One dimensional heat equation

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1 The model

The shifted 1D heat equation is given by

$$z_t = \mu z_{xx} + \alpha z, \quad x \in (0,1) \times (0,T)$$

with boundary conditions

$$z(0,t) = 0, \quad z(1,t) = u(t)$$

and initial condition

$$z(x,0) = z_0(x), \quad x \in (0,1)$$

Here $\alpha \geq 0$ and $\mu > 0$.

1.1 Weak formulation

We assume $z_0 \in L^2(0,1)$. We wish to find $z \in L^2(0,T; H^1_{\{0\}}(0,1))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t),\phi)_{L^2} = -\mu \int_0^1 \frac{\partial z}{\partial x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\, dx + \alpha \int_0^1 z(x,t)\phi\, dx, \quad \forall \phi \in H^1_0(0,1)$$

$$z(1,t) = u(t)$$

$$(z(0),\phi)_{L^2} = (z_0,\phi)_{L^2}$$

1.2 FEM approximation

Consider a regular subdivision of $[0, 1]$

$$0 = x_0 < x_1 < ... < x_N = 1, \quad h = \frac{1}{N}, \quad x_k = kh \quad \forall k \in \{0, 1, ..., N\}$$

We work with the finite dimensional subspaces of $H_0^1(0, 1)$

$$\begin{aligned} Z_h &= \{\phi \in C([0, 1]) : \phi|_{[x_{i-1}, x_i]} \in P_1, \quad \phi(0) = 0\} \\ Z_{h,0} &= \{\phi \in C([0, 1]) : \phi|_{[x_{i-1}, x_i]} \in P_1, \quad \phi(0) = \phi(1) = 0\} \end{aligned}$$

We denote the standard P_1 bases for $Z_{h,0}$ and Z_h by $\{\phi_1, \phi_2, ..., \phi_{N-1}\}$ and $\{\phi_1, \phi_2, ..., \phi_N\}$ respectively, with $\phi_j(x_i) = \delta_{ij}$. The test function are chosen from $Z_{h,0}$. The approximate weak formulation is given as

$$\text{Find } z = \sum_{i=1}^{N-1} z_i\phi_i + u(t)\phi_N \in Z_h, \qquad \text{such that}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t), \phi_k)_{L^2} = -\mu \int_0^1 \frac{\partial z}{\partial x} \frac{\mathrm{d}\phi_k}{\mathrm{d}x} \, dx + \alpha \int_0^1 z(x, t)\phi_k \, dx, \quad \forall k \in \{1, ..., N - 1\}$$

$$(z(0), \phi_k)_{L^2} = (z_0, \phi_k)_{L^2}, \quad \forall k \in \{1, ..., N - 1\}$$

We obtain a a differential system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}_\alpha\mathbf{z} + \mathbf{B}u, \quad \mathbf{M}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1\leq i\leq N-1}$$

where

- $\mathbf{z} = (z_1, z_2, ..., z_{N-1})^\top$
- $\mathbf{M} \in \mathbb{R}^{(N-1)\times(N-1)}, \quad M_{i,j} = \int_0^1 \phi_i\phi_j \, dx$
- $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}, \quad A_{i,j} = -\mu \int_0^1 \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \frac{\mathrm{d}\phi_j}{\mathrm{d}x} \, dx$
- $\mathbf{A}_\alpha = \mathbf{A} + \alpha\mathbf{M}$
- $\mathbf{B} = \frac{\mu}{h}(0, 0, ..., 0, 1)^\top \in \mathbb{R}^{N-1}$
- $u \in \mathbb{R}$

Using exact integration for $\mathbf{M}$ and $\mathbf{A}$ gives

$$\mathbf{M} = h \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix}, \quad \mathbf{A} = \frac{\mu}{h} \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

We use the trapezoidal rule for $u'(t) \int_0^1 \phi_N \phi_{N-1}$ which will evaluate out as zero.

The matrices are computed in the matlab code `matrix_fem.m`

2 Time integration using BDF

Assume for now that the feedback control is of the form

$$u(t) = -\mathbf{K}\mathbf{z}(t), \quad \mathbf{K} \in \mathbb{R}^{1\times(N-1)}$$

We shall use a Backward Differentiation Formula (BDF) for time integration. For the first time step we use BDF1 (backward Euler scheme)

$$\mathbf{M}\frac{\mathbf{z}^1 - \mathbf{z}^0}{\Delta t} = (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\mathbf{z}^1$$

or

$$\left[\frac{\mathbf{M}}{\Delta t} - (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\right]\mathbf{z}^1 = \frac{\mathbf{M}}{\Delta t}\mathbf{z}^0$$

For the remaining time steps we use BDF2

$$\mathbf{M}\frac{\mathbf{z}^{n+1} - \frac{4}{3}\mathbf{z}^n + \frac{1}{3}\mathbf{z}^{n-1}}{\frac{2}{3}\Delta t} = (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\mathbf{z}^{n+1}$$

or

$$\left[\frac{\mathbf{M}}{\frac{2}{3}\Delta t} - (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\right]\mathbf{z}^{n+1} = -\frac{\mathbf{M}}{\frac{2}{3}\Delta t}\left(-\frac{4}{3}\mathbf{z}^n + \frac{1}{3}\mathbf{z}^{n-1}\right)$$

3 Excercises

1. The code in `heat.m` solves the problem with homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$ and initial condition

$$z(x, 0) = x^2(1-x)^3$$

Since $z(1,t) = u(t) = 0$ it is enough to solve

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z}$$

Run the code `heat.m` with $\alpha = 0$

```
alpha = 0;
mu = 1;
```

Are all the eigenvalues stable ? How do the solution and energy behave with time ?

2. Evaluate the decay rate of energy for initial condition $z(x,0) = \sin(\pi x)$

```
P = polyfit(t, log(energy'), 1)
P(1) % Exact rate = -2*mu*pi^2
```

3. With initial condition $z(x,0) = \sin(\pi x)$ run `heat.m` with

```
alpha = 0.4 + pi^2 * mu
```

Are all the eigenvalues stable ? How do the solution and energy behave with time ?

4. For same α as above, compute 10 eigenvalues with largest real part using `eigs` function. For each unstable eigenvalue λ, compute eigenvector $\mathbf{V}$ of $(\mathbf{A}_\alpha^\top, \mathbf{M}^\top)$

$$\mathbf{A}_\alpha^\top \mathbf{V} = \lambda \mathbf{M}^\top \mathbf{V}$$

and check if

$$\mathbf{B}^\top \mathbf{V} \neq 0$$

If the above is true for all unstable eigenvalues, then the system is stabilizable. Is the heat equation system stabilizable ?

5. Copy `heat.m` as `heat1.m` and modify `heat1.m` to implement the following non-homogeneous Dirichlet boundary condition

$$z(1,t) = u(t) = 1$$

and use the same initial condition as before. This means that the code must now solve $\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}$. Use $\mu = 1$ and $\alpha = 0$. The solution should converge to

$$\lim_{t\to\infty} z(x,t) = x$$

4 Minimal norm feedback control

The minimal norm control is given by

$$u(t) = -\mathbf{K}z(t)$$

The feedback matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M}$$

where $\mathbf{P}$ is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}_\alpha^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A}_\alpha - \mathbf{M}^\top\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M} = 0$$

This can be solved using the `care` function

```
[P,L,K] = care(A,B,R,Q,S,M);
```

which solves the more general ARE

$$\mathbf{A}^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A} - (\mathbf{M}^\top\mathbf{P}\mathbf{B} \ + \mathbf{S}^\top)\mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{P}\mathbf{M} + \mathbf{S}) + \mathbf{Q} = 0$$

where $\mathbf{L}$ gives the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K}, \mathbf{M})$. Since $\mathbf{S} = 0$ for our problem, we can simply replace it with `[]` while calling the `care` function. Furthermore, for the minimal norm feedback, we take $\mathbf{Q} = 0$, which can be set in `matrix_fem.m`

4.1 Excercises

1. Set $\mathbf{Q} = 0$ in `matrix_fem.m` which gives minimal norm control and use

   ```
   alpha = 0.4 + pi^2 * mu
   ```

   Run the code `heat_lqr.m`. Use initial condition $z(x, 0) = \sin(\pi x)$. Observe the modification in the eigenvalues; zoom the eigenvalue plot near the origin and locate the unstable eigenvalue and modified eigenvalue; note that the unstable eigenvalue is reflected about the imaginary axis. What about the remaining stable eigenvalues? Is the solution of the heat equation stable ?

2. Save the feedback matrix evaluated

   ```
   save('feedback.mat','K')
   ```

   Copy `heat.m` as `heat2.m`. Load $\mathbf{K}$ into `heat2.m`, and for `alpha = 0` (no shift) introduce control using the feedback matrix

   ```
   load('feedback');
   A = A-B*sparse(K);
   ```

   We now solve $\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{z}$. Find the decay rate in this case using `polyfit` function. We expect it to be atleast $2(0.4 + \mu\pi^2)$.

5 Feedback control using LQR approach

Performance measure

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \int_0^1 z^2(x,t)\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top R u \mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty \mathbf{z}^\top \mathbf{Q}\mathbf{z}\mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top \mathbf{R} u \mathrm{d}t, \qquad \mathbf{Q} = \mathbf{M} = \mathbf{C}^\top\mathbf{C}, \qquad \mathbf{C} = \mathbf{M}^{\frac{1}{2}} \end{aligned}$$

The feedback law

$$u = -\mathbf{K}\mathbf{z}$$

which minimizes J is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M}$$

where $\mathbf{P}$ is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}_\alpha^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A}_\alpha - \mathbf{M}^\top\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M} + \mathbf{Q} = 0$$

5.1 Excercises

1. In `matrix_fem.m`, put $\mathbf{Q} = \mathbf{M}$. Run `heat_lqr.m`. How do the eigenvalues change in LQR as compared to minimal norm control ? Is the solution stable with LQR feedback ?

2. Plot the intial condition and the numerical solution after one time step. How does it differ from the previous Bernoulli (minimal norm) control case ?

3. How does the solution and control vary as `mu` is decreased ?

4. In `matrix_fem.m`, vary the value of $\mathbf{R}$ in $(0.01, 50)$ (choose a couple of values in the range). How does $\mathbf{K}$ change with $\mathbf{R}$?

5. As before, save the feedback matrix and load it into `heat.m`. Is the decay rate increased with LQR feedback as compared to case of minimal norm control ?

6 Control based on stabilizing only the unstable components

Let the us denote by λ_i and $\mathbf{V}_i$ the eigenvalues and normalized eigenvectors such that

$$\mathbf{A}\mathbf{V}_i = \lambda_i\mathbf{M}\mathbf{V}_i$$

We assume that α is chosen such that

$$\lambda_{N-1} < .... < \lambda_{N_\alpha+1} < -\alpha < \lambda_{N_\alpha} < ... < \lambda_1$$

Thus in the shifted system for the heat equation, there will be N_α unstable eigenvalues.

$$\lambda_{N-1} + \alpha < .... < \lambda_{N_\alpha+1} + \alpha < 0 < \lambda_{N_\alpha} + \alpha < ... < \lambda_1 + \alpha$$

The eigenvectors $\mathbf{V}_i$ form a basis for $\mathbb{R}^{N-1}$ and have the property

$$\mathbf{V}_i^\top \mathbf{M} \mathbf{V}_j = \delta_{i,j}, \quad \mathbf{V}_i^\top \mathbf{A} \mathbf{V}_j = \delta_{i,j}\lambda_i$$

Let $\Sigma \in \mathbb{R}^{(N-1)\times(N-1)}$ with $\mathbf{V}_i$ forming the columns, and consider the variable change

$$\mathbf{z} = \Sigma \boldsymbol{\zeta}$$

Thus the shifted system can be written as

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \Lambda_\alpha \boldsymbol{\zeta} + \mathbb{B}u$$

where

$$\Lambda_\alpha = \begin{bmatrix} \alpha+\lambda_1 & 0 & \cdots & 0 \\ 0 & \alpha+\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \alpha+\lambda_{N-1} \end{bmatrix}, \qquad [\mathbb{B}]_i = V_i^\top \mathbf{B}, \qquad 1 \le i \le N-1$$

Next we project the system onto the unstable subspace using the projection operator $\mathbf{\Pi}_{\alpha,\mathbf{u}} \in \mathcal{L}(\mathbb{R}^{N-1}, \mathbb{R}^{N_\alpha})$

$$\frac{\mathrm{d}\boldsymbol{\zeta}_u}{\mathrm{d}t} = \Lambda_{\alpha,u} \boldsymbol{\zeta}_u + \mathbb{B}_u u$$

where

$$\Lambda_{\alpha,u} = \begin{bmatrix} \alpha+\lambda_1 & 0 & \cdots & 0 \\ 0 & \alpha+\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \alpha+\lambda_{N_\alpha} \end{bmatrix}, \qquad [\mathbb{B}_u]_i = V_i^\top \mathbf{B}, \qquad 1 \le i \le N_\alpha$$

and $\boldsymbol{\zeta}_u = \mathbf{\Pi}_{\alpha,\mathbf{u}} \boldsymbol{\zeta}$ are the first N_α components. We find the feedback matrix for this reduced system, by solving the Bernoulli equation

$$\mathbb{P}_u \Lambda_{\alpha,u} + \Lambda_{\alpha,u}^\top \mathbb{P}_u - \mathbb{P}_u \mathbb{B}_u \mathbb{B}_u^\top \mathbb{P}_u = 0 \tag{1}$$

$$\Lambda_{\alpha,u} - \mathbb{B}_u \mathbb{B}_u^\top \mathbb{P}_u \text{ is stable}$$

The corresponding matrix $\mathbf{P} \in \mathcal{L}(\mathbb{R}^{N-1})$ such that $(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{B}\mathbf{B}^\top \mathbf{P}\mathbf{M})$ is stable is given by

$$\mathbf{P} = \Sigma_u \mathbb{P}_u \Sigma_u^\top$$

where Σ_u is the matrix of eigenvectors corresponding to the unstable eigenvalues.

$$\Sigma_u = [V_1, V_2, \ldots, V_{N_\alpha}]$$

This is implemented in `heat_lqru.m`

6.1 Excercises

1. In `matrix_fem.m`, put $\mathbf{Q} = 0$. Run `heat_lqru.m`. How do the eigenvalues change this time compared to minimal norm control ? Is the solution stabilized ?

2. Plot the intial condition and the numerical solution after one time step. How does it differ from the previous two kinds of control ?

3. For the same value of `alpha = 0.4 + mu * pi * pi`, which of the three types of controls discussed so far performs the best, i.e., which one gives th fastest stabilization ?

4. Lecture 5 has the exact solutions for this problem, with `alpha = 10`. Take
   ```
   z0 = sin(pi*x)
   ```
   in the equation and find the expressions for exact control and solution. Plot and compare

   - evolution of the numerical control and the exact control
   - numerical and exact solutions at the final time

5. Save the feedback matrix and load it into `heat.m`. Is the decay rate increased with feedback ? How does it compare with the case where we solve the full Riccati equation as in previous section ?

6. This approach is efficient since we only have to solve a small dimensional Riccati equation. Try to solve the full Riccati equation with a large number of elements, say `ni = 500` and/or `ni = 1000` and find the time taken by matlab. Compare this with the time taken to solve the Riccati equation (1) whose dimension is small and independent of the number of elements.

7 System with noise and partial information

Consider the system with noise in the model and initial condition

$$z_t = \mu z_{xx} + \alpha z + w, \qquad z(x,0) = z_0(x) + \eta$$

where w and η are error/noise terms. The boundary conditions are as before

$$z(0,t) = 0, \quad z(1,t) = u(t)$$

We may not have access to the full state but only some partial information which is also corrupted by noise.

$$y = Hz + v$$

where H is a suitable measure. We shall consider the case where H is given by

$$Hz(t) = \mu z_x(0,t)$$

We have to approximate $z_x(0,t)$ by a finite difference formula. We can use a first order or second order finite differences

$$\begin{aligned} &\text{first order} \qquad && \mu z_x(0,t_n) \approx \mu \frac{z_1^n - z_0^n}{h} = \mu \frac{z_1^n}{h} \\ &\text{second order} \qquad && \mu z_x(0,t_n) \approx \mu \frac{-3z_0^n + 4z_1^n - z_2^n}{2h} = \mu \frac{4z_1^n - z_2^n}{2h} \end{aligned}$$

In the FEM setup, we get the system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}_\alpha \mathbf{z} + \mathbf{B}u + \mathbf{w}, \quad \mathbf{M}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1 \le i \le N-1}$$

$$\mathbf{y} = \mathbf{H}\mathbf{z} + \mathbf{v}$$

where

$$\mathbf{H} = \frac{\mu}{h}(2, 0.5, 0, ..., 0)^\top \in \mathbb{R}^{N-1}$$

7.1 Estimation problem

Linear estimator

$$\frac{\mathrm{d}z_e}{\mathrm{d}t} = A_\alpha z_e + Bu + L(y - Hz_e)$$

We determine the filtering gain L by minimizing

$$J = \frac{1}{2}\int_0^\infty (y - Hz_e)^\top R_v^{-1}(y - Hz_e)\mathrm{d}t + \frac{1}{2}\int_0^\infty w^\top R_w^{-1} w \mathrm{d}t$$

The solution is given by

$$L = P_e H^\top R_v^{-1}$$

where P_e is solution of

$$A_\alpha P_e + P_e A_\alpha^\top - P_e H^\top R_v^{-1} H P_e + R_w = 0$$

The operators R_w and R_v are chosen according to the apriori knowledge we have on the model noise and the measurement noise. In the FEM setup, we have

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{A}_\alpha \mathbf{z}_e + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)$$

$$\mathbf{L} = \mathbf{M}\mathbf{P}_e\mathbf{H}^\top \mathbf{R}_\mathbf{v}^{-1}$$

where $\mathbf{P}_e$ is solution of

$$\mathbf{A}_\alpha \mathbf{P}_e \mathbf{M} + \mathbf{M}\mathbf{P}_e \mathbf{A}_\alpha^\top - \mathbf{M}\mathbf{P}_e \mathbf{H}^\top \mathbf{R}_\mathbf{v}^{-1} \mathbf{H}\mathbf{P}_e \mathbf{M} + \mathbf{R}_\mathbf{w} = 0$$

$$(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{L}\mathbf{H}) \text{ is stable}$$

7.2 Coupled linear system

The feedback is based on estimated solution $u = -\mathbf{K}\mathbf{z}_e$

$$\begin{aligned}
\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= \mathbf{A}_\alpha \mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \mathbf{w} \\
\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} &= \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\mathbf{v}
\end{aligned}$$

or in matrix form

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\alpha & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A}_\alpha - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}$$

The initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0 + \eta, \qquad \mathbf{z}_e(0) = \mathbf{z}_0$$

This is implemented in program `heat_est.m`

7.3 Excercises

1. Study the program `heat_est.m` and run it. We are taking initial condition

$$\mathbf{z}(0) = \sin(\pi x) + 0.1\sin(2\pi x), \qquad \mathbf{z}_e(0) = \sin(\pi x)$$

2. Look at eigenvalues of (A, M), $(A - BK, M)$ and $(A - LH, M)$. Are they stable?

3. By default, the program takes $\mathbf{Q} = 0$. Try the LQR approach using $\mathbf{Q} = \mathbf{M}$.

7.4 Stabilizing the unstable component

We use the variable change to the eigenvector basis

$$\mathbf{z} = \Sigma \boldsymbol{\zeta}$$

The first N_α eigenvalues are unstable; the projection to the unstable space is $\boldsymbol{\zeta}_u = \boldsymbol{\Pi}_{\alpha,\mathbf{u}} \boldsymbol{\zeta}$. As done before, the feedback gain matrix $\mathbf{K}$ can be evaluated by considering the projected system

$$\frac{\mathrm{d}\boldsymbol{\zeta}_u}{\mathrm{d}t} = \Lambda_{\alpha,u} \boldsymbol{\zeta}_u + \mathbb{B}_u u$$

Define the operator $\mathbb{H}$ and $\mathbb{H}_u$ by

$$\mathbb{H}\boldsymbol{\zeta} = \mathbf{H}\mathbf{z}, \qquad \mathbb{H}_u = \mathbb{H}\boldsymbol{\Pi}_{\alpha,\mathbf{u}}$$

The filtering gain $\mathbb{P}_e$ for the reduced projected system is obtained as the solution of the ARE

$$\mathbb{P}_e \Lambda_{\alpha,u}^\top + \Lambda_{\alpha,u} \mathbb{P}_e - \mathbb{P}_e \mathbb{H}_u^\top \mathbf{R}_\mathbf{v}^{-1} \mathbb{H}_u \mathbb{P}_e + \mathbb{Q}_\mathbf{w} = 0$$

$$\Lambda_{\alpha,u} - \mathbb{P}_e \mathbb{H}_u^\top \mathbf{R}_\mathbf{v}^{-1} \mathbb{H}_u \text{ is stable}$$

where $\mathbb{Q}_\mathbf{w} = \Sigma_u^\top \mathbf{R}_\mathbf{w} \Sigma_u$. The corresponding $\mathbf{P}_e$ such that $(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{M}\mathbf{P}_e \mathbf{H}^\top \mathbf{R}_\mathbf{v}^{-1} \mathbf{H})$ is stable is given by

$$\mathbf{P}_e = \Sigma_u \mathbb{P}_e \Sigma_u^\top$$

7.5 Excercises

1. Write a code `heat_estu.m` for the above via the following steps.

   - Copy the contents of `heat_est.m` to a new file called `heat_estu.m`
   - Evaluate the feedback gain $\mathbf{K}$ as done in `heat_lqru.m`
   - Evaluate the projected matrix $\mathbb{H}_u$

     ```
     Hu = full(H*V);
     ```

     where `V` is the matrix of unstable eigenvectors (you would have already evaluated this while obtaining $\mathbf{K}$).
   - Evaluate $\mathbb{Q}_\mathbf{w}$

     ```
     Rwu = full(V'*Rw*V);
     ```

   - Solve the ARE for the reduced system to obtain $\mathbb{P}_e$

     ```
     Peu = care(D',Hu',Rwu,Rv);
     ```

     where `D` is the diagonalized matrix of the reduced system (also evaluated while finding $\mathbf{K}$)

- Find $\mathbf{P}_e$ for the original system
```
Pe = sparse(V*Peu*V');
```

- Finally evaluate $\mathbf{L}$ by
```
L = M*Pe*H'/Rv;
```

- Run `heat_estu.m` with `alpha = 10`. How do the solution and energy behave ?

8 List of Programs

1. `matrix_fem.m`: Computes FEM matrices
2. `feedback_matrix.m`: Computes feedback matrix
3. `heat.m`: Solves the heat equation for a given α, without feedback
4. `heat_lqr.m`: Solves the heat equation for a given α, with feedback
5. `heat_lqr.m`: Solves the heat equation for a given α, stabilizing only the unstable components
6. `heat_est.m`: Solves the coupled estimation and control problem for the heat equation for a given α
7. `heat_estu.m`: Solves the coupled estimation and control problem for the heat equation for a given α, stablilizing only the unstable components.