DGFEM for hyperbolic system of conservation laws

Praveen. C
praveen@tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://praveen.tifrbng.res.in

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System of conservation law

For a set of conserved quantities $\boldsymbol{w} \in \mathbb{R}^m$ with flux $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$, the system of conservation laws can be written as

$$\boldsymbol{w}_t + \boldsymbol{f}(\boldsymbol{w})_x = 0$$

We assume that this system is hyperbolic. This means that the Jacobian of the flux

$$A(\boldsymbol{w}) = \boldsymbol{f}'(\boldsymbol{w}) \in \mathbb{R}^{m \times m}$$

1. has only real eigenvalues
2. and the eigenvectors are linearly independent so that

$$\text{span of eigenvectors } = \mathbb{R}^m$$

Linear system

Consider the linear system of conservation laws

$$\frac{\partial \boldsymbol{w}}{\partial t} + A\frac{\partial \boldsymbol{w}}{\partial x} = 0, \qquad \boldsymbol{f} = A\boldsymbol{w}, \qquad A = \text{constant matrix}$$

Let

$$R = \text{matrix of right eigenvectors}, \qquad L = \text{matrix of left eigenvectors}$$

Then we can diagonalize A

$$A = R\Lambda R^{-1} = R\Lambda L, \qquad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$$

Define the characteristic variables

$$\boldsymbol{c} = R^{-1}\boldsymbol{w} = L\boldsymbol{w}$$

Then we get

$$L\frac{\partial \boldsymbol{w}}{\partial t} + LR\Lambda L\frac{\partial \boldsymbol{w}}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\partial \boldsymbol{c}}{\partial t} + \Lambda\frac{\partial \boldsymbol{c}}{\partial x} = 0$$

Linear system

The is a set of decoupled scalar convection equations

$$\frac{\partial c_i}{\partial t} + \lambda_i \frac{\partial c_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, m$$

We can solve for $c_i(x,t)$ and then obtain

$$\boldsymbol{w} = R\boldsymbol{c}$$

The solution $\boldsymbol{w}$ is a linear combination of the waves c_i.

Euler equations

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix}$$

Internal and total energy per unit volume

$$e = \frac{p}{\gamma - 1}, \qquad E = e + \frac{1}{2}\rho u^2, \qquad \gamma = \frac{C_p}{C_v} > 1$$

Physically admissible states

$$\mathcal{U}_{\text{ad}} = \{\boldsymbol{w} \in \mathbb{R}^3 : \rho(\boldsymbol{w}) > 0,\ p(\boldsymbol{w}) > 0\}$$

where

$$\rho(\boldsymbol{w}) = w_1, \qquad p(\boldsymbol{w}) = (\gamma - 1)\left[w_3 - \frac{1}{2}\frac{w_2^2}{w_1}\right]$$

The set $\mathcal{U}_{\text{ad}}$ is convex.

Euler equations

The Euler equations are hyperbolic. The flux Jacobian $A(\boldsymbol{w}) = \boldsymbol{f}'(\boldsymbol{w})$ has eigenvalues

$$u - a, \qquad u, \qquad u + a$$

where

$$a = \sqrt{\frac{\gamma p}{\rho}} = \text{speed of sound}$$

DG scheme

Divide domain $\Omega = [0,1]$ into cells $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$.

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 1$$

$$x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \qquad \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \qquad h = \max_i \Delta x_i$$

Let us assume that the mesh is regular, i.e., there is a constant $c > 0$ such that

$$\Delta x_i \geq ch$$

Space of broken polynomials

$$\boldsymbol{V}_h^k = \{\boldsymbol{v} : \boldsymbol{v}|_{I_i} \in \boldsymbol{P}_k(I_i),\ 1 \leq i \leq N\}$$

Note that these functions can be discontinuous on the boundary of the elements. Define the left and right limits

$$\boldsymbol{v}_h(x^-) = \lim_{\epsilon \searrow 0} \boldsymbol{v}_h(x - \epsilon), \qquad \boldsymbol{v}_h(x^+) = \lim_{\epsilon \searrow 0} \boldsymbol{v}_h(x + \epsilon)$$

DG scheme

Multiply conservation law by $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$ and integrate by parts

$$\int_{I_i} \frac{\partial \boldsymbol{w}}{\partial t} \cdot \boldsymbol{v}_h \mathrm{d}x - \int_{I_i} \boldsymbol{f}(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{v}_h}{\partial x} \mathrm{d}x$$
$$+ \boldsymbol{f}(\boldsymbol{w}(x^-_{i+\frac{1}{2}}, t)) \cdot \boldsymbol{v}_h(x^-_{i+\frac{1}{2}}) - \boldsymbol{f}(\boldsymbol{w}(x^+_{i-\frac{1}{2}}, t)) \cdot \boldsymbol{v}_h(x^+_{i-\frac{1}{2}}) = 0$$

We approximate the inter-element flux using a numerical flux function

$$\hat{\boldsymbol{f}}_{i+\frac{1}{2}}(t) = \hat{\boldsymbol{f}}(\boldsymbol{w}(x^-_{i+\frac{1}{2}}, t), \boldsymbol{w}(x^+_{i+\frac{1}{2}}, t))$$

This couples the solution in I_i to those in the neighbouring elements I_{i-1} and I_{i+1}.

Properties of numerical flux

1. Consistency

$$\hat{\boldsymbol{f}}(\boldsymbol{w}, \boldsymbol{w}) = \boldsymbol{f}(\boldsymbol{w})$$

2. Lipschitz continuous

$$\|\hat{\boldsymbol{f}}(\boldsymbol{u}_2, \boldsymbol{v}_2) - \hat{\boldsymbol{f}}(\boldsymbol{u}_1, \boldsymbol{v}_1)\| \leq L_1 \|\boldsymbol{u}_2 - \boldsymbol{u}_1\| + L_2 \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|$$

Semi-discrete DG scheme

Find $\boldsymbol{w}_h(\cdot, t) \in \boldsymbol{V}_h^k$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$

$$\int_{I_i} \frac{\partial \boldsymbol{w}_h}{\partial t} \cdot \boldsymbol{v}_h \mathrm{d}x - \int_{I_i} \boldsymbol{f}(\boldsymbol{w}_h) \cdot \frac{\partial \boldsymbol{v}_h}{\partial x} \mathrm{d}x + \hat{\boldsymbol{f}}_{i+\frac{1}{2}}(t) \cdot \boldsymbol{v}_h(x_{i+\frac{1}{2}}^-) - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}(t) \cdot \boldsymbol{v}_h(x_{i-\frac{1}{2}}^+) = 0$$

Enropy stability

See [1]

Limiters and TVD property

Forward difference in time: Find $\boldsymbol{w}_h^{n+1}$ such that

$$\int_{I_i} \frac{\tilde{\boldsymbol{w}}_h^{n+1} - \boldsymbol{w}_h^n}{\Delta t} \cdot \boldsymbol{v}_h \mathrm{d}x - \int_{I_i} \boldsymbol{f}(\boldsymbol{w}_h^n) \cdot \frac{\partial \boldsymbol{v}_h}{\partial x} \mathrm{d}x + \hat{\boldsymbol{f}}_{i+\frac{1}{2}}^n \cdot \boldsymbol{v}_h(x_{i+\frac{1}{2}}^-) - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}^n \cdot \boldsymbol{v}_h(x_{i-\frac{1}{2}}^+) = 0$$

The value $\tilde{\boldsymbol{w}}_h^{n+1}$ may be oscillatory. We limit it to obtain the new solution

$$\boldsymbol{w}_h^{n+1} = \Lambda\Pi_h(\tilde{\boldsymbol{w}}_h^{n+1})$$

Properties of $\Lambda\Pi_h(\cdot)$

1. It should not change the cell average value.
2. It should not affect the accuracy in smooth regions.

There are two possibilities to applying the limiter.

- Apply the limiter componentwise to the conserved variables.
- Apply the limiter componentwise to the characteristic variables.

Characteristic limiter

Let $\overline{\boldsymbol{w}}_i$ denote the cell average value in cell I_i. The solution has the form

$$\boldsymbol{w}_h = \overline{\boldsymbol{w}}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right)\boldsymbol{w}_x + HOT$$

Let

$$R_i = R(\overline{\boldsymbol{w}}_i) = \text{matrix of right eigenvectors of the flux Jacobian } \boldsymbol{f}'(\overline{\boldsymbol{w}}_i)$$

We define the local characteristic variables by

$$\boldsymbol{c}_i = R_i^{-1}\overline{\boldsymbol{w}}_i, \quad \boldsymbol{c}_{i\pm1} = R_i^{-1}\overline{\boldsymbol{w}}_{i\pm1}, \quad \boldsymbol{c}_x = R_i^{-1}\boldsymbol{w}_x$$

Limit the derivative

$$\boldsymbol{c}_x^{(m)} = m(\boldsymbol{c}_x, \boldsymbol{c}_i - \boldsymbol{c}_{i-1}, \boldsymbol{c}_{i+1} - \boldsymbol{c}_i)$$

If $\boldsymbol{c}_x^{(m)} = \boldsymbol{c}_x$ then

$$\Lambda\Pi_h(\boldsymbol{w}_h) = \boldsymbol{w}_h$$

Characteristic limiter

else

$$\Lambda\Pi_h(\boldsymbol{w}_h) = \overline{\boldsymbol{w}}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right)\boldsymbol{w}_x^{(m)}, \qquad \boldsymbol{w}_x^{(m)} = R_i \boldsymbol{c}_x^{(m)}$$

For better acccuracy at smooth extrema, we can use the TVB version of the minmod limiter.

Another version:

$$\boldsymbol{c}_i = R_i^{-1}\overline{\boldsymbol{w}}_i, \quad \boldsymbol{c}_{i\pm1} = R_i^{-1}\overline{\boldsymbol{w}}_{i\pm1}, \quad \boldsymbol{c}^+_{i-\frac{1}{2}} = R_i^{-1}\boldsymbol{w}^+_{i-\frac{1}{2}}, \quad \boldsymbol{c}^-_{i+\frac{1}{2}} = R_i^{-1}\boldsymbol{w}^-_{i+\frac{1}{2}}$$

and the differences

$$\hat{\boldsymbol{c}}_i = \boldsymbol{c}^-_{i+\frac{1}{2}} - \boldsymbol{c}_i, \qquad \check{\boldsymbol{c}}_i = \boldsymbol{c}_i - \boldsymbol{c}^+_{i-\frac{1}{2}}$$

$$\Delta_+\boldsymbol{c}_i = \boldsymbol{c}_{i+1} - \boldsymbol{c}_i, \qquad \Delta_-\boldsymbol{c}_i = \boldsymbol{c}_i - \boldsymbol{c}_{i-1}$$

We cannot modify the cell average value but we can modify the slopes

$$\hat{\boldsymbol{c}}_i^{(m)} = m(\hat{\boldsymbol{c}}_i, \Delta_+\boldsymbol{c}_i, \Delta_-\boldsymbol{c}_i), \qquad \check{\boldsymbol{c}}_i^{(m)} = m(\check{\boldsymbol{c}}_i, \Delta_+\boldsymbol{c}_i, \Delta_-\boldsymbol{c}_i)$$

Characteristic limiter

where m is the minmod function

$$m(a_1, \ldots, a_l) = \begin{cases} s \min(|a_1|, \ldots, |a_l|) & s = \text{sign}(a_1) = \ldots = \text{sign}(a_l) \\ 0 & \text{otherwise} \end{cases}$$

For $k = 1$: We have $\hat{\boldsymbol{c}}_i = \check{\boldsymbol{c}}_i$. The modified trace values are recomputed using the limited slopes

$$\boldsymbol{w}_h^{(m)}(x_{i+\frac{1}{2}}^-) = \overline{\boldsymbol{w}}_i + R_i \hat{\boldsymbol{c}}_i^{(m)}, \qquad \boldsymbol{w}_h^{(m)}(x_{i-\frac{1}{2}}^+) = \overline{\boldsymbol{w}}_i - R_i \check{\boldsymbol{c}}_i^{(m)}$$

This completely specifies the limited polynomial solution inside the cell.

For $k > 1$: If $\hat{\boldsymbol{c}}_i^{(m)} \neq \hat{\boldsymbol{c}}_i$ or $\check{\boldsymbol{c}}_i^{(m)} \neq \check{\boldsymbol{c}}$ then the limiter is active which indicates that we are probably near a discontinuity. In this case we retain only the linear part of the solution. The trace values of the linear part are taken as

$$\boldsymbol{w}_h^{(m)}(x_{i+\frac{1}{2}}^-) = \overline{\boldsymbol{w}}_i + \frac{1}{2} R_i (\check{\boldsymbol{c}}_i^{(m)} + \hat{\boldsymbol{c}}_i^{(m)}), \qquad \boldsymbol{w}_h^{(m)}(x_{i-\frac{1}{2}}^+) = \overline{\boldsymbol{w}}_i - \frac{1}{2} R_i (\check{\boldsymbol{c}}_i^{(m)} + \hat{\boldsymbol{c}}_i^{(m)})$$

Remark: We can use the TVB version of the minmod limiter which leads to a less restrictive limiter.

Linear system

In the case of the linear system

$$\frac{\partial \boldsymbol{w}}{\partial t} + A\frac{\partial \boldsymbol{w}}{\partial x} = 0, \qquad \boldsymbol{f} = A\boldsymbol{w}, \qquad A = \text{constant matrix}$$

TV property [2]

The DG scheme with characteristic limiter is total variation diminishing in the means with the TVD limiter and total variation bounded with the TVB limiter, where the total variation is defined as

$$\mathrm{TVM}(\boldsymbol{w}_h) = \sum_i \sum_{j=1}^{m} |\overline{w}_{i,j} - \overline{w}_{i-1,j}|$$

Some implementation details

We have a vector of unknowns $\boldsymbol{w}_h$. We write each component of $\boldsymbol{w}_h$ in terms of the usual basis functions, e.g., Legendre polynomials. The j'th component of $\boldsymbol{w}_h$ has the form

$$x \in I_i: \qquad w_j(x,t) = \sum_{s=1}^{k+1} w_{i,j,s}(t)\phi_{i,s}(x), \qquad j = 1, \ldots, m$$

where $k =$ degree of polynomial space. We have $(k+1)m$ dofs. The test functions are of the form

$$\boldsymbol{v}_h = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \phi_{i,r} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{There are } (k+1)m \text{ test functions.}$$

Some implementation details

The DG is given by

$$\int_{I_i} \frac{\partial w_j}{\partial t} \phi_{i,r} \mathrm{d}x - \int_{I_i} f_j(\boldsymbol{w}_h) \frac{\partial \phi_{i,r}}{\partial x} \mathrm{d}x + (\hat{\boldsymbol{f}}_{i+\frac{1}{2}})_j \phi_{i,r}(x_{i+\frac{1}{2}}^-) - (\hat{\boldsymbol{f}}_{i-\frac{1}{2}})_j \phi_{i,r}(x_{i-\frac{1}{2}}^+) = 0$$

$$r = 1, 2, \ldots, k+1, \qquad j = 1, 2, \ldots, m$$

For component j define its dofs

$$\vec{w}_{i,j} = [w_{i,j,1}, \ldots, w_{i,j,k+1}]^\top \in \mathbb{R}^{k+1}$$

Then the DG scheme in element I_i can be written as

$$M_i \frac{\mathrm{d}\vec{w}_{i,j}}{\mathrm{d}t} = L_{i,j}(\boldsymbol{w}_h), \qquad j = 1, 2, \ldots, m$$

where

$$M_i \in \mathbb{R}^{(k+1)\times(k+1)} = \text{mass matrix}$$

Some implementation details

with components

$$(M_i)_{rs} = \int_{I_i} \phi_{i,r}\phi_{i,s}\mathrm{d}x, \qquad r,s = 1,2,\ldots,k+1$$

This is a symmetric, positive definite matrix, and usually also diagonal.

The integrals are evaluated using some Gauss quadrature rule. Then the resulting system of ODE can be integrated in time using a Runge-Kutta scheme.

Time step

For degree k polynomials and $(k+1)$-stage Runge-Kutta scheme

$$\Delta t_i = \text{cfl}\frac{(|u_i| + c_i)\Delta x_i}{2k+1}, \qquad 0 < \text{cfl} \leq 1$$

Global time step

$$\Delta t = \min_i \Delta t_i$$

Positivity limiter for Euler equations

To construct a positivity preserving DG scheme, we notice the following properties.

- $\rho(\boldsymbol{w})$ is a linear function.

$$\rho(\theta \boldsymbol{w}_1 + (1-\theta)\boldsymbol{w}_2) = \theta\rho(\boldsymbol{w}_1) + (1-\theta)\rho(\boldsymbol{w}_2), \quad \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{U}_{\mathrm{ad}}, \quad \theta \in [0,1]$$

- $p(\boldsymbol{w})$ is a concave function

$$p(\theta \boldsymbol{w}_1 + (1-\theta)\boldsymbol{w}_2) \geq \theta p(\boldsymbol{w}_1) + (1-\theta)p(\boldsymbol{w}_2), \quad \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{U}_{\mathrm{ad}}, \quad \theta \in [0,1]$$

Positivity limiter for Euler equations

Assumption (Positivity preserving FVM)

Assume that the first order finite volume scheme

$$\boldsymbol{w}_i^{n+1} = \boldsymbol{w}_i^n - \frac{\Delta t}{\Delta x_i}[\hat{\boldsymbol{f}}(\boldsymbol{w}_i^n, \boldsymbol{w}_{i+1}^n) - \hat{\boldsymbol{f}}(\boldsymbol{w}_{i-1}^n, \boldsymbol{w}_i^n)]$$

is positivity preserving, i.e.,

$$\boldsymbol{w}_{i-1}^n, \boldsymbol{w}_i^n, \boldsymbol{w}_{i+1}^n \in \mathcal{U}_{\text{ad}} \quad \Longrightarrow \quad \boldsymbol{w}_i^{n+1} \in \mathcal{U}_{\text{ad}}$$

under a CFL condition

$$\frac{S_i \Delta t}{\Delta x_i} \le 1, \qquad S_i = S(\boldsymbol{w}_{i-1}, \boldsymbol{w}_i, \boldsymbol{w}_{i+1})$$

In the DG scheme, the cell average solution evolves according to

$$\overline{\boldsymbol{w}}_i^{n+1} = \overline{\boldsymbol{w}}_i^n - \frac{\Delta t}{\Delta x_i}[\hat{\boldsymbol{f}}(\boldsymbol{w}_{i+\frac{1}{2}}^-, \boldsymbol{w}_{i+\frac{1}{2}}^+) - \hat{\boldsymbol{f}}(\boldsymbol{w}_{i-\frac{1}{2}}^-, \boldsymbol{w}_{i-\frac{1}{2}}^+)]$$

Positivity limiter for Euler equations

We write the cell average using Q-point Gauss-Lobatto quadrature

$$\overline{\boldsymbol{w}}_i = \sum_{q=1}^{Q} \boldsymbol{w}_{i,q}\omega_q, \qquad \boldsymbol{w}_{i,q} = \boldsymbol{w}_h(x_{i,q}), \qquad 2Q - 3 \geq k$$

Then the equation for the cell average can be written as a convex combination of first order finite volume schemes

$$\overline{\boldsymbol{w}}_i^{n+1} = \sum_{q=1}^{Q} \omega_q \left\{ \boldsymbol{w}_{i,q}^n - \frac{\Delta t}{\omega_q \Delta x_i} [\hat{\boldsymbol{f}}(\boldsymbol{w}_{i,q}^n, \boldsymbol{w}_{i,q+1}^n) - \hat{\boldsymbol{f}}(\boldsymbol{w}_{i,q-1}^n, \boldsymbol{w}_{i,q}^n)] \right\}$$

with the convention that

$$\boldsymbol{w}_{i,0} = \boldsymbol{w}_{i-\frac{1}{2}}^-, \qquad \boldsymbol{w}_{i,Q+1} = \boldsymbol{w}_{i+\frac{1}{2}}^+$$

Positivity limiter for Euler equations

Theorem

Assume that $\overline{\boldsymbol{w}}_i^n \in \mathcal{U}_{\mathrm{ad}}$, $\boldsymbol{w}_{i,q}^n \in \mathcal{U}_{\mathrm{ad}}$. Then under the CFL condition

$$\frac{S_{i,q} \Delta t}{\omega_q \Delta x_i} \le 1, \qquad S_{i,q} = S(\boldsymbol{w}_{i,q-1}, \boldsymbol{w}_{i,q}, \boldsymbol{w}_{i,q+1})$$

we have $\overline{\boldsymbol{w}}_i^{n+1} \in \mathcal{U}_{\mathrm{ad}}$.

At time level n assume that (we suppress the time index)

$$\overline{\rho}_i = \rho(\overline{\boldsymbol{w}}_i) \ge \epsilon, \qquad \overline{p}_i = p(\overline{\boldsymbol{w}}_i) \ge \epsilon$$

We can take $\epsilon = 10^{-13}$ for example in double precision computations. We have to modify $\boldsymbol{w}_h(x)$ so that $\boldsymbol{w}_h(x_{i,q}) \in \mathcal{U}_{\mathrm{ad}}$. In fact we will modify $\boldsymbol{w}_h(x)$ so that $\boldsymbol{w}_h(x_{i,q}) \in \mathcal{U}_{\mathrm{ad}}^{\epsilon}$ where

$$\mathcal{U}_{\mathrm{ad}}^{\epsilon} = \{\boldsymbol{w} \in \mathbb{R}^3 : \rho(\boldsymbol{w}) \ge \epsilon, \quad p(\boldsymbol{w}) \ge \epsilon\}$$

Positivity limiter for Euler equations

Also define

$$\partial\mathcal{U}_{\mathrm{ad}}^{\epsilon} = \{\boldsymbol{w} \in \mathbb{R}^3 : \rho(\boldsymbol{w}) \ge \epsilon, \quad p(\boldsymbol{w}) = \epsilon\}$$

which contains part of the boundary of $\mathcal{U}_{\mathrm{ad}}^{\epsilon}$.

Step 1: Modify $\rho_h(x) \to \hat{\rho}_h(x)$ so that $\hat{\rho}_h(x_{i,q}) \ge \epsilon$. Following the ideas in the scalar case this can be achieved as

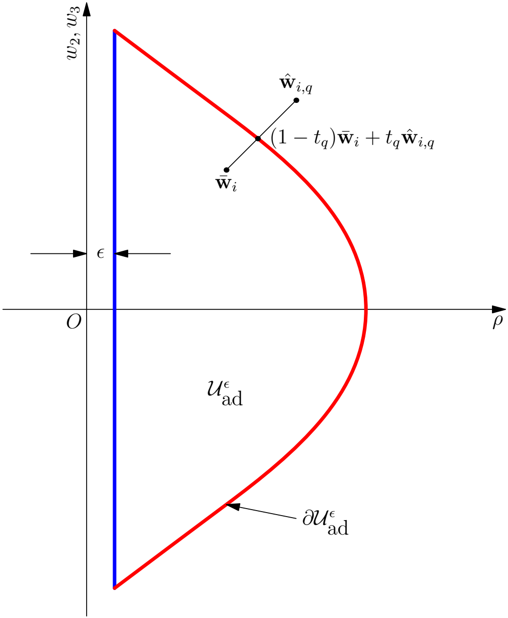
$$\hat{\rho}_h(x) = \theta_1(\rho_h(x) - \overline{\rho}_i) + \overline{\rho}_i, \quad \theta_1 = \min\left\{\frac{|\overline{\rho}_i - \epsilon|}{|\overline{\rho}_i - \rho_m|}, 1\right\}, \quad \rho_m = \min_q \rho_h(x_{i,q})$$

Step 2: Now $\hat{\boldsymbol{w}}_h(x)$ has positive density. We modify it to $\tilde{\boldsymbol{w}}_h(x)$ so that it also has positive pressure. If $p(\hat{\boldsymbol{w}}_{i,q}) < \epsilon$, then find the state between $\overline{\boldsymbol{w}}_i$ and $\hat{\boldsymbol{w}}_{i,q}$ such that (see figure)

$$p((1 - t_q)\overline{\boldsymbol{w}}_i + t_q \hat{\boldsymbol{w}}_{i,q}) = \epsilon$$

This is a quadratic equation for t_q which has unique solution $t_q \in (0, 1)$. If $p(\hat{\boldsymbol{w}}_{i,q}) \ge \epsilon$ then set $t_q = 1$. Then

$$\tilde{\boldsymbol{w}}_h(x) = \theta_2(\hat{\boldsymbol{w}}_h(x) - \overline{\boldsymbol{w}}_i) + \overline{\boldsymbol{w}}_i, \qquad \theta_2 = \min_q t_q$$



w_2, w_3
$\hat{\mathbf{w}}_{i,q}$
$(1 - t_q)\bar{\mathbf{w}}_i + t_q\hat{\mathbf{w}}_{i,q}$
$\bar{\mathbf{w}}_i$
ϵ
O
ρ
$\mathcal{U}_{\mathrm{ad}}^{\epsilon}$
$\partial\mathcal{U}_{\mathrm{ad}}^{\epsilon}$

Lemma

$$\tilde{\boldsymbol{w}}_h(x_{i,q}) \in \mathcal{U}_{\text{ad}}^{\epsilon} \subset \mathcal{U}_{\text{ad}}$$

<u>Proof</u>: $\tilde{\boldsymbol{w}}_h$ is a convex combination of $\overline{\boldsymbol{w}}_i$ and $\hat{\boldsymbol{w}}_h$. Since

$$\rho(\overline{\boldsymbol{w}}_i) \ge \epsilon, \quad \rho(\hat{\boldsymbol{w}}_h(x_{i,q})) \ge \epsilon$$

$$\Longrightarrow \quad \rho(\tilde{\boldsymbol{w}}_h(x_{i,q})) = (1-\theta_2)\rho(\overline{\boldsymbol{w}}_i) + \theta_2 \rho(\hat{\boldsymbol{w}}_h(x_{i,q})) \ge \epsilon$$

Now

$$\begin{aligned}
\tilde{\boldsymbol{w}}_{i,q} &= \theta_2(\hat{\boldsymbol{w}}_{i,q} - \overline{\boldsymbol{w}}_i) + \overline{\boldsymbol{w}}_i \\
&= \frac{\theta_2}{t_q}[(1-t_q)\overline{\boldsymbol{w}}_i + t_q \hat{\boldsymbol{w}}_{i,q}] + \left(1 - \frac{\theta_2}{t_q}\right)\overline{\boldsymbol{w}}_i
\end{aligned}$$

is a convex combination since $\theta_2 \le t_q$. Since $p(\boldsymbol{w})$ is concave we get

$$\begin{aligned}
p(\tilde{\boldsymbol{w}}_{i,q}) &\ge \frac{\theta_2}{t_q} p((1-t_q)\overline{\boldsymbol{w}}_i + t_q \hat{\boldsymbol{w}}_{i,q}) + \left(1 - \frac{\theta_2}{t_q}\right) p(\overline{\boldsymbol{w}}_i) \\
&\ge \frac{\theta_2}{t_q}\epsilon + \left(1 - \frac{\theta_2}{t_q}\right)\epsilon \\
&= \epsilon
\end{aligned}$$

Solution of quadratic equation

$$p((1-t)\overline{\boldsymbol{w}} + t\boldsymbol{w}) = \epsilon$$

Let $\overline{\boldsymbol{w}} = [\overline{\rho}, \overline{m}, \overline{E}]^\top$ and $\boldsymbol{w} = [\rho, m, E]^\top$. Then

$$(1-t)\bar{E} + tE - \frac{[(1-t)\bar{m} + tm]^2}{2[(1-t)\bar{\rho} + t\rho]} = \frac{\epsilon}{\gamma - 1}$$

Define

$$\Delta(\bar{\cdot}) = (\cdot) - (\bar{\cdot})$$

Then

$$2[\bar{\rho} + t\Delta\rho][\bar{E} + t\Delta E] - [\bar{m}^2 + t^2(\Delta m)^2 + 2t\bar{m}\Delta m] = \frac{2\epsilon}{\gamma - 1}[\bar{\rho} + t\Delta\rho]$$

We get a quadratic equation for t

$$t^2[2\Delta\rho\Delta E - (\Delta m)^2] + t[2\bar{E}\Delta\rho + 2\bar{\rho}\Delta E - 2\bar{m}\Delta m - 2\epsilon\Delta\rho/(\gamma - 1)]$$
$$+ 2\bar{\rho}\bar{E} - \bar{m}^2 - \frac{2\epsilon\bar{\rho}}{\gamma - 1} = 0$$

References

T. Barth, Numerical methods for gasdynamic systems on unstructured grids, in: Kroner, Ohlberger, Rohde (Eds.), An introduction to recent developments in theory and numerics for conservation laws, Vol. 5 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, 1998, pp. 198–285.

B. Cockburn, S.-Y. Lin, C.-W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: One-dimensional systems, J. Comput. Phys. 84 (1989) 90–113.