**CPSC 365 / ECON 365: Algorithms** Yale University

Problem Set 2

Out: February 1, 2022 Due: February 8, 2022 at 2:30 pm EST

# Instructions

Write your solution to the following problems carefully. Submit the PDF of your solution via Gradescope. Please start your solution to every question on a new page. Make sure to assign the correct page in your document corresponding to each problem. We recommend writing your solution in Latex. Handwritten solutions are accepted if they are clearly legible. Write your name and SID in your answer to Problem 0; do not write them anywhere else in your solution.

**Collaboration policy:** You may collaborate and discuss these problems with other students. However, you must first make an honest effort to solve these problems by yourself. You may discuss hints and try to solve problems together. But you may not explicitly provide the solution to a problem to other students; nor receive the solution from them. You must write your solution independently, and make sure you understand your solution. You must list all your collaborators (anyone with whom you discussed any part of these problems). You may consult the course textbook and other references related to the class. However, you are forbidden from searching for these problems on the Internet. You must list any resources you consult (beyond the course textbook). You must also follow the Yale academic integrity policy.

# 0 Your Information

On the first page of your submission, include the following information (but do not include these anywhere else in your solution). Also certify that you have followed the collaboration policy. Your solution to the remainder of the problems should start on a new page.

(a) Your name.

(b) Your SID.

(c) A list your collaborators and any outside resources you consulted for this problem set. If none, write “None”.

(d) Certify that you have followed the academic integrity and collaboration policy as written above.

(e) How many hours did you spend in this problem set?

# 1 Stable Matching

## 1.1 Hard example for stable matching

Run the stable matching algorithm (propose and reject with the men proposing) on the following example with $n = 4$ men $(A, B, C, D)$ and 4 women $(1, 2, 3, 4)$.

Men's preference list:

| | | | | |
|---|---|---|---|---|
| $A$ | 1 | 2 | 3 | 4 |
| $B$ | 2 | 3 | 1 | 4 |
| $C$ | 3 | 1 | 2 | 4 |
| $D$ | 1 | 2 | 3 | 4 |

Women's preference list:

| | | | | |
|---|---|---|---|---|
| 1 | $B$ | $C$ | $D$ | $A$ |
| 2 | $C$ | $D$ | $A$ | $B$ |
| 3 | $D$ | $A$ | $B$ | $C$ |
| 4 | $A$ | $B$ | $C$ | $D$ |

## 1.2 A tight bound on the number of proposals

In lecture we showed that the stable matching algorithm must terminate after at most $n^2$ proposals. Prove a sharper bound showing that the algorithm must terminate after at most $n(n-1)+1$ proposals. Conclude that the example in part 1 is a worst case instance for $n = 4$. How many days does the algorithm take on this instance?

*Hint:* Show that there is at most one man who proposes to the last woman in his list.

## 2 Combining Stable Matchings

Suppose that in the stable matching problem with $n$ men and $n$ women, we have found two (possibly different) stable matchings $S$ and $T$. We will show how to combine $S$ and $T$ into two new stable matchings $W$ and $M$, which are the best of both worlds for the women and men respectively.

(a) Suppose each woman is given the name of the man she is matched with in $S$ and the man she is matched with in $T$ (they might be the same man). Of the two names each woman receives, she picks the one she prefers the most.

Prove that no two women would end up picking the same man.

We conclude that if the women are matched with the men they pick as above, the result is a matching. Call this matching $W$.

(b) Prove that $W$ is a stable matching.

This new matching $W$ is called the "best of both worlds" for the women.

(c) Another way of combining $S$ and $T$, is to give each man the names of the women he is matched with in $S$ and $T$, and force each man to pick the woman he prefers the least amongst the two. Prove that this results in the same stable matching $W$ as before.

(d) How would you combine the matchings $S$ and $T$ to create a stable matching $M$ which is the best of both worlds for the men? Provide a brief justification for your answer.

## 3 Checking Connectivity

Let $G = (V, E)$ be an undirected graph with the additional property that every edge has a color, either red or blue. Let $u$ and $v$ be distinct vertices in $G$.

(a) Design an algorithm that decides whether or not there exists a path from $u$ to $v$ such that the path contains only red edges.

Justify correctness of your algorithm and analyze the running time.

(*Hint:* You may use or modify any algorithm we have seen in class. You should describe your algorithm concisely and precisely, either in pseudocode or in words. Make sure your solution includes proof of correctness and running time analysis.)

(b) Design an efficient algorithm that decides whether or not there exists a path from $u$ to $v$ such that within the path, all red edges appear before all blue edges.

Justify the correctness of your algorithm and analyze the running time.