \cdot 89$ is the largest amount you can spend without going on \$20. Thus, $x = 22$ and $y = 2000 - 1958 = 42$, so $xy = 22 \cdot 42 = (32 + 10)(32 - 10) = 32^2 - 10^2 = 924$.

Answer:

924

Reference:

https://www.quantguide.io/questions/big-mac

Subsequent First Ace

Category: Probability
Difficulty: Medium
Companies: Old Mission

Question:

A deck is well-shuffled and cards are dealt out from the top one-by-one. The first ace is the 27th card dealt. Find the probability that the card right after this ace is the 2 of hearts.

Hint:

Note that the first 26 cards must not be aces (since the first should come on the 27th turn) nor the 2 of hearts.

Explanation:

The denominator is the number of ways to obtain the first ace as the 27th card dealt. In particular, we permute 26 of the 48 non-aces to the first 26 spots, which can be done in $P(48, 26)$ ways. Then, we fix the 27th card as an ace, which has 4 options. Then, lastly, we arrange the other 25 cards in any way we want, so there are $P(48, 26) \cdot 4 \cdot 25!$ total arrangements with the 27th card being the first ace.

Now, we need to do the same on the numerator, but we have an additional condition to satisfy. Note that the first 26 cards must not be aces (since the first should come on the 27th turn) nor the 2 of hearts. Therefore, we are permuting 26 of the 47 remaining cards to the first 26 spots, yielding $P(47, 26)$ possibilities. Then, the 27th card can be one of 4 aces. Afterwards, the next card must be fixed as the 2 of hearts. Then, there are 24 cards left, so there are 24! ways to arrange the cards after the 2 of hearts. All together, we get $P(47, 26) \cdot 4 \cdot 24! = 4 \cdot \frac{47!24!}{21!}$. Therefore, our final probability is

$$\frac{4 \cdot P(47, 26) \cdot 24!}{4 \cdot P(48, 26) \cdot 25!} = \frac{\frac{47!}{21!}}{\frac{48!}{22!} \cdot 25} = \frac{1}{48} \cdot \frac{22}{25} = \frac{11}{600}$$

Answer:

11600

Reference:

https://www.quantguide.io/questions/subsequent-first-ace

Horse Results

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

6 equally-skilled swimmers labeled $1-6$ are racing. Find the probability that swimmer 2 ends up in 2nd place and swimmer 5 is in the top 3 somewhere.

Hint:

Since each swimmer is equally-skilled, by the exchangeability of the swimmers, each one ends up in each of the 6 spots with probability $\frac{1}{6}$.

Explanation:

As we know each swimmer is equally-skilled, by the exchangeability of the swimmers, each one ends up in each of the 6 places with probability $\frac{1}{6}$. Swimmer 2 ends up in 2nd place with probability $\frac{1}{6}$. Then, given that, there are 2 spots left in the top 3 for swimmer 5 to go into. As one is taken by swimmer 2, the probability swimmer 5 is in the top 3 is $\frac{2}{5}$. Therefore, our answer is $\frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$.

Answer:

115

Reference:

https://www.quantguide.io/questions/horse-results

Mississippi

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

How many permutations of the word MISSISSIPPI are palindromes? A palindrome means that the string creates reads the same both forwards and backwards.

Hint:

For the word to read the same both forwards and backwards, as Mississippi is 11 letters, the M must be in the middle. Therefore, we have something in the form $-----M-----$, where each $-$ is filled with an S, I, or P.

Explanation:

For the word to read the same both forwards and backwards, as Mississippi is 11 letters, the M must be in the middle. Therefore, we have something in the form $-----M-----$, where each $-$ is filled with an S, I, or P. Note that there are 4 Ss and Is, while there are only 2 Ps. Therefore, for the word to read the same both forwards and backwards, we must have 2 Ss and Is on each side and 1 P on each side. Once we distribute the letters to one side, the other side is also fixed, so the number of ways to do this is $\frac{5!}{2! \cdot 2! \cdot 1!} = 30$, as we must account for the overcounting induced by exchanging the duplicate S and I around.

Answer:

30

Reference:

https://www.quantguide.io/questions/mississippi

Red Blue Equality

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

3 red and n blue socks are in a drawer. We know that when two socks are drawn uniformly at random without replacement, the probability both socks are red is $\frac{1}{2}$. Find n.

Hint:

We want a pair of red socks to show with probability $\frac{1}{2}$. Find the probability when there are n other socks there.

Explanation:

We want a pair of red socks to show with probability $\frac{1}{2}$. The probability the first sock is red is $\frac{3}{n+3}$. The probability the second sock is red is $\frac{2}{n+2}$, as we took out one red sock. Therefore, the probability of both socks being red is $\frac{6}{(n+2)(n+3)}$. This must equal $\frac{1}{2}$, so we need

$$\frac{6}{(n+2)(n+3)} = \frac{1}{2}$$

Equivalently, this means $(n+2)(n+3) = 12$ after rearranging. Expanding this out, we have that $n^2 + 5n - 6 = 0$, so $(n+6)(n-1) = 0$, meaning $n = 1, -6$ solves this. -6 is not possible, so $n = 1$ is our answer.

Note that one can see this quickly by noting that $n \leq 3$, as otherwise, it would be less likely to draw two red than two blue. We can rule out $n = 0$, as that probability would be 1. Therefore, that leaves us three cases to check.

Answer:

1

Reference:

https://www.quantguide.io/questions/red-blue-equality

Matching Jar Colors

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

Jar A has 3 blue and 4 red balls. Jar B has 6 blue and 4 red balls. If one ball is selected uniformly at random from each jar, what is the probability that the two selected balls match in color?

Hint:

We can get matching colors by either both being red or both being blue.

Explanation:

We can get matching colors by either both being red or both being blue. The probability of two reds is $\frac{4}{7} \cdot \frac{4}{10} = \frac{16}{70}$. The probability of two blues is $\frac{3}{7} \cdot \frac{6}{10} = \frac{18}{70}$. Adding these up, we get our total probability is $\frac{16+18}{70} = \frac{17}{35}$.

Answer:

1735

Reference:

https://www.quantguide.io/questions/matching-jar-colors

Head Tail Sequence

Category: Probability
Difficulty: Easy
Companies: Old Mission

Question:

5 fair coins are flipped. Given that you obtained 3 heads, find the probability that you obtained the sequence $THHTH$.

Hint:

How many sequences of length 5 are there with three heads?

Explanation:

There are $\binom{5}{3} = 10$ sequences of length 5 containing exactly three heads. Therefore, the probability of this particular sequence is $\frac{1}{10}$, as all are equally likely.

Answer:

110

Reference:

https://www.quantguide.io/questions/head-tail-sequence

Intersecting Chords

Category: Probability
Difficulty: Medium
Companies: HRT

Question:

10 chords with uniformly randomly chosen endpoints are drawn on a circle. What is the expected number of intersections?

Hint:

Consider indicator random variables on each pair of chords.

Explanation:

Consider indicator random variables on each pair of chords. Any pair of chords will have a total of 4 endpoints, and each of the $\binom{4}{2} = 6$ ways to pair them up into chords are equally likely, of which only 2 result in an intersection. Alternatively, this can be seen by choosing two points for the first chord and computing the probability that two randomly chosen points would lie on the same side of the chord via integrating over the arc angle between the points of the first chord. Hence for each of the $\binom{10}{2} = 45$ pairs of chords, there are $1/3$ expected intersections for a total number of 15.

Answer:

15

Reference:

https://www.quantguide.io/questions/intersecting-chords

Segment Traversal

Category: Probability
Difficulty: Hard
Companies: HRT

Question:

You select a uniformly random starting point on the circumference of a circle, say P_1. Then, you select another uniformly random point on the circumference, P_2, and draw a line segment between P_1 and P_2. You then continue to select uniformly random points on the circumference of the circle, say $P_3, \ldots, P_n$, and draw a line segment between P_i and P_{i+1} for each $1 \leq i \leq n-1$. Lastly, you draw a line segment from P_n back to P_1. Find the expected number of intersections between line segments on the interior of the circle as a function of n. Evaluate this for $n = 12$.

Hint:

In total, there are n line segments when the process is done. How many pairs of line segments are there to have intersections? How many of those pairs can't intersect? For the ones that can, what is the probability of intersection?

Explanation:

In total, there are n line segments when the process is done. This means that there are $\binom{n}{2}$ pairs of line segments that could possibly intersect. However, not all of these have equal probabilities of intersecting. In particular, each of the n points selected represents the intersection of two line segments on the circumference of the circle, meaning that those two segments can't intersect inside the circle. Therefore, we must subtract n possible pairs of line segments that could intersect, as those n pairs can't intersect each other inside the circle. This yields $\binom{n}{2} - n$ possible pairs of line segments that can intersection. By linearity of expectation and the exchangeability of the remaining pairs, we just need to now find the probability of intersection between any two line segments to find the total number of expected intersections.

Our question can be formalized as the following: Suppose that we have X_1, X_2, X_3, and X_4 selected uniformly at random from the circumference of a circle. Draw a chord between X_1 and X_2 and a chord between X_3 and X_4. Find the probability that the chords intersect. Since the circle is rotationally invariant, we can just lock in X_1 at $(1, 0)$. Then, we can consider all possible orderings of X_2, X_3, and X_4 in terms of distance (radians CCW) from the origin. By drawing out the different permutations, you can see that there is an intersection between the segments if X_2 is between X_3 and X_4 (or vice versa). This corresponds to 2 of the $3! = 6$ possible orderings of the random variables, so the probability is $1/3$.

The above yields that the expected number of intersections is $\frac{\binom{n}{2} - n}{3} = \frac{n(n-3)}{6}$. In particular, $n = 12$ here yields 18 intersections on average.

Answer:

18

Reference:

https://www.quantguide.io/questions/segment-traversal

Sum Exceedance III

Category: Probability
Difficulty: Hard
Companies: HRT

Question:

Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID and $N_1 = \min\{n : X_1 + \cdots + X_n > 1\}$. Let $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}[S_{N_1}]$. The answer will be in the form ce for a rational number c. Find c.

Hint:

$S_{N_1} = \sum_{i=1}^{N_1} X_i$ gives the value of the sum after it exceeds 1. N_1 is a stopping time.

Explanation:

By Sum Exceedance I, we know that the expected number of uniforms to exceed 1 is e. Namely, we have that $S_{N_1} = \sum_{i=1}^{N_1} X_i$ gives the value of the sum after it exceeds 1. As N_1 is a stopping time, we can say $\mathbb{E}[S_{N_1}] = \mathbb{E}[N_1]\mathbb{E}[X_i] = \frac{e}{2}$. Therefore, $c = \frac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/sum-exceedance-iii

Doubly Blue

Category: Probability
Difficulty: Easy
Companies: HRT

Question:

Three bowls are presented to you. One has two blue balls, one has two red balls, and the last has one red ball and one blue ball. You select one bowl uniformly at random and note that you drew a blue ball from it. You then draw from the same bowl again after replacing the ball you selected. Find the probability that you would draw a blue ball on this draw.

Hint:

Let BB represent the event of drawing two blues and B represent the event of the first ball being blue. Let U_1 represent the two blue urn, U_2 represent the two red urn, and then U_3 represent the mixed urn. We want

$$\mathbb{P}[BB \mid B] = \frac{\mathbb{P}[BB]}{\mathbb{P}[B]}$$

Condition on the urn you are in.

Explanation:

Let BB represent the event of drawing two blues and B represent the event of the first ball being blue. Let U_1 represent the two blue urn, U_2 represent the two red urn, and then U_3 represent the mixed urn. We want

$$\mathbb{P}[BB \mid B] = \frac{\mathbb{P}[BB]}{\mathbb{P}[B]}$$

The equality above comes from the fact that $BB \subseteq B$. To calculate each, we just condition on the urn we are in. Namely,

$$\mathbb{P}[B] = \mathbb{P}[B \mid U_1]\mathbb{P}[U_1] + \mathbb{P}[B \mid U_2]\mathbb{P}[U_2] + \mathbb{P}[B \mid U_3]\mathbb{P}[U_3]$$

Since the original urn is selected uniformly at random, $\mathbb{P}[U_1] = \mathbb{P}[U_2] = \mathbb{P}[U_3] = \frac{1}{3}$. Furthermore, we have that $\mathbb{P}[B \mid U_1] = 1, \mathbb{P}[B \mid U_2] = 0$, and $\mathbb{P}[B \mid U_3] = \frac{1}{2}$ by the proportions of blue balls in each urn. Since we replace the ball between trials, to get $\mathbb{P}[BB \mid U_i]$, we just have to square $\mathbb{P}[B \mid U_i]$. This means that

$$\mathbb{P}[B] = \frac{1 + 1/2 + 0}{3} = \frac{1}{2}$$

and

$$\mathbb{P}[BB] = \frac{1 + 1/4 + 0}{3} = \frac{5}{12}$$

Therefore, the answer is $\mathbb{P}[BB \mid B] = \frac{5/12}{1/2} = \frac{5}{6}$.

Answer:

56

Reference:

https://www.quantguide.io/questions/doubly-blue

Peeled Dice

Category: Probability
Difficulty: Medium
Companies: HRT

Question:

Suppose you roll a fair 6−sided die and peel that rolled value off the die so that there are five remaining sides with values on them. You roll until you obtain one of the other five values. What is the expected sum of the two rolls?

Hint:

Condition on the value of the first roll.

Explanation:

The value on the second roll is going to depend on the value peeled off from the first roll. Therefore, we condition on the value peeled off on the first roll. You roll each value on the die with equal probability, so if T denotes the total of the two rolls, $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X_1]] = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]$.

$\mathbb{E}[T \mid X_1 = i]$ is now what we need to compute. We know that i is already contributed to our total from the first roll. Then, i is peeled off the die, so that now the remaining values sum to $21 - i$ and there are 5 equally-likely values. Therefore, the expected value of the second roll given the first is i is $\frac{21-i}{5}$ Thus, $\mathbb{E}[T \mid X_1 = i] = \frac{21-i}{5} + i = \frac{21+4i}{5}$.

Plugging this in,

$$
\begin{aligned}
\mathbb{E}[T] &= \frac{1}{6}\sum_{i=1}^{6} \frac{21+4i}{5} \\
&= \frac{7}{10}\sum_{i=1}^{6} 1 + \frac{2}{15}\sum_{i=1}^{6} i \\
&= \frac{42}{10} + \frac{2}{15} \cdot \frac{6(7)}{2} = \frac{21}{5} + \frac{14}{5} = 7
\end{aligned}
$$

Another way to think about this is that the expected value that you remove from the first roll is 3.5, so the expected sum of remaining values is 17.5. As there are 5 equally-likely values on the die, the expected value of the second roll is still 3.5, so adding these two up gives 7 as well.

Answer:

7

Reference:

https://www.quantguide.io/questions/peeled-dice

6 Die Stopper

Category: Probability
Difficulty: Medium
Companies: HRT

Question:

A standard fair 6−sided die is rolled up to (and including) when the first 6 is rolled. Find the probability that the sum of all upfaces viewed before stopping is even.

Hint:

Including or excluding the 6 doesn't matter, as adding 6 does not change the parity of the overall sum. Let p be this probability. Condition on the first roll.

Explanation:

Including or excluding the 6 doesn't matter, as adding 6 does not change the parity of the overall sum. Let p be this probability. We condition on the first roll. If the first roll is a 6, then we have an even sum with probability 1. If the first roll is a 2 or 4, then the sum of all other terms must be even, occurring with probability p. If the first roll is odd, then the sum of the other rolls must be odd, occurring with probability $1 - p$. Writing this out with Law of Total Probability, we see that

$$p = \frac{1}{6} \cdot 1 + \frac{1}{3} \cdot p + \frac{1}{2} \cdot (1 - p)$$

Solving this yields $p = \dfrac{4}{7}$.

Answer:

47

Reference:

https://www.quantguide.io/questions/6-die-stopper

Conditionally Normal

Category: Probability
Difficulty: Medium
Companies: HRT

Question:

Suppose that $X, Y \sim N(0, 1)$ IID. Find $\mathbb{P}[X > 0 \mid X + Y > 0]$.

Hint:

Since X and Y are independent standard normal, exploit the radial symmetry of normal distributions to solve this cleanly.

Explanation:

Since X and Y are independent standard normal, we can exploit the radial symmetry of normal distributions to solve this cleanly. The region $\{X + Y > 0\}$ covers up half of the plane i.e. spans a total of π radians. This can be easily seen by drawing it out in the plane. Of this region, the region $\{X > 0\}$ is $\frac{3\pi}{4}$ radians, also easily seen by picture. Therefore, the conditional probability $X > 0$ is

$$\frac{3\pi/4}{\pi} = \frac{3}{4}$$

Answer:

34

Reference:

https://www.quantguide.io/questions/conditionally-normal

Collecting Toys II

Category: Probability
Difficulty: Hard
Companies: Optiver

Question:

Every box of cereal contains one toy from a group of 5 distinct toys, each of which is mutually independent from the others and is equally likely to be within a given box. How many distinct toys can you expect to collect if you bought 7 boxes?

Hint:

Try to define indicator random variables as a way to utilize the linearity of expectation. Notice that each toy follows the same distribution, despite mutual dependence.

Explanation:

Let X be the number of distinct toys you collect from the set of 5 toys; we are looking for $E[X]$.

Let I_i $(i = 1, 2, ..., 5)$ be the indicator random variable where:

$$I_i = \begin{cases} 1 & \text{if the } i\text{-th toy type exists in the set of seven boxes} \\ 0 & \text{otherwise} \end{cases}$$

We can define X as $X = I_1 + I_2 + ... + I_5 = \sum_{i=1}^{5} I_i$. For a particular toy type, the probability that it is not observed in the first box is $\frac{4}{5}$, and the probability that it is observed in none of the seven boxes is $(\frac{4}{5})^7$ since each box is independent. Therefore, the probability that it is observed in at least one of the seven boxes is $1 - (\frac{4}{5})^7$. Applying the linearity of expectation and the fact that each toy type follows the same distribution, we can solve for $E[X]$:

$$E[X] = E\left[\sum_{i=1}^{5} I_i\right] = 5E[I_1] = 5 \times \left[1 \times \left(1 - \left(\frac{4}{5}\right)^7\right) + 0 \times \left(\frac{4}{5}\right)^7\right] = \frac{61741}{15625}$$

Answer:

6174115625

Reference:

https://www.quantguide.io/questions/collecting-toys-ii

Tennis Deuces II

Category: Probability
Difficulty: Easy
Companies: Optiver

Question:

Andrew and Beth are playing a game of tennis. Tennis scoring works off these rules:. Scoring starts at 0-0 and when someone wins a point, their score goes up by 1 (thus either 1-0 or 0-1). . If a player gets 4 points before the other player, they win unless the score was 3-3 (deuce). If the score was 3-3, a player has to win by two points to win the game. How many ways are there to get to a score of 4-4?

Hint:

What score do you have to always get before reaching 4-4?

Explanation:

From "Tennis Deuces I", we know there are 20 ways to get to 3-3. We know we have to get to 3-3 before getting to 4-4 since any score of 4-x where x isn't a number greater than or equal to 3 means the game ended prematurely. From here, to make 4-4, either Andrew wins a point and then Beth does or vice-versa which is two different outcomes. $20 \cdot 2 = 40$ ways.

Answer:

40

Reference:

https://www.quantguide.io/questions/tennis-deuces-ii

77 Multiple I

Category: Brainteasers
Difficulty: Easy
Companies: Optiver

Question:

What is the smallest multiple of 77 that is at least $70,000$?

Hint:

Consider 77000 and 7700 first.

Explanation:

Since $77000 \cdot 77 \cdot 1000$ and $7700 = 77 \cdot 100$ are multiples of 77, $69300 = 77000 - 7700$ is as well. We also know that $770 = 77 \cdot 10$ is a multiple of 77, so $69300 + 770 = 70070$ is also a multiple of 77. This is the smallest since if we subtract 77, we are below 70000.

Answer:

70070

Reference:

https://www.quantguide.io/questions/77-multiple-i

Valuable Hearts

Category: Probability
Difficulty: Easy
Companies: Optiver

Question:

Suppose that each card in a standard deck is given a value equal to it's face, with $A = 1, J = 11, Q = 12$, and $K = 13$. However, any card that is heart is doubled in value (ex: Ace of Hearts is worth 2). Find the expected value of a uniformly at random selected card from the deck.

Hint:

Consider the sum of all values in the deck.

Explanation:

The sum of all the values of the cards in the deck is $5 \cdot \sum_{i=1}^{13} i$, as each suit contribute $1+2+\cdots+13$ to the total expect hearts, which contributes $2 \cdot (1 + 2 + \cdots + 13)$. There are 52 total cards in the deck, so the expected value of a random card is

$$\frac{5 \cdot \frac{13 \cdot 14}{2}}{52} = \frac{35}{4}$$

Answer:

354

Reference:

https://www.quantguide.io/questions/valuable-hearts

Pick Your Opponent

Category: Brainteasers
Difficulty: Easy
Companies: Optiver

Question:

You are in a tennis tournament with Alice and Bob. The tournament consists of 3 games. You win the tournament if you win two consecutive games. You have a better chance of beating Bob than Alice. If A represents Alice and B represents Bob, would you prefer your three games to be, in order, ABA or BAB? Answer 1 and 2 for ABA and BAB, respectively.

Hint:

Intuitively, to win the tournament, you must win your second game, as you have to win 2 in a row.

Explanation:

Intuitively, to win the tournament, you must win your second game, as you have to win 2 in a row. Therefore, you should play the easier opponent in the middle, meaning you should select ABA. To verify this, let x and y be your probabilities of being Alice and Bob, respectively. We know that $x < y$ by the question.

If you select ABA, your probability of winning the tournament is $xy + xy - x^2y = 2xy - x^2y$ by inclusion-exclusion. If you select BAB, your probability of winning the tournament is $xy + xy - y^2x = 2xy - y^2x$. However, as $x < y$, we have that $y^2x > x^2y$, so your probability of winning with BAB is smaller, as you subtract a larger number.

Answer:

1

Reference:

https://www.quantguide.io/questions/pick-your-opponent

Beer Barrel I

Category: Brainteasers
Difficulty: Medium
Companies: Optiver

Question:

A 120-quart beer barrel was discovered by Anna's parents. Furious, they plan to dump the barrel. Anna begs her parents to let her keep some of the beer. They say that Anna may do so if she is able to measure out an exact quart into each of a 7-quart and 5-quart vessel. Define a transaction as a pour of liquid from one container into another. What is the smallest number of transactions needed to accomplish the challenge? If impossible, respond with -1.

Hint:

We can solve this problem by following these steps in order (These are are the first 4 steps):
. Perform 14 transactions of filling and emptying the 7-quart measure, wasting 98 quarts and leaving 22 quarts in the barrel.. Fill the 7-quart measure, then transfer 5 quarts to the 5-quart measure, leaving 2 quarts in the 7-quart.. Empty the 5-quart measure, then transfer 2 quarts from the 7-quart to the 5-quart.. Fill the 7-quart measure, then fill up the 5-quart from the 7-quart, leaving 4 quarts in the 7-quart.

Explanation:

We can solve this problem by following these steps in order: . Perform 14 transactions of filling and emptying the 7-quart measure, wasting 98 quarts and leaving 22 quarts in the barrel.. Fill the 7-quart measure, then transfer 5 quarts to the 5-quart measure, leaving 2 quarts in the 7-quart.. Empty the 5-quart measure, then transfer 2 quarts from the 7-quart to the 5-quart.. Fill the 7-quart measure, then fill up the 5-quart from the 7-quart, leaving 4 quarts in the 7-quart.. Empty the 5-quart measure, then transfer 4 quarts from the 7-quart to the 5-quart.. Fill the 7-quart measure again, then fill up the 5-quart from the 7-quart, leaving 6 quarts in the 7-quart.. Empty the 5-quart measure, then fill the 5-quart from the 7-quart, leaving 1 quart in the 7-quart.. Empty the 5-quart measure, leaving 1 quart in the 7-quart.. Draw off the remaining 1 quart from the barrel into the 5-quart measure, completing the task in 14 more transactions, totaling 42 transactions with the initial 28.

Answer:

42

Reference:

https://www.quantguide.io/questions/beer-barrel-i

Pairwise Digit Sums II

Category: Brainteasers
Difficulty: Medium
Companies: Optiver

Question:

Let A be the set of 5 digit integers such that all pairwise sums of digits are unique. For example, a three digit number with this property is 174. Let x and y be the minimal and maximal elements of A, respectively. Find $y - x$.

Hint:

Consider working from the largest magnitude digit at the left all the way to the smallest at the right.

Explanation:

We will start with the largest number. Start in the left-most position with 9. Then, the second digit from left should be 8. We can't have replicate digits, as otherwise, we could pair any other digit in the number with either one of the two repeated digits and get the same sum. Then, the next digit is 7, as we still have all unique digit sums. Therefore, our number is in the form $987ab$, with a and b still to be determined. a can't be 6, as $6 + 9 = 7 + 8$. However, with $a = 5$, we still maintain all unique digit sums. Thus, our form is $9875b$. If $b = 4$, then $4 + 9 = 8 + 5$. If $b = 3$, then $3+9 = 5+7$. However, with $b = 2$, we get all unique digit sums. Therefore, $y = 98752$.

For the smallest, the smallest first digit is 1. Then, the smallest second digit is 0. The smallest third digit is 2. Thus, our number is in the form $102cd$. If $c = 3$, $1 + 2 = 3 + 0$, but for $c = 4$, we still maintain unique sums. Therefore, our form is $1024d$. If $d = 5$, then $5 + 0 = 4 + 1$. If $d = 6$, then $6 + 0 = 4 + 2$. However, for $d = 7$, we maintain unique sums, so $x = 10247$. Therefore, $y - x = 88505$.

Answer:

88505

Reference:

https://www.quantguide.io/questions/pairwise-digit-sums-ii

Keg Of Wine

Category: Brainteasers
Difficulty: Easy
Companies: Optiver

Question:

Alex has a 10-gallon keg of wine and a big ladle. On Monday, Alex drew off a ladle-full of wine and filled the keg back up with water. On Tuesday, he repeated the process, making sure to mix the contents in the keg around first. On Wednesday morning, he realized that the keg now contains equal parts of wine and water. How many gallons can the ladle hold? The answer is in the form $a - b\sqrt{c}$ with $a = bc$. Find $a + b + c$.

Hint:

Let x denote the capacity of the jug. Then, the proportion of wine in the keg after Tuesday can be denoted as:

$$\frac{10 - x - \left(\frac{10-x}{10}\right)x}{10}$$

Explanation:

Let x denote the capacity of the jug. Then, the proportion of wine in the keg after Tuesday can be denoted as:

$$\frac{10 - x - \left(\frac{10-x}{10}\right)x}{10} = \frac{1}{2}$$

$$20 - 2x - \frac{10x - x^2}{5} = 10$$

$$\Rightarrow x = 10 - 5\sqrt{2}$$

This means that $a + b + c = 17$.

Answer:

17

Reference:

https://www.quantguide.io/questions/keg-of-wine

Missing Million II

Category: Probability
Difficulty: Easy
Companies: Optiver

Question:

You are on a game show with 3 doors in front of you. One of the doors has \$1 million inside, while the other two are empty. In the final round, the host lets you spin a wheel two times that may reveal which door the \$1 million is in. On each spin, this wheel will tell you the location of the \$1 million 3/5 of the time, independent between spins. Otherwise, it tells you nothing about the location of the money. After two spins, if the wheel has not revealed to you the location of the \$1 million, you must guess uniformly at random. What is the probability you locate the \$1 million door?

Hint:

We do not locate the door with \$1 million exactly when the wheel does not tell us any information both times.

Explanation:

We do not locate the door with \$1 million exactly when the wheel does not tell us any information both times. This occurs with probability 2/5 per trial, so the probability you don't get any information about the location of the money is $\frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$. This means that with probability $\frac{21}{25}$, you are guaranteed to know where the money is located via the wheel. Otherwise, with probability $\frac{4}{25}$, you guess uniformly at random among the doors and locate the money with probability $\frac{1}{3}$. This implies the total probability of finding the money is

$$\frac{21}{25} + \frac{1}{3} \cdot \frac{4}{25} = \frac{67}{75}$$

Answer:

6775

Reference:

https://www.quantguide.io/questions/missing-million-ii

Unknown Starter

Category: Probability
Difficulty: Hard
Companies: Optiver

Question:

You have 3 cards, each labeled n, $n+1$, $n+2$, and you don't know n. All cards start face down. You flip one card and observe the value. If you say "stay", your payout is equal to that card's value. If you don't "stay", then you flip another card. Again, choose to "stay" (and receive payout equal to the 2nd card's value) or flip the final card and receive payout equal to the final card's value. Design the optimal strategy. The expected out payout of this strategy is $n+c$ for some constant c. Find c.

Hint:

For simplicity, we can assume the values of the cards are $1, 2$, and 3, as n is arbitrary and is just a shifting factor. If you make a decision at the first or third card, you have no choice on what the card will be, so the expected value will be 2 in each of those cases. Consider the possible differences between the first two cards.

Explanation:

For simplicity, we can assume the values of the cards are $1, 2$, and 3, as n is arbitrary and is just a shifting factor. If you make a decision at the first or third card, you have no choice on what the card will be, so the expected value will be 2 in each of those cases. Therefore, to do better, we should consider devising a strategy at the second card drawn.

Suppose the two cards you have drawn, in order, are a and b. The possible values for $a-b$ are ± 2 or ± 1. We break up into cases based on the values of $a-b$.

Case 1: If $a-b=\pm 2$, then you know the largest and smallest card among the three, so you just select the larger card in this case and you're done.

Case 2: If $a-b=1$, which occurs with the starting sequence 21 and 32, we would have 3 and 1 as our last card. The expected value of the second card in this case is $3/2$ (either 1 or 2 with equal probability), whereas the expected value of the last card is 2 (3 or 1 with equal probability), so we should select the third card in this case.

Case 3: If $a-b=-1$, which occurs with the starting sequence 12 or 23, we would again have 3 and 1 as our last card. The expected value of the last card is the same still, but the expected value of the second card is now $5/2$, which is larger than 2, so we should stay at the second card.

In summary, for each of the permutations of $1, 2, 3$ under this strategy, we select the following cards:

$$213 : 3, 231 : 3, 321 : 1, 123 : 2, 132 : 3, 312 : 2$$

Therefore, the expected value of the card is

$$\frac{3+3+1+2+3+2}{6} = \frac{8}{3}$$

This was the case where $n = 1$, so to shift with $n, n+1$, and $n+2$, the answer would be $n + 4/3$. This means $c = 4/3$.

Answer:

43

Reference:

https://www.quantguide.io/questions/unknown-starter

Tricky Bob I

Category: Probability
Difficulty: Medium
Companies: Optiver

Question:

Bob proposes a game where Alice and Bob are each given a coin. Each is allowed decide the probability of heads for their coin. Afterwards, they both flip their coins. If it comes up HH, Alice gives Bob \$6. If it comes up TT, Alice gives Bob \$4. Otherwise, Bob gives Alice \$5. However, Alice must tell Bob the success probability p_1 they have chosen before Bob decides his p_2. The interval of values for p_1 such that regardless of what Bob selects, Bob has non-positive expected value on the game is in the form $[a, b]$, where a and b are rational numbers. Find $b - a$.

Hint:

Call Bob's success probability p_2. Write out the expected value as a function of p_1 and p_2. For what values of p_1 does p_2 need to be larger than 1 to obtain a positive expected value?

Explanation:

Call Bob's success probability p_2. Let's write out the expected value as a function of p_1 and p_2. The probability of HH is p_1p_2. The probability of TT is $(1-p_1)(1-p_2)$. The probability of TH or HT is $p_1(1-p_2)+p_2(1-p_1) = p_1+p_2-2p_1p_2$. From Bob's perspective, the respective profits for each of these outcomes are $6, 4$, and -5. Therefore, the expected value is $6p_1p_2+4(1-p_1)(1-p_2)-5(p_1+p_2-2p_1p_2) = 6p_1p_2+4-4p_1-4p_2+4p_1p_2-5p_1-5p_2+10p_1p_2 = 20p_1p_2-9p_2-9p_1+4$. Factoring, this becomes $(20p_1-9)p_2+(4-9p_1)$.

Now, assuming that $p_1 \neq \frac{9}{20}$ (such that we can rearrange and divide), we see that $p_2 > \frac{9p_1-4}{20p_1-9}$. The RHS is 0 when $p_1 = \frac{4}{9}$, as the numerator vanishes. In addition, let's find when the RHS is at least 1. This implies $9p_1-4 \geq 20p_1-9$, which means that $p_1 \leq \frac{5}{11}$. One can verify that the expected value in this case (by plugging into the expected value equation previously) is p_2-1. However, as a probability is at most one 1, unless $p_2 = 1$, Bob has a strictly negative expected value. Similarly, by plugging in $p_1 = \frac{4}{9}$, one can find the expected value to be $-p_2$. Therefore, unless $p_2 = 0$, Bob once again has a strictly negative expected value. Therefore, $a = \frac{4}{9}$ and $b = \frac{5}{11}$, so $b - a = \frac{1}{99}$.

Answer:

199

Reference:

https://www.quantguide.io/questions/tricky-bob-i

Tennis Deuces

Category: Probability
Difficulty: Easy
Companies: Optiver

Question:

Andrew and Beth are playing a game of tennis. Tennis scoring works off these rules:. Scoring starts at 0-0 and when someone wins a point, their score goes up by 1 (thus either 1-0 or 0-1). . If a player gets 4 points before the other player, they win unless the score was 3-3 (deuce). If the score was 3-3, a player has to win by two points to win the game. How many ways are there to get to a score of 4-4?

Hint:

What score do you have to always get before reaching 4-4?

Explanation:

From "Tennis Deuces I", we know there are 20 ways to get to 3-3. We know we have to get to 3-3 before getting to 4-4 since any score of 4-x where x isn't a number greater than or equal to 3 means the game ended prematurely. From here, to make 4-4, either Andrew wins a point and then Beth does or vice-versa which is two different outcomes. $20 \cdot 2 = 40$ ways.

Answer:

40

Reference:

https://www.quantguide.io/questions/tennis-deuces

Lead Count

Category: Probability
Difficulty: Medium
Companies: Optiver

Question:

Suppose you flip a fair coin 10 times. We say a certain outcome (heads or tails) is leading after n flips if there are strictly larger than $\frac{n}{2}$ flips of that outcome within the first n flips. Find the probability that the outcome of the first flip of the coin is leading after 10 flips.

Hint:

Fix the first flip arbitrarily since it is equally likely to be heads or tails. Without loss of generality, say it is heads. Then we would need at least 5 heads in the next 9 flips to be leading.

Explanation:

Fix the first flip arbitrarily since it is equally likely to be heads or tails. Without loss of generality, say it is heads. Then we would need at least 5 heads in the next 9 flips to be leading, as we need strictly more than 5 heads total in the 10 flips and we know the first is heads. The trick here to compute this probability is to note that $\binom{9}{i} = \binom{9}{9-i}$ for each $0 \leq i \leq 9$. Therefore, we have that

$$\binom{9}{0} + \cdots + \binom{9}{4} = \binom{9}{5} + \cdots + \binom{9}{9}$$

As each sequence of heads and tails is equally likely and we note that there and the same number of sequences with at least 5 and strictly less than 5 heads in the 9 flips, we see that our answer is $\frac{1}{2}$.

Answer:

12

Reference:

https://www.quantguide.io/questions/lead-count

Lowest Target

Category: Probability
Difficulty: Easy
Companies: Optiver

Question:

An archer is shooting at targets. There are 4 columns of identical targets, each with $2, 3, 3$, and 4 targets stacked vertically, respectively. The archer first selects a column and then shoots the lowest-hanging target in that column that is not broken. Once a column is out of targets, it is no longer able to be selected. In how many ways can the archer break all the targets?

Hint:

There are 12 total targets and 4 columns. Label the columns $1-4$. How can you identify each unique breaking pattern?

Explanation:

There are 12 total targets and 4 columns. Label the columns $1-4$. Then we are just counting the number of anagrams that can be made with 2 1s, 3 2s, 3 3s, and 4 4s. This is because we can identify a unique breaking order by the column we shoot in at each step. The number of anagrams is just given by the multinomial coefficient $\binom{12}{2,3,3,4} = 277200$

Answer:

277200

Reference:

https://www.quantguide.io/questions/lowest-target

Pairwise Digit Sums I

Category: Brainteasers
Difficulty: Medium
Companies: Optiver

Question:

Let A be the set of 5 digit integers such that all pairwise sums of digits are unique. For example, a three digit number with this property is 174. Let x and y be the minimal and maximal elements of A, respectively. Find $y - x$.

Hint:

Consider working from the largest magnitude digit at the left all the way to the smallest at the right.

Explanation:

We will start with the largest number. Start in the left-most position with 9. Then, the second digit from left should be 8. We can't have replicate digits, as otherwise, we could pair any other digit in the number with either one of the two repeated digits and get the same sum. Then, the next digit is 7, as we still have all unique digit sums. Therefore, our number is in the form $987ab$, with a and b still to be determined. a can't be 6, as $6 + 9 = 7 + 8$. However, with $a = 5$, we still maintain all unique digit sums. Thus, our form is $9875b$. If $b = 4$, then $4 + 9 = 8 + 5$. If $b = 3$, then $3+9 = 5+7$. However, with $b = 2$, we get all unique digit sums. Therefore, $y = 98752$.

For the smallest, the smallest first digit is 1. Then, the smallest second digit is 0. The smallest third digit is 2. Thus, our number is in the form $102cd$. If $c = 3$, $1 + 2 = 3 + 0$, but for $c = 4$, we still maintain unique sums. Therefore, our form is $1024d$. If $d = 5$, then $5 + 0 = 4 + 1$. If $d = 6$, then $6 + 0 = 4 + 2$. However, for $d = 7$, we maintain unique sums, so $x = 10247$. Therefore, $y - x = 88505$.

Answer:

88505

Reference:

https://www.quantguide.io/questions/pairwise-digit-sums-i

Smallest Factorizaiton

Category: Brainteasers
Difficulty: Medium
Companies: Optiver

Question:

The number 1234567890 is not prime, so it can be written in the form ab for two positive integers a and b. Find $a^* + b^*$, where a^* and b^* satisfy $a^*b^* = 1234567890$ and for any other pair (a, b) such that $ab = 1234567890$, $|a - b| \geq |a^* - b^*|$. It may be helpful to consider the prime factorization.

Hint:

With a little bit of work, one can show that the prime factorization of $1234567890 = 2 \cdot 3^2 \cdot 5 \cdot 3607 \cdot 3803$.

Explanation:

With a little bit of work, one can show that the prime factorization of $1234567890 = 2 \cdot 3^2 \cdot 5 \cdot 3607 \cdot 3803$. Therefore, if we were to multiply 3607 by $2 \cdot 5 = 10$ and 3803 by $3^2 = 9$, we should get two factors that are really close. These must be the closest because of the fact that our two factors must be multiples of 3607 and 3803, respectively, so we want to make those multiples as close as possible. The way to do this is by multiplying 3607 by a slightly larger number than 3803 as to close the gap. The arrangement listed the above gives the two factors that are closest together, so our two numbers are $3607 \cdot 10 = 36070$ and $3803 \cdot 9 = 34227$. Therefore, their sum is $36070 + 34227 = 70297$.

Answer:

70297

Reference:

https://www.quantguide.io/questions/smallest-factorizaiton

Meeting On A Grid

Category: Probability
Difficulty: Medium
Companies: Optiver

Question:

Alan is sitting at $(0, 0)$ and Barbara is sitting at $(4, 4)$. Each minute, Alan moves up or to the right one unit with equal probability, unless he is on the border of the 4×4 grid centered at $(2, 2)$, in which case he moves in the only possible direction that keeps him within range. Barbara behaves similarly, but moves down or to the left instead. What is the probability that the two meet before they reach the opposite corner?

Hint:

Can Alan and Barbara meet at any point?

Explanation:

Observe that the sum of Alan's coordinates increases by 1 each minute, and Barbara's decreases by 1. Since this sum starts at 8 for Barbara and 0 for Alan, if they meet at all, they must meet after 4 minutes, and they must do so on the diagonal $y = 4 - x$. During the first 4 minutes, both players are allowed to move in their originally permitted directions unconstrained, so there are 2^4 equally likely ways for each player to reach the diagonal and therefore $2^8 = 256$ ways for both players. There exists a bijection between the number of ways to travel from one corner to the other and the number of viable paths for Alan and Barbara to meet, as any distinct example of one can be viewed uniquely as an example of the other. Hence, we have $\binom{8}{4} = 70$ ways for Alan and Barbara to meet of the original 256, so the probability is $\frac{70}{256}$.

Answer:

70256

Reference:

https://www.quantguide.io/questions/meeting-on-a-grid

Exact Bills

Category: Brainteasers
Difficulty: Easy
Companies: Optiver

Question:

Given the denominations of \$1, \$5, \$20, and \$100 dollar bills, what is the fewest amount of bills needed to form any amount from \$1 to \$100 by taking a subset of the bills? In other words, how many bills do you need to form any number from \$1 to \$100 where you can reuse bills?

Hint:

The amount that requires the most bills will logically be a high number.

Explanation:

Logically speaking, the most difficult amount to form will be very high. For example, any number from \$80 to \$90 can be harder to form by just adding \$10 to it (requires two more bill). Testing a few possibilities out, you'll see that \$99 is the most difficult amount to form as it requires 4 \$20 bills, 3 \$5 bill, and 4 \$1 bills which gives us 11 bills. However, we can't make \$100 yet so we can either another \$1 bill to our collection or a \$100 bill which brings out total to 12 bills.

Answer:

12

Reference:

https://www.quantguide.io/questions/exact-bills

Exact Bills II

Category: Brainteasers
Difficulty: Easy
Companies: Optiver

Question:

What is the fewest amount of bills (denominations of \$1, \$5, \$20, and \$100) you need to always have exact change for a transaction of $\$x$ (paid using a \$100 bill) given that x is a positive integer less than 100?

Hint:

Which amount requires the most amount of bills and how many of each do we need?

Explanation:

Referring to "Exact Bills I", \$99 is the most difficult amount to make using the bills at our disposal. It requires a total of 4 \$20 bills, 3 \$5 bills, and 4 \$1 bills. We also have the exact change for any transactional amount less than \$100 as long as we have those bills and of that quantity as we can just use a subset of those bills.

Answer:

11

Reference:

https://www.quantguide.io/questions/exact-bills-ii

Beer Barrel II

Category: Brainteasers
Difficulty: Medium
Companies: Optiver

Question:

A 120-quart beer barrel was discovered by Anna's parents. Furious, they plan to dump the barrel. Anna begs her parents to let her keep some of the beer. They say that Anna may do so if she is able to measure out an exact quart into each of a 7-quart and 5-quart vessel. You have exactly 1 of each type of vessel. Note that Anna is allowed to pour beer from a vessel back into the beer barrel. Define a transaction as a pour of liquid from one container into another. What is the smallest number of transactions needed to accomplish the challenge? If impossible, respond with -1.

Hint:

To achieve the task in the fewest possible transactions, follow these steps (Here are the first 4):. Fill the 7-quart measure.. Fill the 5-quart measure.. Empty 108 quarts from the barrel.

4. Empty the 5-quart measure into the barrel.

Explanation:

To achieve the task in the fewest possible transactions (17 in total), follow these steps:. Fill the 7-quart measure.. Fill the 5-quart measure.. Empty 108 quarts from the barrel.

4. Empty the 5-quart measure into the barrel.

5. Fill the 5-quart measure from the 7-quart measure.

6. Empty the 5-quart measure into the barrel.

7. Pour 2 quarts from the 7-quart measure into the 5-quart measure.

8. Fill the 7-quart measure from the barrel.

9. Fill up the 5-quart measure from the 7-quart measure.

10. Empty the 5-quart measure into the barrel.

11. Pour 4 quarts from the 7-quart measure into the 5-quart measure.

12. Fill the 7-quart measure from the barrel.

13. Fill up the 5-quart measure from the 7-quart measure.

14. Empty the contents of the 5-quart measure.

15. Fill the 5-quart measure from the barrel.

16. Empty 5 quarts from the 5-quart measure.

17. Empty 1 quart from the barrel into the 5-quart measure.

By following these steps, you can accomplish the task using only 17 transactions, which is the minimum number required.

Answer:

17

Reference:

https://www.quantguide.io/questions/beer-barrel-ii