MATHEMATICAL MODELS FOR CONTROL OF AIRCRAFT AND SATELLITES

THOR I. FOSSEN

Professor of Guidance, Navigation and Control

Department of Engineering Cybernetics
Norwegian University of Science and Technology

January 2011
2nd edition

1st edition published February 1998.
2nd edition published January 2011.

Contents

Figures iii

1 Introduction 1

2 Aircraft Modeling 3
- 2.1 Definition of Aircraft State-Space Vectors . . . 3
- 2.2 Body-Fixed Coordinate Systems for Aircraft . . . 4
  - 2.2.1 Rotation matrices for wind and stability axes . . . 5
- 2.3 Aircraft Equations of Motion . . . 6
  - 2.3.1 Kinematic equations for translation . . . 6
  - 2.3.2 Kinematic equations for attitude . . . 7
  - 2.3.3 Rigid-body kinetics . . . 7
  - 2.3.4 Sensors and measurement systems . . . 8
- 2.4 Perturbation Theory (Linear Theory) . . . 8
  - 2.4.1 Definition of nominal and perturbation values . . . 9
  - 2.4.2 Linearization of the rigid-body kinetics . . . 9
  - 2.4.3 Linear state-space model based using wind and stability axes . . . 11
- 2.5 Decoupling in Longitudinal and Lateral Modes . . . 12
  - 2.5.1 Longitudinal equations . . . 13
  - 2.5.2 Lateral equations . . . 13
- 2.6 Aerodynamic Forces and Moments . . . 14
  - 2.6.1 Longitudinal aerodynamic forces and moments . . . 16
  - 2.6.2 Lateral aerodynamic forces and moments . . . 16
- 2.7 Standard Aircraft Maneuvers . . . 17
  - 2.7.1 Dynamic equation for coordinated turn (bank-to-turn) . . . 17
  - 2.7.2 Dynamic equation for altitude control . . . 18
- 2.8 Aircraft Stability Properties . . . 20
  - 2.8.1 Longitudinal stability analysis . . . 20
  - 2.8.2 Lateral stability analysis . . . 21
- 2.9 Design of flight control systems . . . 22

3 Satellite Modeling 23
- 3.1 Attitude Model . . . 23
  - 3.1.1 Euler's 2nd Axiom Applied to Satellites . . . 23

3.1.2 Skew-symmetric representation of the satellite model 23
3.2 Satellite Model Stability Properties 24
3.3 Design of Satellite Attitude Control Systems 24

4 Matlab Simulation Models 25
4.1 Boeing-767 25
4.1.1 Longitudinal model 25
4.1.2 Lateral model 26
4.2 F-16 Fighter 26
4.2.1 Longitudinal model 27
4.3 F2B Bristol Fighter 28
4.3.1 Lateral model 29

List of Figures

1.1 Sketch showing a modern fighter aircraft (Stevens and Lewis 1992). 1

2.1 Definition of aircraft body axes, velocities, forces, moments and Euler angles (McLean 1990). 4
2.2 Definition of stability and wind axes for an aircraft (Stevens and Lewis 1992). 5
2.3 Control inputs for conventional aircraft. Notice that the two ailerons can be controlled by using one control input: $\delta_A = 1/2(\delta_{A_L} + \delta_{A_R})$. 14
2.4 Aircraft longitudinal eigenvalue configuration plotted in the complex plane. . 21

4.1 Schematic drawing of the Bristol F.2B Fighter (McRuer et al 1973). 28

Chapter 1

Introduction

This note uses a *vectorial notation* to describe aircraft and satellites. The notation is similar to the one used for marine craft (ships, high-speed craft and underwater vehicles). The equations of motion are based on:

Fossen, T. I. (1994). *Guidance and Control of Ocean Vehicles* (John Wiley & Sons Ltd), Chapter 2.

Fossen, T. I. (2011). *Handbook of Marine Craft Hydrodynamics and Motion Control* (John Wiley & Sons Ltd.), Chapters 2 and 3.

The kinematic and kinetic equations of a marine craft can be modified to describe aircraft and satellites by minor adjustments of notation and assumptions.

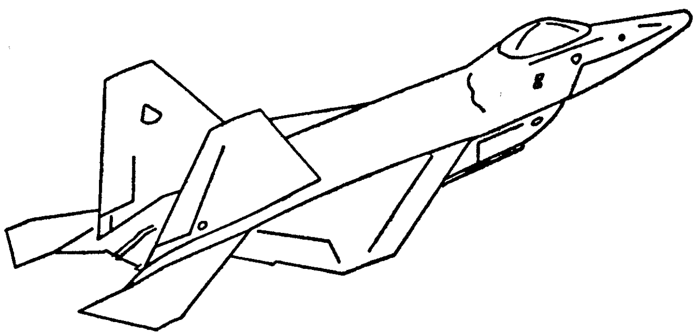

Figure 1.1: Sketch showing a modern fighter aircraft (Stevens and Lewis 1992).

The note is organized according to:

- **Chapter 2: Aircraft Modeling**
- **Chapter 3: Satellite Modeling**
- **Chapter 4: Matlab Simulation Models**

This note is in addition to the textbook *Handbook of Marine Craft Hydrodynamics and Motion Control* in the course *TTK4109 Guidance and Control* that is given at the Department of Engineering Cybernetics, NTNU.

Other useful references on flight control are:

Blakelock, J. H. (1991). *Aircraft and Missiles* (John Wiley & Sons Ltd.)

Etkin, B. and L. D. Reid (1996). Dynamics of Flight: Stability and Control (John Wiley & Sons Ltd.)

McLean, D. (1990). *Automatic Flight Control Systems* (Prentice Hall Inc.)

McRuer, D., D. Ashkenas and A. I. Graham (1973). *Aircraft Dynamics and Automatic Control* (Princeton University Press)

Nelson, R. C. (1998). Flight Stability and Automatic Control (McGraw-Hill int.)

Roskam, J. (1999). *Airplane Flight Dynamics and Automatic Flight Controls* (Darcorporation)

Stevens, B. L. and F. L. Lewis (1992). *Aircraft Control and Simulation* (John Wiley & Sons Ltd.)

Information about the author as well as the graduate courses *TTK4109 Guidance and Control* and *TK8109 Advanced Guidance and Control* are found on the web-pages:

Thor I. Fossen: http://www.itk.ntnu.no/ansatte/Fossen_Thor

TTK4109 Guidance and Control: http://www.itk.ntnu.no/emner/ttk4190

TK8109 Advanced Guidance and Control: http://www.itk.ntnu.no/emner/tk8109

Thor I. Fossen
Trondheim – 3 January 2011

Chapter 2

Aircraft Modeling

This chapter gives an introduction to aircraft modeling. The equations of motion are linearized using perturbation theory and the final results are state-space models for the longitudinal and lateral motions. The models can be used for aircraft simulation and design of flight control systems.

2.1 Definition of Aircraft State-Space Vectors

The aircraft velocity vector is defined according to (see Figure 2.1):

$$\boldsymbol{\nu} := \begin{bmatrix} U \\ V \\ W \\ P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \text{longitudinal (forward) velocity} \\ \text{lateral (transverse) velocity} \\ \text{vertical velocity} \\ \text{roll rate} \\ \text{pitch rate} \\ \text{yaw rate} \end{bmatrix} \tag{2.1}$$

$$\boldsymbol{\eta} := \begin{bmatrix} X_E \\ Y_E \\ Z_E, h \\ \Phi \\ \Theta \\ \Psi \end{bmatrix} = \begin{bmatrix} \text{Earth-fixed } x\text{-position} \\ \text{Earth-fixed } y\text{-position} \\ \text{Earth-fixed } z\text{-position (axis downwards), altitude} \\ \text{roll angle} \\ \text{pitch angle} \\ \text{yaw angle} \end{bmatrix} \tag{2.2}$$

Forces and moments are defined in a similar manner:

$$\begin{bmatrix} X \\ Y \\ Z \\ L \\ M \\ N \end{bmatrix} := \begin{bmatrix} \text{longitudinal force} \\ \text{transverse force} \\ \text{vertical force} \\ \text{roll moment} \\ \text{pitch moment} \\ \text{yaw moment} \end{bmatrix} \tag{2.3}$$

Comment 1: Notice that the capital letters L, M, N for the moments are different from

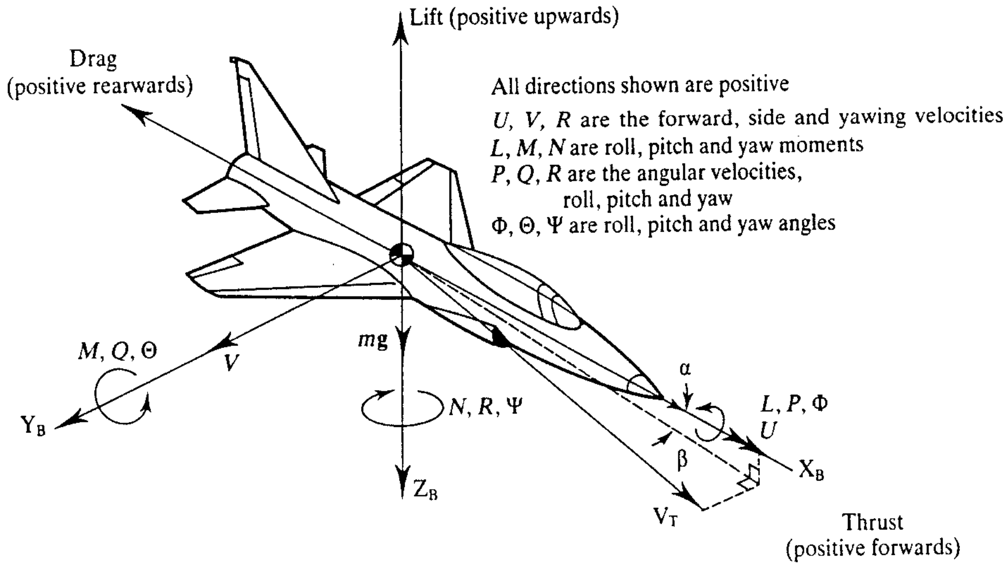

Figure 2.1: Definition of aircraft body axes, velocities, forces, moments and Euler angles (McLean 1990).

those used for marine craft–that is, K, M, N. The reason for this is that L is reserved as length parameter for ships and underwater vehicles.

Comment 2: For aircraft it is common to use capital letters for the states U, V, W, etc. while it is common to use small letters for marine craft.

2.2 Body-Fixed Coordinate Systems for Aircraft

For aircraft it is common to use the following body-fixed coordinate systems:

- Body axes
- Stability axes
- Wind axes

The axis systems are shown in Figure 2.2 where the *angle of attack* α and *sideslip angle* β are defined as:

$$\tan(\alpha) := \frac{W}{U} \tag{2.4}$$

$$\sin(\beta) := \frac{V}{V_T} \tag{2.5}$$

where

$$V_T = \sqrt{U^2 + V^2 + W^2} \tag{2.6}$$

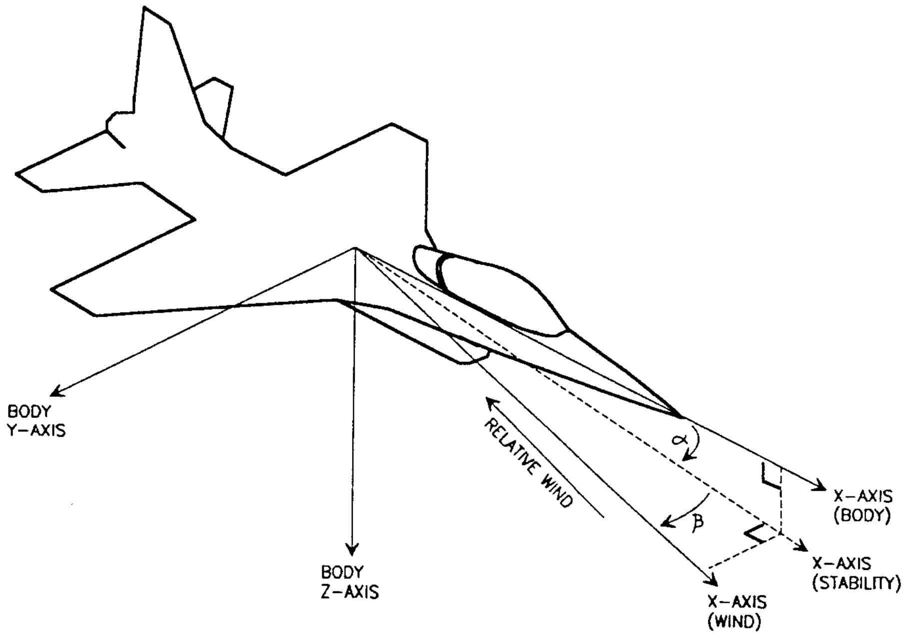

Figure 2.2: Definition of stability and wind axes for an aircraft (Stevens and Lewis 1992).

is the *total speed* of the aircraft. Aerodynamic effects are classified according to the *Mach number*:

$$M := \frac{V_T}{a} \tag{2.7}$$

where $a = 340$ m/s $= 1224$ km/h is the speed of sound in air at a temperature of 20^o C on the ocean surface. The following terminology is speed:

Subsonic speed	$M < 1.0$
Transonic speed	$0.8 \leq M \leq 1.2$
Supersonic speed	$1.0 \leq M \leq 5.0$
Hypersonic speed	$5.0 \leq M$

An aircraft will break the sound barrier at $M = 1.0$ and this is clearly heard as a sharp crack. If you fly at low altitude and break the sound barrier, windows in building will break due to pressure-induced waves.

2.2.1 Rotation matrices for wind and stability axes

The relationship between vectors expressed in different coordinate systems can be derived using rotation matrices. The body-fixed coordinate system is first rotated a negative sideslip angle $-\beta$ about the z-axis. The new coordinate system is then rotated a positive angle of attack α about the new y-axis such that the resulting x-axis points in the direction of the total speed V_T. The first rotation defines the *wind axes* while the second rotation defines the *stability axes*. This can be mathematically expressed as:

$$\begin{aligned}
\mathbf{p}^{\text{wind}} &= \mathbf{R}_{z,-\beta}\mathbf{p}^{\text{stab}} = \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}^{\text{stab}} && (2.8) \\
\mathbf{p}^{\text{stab}} &= \mathbf{R}_{y,\alpha}\mathbf{p}^{\text{body}} = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \mathbf{p}^{\text{body}} && (2.9)
\end{aligned}$$

The rotation matrix becomes:

$$\mathbf{R}^{\text{wind}}_{\text{body}} = \mathbf{R}_{z,-\beta}\mathbf{R}_{y,\alpha} \tag{2.10}$$

Hence,

$$\begin{aligned}
\mathbf{p}^{\text{wind}} &= \mathbf{R}^{\text{wind}}_{\text{body}}\mathbf{p}^{\text{body}} && (2.11) \\
&\Updownarrow \\
\mathbf{p}^{\text{wind}} &= \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \mathbf{p}^{\text{body}} && (2.12) \\
&\Updownarrow \\
\mathbf{p}^{\text{wind}} &= \begin{bmatrix} \cos(\alpha)\cos(\beta) & \sin(\beta) & \sin(\alpha)\cos(\beta) \\ -\cos(\alpha)\sin(\beta) & \cos(\beta) & -\sin(\alpha)\sin(\beta) \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \mathbf{p}^{\text{body}} && (2.13)
\end{aligned}$$

This gives the following relationship between the velocities in body and wind axes:

$$\mathbf{v}^{\text{body}} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = (\mathbf{R}^{\text{wind}}_{\text{body}})^{\top}\mathbf{v}^{\text{wind}} = \mathbf{R}^{\top}_{y,\alpha}\mathbf{R}^{\top}_{z,-\beta} \begin{bmatrix} V_T \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_T\cos(\alpha)\cos(\beta) \\ V_T\sin(\beta) \\ V_T\sin(\alpha)\cos(\beta) \end{bmatrix} \tag{2.14}$$

Consequently,

$$\begin{aligned} U &= V_T\cos(\alpha)\cos(\beta) \\ V &= V_T\sin(\beta) \\ W &= V_T\sin(\alpha)\cos(\beta) \end{aligned} \tag{2.15}$$

2.3 Aircraft Equations of Motion

2.3.1 Kinematic equations for translation

The kinematic equations for translation and rotation of a body-fixed coordinate system ABC with respect to a local geographic coordinate system NED (*North-East-Down*) can be expressed in terms or the Euler angles:

$$\begin{bmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{bmatrix} = \mathbf{R}^{\text{ned}}_{\text{abc}} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{R}_{z,\Psi}\mathbf{R}_{y,\Theta}\mathbf{R}_{x,\Phi} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \tag{2.16}$$

Expanding this expression gives:

$$\begin{bmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{bmatrix} = \begin{bmatrix} c\Psi & -s\Psi & 0 \\ s\Psi & c\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\Theta & 0 & s\Theta \\ 0 & 1 & 0 \\ -s\Theta & 0 & c\Theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\Phi & -s\Phi \\ 0 & s\Phi & c\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$
$$\Updownarrow \tag{2.17}$$
$$\begin{bmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{bmatrix} = \begin{bmatrix} c\Psi c\Theta & -s\Psi c\Phi + c\Psi s\Theta s\Phi & s\Psi s\Phi + c\Psi c\Phi s\Theta \\ s\Psi c\Theta & c\Psi c\Phi + s\Phi s\Theta s\Psi & -c\Psi s\Phi + s\Theta s\Psi c\Phi \\ -s\Theta & c\Theta s\Phi & c\Theta c\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

2.3.2 Kinematic equations for attitude

The attitude is given by:

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_{x,\Phi}^{\top} \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} + \mathbf{R}_{x,\Phi}^{\top}\mathbf{R}_{y,\Theta}^{\top} \begin{bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{bmatrix} \tag{2.18}$$

which gives:

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & s\Phi t\Theta & c\Phi t\Theta \\ 0 & c\Phi & -s\Phi \\ 0 & s\Phi/c\Theta & c\Phi/c\Theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \qquad c\Theta \neq 0 \tag{2.19}$$

2.3.3 Rigid-body kinetics

The aircraft rigid-body kinetics can be expressed as (Fossen 1994, 2011):

$$m(\dot{\boldsymbol{\nu}}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) = \boldsymbol{\tau}_1 \tag{2.20}$$
$$\mathbf{I}_{\mathrm{CG}}\dot{\boldsymbol{\nu}}_2 + \boldsymbol{\nu}_2 \times (\mathbf{I}_{\mathrm{CG}}\boldsymbol{\nu}_2) = \boldsymbol{\tau}_2 \tag{2.21}$$

where $\boldsymbol{\nu}_1 := [U, V, W]^T, \boldsymbol{\nu}_2 := [P, Q, R]^T, \boldsymbol{\tau}_1 := [X, Y, Z]^T$ and $\boldsymbol{\tau}_2 := [L, M, N]^T$. It is assumed that the coordinate system is located in the aircraft center of gravity (CG). The resulting model is written:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{2.22}$$

where

$$\mathbf{M}_{RB} = \begin{bmatrix} m\mathbf{I}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_{\mathrm{CG}} \end{bmatrix}, \quad \mathbf{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\mathbf{S}(\boldsymbol{\nu}_2) & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & -\mathbf{S}(\mathbf{I}_{\mathrm{CG}}\boldsymbol{\nu}_2) \end{bmatrix} \tag{2.23}$$

The inertia tensor is defined as (assume that $I_{xy} = I_{yz} = 0$ which corresponds to $xz-$plane symmetry):

$$\mathbf{I}_{\mathrm{CG}} := \begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix} \tag{2.24}$$

The forces and moments acting on the aircraft can be expressed as:

$$\boldsymbol{\tau}_{RB} = -\mathbf{g}(\boldsymbol{\eta}) + \boldsymbol{\tau} \tag{2.25}$$

where $\boldsymbol{\tau}$ is a generalized vector that includes aerodynamic and control forces. The gravitational force $\mathbf{f}_G = [0\ 0\ mg]^T$ acts in the CG (origin of the body-fixed coordinate system) and this gives the following vector expressed in NED:

$$\mathbf{g}(\boldsymbol{\eta}) = -(\mathbf{R}_{\mathrm{abc}}^{\mathrm{ned}})^{\top} \begin{bmatrix} \mathbf{f}_G \\ \mathbf{O}_{3\times 1} \end{bmatrix} = \begin{bmatrix} mg\sin(\Theta) \\ -mg\cos(\Theta)\sin(\Phi) \\ -mg\cos(\Theta)\cos(\Phi) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.26}$$

Hence, the aircraft model can be written in matrix form as:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) = \boldsymbol{\tau} \tag{2.27}$$

or in component form:

$$\begin{aligned} m(\dot{U} + QW - RV + g\sin(\Theta)) &= X \\ m(\dot{V} + UR - WP - g\cos(\Theta)\sin(\Phi)) &= Y \\ m(\dot{W} + VP - QU - g\cos(\Theta)\cos(\Phi)) &= Z \\ I_x\dot{P} - I_{xz}(\dot{R} + PQ) + (I_z - I_y)QR &= L \\ I_y\dot{Q} + I_{xz}(P^2 - R^2) + (I_x - I_z)PR &= M \\ I_z\dot{R} - I_{xz}\dot{P} + (I_y - I_x)PQ + I_{xz}QR &= N \end{aligned} \tag{2.28}$$

2.3.4 Sensors and measurement systems

It is common that aircraft sensor systems are equipped with three accelerometers. If the accelerometers are located in the CG, the measurement equations take the following form:

$$a_{x_{CG}} = \frac{X}{m} = \dot{U} + QW - RV + g\sin(\Theta) \tag{2.29}$$

$$a_{y_{CG}} = \frac{Y}{m} = \dot{V} + UR - WP - g\cos(\Theta)\sin(\Phi) \tag{2.30}$$

$$a_{z_{CG}} = \frac{Z}{m} = \dot{W} + VP - QU - g\cos(\Theta)\cos(\Phi) \tag{2.31}$$

In addition to these sensors, an aircraft is equipped with gyros, magnetometers and a sensor for altitude h and wind speed V_T. These sensors are used in inertial navigation systems (INS) which again use a Kalman filter to compute estimates of U, V, W, P, Q and R as well as the Euler angles Φ,Θ and Ψ. Other measurement systems that are used onboard aircraft are global navigation satellite systems (GNSS), radar and sensors for angle of attack.

2.4 Perturbation Theory (Linear Theory)

The nonlinear equations of motion can be linearized by using perturbation theory. This is illustrated below.

2.4.1 Definition of nominal and perturbation values

According to linear theory it is possible to write the states as the sum of a *nominal value* (usually constant) and a *perturbation* (deviation from the nominal value). Moreover,

$$\text{Total state} \;=\; \text{Nominal value} \;+\; \text{Perturbation}$$

The following definitions are made:

$$\boldsymbol{\tau} := \boldsymbol{\tau}_0 + \delta\boldsymbol{\tau} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ L_0 \\ M_0 \\ N_0 \end{bmatrix} + \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \\ \delta L \\ \delta M \\ \delta N \end{bmatrix}, \quad \boldsymbol{\nu} := \boldsymbol{\nu}_0 + \delta\boldsymbol{\nu} = \begin{bmatrix} U_0 \\ V_0 \\ W_0 \\ P_0 \\ Q_0 \\ R_0 \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \tag{2.32}$$

Similar, the angles are defined according to:

$$\begin{bmatrix} \Theta \\ \Phi \\ \Psi \end{bmatrix} := \begin{bmatrix} \Theta_0 \\ \Phi_0 \\ \Psi_0 \end{bmatrix} + \begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix} \tag{2.33}$$

Consequently, a linearized state-space model will consist of the following states $u, v, w, p, q, r, \theta, \phi$ and ψ.

2.4.2 Linearization of the rigid-body kinetics

The rigid-body kinetics can be linearized by using perturbation theory.

Equilibrium condition

If the aerodynamic forces and moments, velocities, angles and control inputs are expressed as nominal values and perturbations $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \delta\boldsymbol{\tau}$, $\boldsymbol{\nu} = \boldsymbol{\nu}_0 + \delta\boldsymbol{\nu}$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \delta\boldsymbol{\eta}$, the *aircraft equilibrium point* will satisfy (it is assumed that $\dot{\boldsymbol{\nu}}_0 = \mathbf{0}$):

$$\mathbf{C}_{RB}(\boldsymbol{\nu}_0)\boldsymbol{\nu}_0 + \mathbf{g}(\boldsymbol{\eta}_0) = \boldsymbol{\tau}_0 \tag{2.34}$$

This can be expanded according to:

$$\begin{aligned} m(Q_0W_0 - R_0V_0 + g\sin(\Theta_0)) &= X_0 \\ m(U_0R_0 - P_0W_0 - g\cos(\Theta_0)\sin(\Phi_0)) &= Y_0 \\ m(P_0V_0 - Q_0U_0 - g\cos(\Theta_0)\cos(\Phi_0)) &= Z_0 \\ (I_z - I_y)Q_0R_0 - P_0Q_0I_{xz} &= L_0 \\ (P_0^2 - R_0^2)I_{xz} + (I_x - I_z)P_0R_0 &= M_0 \\ (I_y - I_x)P_0Q_0 + Q_0R_0I_{xz} &= N_0 \end{aligned} \tag{2.35}$$

Perturbed equations

The perturbed equations–that is, the linearized equations of motion are usually derived by a 1st-order Taylor series expansion about the nominal values. Alternatively, it is possible to substitute (2.32) and (2.35) into (2.27) and neglect higher-order terms of the perturbed states. This is illustrated for the first degree of freedom (DOF):

Example 1 (Linearization of surge using perturbation theory)

$$\begin{aligned} m[\dot{U} + QW - RV + g\sin(\Theta)] &= X \\ &\Downarrow \\ m[\dot{U}_0 + \dot{u} + (Q_0+q)(W_0+w) - (R_0+r)(V_0+v) + g\sin(\Theta_0+\theta)] &= X_0 + \delta X \end{aligned} \tag{2.36}$$

This can be written:

$$\sin(\Theta_0+\theta) = \sin(\Theta_0)\cos(\theta) + \cos(\Theta_0)\sin(\theta) \overset{\theta \text{ small}}{\approx} \sin(\Theta_0) + \cos(\Theta_0)\theta \tag{2.37}$$

Since $\dot{U}_0 = 0$ *and*

$$m(Q_0W_0 - R_0V_0 + g\sin(\Theta_0)) = X_0 \tag{2.38}$$

Equation (2.36) is reduced to:

$$m[\dot{u} + Q_0w + W_0q + wq - R_0v - V_0r - vr + g\cos(\Theta_0)\theta] = \delta X \tag{2.39}$$

If it is assumed that the 2nd-order terms wq *and* vr *are negligible, the linearized model becomes:*

$$m[\dot{u} + Q_0w + W_0q - R_0v - V_0r + g\cos(\Theta_0)\theta] = \delta X \tag{2.40}$$

□

Linear state-space model for aircraft

If all DOFs are linearized, the following state-space model is obtained:

$$\begin{aligned} m[\dot{u} + Q_0w + W_0q - R_0v - V_0r + g\cos(\Theta_0)\theta] &= \delta X \\ m[\dot{v} + U_0r + R_0u - W_0p - P_0w - g\cos(\Theta_0)\cos(\Phi_0)\phi + g\sin(\Theta_0)\sin(\Phi_0)\theta] &= \delta Y \\ m[\dot{w} + V_0p + P_0v - U_0q - Q_0u + g\cos(\Theta_0)\sin(\Phi_0)\phi + g\sin(\Theta_0)\cos(\Phi_0)\theta] &= \delta Z \\ I_x\dot{p} - I_{xz}\dot{r} + (I_z - I_y)(Q_0r + R_0q) - I_{xz}(P_0q + Q_0p) &= \delta L \\ I_y\dot{q} + (I_x - I_z)(P_0r + R_0p) - 2I_{xz}(R_0r + P_0p) &= \delta M \\ I_z\dot{r} - I_{xz}\dot{p} + (I_y - I_x)(P_0q + Q_0p) + I_{xz}(Q_0r + R_0q) &= \delta N \end{aligned} \tag{2.41}$$

This can be expressed in matrix form as:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{N}_{RB}\boldsymbol{\nu} + \mathbf{G}\boldsymbol{\eta} = \boldsymbol{\tau} \tag{2.42}$$

where

$$
\mathbf{M}_{RB} = \left[\begin{array}{ccc|ccc} m & 0 & 0 & & & \\ & m & 0 & & \mathbf{0}_{3\times3} & \\ 0 & 0 & m & & & \\ & & & I_x & 0 & -I_{xz} \\ & \mathbf{0}_{3\times3} & & 0 & I_y & 0 \\ & & & -I_{xz} & 0 & I_z \end{array}\right]
$$

$$
\mathbf{N}_{RB} = \left[\begin{array}{cccccc} 0 & -mR_0 & mQ_0 & 0 & mW_0 & -mV_0 \\ mR_0 & 0 & -P_0 & -W_0 & 0 & U_0 \\ -mQ_0 & mP_0 & 0 & mV_0 & -mU_0 & 0 \\ & & & -I_{xz}Q_0 & (I_z-I_y)R_0 - I_{xz}P_0 & (I_z-I_y)Q_0 \\ & \mathbf{0}_{3\times3} & & (I_x-I_z)R_0 & -2I_{xz}P_0 & (I_x-I_z)P_0 - 2I_{xz}R_0 \\ & & & (I_y-I_x)Q_0 & (I_y-I_x)P_0 + I_{xz}R_0 & I_{xz}Q_0 \end{array}\right]
$$

$$
\mathbf{G} = \left[\begin{array}{cccc} & 0 & mg\cos(\Theta_0) & 0 \\ \mathbf{0}_{3\times3} & -mg\cos(\Theta_0)\cos(\Phi_0) & mg\sin(\Theta_0)\sin(\Phi_0) & 0 \\ & mg\cos(\Theta_0)\sin(\Phi_0) & mg\sin(\Theta_0)\cos(\Phi_0) & 0 \\ & & & \\ \mathbf{0}_{3\times3} & & \mathbf{0}_{3\times3} & \\ & & & \end{array}\right]
$$

In addition to this, the kinematic equations must be linearized.

2.4.3 Linear state-space model based using wind and stability axes

An alternative state-space model is obtained by using α and β as states. If it is assumed that α and β are small such that $\cos(\alpha) \approx 1$ and $\sin(\beta) \approx \beta$, Equation (2.15) can be written as:

$$
\begin{array}{lcl} U = V_T & & U = V_T \\ V = V_T\beta & \Rightarrow & \beta = \frac{V}{V_T} \\ W = V_T\alpha & & \alpha = \frac{W}{V_T} \end{array} \tag{2.43}
$$

Furthermore, the state-space vector:

$$
\mathbf{x} = \begin{bmatrix} u \\ \beta \\ \alpha \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \text{surge velocity} \\ \text{sideslip angle} \\ \text{angle of attack} \\ \text{roll rate} \\ \text{pitch rate} \\ \text{yaw rate} \end{bmatrix} \tag{2.44}
$$

is chosen to describe motions in 6 DOF. The relationship between the body-fixed velocity vector:

$$
\boldsymbol{\nu} = [u, v, w, p, q, r]^T \tag{2.45}
$$

and the new state-space vector $\mathbf{x}$ can be written as:

$$
\boldsymbol{\nu} = \mathbf{T}\mathbf{x} = \text{diag}\{1, V_T, V_T, 1, 1, 1, 1\}\mathbf{x} \tag{2.46}
$$

where $V_T > 0$. If the total speed is $V_T = U_0 =$ constant (linear theory), it is seen that:

$$\dot{\alpha} = \frac{1}{V_T}\dot{w} \tag{2.47}$$

$$\dot{\beta} = \frac{1}{V_T}\dot{v} \tag{2.48}$$

$$\dot{V}_T = 0 \tag{2.49}$$

If nonlinear theory is applied, the following differential equations are obtained:

$$\dot{\alpha} = \frac{U\dot{W} - W\dot{U}}{U^2 + W^2} \tag{2.50}$$

$$\dot{\beta} = \frac{\dot{V}V_T - V\dot{V}_T}{V_T^2 \cos\beta} \tag{2.51}$$

$$\dot{V}_T = \frac{U\dot{U} + V\dot{V} + W\dot{W}}{V_T} \tag{2.52}$$

In the linear case it is possible to transform the body-fixed state-space model:

$$\dot{\boldsymbol{\nu}} = \mathbf{F}\boldsymbol{\nu} + \mathbf{G}\mathbf{u} \tag{2.53}$$

to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.54}$$

where

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T}, \qquad \mathbf{B} = \mathbf{T}^{-1}\mathbf{G} \tag{2.55}$$

For $\dot{V}_T \neq 0$ this transformation is much more complicated. The linear state-space transformation is commonly used by aircraft manufactures. An example is the Boeing B-767 model (see Chapter 4).

2.5 Decoupling in Longitudinal and Lateral Modes

For an aircraft it is common to assume that the longitudinal modes (DOFs 1, 3 and 5) are decoupled from the lateral modes (DOFs 2, 4 and 6). The key assumption is that the fuselage is slender–that is, the length is much larger than the width and the height of the aircraft. It is also assumed that the the longitudinal velocity is much larger than the vertical and transversal velocities.

In order to decouple the rigid-body kinetics (2.41) in longitudinal and lateral modes it will be assumed that the states v, p, r and ϕ are negligible in the longitudinal channel while u, w, q and θ are negligible when considering the lateral channel. This gives two sub-systems:

2.5.1 Longitudinal equations

Kinetics:

$$
\begin{aligned}
m[\dot{u} + Q_0 w + W_0 q + g\cos(\Theta_0)\theta] &= \delta X \\
m[\dot{w} - U_0 q - Q_0 u + g\sin(\Theta_0)\cos(\Phi_0)\theta] &= \delta Z \\
I_y \dot{q} &= \delta M
\end{aligned}
\tag{2.56}
$$

$$
\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_y \end{bmatrix}
\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \end{bmatrix}
+
\begin{bmatrix} 0 & mQ_0 & mW_0 \\ -mQ_0 & 0 & -mU_0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} u \\ w \\ q \end{bmatrix}
+
\begin{bmatrix} mg\cos(\Theta_0) \\ mg\sin(\Theta_0)\cos(\Phi_0) \\ 0 \end{bmatrix}
\theta =
\begin{bmatrix} \delta X \\ \delta Z \\ \delta M \end{bmatrix}
\tag{2.57}
$$

Kinematics:

$$
\dot{\theta} = q \tag{2.58}
$$

2.5.2 Lateral equations

Kinetics:

$$
\begin{aligned}
m[\dot{v} + U_0 r - W_0 p - g\cos(\Theta_0)\cos(\Phi_0)\phi] &= \delta Y \\
I_x \dot{p} - I_{xz}\dot{r} + (I_z - I_y)Q_0 r - I_{xz}Q_0 p &= \delta L \\
I_z \dot{r} - I_{xz}\dot{p} + (I_y - I_x)Q_0 p + I_{xz}Q_0 r &= \delta N
\end{aligned}
\tag{2.59}
$$

$$
\begin{bmatrix} m & 0 & 0 \\ 0 & I_x & -I_{xz} \\ 0 & -I_{xz} & I_z \end{bmatrix}
\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix}
+
\begin{bmatrix} 0 & -mW_0 & mU_0 \\ 0 & -I_{xz}Q_0 & (I_z - I_y)Q_0 \\ 0 & (I_y - I_x)Q_0 & I_{xz}Q_0 \end{bmatrix}
\begin{bmatrix} v \\ p \\ r \end{bmatrix}
+
\begin{bmatrix} -mg\cos(\Theta_0)\cos(\Phi_0) \\ 0 \\ 0 \end{bmatrix}
\phi =
\begin{bmatrix} \delta Y \\ \delta L \\ \delta N \end{bmatrix}
\tag{2.60}
$$

Kinematics:

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix}
=
\begin{bmatrix} 1 & \tan(\Theta_0) \\ 0 & 1/\cos(\Theta_0) \end{bmatrix}
\begin{bmatrix} p \\ r \end{bmatrix}
\tag{2.61}
$$

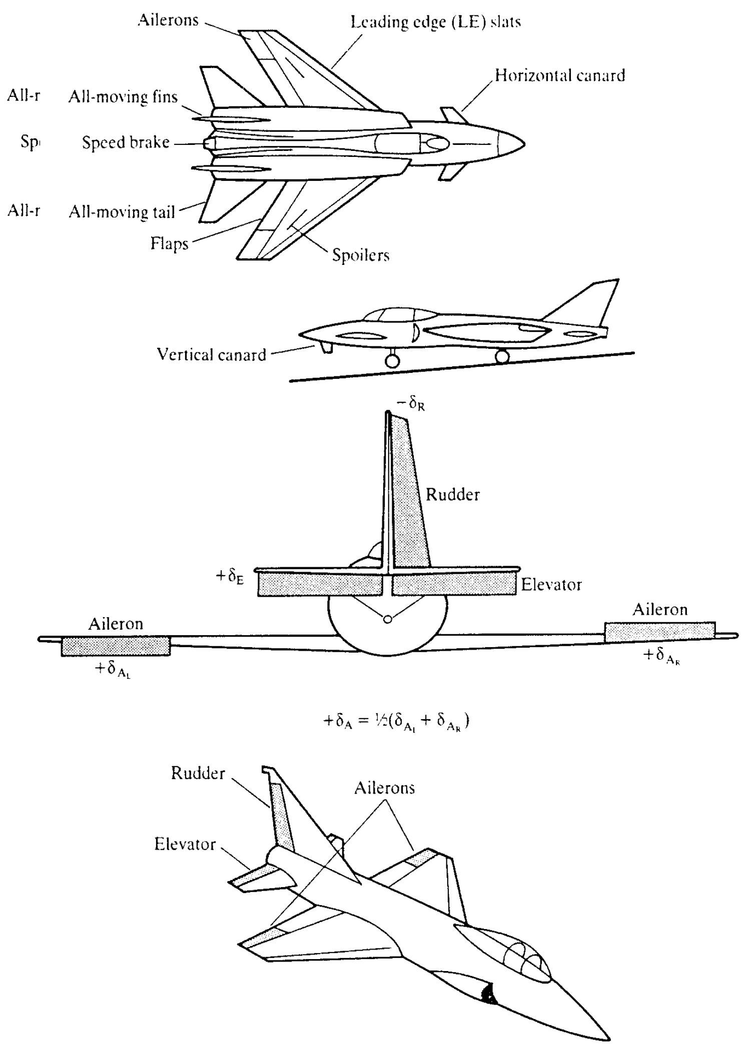

Figure 2.3: Control inputs for conventional aircraft. Notice that the two ailerons can be controlled by using one control input: $\delta_A = 1/2(\delta_{A_L} + \delta_{A_R})$.

2.6 Aerodynamic Forces and Moments

In the forthcoming sections, the following abbreviations and notation will be used to describe the aerodynamic coefficients:

$$\begin{aligned} X_{\text{index}} &= \frac{\partial X}{\partial \text{ index}} & L_{\text{index}} &= \frac{\partial L}{\partial \text{ index}} \\ Y_{\text{index}} &= \frac{\partial Y}{\partial \text{ index}} & M_{\text{index}} &= \frac{\partial M}{\partial \text{ index}} \\ Z_{\text{index}} &= \frac{\partial Z}{\partial \text{ index}} & N_{\text{index}} &= \frac{\partial N}{\partial \text{ index}} \end{aligned}$$

In order to illustrate how control surfaces influence the aircraft, an aircraft equipped with the following control inputs will be considered (se Figure 2.3):

δ_T	Thrust	Jet/propeller
δ_E	Elevator	Control surfaces on the rear of the aircraft used for pitch and altitude control
δ_A	Aileron	Hinged control surfaces attached to the trailing edge of the wing used for roll/bank control
δ_F	Flaps	Hinged surfaces on the trailing edge of the wings used for braking and bank-to-turn
δ_R	Rudder	Vertical control surface at the rear of the aircraft used for turning

Linear theory will be assumed in order to reduce the number of aerodynamic coefficients. Control inputs and aerodynamic forces and moments are written as:

$$\boldsymbol{\tau} = -\mathbf{M}_F\dot{\boldsymbol{\nu}} - \mathbf{N}_F\boldsymbol{\nu} + \mathbf{B}\mathbf{u} \tag{2.62}$$

where $-\mathbf{M}_F$ is aerodynamic added mass, $-\mathbf{N}_F$ ia aerodynamic damping and $\mathbf{B}$ is a matrix describing the actuator configuration including the force coefficients. The actuator dynamics is modeled by a 1st-order system:

$$\dot{\mathbf{u}} = \mathbf{T}^{-1}(\mathbf{u}_c - \mathbf{u}) \tag{2.63}$$

where $\mathbf{u}_c$ is commanded input, $\mathbf{u}$ is the actual control input produced by the actuators and $\mathbf{T} = \text{diag}\{T_1, T_2, ..., T_r\}$ is a diagonal matrix of positive time constants. Substitution of (2.62) into the model (2.42) gives:

$$\begin{aligned}(\mathbf{M}_{RB} + \mathbf{M}_F)\dot{\boldsymbol{\nu}} + (\mathbf{N}_{RB} + \mathbf{N}_F)\boldsymbol{\nu} + \mathbf{G}\boldsymbol{\eta} &= \mathbf{B}\mathbf{u} \\ &\Updownarrow \\ \mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{N}\boldsymbol{\nu} + \mathbf{G}\boldsymbol{\eta} &= \mathbf{B}\mathbf{u}\end{aligned} \tag{2.64}$$

The matrices $\mathbf{M}$ and $\mathbf{N}$ are defined as $\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_F$ and $\mathbf{N} = \mathbf{N}_{RB} + \mathbf{N}_F$. The linearized kinematics takes the following form:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_\nu\boldsymbol{\nu} + \mathbf{J}_\eta\boldsymbol{\eta} \tag{2.65}$$

The resulting state-space models are:

Linear state-space model with actuator dynamics

$$\begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\nu}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_\eta & \mathbf{J}_\nu & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{G} & -\mathbf{M}^{-1}\mathbf{N} & \mathbf{M}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{0} & -\mathbf{T}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\nu} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{T}^{-1} \end{bmatrix} \mathbf{u}_c \tag{2.66}$$

Linear state-space model neglecting the actuator dynamics

$$\begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\nu}} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_\eta & \mathbf{J}_\nu \\ -\mathbf{M}^{-1}\mathbf{G} & -\mathbf{M}^{-1}\mathbf{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\nu} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \mathbf{u} \tag{2.67}$$

2.6.1 Longitudinal aerodynamic forces and moments

McLean [7] expresses the longitudinal forces and moments as:

$$\begin{bmatrix} dX \\ dZ \\ dM \end{bmatrix} = \begin{bmatrix} X_{\dot{u}} & X_{\dot{w}} & X_{\dot{q}} \\ Z_{\dot{u}} & Z_{\dot{w}} & Z_{\dot{q}} \\ M_{\dot{u}} & M_{\dot{w}} & M_{\dot{q}} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_w & X_q \\ Z_u & Z_w & Z_q \\ M_q & M_w & M_q \end{bmatrix} \begin{bmatrix} u \\ w \\ q \end{bmatrix} + \begin{bmatrix} X_{\delta_T} & X_{\delta_E} & X_{\delta_F} \\ Z_{\delta_T} & Z_{\delta_E} & Z_{\delta_F} \\ M_{\delta_T} & M_{\delta_E} & M_{\delta_F} \end{bmatrix} \begin{bmatrix} \delta_T \\ \delta_E \\ \delta_F \end{bmatrix} \quad (2.68)$$

which corresponds to the matrices $-\mathbf{M}_F, -\mathbf{N}_F$ and $\mathbf{B}$ in (2.62). If the aircraft cruise speed $U_0 =$ constant, then $\delta_T = 0$. Altitude can be controlled by using the elevators δ_E. Flaps δ_F can be used to reduce the speed during landing. The flaps can also be used to turn harder for instance by moving one flap while the other is kept at the zero position. This is common in bank-to-turn maneuvers. For conventional aircraft the following aerodynamic coefficients can be neglected:

$$X_{\dot{u}}, X_q, X_{\dot{w}}, X_{\delta_E}, Z_{\dot{u}}, Z_{\dot{w}}, M_{\dot{u}} \quad (2.69)$$

Hence, the model for altitude control reduces to:

$$\begin{bmatrix} dX \\ dZ \\ dM \end{bmatrix} = \begin{bmatrix} 0 & 0 & X_{\dot{q}} \\ 0 & 0 & Z_{\dot{q}} \\ 0 & M_{\dot{w}} & M_{\dot{q}} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_w & 0 \\ Z_u & Z_w & Z_q \\ M_q & M_w & M_q \end{bmatrix} \begin{bmatrix} u \\ w \\ q \end{bmatrix} + \begin{bmatrix} X_{\delta_E} \\ Z_{\delta_E} \\ M_{\delta_E} \end{bmatrix} \delta_E \quad (2.70)$$

If the actuator dynamics is important, aerodynamic coefficients such as $X_{\dot{\delta}_T}, X_{\dot{\delta}_E}, ...$ must be included in the model.

2.6.2 Lateral aerodynamic forces and moments

The lateral model takes the form [7]:

$$\begin{bmatrix} dY \\ dL \\ dN \end{bmatrix} = \begin{bmatrix} Y_{\dot{v}} & Y_{\dot{p}} & Y_{\dot{r}} \\ L_{\dot{v}} & L_{\dot{p}} & L_{\dot{r}} \\ N_{\dot{v}} & N_{\dot{p}} & N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} Y_v & Y_p & Y_r \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_{\delta_A} & Y_{\delta_R} \\ L_{\delta_A} & L_{\delta_R} \\ N_{\delta_A} & N_{\delta_R} \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \quad (2.71)$$

which corresponds to the matrices $-\mathbf{M}_F, -\mathbf{N}_F$ and $\mathbf{B}$ in (2.62). For conventional aircraft the following aerodynamic coefficients can be neglected:

$$Y_{\dot{v}}, Y_p, Y_{\dot{p}}, Y_r, Y_{\dot{r}}, Y_{\delta_A}, L_{\dot{v}}, L_{\dot{r}}, N_{\dot{v}}, N_{\dot{r}} \quad (2.72)$$

This gives:

$$\begin{bmatrix} dY \\ dL \\ dN \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L_{\dot{p}} & 0 \\ 0 & N_{\dot{p}} & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} Y_v & 0 & 0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_R} \\ L_{\delta_A} & L_{\delta_R} \\ N_{\delta_A} & N_{\delta_R} \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \quad (2.73)$$

2.7 Standard Aircraft Maneuvers

The nominal values depends on the aircraft maneuver. For instance:

1. **Straight flight:** $\Theta_0 = Q_0 = R_0 = 0$
2. **Symmetric flight:** $\Psi_0 = V_0 = 0$
3. **Flying with wings level:** $\Phi_0 = P_0 = 0$

Constant angular rate maneuvers can be classified according to:

1. **Steady turn**: $R_0 =$ constant
2. **Steady pitching flight:** $Q_0 =$ constant
3. **Steady rolling/spinning flight:** $P_0 =$ constant

2.7.1 Dynamic equation for coordinated turn (bank-to-turn)

A frequently used maneuver is *coordinated turn* where the acceleration in the y-direction is zero ($\dot{\beta} = 0$), sideslip $\beta = 0$ and zero steady-state pitch and roll angles–that is,

$$\begin{aligned} \beta &= \dot{\beta} = 0 && (2.74) \\ \Theta_0 &= \Phi_0 = 0 && (2.75) \end{aligned}$$

Furthermore it is assumed that $V_T = U_0 =$ constant. Since $\beta = \dot{\beta} = 0$ and $\beta = V/U_0$ it follows that $V = \dot{V} = 0$. This implies that the external forces $\delta Y = 0$. From (2.28) it is seen that:

$$\begin{aligned} m[\dot{V} + UR - WP - g\cos(\Theta)\sin(\Phi)] &= Y && (2.76) \\ &\Downarrow \\ m[(U_0 + u)(R_0 + r) - (W_0 + w)(P_0 + p) - g\cos(\theta)\sin(\phi)] &= Y_0 && (2.77) \end{aligned}$$

Assume that the longitudinal and lateral motions are decoupled–that is, $u = w = q = \theta = 0$. If perturbation theory is applied under the assumption that the 2nd-order terms $ur = pw = 0$,we get:

$$m(U_0R_0 + U_0r - W_0P_0 - W_0p - g\sin(\phi)) = Y_0 \tag{2.78}$$

The equilibrium equation (2.35) gives the steady-state condition:

$$m(U_0R_0 - W_0P_0) = Y_0 \tag{2.79}$$

Substitution of (2.79) into (2.78) gives:

$$m(U_0r - W_0p - g\sin(\phi)) = 0 \tag{2.80}$$

or

$$r = \frac{W_0}{U_0}p + \frac{g}{U_0}\sin(\phi) \tag{2.81}$$

The aircraft is often trimmed such that the angle of attack $\alpha_0 = W_0/U_0 = 0$. This implies that the yaw rate can be expressed as:

$$r = \frac{g}{U_0}\sin(\phi) \overset{\phi\text{ small}}{\approx} \frac{g}{U_0}\phi \tag{2.82}$$

which is a very important result since it states that a roll angle angle ϕ different from zero will induce a yaw rate r which again turns the aircraft (*bank-to-turn*). With other words, we can use a moment in roll, for instance generated by the ailerons, to turn the aircraft. The yaw angle is given by:

$$\dot{\psi} = r \tag{2.83}$$

An alternative method is of course to turn the aircraft by using the rear rudder to generate a yaw moment. The *bank-to-turn* principle is used in many missile control systems since it improves maneuverability, in particular in combination with a rudder controlled system.

Example 2 (Augmented turning model using rear rudders)
Turning autopilots using the rudder δ_R as control input is based on the lateral state-space model. The differential equation for ψ is augmented on the lateral model as shown below:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ 0 \\ 0 \end{bmatrix} \delta_R \tag{2.84}$$

□

Example 3 (Augmented bank-to-turn model using ailerons)
Bank-to-turn autopilots are designed using the lateral state-space model with ailerons as control inputs, for instance $\delta_A = 1/2(\delta_{A_L} + \delta_{A_R})$. This is done by augmenting the bank-to-turn equation (2.82) to the state-space model according to:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g}{U_0} & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ 0 \\ 0 \end{bmatrix} \delta_A \tag{2.85}$$

□

2.7.2 Dynamic equation for altitude control

Aircraft altitude control systems are designed by considering the equation for the the vertical acceleration in the center of gravity expressed in NED coordinates; see (2.28). This gives:

$$a_{z_{CG}} = \dot{W} + VP - QU - g\cos(\Theta)\cos(\Phi) \tag{2.86}$$

If the acceleration is perturbed according to $a_{z_{CG}} = a_{z0} + \delta a_z$ and we assume that $\dot{W}_0 = 0$, the following equilibrium condition is obtained:

$$a_{z0} = V_0 P_0 - Q_0 U_0 - g\cos(\Theta_0)\cos(\Phi_0) \tag{2.87}$$

Equation (2.86) can be perturbed as:

$$a_{z0} + \delta a_z = \dot{w} + (V_0 + v)(P_0 + p) - (Q_0 + q)(U_0 + u) - g\cos(\Theta_0 + \theta)\cos(\Phi_0 + \phi) \tag{2.88}$$

Furthermore, assume that the altitude is changed by symmetric straight-line flight with horizontal wings such that $V_0 = \Phi_0 = \Theta_0 = P_0 = Q_0 = 0$. This gives:

$$a_{z0} + \delta a_z = \dot{w} + vp - q(U_0 + u) - g\cos(\theta)\cos(\phi) \tag{2.89}$$

Assume that the 2nd-order terms vp and uq can be neglected and subtract the equilibrium condition (2.87) from (2.89) such that:

$$\delta a_z = \dot{w} - U_0 q \tag{2.90}$$

Differentiating the altitude twice with respect to time gives the relationship:

$$\ddot{h} = -\delta a_z = U_0 q - \dot{w} \tag{2.91}$$

If we integrate this expression under the assumption that $\dot{h}(0) = U_0\theta(0) - w(0) = 0$, we get:

$$\dot{h} = U_0\theta - w \tag{2.92}$$

The *flight path* is defined as:

$$\gamma := \theta - \alpha \tag{2.93}$$

where $\alpha = w/U_0$. This gives the resulting differential equation for altitude control:

$$\dot{h} = U_0\gamma \tag{2.94}$$

Example 4 (Augmented model for altitude control using ailerons)
An autopilot model for altitude control based on the longitudinal state-space model with states $u, w(\text{alt. } \alpha), q$ and θ is obtained by augmenting the differential equation for h to the state-space model according to:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & U_0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_T \\ \delta_E \end{bmatrix} \tag{2.95}$$

□

2.8 Aircraft Stability Properties

The aircraft stability properties can be investigated by computing the eigenvalues of the system matrix $\mathbf{A}$ using:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0 \tag{2.96}$$

For a matrix $\mathbf{A}$ of dimension $n \times n$ there is n solutions of λ.

2.8.1 Longitudinal stability analysis

For conventional aircraft the characteristic equation is of 4th order. Moreover,

$$(\lambda^2 + 2\zeta_{\mathrm{ph}}\omega_{\mathrm{ph}}\lambda + \omega_{\mathrm{ph}}^2)(\lambda^2 + 2\zeta_{\mathrm{sp}}\omega_{\mathrm{sp}}\lambda + \omega_{\mathrm{sp}}^2) = 0 \tag{2.97}$$

where the subscripts ph and sp denote the following modi:

- **Phugoid mode**
- **Short-period mode**

The phugoid mode is observed as a long period oscillation with little damping. In some cases the Phugoid mode can be unstable such that the oscillations increase with time. The Phugoid mode is characterized by the natural frequency ω_{ph} and relative damping ratio ζ_{ph}.

The short-period mode is a fast mode given by the natural frequency ω_{sp} and relative damping factor ζ_{sp}. The short-period mode is usually well damped.

Classification of eigenvalues

- **Conventional aircraft:** Conventional aircraft have usually two complex conjugated pairs of Phugoid and short-period types. For the B-767 model in Chapter 4, the eigenvalues can be computed using *damp.m* in Matlab:

```
a = [
   -0.0168    0.1121    0.0003   -0.5608
   -0.0164   -0.7771    0.9945    0.0015
   -0.0417   -3.6595   -0.9544         0
         0         0    1.0000         0]

damp(a)
   Eigenvalue        Damping        Freq. (rad/sec)
  -0.0064 + 0.0593i   0.1070             0.0596
  -0.0064 - 0.0593i   0.1070             0.0596
  -0.8678 + 1.9061i   0.4143             2.0943
  -0.8678 - 1.9061i   0.4143             2.0943
```

From this is is seen that $\omega_{\mathrm{ph}} = 0.0596$, $\zeta_{\mathrm{ph}} = 0.1070$, $\omega_{\mathrm{sp}} = 2.0943$ and $\zeta_{\mathrm{sp}} = 0.4143$.

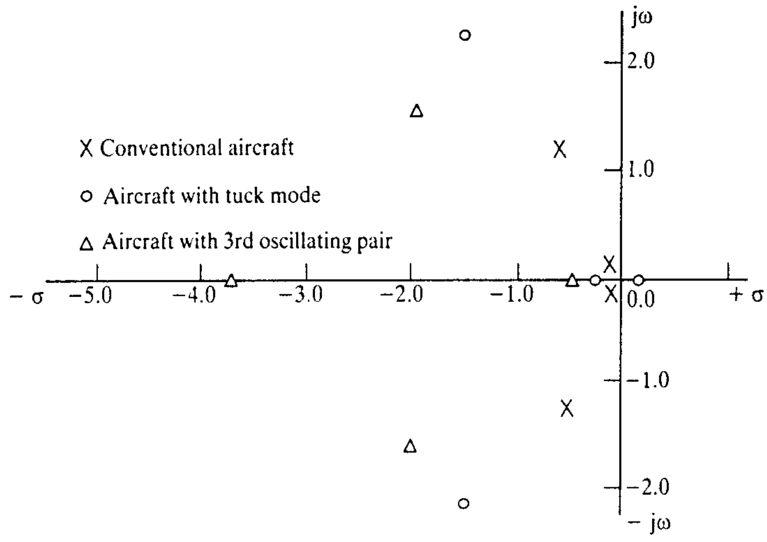

Figure 2.4: Aircraft longitudinal eigenvalue configuration plotted in the complex plane.

- **Tuck Mode:** Supersonic aircraft may have a very large aerodynamic coefficient M_u. This implies that the oscillatory Phugoid equation gives two real solutions where one is positive (unstable) and one is negative (stable). This is referred to as the *tuck mode* since the phenomenon is observed as a downwards pointing nose (*tucking under*) for increasing speed.

- **A third oscillatory mode:** For fighter aircraft the center of gravity is often located behind the neutral point or the aerodynamic center–that is, the point where the the trim moment $M_w w$ is zero. When this happens, the aerodynamic coefficient M_w takes a value such that the roots of the characteristic equation has four real solutions. When the center of gravity is moved backwards one of the roots of the Phugoid and short-period modes become imaginary and they form a new complex conjugated pair. This is usually referred to as the *3rd oscillatory mode*. The locations of the eigenvalues are illustrated in Figure 2.4.

2.8.2 Lateral stability analysis

The lateral characteristic equation is usually of 5th order:

$$\lambda^5 + \alpha_4\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0 \tag{2.98}$$

$$\Downarrow$$

$$\lambda(\lambda + e)(\lambda + f)(\lambda^2 + 2\zeta_D\omega_D\lambda + \omega_D^2) = 0 \tag{2.99}$$

where the term λ in the last equation corresponds to a pure integrator in roll–that is, $\dot{\psi} = r$.

The term $(\lambda + e)$ is the aircraft *spiral/divergence mode.* This is usually a very slow mode. Spiral/divergence corresponds to horizontally leveled wings followed by roll and a diverging spiral maneuver.

The term $(\lambda+f)$ describes the *subsidiary roll mode* while the 2nd-order system is referred to as *Dutch roll.* This is an oscillatory system with a small relative damping factor ζ_D. The natural frequency in *Dutch roll* is denoted ω_D.

If the lateral B-767 Matlab model in Chapter 4 is considered, the Matlab command *damp.m* gives:

```
a = [
  -0.1245  0.0350  0.0414 -0.9962
 -15.2138 -2.0587  0.0032  0.6450
        0  1.0000  0       0.0357
   1.6447 -0.0447 -0.0022 -0.1416

damp(a)
 Eigenvalue         Damping Freq. (rad/sec)
 -0.0143            1.0000  0.0143
 -0.1121 + 1.4996i 0.0745  1.5038
 -0.1121 - 1.4996i 0.0745  1.5038
 -2.0863            1.0000  2.0863
```

In this example we only get four eigenvalues since the pure integrator in yaw is not include in the system matrix. It is seen that the spiral mode is given by $e = -0.0143$ while the subsidiary roll mode is given by $f = -2.0863$. *Dutch roll* is recognized by $\omega_D = 0.0745$ and $\zeta_D = 1.5038$.

2.9 Design of flight control systems

A detailed introduction to design of flight control systems are given by [7], [11], [8] and [12]. The control system are based on linear design techniques using linearized models similar to those discussed in the sections above.

Chapter 3

Satellite Modeling

When stabilizing satellites in geostationary orbits only the attitude of the satellite is of interest since the position is given by the Earth's rotation.

3.1 Attitude Model

3.1.1 Euler's 2nd Axiom Applied to Satellites

The rigid-body kinetics (2.21) gives:

$$\mathbf{I}_{\mathrm{CG}}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_{\mathrm{CG}}\boldsymbol{\omega}) = \boldsymbol{\tau} \tag{3.1}$$

were $\boldsymbol{\omega} = [p, q, r]^\top$ and $\mathbf{I}_{\mathrm{CG}} = \mathbf{I}_{\mathrm{CG}}^\top > 0$ is the inertia tensor about the center of gravity given by:

$$\mathbf{I}_{\mathrm{CG}} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \tag{3.2}$$

If the Euler angles are used to represent attitude, the kinematics become:

$$\dot{\boldsymbol{\theta}} = \mathbf{J}(\boldsymbol{\theta})\boldsymbol{\omega} \tag{3.3}$$

where $\boldsymbol{\theta} = [\Phi, \Theta, \Psi]^\top$ and

$$\mathbf{J}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & \sin(\Phi)\tan(\Theta) & \cos(\Phi)\tan(\Theta) \\ 0 & \cos(\Phi) & -\sin(\Phi) \\ 0 & \sin(\Phi)/\cos(\Theta) & \cos(\Phi)/\cos(\Theta) \end{bmatrix}, \qquad \cos(\Theta) \neq 0 \tag{3.4}$$

The satellite has three controllable moments $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3]^\top$ which can be generated using different actuators.

3.1.2 Skew-symmetric representation of the satellite model

The dynamic model (3.1) can also be written:

$$\mathbf{I}_{\mathrm{CG}}\dot{\boldsymbol{\omega}} - (\mathbf{I}_{\mathrm{CG}}\boldsymbol{\omega}) \times \boldsymbol{\omega} = \boldsymbol{\tau} \tag{3.5}$$

where we have used the fact that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. Furthermore, it is possible to find a matrix $\mathbf{S}(\boldsymbol{\omega})$ such that $\mathbf{S}(\boldsymbol{\omega}) = -\mathbf{S}^\top(\boldsymbol{\omega})$ is skew-symmetric. With other words:

$$\mathbf{I}_{\mathrm{CG}}\dot{\boldsymbol{\omega}} + \mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} = \boldsymbol{\tau} \tag{3.6}$$

The matrix $\mathbf{S}(\boldsymbol{\omega})$ must satisfy the condition:

$$\mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} \equiv -(\mathbf{I}_{\mathrm{CG}}\boldsymbol{\omega}) \times \boldsymbol{\omega} \tag{3.7}$$

A matrix satisfying this condition is:

$$\mathbf{S}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -I_{yz}q - I_{xz}p + I_z r & I_{yz}r + I_{xy}p - I_y q \\ I_{yz}q + I_{xz}p - I_z r & 0 & -I_{xz}r - I_{xy}q + I_x p \\ -I_{yz}r - I_{xy}p + I_y q & I_{xz}r + I_{xy}q - I_x p & 0 \end{bmatrix} \tag{3.8}$$

3.2 Satellite Model Stability Properties

Consider the Lyapunov function candidate:

$$V = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{I}_{\mathrm{CG}}\boldsymbol{\omega} + h(\boldsymbol{\theta}) \tag{3.9}$$

where $h(\boldsymbol{\theta})$ is a positive definite function depending of the attitude. Differentiation of V with respect to time gives:

$$\begin{aligned} \dot{V} &= \boldsymbol{\omega}^T \mathbf{I}_{\mathrm{CG}}\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\theta}}^\top \frac{\partial h}{\partial \boldsymbol{\theta}} && (3.10) \\ &= \boldsymbol{\omega}^\top(-\mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}) + \boldsymbol{\omega}^\top \mathbf{J}^\top(\boldsymbol{\theta})\frac{\partial h}{\partial \boldsymbol{\theta}} && (3.11) \end{aligned}$$

Since $\mathbf{S}(\boldsymbol{\omega}) = -\mathbf{S}^\top(\boldsymbol{\omega})$, it follows that:

$$\boldsymbol{\omega}^\top \mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} = 0 \quad \forall \boldsymbol{\omega} \tag{3.12}$$

This suggests that the control input $\boldsymbol{\tau}$ should be chosen as:

$$\dot{V} = \boldsymbol{\omega}^\top \left(\boldsymbol{\tau} + \mathbf{J}^\top(\boldsymbol{\theta})\frac{\partial h}{\partial \boldsymbol{\theta}}\right) \leq 0 \tag{3.13}$$

One control law satisfying this is:

$$\boldsymbol{\tau} = -\mathbf{K}_d\boldsymbol{\omega} - \mathbf{J}^\top(\boldsymbol{\theta})\frac{\partial h}{\partial \boldsymbol{\theta}} \tag{3.14}$$

where $\mathbf{K}_d > 0$. This finally gives:

$$\dot{V} = -\boldsymbol{\omega}^\top \mathbf{K}_d \boldsymbol{\omega} \leq 0 \tag{3.15}$$

and stability and convergence follow from standard Lyapunov techniques.

3.3 Design of Satellite Attitude Control Systems

Design of nonlinear control systems based on the model (3.6) is quite common in the literature. One example is the nonlinear and passive adaptive attitude control system of Slotine and Di Benedetto [10]. An alternative representation is proposed by Fossen [3].

Chapter 4

Matlab Simulation Models

4.1 Boeing-767

The longitudinal and lateral B-767 state-space models are given below. The state vectors are:

$$\mathbf{x}_{\text{lang}} = \begin{bmatrix} u \text{ (ft/s)} \\ \alpha \text{ (deg)} \\ q \text{ (deg/s)} \\ \theta \text{ (deg)} \end{bmatrix}, \mathbf{x}_{\text{lat}} = \begin{bmatrix} \beta \text{ (deg)} \\ p \text{ (deg/s)} \\ \phi \text{ (deg/s)} \\ r \text{ (deg)} \end{bmatrix} \tag{4.1}$$

$$\mathbf{u}_{\text{lang}} = \begin{bmatrix} \delta_E \text{ (deg)} \\ \delta_T \text{ (\%)} \end{bmatrix}, \mathbf{u}_{\text{lat}} = \begin{bmatrix} \delta_A \text{ (deg)} \\ \delta_R \text{ (deg)} \end{bmatrix} \tag{4.2}$$

Equilibrium point:

Speed	$V_T = 890$ ft/s $= 980$ km/h
Altitude	$h = 35\ 000$ ft
Mass	$m = 184\ 000$ lbs
Mach-number	$M = 0.8$

4.1.1 Longitudinal model

```
a = [ -0.0168    0.1121    0.0003   -0.5608
      -0.0164   -0.7771    0.9945    0.0015
      -0.0417   -3.6595   -0.9544         0
         0         0       1.0000         0];

b = [ -0.0243    0.0519
      -0.0634   -0.0005
      -3.6942    0.0243
         0         0   ];
```

Eigenvalues:

```
lam = [
  -0.8678 + 1.9061i
  -0.8678 - 1.9061i
  -0.0064 + 0.0593i
  -0.0064 - 0.0593i];
```

4.1.2 Lateral model

```
a = [
   -0.1245    0.0350    0.0414   -0.9962
  -15.2138   -2.0587    0.0032    0.6458
         0    1.0000         0    0.0357
    1.6447   -0.0447   -0.0022   -0.1416];

b = [
   -0.0049    0.0237
   -4.0379    0.9613
         0         0
   -0.0568   -1.2168];
```

Eigenvalues:

```
lam = [
 -0.1121 + 1.4996i
 -0.1121 - 1.4996i
 -2.0863
 -0.0143];
```

4.2 F-16 Fighter

The lateral model of the *F-16* fighter aircraft is based on [11], pages 370–371. The state vectors are:

$$\mathbf{x}_{\text{lat}} = \begin{bmatrix} \beta \text{ (ft/s)} \\ \phi \text{ (ft/s)} \\ p \text{ (rad/s)} \\ r \text{ (rad)} \\ \delta_A \text{ (rad)} \\ \delta_R \text{ (rad)} \\ r_w \text{ (rad)} \end{bmatrix}, \mathbf{u}_{\text{lat}} = \begin{bmatrix} u_A \text{ (rad)} \\ u_R \text{ (rad)} \end{bmatrix}, \mathbf{y}_{\text{lat}} = \begin{bmatrix} r_w \text{ (deg)} \\ p \text{ (deg/s)} \\ \beta \text{ (deg)} \\ \phi \text{ (deg)} \end{bmatrix} \tag{4.3}$$

Equilibrium point:

Speed	$V_T = 502$ ft/s $= 552$ km/h
Mach-number	$M = 0.45$

4.2.1 Longitudinal model

```
a = [
    -0.3220    0.0640    0.0364   -0.9917    0.0003    0.0008    0
     0              0         1   -0.0037         0         0    0
   -30.6492         0   -3.6784    0.6646   -0.7333    0.1315    0
     8.5396         0   -0.0254   -0.4764   -0.0319   -0.0620    0
          0         0         0         0     -20.2         0    0
          0         0         0         0         0     -20.2    0
          0         0         0   57.2958         0         0   -1 ];

b = [
          0         0
          0         0
          0         0
          0         0
       20.2         0
          0      20.2
          0         0 ];

c= [
          0         0         0     57.2958         0         0     -1
          0         0   57.2958           0         0         0      0
    57.2958         0         0           0         0         0      0
          0   57.2958         0           0         0         0      0 ];
```

Eigenvalues:

```
lam = [
  -1.0000
  -0.4224+ 3.0633i
  -0.4224- 3.0633i
  -0.0167
  -3.6152
 -20.2000
 -20.2000         ];
```

Notice that the last two eigenvalues correspond to the actuator states.

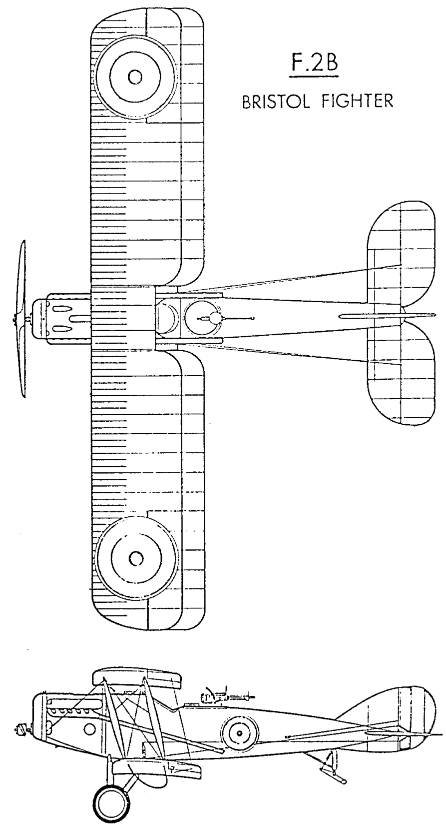

Figure 4.1: Schematic drawing of the Bristol F.2B Fighter (McRuer et al 1973).

4.3 F2B Bristol Fighter

The lateral model of the *F2B Bristol* fighter aircraft is given below [8]. This is a British aircraft from World War I (see Figure 4.1). The aircraft model is for $\beta = 0$ (coordinated turn). The state-space vector is:

$$\mathbf{x}_{\text{lat}} = \begin{bmatrix} p \text{ (deg/s)} \\ r \text{ (deg/s)} \\ \phi \text{ (deg)} \\ \psi \text{ (deg)} \end{bmatrix}, \mathbf{u}_{\text{lat}} = [\delta_A \text{ (deg)}] \tag{4.4}$$

where the dynamics for ψ satisfies the *bank-to-turn* equation:

$$\dot{\psi} = \frac{g}{U_0}\phi = \frac{9.81}{138 \cdot 0.3048}\phi = 0.233\phi \tag{4.5}$$

Equilibrium point:

Speed	$V_T = 138$ ft/s $= 151.4$ km/h
Altitude	$h = 6\ 000$ ft
Mach-number	$M = 0.126$

4.3.1 Lateral model

```
a = [
   -7.1700    2.0600         0         0
   -0.4360   -0.3410         0         0
    1.0000         0         0         0
         0         0    0.2330         0];
b = [
   26.1000
   -1.6600
         0
         0];
```

Eigenvalues:

```
lam = [
         0
         0
   -0.4752
   -7.0358];
```

Bibliography

[1] **Blakelock, J. H. (1991)**. *Aircraft and Missiles* (John Wiley & Sons Ltd.)

[2] **Etkin, B. and L. D. Reid (1996)**. Dynamics of Flight: Stability and Control (John Wiley & Sons Ltd.)

[3] **Fossen, T. I. (1993)**. Comments on "Hamiltonian Adaptive Control of Spacecraft", *IEEE Transactions on Automatic Control*, **TAC**-38(5):671–672.

[4] **Fossen, T. I. (1994).** *Guidance and Control of Ocean Vehicles* (John Wiley & Sons Ltd.)

[5] **Fossen, T. I. (2011).** *Handbook of Marine Craft Hydrodynamics and Motion Control.* (John Wiley & Sons Ltd.)

[6] **Hughes, P. C. (1986)**. *Spacecraft Attitude Dynamics* (John Wiley & Sons Ltd.)

[7] **McLean, D. (1990).** *Automatic Flight Control Systems* (Prentice Hall Inc.)

[8] **McRuer, D., D. Ashkenas og A. I. Graham (1973).** *Aircraft Dynamics and Automatic Control.* (Princeton University Press, New Jersey 1973).

[9] **Nelson R. C. (1998)**. *Flight Stability and Automatic Control* (McGraw-Hill Int.)

[10] **Slotine, J. J. E. og M. D. Di Benedetto (1990).** Hamiltonian Adaptive Control of Spacecraft, *IEEE Transactions on Automatic Control*, **TAC**-35(7):848–852.

[11] **Stevens, B. L. og F. L. Lewis (1992)**. *Aircraft Control and Simulation* (John Wiley & Sons Ltd.)

[12] **Roskam, J. (1999).** *Airplane Flight Dynamics and Automatic Flight Controls* (Darcorporation)