Eigenvalue computation using time-stepper

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1 Eigenvalue problem

Consider the linearized Navier-Stokes (LNS) equation

$$M_1 \frac{\mathrm{d}v}{\mathrm{d}t} = A_1 v + B_1 p, \qquad B_1^\top v = 0$$

Here M_1 is the mass matrix corresponding to the velocity. After eliminating the pressure we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av$$

The exact solution of this equation is

$$v(t) = \exp(At)v_0 = E(t)v_0$$

Let $E_{\Delta t}$ be a numerical scheme to solve the LNS

$$v_{n+1} = E_{\Delta t} v_n, \qquad t_{n+1} = t_n + \Delta t$$

Fix a time $\tau = N\Delta t$ and define

$$E_\tau = (E_{\Delta t})^N$$

i.e., E_τ corresponds to performing N time steps of the numerical scheme.

We want to solve the eigenvalue problem

$$Ae = \lambda e$$

Instead let us consider the eigenvalue problem

$$E_\tau e = \mu e$$

The two eigenvalues are related by

$$\mu = \exp(\lambda \tau)$$

Hence

$$\text{real}(\lambda) = \frac{1}{\tau} \log |\mu|, \qquad \text{imag}(\lambda) = \frac{1}{\tau} \text{arg}(\mu)$$

In the Arnoldi method, the eigenvectors are normalized to unit norm. Since we want to use the L^2 norm we will consider the generalized eigenvalue problem

$$M_1 E_\tau e = \mu M_1 e$$

where M_1 is the mass matrix of velocity. Solving this generalized eigenvalue problem using ARPACK will give us eigenvectors whose $L^2(\Omega)$ norm is unity. Note that we only compute eigenvectors corresponding to velocity.

2 Plane Poiseuille flow

Consider a channel of dimensions $[-1, +1] \times [0, 2\pi]$. The base solution has the velocity $(1 - y^2, 0)$. We use periodic boundary conditions in the x direction.

ARPACK requires following matrix-vector operations.

- `subroutine mx(n, v, w)`
  `imode = 1`

  $$w = M_1 v$$

- `subroutine ax(n, v, w)`
  `imode = 2`

  $$w = M_1 E_\tau v$$

- `subroutine op(n, v, w)`
  `imode = 3`

  $$w = E_\tau v$$

The value of `imode` is passed to the time stepper (`nek5000`) so that it can perform the appropriate operations.