DGFEM for 1-D scalar hyperbolic conservation laws

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Scalar conservation law

A scalar conservation law is of the form

$$u_t + f(u)_x = 0$$

where

- u is called the conserved variable
- $f(u)$ is the flux of u

Conservation principle

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u(x,t)\mathrm{d}x + f(u(b,t)) - f(u(a,t)) = 0$$

Rate of change of total u in $[a,b]$ = Net flux of u into $[a,b]$

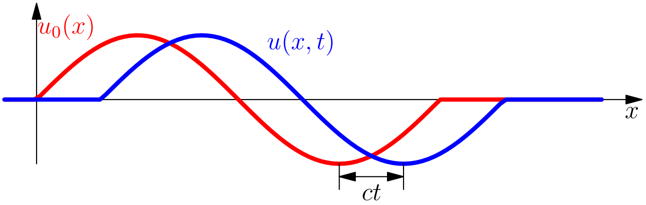
Scalar conservation law

Linear convection equation: $f(u) = cu$, $c =$ constant

$$u_t + cu_x = 0, \qquad u(x,0) = u_0(x)$$

Exact solution

$$u(x,t) = u_0(x - ct)$$



Initial condition is transported with velocity c without change of form.

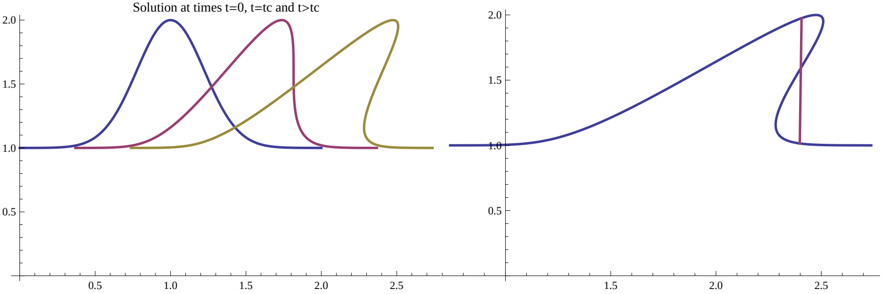
Scalar conservation law

Burger's equation: $f(u) = \frac{1}{2}u^2$

$$u_t + (\frac{1}{2}u^2)_x = 0 \qquad \text{or} \qquad u_t + uu_x = 0 \qquad \text{if } u \text{ is smooth}$$

Exact solution (as long as shocks do not form)

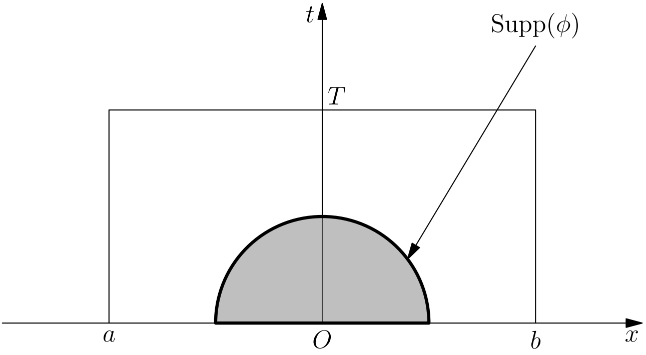
$$u(x,t) = u_0(x - u(x,t)t)$$



Weak solution

Take a smooth test function with compact support, say $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$

$$\int_0^\infty \int_{\mathbb{R}} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \phi \mathrm{d}x\mathrm{d}t = 0$$



Weak solution

Integrate by parts in both terms

Definition: Weak solution

A function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a weak solution of the IVP

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \qquad u(x,0) = u_0(x)$$

together with locally integrable initial data u_0 if u is locally integrable and satisfies

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + f(u)\phi_x)\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0, \qquad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$$

Lemma: Classical solution

Let $u \in C^1(\mathbb{R} \times \mathbb{R}^+)$ be a weak solution. Then it is a classical solution.

Rankine-Hugoniot condition

Suppose there is a discontinuity at $x(t)$ and let

$$s = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \text{shock speed}$$

Then the two values $u(x^-(t), t)$ and $u(x^+(t), t)$ satisfy the RH condition

$$f(u(x^+(t), t)) - f(u(x^-(t), t)) = s(u(x^+(t), t) - u(x^-(t), t))$$

Weak solution

A weak solution u is a piecewise smooth solution which satisfies the RH condition at the points of discontinuity of u.

Kruzkov's result

The scalar Cauchy problem

$$u_t + f(u)_x = 0, \quad f \in C^1(\mathbb{R})$$

with initial condition

$$u(0,x) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R})$$

has a **unique entropy solution**

$$u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$$

which fulfills (important for numerics)

1. Stability: $||u(t,\cdot)||_{L^\infty} \leq ||u_0||_{L^\infty}$, a.e. in $t \in \mathbb{R}_+$
2. Monotone: if $u_0 \geq v_0$ a.e. in $\mathbb{R}$, then

$$u(t,\cdot) \geq v(t,\cdot) \qquad \text{a.e. in } \mathbb{R}, \text{ a.e. in t} \in \mathbb{R}_+$$

Kruzkov's result

3. <u>TV-diminishing</u>: if $u_0 \in BV(\mathbb{R})$ then

$$u(t,\cdot) \in BV(\mathbb{R}) \quad \text{and} \quad TV(u(t,\cdot)) \leq TV(u_0)$$

4. <u>Conservation</u>: if $u_0 \in L^1(\mathbb{R})$ then

$$\int_{\mathbb{R}} u(t,x)\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\mathrm{d}x, \qquad \text{a.e. in } t \in \mathbb{R}_+$$

5. <u>Finite domain of dependence</u>: if u, v are two entropy solutions corresponding to $u_0, v_0 \in L^\infty$ and

$$M = \max_{\phi}\{|f'(\phi)| : |\phi| \leq \max(||u_0||_{L^\infty}, ||v_0||_{L^\infty})\}$$

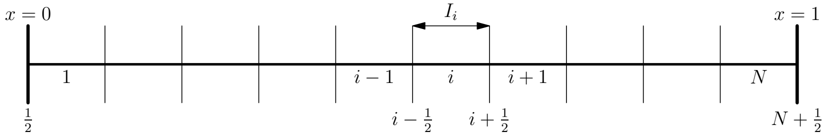
then

$$\int_{|x|\leq R} |u(t,x) - v(t,x)|\mathrm{d}x \leq \int_{|x|\leq R+Mt} |u_0(x) - v_0(x)|\mathrm{d}x$$

DG scheme

Divide domain $\Omega = [0,1]$ into cells $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$.

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 1$$



$$x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \qquad \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \qquad h = \max_i \Delta x_i$$

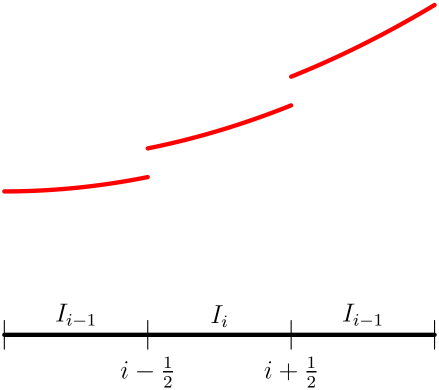
Let us assume that the mesh is regular, i.e., there is a constant $c > 0$ such that

$$\Delta x_i \geq ch$$

Space of broken polynomials

$$V_h^k = \{v \in L^2(\Omega) : v|_{I_i} \in \mathbb{P}_k(I_i),\ 1 \leq i \leq N\}$$

DG scheme



Note that these functions can be discontinuous on the boundary of the elements. Define the left and right limits

$$v_h(x^-) = \lim_{\epsilon \searrow 0} v_h(x-\epsilon)$$

$$v_h(x^+) = \lim_{\epsilon \searrow 0} v_h(x+\epsilon)$$

Multiply conservation law by test function $v_h \in V_h^k$

$$\int_{I_i} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) v_h \mathrm{d}x = 0$$

DG scheme

and integrate by parts on flux derivative term

$$\int_{I_i} \frac{\partial u}{\partial t} v_h \mathrm{d}x - \int_{I_i} f(u) \frac{\partial v_h}{\partial x} \mathrm{d}x \\ + f(x_{i+\frac{1}{2}}, t)) v_h(x_{i+\frac{1}{2}}^-) - f(x_{i-\frac{1}{2}}, t)) v_h(x_{i-\frac{1}{2}}^+) = 0$$

Since u may be discontinuous at $x = x_{i+\frac{1}{2}}$ how to compute $f(x_{i+\frac{1}{2}}, t))$ etc. ?

Numerical flux

We solve the **Riemann problem**

$$\frac{\partial w}{\partial \tau} + \frac{\partial f(w)}{\partial x} = 0, \qquad w(x, \tau = t) = \begin{cases} u(x^-_{i+\frac{1}{2}}, t) & x < x_{i+\frac{1}{2}} \\ u(x^+_{i+\frac{1}{2}}, t) & x > x_{i+\frac{1}{2}} \end{cases}, \qquad \tau \geq t$$

This has a self-similar

$$w(x, \tau) = w_R((x - x_{i+\frac{1}{2}})/(t - \tau); u(x^-_{i+\frac{1}{2}}, t), u(x^+_{i+\frac{1}{2}}, t))$$

and the flux across $x = x_{i+\frac{1}{2}}$ is given by

$$\hat{f}_{i+\frac{1}{2}} = f(w_R(0; u(x^-_{i+\frac{1}{2}}, t), u(x^+_{i+\frac{1}{2}}, t)) \qquad \text{(Godunov flux)}$$

In practice, the Riemann problem is solved approximately.
We approximate the inter-element flux using a **numerical flux function**

$$\hat{f}_{i+\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$$

This couples the solution in I_i to those in the neighbouring elements.

Numerical flux

Properties of numerical flux

1. Consistency

$$\hat{f}(u,u) = f(u)$$

2. Lipschitz continuous

$$|\hat{f}(a_2,b_2) - \hat{f}(a_1,b_1)| \leq L_1|a_2 - a_1| + L_2|b_2 - b_1|$$

3. monotone

$$\hat{f}(a,b), \qquad \text{increasing in } a \text{ and decreasing in } b, \quad f(\uparrow,\downarrow)$$

Numerical flux: Linear convection equation

The flux is linear $f(u) = au$ where $a =$ constant.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

The upwind numerical flux is

$$\hat{f}(u^-, u^+) = \begin{cases} au^- & \text{if } a \ge 0 \\ au^+ & \text{if } a < 0 \end{cases}$$

This can also be written as

$$\hat{f}(u^-, u^+) = \underbrace{\frac{1}{2}[f(u^-) + f(u^+)]}_{\text{centered flux}} - \underbrace{\frac{1}{2}|a|(u^+ - u^-)}_{\text{dissipative flux}}$$

Numerical flux: Non-linear conservation law

Godunov flux:

$$\hat{f} = \begin{cases} \min_{u \in [u^-, u^+]} f(u) & u^- \le u^+ \\ \max_{u \in [u^+, u^-]} f(u) & u^- > u^+ \end{cases}$$

Local Lax-Friedrich's flux:

$$\hat{f} = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}\lambda(u^+ - u^-)$$

where

$$\lambda = \max_{\xi \in (u^-, u^+)} |f'(\xi)|$$

A simple choice is

$$\lambda = \max\{|f'(u^-)|, |f'(u^+)|\}$$

Roe flux:

$$\hat{f} = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}\lambda(u^+ - u^-)$$

where

$$\lambda = \left| \frac{f(u^+) - f(u^-)}{u^+ - u^-} \right|$$

Need to modify Roe scheme to satisfy entropy condition.

Semi-discrete DG scheme

Semi-discrete DG scheme

Find $u_h(\cdot, t) \in V_h^k$ such that for all $v_h \in V_h^k$

$$\int_{I_i} \frac{\partial u_h}{\partial t} v_h \mathrm{d}x - \int_{I_i} f(u_h) \frac{\partial v_h}{\partial x} \mathrm{d}x + \hat{f}_{i+\frac{1}{2}}(t) v_h(x^-_{i+\frac{1}{2}}) - \hat{f}_{i-\frac{1}{2}}(t) v_h(x^+_{i-\frac{1}{2}}) = 0$$

DG and finite volume scheme

If the degree is $k = 0$, then we have only the constant basis function $v_h = 1$, and the solution is piecewise constant

$$u_h = u_i, \qquad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$$

The DG scheme takes the form

$$\Delta x_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + \hat{f}(u_i, u_{i+1}) - \hat{f}(u_{i-1}, u_i) = 0$$

which is also known as finite volume scheme.

DG scheme is conservative

Let degree $k > 0$. Take the test function $v_h \in V_h^k$

$$v_h = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

The DG scheme

$$\int_{I_i} \frac{\partial u_h}{\partial t} v_h \mathrm{d}x - \int_{I_i} f(u_h) \frac{\partial v_h}{\partial x} \mathrm{d}x + \hat{f}_{i+\frac{1}{2}}(t) v_h(x_{i+\frac{1}{2}}^-) - \hat{f}_{i-\frac{1}{2}}(t) v_h(x_{i-\frac{1}{2}}^+) = 0$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{I_i} u_h \mathrm{d}x + \hat{f}_{i+\frac{1}{2}}(t) - \hat{f}_{i-\frac{1}{2}}(t) = 0$$

which is a statement of conservation of u in the element I_i.

Entropy condition

Weak solutions can be non-unique. To obtain a unique weak solution, we need to impose an entropy condition. Let

$$U(u) = \text{ convex entropy function}$$

$$F(u) = \text{ associated entropy flux}$$

such that

$$F'(u) = U'(u) f'(u)$$

If u is smooth, it satisfies additional conservation law

$$U'(u)\frac{\partial u}{\partial t} + U'(u) \underbrace{f'(u)\frac{\partial u}{\partial x}}_{\frac{\partial f}{\partial x}} = 0 \quad \Longrightarrow \quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

For discontinuous solution, we will demand that it satisfy the entropy inequality

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \le 0 \qquad \text{in the sense of distributions}$$

with equality in smooth regions.

Entropy stability of DGFEM

Cell entropy inequality

The solution u_h of semi-discrete DG scheme satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_i} U(u_h)\mathrm{d}x + \hat{F}_{i+\frac{1}{2}}(t) - \hat{F}_{i-\frac{1}{2}}(t) \leq 0$$

for the square entropy $U(u) = \frac{1}{2}u^2$ with some consistent numerical entropy flux $\hat{F}_{i+\frac{1}{2}}(t) = \hat{F}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$.

Proof: Take $v_h = u_h$ in the DG scheme

$$\int_{I_i} \frac{\partial u_h}{\partial t} u_h - \int_{I_i} f(u_h)\frac{\partial u_h}{\partial x} + \hat{f}_{i+\frac{1}{2}}(t) u_h(x^-_{i+\frac{1}{2}}, t) - \hat{f}_{i-\frac{1}{2}}(t) u_h(x^+_{i-\frac{1}{2}}, t) = 0$$

Define

$$\tilde{F}(u) = \int^u f(u)\mathrm{d}u \qquad \Longrightarrow \qquad \tilde{F}'(u) = f(u)$$

Entropy stability of DGFEM

Then

$$\int_{I_i} \frac{\partial U(u_h)}{\partial t} - \tilde{F}(u_h(x^-_{i+\frac{1}{2}}, t)) + \tilde{F}(u_h(x^+_{i-\frac{1}{2}}, t))$$
$$+ \hat{f}_{i+\frac{1}{2}}(t) u_h(x^-_{i+\frac{1}{2}}, t) - \hat{f}_{i-\frac{1}{2}}(t) u_h(x^+_{i-\frac{1}{2}}, t) = 0$$

This can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{I_i} U(u_h) \mathrm{d}x + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}} = 0$$

with consistent numerical entropy flux

$$\hat{F}_{i+\frac{1}{2}} = -\tilde{F}(u_h(x^-_{i+\frac{1}{2}})) + \hat{f}_{i+\frac{1}{2}} u_h(x^-_{i+\frac{1}{2}})$$

and

$$\Theta_{i-\frac{1}{2}} = -\tilde{F}(u_h(x^-_{i-\frac{1}{2}})) + \hat{f}_{i-\frac{1}{2}} u_h(x^-_{i-\frac{1}{2}}) + \tilde{F}(u_h(x^+_{i-\frac{1}{2}})) - \hat{f}_{i-\frac{1}{2}} u_h(x^+_{i-\frac{1}{2}})$$

Entropy stability of DGFEM

This can be written as, for some ξ between u_h^-, u_h^+

$$\begin{aligned}\Theta &= -\tilde{F}(u_h^-) + \hat{f}u_h^- + \tilde{F}(u_h^+) - \hat{f}u_h^+ \\ &= (u_h^+ - u_h^-)(\tilde{F}'(\xi) - \hat{f}) \\ &= (u_h^+ - u_h^-)(f(\xi) - \hat{f}) \geq 0\end{aligned}$$

where the last inequality comes from the monotone property of the numerical flux $\hat{f}$. Thus *the semi-discrete DG scheme satisfies the entropy condition for any order of the basis functions* k.

Remark: To obtain entropy inequality, we can also use the **E-flux** condition

$$(u^+ - u^-)(f(\xi) - \hat{f}(u^-, u^+)) \geq 0, \qquad \forall \xi \text{ between } u^-, u^+$$

This condition can be extended to system of conservation laws.

Remark: The jumps in the solution give stability to the DG scheme. We can see this more clearly if we consider linear conservation law $u_t + au_x = 0$ with the upwind flux.

L^2 stability

For periodic or compactly supported boundary conditions, the semi-discrete DG scheme satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h^2 \mathrm{d}x \leq 0$$

or

$$\|u_h(t)\| \leq \|u_h(0)\|$$

Proof: Adding the cell entropy inequality from all the cells

$$\sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\int_{I_i} U(u_h)\mathrm{d}x + \sum_{i=1}^{N}[\hat{F}_{i+\frac{1}{2}}(t) - \hat{F}_{i-\frac{1}{2}}(t)] \leq 0$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h^2 \mathrm{d}x + \hat{F}_{N+\frac{1}{2}} - \hat{F}_{\frac{1}{2}} \leq 0$$

For periodic case, $\hat{F}_{\frac{1}{2}} = \hat{F}_{N+\frac{1}{2}}$ while for compactly supported case $\hat{F}_{\frac{1}{2}} = \hat{F}_{N+\frac{1}{2}} = 0$, we obtain desired result.

Error estimate: $u_t + cu_x = 0$, semi-discrete scheme

Theorem 2.1 (First L^2-error estimate) *Suppose that the initial condition* u_0 *belongs to* $H^{k+1}(0,1)$. *Let* e *be the approximation error* $u - u_h$. *Then we have,*

$$\| e(T) \|_{L^2(0,1)} \leq C \, | u_0 |_{H^{k+1}(0,1)} (\Delta x)^{k+1/2},$$

where C *depends solely on* k, $|c|$, *and* T.

Theorem 2.2 (Second L^2-error estimate) *Suppose that the initial condition* u_0 *belongs to* $H^{k+2}(0,1)$. *Let* e *be the approximation error* $u - u_h$. *Then we have,*

$$\| e(T) \|_{L^2(0,1)} \leq C \, | u_0 |_{H^{k+2}(0,1)} (\Delta x)^{k+1},$$

where C *depends solely on* k, $|c|$, *and* T.

(B. Cockburn, Lecture notes on *Discontinuous Galerkin methods for convection dominated problem*)

Numerical example

The semi-discrete DG scheme takes the form

$$M \frac{\mathrm{d}U}{\mathrm{d}t} = L(U)$$

We can solve this using an explicit Runge-Kutta scheme (more details later).

- Linear convection equation: smooth initial condition
- Linear convection equation: discontinuous initial condition

Limiters and TVD property

Forward difference in time: Find w_h^{n+1} such that

$$\int_{I_i} \frac{w_h^{n+1} - u_h^n}{\Delta t} v_h - \int_{I_i} f(u_h^n) \frac{\partial v_h}{\partial x} + \hat{f}^n_{i+\frac{1}{2}} v_h(x^-_{i+\frac{1}{2}}) - \hat{f}^n_{i-\frac{1}{2}} v_h(x^+_{i-\frac{1}{2}}) = 0$$

The value w_h^{n+1} may be oscillatory. We limit it to obtain the new solution

$$u_h^{n+1} = \Lambda\Pi_h(w_h^{n+1})$$

Properties of $\Lambda\Pi_h(\cdot)$

1. It should not change the cell average value.
2. It should not affect the accuracy in smooth regions.

Limiters and TVD property

Define **cell average value**

$$\bar{u}_i = \frac{1}{\Delta x_i}\int_{I_i} u_h \mathrm{d}x$$

and

$$\hat{u}_i = u_h(x^-_{i+\frac{1}{2}}) - \bar{u}_i, \qquad \check{u}_i = \bar{u}_i - u_h(x^+_{i-\frac{1}{2}})$$

$$\Delta_+\bar{u}_i = \bar{u}_{i+1} - \bar{u}_i, \qquad \Delta_-\bar{u}_i = \bar{u}_i - \bar{u}_{i-1}$$

We cannot modify the cell average value but we can modify the slopes

$$\hat{u}_i^{(m)} = m(\hat{u}_i, \Delta_+\bar{u}_i, \Delta_-\bar{u}_i), \qquad \check{u}_i^{(m)} = m(\check{u}_i, \Delta_+\bar{u}_i, \Delta_-\bar{u}_i)$$

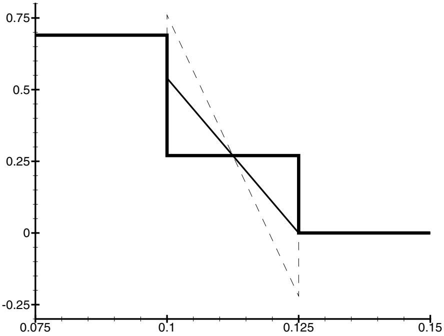
where m is the minmod function

$$m(a_1,\dots,a_l) = \begin{cases} s\min(|a_1|,\dots,|a_l|) & s = \mathrm{sign}(a_1) = \dots = \mathrm{sign}(a_l) \\ 0 & \text{otherwise} \end{cases}$$

For $k = 1$ we have $\hat{u}_i = \check{u}_i$. The trace values are recomputed using the limited slopes

$$u_h^{(m)}(x^-_{i+\frac{1}{2}}) = \bar{u}_i + \hat{u}_i^{(m)}, \qquad u_h^{(m)}(x^+_{i-\frac{1}{2}}) = \bar{u}_i - \check{u}_i^{(m)}$$

Limiters and TVD property



For $k = 0, 1, 2$, this procedure uniquely determines a new polynomial of degree k.

For $k = 1$, let us denote the limited function by $\Lambda\Pi_h^1(u_h)$.

Limiters and TVD property

For $k \geq 3$ there is more freedom; one approach is to determine the remaining dof by an L^2 projection. A simple approach that works well in practice is the following.

1. If
$$u_h^{(m)}(x^+_{i-\frac{1}{2}}) = u_h(x^+_{i-\frac{1}{2}}) \qquad \text{and} \qquad u_h^{(m)}(x^-_{i+\frac{1}{2}}) = u_h(x^-_{i+\frac{1}{2}})$$
then take $u_h^{(m)} = u_h$ for $x \in I_i$.
2. Otherwise, let $u_h^1 \in \mathbb{P}_1(I_i)$ be the L^2 projection of $u_h|_{I_i}$. Take $u_h^{(m)}|_{I_i} = \Lambda\Pi_h^1(u_h^1)$.

Limiters and TVD property

Lemma (Harten)

If a scheme can be written as

$$\bar{u}_i^{n+1} = \bar{u}_i^n + C_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n - D_{i-\frac{1}{2}}^n \Delta_- \bar{u}_i^n$$

Assume that boundary conditions are periodic or compactly supported. If

$$C_{i+\frac{1}{2}} \geq 0, \qquad D_{i+\frac{1}{2}} \geq 0, \qquad C_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \leq 1$$

then the scheme is TVD

$$\mathrm{TV}(u^{n+1}) \leq \mathrm{TV}(u^n)$$

where the total variation is defined as

$$\mathrm{TV}(u) = \sum_i |\Delta_+ \bar{u}_i|$$

For DG solutions, let us define the **total variation of the means**

$$\mathrm{TVM}(u_h) = \sum_i |\Delta_+ \bar{u}_i|$$

Limiters and TVD property

Proposition

For periodic or compactly supported boundary conditions, the DG scheme with limiter is TVD in the means, i.e.,

$$\mathrm{TVM}(u_h^{n+1}) \leq \mathrm{TVM}(u_h^n)$$

Proof: Taking $v_h = 1$ for $x \in I_i$

$$w_i = \bar{u}_i - \lambda_i[\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_{i-1} + \hat{u}_{i-1}, \bar{u}_i - \check{u}_i)]$$

We can write this incremental form with

$$C_{i+\frac{1}{2}} = -\lambda_i \frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i)}{\Delta_+ \bar{u}_i}$$

$$D_{i-\frac{1}{2}} = \lambda_i \frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i) - \hat{f}(\bar{u}_{i-1} + \hat{u}_{i-1}, \bar{u}_i - \check{u}_i)}{\Delta_- \bar{u}_i}$$

Limiters and TVD property

Rewrite the coefficient

$$C_{i+\frac{1}{2}} = \underbrace{-\lambda_i \hat{f}_2}_{\geq 0} \left(1 - \frac{\check{u}_{i+1}}{\Delta_+ \bar{u}_i} + \frac{\check{u}_i}{\Delta_+ \bar{u}_i}\right)$$

where

$$0 \leq -\hat{f}_2 = -\frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i)}{(\bar{u}_{i+1} - \check{u}_{i+1}) - (\bar{u}_i - \check{u}_i)} < L_2$$

Since u_h^n has been pre-processed by the minmod limiter, we have

$$0 \leq \frac{\check{u}_{i+1}}{\Delta_+ \bar{u}_i} \leq 1, \qquad 0 \leq \frac{\check{u}_i}{\Delta_+ \bar{u}_i} \leq 1$$

and hence

$$0 \leq C_{i+\frac{1}{2}} \leq 2\lambda_i L_2$$

and similarly

$$0 \leq D_{i+\frac{1}{2}} \leq 2\lambda_{i+1} L_1$$

Limiters and TVD property

If the time step satisfies the condition

$$2(\lambda_i L_2 + \lambda_{i+1} L_1) \leq 1 \qquad \text{or} \qquad \Delta t \leq \frac{1}{2(\frac{L_1}{\Delta x_{i+1}} + \frac{L_2}{\Delta x_i})}$$

Then from Hartens's Lemma, we have

$$\mathrm{TVM}(w_h^{n+1}) \leq \mathrm{TVM}(u_h^n)$$

Since the limiter does not change the TVM, we have $u_h^{n+1} = \Lambda\Pi_h(w_h^{n+1})$ and

$$\mathrm{TVM}(u_h^{n+1}) = \mathrm{TVM}(w_h^{n+1}) \leq \mathrm{TVM}(u_h^n)$$

□

$$\boxed{Numerical\ example}$$

TVB Limiter

In smooth regions of the solution

$$\hat{u}_i = \frac{1}{2} u_x(x_i)\Delta x_i + \mathcal{O}\left(h^2\right), \qquad \check{u}_i = \frac{1}{2} u_x(x_i)\Delta x_i + \mathcal{O}\left(h^2\right)$$

$$\Delta_+ \bar{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i+1}) + \mathcal{O}\left(h^2\right)$$

$$\Delta_- \bar{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i-1}) + \mathcal{O}\left(h^2\right)$$

If the solution is smooth and monotone around I_i, all quantities have same sign, and

$$\hat{u}_i^{(m)} = \hat{u}_i, \qquad \check{u}_i^{(m)} = \check{u}_i$$

However, if there is a smooth extrema, then the limiter degrades the accuracy. In this case a TVB limiter can be used

$$\tilde{m}(a_1, a_2, \ldots, a_l) = \begin{cases} a_1 & \text{if } |a_1| \le Mh^2 \\ m(a_1, a_2, \ldots, a_l) & \text{otherwise} \end{cases}$$

TVB Limiter

TVB property

With the TVB limiter, if the CFL condition

$$\Delta t \leq \frac{1}{2(\frac{L_1}{\Delta x_{i+1}} + \frac{L_2}{\Delta x_i})}, \qquad \forall i$$

is satisfied, then

$$\mathrm{TVM}(u_h^{n+1}) \leq \mathrm{TVM}(u_h^n) + CMh$$

Proof: See [1]

The quantity M is an estimate of the second derivative of the solution at smooth extrema. This can be based on the initial condition, e.g.,

$$M = \max_x \{|u_0''(x)| : u_0'(x) = 0\}$$

Ideally M should be estimated from the numerical solution. In practice, the value of M does not seem to be too important, as long as we dont take it too small. But this is still a weak point of the TVB limiter.

TVB Limiter

LEMMA 2.2. *If*

$$M = \frac{2}{3} M_2, \tag{2.25a}$$

or

$$M = M_j = \frac{2}{9}(3 + 10M_2) \cdot M_2 \cdot \frac{h^2}{h^2 + |\Delta_+ u_j^{(0)}| + |\Delta_- u_j^{(0)}|}, \tag{2.25b}$$

then the limiting (2.21)–(2.24) *does not affect accuracy in any region where* $u \in C^2$ *and* $|u_{xx}| \leq M_2$.

Proof: See [1]

$\boxed{\textit{Numerical example}}$

Strong stability preserving RK schemes

Shu, Osher, Gottlieb

Consider an ODE

$$\frac{\mathrm{d}U}{\mathrm{d}t} = L(U)$$

- 2-stage, second order RK

$$\begin{aligned} U^{(1)} &= U^n + \Delta t L(U^n) \\ U^{n+1} &= \frac{1}{2}U^n + \frac{1}{2}[U^{(1)} + \Delta t L(U^{(1)})] \end{aligned}$$

- 3-stage, third order RK

$$\begin{aligned} U^{(1)} &= U^n + \Delta t L(U^n) \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}[U^{(1)} + \Delta t L(U^{(1)})] \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}[U^{(2)} + \Delta t L(U^{(2)})] \end{aligned}$$

Strong stability preserving RK schemes

Shu, Osher, Gottlieb

SSP Runge-Kutta

If the forward Euler scheme is stable

$$\Delta t \le \Delta t_c \qquad \Longrightarrow \qquad \|U + \Delta t L(U)\| \le \|U\|$$

then the SSPRK scheme is stable under a CFL condition $\Delta t \le \alpha \Delta t_c$. For the second and third order schemes, $\alpha = 1$.

Remark: The above SSPRK scheme require storage for three steps of vectors,

- U^n
- current stage solution $U^{(s)}$
- Residual L

Remark: There exist higher order SSPRK schemes but they need more steps. There is a 5-stage, 4-th order SSPRK scheme [2].

Basis functions

We have to construct basis functions for V_h^k for which there are two approaches: **nodal** and **modal**. The DG solution has the form

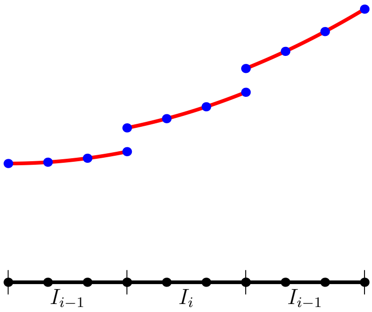
$$x \in I_i : \qquad u_h(x,t) = \sum_{j=1}^{N} u_{ij}(t)\phi_{ij}(x), \qquad N = \dim(V_h^k) = k+1$$

In the **nodal** approach, there are certain node locations

$$x_{ij} \in I_i, \qquad j = 1, \ldots, N \qquad \text{such that} \quad u_h(x_{ij}) = u_{ij}$$

This means that the basis functions have the interpolation property

$$\phi_{ij}(x_{il}) = \delta_{jl}, \qquad 1 \le j, l \le N$$



Basis functions

A degree k polynomial is determined by $N = k + 1$ values. The location of the N nodal points x_{ij} can be

- uniformly distributed inside I_i
  - Runge phenomenon for high degree polynomials
  - full mass matrix, ill-condition for large degree k
- Chebyshev points: In $[-1, +1]$ they are given by

$$x_j = \cos\left(\frac{2j-1}{2N}\pi\right), \qquad j = 1, 2, \ldots, N$$

- based on Gauss-Legendre or Gauss-Lobatto integration points

Basis functions

Once the nodal points are chosen, the basis functions can be obtained from Lagrange interpolation.

$\underline{k=0}$: One dof per element

$$\phi_{i1}(x) = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

This is the classical finite volume method.

$\underline{k=1}$: Two dof per element; $x_{i1} = x_{i-\frac{1}{2}}$, $x_{i2} = x_{i+\frac{1}{2}}$

$$\phi_{i1}(x) = \begin{cases} \frac{x_{i+\frac{1}{2}}-x}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}, \qquad \phi_{i2}(x) = \begin{cases} \frac{x-x_{i-\frac{1}{2}}}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

$\underline{k=2}$: Three dof per element; $x_{i1} = x_{i-\frac{1}{2}}$, $x_{i2} = x_i$, $x_{i3} = x_{i+\frac{1}{2}}$

Basis functions

In general: We choose N distinct nodes $\{x_{i1}, x_{i2}, \dots, x_{i,N}\} \subset I_i$

$$\phi_{ij}(x) = \frac{(x - x_{i,1}) \dots (x - x_{i,j-1})(x - x_{i,j+1}) \dots (x - x_{i,N})}{(x_{i,j} - x_{i,1}) \dots (x_{i,j} - x_{i,j-1})(x_{i,j} - x_{i,j+1}) \dots (x_{i,j} - x_{i,N})}$$

Remark: If nodes are located at the element boundaries, then we have multiple dofs at the boundary since the solution is in general discontinuous.

Taylor basis functions

Here we do not use nodal basis functions, but use Taylor series to generate the basis functions. The cell average value u_i is one of the degrees freedom. The other dof are the gradient, hessian, etc.

Define the **moments**

$$m_{is} = \frac{1}{s!\Delta x_i} \int_{I_i} \left(\frac{x - x_i}{\Delta x_i} \right)^s \mathrm{d}x, \qquad s = 1, 2, \ldots$$

$\underline{k = 1}$: Dof are (u_i, s_i)

$$u_h(x,t) = u_i(t) + \frac{x - x_i}{\Delta x_i} s_i(t)$$

$$\phi_{i1}(x) = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}, \qquad \phi_{i2}(x) = \begin{cases} \frac{x - x_i}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

$\underline{k = 2}$: Dof are (u_i, s_i, q_i)

$$u_h(x,t) = u_i(t) + \frac{x - x_i}{\Delta x_i} s_i(t) + \left[\frac{1}{2} \left(\frac{x - x_i}{\Delta x_i} \right)^2 - m_{i2} \right] q_i(t)$$

Taylor basis functions

$$\phi_{i3}(x) = \begin{cases} \frac{1}{2}\left(\frac{x-x_i}{\Delta x_i}\right)^2 - m_{i2} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

The basis functions are chosen such that

$$\int_{I_i} \phi_{i1} = 1, \qquad \int_{I_i} \phi_{ij} = 0, \quad j = 2, 3, \ldots$$

- Can be extended to higher degrees using Taylor series
- Heirarchical representation
- Diagonal mass matrix for $k \leq 2$
- Extension to multi-dimensions on arbitrary polygonal elements; orthogonalize using Gram-Schmidt process

Orthogonal polynomials (Modal approach)

- Legendre polynomials: Solution of Legendre's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi^2)\frac{\mathrm{d}}{\mathrm{d}\xi}P_n(\xi)\right] + n(n+1)P_n(\xi) = 0$$

$$\begin{array}{ll} P_0(\xi) = 1 & P_1(\xi) = \xi \\ P_2(\xi) = \frac{1}{2}(3\xi^2 - 1) & P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi) \\ P_4(\xi) = \frac{1}{8}(35\xi^4 - 30\xi^2 + 3) & P_5(\xi) = \frac{1}{8}(63\xi^5 - 70\xi^3 + 15\xi) \end{array}$$

$$(n+1)P_{n+1}(\xi) = (2n+1)\xi P_n(\xi) - nP_{n-1}(\xi)$$

- Orthogonality

$$\int_{-1}^{+1} P_j(\xi)P_k(\xi)\mathrm{d}\xi = \begin{cases} 0 & j \neq k \\ \frac{2}{2j+1} & j = k \end{cases}$$

Orthogonal polynomials (Modal approach)

- Basis functions: $j = 0, 1, 2, \ldots$

$$\phi_{ij}(x) = \sqrt{2j+1} P_j\left(\frac{x - x_i}{\Delta x_i/2}\right), \qquad \int_{I_i} \phi_{ij}\phi_{ik} \mathrm{d}x = \begin{cases} 0 & j \neq k \\ \Delta x_i & j = k \end{cases}$$

In `deal.II`, the `FE_DGP` space makes use of this basis functions.

- Most authors use the following normalization

$$\phi_{ij}(x) = P_j\left(\frac{x - x_i}{\Delta x_i/2}\right), \qquad \int_{I_i} \phi_{ij}\phi_{ik} \mathrm{d}x = \begin{cases} 0 & j \neq k \\ \frac{\Delta x_i}{2j+1} & j = k \end{cases}$$

The semi-discrete DG scheme takes the form

$$\frac{\Delta x_i}{2j+1}\frac{\mathrm{d}u_{ij}}{\mathrm{d}t} - \int_{I_i} f(u_h)\frac{\mathrm{d}\phi_{ij}}{\mathrm{d}x}\mathrm{d}x + \hat{f}_{i+\frac{1}{2}}(t) - (-1)^j \hat{f}_{i-\frac{1}{2}}(t) = 0, \quad j = 0, 1, \ldots, k$$

Quadrature rules

The DG scheme involves integrals which must be approximated by quadrature. Let

$$f : [-1, +1] \to \mathbb{R}$$

Choose n quadrature nodes $\{\xi_1, \xi_2, \dots, \xi_n\} \subset [-1, +1]$.

$$\int_{-1}^{+1} f(\xi) \mathrm{d}\xi \approx \sum_{q=1}^{n} \omega_q f(\xi_q)$$

- Gauss-Legendre quadrature
  - Nodes $\{\xi_q\}$ are roots of Legendre polynomial $P_n(\xi)$
  - n-point rule is exact for any $f \in \mathbb{P}_{2n-1}$
- Gauss-Lobatto-Legendre quadrature
  - Nodes include $\{-1, +1\}$ and the roots of $P'_{n-1}(\xi)$
  - n-point rule is exact for any $f \in \mathbb{P}_{2n-3}$

Quadrature rules

For function on general interval $f : [a, b] \to \mathbb{R}$, do change of variable

$$x(\xi) = \frac{1-\xi}{2}a + \frac{1+\xi}{2}b, \qquad \xi \in [-1, +1]$$

and

$$\begin{aligned}
\int_a^b f(x)\mathrm{d}x &= \frac{1}{2}(b-a)\int_{-1}^{+1} f(x(\xi))\mathrm{d}\xi \\
&\approx \frac{1}{2}(b-a)\sum_{q=1}^{n} \omega_q f(x(\xi_q)) \\
&= \sum_{q=1}^{n} \tilde{\omega}_q f(x(\xi_q)), \qquad \tilde{\omega}_q = \frac{1}{2}(b-a)\omega_q
\end{aligned}$$

Limiters: Implementation

Degree $k = 1$: Let us write the solution in terms of Taylor or Legendre basis

$$u_h = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i} \right) u_x$$

Limit the "derivative"

$$u_x^{(m)} = \operatorname{minmod}\left(u_x, \bar{u}_i - \bar{u}_{i-1}, \bar{u}_{i+1} - \bar{u}_i\right)$$

The limited solution is

$$\Lambda\Pi_h^1(u_h) = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i} \right) u_x^{(m)}$$

Limiters: Implementation

Degree $k > 1$: Let us write the solution in terms of Taylor or Legendre basis

$$u_h = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i} \right) u_x + HOT$$

Limit the "derivative"

$$u_x^{(m)} = \mathrm{minmod}\left(u_x, \bar{u}_i - \bar{u}_{i-1}, \bar{u}_{i+1} - \bar{u}_i\right)$$

If $u_x^{(m)} = u_x$ then

$$\Lambda\Pi_h(u_h) = u_h$$

else

$$\Lambda\Pi_h(u_h) = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i} \right) u_x^{(m)}$$

ODE system

Degrees of freedom associated with I_i

$$U^{(i)} = [u_{i1}, \ldots, u_{iN}]^\top$$

ODE system for element I_i

$$M^{(i)} \frac{\mathrm{d}U^{(i)}(t)}{\mathrm{d}t} = L^{(i)}(U(t))$$

Mass matrix: $M^{(i)} \in \mathbb{R}^{N\times N}$ symmetric, positive definite

$$M^{(i)}_{jk} = \int_{I_i} \phi_{ij}\phi_{ik} = \sum_q \omega_{iq}\phi_{ij}(x_{iq})\phi_{ik}(x_{iq}), \quad 1 \le j,k \le N$$

Right hand side: $1 \le j \le N$

$$\begin{aligned} L^{(i)}(U(t))_j &= \int_{I_i} f(u_h)\phi_{ij}' - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \\ &\approx \sum_q \omega_{iq} f(u_h(x_{iq},t))\phi_{ij}'(x_{iq}) - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) \\ &\quad + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \end{aligned}$$

ODE system

where

$$\omega_{iq} = \frac{1}{2}\omega_q \Delta x_i, \qquad x_{iq} = \frac{1-\xi_q}{2} x_{i-\frac{1}{2}} + \frac{1+\xi_q}{2} x_{i+\frac{1}{2}}$$

The quadrature rule must be chosen such that mass matrix is evaluated exactly.

- $(k+1)$-point Gauss-Legendre, exact for $\mathbb{P}_{2k+1}$
- $(k+2)$-point Gauss-Lobatto-Legendre, exact for $\mathbb{P}_{2k+1}$

Same quadrature rule can be used for right hand side also.

The mass matrix has block diagonal structure; each block corresponds to one element and can be inverted independently of the others.

CFL condition

- Explicit time integration schemes are stable only under a restriction on the time step Δt.
- For $k \geq 1$, the forward Euler scheme (RK1) is known to be unconditionally unstable in L^2 (Chavent and Cockburn, 1989).
- For DG space discretizations using polynomials of degree k, and a $(k+1)$-stage RK method of order $k+1$, a Von-Neumann stability analysis for the one-dimensional linear case

$$f(u) = cu$$

with upwind flux gives the CFL condition

$$|c|\frac{\Delta t}{\Delta x} \leq \frac{1}{2k+1}$$

Theoretical proof of this is available only for $k = 0, 1, 2$. For $k \geq 3$ the above condition is close to the numerically determined values of CFL numbers.

CFL condition

Remark: The CFL number for SSPRK scheme to be TVDM is higher than that required for L^2 stability. However, to control round-off errors, the smaller CFL condition from L^2 stability has to be used in practical computations.

Table 2.3. CFL Numbers for RKDG Methods of Order $k+1$

k	0	1	2
$\mathrm{CFL_{TV}}$	1	1/2	1/2
$\mathrm{CFL_{L^2}}$	1	1/3	1/5

Boundary condition

- Boundary conditions can be specified if the characteristics are entering the domain (inflow boundary), e.g., at $x = 0$ if $f' > 0$ then a boundary condition on u can be specified.
- In general let us take the boundary conditions

$$u(0,t) = a(t), \qquad u(1,t) = b(t)$$

- The boundary conditions are incorporated in the DG scheme via the boundary fluxes

$$\hat{f}_{\frac{1}{2}}(t) = \hat{f}(a(t), u_h(0^+, t)), \qquad \hat{f}_{N+\frac{1}{2}}(t) = \hat{f}(u_h(1^-, t), b(t))$$

- To apply the limiter, the cell average values are extrapolated

$$u_0 = 2a(t) - u_1, \qquad u_{N+1} = 2b(t) - u_N$$

- Periodic boundary

$$\hat{f}_{\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{N+\frac{1}{2}}, t), u_h(x^+_{\frac{1}{2}}, t)), \qquad \hat{f}_{N+\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{N+\frac{1}{2}}, t), u_h(x^+_{\frac{1}{2}}, t))$$

Setting initial condition

Suppose the initial condition is

$$u(x, 0) = u_0(x)$$

The degree k solution polynomial is

$$x \in I_i : \qquad u_h(x, t) = \sum_{j=1}^{N} u_{ij}(t)\phi_{ij}(x), \qquad N = k + 1$$

- Nodal basis: Interpolate initial condition

$$u_{ij}(0) = u_h(x_{ij}, 0) = u_0(x_{ij})$$

Setting initial condition

- Modal basis: Do an L^2 projection

$$\min \int_{I_i} (u_h(x,0) - u_0(x))^2 \mathrm{d}x \quad \text{wrt} \quad u_{i1}, \ldots, u_{iN}$$

Optimality condition

$$\frac{\mathrm{d}}{\mathrm{d}u_{ik}} \int_{I_i} (u_h(x,0) - u_0(x))^2 \mathrm{d}x = 0, \qquad k = 1, 2, \ldots, N$$

$$\int_{I_i} u_h \phi_{ik} \mathrm{d}x = \int_{I_i} u_0 \phi_{ik} \mathrm{d}x, \qquad k = 1, 2, \ldots, N$$

Using orthogonality and quadrature rule

$$u_{ik} \Delta x_i = \sum_q u_0(x_{iq}) \phi_{ik}(x_{iq}) \omega_{iq}$$

Algorithm

- Compute and store the mass matrix
- Determine w_h^0 from initial condition u_0 by an L^2-projection
- Find u_h^0 from w_h^0 by applying the limiter, $u_h^0 = \Lambda\Pi_h(w_h^0)$
- For $n = 0, 1, \ldots$
  - Compute time step from CFL condition
  - Set $u_h^{n,0} = u_h^n$
  - RK stages: For $r = 0, 1, \ldots, N_{rk} - 1$
    - Compute right hand side $L_h(u_h^{n,r})$
    - Update solution to next RK stage

$$u_h^{n,r} \to w_h^{n,r+1}$$

    - Apply limiter

$$u_h^{n,r+1} = \Lambda\Pi_h(w_h^{n,r+1})$$

A first example

Let us consider a simple example

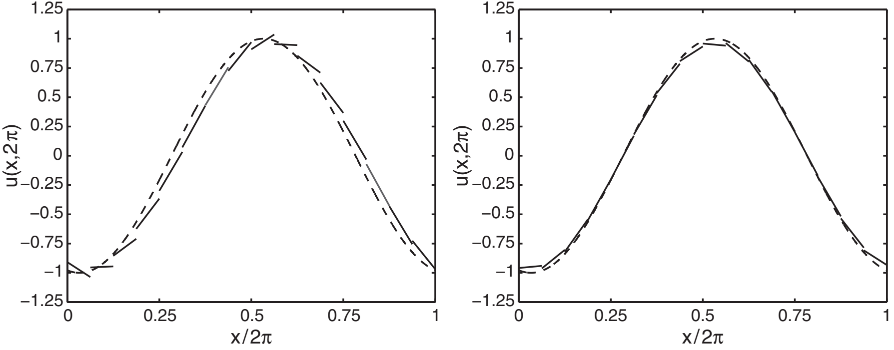
$$\frac{\partial u}{\partial t} - 2\pi \frac{\partial u}{\partial x} = 0, \quad x \in [0, 2\pi], \qquad u(x,0) = \sin(lx), \quad l = \frac{2\pi}{\lambda},$$

N\ K	2	4	8	16	32	64	Convergence rate
1	–	4.0E-01	9.1E-02	2.3E-02	**5.7E-03**	1.4E-03	2.0
2	2.0E-01	4.3E-02	**6.3E-03**	8.0E-04	1.0E-04	1.3E-05	3.0
4	**3.3E-03**	3.1E-04	9.9E-06	3.2E-07	1.0E-08	3.3E-10	5.0
8	2.1E-07	2.5E-09	4.8E-12	2.2E-13	5.0E-13	6.6E-13	$\simeq$ 9.0

The error clearly behaves as

$$\|u - u_h\|_{\Omega,h} \leq C h^{N+1}.$$

(Hesthaven)



Central flux

Upwind flux

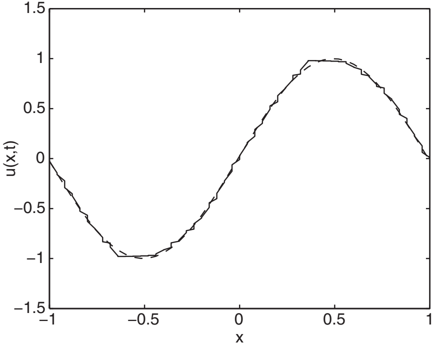
(Hesthaven)

Limiting

Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x \in [-1, 1],$$

Smooth initial condition



Reduction to 1st order at local smooth extrema

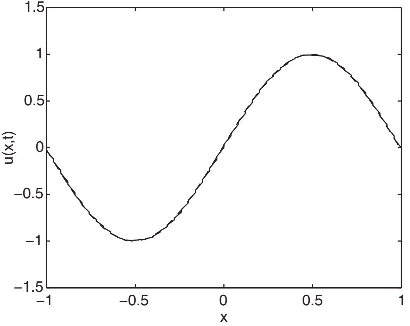
(Hesthaven)

Limiting

Introduce the TVB minmod

$$\bar{m}(a_1, \ldots, a_m) = m\left(a_1, a_2 + Mh^2\text{sign}(a_2), \ldots, a_m + Mh^2\text{sign}(a_m)\right),$$

M estimates maximum curvature



(Hesthaven)

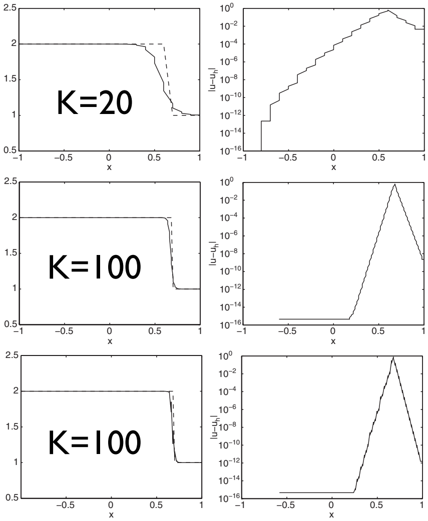
Limiting

Consider Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1],$$

$$u_0(x) = u(x,0) = \begin{cases} 2, & x \leq -0.5 \\ 1, & x > -0.5. \end{cases}$$

$$u(x,t) = u_0(x - 3t),$$



Too dissipative limiting leads to severe smearing.

.. but no oscillations!

(Hesthaven)

Linear convection equation

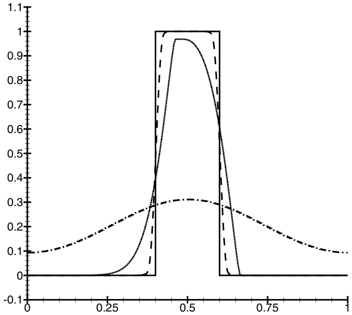


Fig. 1 Effect of the polynomial degree on the dissipation of the DG method. The exact solution u (solid line) is contrasted against the approximate solution obtained on a mesh of 160 elements with piecewise-constant (dash-point line), piecewise-linear (dotted line) and piecewise-quadratic (dashed line) approximations.

Increasing polynomial degree gives more accurate solutions

Linear convection equation

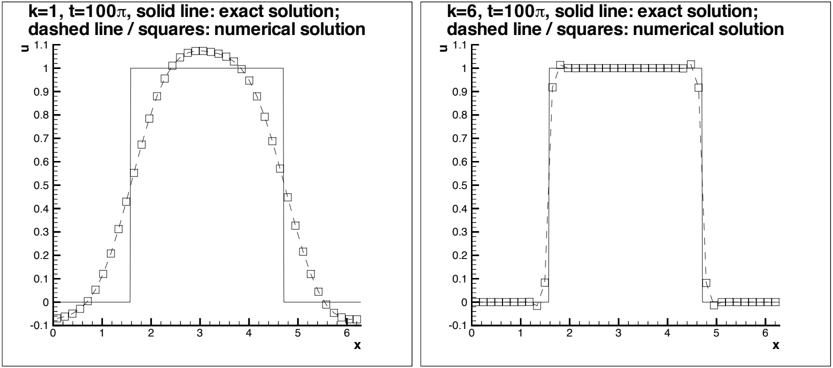


Figure 3.1: Transport equation: Comparison of the exact and the RKDG solutions at $T = 100\pi$ with second order (P^1, left) and seventh order (P^6, right) RKDG methods. One dimensional results with 40 cells, exact solution (solid line) and numerical solution (dashed line and symbols, one point per cell).

Limiting

General remarks on limiting

- ✓ The development of a limiting technique that avoid local reduction to 1st order accuracy is likely the most important outstanding problem in DG
- ✓ There are a number of techniques around but they all have some limitations -- restricted to simple/ equidistant grids, not TVD/TVB etc
- ✓ The extensions to 2D/3D and general grids are challenging

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References

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