Nontrivial Time-Periodic Solutions to the Whitham Equation

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June 2020

Abstract

We present our results from applying the method introduced by Ambrose and Wilkening to locate nontrivial time-periodic solutions to the Whitham equation [2]. The method is based on minimizing a functional of an initial condition and its temporal period, and we assume all solutions have a spatial period of 2π. We solve the original partial differential equation (PDE) with a stationary wave solution perturbed in the direction of a nontrivial time-periodic solution, and then evolve according to the adjoint PDE to compute the gradient of this functional with respect to the initial condition. The gradient is then used to iteratively minimize the functional with a quasi-Newton's method. We are able to fix the spatial and temporal phases of the Fourier modes by assuming a nontrivial time-periodic solution can be phase shifted so that its starting position has even symmetry in space and time. A penalty function is implemented to prevent the method from converging back onto traveling-wave solutions.

1 Introduction

Euler's equations can be derived as a result of the physical forces acting on a fluid. However, if the primary interest is the propagation of small amplitude long waves in relatively shallow water, then the Korteweg-de Vries equation would be a candidate model [12]. Furthermore, if the goal is to find a model for small amplitude waves of all wavelengths in relatively shallow water with unidirectional linear phase velocity identical to Euler's equations, then the Whitham equation is an ideal candidate.

Time-periodic solutions are of particular interest for wave propagation models. John Scott Russel studied soliton waves in the early 1800s [12], and by the end of the century Korteweg and de Vries published their paper which introduced the equation that bears their names. Stationary and traveling-wave solutions are examples of time-periodic solutions, but there exist solutions that fall into neither of those classifications that are also truly periodic in both space and time.

Standing waves do not propagate through space, but they do occupy some spatial period and return to their original shape after some temporal period.

The particles on the surface of a standing wave traverse paths perpendicular to the surface and oscillate over some temporal period. Traveling-wave solutions propagate through space over some spatial and temporal period. Traveling-waves are key for the variational method to be applied to the Whitham equation, so the focus shall begin with such solutions. A spectral description of a traveling-wave could be a solution whose Fourier modes maintain a constant magnitude and whose phases change with some constant rate.

The subject of this paper is to find time-periodic solutions to the Whitham equation whihch are not traveling-wave solutions. These solutions can be thought of as small amplitude waves traveling on a larger carrier wave. In [2], Ambrose and Wilkening present an algorithm for computing such solutions for the Benjamin-Ono equation. In addition to their numerical algorithm and results for the Benjamin-Ono equation, they conjecture that the method is not restricted to just one equation, but that this method should be applicable to any PDE that permits wave solutions. The process involves two steps to adapt to a new PDE: (1) establish the variational equation related to the linearization about stationary solutions, (2) derive the adjoint PDE and account for time reversal.

2 Governing Equations

Euler's equation is the result of applying Newton's Second Law to an inviscid fluid, and it is defined as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1a}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\,\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - g, \tag{1b}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \epsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \tag{2}$$

There is also the surface and bottom boundary conditions. The surface boundary condition is composed of the kinematic and dynamic conditions

$$w = \frac{\partial \eta}{\partial t} + \epsilon\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) \qquad \text{on } z = 1 + \epsilon\eta \tag{3a}$$

$$p = \eta \qquad \text{on } z = 1 + \epsilon\eta, \tag{3b}$$

and the bottom boundary condition is

$$w = b_t + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)b \quad \text{on} \quad z = b. \tag{4}$$

Here u, v, w are the fluid velocities in the x, y, z dimensions respectively, t is time, ρ is the fluid density, p is pressure, g is gravity, h is the free surface, and

b is the bathemtry [12]. Euler's equation is a complex system of equations that involves nonlinearities and a free surface.

In order to simplify this model some assumptions about the regime of particular waves of interest can be introduced. Specifically, small amplitude and long waves in relatively shallow fluid. The Korteweg-de Vries (KdV) equation is derived from Euler's equations by making these assumptions and applying asymptotic principles. For derivation details see Johnson [12]. KdV is given as

$$u_t = -u_{xxx} - 2uu_x, \tag{5}$$

where u now refers to the surface displacement.

The Whitham equation is an alternate model equation for the propagation of unidirectional small-amplitude surface waves of an inviscid fluid. The Whitham equation is given by,

$$u_t = -\frac{1}{2\pi} \int_{-\infty}^{\infty} ik \sqrt{\frac{\tanh k}{k}}\ \hat{u}(k,t)\ e^{ikx}\ dk - 2uu_x, \tag{6}$$

We can instead represent the Whitham equation in the more concise form

$$u_t = -K * u_x - 2uu_x, \tag{7}$$

where $*$ is the convolution operator, and the kernel of the convolution is rewritten as a Fourier multiplier,

$$\hat{K}(k) = \sqrt{\frac{\tanh(k)}{k}}. \tag{8}$$

There are no known nontrivial solutions to the Whitham equation.

In [2], Ambrose and Wilkening studied nontrivial time-periodic solutions for the Benjamin-Ono equation

$$u_t = H * u_{xx} - uu_x \tag{9}$$

where H is the Hilbert transform defined as $\hat{H}(k) = -i\ \mathrm{sgn}(k)$. There are stationary solutions for which closed form solutions are known for (9). Traveling-wave solutions to (9) are simply vertical translations of stationary solutions by some constant c. This results in the solution to traverse the periodic domain with speed c

$$u_{tw} = u_{stat} + c \tag{10}$$

The Benjamin-Ono and Whitham equations share several properties. They are both nonlinear and nonlocal, and differ only in their respective convolution terms. The challenge the Whitham equation presents is that there are no known closed form solutions for nontrivial solutions. Whereas the Benjamin-Ono equation has known closed form solutions for stationary waves.

3 Stationary and Traveling-Wave Solutions

The aforementioned traveling-wave solutions are always real and have the property that they can be phase shifted so that at $t = 0$ the function is even. This means that the search for traveling-wave solutions can be restricted to solutions whose intial condition has strictly real Fourier modes. An approach to finding these solutions is to use Newton's method to solve for the positive Fourier modes.[1]

The search for time-periodic solutions begins with traveling-wave solutions with spatial period 2π. To find these solutions, assume the solutions have the form

$$u(x,t) = f(x - ct) = f(z). \tag{11}$$

Substituting this into (7), and then integrating once gives

$$-cf + K * f + f^2 = B, \tag{12}$$

where B is the constant of integration. In [9], Ehrnström and Kalisch show that the Whitham equation has solutions of this form. Assuming the solutions are nice and spatially periodic, any introductory Fourier Analysis textbook, such as Davis [6], will show that the solutions can be represented as an infinite Fourier series

$$f(z) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{-ikz}. \tag{13}$$

This gives us the following system of equations

for $k = 0$,

$$F_0 := \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{f}(-n) - B = 0, \tag{14a}$$

for $k \neq 0$,

$$F_k := \Big(- c + \hat{K}(k) \Big) \hat{f}(k) + \sum_{n=-\infty}^{\infty} \Big(\hat{f}(k-n)\hat{f}(n) + \hat{f}(k+n)\hat{f}(n) \Big) = 0. \tag{14b}$$

By restricting the search for solutions with zero mean, the zero mode is assumed to be zero in the following simplifications. This has the consequence that only the positive modes are needed to compute B, and thus B can be recovered without explicitly including it in the root-finding routine. By choosing some cut-off frequency the solution space is constrained to $M/2$ degrees of freedom for a solution with M spatial points as shown below

[1] In our implementation choose a sufficiently large number of Fourier modes for the spatial resolution and use Newton's method to find the spectral composition of the real positive modes. We double the number of modes whenever the upper-half of the spectrum exceeds some tolerance that we establish for zero.

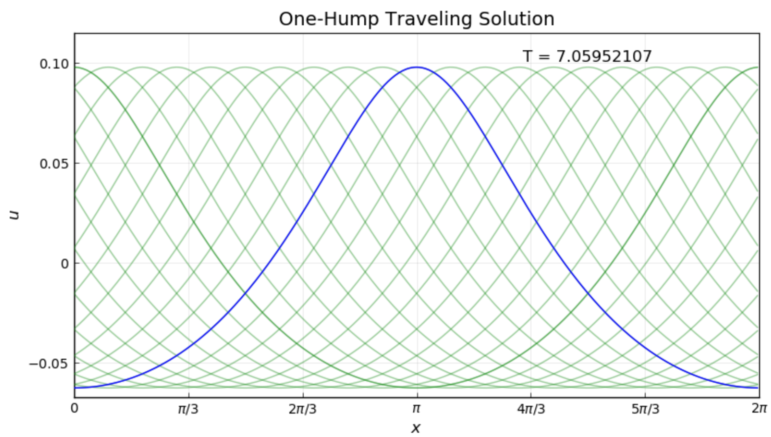

Figure 1: A zero mean traveling-wave solution to the Whitham equation. The green curves represent the position of the solution at 20 evenly spaced time slices over T, and the blue curve is the solution at $T/2$.

for $0 < k \leq M/2$,

$$F_k := \Big(-c + \hat{K}(k) \Big) \hat{f}(k) + \sum_{n=1}^{M/2} \Big(\hat{f}(k-n)\hat{f}(n) + \hat{f}(k+n)\hat{f}(n) \Big) = 0. \quad (15)$$

To verify that the solutions we computed are in fact numerically accurate we can find the error-norm after some integer multiple of the temporal period. Checking the difference of the conserved quantities at $t = 0$ and $t = T$ serves as an additional check. Sanford et al [18] list the first three, and they are given as

$$Q_1 = \int_{\infty}^{\infty} u \; dx \quad (16a)$$

$$Q_2 = \int_{-\infty}^{\infty} u^2 \; dx \quad (16b)$$

$$Q_3 = \int_{-\infty}^{\infty} uK * u - u^3 \; dx \quad (16c)$$

In Figure 1 we see a traveling-wave that was computed with the algorithm laid out above. It was achieved by iterating the routine while incriminating c after Newton's method converged to a traveling-wave solution. It should be noted that this routine can be reformulated into an optimization problem, but for the purposes of this project a root-finding routine worked fine and lowered the complexity of the problem.

When this algorithm is implemented additional traveling-wave solutions can be found by choosing a variable as a bifurcation parameter. In the work presented, speed c was used as the bifurcation parameter.

3.1 Properties of Traveling-Wave Solutions

In [9], Ehrnström and Kalisch show that the Whitham equation only has constant solutions of non-zero mean and non-zero speed. Mean α_0 describes the net mass displaced, and this is represented by

$$\alpha_0 = \frac{1}{L}\int_0^L u\,dx. \tag{17}$$

Consequently without proof, there is no stationary solution with zero mean. Solutions that possess a non-zero mean are not typical physical interpretations of water in nature since this would mean their is more mass above or below the the boundary where mass would be found if the fluid was at rest. As such, solutions with zero mean are of primary concern.

Ehrnström and Kalisch [9] also show that traveling-wave solutions for (7) are invariant under the following transformation,

$$f \to f + \gamma, \quad c \to c + 2\gamma, \quad B \to B + \gamma(1 - c - \gamma), \tag{18}$$

where γ is some constant. Traveling-wave solutions can be considered stationary waves if a moving frame of reference is introduced. By introducing the new coordinates

$$\xi = x - ct \text{ and } \tau = t \tag{19}$$

Substituting this back into (7)

$$u_\tau = c\,u_\xi - K * u_\xi - 2uu_\xi. \tag{20}$$

Using (18) on the previous traveling-wave solution produces the following stationary wave.

Yet another property is that if $u(x,t)$ is a solution to (7), then so is $u(-x,-t)$.

4 Linear Theory

Similarly stated in [2], to find time-periodic solutions there must be initial conditions u_0 with some temporal period T, such that the following equation is satisfied, $F(u_0, T) = 0$ where $F : H^1 \times \mathbb{R} \to H^1$ by

$$F(u_0, T) = u(\cdot, T) - u_0, \quad u_t = -K * u_x - 2uu_x, \quad u(\cdot, 0) = u_0, \tag{21}$$

where H^1 is a Sobolev function space. A thorough definition for this function space can be found in any functional analysis textbook such as Debnath and Mikusiński [7]. By linearizing F about traveling-wave solutions and use solutions of the linearized problem as the initial condition to find nontrivial time-periodic solutions to the nonlinear problem. However, to reduce the complexity of this problem, traveling-wave solutions will be linearized with respect to the moving frame of reference.

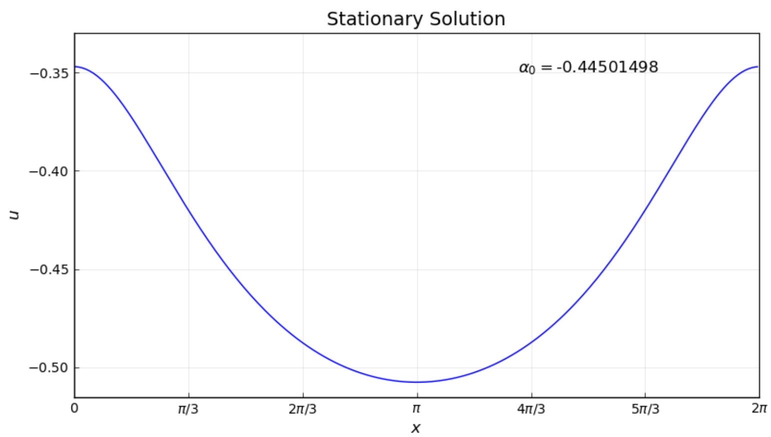

Figure 2: By translating the the traveling-wave solution to have zero-speed, it now has nonzero mean.

4.1 Linear Stability Problem

Let $u(\xi)$ be a traveling-wave solution, and if $u(\xi) + v(\xi, \tau)$ satisfies (7) to the first-order in ϵ, within a traveling frame of reference, then v will satisfy

$$(u + \epsilon v)_\tau = c(u + \epsilon v)_\xi - K * (u + \epsilon v)_\xi + 2(u + \epsilon v)(u + \epsilon v)_\xi. \tag{22}$$

Since the $\mathcal{O}(\epsilon^0)$ terms are equivalent to (7), these terms disappear, and $\mathcal{O}(\epsilon^2)$ terms are assumed to be negligible. Taking only the $\mathcal{O}(\epsilon)$ terms, ignoring higher order terms, and dropping ϵ gives

$$v_\tau = cv_\xi - K * v_\xi - 2(uv)_\xi. \tag{23}$$

This can be rewritten as

$$v_\tau = BAv, \tag{24}$$

where the operators A and B on H^1 are defined as

$$A = -c + K * +2u, \tag{25a}$$

$$B = -\partial_\xi. \tag{25b}$$

Without loss of generality, $v(x, t)$ can be written as

$$v(\xi, \tau) = z(\xi)e^{\omega\tau} + c.c., \tag{26}$$

where $c.c.$ is the complex conjugate. This is equivalent to solving the eigenvalue problem

$$BAz = \omega z. \tag{27}$$

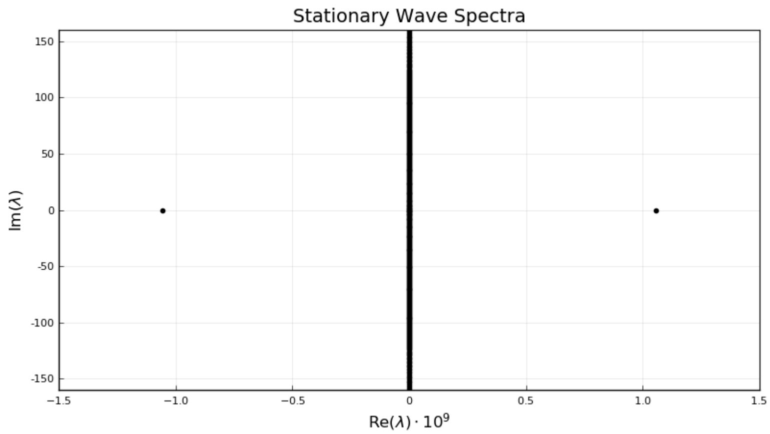

Figure 3: Using the stationary solution from Figure 2, 384 Fourier modes are used to represent the solution spectrally, and the spectra is computed for the operators in (25).

A thorough explanation of computing the spectra of linear operators is given in [8] by Deconick and Kutz. The eigenvalue problem (27) is solved numerically by choosing a cut-off frequency for the discrete spectra, and then an eigenvalue routine can be applied to the resulting matrix to spectrally decompose the operators into eigenvalue- eigenvectors pairs. The spectra of the stationary wave from Figure 2 is show in Figure 3. Sanford et al. [18] as well as Carter and Rozman [4] look at the spectrum of these operators in greater detail.

4.2 Bifurcations from Stationary-Waves

By considering $u(\xi)$ such that

$$DF = (D_1 F, D_2 F) : H^1 \times \mathbb{R} \to H^1 \tag{28}$$

satifies

$$\begin{aligned} D_1 F(u_0, T)v_0 &= \frac{\partial}{\partial \epsilon}|_{\epsilon=0} F(u_0 + \epsilon v_0, T) = v(\cdot, T) - v_0 = (e^{iBAT} - I)v_0, \\ D_2 F(u_0, T)\sigma &= \frac{\partial}{\partial \epsilon}|_{\epsilon=0} F(u_0, T + \epsilon\sigma) = 0. \end{aligned} \tag{29}$$

We can equate DF to be the variational derivative of (21). This says that for $v_0 \in \ker D_1 F(u, T)$, then $v(\xi, \tau)$ is periodic with temporal period T. Which is to say that solutions v that are minimizers of (21) force the variational derivative to be zero.

5 Numerics

In [2], the foundation for this variational method is laid down. Ambrose and Wilkening optimized their algorithm and extended the bifurcation theory from stationary solutions to traveling-wave solutions in [3]. Additional optimizations were laid out in [1] when the authors adopted their method to the vortex-sheet problem. The following method presented is the implementation of the algorithm given in [2] and is adapted for the Whitham equation. It should also be highlighted that the optimizations introduced in [3] and [1] are also used by this implementation. The implementation was developed in the Julia programming language and will eventually be available on GitHub: https://github.com/cpross90.

5.1 Method Setup

In [2], Ambrose and Wilkening note that all bifurcations can be phase shifted to have even symmetry at $t = 0$, we believe that this same assumption which will allow us to simplify several implementation details is also true for Whitham, but we cannot verify this claim at this time. The variational approach of this method relies on finding an initial condition u_0 with even symmetry and a temporal period T such that it minimizes the functional

$$G_{tot}(u_0, T) = G(u_0, T) + G_{pen}(\hat{u}_0, T), \tag{30}$$

where $\hat{u}_0(k)$ is the Fourier transform of $u_0(x)$, and G,G_{pen} are defined by

$$\begin{aligned} G(u_0, T) &= \frac{1}{2}\int_0^{2\pi} \Big[u(x, T/2) - u(2\pi - x, T/2)\Big]^2\, dx, \\ G_{pen}(\hat{u}_0, T) &= \frac{1}{2}\bigg(\big[\operatorname{Re}\{\hat{u}_0(0)\} - \alpha_0\big]^2 + \big[\operatorname{Re}\{\hat{u}_0(\tilde{k})\} - \rho\big]^2\bigg). \end{aligned} \tag{31}$$

Without the inclusion of the penalty function the routine would most likely converge back onto stationary and traveling-wave solutions. By imposing restrictions on the mean of the solution, the real part of the $\tilde{k}^{\text{th}}$ Fourier mode, and restricting the search to solutions even in space and time at $t = 0$ Ambrose and Wilkening found that they could bifurcate to other nontrivial time-periodic solutions without immediately converging back to traveling-wave solutions. The value ρ is used as the bifurcation parameter and allows for additional nontrivial time-periodic solutions to be computed by varying the $\tilde{k}^{\text{th}}$ Fourier mode, and the initial mean, α_0, can be used to enforce a particular or as an additional bifurcation parameter. The $\tilde{k}^{\text{th}}$ Fourier mode is typically chosen to be the first nonzero Fourier mode in the chosen eigenfunction used for constructing the perturbation.

To minimize the functional, the variational derivative of G must be computed

$$\begin{aligned} \frac{\delta G}{\delta u_0} &= 2w(x, T/2), \\ w_0(x) &= u(x, T/2) - u(2\pi - x, T/2), \end{aligned} \tag{32}$$

where w is evolved according to the adjoint PDE

$$w_s(x,s) = -K * w_x - 2u(x, \frac{T}{2} - s)w_x(x,s), \tag{33}$$

and u is evolved according to (7).

Since it is assumed that u has even symmetry at $t = 0$, then

$$u(2\pi - x, T/2) = u(x, -T/2).$$

This means that u and w only need to be evolved to time $T/2$. Doing so takes half as much computational time, and decreases that amount of error propagation in calculations. Had the aforementioned property and the evenness not been taken advantage of, then u and w would need to be evolved all the way to time T. The penalty function would also need to fix the spatial and temporal phases of the Fourier modes for $\hat{u}_0$.

5.2 Implementation Details

We represent the initial condition u_0 with M Fourier modes, but we only store the first $M/4$ non-negative real Fourier modes and the temporal period (T) in an array q for a total of $M/4 + 1$ elements

$$q = [a_0, a_1, ..., a_{M/4-1}, T], \qquad \hat{u}_0(k) = a + ib.$$

We evolve u and w in physical space according to their respective equations (7) and (33). Similar to [2], we implement a pseudospectral method for evolving initial conditions to (7) and (33). We compute spatial derivatives in Fourier space via the FFT[2] [5] and nonlinear terms in physical space. In the evolution of u and w we zero out the Nyquist frequency in Fourier space before returning to physical space in the linear and nonlinear parts of the solver. The explicit singly diagonally implicit Runge-Kutta (ESDIRK) Kennedy-Carpenter [14] routine is used to evolve initial conditions for (7) and (33). The 'KenCarp4' routine in the DifferentialEquations.jl [17] package provides a fourth-order interpolation routine for intermediate time steps for use in the non-autonomous term in (33).

For the optimization routine, L-BFGS from the Optim.jl package [16] is used to find an initial condition to (7) that minimizes the functional (30). The BFGS routine computes G, $\frac{\delta G}{\delta u_0}$, and $\frac{\delta G}{\delta T}$ during each iteration of the optimization routine. The reason for using BFGS over Newton is that it takes longer to compute (30) than the linear algebra involved to update the approximate Hessian matrix, and therefore computing the Hessian directly would be unreasonably computationally expensive for this use-case. Additional nontrivial time-periodic solutions can be obtained by increasing the bifurcation parameter ρ.[3] The L-BFGS algorithm was initially developed for low-memory systems,

[2]We use the r2c version of the FFT. In Julia this is implemented as rfft.

[3]Using the previously computed Hessian instead of the identity leads to faster convergence when bifurcating in accordance with the continuation scheme.

but by storing twice as many updates as there are degrees of freedom can allow for BFGS to converge several orders of magnitude faster.

To avoid repeat computations inside of the optimization routine, there is an option to pass a single objective function and declare which areas are needed for the functional and gradient. This allows the linesearch algorithm to determine the optimal step size and direction[4]. The functional and $\frac{\partial G}{\partial T}$ are computed with the trapezoid rule in physical space. After G is computed, $\nabla_{u_0} G$ is computed by evolving w to $T/2$ and then taking the FFT of $\frac{\delta G}{\delta u_0}$ and scaling its Fourier modes to account for normalization.[5] The equations for the gradient are as follows

$$\begin{aligned} \frac{\partial G}{\partial a_0} &= \int_0^{2\pi} \frac{\delta G}{\delta u_0}\, dx = 2\pi \Big(\frac{\delta G}{\delta u_0}\Big)^{\wedge}_0, \\ \frac{\partial G}{\partial a_k} &= \int_0^{2\pi} \frac{\delta G}{\delta u_0} \big(e^{ikx} + e^{-ikx}\big)\, dx = 4\pi \operatorname{Re}\left\{ \Big(\frac{\delta G}{\delta u_0}\Big)^{\wedge}_k \right\}, \\ \frac{\partial G}{\partial T} &= \frac{1}{2}\int_0^{2\pi} \Big[u(x,T/2) - u(2\pi - x, T/2)\Big]\Big[u_t(x,T/2) - u_t(2\pi - x, T/2)\Big] dx. \end{aligned} \tag{34}$$

However, since we minimize (30) we need to calculate $\nabla_{u_0} G_{tot}$. Calculating $\partial_{a_0} G_{pen}$ and $\partial_{a_{\tilde{k}}} G_{pen}$ is straightforward, and since G_{pen} is independent of T there is no contribution to $\partial_T G_{tot}$ from G_{pen}.

5.3 Nontrivial Time-Periodic Solutions

We now describe the numerical results that we were able to achieve by executing the various procedures laid out in this paper.

To demonstrate these procedures numerically, we first choose a traveling-wave solution for (7) that has been computed using the method outlined in Section 3. A solution with temporal period $T = 7.05952107$ was chosen; which corresponds to Figure 1. Using (18) we vertically translate the traveling-wave solution by $-c/2$ to get the stationary solution in Figure 2. Applying the method outlined in Section 4, we compute the spectra of the linear operators to find admissible perturbations and their associated speeds, and we can see this in Figure 3. The eigenfunction will dictate the kind of small-amplitude wave we see traversing the original stationary wave, and we decided to choose the eigenvalue with zero real-part and the smallest positive imaginary-part, and this is shown in upper plot in Figure 4 for $\omega = 0.35535515$. Before we could execute the search routine, we needed to construct the perturbation first. We selected the perturbation corresponding to $\operatorname{Re}(z(\xi)e^{i\omega t})$. By inspecting the Fourier modes we noticed that $\tilde{k} = 1$ is the first nonzero mode. This also tells us we should vary $\rho = \hat{u}_0(1) = 0.03778142$ towards 0. By continuing this bifurcation scheme we eventually arrive to a solution such that $\hat{u}_0(1) = 0.02778142$, and this solution is shown in lower plot in Figure 4. As long as we kept the bifurcation parameter

[4] We found that the limited memory conjugate gradient linesearch algorithm introduced by Hager and Zhang [11] performed particularly well.

[5] This is $1/M$ for the FFTW implementation in Julia. If the brfft interface is used, then there is no need for normalization.

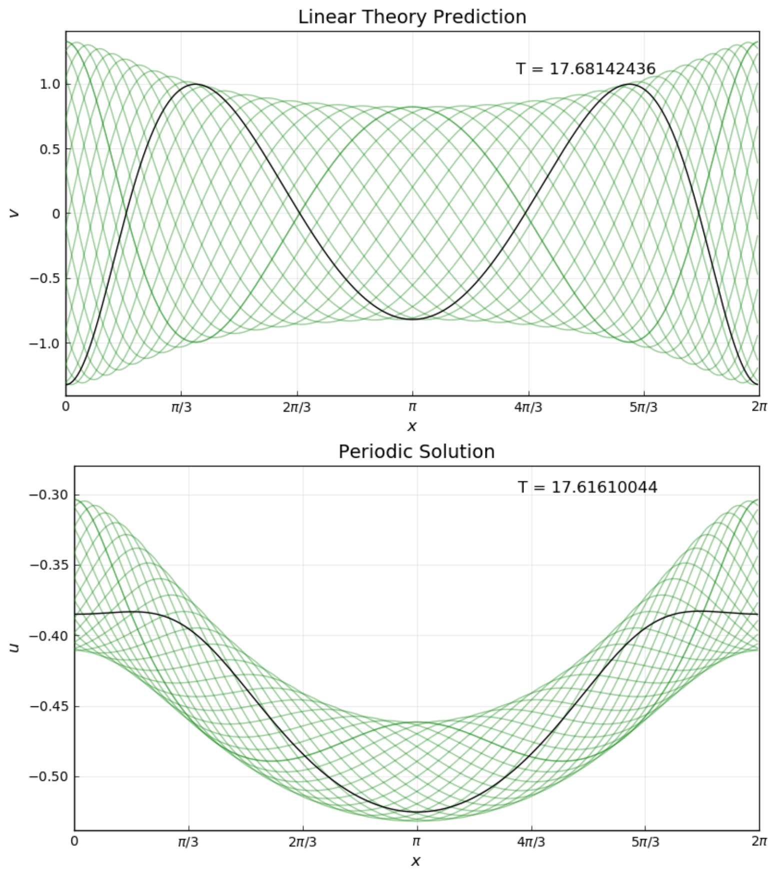

Figure 4: The eigenfunction associated with $\omega = 0.35535515$, used as a perturbation/correction in the search for time-periodic solutions which are not traveling-wave solutions. The solution to (23) is shown at 20 evenly spaced positions in time, and the black curve is meant to differentiate from the blue stationary/traveling-wave curve at $t = T/2$. Below it is the nontrivial time-periodic solution when $\hat{u}_0(\tilde{k}) = 0.02778142$.

ρ small we were able to find new solutions in a reasonable amount of time on a 6-core 3.8 GHz desktop. Depending on the complexity of the solution and the step of ρ, new nontrivial solutions could be computed as quickly as < 1 minutes and as long as $5 - 15$ minutes. One of the optimizations that we have not taken advantage of yet is the tracking of past Hessians. So, currently new nontrivial solutions are using the identity as the initial approximate Hessian, and must build up a more accurate approximate Hessian as BFGS iterates over new gradients. In communicating with a Julia community member, it was suggest to try altering this algorithm for a nonlinear deflation scheme similar to one found in the BifurcationKit.jl [19] by the package developer Veltz.

In [2] and [3], Ambrose and Wilkening found that bifurcations from stationary and traveling-wave solutions acted like "rungs on a ladder" to other stationary and traveling-wave solutions. Initial results indicate that something similar exists for the Whitham equation, but we would like to spend more time looking at our results before making a similar claim.

6 Conclusion

This has been our implementation of the method for finding nontrivial time-periodic solutions to the Whitham equation. We attempt to find traveling-wave solutions and solutions to the linearized problem with 85+ bits of precision with the DoubleFloats.jl package [10]. This means that numerical instabilities should only arise from the BFGS routine where we are currently only using double precision due to some particularities with the Julia implementation of the FFT algorithm.

There are some challenges that we are still presented with, 1) we are having some difficulties bifurcating from traveling-wave solutions, 2) we need to implement a routine to retain the approximate Hessian for future solutions, and 3) we would like to workout some of the particulars without having access to closed form solutions. There are many similarities between the Benjamin-Ono and Whitham equations, and we hope that past work with Benjamin-Ono can continue to point us in a fruitful direction. Unfortunately, because we have yet to implement code for challenge 2), bifurcations after a certain point become more and more computationally expensive, and will sometimes fail to converge if BFGS is unable to build an accurate Hessian.

7 Acknowledgements

This work would not have been possible without the contributions and motivations by Ambrose and Wilkening as well as the many authors they used as inspiration for their work. Members of the Julia programming community also helped guide us along with difficulties we encountered alongside the rapid development of the programming language.

This material is based upon work supported by the National Science Foundation under grant DMS-1716120 as well as sponsorship from the Seattle University College of Science and Engineering.

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