Discontinuous Galerkin Method

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*Starred subsections are optional.

Chapter One

Scalar conservation law

1 Scalar conservation law

A scalar conservation law is a partial differential equation of the form

$$u_t + f(u)_x = 0$$

where

- u is called the conserved variable
- $f(u)$ is the flux of u

Such an equation is a conservation law because, if we integrate the equation on any interval $[a, b]$, we obtain

$$\frac{d}{dt}\int_a^b u(x,t)dx + f(u(b,t)) - f(u(a,t)) = 0$$

This says that

$$\text{Rate of change of total } u \text{ in } [a,b] = \text{Net flux of } u \text{ into } [a,b]$$

If the net flux is zero, then the total quantity is conserved. Moreover, these are hyperbolic conservation laws since their solutions have wave-like properties.

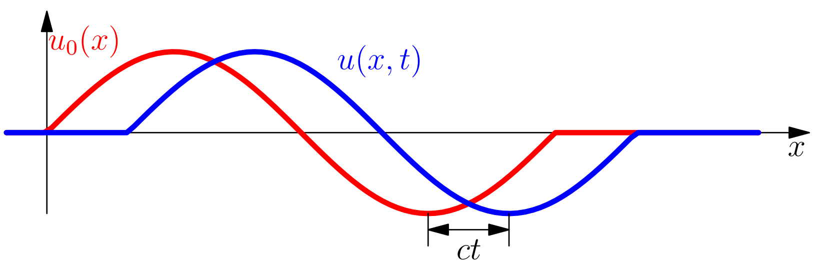
1.1 Linear convection equation

The flux is a linear function of the conserved variable, $f(u) = cu$, where $c =$ constant

$$u_t + cu_x = 0, \qquad u(x,0) = u_0(x)$$

The exact solution is given by

$$u(x,t) = u_0(x - ct)$$



The initial condition is transported with velocity c without change of form.

1.2 Burger's equation

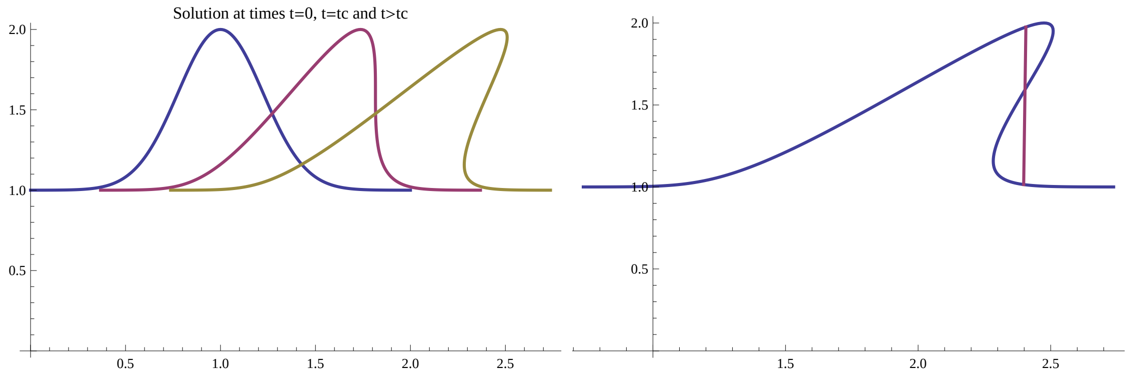
If the flux is of the form $f(u) = \frac{1}{2}u^2$, then the conservation law is

$$u_t + (\frac{1}{2}u^2)_x = 0 \qquad \text{or} \qquad u_t + uu_x = 0 \qquad \text{if } u \text{ is smooth}$$

As long as the solution is smooth, it is given by

$$u(x,t) = u_0(x - u(x,t)t)$$

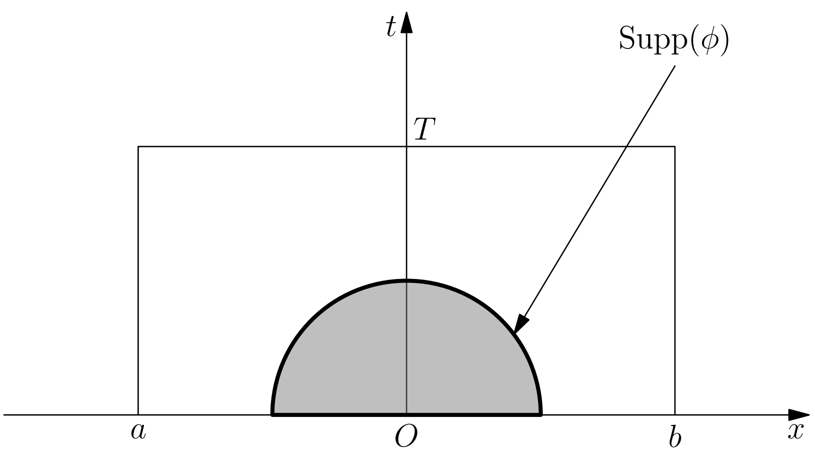
which is an implicit solution. In many cases, even if the initial condition is infinitely smooth, the solution at future times cannot remain smooth and discontinuities can develop in finite time.



2 Weak solution

Take a smooth test function with compact support, say $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$

$$\int_0^\infty \int_{\mathbb{R}} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \phi dx dt = 0$$



Perform an integration by parts on both terms in the equation so that the derivatives are transfered to the test function. This motivates the following definition.

1.1 Definition (Weak solution) A function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a weak solution of the IVP

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \qquad u(x,0) = u_0(x)$$

together with locally integrable initial data u_0, if u is locally integrable and satisfies

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + f(u)\phi_x) dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0, \qquad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$$

1.2 Lemma (Classical solution) Let $u \in C^1(\mathbb{R} \times \mathbb{R}^+)$ be a weak solution. Then it is a classical solution.

2.1 Rankine-Hugoniot condition

Suppose there is a discontinuity at $x(t)$ and let

$$s = \frac{d}{dt}x(t) = \text{shock speed}$$

Then the two values $u(x^-(t),t)$ and $u(x^+(t),t)$ satisfy the RH condition

$$f(u(x^+(t),t)) - f(u(x^-(t),t)) = s(u(x^+(t),t) - u(x^-(t),t))$$

1.1 Definition (Weak solution) A weak solution u is a piecewise smooth solution which satisfies the RH condition at the points of discontinuity of u.

3 Kruzkov's result

The scalar Cauchy problem for

$$u_t + f(u)_x = 0, \quad f \in C^1(\mathbb{R})$$

with initial condition

$$u(0,x) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R})$$

has a **unique entropy solution**

$$u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$$

which fulfills (important for numerics)

(1) Stability: $||u(t,\cdot)||_{L^\infty} \leqslant ||u_0||_{L^\infty}$, a.e. in $t \in \mathbb{R}_+$

(2) Monotone: if $u_0 \geqslant v_0$ a.e. in $\mathbb{R}$, then

$$u(t,\cdot) \geqslant v(t,\cdot) \quad \text{a.e. in } \mathbb{R}, \text{ a.e. in } t \in \mathbb{R}_+$$

(3) TV-diminishing: if $u_0 \in BV(\mathbb{R})$ then

$$u(t, \cdot) \in BV(\mathbb{R}) \quad \text{and} \quad TV(u(t, \cdot)) \leqslant TV(u_0)$$

(4) Conservation: if $u_0 \in L^1(\mathbb{R})$ then

$$\int_{\mathbb{R}} u(t, x)dx = \int_{\mathbb{R}} u_0(x)dx, \qquad \text{a.e. in } t \in \mathbb{R}_+$$

(5) Finite domain of dependence: if u, v are two entropy solutions corresponding to $u_0, v_0 \in L^\infty$ and

$$M = \max_{\phi}\{|f'(\phi)| : |\phi| \leqslant \max(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty})\}$$

then

$$\int_{|x| \leqslant R} |u(t, x) - v(t, x)|dx \leqslant \int_{|x| \leqslant R+Mt} |u_0(x) - v_0(x)|dx$$

Chapter Two

DG scheme in 1-D

In finite volume methods, there is one solution variable per cell, the cell average value, and the solution is assumed to be piecewise constant. To obtain high order accuracy, we reconstruct the solution by a polynomial inside each cell by making use of the cell averages in a small stencil around the current cell. IN DG methods we start with a polynomials solution in each cell which evolved forward in time by the scheme. DG methods are finite element methods, where we approximate some function in terms of certain basis functions with compact support. The basis functions are taken to be polynomials and are allowed to be discontinuous. The compact support property ensures that the stencil of the scheme is small which leads to efficient methods for solution. Let us consider a general conservation law of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

and try to construct the DG scheme for this problem.

1 Mesh and approximation space

Divide domain $\Omega = [0, 1]$ into disjoint cells $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ using the parition

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 1$$

as shown in figure (Two.1). Define the cell center and cell size as

$$x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \qquad \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \qquad h = \max_i \Delta x_i$$

Define the space of broken polynomials

$$V_h^k = \{v \in L^2(\Omega) : v|_{I_i} \in \mathbb{P}_k(I_i),\ 1 \leqslant i \leqslant N\}$$

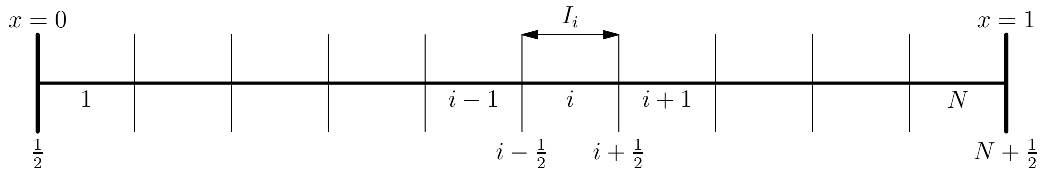


Figure Two.1: Mesh for DG scheme

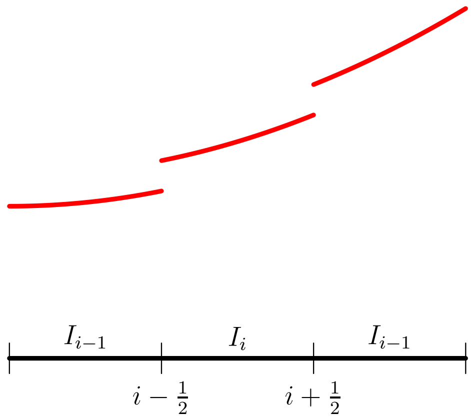


Figure Two.2: Discontinuous solution

Note that these functions can be discontinuous on the boundary of the elements as shown in the figure (Two.2). Define the left and right limits

$$v_h(x^-) = \lim_{\epsilon \searrow 0} v_h(x-\epsilon), \qquad v_h(x^+) = \lim_{\epsilon \searrow 0} v_h(x+\epsilon)$$

2 Semi-discrete DG scheme

Multiply conservation law by a smooth test function v

$$\int_{I_i} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) v dx = 0$$

and integrate by parts on flux derivative term

$$\int_{I_i} \frac{\partial u}{\partial t} v dx - \int_{I_i} f(u) \frac{\partial v}{\partial x} dx \\ + f(x_{i+\frac{1}{2}}, t) v(x^-_{i+\frac{1}{2}}) - f(x_{i-\frac{1}{2}}, t) v(x^+_{i-\frac{1}{2}}) = 0$$

We now want to replace u with u_h and v with v_h. At $x = x_{i+\frac{1}{2}}$, u_h may be discontinuous, i.e., $u_h(x^-_{i+\frac{1}{2}}, t) \neq u_h(x^+_{i+\frac{1}{2}}, t)$. In this case, how to compute the flux $f(x_{i+\frac{1}{2}}, t)$? Following the finite volume method, we will approximate this flux by a *numerical flux function* denoted by $\hat{f}_{i+\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$ leading to

2.1 Definition (Semi-discrete DG scheme) Find $u_h(\cdot, t) \in V_h^k$ such that for all $v_h \in V_h^k$

$$\int_{I_i} \frac{\partial u_h}{\partial t} v_h dx - \int_{I_i} f(u_h) \frac{\partial v_h}{\partial x} dx \\ + \hat{f}_{i+\frac{1}{2}}(t) v_h(x^-_{i+\frac{1}{2}}) - \hat{f}_{i-\frac{1}{2}}(t) v_h(x^+_{i-\frac{1}{2}}) = 0 \tag{Two.1}$$

The initial condition is obtained by an L^2 projection onto the finite element solution space V_h^k, i.e.,

$$\int_{I_i} u_h(x, 0) v_h(x) dx = \int_{I_i} u(x, 0) v_h(x) dx, \qquad \forall\ v_h \in V_h^k$$

Note that the numerical flux couples the solution in I_i to those in the neighbouring elements.

3 Relation to finite volume scheme

If the degree is $k = 0$, then we have only the constant basis function $v_h = 1$, and the solution is piecewise constant

$$u_h(x) = u_i, \qquad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$$

The DG scheme takes the form

$$\Delta x_i \frac{du_i}{dt} + \hat{f}(u_i, u_{i+1}) - \hat{f}(u_{i-1}, u_i) = 0$$

which is the finite volume scheme. Thus the DG scheme can be considered as a generalization of finite volume scheme to higher order of accuracy. In FVM only the cell average values are updated by the scheme and to achieve higher order accuracy, the solution has to be reconstructed by piecewise polynomials inside each cell using the cell average values. In DG schemes, we start with a polynomial solution representation inside each cell and evolve the entire polynomial forward in time by using the DG scheme.

4 Conservation property

Finite volume schemes are conservative which is necessary to compute correct weak solutions. We will now show that DG schemes are also conservative. Let degree $k > 0$. Take the test function $v_h \in V_h^k$ of the form

$$v_h = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

The DG scheme (Two.1) becomes

$$\frac{d}{dt}\int_{I_i} u_h dx + \hat{f}_{i+\frac{1}{2}}(t) - \hat{f}_{i-\frac{1}{2}}(t) = 0$$

which is a statement of conservation of u in the element I_i.

Chapter Three

Numerical flux

The numerical flux is an important ingredient of the DG scheme. We can make use of the developments in finite volume methods to compute the numerical flux and there are many possible schemes for this. In any case, we desire some common properties of the numerical flux as follows.

1 Properties of numerical flux

(1) Consistency

$$\hat{f}(u, u) = f(u)$$

(2) Locally Lipschitz continuous

$$|\hat{f}(a_2, b_2) - \hat{f}(a_1, b_1)| \leqslant L_1|a_2 - a_1| + L_2|b_2 - b_1|$$

(3) monotone

$$\hat{f}(a, b), \qquad \text{increasing in } a \text{ and decreasing in } b, \quad f(\uparrow, \downarrow)$$

In terms of the derivatives, we can state this property as

$$\frac{\partial}{\partial a}\hat{f}(a, b) \geqslant 0, \qquad \frac{\partial}{\partial b}\hat{f}(a, b) \leqslant 0$$

2 Riemann solver

One way to obtain a numerical flux is to solve a **Riemann problem**. To compute the flux $\hat{f}_{i+\frac{1}{2}}$, we solve the Riemann problem

$$\frac{\partial w}{\partial \tau} + \frac{\partial f(w)}{\partial x} = 0, \qquad w(x, \tau = t) = \begin{cases} u(x^-_{i+\frac{1}{2}}, t) & x < x_{i+\frac{1}{2}} \\ u(x^+_{i+\frac{1}{2}}, t) & x > x_{i+\frac{1}{2}} \end{cases}, \qquad \tau \geqslant t$$

This has a self-similar solution

$$w(x, \tau) = w_R((x - x_{i+\frac{1}{2}})/(t - \tau); u(x^-_{i+\frac{1}{2}}, t), u(x^+_{i+\frac{1}{2}}, t))$$

and the flux across $x = x_{i+\frac{1}{2}}$ is given by

$$\hat{f}_{i+\frac{1}{2}} = f(w_R(0; u(x^-_{i+\frac{1}{2}}, t), u(x^+_{i+\frac{1}{2}}, t)) \qquad \text{(Godunov flux)}$$

In practice, the Riemann problem is solved approximately.

3 Numerical flux: Linear convection equation

The flux is linear $f(u) = au$ where $a =$ constant and the equation is of the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

The upwind numerical flux is given by

$$\hat{f}(u^-, u^+) = \begin{cases} au^- & \text{if } a \geqslant 0 \\ au^+ & \text{if } a < 0 \end{cases}$$

This can also be written as

$$\hat{f}(u^-, u^+) = \underbrace{\frac{1}{2}[f(u^-) + f(u^+)]}_{\text{centered flux}} - \underbrace{\frac{1}{2}|a|(u^+ - u^-)}_{\text{dissipative flux}}$$

The second part of the flux contributes to the numerical dissipation that gives rise to stability.

4 Numerical flux: Non-linear conservation law

Here we list some standard numerical fluxes and the reader must consult the literature on finite volume schemes for more details.

4.1 Godunov flux

This is obtained by exactly solving the Riemann problem. The general formula is

$$\hat{f}(u^-, u^+) = \begin{cases} \displaystyle\min_{u \in [u^-, u^+]} f(u), & \text{if} \quad u^- \leqslant u^+ \\ \displaystyle\max_{u \in [u^+, u^-]} f(u), & \text{if} \quad u^- > u^+ \end{cases}$$

For convex flux, we can give simpler expression as follows. Let u^* be the only sonic point, i.e., $f'(u^*) = 0$. Then the Godunov flux is given by

$$\hat{f} = \max\{f(\max\{u^*, u^-\}), f(\min\{u^*, u^+\})\}$$

4.2 Lax-Friedrich flux

This may be considered as a generalization of the upwind flux formula by using the wave speed to be $\Delta x/\Delta t$

$$\hat{f}(u^-, u^+) = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}\frac{\Delta x}{\Delta t}(u^+ - u^-)$$

4.3 Rusanov flux

This flux is also refered to as a local Lax-Friedrich flux and it makes use of a local wave speed estimate in the dissipative flux

$$\hat{f} = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}\lambda(u^+ - u^-)$$

where

$$\lambda = \max_{\xi \in (u^-, u^+)} |f'(\xi)|$$

A simple choice, which is exact for convex fluxes, is

$$\lambda = \max\{|f'(u^-)|, |f'(u^+)|\}$$

4.4 Roe flux

This flux is based on a approximate Riemann solver. The non-linear Riemann problem is replaced by the linear problem

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0$$

and the exact solution of this problem gives the flux

$$\hat{f} = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}\lambda(u^+ - u^-)$$

where

$$\lambda = \left| \frac{f(u^+) - f(u^-)}{u^+ - u^-} \right|$$

We have to modify the Roe scheme to satisfy entropy condition.

3.1 Remark Many of the numerical fluxes have the upwind property in the sense that if $f' \geqslant 0$ in the Riemann problem, then $\hat{f}(u^-, u^+) = f(u^-)$. This is not the case for Lax-Friedrich and Rusanov fluxes, which are hence called central fluxes.

Chapter Four

Stability

1 Linear convection equation

Let us consider the linear convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad x \in (0,1)$$

The semi-discrete DG scheme is given by

$$\int_{I_i} \frac{\partial u_h}{\partial t} v_h dx - \int_{I_i} a u_h \frac{\partial v_h}{\partial x} dx + \hat{f}_{i+\frac{1}{2}}(t) v_h(x_{i+\frac{1}{2}}^-) - \hat{f}_{i-\frac{1}{2}}(t) v_h(x_{i-\frac{1}{2}}^+) = 0$$

Taking $v_h = u_h$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{I_i} u_h^2 dx - &\frac{a}{2}\left[u_h^2(x_{i+\frac{1}{2}}^-, t) - u_h^2(x_{i-\frac{1}{2}}^+, t)\right] \\ &+\hat{f}_{i+\frac{1}{2}}(t) u_h(x_{i+\frac{1}{2}}^-, t) - \hat{f}_{i-\frac{1}{2}}(t) u_h(x_{i-\frac{1}{2}}^+, t) = 0\end{aligned}$$

Summing over all the cells, we obtain the energy equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + &\frac{a}{2}u_h^2(x_{\frac{1}{2}}, t) - \hat{f}_{\frac{1}{2}}(t) u_h(x_{\frac{1}{2}}, t) + \sum_{i=1}^{N-1}\left[\frac{a}{2}[\![u_h^2]\!]_{i+\frac{1}{2}} - \hat{f}_{i+\frac{1}{2}}(t)[\![u_h]\!]_{i+\frac{1}{2}}\right] \\ &-\frac{a}{2}u_h^2(x_{N+\frac{1}{2}}, t) + \hat{f}_{N+\frac{1}{2}}(t) u_h(x_{N+\frac{1}{2}}, t) = 0\end{aligned}$$

where the jump term is defined as

$$[\![v]\!]_{i+\frac{1}{2}} = v(x_{i+\frac{1}{2}}^+) - v(x_{i+\frac{1}{2}}^-)$$

For the upwind numerical flux, which is given by

$$\hat{f}_{i+\frac{1}{2}} = \frac{a}{2}[u_h(x_{i+\frac{1}{2}}^-, t) + u_h(x_{i+\frac{1}{2}}^+, t)] - \frac{|a|}{2}[\![u_h]\!]_{i+\frac{1}{2}}$$

we can write

$$\frac{a}{2}[\![u_h^2]\!]_{i+\frac{1}{2}} - \hat{f}_{i+\frac{1}{2}}(t)[\![u_h]\!]_{i+\frac{1}{2}} = \frac{|a|}{2}[\![u_h]\!]_{i+\frac{1}{2}}^2$$

so that the energy equation is

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + &\frac{a}{2}u_h^2(x_{\frac{1}{2}}, t) - \hat{f}_{\frac{1}{2}}(t) u_h(x_{\frac{1}{2}}, t) + \sum_{i=1}^{N-1}\frac{|a|}{2}[\![u_h]\!]_{i+\frac{1}{2}}^2 \\ &-\frac{a}{2}u_h^2(x_{N+\frac{1}{2}}, t) + \hat{f}_{N+\frac{1}{2}}(t) u_h(x_{N+\frac{1}{2}}, t) = 0\end{aligned} \qquad \text{(Four.1)}$$

1.1 Periodic boundary conditions

In this case

$$\hat{f}_{\frac{1}{2}} = \hat{f}_{N+\frac{1}{2}} = \hat{f}(u_h(x_{N+\frac{1}{2}}, t), u_h(x_{\frac{1}{2}}, t))$$

and (Four.1) simpifies to

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \frac{|a|}{2}\sum_{i=1}^{N} [\![u_h]\!]^2_{i+\frac{1}{2}} = 0$$

where

$$[\![u_h]\!]_{N+\frac{1}{2}} = u_h(x_{N+\frac{1}{2}}) - u_h(x_{\frac{1}{2}})$$

Integrating over time

$$\|u_h(T)\|^2 + |a|\sum_{i=1}^{N}\int_0^T [\![u_h]\!]^2_{i+\frac{1}{2}} dt = \|u_h(0)\|^2$$

This immediately implies that

$$\|u_h(t)\| \leqslant \|u_h(0)\|$$

The use of upwind scheme together with discontinuous basis functions leads to L^2 stability. If we use the central flux

$$\hat{f}_{i+\frac{1}{2}} = \frac{a}{2}[u_h(x^-_{i+\frac{1}{2}}, t) + u_h(x^+_{i+\frac{1}{2}}, t)]$$

then we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 = 0$$

and in this case, the energy is conserved, $\|u_h(T)\| = \|u_h(0)\|$. The scheme does not have any dissipation and is neutrally stable. However, the central flux should not be used in practice since it gives less accurate solutions than the upwind flux and will be unstable for non-linear problems.

1.2 Dirichlet condition

Assume that $a > 0$. In this case, we can specify boundary conditions only at $x = x_{\frac{1}{2}} = 0$, i.e.,

$$u(0, t) = g(t)$$

We take the numerical flux at the boundaries as

$$\hat{f}_{\frac{1}{2}}(t) = ag(t), \qquad \hat{f}_{N+\frac{1}{2}}(t) = au_h(x_{N+\frac{1}{2}}, t)$$

which is an upwind flux. This corresponds to a weak implementation of the boundary condition since the solution does not satisfy $u_h(x_{\frac{1}{2}}, t) = g(t)$. The energy equation is

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \frac{a}{2}u_h^2(x_{\frac{1}{2}}, t) - ag(t)u_h(x_{\frac{1}{2}}, t) + \frac{|a|}{2}\sum_{i=1}^{N-1}[\![u_h]\!]^2_{i+\frac{1}{2}} + \frac{a}{2}u_h^2(x_{N+\frac{1}{2}}, t) = 0$$

The boundary terms can be rearranged as

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \frac{a}{2}[g(t) - u_h(x_{\frac{1}{2}},t)]^2 - \frac{a}{2}g^2(t) + \frac{|a|}{2}\sum_{i=1}^{N-1}[\![u_h]\!]^2_{i+\frac{1}{2}} + \frac{a}{2}u_h^2(x_{N+\frac{1}{2}},t) = 0$$

Integrating in time, we obtain the energy equation

$$\|u_h(T)\|^2 + \int_0^T \left\{ \frac{a}{2}[g(t) - u_h(x_{\frac{1}{2}},t)]^2 + \frac{|a|}{2}\sum_{i=1}^{N-1}[\![u_h]\!]^2_{i+\frac{1}{2}} + \frac{a}{2}u_h^2(x_{N+\frac{1}{2}},t) \right\} dt$$
$$= \|u_h(T)\|^2 + \frac{a}{2}\int_0^T g^2(t)dt$$

The jump terms at the boundary and interior faces lead to dissipation of energy giving the inequality

$$\|u_h(T)\|^2 \leqslant \|u_h(0)\|^2 + \frac{a}{2}\int_0^T g^2(t)dt \leqslant \|u(0)\|^2 + \frac{a}{2}\int_0^T g^2(t)dt$$

The energy in the numerical solution is bounded by the energy in the exact solution. If $g(t) \equiv 0$ then the energy will decrease with time and we have L^2 stability.

2 Non-linear conservation law

Weak solutions of non-linear conservation laws can be non-unique. To obtain a unique weak solution, we need to impose an entropy condition.

2.1 Entropy condition

Let $U(u)$ be a convex entropy function and let $F(u)$ be an associated entropy flux such that

$$F'(u) = U'(u)f'(u) \tag{Four.2}$$

If u is a smooth solution, then it satisfies the equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0$$

Multiplying throughout by $U'(u)$

$$U'(u)\frac{\partial u}{\partial t} + U'(u)\underbrace{f'(u)\frac{\partial u}{\partial x}}_{\frac{\partial f}{\partial x}} = 0 \quad\Longrightarrow\quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

we see that the smooth solution satisfies an additional conservation law. For a discontinuous solution, we will demand that it satisfy the entropy inequality

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \leqslant 0$$

in the sense of distributions with equality in smooth regions. Then Kruzkov theory shows that the weak solution is unique.

4.1 Theorem (Cell entropy inequality) The solution u_h of the semi-discrete DG scheme satisfies

$$\frac{d}{dt}\int_{I_i} U(u_h)dx + \hat{F}_{i+\frac{1}{2}}(t) - \hat{F}_{i-\frac{1}{2}}(t) \leqslant 0$$

for the square entropy $U(u) = \frac{1}{2}u^2$ with some consistent numerical entropy flux $\hat{F}_{i+\frac{1}{2}}(t) = \hat{F}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$.

Proof: Take $v_h = u_h$ in the DG scheme

$$\int_{I_i} \frac{\partial u_h}{\partial t} u_h dx - \int_{I_i} f(u_h)\frac{\partial u_h}{\partial x}dx + \hat{f}_{i+\frac{1}{2}}(t)u_h(x^-_{i+\frac{1}{2}}, t) - \hat{f}_{i-\frac{1}{2}}(t)u_h(x^+_{i-\frac{1}{2}}, t) = 0$$

Define

$$\tilde{F}(u) = \int_0^u f(s)ds \qquad \Longrightarrow \qquad \tilde{F}'(u) = f(u)$$

Integrating the compatibility condition (Four.2), we get

$$F(u) - F(0) = uf(u) - \int_0^u f(s)ds = uf(u) - \tilde{F}(u)$$

Ignoring the constant term $F(0)$ we have $F(u) = uf(u) - \tilde{F}(u)$. Then

$$-\int_{I_i} f(u_h)\frac{\partial u_h}{\partial x}dx = -\int_{I_i} \tilde{F}'(u_h)\frac{\partial u_h}{\partial x}dx = -\tilde{F}(u_h(x^-_{i+\frac{1}{2}}, t)) + \tilde{F}(u_h(x^+_{i-\frac{1}{2}}, t))$$

so that the entropy equation becomes

$$\int_{I_i} \frac{\partial U(u_h)}{\partial t} - \tilde{F}(u_h(x^-_{i+\frac{1}{2}}, t)) + \tilde{F}(u_h(x^+_{i-\frac{1}{2}}, t)) + \hat{f}_{i+\frac{1}{2}}(t)u_h(x^-_{i+\frac{1}{2}}, t) - \hat{f}_{i-\frac{1}{2}}(t)u_h(x^+_{i-\frac{1}{2}}, t) = 0$$

This can be re-written as

$$\frac{d}{dt}\int_{I_i} U(u_h)dx + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}} = 0$$

with consistent numerical entropy flux

$$\hat{F}_{i+\frac{1}{2}} = -\tilde{F}(u_h(x^-_{i+\frac{1}{2}})) + \hat{f}_{i+\frac{1}{2}}u_h(x^-_{i+\frac{1}{2}})$$

and

$$\Theta_{i-\frac{1}{2}} = -\tilde{F}(u_h(x^-_{i-\frac{1}{2}})) + \hat{f}_{i-\frac{1}{2}}u_h(x^-_{i-\frac{1}{2}}) + \tilde{F}(u_h(x^+_{i-\frac{1}{2}})) - \hat{f}_{i-\frac{1}{2}}u_h(x^+_{i-\frac{1}{2}})$$

Ignoring the suffixes, the quantity Θ can be written as

$$\begin{aligned} \Theta &= \tilde{F}(u^+) - \tilde{F}(u^-) + (u^- - u^+)\hat{f} \\ &= (u^+ - u^-)(\tilde{F}'(\xi) - \hat{f}), \qquad \min\{u^-, u^+\} \leqslant \xi \leqslant \max\{u^-, u^+\} \\ &= (u^+ - u^-)(f(\xi) - \hat{f}) \end{aligned}$$

Now assume that $u^+ \geqslant \xi \geqslant u^-$. Then since $\hat{f}$ is a monotone flux

$$f(\xi) = \hat{f}(\xi, \xi) \geqslant \hat{f}(u^-, \xi) \geqslant \hat{f}(u^-, u^+)$$

and hence $\Theta \geqslant 0$. In the case $u^+ \leqslant \xi \leqslant u^-$ we can again show that $\Theta \geqslant 0$. Thus *the semi-discrete DG scheme satisfies the entropy condition for any order of the basis functions* k. QED

4.2 Remark To obtain entropy inequality, we can also use the **E-flux** condition

$$(u^+ - u^-)(f(\xi) - \hat{f}(u^-, u^+)) \geqslant 0, \qquad \forall \xi \text{ between } u^-, u^+$$

This condition can be extended to system of conservation laws (Barth).

4.3 Corollary **(L^2 stability)** For periodic or compactly supported boundary conditions, the semi-discrete DG scheme satisfies

$$\frac{d}{dt}\int_\Omega u_h^2 dx \leqslant 0$$

which implies that

$$\|u_h(t)\| \leqslant \|u_h(0)\| \leqslant \|u(0)\|$$

Proof: Adding the cell entropy inequality from all the cells

$$\sum_{i=1}^{N} \frac{d}{dt}\int_{I_i} U(u_h) dx + \sum_{i=1}^{N} [\hat{F}_{i+\frac{1}{2}}(t) - \hat{F}_{i-\frac{1}{2}}(t)] \leqslant 0$$

the internal fluxes cancel one another, leading to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \hat{F}_{N+\frac{1}{2}} - \hat{F}_{\frac{1}{2}} \leqslant 0$$

For periodic case, $\hat{F}_{\frac{1}{2}} = \hat{F}_{N+\frac{1}{2}}$, while for compactly supported case $\hat{F}_{\frac{1}{2}} = \hat{F}_{N+\frac{1}{2}} = 0$, we obtain desired result. QED

Chapter Five

Error estimates

1 Error estimate: $u_t + cu_x = 0$, semi-discrete scheme

Theorem 2.1 (First L^2-error estimate) *Suppose that the initial condition u_0 belongs to $H^{k+1}(0,1)$. Let e be the approximation error $u - u_h$. Then we have,*

$$\| e(T) \|_{L^2(0,1)} \leq C \, | u_0 |_{H^{k+1}(0,1)} (\Delta x)^{k+1/2},$$

where C depends solely on k, $|c|$, and T.

Theorem 2.2 (Second L^2-error estimate) *Suppose that the initial condition u_0 belongs to $H^{k+2}(0,1)$. Let e be the approximation error $u - u_h$. Then we have,*

$$\| e(T) \|_{L^2(0,1)} \leq C \, | u_0 |_{H^{k+2}(0,1)} (\Delta x)^{k+1},$$

where C depends solely on k, $|c|$, and T.

(B. Cockburn, Lecture notes on *Discontinuous Galerkin methods for convection dominated problem*)

Chapter Six

Basis functions

We have to construct basis functions for V_h^k for which there are two approaches: **nodal** and **modal**. The DG solution has the form

$$x \in I_i : \qquad u_h(x,t) = \sum_{j=1}^{N} u_{ij}(t)\phi_{ij}(x), \qquad N = \dim(V_h^k) = k+1$$

and we refer to the set of values $\{u_{ij} : 1 \leqslant j \leqslant N\}$ as the *degrees of freedom* or *dof* associated with the i'th cell.

1 Nodal basis functions

A degree k polynomial is determined by $N = k+1$ values. In the **nodal** approach, we choose N distinct nodes in each cell

$$x_{ij} \in I_i, \qquad j = 1, 2, \ldots, N$$

These nodes can be used to define the Lagrange polynomials of degree k which have the interpolation property

$$\phi_{ij}(x_{il}) = \delta_{jl}, \qquad 1 \leqslant j, l \leqslant N$$

This property implies that

$$u_h(x_{ij}) = u_{ij}, \qquad 1 \leqslant j \leqslant N$$

so that the dofs in this case as the solution values at the nodes as shown in figure (Six.1). The location of the N nodal points x_{ij} can be

- uniformly distributed inside I_i
  - We may encounter Runge phenomenon for high degree polynomials
  - The mass matrix is full and ill-conditioned for large degree k
- based on Gauss-Legendre or Gauss-Lobatto integration points

Once the nodal points are chosen, the basis functions can be obtained from Lagrange interpolation.

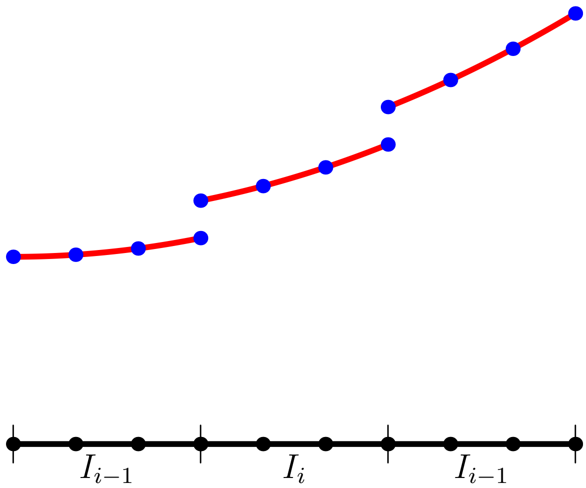


Figure Six.1: Nodal solution representation

- $\underline{k=0}$: One dof per element

$$\phi_{i1}(x) = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

- $\underline{k=1}$: Two dof per element; $x_{i1} = x_{i-\frac{1}{2}}$, $x_{i2} = x_{i+\frac{1}{2}}$

$$\phi_{i1}(x) = \begin{cases} \frac{x_{i+\frac{1}{2}}-x}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}, \qquad \phi_{i2}(x) = \begin{cases} \frac{x-x_{i-\frac{1}{2}}}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

- $\underline{k=2}$: Three dof per element; $x_{i1} = x_{i-\frac{1}{2}}$, $x_{i2} = x_i$, $x_{i3} = x_{i+\frac{1}{2}}$

- In general: We choose N distinct nodes $\{x_{i1}, x_{i2}, \ldots, x_{i,N}\} \subset I_i$

$$\phi_{ij}(x) = \frac{(x-x_{i,1})\ldots(x-x_{i,j-1})(x-x_{i,j+1})\ldots(x-x_{i,N})}{(x_{i,j}-x_{i,1})\ldots(x_{i,j}-x_{i,j-1})(x_{i,j}-x_{i,j+1})\ldots(x_{i,j}-x_{i,N})}$$

6.1 Remark It is efficient to compute the shape functions on a reference cell. Let us map cell I_i to $[-1,+1]$ by

$$\xi = \frac{x-x_i}{\frac{1}{2}\Delta x_i}, \qquad x = \frac{1-\xi}{2}x_{i-\frac{1}{2}} + \frac{1+\xi}{2}x_{i+\frac{1}{2}}$$

Choose a set of distinct nodes $\xi_0, \xi_1, \ldots, \xi_k \in [-1,+1]$. The j'th basis function is given by

$$\phi_{ij}(x) = \hat{\varphi}_j(\xi) = \frac{(\xi-\xi_0)\ldots(\xi-\xi_{j-1})(\xi-\xi_{j+1})\ldots(\xi-\xi_k)}{(\xi_j-\xi_0)\ldots(\xi_j-\xi_{j-1})(\xi_j-\xi_{j+1})\ldots(\xi_j-\xi_k)}$$

6.2 Remark If nodes are located at the element boundaries, then we have multiple dofs at the boundary since the solution is in general discontinuous.

2 Taylor basis functions

Here we do not use nodal basis functions, but use Taylor series to generate the basis functions. The cell average value u_i is one of the degrees freedom. The other dof are the gradient, hessian, etc.

Define the **moments**

$$m_{is} = \frac{1}{s!\Delta x_i} \int_{I_i} \left(\frac{x - x_i}{\Delta x_i} \right)^s dx, \qquad s = 1, 2, \ldots$$

Then the solution for different degree are taken as follows.

$\underline{k = 1}$: Dof are (u_i, s_i)

$$u_h(x,t) = u_i(t) + \frac{x - x_i}{\Delta x_i} s_i(t)$$

$$\phi_{i1}(x) = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise} \end{cases}, \qquad \phi_{i2}(x) = \begin{cases} \frac{x - x_i}{\Delta x_i} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

$\underline{k = 2}$: Dof are (u_i, s_i, q_i)

$$u_h(x,t) = u_i(t) + \frac{x - x_i}{\Delta x_i} s_i(t) + \left[\frac{1}{2} \left(\frac{x - x_i}{\Delta x_i} \right)^2 - m_{i2} \right] q_i(t)$$

$$\phi_{i3}(x) = \begin{cases} \frac{1}{2} \left(\frac{x - x_i}{\Delta x_i} \right)^2 - m_{i2} & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

In general, the basis functions are chosen such that

$$\int_{I_i} \phi_{i1} = \Delta x_i, \qquad \int_{I_i} \phi_{ij} = 0, \quad j = 2, 3, \ldots$$

- Can be extended to higher degrees using Taylor series
- Heirarchical representation
- Diagonal mass matrix for $k \leqslant 2$
- Extension to multi-dimensions on arbitrary polygonal elements; orthogonalize using Gram-Schmidt process

3 Orthogonal polynomials (Modal approach)

The Legendre polynomials are the solution of Legendre's differential equation

$$\frac{d}{d\xi} \left[(1 - \xi^2) \frac{d}{d\xi} P_n(\xi) \right] + n(n+1) P_n(\xi) = 0, \qquad n = 0, 1, 2, \ldots$$

The first few polynomials are given by

$$\begin{array}{ll} P_0(\xi) = 1 & P_1(\xi) = \xi \\ P_2(\xi) = \frac{1}{2}(3\xi^2 - 1) & P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi) \\ P_4(\xi) = \frac{1}{8}(35\xi^4 - 30\xi^2 + 3) & P_5(\xi) = \frac{1}{8}(63\xi^5 - 70\xi^3 + 15\xi) \end{array}$$

These polynomials satisfy the following recurrence relation which is useful for numerical computation

$$P_n(\xi) = \left(\frac{2n-1}{n}\right)\xi P_{n-1}(\xi) - \left(\frac{n-1}{n}\right)P_{n-2}(\xi), \qquad n = 2, 3, \dots$$

A very useful property of these polynomials is that they are mutually orthogonal, i.e.,

$$\int_{-1}^{+1} P_j(\xi) P_k(\xi) d\xi = \begin{cases} 0 & j \neq k \\ \frac{2}{2j+1} & j = k \end{cases}$$

Using these polynomials, we can define our basis functions as: $j = 0, 1, 2, \dots$

$$\phi_{ij}(x) = \sqrt{2j+1} P_j\left(\frac{x - x_i}{\Delta x_i/2}\right), \qquad \int_{I_i} \phi_{ij}\phi_{ik} dx = \begin{cases} 0 & j \neq k \\ \Delta x_i & j = k \end{cases}$$

In `deal.II`, the `FE_DGP` space makes use of these basis functions. It is convenient to define the functions on the reference cell $[-1, +1]$ as

$$\phi_{ij}(x) = \hat{\varphi}_j(\xi) = \sqrt{2j+1} P_j(\xi), \qquad \xi = \frac{x - x_i}{\frac{1}{2}\Delta x_i}$$

Many authors use the following definition for the basis functions

$$\phi_{ij}(x) = P_j\left(\frac{x - x_i}{\Delta x_i/2}\right), \qquad \int_{I_i} \phi_{ij}\phi_{ik} dx = \begin{cases} 0 & j \neq k \\ \frac{\Delta x_i}{2j+1} & j = k \end{cases}$$

Chapter Seven

Implementation in 1-D

The solution inside cell I_i is a polynomial of degree k and is of the form

$$u_h(x,t) = \sum_{j=1}^{N} u_{ij}(t)\phi_{ij}(x), \qquad N = k+1$$

The semi-discrete DG scheme is given by

$$\int_{I_i} \frac{\partial u_h}{\partial t}\phi_{ij} dx - \int_{I_i} f(u_h)\frac{\partial \phi_{ij}}{\partial x} dx \\ + \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) - \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) = 0, \qquad 1 \leqslant j \leqslant N$$

1 ODE system

Plugging the solution u_h in the DG scheme, the first term is of the form

$$\int_{I_i} \frac{\partial u_h}{\partial t}\phi_{ij} dx = \sum_{k=1}^{N} \frac{du_{ik}}{dt}\int_{I_i} \phi_{ij}\phi_{ik} dx = \sum_{k=1}^{N} M^{(i)}_{jk}\frac{du_{ik}}{dt}$$

The quantities $M^{(i)}_{jk}$ form the elements of the **mass matrix** $M^{(i)} \in \mathbb{R}^{N\times N}$. This is evaluated using a q-point quadrature rule as

$$M^{(i)}_{jk} = \int_{I_i} \phi_{ij}\phi_{ik} = \sum_{q} \omega_{iq}\phi_{ij}(x_{iq})\phi_{ik}(x_{iq}), \quad 1 \leqslant j,k \leqslant N$$

where $\{x_{iq}\}$ are the quadrature nodes in element I_i and ω_{iq} are the corresponding weights. The remaining terms which we will put on the right hand side are computed as

$$\begin{aligned} L^{(i)}(U(t))_j &= \int_{I_i} f(u_h)\phi'_{ij} - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \\ &\approx \sum_{q} \omega_{iq} f(u_h(x_{iq},t))\phi'_{ij}(x_{iq}) - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) \\ &\quad + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \end{aligned}$$

Let us put the degrees of freedom associated with I_i in a vector

$$U^{(i)} = [u_{i1}, \ldots, u_{iN}]^\top \in \mathbb{R}^N$$

Then we obtain the following system of ODE for element I_i

$$M^{(i)}\frac{dU^{(i)}(t)}{dt} = L^{(i)}(U(t))$$

The mass matrix is obviously symmetric, and it is also positive definite. For any vector $U = [u_1, \ldots, u_N]^\top \in \mathbb{R}^N$

$$U^\top M^{(i)} U = \int_{I_i} |\sum_{j=1}^{n} u_j \phi_{ij}|^2 dx > 0, \qquad U \neq 0$$

Hence we can invert the mass matrix and write the ODE as

$$\frac{dU^{(i)}(t)}{dt} = [M^{(i)}]^{-1} L^{(i)}(U(t)) = R^{(i)}(U(t))$$

2 Quadrature rules

The quadrature rule must be chosen such that mass matrix is evaluated exactly. Moreover, in order to achieve the optimal convergence rate, the integral inside the cell must be computed with a quadrature rule which is exact for polynomials of degree 2k. These requirements are satisfied by using

- $(k+1)$-point Gauss-Legendre, which is exact for $\mathbb{P}_{2k+1}$
- $(k+2)$-point Gauss-Lobatto-Legendre, which is exact for $\mathbb{P}_{2k+1}$

The quadrature rules are explained in section (1). The elements of the mass matrix can be computed as

$$M_{jk}^{(i)} = \int_{I_i} \phi_{ij}\phi_{ik} = \sum_q \omega_{iq}\phi_{ij}(x_{iq})\phi_{ik}(x_{iq}), \quad 1 \leqslant j, k \leqslant N$$

where

$$\omega_{iq} = \frac{1}{2}\omega_q \Delta x_i, \qquad x_{iq} = \frac{1-\xi_q}{2} x_{i-\frac{1}{2}} + \frac{1+\xi_q}{2} x_{i+\frac{1}{2}}$$

and $\{\xi_q\}$ are the quadrature points in the reference element $[-1, +1]$. If we use the orthogonal Legendre basis, then the mass matrix can be computed directly as

$$M_{jk}^{(i)} = \int_{I_i} \phi_{ij}\phi_{ik} dx = \Delta x_i \delta_{jk}$$

In this case, the mass matrix is diagonal, i.e., $M^{(i)} = \Delta x_i I_{N\times N}$. If we use nodal Lagrange basis at Gauss-Legendre points and if the quadrature is exact, then the mass matrix is again diagonal. Using $(k+1)$-point Gauss-Legendre quadrature, which should give the exact mass matrix, we get

$$\begin{aligned} M_{jk}^{(i)} &= \int_{I_i} \phi_{ij}\phi_{ik} = \sum_q \omega_{iq}\phi_{ij}(x_{iq})\phi_{ik}(x_{iq}), \quad 1 \leqslant j, k \leqslant N \\ &= \sum_q \omega_{iq}\delta_{jq}\delta_{kq} = \omega_{ij}\delta_{jk} \end{aligned}$$

The right hand side is also computed using a quadrature rule

$$\begin{aligned} L^{(i)}(U(t))_j &= \int_{I_i} f(u_h)\phi'_{ij}dx - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \\ &\approx \sum_q \omega_{iq} f(u_h(x_{iq},t))\phi'_{ij}(x_{iq}) - \hat{f}_{i+\frac{1}{2}}(t)\phi_{ij}(x^-_{i+\frac{1}{2}}) \\ &\quad + \hat{f}_{i-\frac{1}{2}}(t)\phi_{ij}(x^+_{i-\frac{1}{2}}) \end{aligned}$$

and if $f(u)$ is non-linear, then the quadrature is not exact. The derivative of the test function is given by

$$\frac{\partial}{\partial x}\phi_{ij}(x_{iq}) = \frac{d\xi}{dx}\frac{d}{d\xi}\hat{\varphi}_j(\xi_q) = \frac{1}{2}\Delta x_i \hat{\varphi}'_j(\xi_q)$$

and hence the quadrature is given by

$$\int_{I_i} f(u_h)\phi'_{ij}dx \approx \sum_q \omega_{iq} f(u_h(x_{iq},t))\phi'_{ij}(x_{iq}) = \frac{1}{2}\sum_q \omega_q f(u_h(\xi_q))\hat{\varphi}'_j(\xi_q)$$

The solution at the quadrature points are given by

$$u_h(\xi_q) = \sum_{j=1}^{N} u_{ij}\hat{\varphi}_j(\xi_q)$$

Thus, all of computations involving basis functions can be done on the reference cell which is common to all the cells.

3 Setting initial condition

Suppose the initial condition is

$$u(x,0) = u_0(x)$$

- Nodal basis: We can interpolate initial condition

  $$u_{ij}(0) = u_h(x_{ij},0) = u_0(x_{ij})$$

  Alternately, we can also set the initial condition by performing an L^2 projection.

- Modal basis: Do an L^2 projection, i.e.,

  $$\min \int_{I_i} (u_h(x,0) - u_0(x))^2 dx \quad \text{wrt} \quad u_{i1},\ldots,u_{iN}$$

  The first order optimality condition is

  $$\begin{aligned} \frac{d}{du_{ik}}\int_{I_i}(u_h(x,0)-u_0(x))^2dx = 0, \qquad k = 1,2,\ldots,N \\ \Longrightarrow \int_{I_i} u_h\phi_{ik}dx = \int_{I_i} u_0\phi_{ik}dx, \qquad k = 1,2,\ldots,N \end{aligned}$$

  Using the orthogonality of the basis functions, and a quadrature rule for the right hand side, we get

  $$u_{ik}\Delta x_i = \sum_q u_0(x_{iq})\phi_{ik}(x_{iq})\omega_{iq}$$

  which determines the dofs of the initial solution.

4 Boundary condition

We can specify Dirichlet boundary condition at some boundary point if the characteristics are entering the domain (inflow boundary) at that point. For example, at $x = 0$ if $f' > 0$, then we can specify the boundary condition on u. In general let us take the boundary conditions

$$u(0, t) = a(t), \qquad u(1, t) = b(t)$$

The boundary conditions are incorporated in the DG scheme via the boundary fluxes

$$\hat{f}_{\frac{1}{2}}(t) = \hat{f}(a(t), u_h(0^+, t)), \qquad \hat{f}_{N+\frac{1}{2}}(t) = \hat{f}(u_h(1^-, t), b(t))$$

In case of periodic boundaries, the fluxes at the boundary are computed as

$$\hat{f}_{\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{N+\frac{1}{2}}, t), u_h(x^+_{\frac{1}{2}}, t)), \qquad \hat{f}_{N+\frac{1}{2}}(t) = \hat{f}(u_h(x^-_{N+\frac{1}{2}}, t), u_h(x^+_{\frac{1}{2}}, t))$$

5 Strong stability preserving RK schemes

Consider an ODE of the form

$$\frac{dU}{dt} = R(U, t)$$

A standard method to solve ODEs is the Runge-Kutta method. We will consider the strong stability preserving RK schemes, examples of which are given below.

- 1-stage, first order scheme

$$U^{n+1} = U^n + \Delta t\ R(U^n, t_n)$$

- 2-stage, second order RK

$$\begin{aligned} U^{(1)} &= U^n + \Delta t\ R(U^n, t_n) \\ U^{n+1} &= \frac{1}{2}U^n + \frac{1}{2}[U^{(1)} + \Delta t\ R(U^{(1)}, t_n + \Delta t)] \end{aligned}$$

- 3-stage, third order RK

$$\begin{aligned} U^{(1)} &= U^n + \Delta t\ R(U^n, t_n) \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}[U^{(1)} + \Delta t\ R(U^{(1)}, t_n + \Delta t)] \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}[U^{(2)} + \Delta t\ R(U^{(2)}, t_n + \tfrac{1}{2}\Delta t)] \end{aligned}$$

7.1 Lemma (SSP Runge-Kutta) If the forward Euler scheme is stable in some norm and under some time step condition, i.e.,

$$\Delta t \leqslant \Delta t_c(U) \qquad \Longrightarrow \qquad \|U + \Delta t\ R(U)\| \leqslant \|U\|$$

then the SSPRK scheme is stable under a CFL condition $\Delta t \leqslant \alpha \Delta t_c$. For the second and third order schemes, $\alpha = 1$.

Proof: Let us show this for the 2-stage scheme. Since the forward Euler scheme is stable if $\Delta t \leqslant \Delta t_c$, and as first stage resembles a forward Euler scheme, we get

$$\|U^{(1)}\| = \|U^n + \Delta t \ R(U^n)\| \leqslant \|U^n\|$$

For the second stage, we apply triangle inequality

$$\|U^{n+1}\| \leqslant \frac{1}{2}\|U^n\| + \frac{1}{2}\|U^{(1)} + \Delta t \ R(U^{(1)})\| \leqslant \frac{1}{2}\|U^n\| + \frac{1}{2}\|U^{(1)}\| \leqslant \|U^n\|$$

since the second term has the form of a forward Euler scheme. QED

7.2 Remark The allowed time step Δt_c depends on the current solution. The above proof is correct provided

$$\Delta t \leqslant \min\{\Delta t_c(U^n), \Delta t_c(U^{(1)})\}$$

But we do not know $U^{(1)}$ at the beginning of the time step. In practice, we can put a margin of safety, e.g., $\Delta t = \alpha \Delta t_c(U^n)$ where $\alpha = 0.9$ for example, which usually works in practice. If it turns out after the first stage, that the Δt we have chosen is larger than $\Delta t_c(U^{(1)})$, then we can reduce it further and restart the time step from the first stage.

7.3 Remark The above SSPRK scheme require storage for three steps of vectors,

- U^n
- current stage solution $U^{(s)}$
- Residual R

7.4 Remark There exist higher order SSPRK schemes but they need more steps. There is a 5-stage, 4-th order SSPRK scheme [2].

6 CFL condition

Explicit time integration schemes are stable only under a restriction on the time step Δt. For $k = 0$, the scheme is L^2 stable under CFL number of one. For $k \geqslant 1$, the forward Euler scheme (RK1) is known to be unconditionally unstable in L^2 (Chavent and Cockburn, 1989) if the CFL number is of order unity. It is L^2 stable for finite time intervals if the CFL number is $\mathcal{O}\left(h^{1/2}\right)$ which means that $\Delta t = \mathcal{O}\left(h^{3/2}\right)$ which is very restrictive. For DG space discretizations using polynomials of degree k, and a $(k+1)$-stage RK method of order $k+1$, a Von-Neumann stability analysis for the one-dimensional linear case

$$f(u) = cu$$

with upwind flux gives the CFL condition

$$|c|\frac{\Delta t}{\Delta x} \leqslant \frac{1}{2k+1}$$

Theoretical proof of this is available only for $k = 0, 1, 2$. For $k \geqslant 3$ the above condition is close to the numerically determined values of CFL numbers.

7.1 Remark The CFL number for SSPRK scheme to be TVDM is higher than that required for L^2 stability. However, to control round-off errors, the smaller CFL condition from L^2 stability has to be used in practical computations.

Table 2.3. CFL Numbers for RKDG Methods of Order $k+1$

k	0	1	2
$\mathrm{CFL}_{\mathrm{TV}}$	1	1/2	1/2
$\mathrm{CFL}_{\mathrm{L}^2}$	1	1/3	1/5

7 Algorithm

Let us now summarize the main steps in the DG scheme.

- Compute and store the mass matrix if needed.
- Determine u_h^0 from initial condition u_0 by an L^2-projection or interpolation
- Set time counter $t = 0$
- For $n = 0, 1, \ldots$
  - Compute time step Δt from CFL condition
  - Set $u_h^{n,0} = u_h^n$
  - RK stages: For $r = 0, 1, \ldots, N_{rk} - 1$
    * Compute right hand side $L_h(u_h^{n,r})$
    * Update solution to next RK stage

$$u_h^{n,r} \to u_h^{n,r+1}$$

  - Increment time counter $t = t + \Delta t$

8 Numerical example

Let us apply the DG scheme to linear and non-linear problems with smooth and discontinuous initial conditions.

- Linear convection equation: smooth initial condition with periodic boundary conditions

$$u(x, 0) = \sin(\pi x), \qquad x \in [-1, +1]$$

- Linear convection equation: continuous initial condition with periodic boundary conditions

$$u(x,0) = \begin{cases} 1+2x, & -\frac{1}{2} \leqslant x \leqslant 0 \\ 1-2x, & 0 \leqslant x \leqslant \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}, \qquad x \in [-1,+1]$$

- Linear convection equation: discontinuous initial condition with periodic boundary conditions

$$u(x,0) = \begin{cases} 1, & |x| < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}, \qquad x \in [-1,+1]$$

- Burgers equation: smooth initial condition

$$u(x,0) = \sin(2\pi x), \qquad x \in [0,1]$$

Chapter Eight

Limiters and TVD property

When the solution is discontinuous or has large gradients, the higher order DG scheme produces oscillatory solution. This situation is similar to high order finite volume schemes and is related to loss of TVD property. The oscillatory numerical solution has more total variation than the initial condition. In the case of finite volume schemes, this problem is resolved by reducing the slope of the reconstructed solution by appropriate limiter functions so that the scheme becomes TVD. In DG schemes we do not have to perform any reconstruction since we have a polynomial inside each cell. But we can borrow the limiter idea and reduce the slope of the solution in each cell to achieve TVD property. The approach we will take is to construct a limiter so that the DG scheme with forward Euler discretization is TVD. The use of an SSPRK scheme then automatically gives TVD property for higher order versions of the DG scheme.

1 Limiter for DG scheme

Consider the forward Euler time discretization, i.e., find w_h^{n+1} such that

$$\int_{I_i} \frac{w_h^{n+1} - u_h^n}{\Delta t} v_h dx - \int_{I_i} f(u_h^n) \frac{\partial v_h}{\partial x} dx + \hat{f}^n_{i+\frac{1}{2}} v_h(x^-_{i+\frac{1}{2}}) - \hat{f}^n_{i-\frac{1}{2}} v_h(x^+_{i-\frac{1}{2}}) = 0$$

The solution w_h^{n+1} may be oscillatory. We will treat this as a provisional solution and limit it to obtain the solution at the next time step

$$u_h^{n+1} = \Lambda\Pi_h(w_h^{n+1})$$

We require that the limiter $\Lambda\Pi_h()$ satisfy some basic properties as follows.

Properties of $\Lambda\Pi_h(\cdot)$

(1) It should not change the cell average value.

(2) It should not affect the accuracy in smooth regions.

Define the **cell average value**

$$\bar{u}_i = \frac{1}{\Delta x_i} \int_{I_i} u_h dx$$

the forward and backward differences

$$\hat{u}_i = u_h(x^-_{i+\frac{1}{2}}) - \bar{u}_i, \qquad \check{u}_i = \bar{u}_i - u_h(x^+_{i-\frac{1}{2}})$$

$$\Delta_+\bar{u}_i = \bar{u}_{i+1} - \bar{u}_i, \qquad \Delta_-\bar{u}_i = \bar{u}_i - \bar{u}_{i-1}$$

For $k = 1$ we have $\hat{u}_i = \check{u}_i$. We cannot modify the cell average value but we can modify the slopes

$$\hat{u}_i^{(m)} = m(\hat{u}_i, \Delta_+\bar{u}_i, \Delta_-\bar{u}_i), \qquad \check{u}_i^{(m)} = m(\check{u}_i, \Delta_+\bar{u}_i, \Delta_-\bar{u}_i)$$

where m is the minmod function

$$m(a_1, \ldots, a_l) = \begin{cases} s\min(|a_1|, \ldots, |a_l|) & s = \text{sign}(a_1) = \ldots = \text{sign}(a_l) \\ 0 & \text{otherwise} \end{cases}$$

If all arguments have same sign, the minmod function returns the one with smallest magnitude, otherwise it returns zero. The trace values are recomputed using the limited slopes

$$u_h^{(m)}(x^-_{i+\frac{1}{2}}) = \bar{u}_i + \hat{u}_i^{(m)}, \qquad u_h^{(m)}(x^+_{i-\frac{1}{2}}) = \bar{u}_i - \check{u}_i^{(m)}$$

For $k = 1$, we have $\hat{u}_i^{(m)} = \check{u}_i^{(m)}$. The effect of the limiter is to reduce the slope of

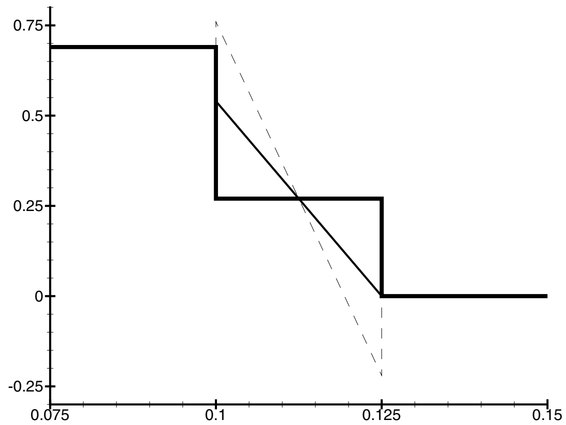


Figure Eight.1: Effect of applying TVD limiter is to reduce the slope of the solution so that it is less oscillatory

the solution in case it is larger than the finite difference slope of the cell average values as shown in figure (Eight.1). If the cell averages are monotone around cell i, then the limited linear polynomial satisfies

$$\min\{\bar{u}_{i-1}, \bar{u}_{i+1}\} \leqslant u_h^{(m)}(x) \leqslant \max\{\bar{u}_{i-1}, \bar{u}_{i+1}\}, \qquad x \in I_i$$

For $k = 0, 1, 2$, this procedure uniquely determines a new polynomial of degree k. For $k = 1$, let us denote the limited function by $\Lambda\Pi_h^1(u_h)$. For $k \geqslant 3$ there is more freedom since the cell average and the two trace values do not completely determine the polynomial. One approach is to determine the remaining dofs by an L^2 projection. A more simple approach that works well in practice is the following.

(1) If the limiter does not modify the trace values, i.e.,

$$u_h^{(m)}(x_{i-\frac{1}{2}}^+) = u_h(x_{i-\frac{1}{2}}^+) \qquad \text{and} \qquad u_h^{(m)}(x_{i+\frac{1}{2}}^-) = u_h(x_{i+\frac{1}{2}}^-)$$

then take $u_h^{(m)} = u_h$ for $x \in I_i$.

(2) Otherwise, let $u_h^1 \in \mathbb{P}_1(I_i)$ be the L^2 projection of $u_h|_{I_i}$. Take $u_h^{(m)}|_{I_i} = \Lambda\Pi_h^1(u_h^1)$.

If the DG solution has been processed by the above limiter, then we can prove a TVD property. Let us first recall Harten's sufficient conditions for a finite volume scheme to be total variation diminishing.

8.1 Lemma (Harten) If a scheme can be written as

$$\bar{u}_i^{n+1} = \bar{u}_i^n + C_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n - D_{i-\frac{1}{2}}^n \Delta_- \bar{u}_i^n$$

Assume that boundary conditions are periodic or compactly supported. If

$$C_{i+\frac{1}{2}} \geqslant 0, \qquad D_{i+\frac{1}{2}} \geqslant 0, \qquad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leqslant 1$$

then the scheme is TVD

$$TV(u_h^{n+1}) \leqslant TV(u_h^n)$$

where the total variation is defined as

$$TV(u_h) = \sum_i |\Delta_+ \bar{u}_i|$$

Proof: Take the update equation at i and $i+1$

$$\begin{aligned} \bar{u}_{i+1}^{n+1} &= \bar{u}_{i+1}^n + C_{i+\frac{3}{2}}^n \Delta_+ \bar{u}_{i+1}^n - D_{i+\frac{1}{2}}^n \Delta_- \bar{u}_{i+1}^n \\ \bar{u}_i^{n+1} &= \bar{u}_i^n + C_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n - D_{i-\frac{1}{2}}^n \Delta_- \bar{u}_i^n \end{aligned}$$

and subtracting them yields

$$\begin{aligned} \Delta_+ \bar{u}_i^{n+1} &= \Delta_+ \bar{u}_i^n + C_{i+\frac{3}{2}}^n \Delta_+ \bar{u}_{i+1}^n - D_{i+\frac{1}{2}}^n \Delta_- \bar{u}_{i+1}^n - C_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n + D_{i-\frac{1}{2}}^n \Delta_- \bar{u}_i^n \\ &= \Delta_+ \bar{u}_i^n + C_{i+\frac{3}{2}}^n \Delta_+ \bar{u}_{i+1}^n - D_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n - C_{i+\frac{1}{2}}^n \Delta_+ \bar{u}_i^n + D_{i-\frac{1}{2}}^n \Delta_+ \bar{u}_{i-1}^n \\ &= (1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) \Delta_+ \bar{u}_i^n + C_{i+\frac{3}{2}}^n \Delta_+ \bar{u}_{i+1}^n + D_{i-\frac{1}{2}}^n \Delta_+ \bar{u}_{i-1}^n \end{aligned}$$

Applying triangle inequality and noting that all the coefficients are positive according to our assumption

$$|\Delta_+ \bar{u}_i^{n+1}| \leqslant (1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) |\Delta_+ \bar{u}_i^n| + C_{i+\frac{3}{2}}^n |\Delta_+ \bar{u}_{i+1}^n| + D_{i-\frac{1}{2}}^n |\Delta_+ \bar{u}_{i-1}^n|$$

Summing over all cells, all the terms on the right cancel except the first one, yielding

$$\sum_i |\Delta_+ \bar{u}_i^{n+1}| \leqslant \sum_i |\Delta_+ \bar{u}_i^n|$$

which proves the lemma. QED

8.2 Remark Consider the first order upwind finite volume scheme for $u_t + au_x = 0$ which is identical to the DG scheme for degree $k = 0$. This scheme can be written in the incremental form with

$$C_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\frac{|a| - a}{2}, \qquad D_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\frac{|a| + a}{2}$$

The coefficients and the condition

$$C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} = \frac{|a|\Delta t}{\Delta x} \leqslant 1$$

leads to the CFL condition.

8.3 Definition (TVM) For DG solution u_h, let us define the **total variation of the means** by

$$\mathrm{TVM}(u_h) = \sum_i |\Delta_+ \bar{u}_i|$$

We will show that the TVM does not increase with time if the limiter is applied. For this we make use of the monotone flux property. Since $\hat{f}(u, v)$ is increasing function of u and decreasing function of v, we have

$$\frac{\hat{f}(a,b) - \hat{f}(c,b)}{a - c} \geqslant 0, \qquad \frac{\hat{f}(a,b) - \hat{f}(a,c)}{b - c} \leqslant 0$$

We can bound the above ratios as follows. Since

$$\frac{\hat{f}(a,b) - \hat{f}(c,b)}{a - c} = \frac{\partial}{\partial u}\hat{f}(\xi, b), \qquad \xi \in I(a,c)$$

we have

$$\frac{\hat{f}(a,b) - \hat{f}(c,b)}{a - c} \leqslant \max_{\xi \in I(b,c)} \frac{\partial}{\partial u}\hat{f}(\xi, b) =: L_1(b)$$

Similarly

$$-\frac{\hat{f}(a,b) - \hat{f}(a,c)}{b - c} = -\frac{\partial}{\partial v}\hat{f}(a, \eta), \qquad \eta \in I(b,c)$$

so that

$$-\frac{\hat{f}(a,b) - \hat{f}(a,c)}{b - c} \leqslant \max_{\eta \in I(b,c)} \left|\frac{\partial}{\partial v}\hat{f}(a, \eta)\right| =: L_2(a)$$

8.4 Theorem For periodic or compactly supported boundary conditions, the DG scheme with the minmod limiter is TVD in the means, i.e.,

$$\mathrm{TVM}(u_h^{n+1}) \leqslant \mathrm{TVM}(u_h^n)$$

Proof: Taking $v_h = 1$ for $x \in I_i$

$$\begin{aligned} w_i &= \bar{u}_i - \lambda_i[\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}], \qquad \lambda_i = \frac{\Delta t}{\Delta x_i} \\ &= \bar{u}_i - \lambda_i[\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_{i-1} + \hat{u}_{i-1}, \bar{u}_i - \check{u}_i)] \end{aligned}$$

We can write this in incremental form with

$$C_{i+\frac{1}{2}} = -\lambda_i \frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i)}{\Delta_+ \bar{u}_i}$$

$$D_{i-\frac{1}{2}} = \lambda_i \frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i) - \hat{f}(\bar{u}_{i-1} + \hat{u}_{i-1}, \bar{u}_i - \check{u}_i)}{\Delta_- \bar{u}_i}$$

Rewrite the coefficient

$$C_{i+\frac{1}{2}} = \underbrace{-\lambda_i \hat{f}_2}_{\geqslant 0} \left(1 - \frac{\check{u}_{i+1}}{\Delta_+ \bar{u}_i} + \frac{\check{u}_i}{\Delta_+ \bar{u}_i} \right)$$

where

$$0 \leqslant -\hat{f}_2 = -\frac{\hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_{i+1} - \check{u}_{i+1}) - \hat{f}(\bar{u}_i + \hat{u}_i, \bar{u}_i - \check{u}_i)}{(\bar{u}_{i+1} - \check{u}_{i+1}) - (\bar{u}_i - \check{u}_i)} \leqslant L_2$$

Since u_h^n has been pre-processed by the minmod limiter, we have

$$0 \leqslant \frac{\check{u}_{i+1}}{\Delta_+ \bar{u}_i} \leqslant 1, \qquad 0 \leqslant \frac{\check{u}_i}{\Delta_+ \bar{u}_i} \leqslant 1 \quad \Longrightarrow \quad 1 - \frac{\check{u}_{i+1}}{\Delta_+ \bar{u}_i} + \frac{\check{u}_i}{\Delta_+ \bar{u}_i} \leqslant 2$$

and hence

$$0 \leqslant C_{i+\frac{1}{2}} \leqslant 2\lambda_i L_2$$

and similarly

$$0 \leqslant D_{i+\frac{1}{2}} \leqslant 2\lambda_{i+1} L_1$$

If the time step satisfies the condition

$$C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} = 2(\lambda_i L_2 + \lambda_{i+1} L_1) \leqslant 1 \qquad \text{or} \qquad \Delta t \leqslant \frac{1}{2(\frac{L_1}{\Delta x_{i+1}} + \frac{L_2}{\Delta x_i})}$$

then from Hartens's Lemma, we have

$$\mathrm{TVM}(w_h^{n+1}) \leqslant \mathrm{TVM}(u_h^n)$$

We now apply the limiter to obtain $u_h^{n+1} = \Lambda\Pi_h(w_h^{n+1})$, and we know that u_h^{n+1} and w_h^{n+1} have the same cell average values. Hence

$$\mathrm{TVM}(u_h^{n+1}) = \mathrm{TVM}(w_h^{n+1}) \leqslant \mathrm{TVM}(u_h^n)$$

which proves the theorem. QED

8.5 Remark In the case of linear convection equation and upwind flux, we have $L_1 = \frac{1}{2}(|a| + a)$, $L_2 = \frac{1}{2}(|a| - a)$ so that the time step on a uniform mesh should satisfy the condition $\frac{|a|\Delta t}{\Delta x} \leqslant \frac{1}{2}$.

8.6 Remark On non-uniform grids, we can define the limiter as

$$\begin{aligned} \hat{u}_i^{(m)} &= \Delta x_i m \left(\frac{\hat{u}_i}{\Delta x_i}, \frac{\Delta_+ \bar{u}_i}{\frac{1}{2}(\Delta x_i + \Delta x_{i+1})}, \frac{\Delta_- \bar{u}_i}{\frac{1}{2}(\Delta x_i + \Delta x_{i-1})} \right) \\ \check{u}_i^{(m)} &= \Delta x_i m \left(\frac{\check{u}_i}{\Delta x_i}, \frac{\Delta_+ \bar{u}_i}{\frac{1}{2}(\Delta x_i + \Delta x_{i+1})}, \frac{\Delta_- \bar{u}_i}{\frac{1}{2}(\Delta x_i + \Delta x_{i-1})} \right) \end{aligned}$$

The factor in $C_{i+\frac{1}{2}}$ is

$$1 - \frac{\check{u}_{i+1}}{\Delta_+\bar{u}_i} + \frac{\check{u}_i}{\Delta_+\bar{u}_i} \leqslant 1 + \frac{\Delta x_i + \Delta x_{i+1}}{2\Delta x_i}$$

and in $D_{i+\frac{1}{2}}$ is

$$1 - \frac{\hat{u}_i}{\Delta_-\bar{u}_{i+1}} + \frac{\hat{u}_{i+1}}{\Delta_-\bar{u}_{i+1}} \leqslant 1 + \frac{\Delta x_i + \Delta x_{i+1}}{2\Delta x_{i+1}}$$

The CFL condition is

$$C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leqslant (1 + \frac{\Delta x_i + \Delta x_{i+1}}{2\Delta x_i})\lambda_i L_1 + (1 + \frac{\Delta x_i + \Delta x_{i+1}}{2\Delta x_{i+1}})\lambda_{i+1} L_2 \leqslant 1$$

2 Limiters: Implementation

Degree $k = 1$: Let us write the solution in terms of Taylor or Legendre basis

$$u_h = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right) s_i$$

Note that $u_h(x^-_{i+\frac{1}{2}}) = \bar{u}_i + s_i$ and $u_h(x^+_{i-\frac{1}{2}}) = \bar{u}_i - s_i$. We limit the slope with the minmod function

$$s_i^{(m)} = \text{minmod}\,(s_i, \bar{u}_i - \bar{u}_{i-1}, \bar{u}_{i+1} - \bar{u}_i)$$

The limited solution is

$$\Lambda\Pi_h^1(u_h) = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right) s_i^{(m)}$$

Degree $k > 1$: Let us write the solution in terms of Taylor or Legendre basis

$$u_h = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right) s_i + \text{HOT}$$

We obtain a limited slope

$$s_i^{(m)} = \text{minmod}\,(s_i, \bar{u}_i - \bar{u}_{i-1}, \bar{u}_{i+1} - \bar{u}_i)$$

If $s_i^{(m)} = s_i$ then

$$\Lambda\Pi_h(u_h) = u_h \in \mathbb{P}_k$$

else

$$\Lambda\Pi_h(u_h) = \bar{u}_i + \left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right) s_i^{(m)} \in \mathbb{P}_1$$

8.1 Remark If we use the Legendre polynomials to form the basis functions as $\hat{\varphi}_j(\xi) = \sqrt{2j+1}P_j(\xi)$, then the linear solution is of the form

$$u_h = \bar{u}_i + \sqrt{3}\left(\frac{x - x_i}{\frac{1}{2}\Delta x_i}\right) s_i$$

The limited slope is given by

$$s_i^{(m)} = \frac{1}{\sqrt{3}}\text{minmod}\left(\sqrt{3}s_i, \bar{u}_i - \bar{u}_{i-1}, \bar{u}_{i+1} - \bar{u}_i\right)$$

3 Numerical example

Discontinuous solution and smooth solution

$$\boxed{\text{Numerical example}}$$

If the solution in cell I_i has an extremum, see figure (xxx), then the slope in the cell has a different sign compared to $\Delta_-\bar{u}_i$, $\Delta_+\bar{u}_i$ and the limiter returns zero slope. The limited solution becomes constant and equal to the cell average value. Thus there is loss of accuracy at smooth extrema due to TVD limiter. We have to modify the limiter so that it does not change the solution at smooth extrema.

4 TVB Limiter

In smooth regions of the solution, the differences inside the cell are

$$\hat{u}_i = \frac{1}{2}u_x(x_i)\Delta x_i + \mathcal{O}\left(h^2\right), \qquad \check{u}_i = \frac{1}{2}u_x(x_i)\Delta x_i + \mathcal{O}\left(h^2\right)$$

while the differences of cell averages are

$$\Delta_+\bar{u}_i = \frac{1}{2}u_x(x_i)(\Delta x_i + \Delta x_{i+1}) + \mathcal{O}\left(h^2\right)$$

$$\Delta_-\bar{u}_i = \frac{1}{2}u_x(x_i)(\Delta x_i + \Delta x_{i-1}) + \mathcal{O}\left(h^2\right)$$

If the solution is smooth and monotone around I_i, and since the $\mathcal{O}\left(h^2\right)$ term is small, all the above quantities have the same sign, so that the limiter yields

$$\hat{u}_i^{(m)} = \hat{u}_i, \qquad \check{u}_i^{(m)} = \check{u}_i$$

Since the solution is not modified by the limiter, we obtain the full accuracy of the scheme. However, if there is a smooth extremum in cell I_i, then $u_x(x_i) \approx 0$ and the sign is determined by the second derivative. If M is the magnitude of the second derivative at smooth extrema, then $\hat{u}_i = \mathcal{O}\left(Mh^2\right)$. This motivates the definition of the TVB limiter function as

$$\tilde{m}(a_1, a_2, \ldots, a_l) = \begin{cases} a_1 & \text{if } |a_1| \leqslant Mh^2 \\ m(a_1, a_2, \ldots, a_l) & \text{otherwise} \end{cases}$$

If we are near a smooth extremum, the TVB limiter returns the original slope and the solution is not modified in that cell. With the above limiter, the scheme is no longer TVDM and the TVM can increase. However this violation is small and of the other of the mesh size.

8.1 Lemma (TVB property) With the TVB limiter, if the CFL condition

$$\Delta t \leqslant \frac{1}{2(\frac{L_1}{\Delta x_{i+1}} + \frac{L_2}{\Delta x_i})}, \qquad \forall i$$

is satisfied, then

$$\mathrm{TVM}(u_h^{n+1}) \leqslant \mathrm{TVM}(u_h^n) + CMh$$

Proof: See [1]

8.2 Remark If we are interested in the solution in a finite time interval $[0, T]$, then the TVB limiter ensures that

$$\text{TVM}(u_h^n) \leqslant \text{TV}(u_0) + CMT, \qquad \forall\; n\Delta t \leqslant T$$

so that the numerical solutions have bounded variation. This is sufficient to prove convergence to a weak solution.

8.3 Remark The quantity M is an estimate of the second derivative of the solution at smooth extrema. This can be based on the initial condition, e.g.,

$$M = \max_x \{|u_0''(x)| : u_0'(x) = 0\}$$

Ideally M should be estimated from the numerical solution but there is no reliable way to do this. The solution may have several extrema with different magnitude of the second derivatives. In practice people choose the value of M by some trial and error. But this is still a weak point of the TVB limiter.

$\boxed{\mathcal{Numerical\ example}}$

5 Algorithm

The DG scheme together with the TVD/TVB limiter is as follows.

- Compute and store the mass matrix
- Determine w_h^0 from initial condition u_0 by an L^2-projection
- Find u_h^0 from w_h^0 by applying the limiter, $u_h^0 = \Lambda\Pi_h(w_h^0)$
- For $n = 0, 1, \ldots$
  - Compute time step from CFL condition
  - Set $u_h^{n,0} = u_h^n$
  - RK stages: For $r = 0, 1, \ldots, N_{rk} - 1$
    * Compute right hand side $L_h(u_h^{n,r})$
    * Update solution to next RK stage

      $$u_h^{n,r} \to w_h^{n,r+1}$$

    * Apply limiter

      $$u_h^{n,r+1} = \Lambda\Pi_h(w_h^{n,r+1})$$

Appendix A

Quadrature rules

1 Quadrature in 1-D

The DG scheme involves integrals which must be approximated by quadrature. Let

$$f : [-1, +1] \to \mathbb{R}$$

Choose n quadrature nodes $\{\xi_1, \xi_2, \dots, \xi_n\} \subset [-1, +1]$.

$$\int_{-1}^{+1} f(\xi) d\xi \approx \sum_{q=1}^{n} \omega_q f(\xi_q)$$

- Gauss-Legendre quadrature
  - Nodes $\{\xi_q\}$ are roots of Legendre polynomial $P_n(\xi)$
  - n-point rule is exact for any $f \in \mathbb{P}_{2n-1}$
- Gauss-Lobatto-Legendre quadrature
  - Nodes include $\{-1, +1\}$ and the roots of $P'_{n-1}(\xi)$
  - n-point rule is exact for any $f \in \mathbb{P}_{2n-3}$

For function on general interval $f : [a, b] \to \mathbb{R}$, do change of variable

$$x(\xi) = \frac{1-\xi}{2} a + \frac{1+\xi}{2} b, \qquad \xi \in [-1, +1]$$

and

$$\begin{aligned}
\int_a^b f(x) dx &= \frac{1}{2}(b-a) \int_{-1}^{+1} f(x(\xi)) d\xi \\
&\approx \frac{1}{2}(b-a) \sum_{q=1}^{n} \omega_q f(x(\xi_q)) \\
&= \sum_{q=1}^{n} \tilde{\omega}_q f(x(\xi_q)), \qquad \tilde{\omega}_q = \frac{1}{2}(b-a)\omega_q
\end{aligned}$$

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