SUB-RIEMANNIAN METRICS ON STEP 2 DISTRIBUTIONS WITH THE SAME GEODESICS: DIRECT PRODUCT STRUCTURE PHENOMENA

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ABSTRACT. This paper extends the classical results of *Eisenhart (1923)* and *Levi-Civita (1898)* in Riemannian geometry concerning the classification of pairs of germs of Riemannian metrics—those having the same geodesics either as parametrized curves (*affine equivalence*) or as unparametrized curves (*projective equivalence*)—to the setting of *sub-Riemannian metrics* on arbitrary *step-2 distributions* at a generic point. This generalizes the recent result of *Lin–Zelenko (2023)* [8], obtained under the more restrictive condition of *ad-surjectivity*. In the affine case, we prove that if a sub-Riemannian metric from the considered class admits another metric sharing the same geodesics, but not constantly proportional to it, then locally it decomposes as a *product of two sub-Riemannian metrics*. In the projective case, we establish a slightly weaker version of the *2019 conjecture of Jean–Maslovskaya–Zelenko* [6] on Levi-Civita pairs, showing that the underlying distribution admits a *product structure*, that the *squares of each component* are *involutive*, and that the restrictions of each sub-Riemannian metric in the pair to each leaf of the corresponding foliation are *conformal* to each other under the appropriate pullback. The proof is based on a new *graph-theoretic formalism*, in which appropriately compatible moving frames are represented by the *vertices and edges* of certain graphs. As a byproduct, we introduce new *discrete invariants* of step-2 nilpotent graded Lie algebras, called the *characteristic tuple*.

1. INTRODUCTION

This paper is devoted to the classification of pairs of germs of sub-Riemannian metrics on step-2 distributions that share the same (variational) geodesics—either as parametrized curves (*affine equivalence*) or as unparametrized curves (*projective equivalence*) (see Theorem 4.4 below). These results extend the classical theorems of *Eisenhart (1923)* and *Levi-Civita (1898)* for Riemannian metrics, which can be viewed as a special case of sub-Riemannian metrics where the underlying distribution coincides with the tangent bundle. In particular, our results confirm the phenomenon of direct product (or separation of variables), fully in the affine case and at least partially in the projective case, previously observed in the Riemannian setting.

Although step 2 distributions still form a rather restrictive subclass among all distributions, the results of this paper represent substantial generalizations of the known classification results previously established for contact and even-contact distributions in [12], as well as for the so-called ad-surjective step-2 distributions in [8]. Moreover, the graph-theoretic approach developed in this work both significantly simplifies and clarifies the proof in the particular ad-surjective case given in [12], and suggests that the method may be further extended to distributions of higher rank through the use of *hypergraphs*.

In more details, recall that a distribution D on a manifold M is a subbundle of the tangent bundle TM. A sub-Riemannian manifold/structure is a triple (M, D, g), where M is a smooth manifold, D is a bracket-generating distribution, and for any q, $g(q)$ is an inner product on $D(q)$ which depends smoothly on q. We say that g is a sub-Riemannian metric on (M, D). As already mentioned, a Riemannian manifold/structure/metric appears as the particular case where $D = TM$.

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In the sequel, we will assume that the distribution D is bracket-generating, i.e. for every point $q \in M$ the iterative Lie brackets of vector fields tangent to a distribution D (i.e., of sections of D) span the tangent space T_qM. In more details, one can define a filtration

filtr
$$D = D^1 \subset D^2 \subset \dots D^j \subset \dots$$

of the tangent bundle, called a *weak derived flag*, as follows: set $D = D^1$ and define recursively

Dpower1
$$D^j = D^{j-1} + [D, D^{j-1}], \quad j > 1.$$

If $X_1, \dots, X_m$ are m vector fields constituting a local basis of a distribution D, then $D^j(q)$ is the linear span of all iterated Lie brackets of these vector fields, of length not greater than j, evaluated at a point q,

Dpower2
$$D^j(q) = \operatorname{span}\{\big[X_{i_1}(q), \dots [X_{i_{s-1}}, X_{i_s}](q) \dots\big] : (i_1, \dots, i_s) \in [1:m]^s, s \in [1:j]\}$$

(here given a positive integer n we denote by $[1:n]$ the set $\{1, \dots, n\}$). A distribution D is called *bracket-generating* (or *completely nonholonomic*) if for any q there exists $\mu(q) \in \mathbb{N}$ such that $D^{\mu(q)}(q) = T_qU$. The number $\mu(q)$ is called the *degree of nonholonomy* of D at a point q. If the degree of nonholonomy is equal to a constant μ at every point, one says that D is *step μ distribution.*

Since we work locally, the assumption of bracket-genericity is not too restrictive: if a distribution is not bracket-generating then in a neighborhood U of a generic point there exists a positive integer μ such that $D^{\mu+1} = D^\mu \subsetneqq TM$. So, D^μ is a proper involutive subbundle of TU and the distribution D is bracket-generating on each integral submanifold of D^μ in U. So, we can restrict ourselves to these integral submanifolds instead of U.

In the present paper, we consider geodesics defined via the variational approach, which is natural from the optimal-control perspective. For a comparison with the differential-geometric viewpoint, i.e., "shortest" versus "straightest" curves, see [3]; note that both notions coincide only in the Riemannian case.

A horizontal curve $\gamma : [a, b] \to M$ is an absolutely continuous curve tangent to D, i.e., $\gamma'(t) \in D(\gamma(t))$. In the sequel, the manifold M is assumed to be connected. By the Rashevskii-Chow theorem the assumption that D is bracket-generating guarantees that the space of horizontal curves connecting two given points q_0 and q_1 is not empty. The following energy-minimizing problem:

energy
$$\begin{aligned} E(\gamma) &= \int_a^b g\big(\gamma'(t), \gamma'(t)\big)\, dt \to \min, \\ \gamma'(t) &\in D(\gamma(t)) \qquad \text{a.e. } t, \\ \gamma(a) &= q_0, \quad \gamma(b) = q_1. \end{aligned} \tag{1.1}$$

can be solved using the Pontryagin Maximum Principle ([2, 10]) in optimal control theory that defines special curves in the cotangent bundle T^*M, called the *Pontryagin extremals*, so that a minimizer of the optimal control problem (1.1) is a projection from T^*M to M of some *Pontryagin extremal* (for more explicit description of Pontryagin extremals see the beginning of section (5) below).

geod **Definition 1.1.** The *(variational) sub-Riemannian geodesics* are projections of the Pontryagin extremals of the optimal control problem (1.1).

Note that in the Riemannian case geodesics given by Definition 1.1 coincides with the usual Riemannian geodesics.

Definition 1.2. Let M be a manifold and D be a bracket-generating distribution on M. Two sub-Riemannian metrics g_1 and g_2 on (M, D) are called *projectively equivalent* at $q_0 \in M$ if they have the same geodesics, up to a reparameterization, in a neighborhood of q_0 or, equivalently, for every geodesic $\gamma(t)$ of g_1 there exists a reparametrization $t = \varphi(\tau)$ such that $\gamma\big(\varphi(\tau)\big)$ is a geodesic of g_2.They are called *affinely equivalent* at q_0 if they are projectively equivalent at q_0 and the reparametrizations $\varphi(\tau)$ above are affine functions, i.e., they are of the form $\varphi(\tau) = a\tau + b$.

We will write $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$ in the case of projective and affine equivalence, respectively. Obviously, for a sub-Riemannian metric g on (M, D) and a positive constant c the metrics cg and g are affinely equivalent. The metric cg will be said a *constantly proportional metric* to the metric g.

Definition 1.3. A sub-Riemannian metric g on (M, D) is called *affinely rigid* if the sub-Riemannian metrics constantly proportional to it are the only sub-Riemannian metrics on (M, D) that are affinely equivalent to g.

Examples of affinely nonrigid sub-Riemannian structures can be constructed with the help of the appropriate notion of product structure. For this we first have to define distributions admitting product structure as follows:

Definition 1.4. A distribution D on a manifold M *admits a product structure* if there exist two manifold M_1 and M_2 of positive dimension endowed with two distributions D_1 and D_2 of positive rank (on M_1 and M_2, respectively) such that the following two conditions holds:

(1) $M = M_1 \times M_2$;
(2) If $\pi_i : M \to M_i$, $i = 1, 2$, are the canonical projections and $\pi_i^* D_i$ denotes the pullback of the distribution D_i from M_i to M, i.e.,
$$\pi_i^* D_i(q) = \{v \in T_q M : d\pi_i(q)v \in D_i\big(\pi_i(q)\big)\},$$
then

proddist
$$D(q) = \pi_1^* D_1(q) \cap \pi_2^* D_2(q).$$

In this case, we will write that $(M, D) = (M_1, D_1) \times (M_2, D_2)$.

Definition 1.5. A sub-Riemannian structure (M, D, g) *admits a product structure* if there exist (nonempty) sub-Riemannian structures (M_1, D_1, g_1) and (M_2, D_2, g_2) such that $(M, D) = (M_1, D_1) \times (M_2, D_2)$ and if $\pi_i : M \mapsto M_1$ are the canonical projections then

prodmetric
$$g = \pi_1^* g_1 + \pi_2^* g_2.$$

In this case we will write that $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$.

It is easy to see that if $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$, then this sub-Riemannian metric is affinely equivalent to

onconstprop
$$(M_1, D_1, c_1 g_1) \times (M_2, D_2, c_2 g_2)$$

for every two positive constants c_1 and c_2, but the latter metric is not constantly proportional to (M, D, g) , if $c_1 \neq c_2$, i.e., a *sub-Riemannian metric admitting product structure* is not affinely rigid. The main question is whether or not the converse of this statement, at least in a local setting, i.e., the analog of the Eisenhart theorem (Theorem **??**) holds.

conjecture
Conjecture 1.6 ([6])**.** *If a sub-Riemannian metric g is not affinely rigid near a point q_0, i.e., admits a locally affinely equivalent non-constantly proportional sub-Riemannian metric in a neighborhood of a point q_0, then the metric g is the direct product of two sub-Riemannian metrics in a neighborhood of q_0.*

In this paper, we prove this conjecture for sub-Riemannian metrics on a class of step 2 distributions, see Theorem 4.4.

2. Characteristic tuple of step-2 fundamental Lie algebras

Definition 2.1. Suppose $\mathfrak{m} = \bigoplus\limits_{k \in \mathbb{Z}_-} \mathfrak{m}_k$ is a $\mathbb{Z}_-$-graded Lie algebra with finitely many nonzero homogeneous components. The algebra $\mathfrak{m}$ is called *fundamental* if $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$. Moreover, if μ is the smallest degree among non-zero homogeneous components, we say that $\mathfrak{m}$ is a *step*-$|\mu|$ fundamental Lie algebra. In what follows, we will always assume that $\mathfrak{m}$ is fundamental and omit the term for brevity.

For a given step-2 Lie algebra, $\mathfrak{m}$, we will exhibit a novel filtration of $\mathfrak{m}$ by certain ideals and an associated adapted local frame of TM which will simplify our analysis.

To that end, suppose $\mathfrak{m}$ is a step-2 Lie algebra and $X \in \mathfrak{m}_{-1}$. We view $\mathrm{ad}X : \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$ as a linear map from $\mathfrak{m}_{-1}$ to $\mathfrak{m}_{-2}$ and assume that

max_r
$$r = \max\{\mathrm{rank}\ \mathrm{ad}\widetilde{X} : \widetilde{X} \in \mathfrak{m}_{-1}\} \tag{2.1}$$

ad+max **Definition 2.2.** We say that $X \in \mathfrak{m}_{-1}$ is *ad-maximal*, if $\mathrm{rank}\ \mathrm{ad}X = r$, where r is as in (2.1).

Note that the set of ad-maximal elements is (non-empty) Zariski open in $\mathfrak{m}_{-1}$. For step-2 Lie algebras, the kernel (of the map $\mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$) and image of the map $\mathrm{ad}X : \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$ for an ad-maximal X are particularly well behaved, as shown in the following:

a1 **Lemma 2.3.** *Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be a step-2 Lie algebra. If $X \in \mathfrak{m}_{-1}$ is ad-maximal, then* $[\ker \mathrm{ad}X, \mathfrak{m}_{-1}] \subset \mathrm{Im}\ \mathrm{ad}X$ *or, equivalently* [1], $\ker \mathrm{ad}X \oplus \mathrm{Im}\ \mathrm{ad}X$ *is an ideal of* $\mathfrak{m}$.

Proof. Let $r = \mathrm{rank}\ \mathrm{ad}X$. By way of contradiction, suppose that there exists

Y, Z
$$Y \in \ker \mathrm{ad}X, \quad Z \in \mathfrak{m}_{-1} \tag{2.2}$$

such that

Y, Z, W
$$[Z, Y] = W \in \mathfrak{m}_{-2} \backslash \mathrm{Im}\ \mathrm{ad}X. \tag{2.3}$$

Since r satisfies (2.1), it follows that $\mathrm{rank}\ \mathrm{ad}(X + tZ)$ is equal to r for sufficiently small t. For such t, the subspaces $\mathrm{Im}\ \mathrm{ad}(X + tZ)$ converge to $\mathrm{Im}\ \mathrm{ad}X$ in $\mathbf{Gr}_r(\mathfrak{m}_{-2})$. Thus, by (2.3), we have that $W \notin \mathrm{Im}\ \mathrm{ad}(X + tZ)$ for sufficiently small t. On the other hand, by (2.2), we have $[X + tZ, Y] = tW$, i.e. $W \in \mathrm{Im}\ \mathrm{ad}(X + tZ)$ for $t \neq 0$ – a contradiction. □

Before we proceed, we introduce new notation that will ease the task of indexing in our constructions. In lieu of developing a complete set of numerical indices for an adapted frame, we instead relate elements of a basis to vertices and edges of a particular graph. As will be seen shortly, virtually all of the relevant information for our purposes can be encoded by a graph derived from ad-maximal elements of $\mathfrak{m}$ and its quotients by ideals.

To that end, suppose $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ is a step-2 Lie algebra. We introduce the basic, discrete invariants of this algebra called the *characteristic tuple*: Begin by setting

Need to find different notation for index of quotients, suggest $\mathfrak{m}^{(\omega)}$

m0
$$\mathfrak{m}^0 := \mathfrak{m}.$$
$$\mathfrak{m}^0_{-j} := \mathfrak{m}_{-j}, j \in \{1, 2\}$$

and let $m = \dim \mathfrak{m}_{-1}$. Choose a vector $Y_0 \in \mathfrak{m}^0_{-1}$ such that $\mathrm{ad}Y_0 : \mathfrak{m}^0_{-1} \to \mathfrak{m}^0_{-2}$ is ad-maximal. It follows from Lemma 2.3 and the fact that $\mathfrak{m}$ is step-2 that $\ker \mathrm{ad}Y_0 \oplus \mathrm{Im}\ \mathrm{ad}Y_0$ is an ideal of $\mathfrak{m}^0$. Set

M^1
$$\mathfrak{m}^1(Y_0) := \mathfrak{m}^0/(\ker \mathrm{ad}Y_0 \oplus \mathrm{Im}\ \mathrm{ad}Y_0)$$

and proceed inductively to define the sequence of algebras

mj
$$\mathfrak{m}^{j+1}(Y_0, \ldots, Y_j) := \mathfrak{m}^j(Y_0, \ldots Y_{j-1})/(\ker \mathrm{ad}Y_j \oplus \mathrm{Im}\ \mathrm{ad}Y_j),$$

whereby each $\mathfrak{m}^{j+1}(Y_0, \ldots, Y_j)$ inherits a grading from its predecessor, namely,

$$\begin{aligned}\mathfrak{m}^{j+1}(Y_0, \ldots, Y_j) &= (\mathfrak{m}^j_{-1}(Y_0, \ldots, Y_{j-1})/\ker \mathrm{ad}Y_j) \oplus (\mathfrak{m}^j_{-2}((Y_0, \ldots, Y_{j-1})/\mathrm{Im}\ \mathrm{ad}Y_j)\\ &=: \mathfrak{m}^{j+1}_{-1}(Y_0, \ldots, Y_j) \oplus \mathfrak{m}^{j+1}_{-2}(Y_0, \ldots, Y_j),\end{aligned}$$

and each $Y_j \in \mathfrak{m}^j_{-1}$ is chosen to be ad-maximal for $\mathfrak{m}^j(Y_0, \ldots, Y_{j-1})$.

[1]Here we use that $\mathfrak{m}$ is a step-2 Lie algebra.

Let $r_j(Y_0,\dots,Y_{j-1})$ be the rank of the linear map $\mathrm{ad}Y_j : \mathfrak{m}^j_{-1} \to \mathfrak{m}^j_{-2}$:

r_j

$$r_j(Y_0,\dots,Y_{j-1}) = \operatorname{rank}\mathrm{ad}Y_j \tag{2.4}$$

and set $d = \dim \mathfrak{m}_{-2}$. Since $\dim \mathfrak{m}^{j+1}_{-1} = r_j(Y_0,\dots,Y_{j-1})$, it follows that $r_{j+1}(Y_0,\dots,Y_j) \neq r_j(Y_0,\dots,Y_{j-1})$, otherwise $\mathrm{ad}Y_{j+1}$ would have nonzero rank on the subspace spanned by Y_{j+1}. Thus, by construction, we have strict inequality

$$r_j(Y_0,\dots,Y_{j-1}) > r_{j+1}(Y_0,\dots,Y_j) \geq 0.$$

Moreover, since $\mathfrak{m}$ is fundamental, there exist ω (depending on the choice of $Y_0,\dots,Y_\omega$) such that the sequence of quotient algebras

resolution

$$\mathfrak{m} = \mathfrak{m}^0 \to \mathfrak{m}^1(Y_0)\dots \to \mathfrak{m}^\omega(Y_0,\dots,Y_{\omega-1}) \to \mathfrak{m}^{\omega+1}(Y_0,\dots,Y_\omega) \tag{2.5}$$

terminates with the non-zero abelian algebra $\mathfrak{m}^{\omega+1}(Y_0,\dots,Y_\omega)$, or, equivalently, when

$$r_{\omega+1}(Y_0,\dots,Y_\omega) = 0.$$

This happens precisely when

sum_r

$$\sum_{j=0}^{\omega} r_j(Y_0,\dots,Y_{j-1}) = d, \quad r_j(Y_0,\dots,Y_{j-1}) > 0,\ \forall\, 0 \leq j \leq \omega.$$

In particular, $\omega \leq d-1$ independently of the choice of $Y's$.

Although the choice of tuple or ranks $(r_0, r_1, \dots, r_\omega)$ (and even its length ω) depends on the choice of Y's in the above construction, for the generic choice of Y's this tuple is maximal in the lexicographic order [2] among all such tuples arising from different choices of ad-maximal Y's.

r_tuple_def

Definition 2.4. The maximal tuple of ranks among all such tuples (with respect to lexicographic order) arising from different choices of ad-maximal Y's is called the *characteristic tuple* of the step-2 Lie algebra $\mathfrak{m}$.

In the sequel, we assume that the sequence of ad-maximal Y's is chosen such that the tuple of ranks defined by (2.4) forms the characteristic tuple of the algebra $\mathfrak{m}$, and, for brevity, we denote the corresponding algebras $\mathfrak{m}^j(Y_0,\dots Y_{j-1})$ from (2.5) by $\mathfrak{m}^j$. The sequence of quotient algebras $\{\mathfrak{m}^j\}_{j=0}^{\omega+1}$ naturally defines a filtration on $\mathfrak{m}$ by ideals. Indeed, if

_projection

$$\varphi_j : \mathfrak{m}^j \to \mathfrak{m}^{j+1} \tag{2.6}$$

denotes the canonical projection to a quotient space, then $\varphi_j^{-1}(\ker \mathrm{ad}Y_{j+1} \oplus \mathrm{Im}\ \mathrm{ad}Y_{j+1})$ is an ideal of $\mathfrak{m}^j$. Set

h0

$$\mathfrak{h}_0 := \ker \mathrm{ad}Y_0 \oplus \mathrm{Im}\ \mathrm{ad}Y_0$$

$\mathfrak{h}_{\omega+1} = \mathfrak{m}$, and let

hj

$$\mathfrak{h}_j = (\varphi_{j-1} \circ \dots \circ \varphi_0)^{-1}(\ker \mathrm{ad}Y_j \oplus \mathrm{Im}\ \mathrm{ad}Y_j) \subset \mathfrak{m}, \quad 1 \leq j \leq \omega. \tag{2.7}$$

Note that each $\mathfrak{h}_j$ is an ideal of $\mathfrak{m}$, and such a collection defines a filtration

filt

$$\mathfrak{h}_0 \subset \mathfrak{h}_1 \dots \subset \mathfrak{h}_\omega \subset \mathfrak{h}_{\omega+1} = \mathfrak{m} \tag{2.8}$$

of the algebra $\mathfrak{m}$ by the ideals $\{\mathfrak{h}_j\}_{j=0}^{\omega+1}$. We say that this filtration is *associated with the tuple* $\{Y_j\}_{j=0}^{\omega}$.

We wish to assign a class of bases to the algebra $\mathfrak{m}$ which are compatible with the filtration (2.8). As mentioned above, it is convenient to index the elements of such a basis by vertices and edges of a certain graph. In more detail, let Γ be a simple, acyclic directed graph with m vertices and d edges. We denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of all vertices and all edges of the graph Γ, respectively. The symbol $\overrightarrow{ab}$ represents the directed edge which connects the initial vertex a to the terminal vertex b. For a given vertex $a \in V(\Gamma)$, we denote by Out(a) the set of all edges of the form $\overrightarrow{ab}$ such that $\overrightarrow{ab} \in E(\Gamma)$ and by In(a) the set of all edges of the form

[2] To compare two tuples of different lengths lexicographically, we pad the shorter tuple with zeros to match the length of the longer one before performing the comparison.

$\overrightarrow{ba}$ such that $\overrightarrow{ba} \in E(\Gamma)$. We call $|\mathrm{Out}(a)|$ and $|\mathrm{In}(a)|$ the *outgoing index* and *incoming index*, respectively, and their sum the *total index* of the vertex a.

Definition 2.5. If $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ is a step-2 Lie algebra, then we say $\mathfrak{m}$ and the graph Γ are *compatible* if there exists a basis $\{X_v\}_{v \in V(\Gamma)}$ of $\mathfrak{m}_{-1}$ and a basis $\{X_e\}_{e \in E(\Gamma)}$ of $\mathfrak{m}_{-2}$ such that, for any $e \in E(\Gamma)$ with $e = \overrightarrow{v_1 v_2}$, we have

es_vertices

$$X_{\overrightarrow{v_1 v_2}} := [X_{v_1}, X_{v_2}].$$

In this case, the basis $\{X_s\}_{s \in \{V(\Gamma) \cup E(\Gamma)\}}$ of $\mathfrak{m}$ is also said to be *compatible* with the graph Γ.

def_type_r **Definition 2.6.** Let Γ be a graph with m vertices. If there exists $v \in V(\Gamma)$ with outgoing index $r > 0$ and Γ has $m - r - 1$ isolated vertices (i.e., vertices with total index 0), then we say that the graph Γ is *normal of degree* r with *root* v.

Remark 2.7. Note that the the root v in (2.6) is unique and necessarily has incoming index 0 since Γ is acyclic.

reduce **Definition 2.8.** If Γ is normal and $v \in V(\Gamma)$ a root, then the *reduction of* Γ *by* v is the graph $R_v(\Gamma)$ obtained from Γ by the deletion of the vertex v, the edges $\mathrm{Out}(v)$, and the isolated vertices of Γ.

creduce **Definition 2.9.** The graph Γ is called *normalizable* if there exists vertices $v_0, v_1, \dots, v_\omega$ of Γ and positive integers $(r_0, r_1, \dots r_\omega)$ such that the following hold:

(1) The graph Γ is normal of degree r_0 with root v_0.
(2) For $1 \le j \le \omega$, the graph $R_{v_{j-1}} \circ R_{v_{j-2}} \circ \cdots \circ R_{v_0}(\Gamma)$, obtained from Γ by successive reductions $R_{v_0}, \dots, R_{v_{j-1}}$, is normal of degree r_j with root v_j.
(3) The set of edges of the graph $R_{v_\omega} \circ R_{v_{\omega-1}} \circ \cdots \circ R_{v_0}(\Gamma)$ is empty.

root_def **Definition 2.10.** If a graph Γ is normalizable, then we call the numbers $(r_0, r_1, \dots r_\omega)$ the *characteristic tuple* of Γ. The vertex v_j from item (2) of Definition 2.9 is called the j*-th root* of Γ and the tuple $\{v_j\}_{j=0}^{\omega}$ is called the *tuple of roots* of Γ.

3. Construction of the Adapted Frame

Let $I_0(\Gamma) \subset V(\Gamma)$ denote the set of vertices of Γ which are deleted from Γ by R_{v_0}. Similarly, for $1 \le j \le \omega$, let $I_j(\Gamma) \subset V(\Gamma)$ denote the set of vertices which are deleted from $R_{v_{j-1}} \circ R_{v_{j-2}} \circ \cdots \circ R_{v_0}(\Gamma)$ to obtain $R_{v_j} \circ R_{v_{j-1}} \circ \cdots \circ R_{v_0}(\Gamma)$, and , finally, let $I_{\omega+1}$ be the set of vertices of $R_{v_\omega} \circ R_{v_{\omega-1}} \cdots \circ R_{v_0}(\Gamma)$. Note that

union

$$\bigcup_{j=0}^{\omega+1} I_j(\Gamma) = V(\Gamma).$$

ertex_depth **Definition 3.1.** We say that a vertex of a normalizable graph Γ belonging to $I_j(\Gamma)$ is a vertex of *depth* j. We denote by $\mathrm{depth}(v)$ the depth of a vertex v in Γ.

r_depth_rem *Remark* 3.2. By construction, the number r_p from the characteristic tuple is equal to the number of vertices of depth greater than p.

Using the constructions above, we may now state the main result of this section.

ccreduce **Theorem 3.3.** *If* $\mathfrak{m}$ *is a step-2 Lie algebra with characteristic tuple* $(r_0, \dots, r_\omega)$ *and tuple of roots* $\{v_j\}_{j=0}^{\omega}$*, then* $\mathfrak{m}$ *is compatible with a normalizable graph* Γ *with characteristic tuple* $(r_0, \dots, r_\omega)$*. Moreover, a basis* $\{X_s\}_{s \in \{V(\Gamma) \cup E(\Gamma)\}}$*, compatible with* Γ*, can be chosen such that*

$$\mathrm{span}\left\{ X_s : s \in \bigcup_{k=0}^{j} \big(I_k(\Gamma) \cup \mathrm{Out}(v_k)\big) \right\} = \mathfrak{h}_j, \quad 0 \le j \le \omega,$$

where the filtration $\{\mathfrak{h}_j\}_{j=0}^{\omega+1}$ *of* $\mathfrak{m}$ *is associated with the tuple* $\big\{(\varphi_{j-1} \circ \ldots \circ \varphi_0)(X_{v_j})\big\}_{j=0}^{\omega}$*. Furthermore, if* $\mathfrak{m}_{-1}$ *is endowed with an inner product, then the basis* $\{X_s\}_{s \in \{V(\Gamma) \cup E(\Gamma)\}}$ *can be chosen such that* $\{X_v\}_{v \in V(\Gamma)}$ *is an orthonormal basis of* $\mathfrak{m}_{-1}$*.*

Proof. Suppose that $\mathfrak{m}_{-1}$ is endowed with an inner product and consider the ad-maximal tuple $\{Y_j\}_{j=0}^{\omega}$ as above. Set $X_{v_0} = Y_0$ and complete $\{X_{v_0}\}$ to an orthonormal basis $\{X_v\}_{v\in I_0(\Gamma)}$ of $\ker \mathrm{ad} X_{v_0}$. There is an isomorphism

$$\psi_0 : (\ker \mathrm{ad} X_0)^{\perp} \to \mathfrak{m}_{-1}^{1}$$

such that $\varphi_0 \circ \psi_0^{-1} = \mathrm{id}|_{\mathfrak{m}_{-1}^{1}}$. The isomorphism ψ_0 induces an inner product on $\mathfrak{m}_{-1}^{1}$ (i.e., the inner product which makes ψ_0 an isometry). Thus, we have a splitting

$$\mathfrak{m}_{-1}^{1} = \langle Y_1\rangle \oplus \big(\langle Y_1\rangle^{\perp} \cap \ker \mathrm{ad} Y_1\big) \oplus (\ker \mathrm{ad} Y_1)^{\perp},$$

where $\perp$ denotes the orthogonal complement with respect to the induced inner product on $\mathfrak{m}_{-1}^{1}$. Complete $\{Y_1\}$ to an orthonormal basis of $\ker \mathrm{ad} Y_1$ and let $\{X_v\}_{v\in I_1(\Gamma)}$ denote the preimage of this basis under ψ_0 such that $X_{v_1} = \psi_0^{-1}(Y_1)$. Proceeding inductively, we construct an orthonormal basis of $\mathfrak{m}_{-1}$ indexed by $V(\Gamma)$ such that the following hold:

(1) The vectors $\{X_{v_j}\}_{j=0}^{\omega}$ are the preimages in $\mathfrak{m}_{-1}$ of the ad-maximal elements $\{Y_j\}_{j=0}^{\omega}$ under the respective isomorphisms $\{\psi_k\}$ and are indexed by the roots $v_0, v_1, \dots, v_\omega \in V(\Gamma)$(we also refer to such X_{v_j} as ad-maximal elements of $\mathfrak{m}$).
(2) For $1 \le j \le \omega$, the vectors $\{X_v\}_{v\in I_j(\Gamma)}$ are the preimages in $\mathfrak{m}_{-1}$ of the chosen orthonormal basis of $\ker \mathrm{ad} Y_j$ under the respective isomorphisms $\{\psi_k\}$.
(3) The vectors $\{X_v\}_{v\in I_{\omega+1}(\Gamma)}$ are the preimages of the chosen orthonormal basis of $\mathfrak{m}_{-1}^{\omega+1}$ under the respective isomorphisms $\{\psi_k\}$

For each root v_i of Γ, if $\overrightarrow{v_i, v} \in \mathrm{Out}(v_i)$, then we associate the vector

out (3.1) $$X_{\overrightarrow{v_i,v}} = [X_{v_i}, X_v]$$

to the edge $\overrightarrow{v_i, v}$. Note that the vectors X_e with $e \in \bigcup_{j=0}^{\omega} E_j(\Gamma)$ form a basis of $\mathfrak{m}_{-2}$ by construction. In this way, the construction of the required basis $\{X_s\}_{s\in\{V(\Gamma)\cup E(\Gamma)\}}$ is completed. $\square$

The proof of Theorem 3.3 shows that any the step-2 Lie algebra $\mathfrak{m}$ is compatible with the normalizable graph $\Gamma(\mathfrak{m})$ which is uniquely determined, up to isomorphism, by its characteristic tuple.

graph_m **Definition 3.4.** The normalizable graph $\Gamma(\mathfrak{m})$ is called the normalizable graph of the step-2 algebra $\mathfrak{m}$.

In what follows, we will label the roots $v_0, v_1, \dots, v_\omega$ by $\rho_0, \rho_1, \dots, \rho_\omega$. When there are multiple graphs indexed by a certain indexing set $\mathcal{A}$, say $\{\Gamma_i\}_{i\in\mathcal{A}}$, we will label the roots of a given indexed graph Γ_i by $\rho_0^i, \dots, \rho_{\omega_i}^i$ to distinguish its roots from those of other graphs in the collection.

Notice that the above construction implies that

decrease (3.2) $$\mathrm{ad} X_{\rho_j}(\mathfrak{h}_j) \subset \mathfrak{h}_{j-1}.$$

for any root ρ_j (here we set $\mathfrak{h}_{-1} = 0$).

ad_rem1 *Remark* 3.5. Let $\{X_s\}_{s\in\{V(\Gamma)\cup E(\Gamma)\}}$ be a basis of $\mathfrak{m}$ as in Theorem 3.3. If v is a vertex of depth p in Γ, then, from the filtration (2.8) by ideals and the definition of $\mathfrak{h}_j$ as in (2.7), it follows that

d_X_depth_p (3.3) $$\mathrm{Im}\ \mathrm{ad} X_v \subset \bigoplus_{j=0}^{p} \mathrm{Im}\ \mathrm{ad} X_{\rho_j}.$$

4. The Tanaka Symbol and The Fundamental Algebraic System

Now we specialize to the case when $\mathfrak{m}(q) = \mathfrak{m}_{-1}(q) \oplus \mathfrak{m}_{-2}(q)$ is the Tanaka symbol at a point $q \in M$ of a step-2, bracket generating distribution D.

Definition 4.1. Given two distinct sub-Riemannian metrics g and h on D, we define the *transition operator* to be the linear map $S : D(q) \to D(q)$ such that $g(X, Y) = h(SX, Y)$.

Now, if D admits a pair of projectively equivalent sub-Riemannian metrics g_1, g_2 and the transition operator S has a constant number of distinct eigenvalues in a neighborhood of q, then [6, Proposition ??] shows that the Tanaka symbol $\mathfrak{m}$ of D is the direct sum of step-2 Lie algebras:

igen_decomp

$$\mathfrak{m}(q) = \bigoplus_{i=1}^{k} \mathfrak{m}^i(q). \tag{4.1}$$

In particular, each summand is of the form $\mathfrak{m}^i(q) = \mathfrak{m}^i_{-1}(q) \oplus \mathfrak{m}^i_{-2}(q)$ with

$$\mathfrak{m}^i_{-1}(q) = D_i(q)$$

$$\mathfrak{m}^i_{-2}(q) = D_i^2(q)/(D_i^2(q) \cap D(q)),$$

where D_i is the eigenspace of D with respect to the transition operator S which corresponds to the eigenvalue α_i. It is important to note that $\mathfrak{m}^i(q)$ is not *a priori* equivalent to the Tanaka symbol of D_i at q since they differ in their definition of -2 components (though, it will be shown later that they necessarily coincide). Before proceeding, we need to make a certain regularity assumption about the symbol $\mathfrak{m}$.

reg_assump

Assumption 4.2. Assume that we work on an open set $\mathcal{U} \subset M$ such that for all $1 \leq i \leq k$, the characteristic tuples of the algebras $\mathfrak{m}^i(q)$ are constant for all $q \in \mathcal{U}$.

Now, let Γ_i be the normalizable graph of $\mathfrak{m}^i(q)$, i.e. $\Gamma_i := \Gamma\big(\mathfrak{m}^i(q)\big)$ and let

raph_decomp

$$\Gamma = \bigsqcup_{i=1}^{k} \Gamma_i. \tag{4.2}$$

mponent_def

Definition 4.3. We say that the graph Γ_i is the *ith component of the graph* Γ with respect to decompositon (4.1). [3].

We are now in a position to state the main theorem of this paper which will be proved in the following sections:

mainthm

Theorem 4.4. *Assume that (M, D, g) is a sub-Riemannian structure such that D is a bracket generating, step 2 distribution whose Tanaka symbol satisfies Assumption 4.2. Then the following statements hold:*

1. *If (M, D, g) is not affinely rigid, then it admits a product structure (locally), i.e. the Eisenhart type theorem is valid in this case;*
2. *If (M, D, g) admits a projectively equivalent sub-Riemannian metric which is not conformal to it, then the distribution D admits a product structure (locally).*

orb_sec

5. Orbital equivalence and fundamental algebraic system

In general, there are two types of Pontryagin extremals for optimal control problems, *normal* and *abnormal* ([1, 2]): for the former, the Lagrange multiplier near the functional is non-zero, and, for the latter, it is zero. In particular, abnormal extremals, as unparametrized curves, depend only on the distribution D and not on a metric g defined thereon. This indicates that only normal extremals are essential for the considered problems of affine/projective equivalence (see Proposition 5.2 for the precise formulation). Therefore, we give an explicit description of normal extremals only. They are the integral curves of the Hamiltonian vector field $\vec{h}$ on T^*M corresponding to the Hamiltonian

Hamiltonian

$$h(p, q) = ||\, p|_{D(q)}||^2, \quad q \in M, p \in T_q^*M, \tag{5.1}$$

and lying on a nonzero level set of h. Here $||\, p|_{D(q)}||$ denotes the operator norm of the functional $p|_{D(q)}$, i.e.,

[3] Note that the notion of component here is completely different from a connected component because the graphs Γ_i are not connected in general: they may have isolated vertices (equivalently, the vertices of depth 0).

$$||\, p|_{D(q)}|| = \max\{p(v) : v \in D(q), g(q)(v,v) = 1\}.$$

The Hamiltonian h defined by (5.1) is called the *Hamiltonian, associated with the metric g* or shortly the *sub-Riemannian Hamiltonian.* In [6], following [12], the problems of projectively and affine equivalence of sub-Riemannian metric were reduced to the study of the orbital equivalence of the corresponding sub-Riemannian Hamiltonian systems for normal Pontryagin extremals of the energy minimizing problem (1.1), which in turn is reduced to the study of solvability of a special linear algebraic system with coefficients being polynomial in the fibers, called the *fundamental algebraic system* ([6, Proposition 3.10]). In this section we summarize all information from [6] we need for the proof of Theorem 4.4.

As before, fix a connected manifold M and a bracket-generating equiregular distribution D on M, and consider two sub-Riemannian metrics g_1 and g_2 on (M, D). We denote by h_1 and h_2 the respective sub-Riemannian Hamiltonians of g_1 and g_2, as defined in (5.1). Let the annihilator $D^\perp$ of D in T^*M be defined as follows:

perp

$$D^\perp = \{(p,q) \in T^*M : p|_{D(q)} = 0\}$$

It coincides with the zero level set of the sub-Riemannian Hamiltonian h from (5.1).

Definition 5.1. We say that $\vec{h}_1$ and $\vec{h}_2$ are *orbitally diffeomorphic* on an open subset V_1 of $T^*M\backslash D^\perp$ if there exists an open subset V_2 of $T^*M\backslash D^\perp$ and a diffeomorphism $\Phi : V_1 \to V_2$ such that Φ is fiber-preserving, i.e., $\pi(\Phi(\lambda)) = \pi(\lambda)$, and Φ sends the integral curves of $\vec{h}_1$ to the reparameterized integral curves of $\vec{h}_2$, i.e., there exists a smooth function $s = s(\lambda, t)$ with $s(\lambda, 0) = 0$ such that $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$ for all $\lambda \in V_1$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined. Equivalently, there exists a smooth function $a(\lambda)$ such that

orbeq

$$d\Phi\, \vec{h}_1(\lambda) = a(\lambda)\vec{h}_2(\Phi(\lambda)). \tag{5.2}$$

The map Φ is called an *orbital diffeomorphism* between the extremal flows of g_1 and g_2.

The reduction of projective (respectively, affine) equivalence of sub-Riemannian metrics to the orbital (respectively, a special form of orbital) equivalence of the corresponding sub-Riemannian Hamiltonian systems is given by the following:

orb.to.proj

Proposition 5.2. [6, a combination of Proposition 3.4 and Theorem 2.10 there] *Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $U \subset M$ and let $\pi : T^*M \to M$ be the canonical projection. Then, for generic point $\lambda_1 \in \pi^{-1}(U)\backslash D^\perp$, $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood V_1 of λ_1 in T^*M. Moreover, if g_1 and g_2 are affinely equivalent in a neighborhood $U \subset M$, then the function $a(\lambda)$ in* (5.2) *satisfies $\vec{h}_1(a) = 0$.*

By Theorem 3.3 and our assumption on the constancy of the characteristic tuple, it follows that we can choose a smooth frame $\{X_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ in $\mathcal{U}$ such that if

pr_def

$$\mathrm{pr}_q : T_qM \to T_qM/D(q) \tag{5.3}$$

is the canonical projection, then for every $1 \le i \le k$ and $q \in \mathcal{U}$ the tuple

$$\big\{\{X_v(q)\}_{v\in V(\Gamma_i)}, \{\mathrm{pr}_q\big(X_e(q)\big)\}_{e\in E(\Gamma)}\big\}$$

is a compatible basis of $\mathfrak{m}_i(q)$ with respect to the graph Γ_i and

inorm_basis

$$[X_v, X_w] = X_{\overrightarrow{vw}}, \qquad \text{for every } v, w \in V(\Gamma) \text{ such that } \overrightarrow{vw} \in E(\Gamma). \tag{5.4}$$

We call such frame a *quasi-normal frame adapted to the pair of sub-Riemannian metrics g_1 and g_2*, or, shortly a *quasi-normal frame.*

In the sequel we will work with quasi-normal frames only. The *structure functions* of the frame $\{X_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ are the real-valued functions c_{sv}^w, $s, v, w \in V(\Gamma) \cup E(\Gamma)$ defined near q by

ct_function

$$[X_s, X_v] = \sum_{w\in V(\Gamma)\cup E(\Gamma)} c_{sv}^w X_w. \tag{5.5}$$

In particular, from (5.4) it follows that

t_quasinorm
$$\text{if } \overrightarrow{vw} \in E(\Gamma), \text{ then } c^{e}_{v,w} = \begin{cases} 1 & \text{if } e = \overrightarrow{vw}, \\ 0 & \text{if } e \in V(\Gamma) \cup \big(E(\Gamma)\backslash\{\overrightarrow{vw}\}\big) \end{cases} \tag{5.6}$$

ad_rem2 *Remark* 5.3. Let $\{X_s\}_{s\in\{V(\Gamma)\cup E(\Gamma)\}}$ be a quasi-normal frame. By Remark 3.5 (especially relation (3.3)) and the decomposition (4.1) of the Tanaka symbol $\mathfrak{m}$, if v as a vertex of depth p in Γ_i, it follows that

X_depth_p_i
$$\text{Im ad}X_v \subset \bigoplus_{j=0}^{p} \text{Im ad}X_{\rho_j}|_{D_i} \mod D. \tag{5.7}$$

The quasi-normal frame also induces a frame on each fiber of $T^*\mathcal{U}$. To show this, we first define the following functions called *quasi-impulses*:

ui
$$u_s(q,p) = p\left(X_s(q)\right), \quad s \in V(\Gamma) \cup E(\Gamma),\, q \in \mathcal{U},\, p \in T^*_q\mathcal{U}.$$

Given $q \in \mathcal{U}$, the tuple $\{u_s(q,\cdot)_{s\in V(\Gamma)\cup E(\Gamma)}\}$ forms a coordinate system on the fiber $T^*_q\mathcal{U}$ with corresponding coordinate frame $\{\partial_{u_s}\}_{s\in V(\Gamma)\cup E(\Gamma)}$ on the fibers of $T^*\mathcal{U}$. Finally, lift X_s to the local section Y_s of T^*M so that $\pi_* Y_s = X_s$ (where π denotes the canonical projection) and $du_w(Y_s) = 0$ for any $w \in V(\Gamma) \cup E(\Gamma)$. The collection $\{Y_s, \partial_{u_s}\}_{s\in V(\Gamma)\cup E(\Gamma)}$ constitutes a frame on $T^*\mathcal{U}$.

The expression for the sub-Riemannian Hamiltonian then becomes

$$h_1 = \frac{1}{2} \sum_{v\in V(\Gamma)} u_v^2$$

Therefore,

eq:vech
$$\vec{h}_1 = \sum_{v\in V(\Gamma)} u_v \vec{u}_v = \sum_{v\in V(\Gamma)} u_v Y_v + \sum_{v\in V(\Gamma)} \sum_{w,s\in V(\Gamma)\cup E(\Gamma)} c^s_{vw} u_v u_s \partial_{u_w}. \tag{5.8}$$

The proof of (5.8) follows from the relations

poison1
$$\overrightarrow{H_{Z_1}}(H_{Z_2}) = d\,H_{Z_2}\big(\overrightarrow{H_{Z_1}}\big) = H_{[Z_1,Z_2]}. \tag{5.9}$$

Where $H_Z(p,q) := p(Z(q)), p \in T^*_q M$. From this and (5.5), it follows immediately that

Ham_lift
$$\vec{u_v} = Y_v + \sum_{w\in V(\Gamma)\cup E(\Gamma)} \vec{u_v}(u_w)\partial_{u_w} = Y_v + \sum_{w,s\in V(\Gamma)\cup E(\Gamma)} c^s_{vw} u_s \partial_{u_w},$$

which, in turn, implies (5.8).

Define the function $\alpha : V(\Gamma) \cup E(\Gamma) \to \mathbb{R}$ as follows:

alpha_of
$$\alpha(l) := \alpha_i, \quad \forall l \in V(\Gamma_i) \cup E(\Gamma_i), \tag{5.10}$$

where, as before, α_i denotes the eigenvalue of the eigenspace corresponding to the graph Γ_i (i.e., D_i).

If $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic via Φ near $\lambda_0 \in H_1 \cap \pi^{-1}(q_0)$, then we denote by $\Phi_w := u_w \circ \Phi(\lambda), w \in V(\Gamma) \cup E(\Gamma)$ the coordinates of Φ on a given fiber. Then it is easy to see [12, Lemma 1] that the function α from (5.2) satisfies

eq:2
$$a^2 = \frac{\sum\limits_{v\in V(\Gamma)} \alpha(v)^2 u_v^2}{\sum\limits_{v\in V(\Gamma)} u_v^2} \tag{5.11}$$

and

$$\Phi_v = \frac{\alpha(v)^2 u_v}{a}, \forall v \in V(\Gamma).$$

We find the remaining Φ_w, for the edges $w \in E(\Gamma)$, by successive differentiation and subsequent substitution for the derivative of Φ given by the orbital diffeomorphism condition. This prolongation procedure produces an algebraic system (5.14) called *the fundamental algebraic system*, which we describe in the following proposition:

fda

Proposition 5.4. *(*[6, a combination of Proposition 3.4, Proposition 3.10, Proposition 4.2, Lemma 4.4] *and Appendix A) Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $\mathcal{U} \subset M$ and let Φ be the corresponding orbital diffeomorphism between the normal extremal flows of g_1 and g_2 with coordinates Φ_w, $w \in V(\Gamma) \cup E(\Gamma)$. Set*

$$\widetilde{\Phi}_w = a\Phi_w, \quad \forall\, w \in E(\Gamma),$$

qjk

$$q_{ws} = \sum_{v\in V(\Gamma)} c^s_{vw} u_v, \quad w, s \in V(\Gamma) \cup E(\Gamma), \tag{5.12}$$

$$Q = \sum_{w\in V(\Gamma)} \frac{X(\alpha(w)^2)}{\alpha(w)^2} u_w$$

and

RJ

$$R_w = \alpha(w)^2 \vec{h}_1(u_w) - \sum_{s,v\in V(\Gamma)} \alpha(s)^2 c^s_{vw} u_v u_s.\text{[4]} \tag{5.13}$$

Then $\widetilde{\Phi}$ satisfies a linear system of equations,

A.phi.B

$$A\widetilde{\Phi} = b, \tag{5.14}$$

where A is a matrix with $|E(\Gamma)| = n - m$ columns and an infinite number of rows, and b is a column vector with an infinite number of rows. These infinite matrices can be decomposed in layers of $|V(\Gamma)| = m$ rows each as

eq:A_and_b

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^p \\ \vdots \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^p \\ \vdots \end{pmatrix}, \tag{5.15}$$

where the coefficients a^p_{vw} of the $(m \times (n-m))$ matrix A^p, $p \in \mathbb{N}$, are defined by induction as

elem.A

$$\begin{cases} a^1_{v,w} = q_{vw}, & v \in V(\Gamma),\ w \in E(\Gamma), \\ a^{p+1}_{v,w} = \vec{h}_1(a^p_{vw}) + \displaystyle\sum_{l\in E(\Gamma)} a^p_{v,l} q_{lw}, & v \in V(\Gamma),\ w \in E(\Gamma), \end{cases} \tag{5.16}$$

and the entries b^p_v, $v \in V(\Gamma)$, of the vector $b^p \in \mathbb{R}^m$ are defined by

.b.rec.form

$$\begin{cases} b^1_v = R_v, \\ b^{p+1}_v = \vec{h}_1(b^p_v) - \dfrac{\vec{h}(a)}{a} b^p_v - \displaystyle\sum_{l\in E(\Gamma)} \sum_{w\in V(\Gamma)} a^p_{v,l} u_w \left(\alpha(w)^2 q_{lw} + \frac{X_l\big(\alpha(w)^2\big)}{2} \cdot u_w \right). \end{cases} \tag{5.17}$$

Finally, by [6, Proposition 4.2 and Lemma 4.4], $\dfrac{\vec{h}(a)}{a}$ *is a linear form on the fibers T^*M depending on u_v with $v \in V(\Gamma)$ only.*[5]

---

[4]In previous papers ([6], [8]), the form of R_w differed significantly between the affinely equivalent and projectively equivalent cases, however, a lengthy calculation (included in the AppendixA below) shows that they are both equal to what is presented here.

[5]In more details, in [6, Proposition 4.2 and Lemma 4.4] it was shown that $\dfrac{\vec{h}(a^2)}{a^2}$ has this property, called the *first divisibility condition*, following [12]. However, the same is true for $\dfrac{\vec{h}(a)}{a} = \dfrac{\vec{h}(a^2)}{2a^2}$.

Remark 5.5. The column vector b in (5.14) here corresponds to $a \cdot b$ in the notations of the fundamental algebraic system in [6, Proposition 3.10]. Similar replacement of b with $a \cdot b$ was made in [8, Proposition 3.7] in the affine case in which $\vec{h}(a) = 0$. In the general projective case, $\vec{h}(a)$ is not necessarily zero. And additional term appears here compared to [8, Proposition 3.7] on the right-hand side of the second line of (5.17). The reason for redefining b in this way compared to [6] is that all components of the new b are polynomials on the fibers of T^*M. Despite the fact the the term $\dfrac{\vec{h}(\alpha)}{\alpha}$ appears rational, this term is still a polynomial on the fibers of T^*M by the first divisibility condition mentioned in the last footnote.

The sub-matrix A^p of the matrix A as in (5.15) will be called the *pth layer of* A and the subcolumn b^p of the coulumn b there will be called the *pth layer of* b.

The rows (columns) of A^p are indexed by vertices (respectively edges) of Γ. In the sequel, for brevity, we call the row (respectively, the column) of A^p indexed by a vertex v of Γ the *v-row* of $\mathfrak{A}^p$.

block_q *Remark* 5.6. From the decomposition (4.1) and the formula (5.12) it follows that if $w \in V(\Gamma_i)$ and $l \in E(\Gamma_j)$ for $i \neq j$, then $q_{vw} = 0$. In fact, later we will prove much more general statement regarding higher layers, see Corollary 6.11, item (1).

The matrix A has $n - m$ columns and infinitely many rows and b is the infinite-dimensional column vector. So, the fundamental algebraic system (5.14) is an over-determined linear system on $(\Phi_{m+1}, \dots, \Phi_n)$, and by (5.16), (5.17), and the last sentence of Proposition 5.4 all entries of A and b are polynomials in u-variables , i.e., on the fibers of T^*M. Therefore, all $(n - m + 1) \times (n - m + 1)$ minors of the augmented matrix $[A|b]$ must be equal to zero. Since all of the minors are polynomials in u_j's, the coefficient of every monomial of these polynomials must be equal to zero. The result is a large collection of constraints on the structure coefficient c_{ij}^k. By discovering and analyzing the monomials with the "simplest" coefficients, and since they necessarily must vanish by the previous comments, we are able to prove our main Theorem 4.4.

6. Proof of the main theorem

In the fist step of the proof, we will show that the coefficients of monomials of minimal degree with respect to u-variables indexed by vertices of Γ in the $(n-m+1) \times (n-m+1)$ minors of the augmented matrix $[A|b]$ are equal to the coefficients of the same monomials in the corresponding minors of a more simple augmented matrix, see Corollary6.7 below. This is important because as will be shown in the later stages of the proof, in order to proof our main Theorem 4.4 it is enough to consider the coefficients of these monomials only.

To begin with, let us introduce some terminology and notation. Let U be the tuple

$$U = \{u_s\}_{s \in V(\Gamma) \cup E(\Gamma)}$$

U-poly **Definition 6.1.** A *U-polynomial* is a polynomial in the variables of the tuple U with coefficient which are smooth functions on an open set $\mathcal{U}$ of the manifold M, where the affine equivalence or our sub-Riemannian metrics g_1 and g_2 holds and the local quasi-normal frame $\{X_s\}_{s \in V(\Gamma) \cup E(\Gamma)}$ is defined.

The degree of a monmial of a U-polynomial and the notion of homogeneity is defined in an obvuious way.

mogen_total *Remark* 6.2. Since U-polymoials q_{vs} and R_w are linear and quadratic, respectively, from recursive formulas (5.16) and (5.17) it follows by induction that the entries of A^p and b^p are homogeneous U-polynomials of degree p and degree $p + 1$, respectively.

var_def **Definition 6.3.** The variable u_s is called a *vertex variable* if $s \in V(\Gamma)$ and and an *edge variable* if $s \in E(\Gamma)$. A vertex (respectively, ad edge) variable u_s is called *Γ_i-vertex variable* if $s \in V(\Gamma_i)$ (respectivly, an *Γ_i-edge variable* if $s \in E(\Gamma_i)$).

Definition 6.4. Given a U-monomial define its *vertex degree* (respectively, *edge degree*) as the number of vertex (respectively edge) variables factors in it counting repetitions. In the same way one can define Γ_i -vertex (respectively, Γ_i-edge) degree of a monomial. A U-polynomial f is called vertex (respectively, edge) homogeneous of degree l if all its monomials have l vertex degree l (respectively, edge degree l) variables. In the same way, we define Γ_i-vertex (respectively, Γ_i-edge) homogeneity.

Any U- polynomial f can be uniquely decomposed into vertex homogeneous polynomials:

$$f = \sum_{j=0}^{\mu} f_{(j)},$$

for some nonnegative integer μ, where $f_{(j)}$ a vertex homogeneous of degree j. We call $f_{(j)}$ the *jth vertex-homogeneous component of f*. We say that $j \geq 0$ is the *minimal vertex degree of f* if j is the minimal nonnegative integer such that $f_{(j)}(U) \neq 0$[6]. In this case, $f_{(j)}$ is called the *principal homogeneous part of f*.

Maybe the text in red is not needed.

Given a matrix L with entries being U-polynomials denote by $L_{(j)}$ the matrix with entries equal to j vertex homogeneous components of the corresponding entries of L. The matrix $L_{(j)}$ will be called the *j vertex homogeneous part of the matrix L*. The notion of j vertex homogeneous and minimal vertex homogeneity for the matrix L with polynomial in U entries is defined in an obvious way (analogously to the same notion for a single polynomial in the previous paragraph). We also call $L_{(1)}$ (respectively $L_{(2)}$) the *linear* (respectively, *quadratic*) vertex homogeneous part of L. Finally, 1 vertex homogeneous (respectively, 2 vertex homogeneous) polynomials or matrices with polynomial entries in U will be called linearly (respectively, quadratically) vertex homogeneous.

Further, define a map $\mathcal{C} : V(\Gamma) \cup E(\Gamma) \mapsto [1 : k]$ as follows:

mponent_map

$$C(s) = i, \quad \text{if } s \in V(\Gamma_i) \cup E(\Gamma_i), \tag{6.1}$$

where Γ_i is as in (4.2).

h_1_degree

Lemma 6.5. *If j is the minimal vertex degree of a U-polynomial f (respectively, a matrix L with U-polynomial entries), then the minimal vertex degree of $\vec{h}_1(F)$ (respectively, $\vec{h}_1(L)$) is at least j. Moreover, $(\vec{h}_1(f))_{(j)}$ is equal to the sum of all possible terms obtained from the summands of $f_{(j)}$ by replacing one of their vertex variables factor by the 1-vertex homogeneous component of the action of $\vec{h}_1$ on this variable.*

Proof. First that from (5.8), (5.9), and (5.5), it easily follows that

n_vf_action

$$\vec{h}_1(u_v) = \sum_{s\in V(\Gamma)\cup E(\Gamma)} q_{vs}u_s = \sum_{j\in V(\Gamma)} \sum_{s\in V(\Gamma)\cup E(\Gamma)} c^s_{jv}u_ju_s, \tag{6.2}$$

where q_{ws} is as in (5.12). It implies that By the product rule, the action of $\vec{h}_1$ on any summand S of a U-polynomial f is equal to the sum of terms in which one of the factors of S is replaced by the action of $\vec{h}_1$ on this factor. There are three type of factors of S:

(1) a vertex variable u_v: its vertex degree is 1 and the vertex degree of $\vec{h}_1(u_v)$ is at least 1 by (6.2);
(2) and edge variable u_e: its vertex degree is 0 and the vertex degree of $\vec{h}_1(u_e)$ is at least 1 by (6.2);
(3) a function K on an open set $\mathcal{U}$ of M: The vertex degree of K is zero, but the vertex degree of $\vec{h}_1(K)$ is at least 1 by (5.8).

So, the first statement of the lemma follows from the fact that in all three cases above the minimal vertex degree is not decreased by the action of $\vec{h}_1$, and the second statemnt of the

[6]When f is identically equal to zero, the minimal vertex degree is set t be equal to ∞.

lemma follows from the fact that only in the case (1) the minimal vertex degree can be preserved by this action. □

Given $l \in E(\Gamma)$ let A_l be the column of A corresponding to the edge l. Since we are concerned with the minors of the augmented matrix $[A|b]$ of the fundamental algebraic system (5.14), we can replace the column b of this matrix with a column obtained from it by a column operation, i.e. subtracting from b any linear combination of columns of A. In order to use the vertex homogeneity the following column operation is convenient

eq:1
$$\tilde{b} = b - \sum_{l \in E(\Gamma)} \alpha^2(l) u_l A_l, \tag{6.3}$$

where $\alpha : V(\Gamma) \cup E(\Gamma) \to \mathbb{R}$ is defined as in (5.10). Let $\tilde{b}^p$ be the sub-column of b corresponding to the pth layer. The following lemma and its corollary justify the introduced terminology

in_quad_lem **Lemma 6.6.** *If A and b are as in the fundamental algebraic system* (5.14) *and $\tilde{b}$ is as* (6.3), *then the minimal vertex degree of A is* 1, *and the minimal vertex degree of $\tilde{b}$ is* 2.

Proof. The proof is by induction on p-layers A^p and $\tilde{b}^p$ of A and b:

For the first layer A^1 of A from (5.16) and (5.12)it follows that A^1 is linearly vertex homogeneous or zero [7], which implies the statement of the Lemma for A^1.

We probably should exclude the case $D = TM$ from the very beginning and we consider "proper" step-2 distributions.

The statement for the first layer $\tilde{b}^1$ of $\tilde{b}$ follows immediately from the following formula (also proved in [8], see Lemma 2 and Remark 7 there)

tb1
$$\tilde{b}^1_w = \sum_{l \in V(\Gamma)} (\alpha(w)^2 - \alpha(l)^2) q_{wl} u_l = \sum_{\substack{l, v \in V(\Gamma) \\ \mathcal{C}(l) \neq \mathcal{C}(w)}} (\alpha(w)^2 - \alpha(l)^2) c^l_{vw} u_v u_l, \tag{6.4}$$

where $\mathcal{C}$ is as in (6.1).

To prove (6.4) first, recall that $b^1_w = R_w$. Plugging (6.2) to (5.13) and then using (6.3), we obtain

tb1_0
$$\tilde{b}^1_w = \sum_{l \in V(\Gamma) \cup E(\Gamma)} (\alpha(w)^2 - \alpha(l)^2) q_{wl} u_l,$$

where we also take l such that $\mathcal{C}(l) \neq \mathcal{C}(w)$ since $\alpha(w)^2 - \alpha(l)^2 = 0$ otherwise. Moreover, Remark 5.6 implies that the terms of this sum corresponding to $l \in E(\Gamma)$ vanish, which implies (6.4).

The induction step for the $(p+1)$ layer A^{p+1} (assuming that the statement holds for the pth layer A^p) follows from the second line of (5.16) of A by using the induction hypothesis and Lemma 6.5 for the first term on the right-hand side of the equation, and, in addition to the induction hypothesis, the vertex linearity of q_{lw} from (5.12) for the second term.

Denote by $\mathfrak{V}^j$ the space of all U-polynomials whose minimal vertex degree not less that j. The induction step for the $(p+1)$ layer b^{p+1} (assuming that statement holds for both the pth layer A^p and b^p) follows from the following formula (proved below):

tbs
$$\tilde{b}^{p+1}_v = \vec{h}_1(\tilde{b}^p_v) + \sum_{l \in E(\Gamma)} \sum_{\substack{w \in E(\Gamma) \\ \mathcal{C}(w) \neq \mathcal{C}(l)}} (\alpha(l)^2 - \alpha(w)^2) a^p_{vl} q_{lw} u_w \qquad \mathrm{mod}\ \mathfrak{V}^3, \tag{6.5}$$

where $p \geq 1$. Indeed, the minimal vertex degree of the right side of (6.5) is at least 2: for the first summand it is because of Lemma 6.5 and the induction hypothesis for b^p, for the second summabd it is by the induction hypothesis for A^p and (5.12), and for the part being modded out it is by definition.

[7] the latter is in fact impossible from (5.12), if $D \neq TM$, as D is not involutive in this case.

It remains to prove the formula (6.5). From (6.3), we have $b_v^p = \tilde{b}_v^p + \sum\limits_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^p$. We apply $\vec{h}_1$ to this equation to obtain

h1bs
$$\vec{h}_1(b_v^p) = \vec{h}_1(\tilde{b}_v^p) + \vec{h}_1\left(\sum_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^p\right). \tag{6.6}$$

By the product rule,

eq:2
$$\vec{h}_1\left(\sum_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^p\right) = \sum_{l\in E(\Gamma)} \alpha(l)^2 \vec{h}_1(u_l) a_{vl}^p + \sum_{l\in E(\Gamma)} \alpha(l)^2 u_l \vec{h}_1(a_{vl}^p). \tag{6.7}$$

The term $\vec{h}_1(a_{vl}^p)$ can be replaced by $a_{vl}^{p+1} - \sum\limits_{r\in E(\Gamma)} a_{vr}^p q_{rl}$ as in the recursive definition (5.16), and, using (6.2), we obtain the following for the right side of (6.7):

eq:3
$$\sum_{l\in E(\Gamma)} \alpha(l)^2 a_{vl}^p \sum_{w\in\Gamma} q_{lw} u_w + \sum_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^{p+1} - \sum_{l,r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} \tag{6.8}$$

If we rewrite the last summation in (6.8) so that l ranges over all values in $V(\Gamma)\cup E(\Gamma)$, then we obtain

eq:4
$$-\sum_{l,r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} = -\sum_{l\in V(\Gamma)\cup E(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} + \sum_{l\in V(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl}$$

Substituting everything into (6.6), we finally obtain

$$\begin{aligned}\vec{h}_1(b_v^p) = \vec{h}_1(\tilde{b}_v^p) &+ \sum_{l\in E(\Gamma)} \alpha(l)^2 a_{vl}^p \sum_{w\in V(\Gamma)\cup E(\Gamma)} q_{lw} u_w + \sum_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^{p+1} \\ &- \sum_{l\in V(\Gamma)\cup E(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} + \sum_{l\in V(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl},\end{aligned}$$

which, after rearranging terms, can be written as

eq:6
$$\begin{aligned}&\vec{h}_1(b_v^p) - \sum_{l\in V(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} - \sum_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^{p+1} \\ &= \vec{h}_1(\tilde{b}_v^p) + \sum_{l\in E(\Gamma)} \alpha(l)^2 a_{vl}^p \sum_{w\in V(\Gamma)\cup E(\Gamma)} q_{lw} u_w - \sum_{l\in V(\Gamma)\cup E(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl}.\end{aligned} \tag{6.9}$$

From recursive formula (5.17) it follows that

$$\vec{h}_1(b_v^p) - \sum_{l\in V(\Gamma)} \sum_{r\in E(\Gamma)} \alpha(l)^2 u_l a_{vr}^p q_{rl} = b_v^{p+1} \mod \mathfrak{V}^3,$$

because $\dfrac{\vec{h}(a)}{a} b_v^p$ and $a_{v,l}^p \dfrac{X_l\big(\alpha(w)^2\big)}{2} \cdot u_w^2$, $w \in V(\Gamma)$, belong to $\mathfrak{V}^3$ by the induction hypothesis for the first term and the last sentence of Proposition 5.4 (the first divisibility condition) for the second term. Since $\sum\limits_{l\in E(\Gamma)} \alpha(l)^2 u_l a_{vl}^{p+1}$ is exactly the column operation (6.3) for the $(p+1)$st layer A^{p+1} of A, the left hand side of (6.9) is equal to $\tilde{b}_v^{p+1} \mod \mathfrak{V}^3$. After re-indexing the right side of (6.9), we obtain (6.5) (again we can take l with $\mathcal{C}(l) \neq \mathcal{C}(l)$, because $\alpha(w)^2 - \alpha(l)^2 = 0$ otherwise).

□

By Lemma 6.6 the minimal homogeneous degree of any $(n-m+1)\times(n-m+1)$ minor of the augmented matrix $(A|\tilde{b})$ is at least $m-n+2$. This immediately implies the following

frak_FAL **Corollary 6.7.** *The $(n-m+2)$ vertex homogeneous component of any $(n-m+1)\times(n-m+1)$ minor of the augmented matrix $(A|b)$ of the fundamental algebraic system is equal to the corresponding minor of $(A_{(1)})|\tilde{b}_{(2)})$.*

Corollary 6.7 shows that the analysis of coefficients of the $(n-m+2)$ vertex homogeneous component in the minors of the augmented matrix $(A|b)$ of the fundamental algebraic system can be done through the analysis of the same monomials in the minors of much more simple matrix $(A_{(1)}|\tilde{b}_{(2)})$. Because of the importance and frequency of use of matrices $A_{(1)}$ and $\tilde{b}_{(2)}$, and for simplicity of notation, we denote them by $\mathfrak{A}$ and $\tilde{\mathfrak{b}}$, respectively.

Definition 6.8. We call the system

.phi.B.frak

$$\mathfrak{A}\Psi = \tilde{\mathfrak{b}},$$

the *principal part* of the fundamental algebraic system (5.14).

As before, we write $\mathfrak{A}^p$ (respectively, $\tilde{\mathfrak{b}}^p$) for the linear vertex homogeneous component of the pth layer of the matrix $\mathfrak{A}$ (respectively, for the quadratic vertex homogeneous component of the column of $\mathfrak{b}$), the corresponding entries of $\mathfrak{A}^p$ are labeled by $\mathfrak{a}^p_{ws}$, where $w \in V(\Gamma)$ and $s \in E(\Gamma)$ (respectively, $\tilde{\mathfrak{b}}^p_w$), and we called $\mathfrak{A}$ the pth layer of $\mathfrak{A}$ (respectively, $\tilde{\mathfrak{b}}^p$) the pth layer of $\mathfrak{b}$.

Remark 6.9. Similarly to the proof of Lemma 6.6, one can show by induction on the layer p that the minimal vertex degree of the second terms in the second lines of (5.16) and (5.17) are at least two and three, respectively. Therefore, $\mathfrak{A}^p$ satisfy the following recursive relations:

elem.A.frak

$$\begin{cases} \mathfrak{a}^1_{v,w} = a^1_{v,w} = q_{vw}, & v \in V(\Gamma),\ w \in E(\Gamma), \\ \mathfrak{a}^{p+1}_{v,w} = \left(\vec{h}_1(\mathfrak{a}^p_{vw})\right)_{(1)}, & v \in V(\Gamma),\ w \in E(\Gamma), \end{cases} . \tag{6.10}$$

The recursive formulas for $\mathfrak{b}^p$ is derived in Lemma 6.13 below, formula (6.14) there.

minvert

Lemma 6.10. *If*

lin_vertex

$$Ku_{e_1}u_{e_2}\dots u_{e_{p-1}}u_v$$

is a monomial appearing in the entry $\mathfrak{a}^p_{ws}$ of the linear vertex homogeneous part $\mathfrak{A}^p$ of A^p, where $\{e_j\}_{j=1}^{p-1} \in E(\Gamma)$ and $v \in V(\Gamma)$, then the coefficient K is a sum of terms of the form

cprod

$$c^s_{v_{p-1}w}\prod_{j=1}^{p-1} c^{e_j}_{v_{j-1}v_j}, \qquad \textit{with } v_0 := v, \tag{6.11}$$

ranging over all possible vertices $v_1, \dots, v_{p-1}$.

Proof. The proof is by induction on the layer p of $\mathfrak{A}^p$. For $\mathfrak{A}^1$ the statement follows immediately from the first line of (6.10) and (5.12).

Suppose by induction that the conclusion of the lemma holds for $\mathfrak{A}^p$ with $p > 1$, and consider the entries of $\mathfrak{A}^{p+1}$, namely, the terms of the second line of (6.10). Using the expression for $\vec{h}_1$ given in (5.8), it follows that such a term comes only from application of $\vec{h}_1$ to the linear vertex homogeneous part of a^p_{vw}. By the induction hypothesis, the entry $\mathfrak{a}^p_{ws}$ is equal to the sum of the terms

induction_p

$$c^s_{v_{p-1}w}u_v\prod_{j=1}^{p-1} c^{e_j}_{v_{j-1}v_j}u_{e_j} \text{ with } v_0 := v,$$

ranging over all possible vertices $v_1, \dots, v_{p-1}$. The second line of (6.10) shows that the entry $\mathfrak{a}^{p+1}_{ws}$ is the sum of the terms of the form

$$\left(\vec{h}_1\Big(c^s_{v_{p-1}w}u_v\prod_{j=1}^{p-1} c^{e_j}_{v_{j-1}v_j}u_{e_j}\Big)\right)_{(1)} .$$

Now, the second statement of Lemma 6.5 implies that

h_1_p_lin (6.12)
$$\left(\vec{h}_1\Big(c^s_{v_{p-1}w}u_v\prod_{j=1}^{p-1}c^{e_j}_{v_{j-1}v_j}u_{e_j}\Big)\right)_{(1)} = c^s_{v_{p-1}w}\prod_{j=1}^{p-1}c^{e_j}_{v_{j-1}v_j}u_{e_j}\left(\vec{h}_1(u_v)\right)_{(1)}$$

From (6.2), it follows that

nian_vf_lin (6.13)
$$\left(\vec{h}_1(u_{v_1})\right)_{(1)} = \sum_{j\in V(\Gamma)}\sum_{s\in E(\Gamma)} c^s_{jv_1}u_ju_s$$

Substituting (6.13) to (6.12), we get

$$c^s_{v_{p-1}w}\prod_{j=1}^{p-1}c^{e_j}_{v_{j-1}v_j}u_{e_j}\left(\vec{h}_1(u_v)\right)_{(1)} = c^s_{v_{p-1}w}\prod_{j=1}^{p-1}c^{e_j}_{v_{j-1}v_j}u_{e_j}\sum_{j\in V(\Gamma),l\in E(\Gamma)} c^l_{jv}u_ju_l$$

Re-indexing $l\to e_1$, $j\to v\ (=v_0)$, $v\to v_1$, $e_j\to e_{j+1}$, and $v_j\to v_{j+1}$ for all $j\in[1:p-1]$ yields the desired form for $(p+1)$st layer and completes the induction. □

block_diag **Corollary 6.11.** *The following two statements hold:*

1. *If $w\in V(\Gamma)$ and $s\in E(\Gamma)$ such that $\mathcal{C}(w)\neq\mathcal{C}(s)$, then $\mathfrak{a}^p_{ws}=0$.*
2. *If $w\in V(\Gamma_i)$ and $s\in E(\Gamma_i)$, then $\mathfrak{a}^p_{ws}$ depends on Γ_i-vertex variables and on Γ_i-edges variables only. Moreover, it is linear in Γ_i-vertex variables and has Γ_i-edge degree equal to $p-1$.*

Proof. The corollary immediately follows from Lemma 6.10 and the fact that if the product (6.11) does not vanish, then vertices $w,v_1,\dots,v_{p-1},v$ and edges $s,e_1,\dots,e_{p-1}$ belong to the same component of the graph Γ. The later fact follows from the decomposition (4.1) of the Tanaka symbol $\mathfrak{m}$. □

ck_diag_rem *Remark* 6.12. Recalling that the rows (columns) of $\mathfrak{A}^p$ are indexed by vertices (respectively edges) of Γ, $\mathfrak{A}^p$ has a block diagonal form according to the decomposition (4.2).

ilde_b_frak **Lemma 6.13.** *For $p\geq 1$ the following recursive formula holds for any $v\in V(\Gamma)$:*

tbs_frak (6.14)
$$\tilde{\mathfrak{b}}^{p+1}_v = \left(\vec{h}_1(\tilde{\mathfrak{b}}^p_v)\right)_{(2)} + \sum_{\substack{l,w\,\in\,E(\Gamma)\\ \mathcal{C}(l)=\mathcal{C}(v),\mathcal{C}(w)\neq\mathcal{C}(l)}} (\alpha(l)^2-\alpha(w)^2)\mathfrak{a}^p_{vl}q_{lw}u_w,$$

Proof. Since by definition $\tilde{\mathfrak{b}}^{p+1}_v=(\tilde{b}^{p+1}_v)_2$ we can get (6.14) by analyzing separately two terms in the right side of the recursive formula (6.5):

1. Regarding the first term of (6.5), using Remark 6.2 and Lemma 6.6, we have $\left(\vec{h}_1(\tilde{b}^p_v)\right)_{(2)} = \left(\vec{h}_1(\tilde{\mathfrak{b}}^p_v)\right)_{(2)}$, as action of $\vec{h}_1$ on components of vertex degree greater than 2 of $\tilde{b}^p_v$ does not contribute to the quadratic vertex homogeneous component of $\vec{h}_1(\tilde{b}^p_v)$.
2. Regarding the second term of (6.5), since the minimal vertex degree of a summand $(\alpha(l)^2-\alpha(w)^2)\mathfrak{a}^p_{vl}q_{lw}u_w$ with $w\in V(\Gamma)$ is at least 3 we get

ssb (6.15)
$$\left(\sum_{l\in E(\Gamma)}\sum_{\substack{w\,\in\,V(\Gamma)\cup E(\Gamma)\\ \mathcal{C}(w)\neq\mathcal{C}(l)}}(\alpha(l)^2-\alpha(w)^2)a^p_{vl}q_{lw}u_w\right)_{(2)} = \sum_{\substack{l,w\,\in\,E(\Gamma)\\ \mathcal{C}(l)=\mathcal{C}(v),\\ \mathcal{C}(w)\neq\mathcal{C}(l)}}(\alpha(l)^2-\alpha(w)^2)a^p_{vl}q_{lw}u_w$$

   The condition $\mathcal{C}(l)=\mathcal{C}(v)$ in the summation in (6.15) follows from item (1) of Corollary 6.11.

Combining both items, we get (6.14). □

In the sequel, we need the following proposition, which is Proposition 11 from [8], adapted to the notation and terminology of the present paper. Originally the properties there were proved in the case of projective equivalence in [12, Proposition 6] as consequences of the so-called *first divisibility condition*, proved in full generality in [6], and Proposition 11 in [8] is the version of them in the case of affine equivalence (see the footnotes in item below).

first divi **Proposition 6.14.** *Suppose g_1 and g_2 are non-conformal, projectively equivalent sub-Riemannian metrics on a distribution D. If $\{X_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ is an adapted frame with respect to this pair and $\mathcal{C}$ as in* (6.1)*, then the following hold:*

(1) $X_v\left(\frac{\alpha(w)^2}{\alpha(v)^2}\right) = 2c^w_{vw}\left(1-\frac{\alpha(w)^2}{\alpha(v)^2}\right)$ *for $v, w \in V(\Gamma)$*
(2) $X_v\left(\frac{\alpha(w)^2}{\alpha(v)}\right) = 0$ *whenever $\mathcal{C}(w) \neq \mathcal{C}(v)$*
(3) $c^l_{vw} = -c^w_{vl}$ *whenever $v, w, l \in V(\Gamma)$ such that $\mathcal{C}(w) = \mathcal{C}(l) \neq \mathcal{C}(v)$ and $w \neq l$.*
(4) $(\alpha(v)^2-\alpha(w)^2)c^l_{vw}+(\alpha(w)^2-\alpha(l)^2)c^v_{wl}+(\alpha(v)^2-\alpha(l)^2)c^w_{vl} = 0$ *whenever $v, l, w \in V(\Gamma)$ are pairwise distinct vertices.*

Note that item (3) above is the consequence of item (4) applied to the case when one pair (for example l and w) of vertices in the triple $\{l, v, w\}$ belongs to the same component of the graph Γ (see Definition 4.3 above)

In the sequel, we will assume that the relations given in items (1)-(3) of Proposition 6.14 hold. The following lemma is the generalization of Lemma 3 from [8], where it is proved for components of the first layer $\tilde{\mathfrak{b}}^1$ of $\tilde{\mathfrak{b}}$ only.

bu2 **Lemma 6.15.** *A monomial appearing in an entry of $\tilde{\mathfrak{b}}$ cannot depend on u-variables indexed by vertices and edges of a single component of Γ only.*

Proof. We will prove the lemma by induction on the layer p. For $p = 1$ by (6.4), the coefficient of the monomial $u_v u_l$ in the entry $\tilde{\mathfrak{b}}^1_w$ is equal to

b1tilde (6.16)
$$\tilde{\mathfrak{b}}^1_w = (\alpha(w)^2-\alpha(l)^2)(c^l_{vw}+c^v_{lw}),$$

if $v \neq l$, and equal to the half of it if $v = l$. Clearly, if w and l belong to the same component Γ_i, then this coefficient vanishes since $(\alpha(w)^2-\alpha(l)^2) = 0$ in that case. Now if v and l belong to the same component of Γ which is different from the component of w, then (6.16) vanishes since $c^l_{vw}+c^v_{lw} = 0$ by items (1) and (2) of Proposition 6.14.

Now assume by induction that the statement of the lemma holds for $\tilde{\mathfrak{b}}^p$ with $p \geq 1$ and prove it for $\tilde{\mathfrak{b}}^p$. For this analyze separately two terms in the right side of recursive formula (6.14):

(1). Analysis of the second term in the right side of(6.14) does not depend on $\tilde{\mathfrak{b}}^p$, so it does not require an induction hypothesis. Because of the condition $\mathcal{C}(l) = \mathcal{C}(v)$ in the summation in this term, we can imply item (2) of Corollary 6.11 to conclude that a^p_{vl} depends on u-variables indexed by vertices and edges of the component containing vertex v. On the other hand, we also have the condition $\mathcal{C}(w) \neq \mathcal{C}(l)$ in the same summation. Therefore, every monomial in a summand $(\alpha(l)^2-\alpha(w)^2)\mathfrak{a}^p_{vl}q_{lw}u_w$ of the second term of (6.14) depend on u-variables indexed by vertices and edges of at least two component of Γ, i.e., the component containing the vertex v and the component containing the edge w, i.e. satisfy the conclusion of the lemma.

(2). For the first term of the right side of (6.14), assume that $m(U)$ is a monomial appearing in $\tilde{\mathfrak{b}}^p_w$ depending on vertex variables u_{v_1} and u_{v_2}. By the second statement of Lemma 6.5, the contribution to $\left(\vec{h}_1\big(m(U)\big)\right)_{(2)}$ comes only from the terms in the product rule expansion of $h_1\big(m(u)\big))$ containing factors $\vec{h}_1(u_{v_1})$ or $\vec{h}_1(u_{v_1})$. Consider two sub-cases:

(2a) *Vertices v_1 and v_2 lie in the same component Γ_i of Γ.* For $p = 1$, this case is excluded by the induction hypothesis. If $p > 1$, then the term of $\vec{h}_1\big(m(U)\big)$ that contains the factor $\vec{h}(u_{v_1})$ (respectively, $\vec{h}(u_{v_2})$) will also contain u_{v_2} (respectively, u_{v_1}), along with other edge variables, which–by the induction hypothesis–are indexed by a component of Γ different from Γ_i.

(2b) *Vertices v_1 and v_2 lie in different components of Γ.* Consider the term in the expansion of $\vec{h}_1\big(m(U)\big)$ that contains factor $\vec{h}(u_{v_1})$ (the analysis of the term containing the factor $\vec{h}(u_{v_1})$

is completely analogous). The contribution to $\big(\vec{h}_1\big(m(U)\big)\big)_{(2)}$ comes from $\big(\vec{h}(u_{v_1})\big)_{(1)}$ only. From (6.13) and decomposition (4.1) of the Tanaka symbol $\mathfrak{m}$ it follows that $\big(\vec{h}_1(u_{v_1})\big)_{(1)}$ depends on variables indexed by vertices and edges of the component of Γ containing v_1 only. Since the term in the expansion of $\vec{h}_1\big(m(U)\big)$ containing the factor $\vec{h}(u_{v_1})$, also depends on u_{v_2}, every its monomial depend on u- variables indexed by vertices and edges of at least two components of Γ.

This completes the proof of the lemma.

□

We now introduce a few definitions that will simplify the statement and proof of the following lemmas. In the sequel, we denote by ρ^i_j the jth-root of the graph Γ_i, see Definition 2.10.

t_monom_def **Definition 6.16.** We call a U-monomial a *path monomial* of the graph Γ_i if the monomial contains the variable $u_{\rho^i_0}$ and all of the remaining variables are edge-variables indexed by edges from a simple path in Γ_i starting at ρ^i_0. In addition, if a path monomial corresponds to a path in Γ_i which traverses only roots in consecutive order (i.e. the jth edges connects the $(j-1)$'st root of Γ_i with the jth root of Γ_i), then we call such a monomial a *root monomial* and the corresponding path a *root path.*[8]

Since the U-polynomials appearing as entries of $\mathfrak{A}^p$ are homogeneous of degree p, it follows that root monomials in the entries of the pth layer $\mathfrak{A}^p$ of the fundamental system are of the form $Cu_{\rho_i}u_{\overrightarrow{\rho^i_0\rho^i_1}}\dots u_{\overrightarrow{\rho^i_{p-2},\rho^i_{p-1}}}$ for some $i\in[1:k]$, $p\in[1:\omega_i+1]$.

(here given a positive integer n I denote by $[1:n]$ the set $\{1,\dots,n\}$. We probably have to introduce this notation earlier.

Note that path and root monomials satisfy Lemma 6.10. In particular, the coefficient C of a root monomial is a sum of terms of the form

prelim_prod
$$c^s_{v_{p-1}w}\prod_{j=1}^{p-1}c_{v_{j-1}v_j}^{\overrightarrow{\rho^i_{j-1}\rho^i_j}}.$$

However, it turns out that the remaining indices which aren't fixed by (6.10) in the coefficient of a root monomial (i.e. $v_1,\dots,v_{p-1}$) are completely determined for suitably chosen rows of the fundamental system, and the sum of terms in this coefficient collapses to a single term.

root_edge **Lemma 6.17.** *Let $c_{xy}^{\overrightarrow{\rho^i_k\rho^i_{k+1}}}$ be such that $x,y\in V(\Gamma_i)$, then we have the following:*

(1) *If $x=\rho^i_k$, then $c_{xy}^{\overrightarrow{\rho^i_k\rho^i_{k+1}}}=0$ unless $y=\rho^i_{k+1}$;*
(2) *If either x or y is of depth less than k, then $c_{xy}^{\overrightarrow{\rho^i_k\rho^i_{k+1}}}=0$.*

Proof. For $k=0$, the conclusion of the Lemma follows since ρ^i_0 is adjacent to every non-isolated vertex and since the isolated vertices correspond precisely to $\ker \operatorname{ad} X_{\rho^i_0}$. If $k>0$, then by (5.4) we have that

$$X_{\overrightarrow{\rho^i_k,\rho^i_{k+1}}}=[X_{\rho^i_k},X_{\rho^i_{k+1}}].$$

Let $\{\mathfrak{h}^i_l\}_{l=0}^{\omega_i+1}$ be the filtration as in (2.8) associated with the tuple $\Big\{(\varphi^i_{j-1}\circ\dots\circ\varphi^i_0)(X_{\rho^i_j})\Big\}_{j=0}^{\omega_i}$, where $\varphi^i_j:\mathfrak{m}^j_i\to\mathfrak{m}^{j+1}_i$ are the canonical projections as in (2.6). Pointwise, by construction, we have

$$\operatorname{pr}\big(X_{\overrightarrow{\rho^i_k,\rho^i_{k+1}}}\big)\in\mathfrak{h}^i_k,\ \text{ but } \operatorname{pr}\big(X_{\overrightarrow{\rho^i_k,\rho^i_{k+1}}}\big)\notin\mathfrak{h}^i_l, \text{for } l<k,$$

[8]There are ω_i+1 different root monomials of Γ_i, one for each degree from 1 to ω_i+1, namely $u_{\rho^i_0},u_{\rho^i_0}u_{\overrightarrow{\rho^i_0\rho^i_1}},\dots,u_{\rho^i_0}u_{\overrightarrow{\rho^i_0\rho^i_1}}\dots u_{\overrightarrow{\rho^i_{\omega_i-2},\rho^i_{\omega_i-1}}}$.

where pr is as in (5.3). Thus, since the $\mathfrak{h}_j^i$ are nested ideals, we have that $c_{xy}^{\overrightarrow{\rho_k^i,\rho_{k+1}^i}} = 0$ whenever vertices $x, y \in V(\Gamma_i)$ are of depth smaller than k, see Definition 3.1. Now suppose $x = \rho_k^i$. If $X_y \in \mathfrak{h}_k^i$, then $c_{\rho_k^i y}^{\overrightarrow{\rho_k^i,\rho_{k+1}^i}} = 0$ for any root ρ_k^i by (3.2). On the other hand, since $y \in V(\Gamma_i)$, (3.1) shows that $[X_{\rho_k^i}, X_y] = X_{\overrightarrow{\rho_k^i,y}}$ whenever y has depth greater than k. This implies that $c_{\rho_k^i,y}^{\overrightarrow{\rho_k^i\rho_{k+1}^i}} = 0$ unless $y = \rho_{k+1}^i$. The Lemma follows from anti-symmetry in the lower indices. $\square$

root **Lemma 6.18.** *Assume that $f(U) = K u_{e_1} u_{e_2} \dots u_{e_{p-1}} u_{\rho_0^i}$ is a monomial which appears in the $\mathfrak{a}_{ws}^p$ entry of $\mathfrak{A}^p$ such that each e_j is an edge belonging to a root path in Γ_i (see Definition 6.16). Then the following two statements hold*

1. *For $p \geq 2$, $u_{\overrightarrow{\rho_0^i\rho_1^i}}$ divides $f(U)$;*
2. *For $p \geq 3$ if for some $0 \leq j \leq p-3$, $u_{\overrightarrow{\rho_k^i\rho_{k+1}^i}}$ appears linearly in $f(U)$ for all $0 \leq k \leq j$, then $u_{\overrightarrow{\rho_{j+1}^i\rho_{j+2}^i}}$ appear in $f(U)$.*

Moreover, if $f(U)$ divides a root monomial of Γ_i, then

f_div
$$f(U) = c_{\rho_{p-1}^i,w}^s u_{\rho_0^i} u_{\overrightarrow{\rho_0^i\rho_1^i}} \dots u_{\overrightarrow{\rho_{p-2}^i,\rho_{p-1}^i}}, \tag{6.17}$$

and, in particular, it is a root monomial.

Proof. Since all non-zero monomials which appear in the entries of $\mathfrak{A}$ have exactly one vertex variable, and since f divides a root monomial, it follows that this single vertex variable of f must be $u_{\rho_0^i}$. From Lemma 6.10, the coefficient of such a monomial is a sum of terms of the form

$$c_{v_{p-1}w}^s \prod_{j=1}^{p-1} c_{v_{j-1}v_j}^{e_j}$$

(note that $v_0 = \rho_0^i$ in order to accommodate the necessary occurrence of $u_{\rho_0^i}$).

For $p \geq 2$, consider the factor $c_{\rho_0^i v_1}^{e_1}$ appearing in the coefficient of f. By assumptions, e_1 must be of the form $e_1 = \overrightarrow{\rho_k^i\rho_{k+1}^i}$ for some k. By Lemma 6.17, $c_{\rho_0^i v_{p-1}}^{\overrightarrow{\rho_k^i,\rho_{k+1}^i}} \neq 0$ if and only if $k = 0$ and $v_1 = \rho_1^i$. Hence $f(U)$ is divisible by $u_{\overrightarrow{\rho_0\rho_1^i}}$, and we get Statement (1) of the lemma. Moreover, in that case $c_{\rho_0^i\rho_1^i}^{\overrightarrow{\rho_0^i,\rho_1^i}} = 1$. Thus, the terms in the summand of the expression of the coefficient of f must be of the form $c_{v_{p-1}w}^s c_{v_{p-2}v_{p-1}}^{e_{p-1}} \dots c_{\rho_1^i v_2}^{e_2} c_{\rho_0^i\rho_1^i}^{\overrightarrow{\rho_0^i\rho_1}} = c_{v_{p-1}w}^s c_{v_{p-2}v_{p-1}}^{e_{p-1}} \dots c_{\rho_1^i v_2}^{e_2}$.

Further, for $p \geq 3$, by Lemma 6.17if $c_{\rho_1^i v_2}^{e_2} \neq 0$, then either $e_2 = \overrightarrow{\rho_1^i\rho_2^i}$ or $e_2 = \overrightarrow{\rho_0^i\rho_1^i}$. If $u_{\overrightarrow{\rho_0^i\rho_1^i}}$ appears in $f(U)$ linearly, then the latter case is excluded, so $e_2 = \overrightarrow{\rho_1^i\rho_2^i}$ and $f(U)$ is divisible by $u_{\overrightarrow{\rho_1^i\rho_2^i}}$. This proves Statement (2) of the lemma for $j = 0$.

Moreover, Lemma 6.17 implies that $v_2 = \rho_2^i$. Thus, the terms in the sum of the coefficient of f are all of the form $c_{v_{p-1}w}^s c_{v_{p-2}v_{p-1}}^{e_{p-1}} \dots c_{\rho_2^i v_3}^{e_3} c_{\rho_1^i\rho_2^i}^{\overrightarrow{\rho_1^i\rho_2}} c_{\rho_0^i\rho_1^i}^{\overrightarrow{\rho_0^i\rho_1}} = c_{v_{p-1}w}^s c_{v_{p-2}v_{p-1}}^{e_{p-1}} \dots c_{\rho_2^i v_3}^{e_3}$.

Assume by way of induction that (for $\bar{j} \geq 0$ and $p \geq \bar{j} + 3$) if $u_{\overrightarrow{\rho_k^i\rho_{k+1}^i}}$ appears linearly in $f(U)$ for all $0 \leq k \leq \bar{j}$, then $v_k = \rho_k^i$ and $e_k = \overrightarrow{\rho_{k-1}^i, \rho_k^i}$ for all $0 \leq k \leq \bar{j}$. By Lemma 6.17, if $c_{\rho_{\bar{j}+1}^i v_{p-\bar{j}-2}}^{e_{p-\bar{j}-2}} \neq 0$, then either $e_{p-\bar{j}-2} = \overrightarrow{\rho_{\bar{j}+1}^i\rho_{\bar{j}+2}^i}$ or $e_{p-\bar{j}-2} = \overrightarrow{\rho_{\bar{j}}^i\rho_{\bar{j}+1}^i}$. By assumption of linearity of $u_{\overrightarrow{\rho_{\bar{j}}^i\rho_{\bar{j}+1}^i}}$ in $f(U)$, the latter case is excluded, so $e_{p-\bar{j}-2} = \overrightarrow{\rho_{\bar{j}+1}^i\rho_{\bar{j}+2}^i}$ and $f(U)$ is divisible by $u_{\overrightarrow{\rho_{\bar{j}+1}^i\rho_{\bar{j}+2}^i}}$. This proves Statement (2) of the lemma for $j = \bar{j}$.

Moreover, Lemma 6.17 implies that $v_{p-\bar{j}-2} = \rho^i_{\bar{j}+2}$. Thus, the terms in the sum of the coefficient of f are all of the form

coef_bar j (6.18)
$$c^s_{v_1w}c^{e_1}_{v_2v_1}\cdots c^{e_{p-\bar{j}-3}}_{\rho^i_{\bar{j}+2}v_{p-\bar{j}-3}}\, c^{\overrightarrow{\rho^i_{\bar{j}+1}\rho^i_{\bar{j}+2}}}_{\rho^i_{\bar{j}+1}\rho^i_{\bar{j}+2}}\cdots c^{\overrightarrow{\rho^i_0\rho^i_1}}_{\rho^i_0\rho^i_1} = c^s_{v_1w}c^{e_1}_{v_2v_1}\cdots c^{e_{p-\bar{j}-3}}_{\rho^i_{\bar{j}+2}v_{p-\bar{j}-3}}.$$

Finally, if $f(U)$ divides a root monomial of Γ_i, then all u's will appear linearly in it. Applying inductively Statement (2) of the lemma (with Statement (1) as a basis of induction), we can conclude that $f(U)$ will satisfy the assumptions of this Statement with $j = p-3$ which together with (6.18) implies (6.17). □

We begin the task of analyzing maximal minors of the fundamental system. We will fix a particular $(n-m)\times(n-m)$ sub-matrix T of $\mathfrak{A}$ by choosing rows of $\mathfrak{A}$ such that the coefficients of root monomials in their entries are identically 1. Every $i \in [1:k]$ contributes $\dim \mathfrak{m}^i_{-2}$ rows of A to T as follows: for each $p \in [1:\omega_i+1]$, choose the rows of $\mathfrak{A}^p$ which correspond to the vertices of Γ_i of depth greater than $p-1$ (see Definition 3.1). Note that by Remark 3.2 there are r^i_{p-1} such vertices.

degree_rk *Remark* 6.19. By item (2) of Corollary 6.11, $\det T$ is a Γ_i-vertex homogeneous (of degree d_i) and Γ_i-edge homogeneous (of degree $\sum_{j=1}^{\omega_i} j r^i_j$).

coeff=1 **Lemma 6.20.** *The root monomials (of some Γ_i) appear exactly once in each row and each column of T and its coefficient is equal to 1. In more detail, if w is a vertex of depth greater than $p-1$ in Γ_i, then in the w-row of T in $\mathfrak{A}^p$ the root monomial appears in the $\overrightarrow{\rho^i_{p-1}, w}$-column.*

Proof. By Lemma 6.18 the coefficient of a root monomial in the entry $\mathfrak{a}^p_{ws}$ is given by $c^s_{\rho^i_{p-1},w}$. Since the columns and rows of $\mathfrak{A}^p$ are naturally parameterized by $E(\Gamma)$ and $V(\Gamma)$, respectively, the decomposition (4.1) of the symbol $\mathfrak{m}$ implies that $c^s_{\rho^i_{p-1},w}$ vanishes whenever s and w belong to different graphs. On the other hand, for a fixed Γ_i, if $s \in E(\Gamma_i)$ and $w \in V(\Gamma_i)$, then (5.6) and our choice of rows in T imply $c^s_{\rho^i_{p-1},w} = 1$ when $s = \overrightarrow{\rho^i_{p-1}, w}$ and vanishes otherwise, which completes the proof. □

The previous lemma implies that the product of all of the root monomials in the entries of T will appear (with coefficient ± 1) in the Leibniz expansion [9] for $\det T$.

For a given Γ_i, we denote the product of all of the root monomials of Γ_i which appear in the entries of T by $p_i(U)$. By Lemma 6.20, the root monomial in the entry $\mathfrak{a}^p_{w,\overrightarrow{\rho^i_{p-1},w}}$ is equal to $u_{\rho^i_0}u_{\overrightarrow{\rho^i_0\rho^i_1}}\cdots u_{\overrightarrow{\rho^i_{p-2},\rho^i_{p-1}}}$. Therefore, setting $d_i = \dim \mathfrak{m}^i_{-2}$, we get

p_i (6.19)
$$p_i(U) = u^{d_i}_{\rho^i_0}\prod_{j=1}^{\omega_i}\left(u_{\overrightarrow{\rho^i_{j-1}\rho^i_j}}\right)^{d_i-r^i_0-\ldots-r^i_{j-1}}.$$

If

p (6.20)
$$p(U) = \prod_{i=1}^{k} p_i(U)$$

is the product of all root monomials appearing in the entries of T, then the coefficient of p in $\det T$ is particularly simple.

[9]From now on, the classical formula for determinants in terms of permutations of the matrix elements is referred as the *Leibniz expansion* for determinants.

pcoeff1 **Lemma 6.21.** *The monomial* $p(U)$ *appears in the Leibnitz exapnsion of* $\det T$ *only in the term*

_term_det_T
$$\prod_{i=1}^{k}\prod_{p=1}^{\omega_i+1}\prod_{\substack{v\in V(\Gamma_i),\\ \mathrm{depth}(v)>p-1}} \mathfrak{a}^p_{\overrightarrow{v\rho^i_{p-1},v}}.$$

Moreover, the coefficient of $p(U)$ *in* $\det T$ *is* ± 1.

Proof. Let $d_i = \dim \mathfrak{m}^i_{-2}$. Since the Tanaka symbol $\mathfrak{m}$ is decomposed, T can be made to be block diagonal by the interchange of columns or rows such that a given block is a $d_i \times d_i$ sub-matrix corresponding to a particular Γ_i. Since the sign of $\det T$ is immaterial to us, we will assume that T has been made to have this block diagonal form. Thus it suffices to consider the coefficient of a particular $p_i(U)$ in the determinant of the block corresponding to Γ_i.

Suppose that

prod
$$\prod_{k=1}^{d_i} f_k(U) = Cp_i(U) \tag{6.21}$$

appears in the Leibniz expansion of the determinant of the block corresponding to Γ_i and $\{f_k(U)\}_{k=1}^{d_i}$ are monomials which appear in the entries of this block.

This decription regarding the blocks has to be revised.

We will show that the polynomials $f_k(U)$ in (6.21) have no repeated factors. Then, by Lemma 6.18, each $f_k(U)$ must be a root monomial. Finally, applying Lemma 6.20, we conclude the statement of the present lemma.

First, every monomial appearing in the entries of T must contain at least one factor of u which is indexed by $V(\Gamma)$. Since from (6.19) it follows that each $f_k(U)$ must have exactly one factor $u_{\rho^i_0}$, otherwise by (6.21) the degree of $u_{\rho^i_0}$ in p_i would be larger then d_i, which will contradict (6.19).

Further, by Remark 3.2 in the list $\{f_k(U)\}_{k=1}^{d_i}$ there are precisely r^i_{p-1} monomials which come from the matrix $\mathfrak{A}^p$. Moreover, the monomials appearing in the entries of a given layer $\mathfrak{A}^p$ are of degree p.

Now consider the monomials in the list $\{f_k(U)\}_{k=1}^{d_i}$ of degree greater than 1 (or, equivalently, coming from $\mathfrak{A}^p$ with $1 < p \leq \omega_i + 1$). By Statement (1) of Lemma 6.18 all such monomials are divisible by $u_{\overrightarrow{\rho^i_0,\rho^i_1}}$. By the previous paragraph there are exactly $d_i - r^i_0$ such monomials. Then it follows that each $f_k(U)$ must have exactly one factor $u_{\overrightarrow{\rho^i_0,\rho^i_1}}$, otherwise by (6.21) the degree of $u_{\overrightarrow{\rho^i_0,\rho^i_1}}$ in p_i would be larger then $d_i - r^i_0$, which will contradict (6.19).

Assume by induction that for some $1 \leq j \leq \omega_i$ in all the monomials in the list $\{f_k(U)\}_{k=1}^{d_i}$ of degree greater than j (or, equivalently , coming from $\mathfrak{A}^p$ with $j < p \leq \omega_i + 1$) the variables $u_{\overrightarrow{\rho^i_k,\rho^i_{k+1}}}$ appear linearly for all $0 \leq k \leq j$. By Statement (2) of Lemma 6.18 all such monomials are divisible by $u_{\overrightarrow{\rho^i_{j+1},\rho^i_{j+2}}}$. Besides, by above there are exactly $d_i - r^i_0 - \ldots r^i_{j+1}$ such monomials. Then it follows that each $f_k(U)$ must have exactly one factor $u_{\overrightarrow{\rho^i_{j+1},\rho^i_{j+2}}}$, otherwise by (6.21) the degree of $u_{\overrightarrow{\rho^i_{j+1},\rho^i_{j+2}}}$ in p_i would be larger then $d_i - r^i_0 - \ldots - r^i_{j+1}$, which will contradict (6.19). $\square$

The maximal minors of the fundamental algebraic system are determinants of $(n-m+1) \times (n-m+1)$ sub-matrices. Since the matrix T, constructed above, is a particular $(n-m)\times(n-m)$ sub-matrix of $[\mathfrak{A}|\tilde{\mathfrak{b}}]$, we must augment T by an additional row and column of the fundamental system if its determinant is to be a maximal minor of the fundamental system. There is no freedom in choosing a column as the only remaining column is $\tilde{b}$ itself. However, the choice of row in $\mathfrak{A}$ used to augment T is crucial for our analysis.

Definition 6.22. Given an integer $p \geq 1$ and a vertex w of the graph Γ, the sub-matrix of $[\mathfrak{A}|\tilde{\mathfrak{b}}]$ obtained by augmenting T by the column $\tilde{\mathfrak{b}}$ and the w–row of $\mathfrak{A}^p$ is called the (p,w)–*augmentation of the matrix* T and will be denoted by W_w^p .

cofterm **Proposition 6.23.** *Let $v, l \in V(\Gamma)$ and $w \in V(\Gamma_i)$ be a depth 0 vertex. The monomial $u_v u_l p(U)$ appears only in the term $\tilde{\mathfrak{b}}_w^1 \det T$ of the cofactor expansion of $\det W_w^1$.*

Proof. We will analyze $\det W_w^1$ by two consecutive cofactor expansions: first along the column corresponding to $\tilde{\mathfrak{b}}$ and then along the w-row for each cofactor of the first step, except the one obtained from removing this column and the w-row. In more detail, given $p \geq 1$, a vertex j of depth greater than $p-1$, and an edge y, denote by $T^p_{\hat{j}\hat{y}}$ the matrix obtained from the matrix T by removing the j-row of $\mathfrak{A}^p$ and the y-column of $\mathfrak{A}$. Then, up to a sign, the general term in the resulting expansion is given by

cofactor
$$\tilde{\mathfrak{b}}_j^p \mathfrak{a}^1_{wy} \det T^p_{\hat{j}\hat{y}}, \quad j \neq w,\ \mathrm{depth}(j) > p-1, \tag{6.22}$$

or

cofactor1
$$\tilde{\mathfrak{b}}_w^1 \det T. \tag{6.23}$$

We proceed by excluding terms of the form (6.22). By item (1) of Corollary 6.11, it follows that $\mathfrak{a}^1_{wy} = 0$ unless y belongs to the same component Γ_i as w. Using also Remark 6.12, one gets that $\det T^p_{\hat{j}\hat{y}} = 0$ unless y and j belong to the same component Γ_i. The latter conclusion follows from the fact that a cofactor of a block-diagonal matrix corresponding to an entry off the block-diagonal block is equal to zero. In summary, the terms (6.22) vanish unless the vertex j and the edge y belong to the same component Γ_i (as the vertex w).

Moreover, since by the first line of (6.10) and (5.12) $\mathfrak{a}^1_{wy} = \sum\limits_{x \in V(\Gamma)} c^y_{xw} u_x$, from (5.7) and the assumption that w has depth 0 it follows that (6.22) also vanishes unless y is of the form

y_bar z
$$y = \overrightarrow{\rho_0^i \bar{z}}, \quad \text{for some } \bar{z} \in V(\Gamma_i) \text{ with } \mathrm{depth}(\bar{z}) > 0. \tag{6.24}$$

Since j and y belong to the same component Γ_i, by second item of Corollary 6.11 Γ_i-vertex degree of $\det T^p_{\hat{j}\hat{y}}$ is equal to $d_i - 1$. By Lemma 6.20 the monomial $u_{\rho_0^i}$ appears exactly in one entry of each row of the matrix T belonging to $\mathfrak{A}^1$. Moreover, in the $\bar{z}$-row of $\mathfrak{A}^1$ it appears precisely in the entry $\mathfrak{a}^1_{\bar{z}\overrightarrow{\rho_0^i \bar{z}}}$. Consider the following two cases separately:

Case 1. *In* (6.22) *either $p > 1$ or $p = 1$ and $j \neq \bar{z}$ (where $\bar{z}$ is as in* (6.24)*)* In this case $\bar{z}$-row of $\mathfrak{A}^1$, with the entry $\mathfrak{a}^1_{\bar{z}\overrightarrow{\rho_0^i \bar{z}}}$ excluded, is a row of the matrix $T^p_{\hat{j}\hat{y}}$. Consequently, this row does not depend on $u_{\rho_0^i}$. Therefore, monomials in the Leibniz expansion of $\det T^p_{\hat{j}\hat{y}}$ will contain at most $d_i - 2$ factors of $u_{\rho_0^i}$. On the other hand, by (6.19) and (6.20) the monomial (6.25) contains at least d_i such factors. Hence, the monomials of the factor $\tilde{\mathfrak{b}}_j^p \mathfrak{a}^1_{wy}$ of (6.22), contributing to the monomial (6.25), must be divisible by $u^2_{\rho_0^i}$. Since w is of depth 0, using the first line of (6.10) and (5.12), $\mathfrak{a}^1_{wy}$ is independent of $u_{\rho_0^i}$. Therefore, the monomials of the factor of $\tilde{\mathfrak{b}}_j^p$ of(6.22) contributing to the monomial (6.25) must be divisible by $u^2_{\rho_0^i}$.

Now consider two sub-cases:

Case 1.1. $p = 1$*:* By Lemma 6.15, $\tilde{\mathfrak{b}}_j^1$ does not contain the monomial $u^2_{\rho_0^i}$, so this case does not contribute to monomial (6.25).

Case 1.2 $p > 1$*:* By Lemma 6.15, if $u^2_{\rho_0^i}$ divides a monomial of $\tilde{\mathfrak{b}}_j^p$, then that monomial has to be dependent on an edge variable indexed by a different component $\Gamma_{t'}$ of Γ with $t' \neq i$. In other words, by Remark 6.2 and Lemma 6.6, the Γ_i-edge degree of this monomial of $\tilde{\mathfrak{b}}_j^p$ is strictly smaller than $p-1$.

Let us show that in this case the term (6.22) cannot contribute to the monomial (6.25). Indeed, by (6.19) and (6.20) the monomial (6.25) has Γ_i-edge degree equal to $\sum_{j=1}^{\omega_i} j r_j^i$. On the other hand, by Remark 6.19, $\det T^p_{\hat{j}\hat{y}}$ has Γ_i-edge degree equal to $\sum_{j=1}^{\omega_i} j r_j^i - (p-1)$, because from the second item of lemma 6.11, the Γ_i-edge degree of non-zero entries of the j-row of $\mathfrak{A}^p$ is equal to $p-1$. Besides, by Remark 6.2 and Lemma 6.6 again, the (total) edge degree of the entry $\tilde{\mathfrak{b}}_j^p$ is equal to $p-1$, and $\mathfrak{a}_{wy}^1$ does not depend on edge variables. Therefore, if the monomial of the factor $\tilde{\mathfrak{b}}_j^p$ of (6.22) contributes to the monomial (6.25), then the Γ_i-edge degree of this monomial (of $\tilde{\mathfrak{b}}_j^p$) must be exactly equal to $p-1$, which proves the statement in the first sentence of the paragraph.

Case 2. *In* (6.22) $p=1$ *and* $j=\bar{z}$ In this case, the monomials in the Leibniz expansion of $\det T^1_{\hat{j}\hat{y}}$ will contain at most d_i-1 factors of $u_{\rho_0^i}$, and by the same arguments as in the first paragraph of Case 1 above, the monomial from the factor $\tilde{\mathfrak{b}}_j^1$ of(6.22) contributing to (6.25) must contain $u_{\rho_0^i}$. Recall from (6.4) that the entry $\tilde{\mathfrak{b}}_{\bar{z}}^1$ is given by

$$\sum_{\substack{s,x\in V(\Gamma)\\ \mathcal{C}(s)\neq\mathcal{C}(\bar{z})}} (\alpha(\bar{z})^2-\alpha(s)^2)c^s_{x\bar{z}}u_su_x.$$

First this implies that $s\neq\rho_0^i$. On the other hand, if $x=\rho_0^i$, then $c^s_{x\bar{z}}=0$ by (5.6). Thus $\tilde{\mathfrak{b}}_{\bar{z}}^1$ is independent of $u_{\rho_0^i}$ and $u_vu_lp(U)$ doesn't appear.

We conclude that no terms of the form (6.22) contribute to the coefficient of the monomial (6.25) in $\det W^1_w=0$.

□

In the sequel, given a collection of smooth functions on an open set $\mathcal{U}$ of the manifold M, where the affine equivalence or our sub-Riemannian metrics g_1 and g_2 holds and the local quasi-normal frame $\{X_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ is defined, denote by $\mathrm{Ideal}(\mathcal{F})$ the ideal in $C^\infty(\mathcal{U})$ generated by the collection $\mathcal{F}$.

c0 **Proposition 6.24.** *If $v\in V(\Gamma_i)$ is a depth 0 vertex and $l\in V(\Gamma_t)$ such that $t\neq i$, then the coefficient of the monomial*

vl1_monom
$$u_vu_lp(U) \tag{6.25}$$

in the determinant of the $(1,\rho_0^i)$–augmentation $W^1_{\rho_0^i}$ of the matrix T is equal, up to a sign, to

c0v
$$\begin{cases} (\alpha(\rho_0^i)^2-\alpha(l)^2)c^l_{\rho_0^iv}, & \text{if } l=\rho_0^t,\\ (\alpha(\rho_0^i)^2-\alpha(l)^2)c^l_{\rho_0^iv} \mod \mathrm{Ideal}(c^{\rho_0^t}_{\rho_0^iv}), & \text{if } l\in V(\Gamma_t), l\neq\rho_0^t \end{cases} \tag{6.26}$$

(here $p(U)$ is as in (6.20)*).*

Therefore, if $\det W^1_{\rho_0^i}=0$, *then*

crt_depth0
$$c^l_{\rho_0^iv}=0,\quad v\in V(\Gamma_i), \mathrm{depth}(v)=0, l\in V(\Gamma_t), t\neq i^{10}. \tag{6.27}$$

Proof. By Proposition 6.23, we only need to consider the coefficient of the monomial (6.25) in the term

only_term1
$$\tilde{\mathfrak{b}}^1_{\rho_0^i}\det T. \tag{6.28}$$

of the cofactor expansion of $\det W^1_{\rho_0^i}$. By (6.19) and (6.20), the following vertex variables appear

vl_vertex
$$\{u_{\rho_0^j}\}_{j=1}^k, u_v, \text{and } u_l \tag{6.29}$$

in the monomial (6.25). Further, in this monomial the number of Γ_j-vertex variable factors (see Definition 6.3) is equal to d_j+1 if $j\in\{i,t\}$ and to d_j otherwise, while Remark 6.19 implies that in the monomials appearing in $\det T$ the number of Γ_j-vertex variable factors is equal to d_j for all $j\in[1:k]$. Therefore if a monomial of $\tilde{\mathfrak{b}}^1_{\rho_0^i}$ contributes to (6.25) in (6.28), then one

[10]Note that if we replace here the assumption $\mathrm{depth}(v)=0$ by $\mathrm{depth}(v)>0$, $c^l_{\rho_0^iv}=0$ by (5.6) and the fact that $\overrightarrow{\rho_0^iv}\in E(\Gamma)$ in this case.

of its factor is Γ_i -vertex variables, i.e., by (6.29), either $u_{\rho_0^i}$ or u_v and the other is Γ_t vertex variable, i.e. either $u_{\rho_0^t}$ or u_l.

By (6.4)

tb1_rho

$$\tilde{\mathfrak{b}}^1_{\rho_0^i} = \sum_{s,x\in V(\Gamma),\mathcal{C}(l)\neq i} (\alpha(\rho_0^i)^2 - \alpha(s)^2)c^s_{x\rho_0^i} u_x u_s. \tag{6.30}$$

Since $\mathcal{C}(s) \neq i$, by the previous paragraph we have:

(1) $s \in \Gamma_t$, therefore $s \in \{\rho_0^t, l\}$;
(2) $x \in \Gamma_i$, therefore $x \in \{\rho_0^i, v\}$, but since $c^s_{\rho_0^i\rho_0^i} = 0$, we conclude that $x = v$.

According to item (1) above we have the following two cases:

(A) $s = l$. In this case $\tilde{b}^1_{\rho_0^i}$ contributes to monomial (6.25) in the product (6.28) the coefficient of $u_v u_l$, which by (6.30) is equal, up to a sign, to

_rt_contrib

$$\big(\alpha(\rho_0^i)^2 - \alpha(l)^2\big)c^l_{\rho_0^i v}. \tag{6.31}$$

Therefore, the contribution of $\det T$ to the coefficient of monomial (6.25) in the same product comes from the coefficient of $p(U)$ in $\det T$. By Lemma 6.21, the latter coefficient is equal to ± 1. Hence, the contribution of the considered case to the coefficient of monomial (6.25) in the product (6.28) is equal, up to a sign, to the (6.31).

(B) $s = \rho_0^t$. In this case $\tilde{b}^1_{\rho_0^i}$ contributes to monomial (6.25) in the product (6.28) the coefficient of $u_v u_{\rho_0^t}$ and this coefficient contains $c^{\rho_0^t}_{\rho_0^i v}$ as a factor. Therefore, the contribution of the considered case to the coefficient of monomial (6.25) in the product (6.28) belongs to $\mathrm{Ideal}(c^{\rho_0^t}_{\rho_0^i v})$.

Taking into account that for $l = \rho_0^t$ cases (A) and (B) coincide, we get (6.36). Finally, to get (6.27) from $\det W^1_{\rho_0^i} = 0$, first we get it from he first line of (6.36) for $l = \rho_0^t$ and then from the second line of (6.36) for the rest of ls.

□

brt0 **Corollary 6.25.** *If relation*(6.27) *holds, then the entries $\tilde{\mathfrak{b}}^1_j$ for $j \in V(\Gamma_i)$ do not depend on $u_{\rho_0^i}$*

Proof. This follows immediately from the assumption (6.27) and equations (6.4) and (5.6). □

c1 **Proposition 6.26.** *If $v, w \in V(\Gamma_i)$ with w of depth 0, $l \in V(\Gamma_t)$ with $i \neq t$ and* (6.27) *holds, then the coefficient of the monomial* (6.25) *in the determinant of the $(1, w)$–augmentation W^1_w of the matrix T is equal, up to a sign, to*

c0v

$$(\alpha(w)^2 - \alpha(l)^2)c^l_{wv}. \tag{6.32}$$

In addition, if $\det W^1_w = 0$, *then*

cw_depth0

$$c^l_{wv} = 0, \quad v, w \in V(\Gamma_i), \mathrm{depth}(w) = 0, i \in V(\Gamma_t), t \neq i.$$

Proof. By Proposition 6.23, we only need to consider the term $\tilde{\mathfrak{b}}^1_w \det T$ in our analysis of $\det W^1_w$. Corollary 6.25 shows that $\tilde{\mathfrak{b}}^1_w$ does not depend on $u_{\rho_0^i}$, so the only monomial of the factor $\tilde{\mathfrak{b}}^1_w$ in (6.23) contributing to the monomial (6.25) is $u_v u_l$, whose coefficient is equal exactly to (6.36) by (6.4). Consequently, the only monomial of the factor $\det T$ in (6.23) contributing to the monomial (6.25) is $p(U)$, whose coefficient is equal to ± 1 by Lemma 6.21. This concludes the proof of the proposition. □

aref **Lemma 6.27.** *The following two statements hold:*

(1) *In the ρ_0^i-row of $\mathfrak{A}^2$, the only non-zero entries are entries of the form $\mathfrak{a}^2_{\rho_0^i\overrightarrow{\rho_0^i z}}$, where $z \in V(\Gamma_i)$ has depth greater than 0;*
(2) *$u_{\rho_0^i} u_{\overrightarrow{\rho_0^i z}}$ is the unique monomial in the entry $\mathfrak{a}^2_{\rho_0^i\overrightarrow{\rho_0^i z}}$ which has a factor $u_{\rho_0^i}$.*

Proof. By Lemma (6.10) , in particular, by equation (6.11), the entry $\mathfrak{a}^2_{\rho_0^i y}$ is equal to

a^2_lin
$$\sum_{x,z\in V(\Gamma), e\in E(\Gamma)} c^y_{z\rho_0^i} c^e_{xz} u_x u_e. \tag{6.33}$$

By (5.6), $c^y_{z\rho_0^i}$ vanishes unless $y = \overrightarrow{\rho_0^i z}$ (in which case $c^{\overrightarrow{\rho_0^i z}}_{z\rho_0^i} = -1$). This proves item (1) of the lemma.

Further, if the entry $\mathfrak{a}^2_{\rho_0^i \overrightarrow{\rho_0^i z}}$ depends on the variable $u_{\rho_0^i}$, then it corresponds to $x = \rho_0^i$ in (6.33), in which case by (5.6), $c^e_{xz} = 0$ unless $e = \overrightarrow{\rho_0^i z}$. This proves item (2) of the lemma. □

cofterm2

Proposition 6.28. *Let $v, l \in V(\Gamma)$ and $w \in V(\Gamma_i)$ be a vertex with depth greater than 0. The monomial $u_v u_l u_{\overrightarrow{\rho_0^i w}} p(U)$ appears only in the term $\tilde{\mathfrak{b}}^1_w \mathfrak{a}^2_{\rho_0^i \overrightarrow{\rho_0^i w}} \det T^1_{\hat{w}\widehat{\overrightarrow{\rho_0^i w}}}$ of the cofactor expansion of* $\det W^2_{\rho_0^i}$.

Proof. As before we will analyze $\det W^2_{\rho_0^i}$ by two consecutive cofactor expansions: first along the column corresponding to $\tilde{\mathfrak{b}}$ and then along the row of $\mathfrak{A}^2$ corresponding to ρ_0^i for each cofactor from the first step. Up to a sign, the general term of this expansion is given by

ba2t
$$\tilde{\mathfrak{b}}^p_j \mathfrak{a}^2_{\rho_0^i y} \det T^p_{\hat{j}\hat{y}}, \quad \operatorname{depth}(j) > p-1 \tag{6.34}$$

and

b2tt
$$\tilde{\mathfrak{b}}^2_{\rho_0^i} \det T. \tag{6.35}$$

We analyze the two cases separately:

Case 1. *Terms of the form* (6.35). From equation (6.4), the coefficient of the monomial $u_x u_z$ for $x, z \in V(\Gamma)$ in $\tilde{\mathfrak{b}}^1_{\rho_0^i}$ is equal to $(\alpha(\rho_0^i)^2 - \alpha(z)^2) c^z_{x\rho_0^i}$. By Proposition 6.24 and (5.6), this term vanishes whenever $z \notin V(\Gamma_i)$ and $x \in V(\Gamma_i)$. On the other hand, when $x, z \in V(\Gamma_i)$, it vanishes because $(\alpha(\rho_0^i)^2 - \alpha(z)^2) = 0$. Thus, x and z must both come from a component of Γ different from Γ_i, say $\Gamma_{t'}$ and $\Gamma_{t''}$, respectively. In that case, by Lemma 6.5, equation (6.2) and the decomposition of Tanaka symbol (4.1), $\vec{h}_1(\tilde{\mathfrak{b}}^1_{\rho_0^i})_{(2)}$ has only edge variables from either $\Gamma_{t'}$ or $\Gamma_{t''}$. Therefore, using Remark 6.19 the $\Gamma_{t'}$-edge degree and $\Gamma_{t''}$-edge degree of the terms $\vec{h}_1(\tilde{\mathfrak{b}}^1_{\rho_0^i})_{(2)} \det T$ are strictly greater than than those of the monomial $u_v u_l u_{\overrightarrow{\rho_0^i w}} p(u)$, so the only terms of $\tilde{\mathfrak{b}}^2_{\rho_0^i}$ which may contribute to the coefficient of our monomial come from the second term in the right side of (6.14), namely,

$$\sum_{\substack{x,\, s\, \in\, E(\Gamma) \\ \mathcal{C}(x) = \mathcal{C}(\rho_0^i), \mathcal{C}(s) \neq \mathcal{C}(x)}} (\alpha(x)^2 - \alpha(s)^2) q_{\rho_0^i x} q_{xs} u_s.$$

However, since $\mathcal{C}(\rho_0^i) \neq \mathcal{C}(s)$, it must be that u_s is indexed by some component $\Gamma_{t'}$ different from Γ_i. Thus, the $\Gamma_{t'}$-edge degree in the terms (6.35) is strictly greater than that of $u_v u_l u_{\overrightarrow{\rho_0^i w}} p(u)$ and we conclude that the terms of the form (6.35) do not contribute to the coefficient of $u_v u_l u_{\overrightarrow{\rho_0^i w}} p(u)$.

Case 2. *Terms of the form* (6.34). By statement (1) of Lemma 6.27, $\mathfrak{a}^2_{\rho_0^i y}$, and therefore the terms (6.34), are zero unless y is as in (6.24), i.e. $y = \overrightarrow{\rho_0^i \bar{z}}$ for some $\bar{z} \in V(\Gamma_i)$ of depth greater than 0. Moreover, by the arguments in the first paragraph of Case 1 in the proof of Proposition 6.29, $\det T^p_{\hat{j}\hat{y}}$, and therefore (6.34), vanish unless j belongs to the same component of Γ as y, namely to Γ_i.

Sub-case 1.1. *p=1.* First note that the only Γ_i-vertex variables in the monomial (6.37) are $u_{\rho_0^i}$ and u_v, and the only Γ_t-vertex variables in (6.37) are $u_{\rho_0^t}$ and u_l. Furthermore, by

Corollary 6.25, $\tilde{\mathfrak{b}}^1_j$ is independent of $u_{\rho_0^i}$, and by Lemma 6.15 the monomials of $\tilde{\mathfrak{b}}^1_j$ are quadratic in U-variables with both U-variables being vertex-variables from two different components of Γ. Therefore, the only possible contributions of the factor $\tilde{\mathfrak{b}}^1_j$ of (6.34) to the coefficient of the monomial (6.37) may come from the coefficient of either the monomials $u_v u_{\rho_0^t}$ or $u_v u_l$ in $\tilde{\mathfrak{b}}^1_j$.

Sub-case 1.1 $p > 1$. We follow the same arguments as in Sub-case 1.1.2 of Proposition 6.29 to exclude terms for which $p > 1$.

Sub-case 1.2 $p = 1$ *and* $j \neq w$. By the same arguments as in Sub-case 1.1 of Proposition 6.29, we conclude that the degree of $u_{\rho_0^i}$ in the terms of $\det T^p_{\hat{j}\hat{y}}$ is at most $d_i - 2$. It follows that $u^2_{\rho_0^i}$ must divide some term in the product $\tilde{\mathfrak{b}}^1_j \mathfrak{a}^2_{\overrightarrow{\rho_0^i \rho_0^i y}}$ to obtain a contribution to the coefficient of the considered monomial. However, by Corollary 6.25, $\tilde{\mathfrak{b}}^1_j$ is independent of $u_{\rho_0^i}$. Finally, by construction, the entry $\mathfrak{a}^2_{\rho_0^i y}$ does not contain any terms which are quadratic in vertex variables. Therefore, terms with $p = 1$ and $j \neq w$ do not contribute to the coefficient of our monomial.

$\square$

c1 **Proposition 6.29.** *If $v, w \in V(\Gamma_i)$ with w a depth 0 vertex, $l \in V(\Gamma_t)$ with $i \neq t$ and* (6.27) *holds, then the coefficient of the monomial* (6.25) *in the determinant of the* $(1, w)$*–augmentation W^1_w of the matrix T is equal, up to a sign, to*

c0v
$$(\alpha(w)^2 - \alpha(l)^2) c^l_{wv}. \tag{6.36}$$

In addition, if $\det W^1_w = 0$*, then*

cw_depth0
$$c^l_{wv} = 0, \quad v, w \in V(\Gamma_i), \mathrm{depth}(w) = 0, i \in V(\Gamma_t), t \neq i.$$

Proof. First note that Proposition 6.23 shows that we need only consider the term $\tilde{\mathfrak{b}}^1_w \det T$ in our analysis of $\det W^1_w$. By Corollary 6.25, $\tilde{\mathfrak{b}}^1_w$ does not depend on $u_{\rho_0^i}$, so the only monomial of the factor $\tilde{\mathfrak{b}}^1_w$ in (6.23) contributing to the monomial (6.25) is $u_v u_l$, whose coefficient is equal exactly to (6.36) by (6.4). Consequently, the only monomial of the factor $\det T$ in (6.23) contributing to the monomial (6.25) is $p(U)$, whose coefficient is equal to ± 1 by Lemma 6.21. This concludes the proof of the proposition. $\square$

c2 **Proposition 6.30.** *If $v, w \in V(\Gamma_i)$, w has depth greater than* 0*, and $l \in V(\Gamma_t)$ with $i \neq t$ and* (6.27) *holds , then the coefficient of the monomial*

vl2_monom
$$u_v u_l u_{\overrightarrow{\rho_0^i w}} p(U) \tag{6.37}$$

in the determinant of the $(2, \rho_0^i)$*–augmentation $W^2_{\rho_0^i}$ of the matrix T is equal, up to sign, to*

c2v
$$(\alpha(w)^2 - \alpha(l)^2) c^l_{wv}. \tag{6.38}$$

So, if in addition $\det W^2_{\rho_0^i}$ *=0, then*

cw_depth0
$$c^l_{wv} = 0, \quad v, w \in V(\Gamma_i), \mathrm{depth}(w) > 0, l \in V(\Gamma_t), t \neq i.$$

Proof. By Proposition 6.28, we only need to consider the term $\tilde{\mathfrak{b}}^1_w \mathfrak{a}^2_{\overrightarrow{\rho_0^i \rho_0^i w}} \det T^1_{\hat{w}\overrightarrow{\rho_0^i w}}$ in our analysis of $\det W^2_{\rho_0^i}$. First, we specialize to the case when $l = \rho_0^t$. Notice that when $j = w$, $p = 1$, and $y = \overrightarrow{\rho_0^i w}$, the monomial $u_v u_{\rho_0^t} u_{\overrightarrow{\rho_0^i w}} p(U)$ appears in (6.34) with the coefficient (6.38) by Lemmas 6.27 and 6.21 and equation (6.4). Moreover, by Lemma 6.21 and the previous cases, this is the only contribution. We conclude that $c^{\rho_0^t}_{vw} = 0$ if $\det W^2_{\rho_0^i} = 0$. The general case when $l \neq \rho_0^t$ follows in much the same way. Indeed, if $\det W^2_{\rho_0^i} = 0$, then $c^{\rho_0^t}_{vw} = 0$ so that $\tilde{\mathfrak{b}}^1_w \mathfrak{a}^2_{\overrightarrow{\rho_0^i \rho_0^i w}}$ is independent of $u_{\rho_0^t}$ and the only contribution occurs when the factors $u_v u_l$ come from the term $\tilde{\mathfrak{b}}^1_w$. $\square$

c3 **Proposition 6.31.** *Let $\Gamma_i, \Gamma_t, \Gamma_{t'}$ be distinct components of the graph Γ such that $w \in V(\Gamma_i)$, $v \in V(\Gamma_t)$, and $l \in V(\Gamma_{t'})$. Up to a sign, the coefficient of the monomial*

$$u_l u_v p(U)$$

in $\det W^1_w$ *is*

$$(\alpha(w)^2 - \alpha(v)^2) c^v_{lw} + (\alpha(w)^2 - \alpha(l)^2) c^l_{vw}$$

and that of

$$u_v u_l u_{\overrightarrow{\rho^i_0 w}} p(U)$$

in $\det W^2_{\rho^i_0}$ *is*

$$(\alpha(w)^2 - \alpha(l)^2) c^l_{vw} + (\alpha(w)^2 - \alpha(v)^2) c^v_{lw}$$

Moreover, if the fundamental system has a solution, then

$$c^l_{vw} = c^v_{lw} = c^w_{lv} = 0.$$

Proof. The proof of the proposition includes many similar, yet distinct cases:
Case 1. $v = \rho^t_0, l = \rho^{t'}_0, w = \rho^i_0$. For this case, we consider the monomials $u_{\rho^i_0} u_{\rho^{t'}_0} p(U)$, $u_{\rho^t_0} u_{\rho^i_0} p(U)$, and $u_{\rho^t_0} u_{\rho^{t'}_0} p(U)$ in $\det W^1_{\rho^t_0}$, $\det W^1_{\rho^{t'}_0}$, and $\det W^1_{\rho^i_0}$, respectively. Again, using Proposition 6.23, we only consider the terms $\tilde{\mathfrak{b}}^1_{\rho^x_0} \det T$ for $x \in \{i, t, t'\}$.

By the same arguments as in the third paragraph of the proof of Proposition 6.24, if a monomial of $\tilde{\mathfrak{b}}^1_{\rho^i_0}$ contributes to (6.25) in (6.28), then one of its factor is Γ_t -vertex variables, i.e., by (6.29), either $u_{\rho^0_t}$ or u_v and another one is $\Gamma_{t'}$ vertex variable, i.e. either $u_{\rho^{t'}_0}$ or u_l.

In particular, when $v = \rho^t_0, l = \rho^{t'}_0$ (and vice versa) in (6.30), we see that the total coefficient of $u_{\rho^t_0} u_{\rho^{t'}_0} p(U)$ is

$$(\alpha(\rho^i_0)^2 - \alpha(\rho^t_0)^2) c^{\rho^t_0}_{\rho^{t'}_0 \rho^i_0} + (\alpha(\rho^i_0)^2 - \alpha(\rho^{t'}_0)^2) c^{\rho^{t'}_0}_{\rho^t_0 \rho^i_0}.$$

By permuting indices, we similarly conclude that the coefficients of $u_{\rho^t_0} u_{\rho^i_0} p(U)$ and $u_{\rho^i_0} u_{\rho^{t'}_0} p(U)$ in $\det W^1_{\rho^{t'}_0}$ and $\det W^1_{\rho^t_0}$, respectively, are

$$(\alpha(\rho^{t'}_0)^2 - \alpha(\rho^t_0)^2) c^{\rho^t_0}_{\rho^i_0 \rho^{t'}_0} + (\alpha(\rho^{t'})^2 - \alpha(\rho^i_0)^2) c^{\rho^i_0}_{\rho^t_0 \rho^{t'}_0}$$

and

$$(\alpha(\rho^t_0)^2 - \alpha(\rho^{t'}_0)^2) c^{\rho^{t'}_0}_{\rho^i_0 \rho^t_0} + (\alpha(\rho^t)^2 - \alpha(\rho^i_0)^2) c^{\rho^i_0}_{\rho^{t'}_0 \rho^t_0}.$$

If all three determinants are zero, then the three sets of coefficients above define a homogeneous system of equations. By anti-symmetry in the lower indices, this system of equations may be written in terms of the structure functions $c^{\rho^{t'}_0}_{\rho^t_0 \rho^i_0}$, $c^{\rho^t_0}_{\rho^{t'}_0 \rho^i_0}$, and $c^{\rho^i_0}_{\rho^{t'}_0 \rho^t_0}$ only. In particular, we arrive at the following system of equations:

$$\begin{cases} (\alpha(\rho^i_0)^2 - \alpha(\rho^t_0)^2) c^{\rho^t_0}_{\rho^{t'}_0 \rho^i_0} + (\alpha(\rho^i_0)^2 - \alpha(\rho^{t'}_0)^2) c^{\rho^{t'}_0}_{\rho^t_0 \rho^i_0} = 0 \\ -(\alpha(\rho^{t'}_0)^2 - \alpha(\rho^t_0)^2) c^{\rho^t_0}_{\rho^{t'}_0 \rho^i_0} - (\alpha(\rho^{t'})^2 - \alpha(\rho^i_0)^2) c^{\rho^i_0}_{\rho^{t'}_0 \rho^t_0} = 0 \\ -(\alpha(\rho^t_0)^2 - \alpha(\rho^{t'}_0)^2) c^{\rho^{t'}_0}_{\rho^t_0 \rho^i_0} + (\alpha(\rho^t)^2 - \alpha(\rho^i_0)^2) c^{\rho^i_0}_{\rho^{t'}_0 \rho^t_0} = 0 \end{cases}$$

The coefficient matrix

$$\begin{pmatrix} (\alpha(\rho^i_0)^2 - \alpha(\rho^t_0)^2) & (\alpha(\rho^i_0)^2 - \alpha(\rho^{t'}_0)^2) & 0 \\ -(\alpha(\rho^{t'}_0)^2 - \alpha(\rho^t_0)^2) & 0 & -(\alpha(\rho^{t'})^2 - \alpha(\rho^i_0)^2) \\ 0 & -(\alpha(\rho^t_0)^2 - \alpha(\rho^{t'}_0)^2) & (\alpha(\rho^t)^2 - \alpha(\rho^i_0)^2) \end{pmatrix}$$

of this homogeneous system of equations has determinant equal to

$$-2(\alpha(\rho^t_0)^2 - \alpha(\rho^{t'}_0)^2)(\alpha(\rho^i_0)^2 - \alpha(\rho^t_0)^2)(\alpha(\rho^{t'}_0)^2 - \alpha(\rho^i_0)^2)$$

which is non-zero because $\alpha(\rho_0^i)$, $\alpha(\rho_0^t)$, and $\alpha(\rho_0^{t'})$ are pairwise distinct by assumptions. The conclusion follows.

Case 2. $v \neq \rho_0^t, l = \rho_0^{t'}, w = \rho_0^i$. For this case, we consider the monomials $u_{\rho_0^{t'}} u_v p(U)$ and $u_{\rho_0^i} u_v p(U)$ in $\det W^1_{\rho_0^i}$ and $\det W^1_{\rho_0^{t'}}$, respectively. The conclusion of Case 1 shows that the entries $\tilde{\mathfrak{b}}^1_{\rho_0^i}$ and $\tilde{\mathfrak{b}}^1_{\rho_0^{t'}}$ do not contain the monomials $u_{\rho_0^{t'}} u_{\rho_0^t}$ and $u_{\rho_0^i} u_{\rho_0^t}$, respectively. Using this, together with Proposition 6.23, we see that the coefficient of $u_v u_{\rho_0^{t'}} p(U)$ in $\det W^1_{\rho_0^i}$ is

$$(\alpha(\rho_0^i)^2 - \alpha(v)^2) c^v_{\rho_0^{t'} \rho_0^i} + (\alpha(\rho_0^i)^2 - \alpha(\rho_0^{t'})^2) c^{\rho_0^{t'}}_{v \rho_0^i}.$$

Similarly, by permuting indices we find that the coefficient of $u_v u_{\rho_0^i} p(U)$ in $\det W^1_{\rho_0^{t'}}$ is

$$(\alpha(\rho_0^{t'})^2 - \alpha(v)^2) c^v_{\rho_0^i \rho_0^{t'}} + (\alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2) c^{\rho_0^i}_{v \rho_0^{t'}}.$$

If (**??**) holds, then the coefficients above define a system of homogeneous equations as before. In particular, using the anti-symmetry of the lower indices, we obtain

rv_rel1
$$(\alpha(\rho_0^i)^2 - \alpha(v)^2) c^v_{\rho_0^{t'} \rho_0^i} + (\alpha(\rho_0^i)^2 - \alpha(\rho_0^{t'})^2) c^{\rho_0^{t'}}_{v \rho_0^i} = 0 \tag{6.39}$$

rv_rel2
$$-(\alpha(\rho_0^{t'})^2 - \alpha(v)^2) c^v_{\rho_0^{t'} \rho_0^i} + (\alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2) c^{\rho_0^i}_{v \rho_0^{t'}} = 0. \tag{6.40}$$

On the other hand, by item (3) of Proposition 6.14 we have

irst_divi_3
$$(\alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2) c^v_{\rho_0^{t'} \rho_0^i} + (\alpha(v)^2 - \alpha(\rho_0^i)^2) c^{\rho_0^{t'}}_{v \rho_0^i} + (\alpha(v)^2 - \alpha(\rho_0^{t'})^2) c^{\rho_0^i}_{v \rho_0^{t'}} = 0. \tag{6.41}$$

The coefficient matrix

$$\begin{pmatrix} \alpha(\rho_0^i)^2 - \alpha(v)^2 & \alpha(\rho_0^i)^2 - \alpha(\rho_0^{t'})^2 & 0 \\ -(\alpha(\rho_0^{t'})^2 - \alpha(v)^2) & 0 & \alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2 \\ \alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2 & \alpha(v)^2 - \alpha(\rho_0^i)^2 & \alpha(v)^2 - \alpha(\rho_0^{t'})^2 \end{pmatrix}$$

of the homogeneous system of equations (6.39),(6.40), and (6.41) has determinant equal to [11]

det_line1
$$(\alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2)\big((\alpha(v)^2 - \alpha(\rho_0^{t'})^2)^2 + (\alpha(v)^2 - \alpha(\rho_0^i)^2)^2 - (\alpha(\rho_0^i)^2 - \alpha(\rho_0^{t'}))^2\big) \tag{6.42}$$

Using the elementary identity $a^2 + b^2 - (b-a)^2 = 2ab$ applied to $a = \alpha(v)^2 - \alpha(\rho_0^{t'})^2$ and $b = \alpha(v)^2 - \alpha(\rho_0^i)^2)$ we get that this determinant is equal to

det_line2
$$2(\alpha(\rho_0^{t'})^2 - \alpha(\rho_0^i)^2)(\alpha(v)^2 - \alpha(\rho_0^{t'})^2)(\alpha(v)^2 - \alpha(\rho_0^i)^2), \tag{6.43}$$

which is not zero because $\alpha(\rho_0^i)$, $\alpha(v)$, and $\alpha(\rho_0^{t'})$ are pairwise distinct by assumption. This completes the proof for this case.

Case 3. $v \neq \rho_0^t, l \neq \rho_0^{t'}, w = \rho_0^i$ *and v is a depth* 0 *vertex.* In this case, we will consider the monomials $u_v u_l p(U)$ and $u_{\rho_0^i} u_l p(U)$ in $\det W^1_{\rho_0^i}$ and $\det W^1_v$, respectively. We invoke Proposition 6.23 once more to limit our consideration to the terms $\tilde{\mathfrak{b}}^1_{\rho_0^i} \det T$ and $\tilde{\mathfrak{b}}^1_v \det T$. Note that the previous cases implies that $u_{\rho_0^{t'}}$ and $u_{\rho_0^t}$ do not appear in $\tilde{\mathfrak{b}}^1_{\rho_0^i}$. It follows that the factor $u_v u_l$ must come from the term $\tilde{\mathfrak{b}}^1_{\rho_0^i}$ (otherwise u_v or u_l must come from $\det T$ thereby forcing $u_{\rho_0^{t'}}$ or $u_{\rho_0^t}$ to come from $\tilde{\mathfrak{b}}^1_{\rho_0^i}$). Thus, we conclude that the total coefficient of $u_v u_l p(U)$ in $\det W^1_{\rho_0^i}$ is

$$(\alpha(\rho_0^i)^2 - \alpha(v)^2) c^v_{l \rho_0^i} + (\alpha(\rho_0^i)^2 - \alpha(l)^2) c^l_{v \rho_0^i}.$$

[11] Up to this point, the proof, at least for the same monomial, follows exactly as in Lemma 4.16 of [8], which addresses a more specific case. However, we noticed that in [8], a sign error in one of the entries of the corresponding 3×3 matrix led to an incorrect expression for the determinant: the second factor in the expression analogous to (6.42) was mistakenly written as a sum of three squares. This discrepancy renders the subsequent arguments in [8] invalid. In contrast, subsequent expression (6.43) for the determinant demonstrates how to correctly modify the arguments to complete the proof.

Moving now to $\det W^1_v$, the same arguments with respect to the Tanaka symbol apply and we conclude that the first row is independent of variables from components different from Γ_t except the entry $\tilde{\mathfrak{b}}^1_v$. In the same way as before, this implies that the only contributions to the coefficient of $u_{\rho^i_0}u_l p(U)$ may come from the term $\tilde{\mathfrak{b}}^1_v \det T$ in the cofactor expansion along the v-row of W^1_v. Moreover, Proposition **??** implies that $c^{\rho^i_0}_{\rho^{t'}_0 v} = c^{\rho^{t'}_0}_{\rho^i_0 v} = 0$. Thus, $u_{\rho^i_0}$ and $u_{\rho^{t'}_0}$ cannot be simultaneous factors of a non-zero monomial appearing in the entry $\tilde{\mathfrak{b}}^1_v$. It follows that the factor $u_{\rho^i_0}u_l$ must come from $\tilde{\mathfrak{b}}^1_v$, hence, the total coefficient of $u_{\rho^i_0}u_l p(U)$ in $\det W^1_v$ is

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^{\rho^i_0}_{lv} + (\alpha(v)^2 - \alpha(l)^2)c^l_{\rho^i_0 v}.$$

If $\det W^1_{\rho^i_0} = \det W^1_v = 0$, then the above coefficients define a system of homogeneous equations. As before, we obtain a third equation from (3) of Proposition 6.14, namely

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^l_{v\rho^i_0} + (\alpha(\rho^i_0)^2 - \alpha(l)^2)c^v_{\rho^i_0 l} + (\alpha(v)^2 - \alpha(l)^2)c^{\rho^i_0}_{vl} = 0$$

Using anti-symmetry in the lower indices, this system of equations becomes

$$(\alpha(\rho^i_0)^2 - \alpha(v)^2)c^v_{l\rho^i_0} + (\alpha(\rho^i_0)^2 - \alpha(l)^2)c^l_{v\rho^i_0} = 0$$

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^{\rho^i_0}_{lv} - (\alpha(v)^2 - \alpha(l)^2)c^l_{v\rho^i_0} = 0$$

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^l_{v\rho^i_0} - (\alpha(\rho^i_0)^2 - \alpha(l)^2)c^v_{l\rho^i_0} - (\alpha(v)^2 - \alpha(l)^2)c^{\rho^i_0}_{lv} = 0.$$

The coefficient matrix of this system of equations is therefore given by

$$\begin{pmatrix} (\alpha(\rho^i_0)^2 - \alpha(v)^2) & (\alpha(\rho^i_0)^2 - \alpha(l)^2) & 0 \\ 0 & -(\alpha(v)^2 - \alpha(l)^2) & -(\alpha(\rho^i_0)^2 - \alpha(v)^2) \\ -(\alpha(\rho^i_0)^2 - \alpha(l)^2) & -(\alpha(\rho^i_0)^2 - \alpha(v)^2) & (\alpha(v)^2 - \alpha(l)^2) \end{pmatrix}$$

and whose determinant is equal to

$$-(\alpha(\rho^i_0)^2 - \alpha(v)^2)[(\alpha(\rho^i_0)^2 - \alpha(v)^2)^2 + (\alpha(v)^2 - \alpha(l)^2)^2 - (\alpha(\rho^i_0)^2 - \alpha(l)^2)].$$

By the same arguments presented in the final paragraph of the proof of Proposition **??**, this determinant is non-zero, and so the claim of the proposition follows for this case.

Case 4. *$v \neq \rho^t_0, l \neq \rho^{t'}_0, w = \rho^i_0$ and v is a vertex with depth greater than* 0. In this case, we will consider the monomials $u_v u_l p(U)$ and $u_{\rho^i_0}u_l u_{\overrightarrow{\rho^t_0 v}}p(U)$ from $\det W^1_{\rho^i_0}$ and $\det W^2_{\rho^t_0}$, respectively. As in Case 3 above, we will use item (3) of Proposition 6.14 and the same cofactor expansion for $\det W^1_{\rho^i_0}$. Since the arguments presented there do not rely on the depths of the vertices, we immediately obtain the same equations as before (assuming $\det W^1_{\rho^i_0} = 0$), namely

$$(\alpha(\rho^i_0)^2 - \alpha(v)^2)c^v_{l\rho^i_0} + (\alpha(\rho^i_0)^2 - \alpha(l)^2)c^l_{v\rho^i_0} = 0$$

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^l_{v\rho^i_0} - (\alpha(\rho^i_0)^2 - \alpha(l)^2)c^v_{l\rho^i_0} - (\alpha(v)^2 - \alpha(l)^2)c^{\rho^i_0}_{lv} = 0.$$

Now we consider $\det W^2_{\rho^t_0}$. By Proposition 6.28, we see that contributions to the coefficient of $u_{\rho^i_0}u_l u_{\overrightarrow{\rho^t_0 v}}p(U)$ may only come from the term $\tilde{\mathfrak{b}}^1_v \mathfrak{a}^2_{\rho^t_0 \overrightarrow{\rho^t_0 v}} \det T_{\hat{v}\overrightarrow{\rho^t_0 v}}$. Using the conclusion of Case 2, we conclude the factor $u_{\rho^i_0}u_l$ must come from $\tilde{\mathfrak{b}}^1_v$. Thus, the total coefficient of $u_{\rho^i_0}u_l u_{\overrightarrow{\rho^t_0 v}}p(U)$ in $\det W^2_{\rho^t_0}$ must be

$$(\alpha(v)^2 - \alpha(\rho^i_0)^2)c^{\rho^i_0}_{lv} + (\alpha(v)^2 - \alpha(l)^2)c^l_{\rho^i_0 v}.$$

If $\det W^2_{\rho_0^t} = 0$, then we obtain the same system of equations as in Case 1 above. The conclusion for this case then follows by the same arguments.

Case 5. *$v \neq \rho_0^t, l \neq \rho_0^{t'}, w \neq \rho_0^i$ and v, w are both depth 0 vertices.* We consider the monomials $u_vu_lp(U)$ and $u_wu_lp(U)$ in $\det W^1_w$ and $\det W^1_v$, respectively. Proposition 6.23 shows that contributions to the coefficients of the monomials $u_vu_lp(U)$ and $u_wu_lp(U)$ only come from the terms $\tilde{\mathfrak{b}}^1_w \det T$ and $\tilde{\mathfrak{b}}^1_v \det T$ in the cofactor expansions of $\det W^1_w$ and $\det W^1_v$, respectively. From (6.4) and the conclusions of the previous cases, it follows that the total coefficients of $u_vu_lp(U)$ and $u_wu_lp(U)$ in $\det W^1_w$ and $\det W^1_v$ are

$$(\alpha(w)^2 - \alpha(v)^2)c^v_{lw} + (\alpha(w)^2 - \alpha(l)^2)c^l_{vw}$$

and

$$(\alpha(v)^2 - \alpha(l)^2)c^l_{wv} + (\alpha(v)^2 - \alpha(w)^2)c^w_{lv},$$

respectively. With item (3) of Corollary 6.14 and the assumptions that $\det W^1_w = \det W^1_v = 0$, we arrive at a similar set of homogeneous equations as in the previous cases. The remainder of the proof for this case is so similar to those that it is omitted for brevity.

Case 6. *$v \neq \rho_0^t, l \neq \rho_0^{t'}, w \neq \rho_0^i$, v is depth 0, and w has depth greater than 0.* In this case we examine the monomials $u_vu_lu_{\overrightarrow{\rho_0^iw}}p(U)$ and $u_wu_lp(U)$ from $\det W^2_{\rho_0^i}$ and $\det W^1_v$, respectively. From Case 1, we immediately have that the coefficient of $u_wu_lp(U)$ in $\det W^1_v$ is

$$(\alpha(v)^2 - \alpha(l)^2)c^l_{wv} + (\alpha(v)^2 - \alpha(w)^2)c^w_{lv}.$$

Similarly, we conclude that the coefficient of $u_vu_lu_{\overrightarrow{\rho_0^iw}}p(U)$ in $\det W^2_{\rho_0^i}$ is

$$(\alpha(w)^2 - \alpha(v)^2)c^v_{lw} + (\alpha(w)^2 - \alpha(l)^2)c^l_{vw}.$$

As before, this case follows from the assumptions that $\det W^1_v = \det W^2_{\rho_0^i} = 0$ and item (3) of Corollary 6.14.

Case 7. *$v \neq \rho_0^t, l \neq \rho_0^{t'}, w \neq \rho_0^i$ and both v, w are vertices with depth greater than 0.* In this case we consider the monomials $u_vu_lu_{\overrightarrow{\rho_0^iw}}p(U)$ and $u_wu_lu_{\overrightarrow{\rho_0^tv}}p(U)$ in $\det W^2_{\rho_0^i}$ and $\det W^2_{\rho_0^t}$, respectively. We immediately obtain the coefficients

$$(\alpha(w)^2 - \alpha(v)^2)c^v_{lw} + (\alpha(w)^2 - \alpha(l)^2)c^l_{vw}$$

and

$$(\alpha(v)^2 - \alpha(l)^2)c^l_{wv} + (\alpha(v)^2 - \alpha(w)^2)c^w_{lv}$$

by the same arguments as the previous cases. Using once more item (3) of Corollary 6.14 and the assumptions $\det W^2_{\rho_0^i} = \det W^2_{\rho_0^t} = 0$, the conclusion follows in the same manner. □

b10 **Corollary 6.32.** *If the maximal minors of the fundamental system vanish, then the entries of $\tilde{\mathfrak{b}}^1$ are identically 0.*

In fact, more can be said about $\tilde{\mathfrak{b}}$. Having established Corollary 6.32, one can show the following:

b00 **Corollary 6.33.** [8, *Proof of Proposition 4.4*, p.25] *The entries of $\tilde{\mathfrak{b}}$ are identically 0.*

ady **Proposition 6.34.** *The following inclusions hold for $i, j \in [1 : k]$:*

(1) $\mathrm{ad}X_y(D^2_i) \subset D^2_i + D^2_t$ *whenever* $y \in V(\Gamma_t), t \neq i$
(2) $\mathrm{ad}X_y(D^2_i) \subset D^2_i$ *whenever* $y \in V(\Gamma_i)$

Proof. Together with Propositions 6.29, 6.30, and 6.31, the conclusion will follow by showing that structure functions of the form c^w_{yl} vanish whenever $l, w \in E(\Gamma)$ such that $\mathcal{C}(w) \neq \mathcal{C}(l) = i$

for (2) and when $C(w) \neq i, t$ for (1). To that end, we will analyze the entries of the column vector $\tilde{\mathfrak{b}}^2$. By (6.14) and Corollary 6.32, it follows that an entry $\tilde{\mathfrak{b}}^2_j$ of $\tilde{\mathfrak{b}}^2$ is given by

$$\sum_{\substack{l, w \in E(\Gamma) \\ \mathcal{C}(l) = \mathcal{C}(j), \mathcal{C}(w) \neq \mathcal{C}(l)}} (\alpha(l)^2 - \alpha(w)^2) q_{jl} q_{lw} u_w.$$

Which, by (5.12), is equal to

$$\sum_{x,y \in V(\Gamma)} \sum_{\substack{l, w \in E(\Gamma) \\ \mathcal{C}(l) = \mathcal{C}(j), \mathcal{C}(w) \neq \mathcal{C}(l)}} (\alpha(l)^2 - \alpha(w)^2) c^l_{xj} c^w_{yl} u_w u_x u_y.$$

Since $\mathcal{C}(l) = i$, the above expression vanishes unless both $j, x \in V(\Gamma_i)$ by (4.1). Thus, the sum over $x \in V(\Gamma)$ collapses to a sum over $x \in V(\Gamma_i)$.

It is advantageous to split the sum into two parts, one in which $y \in V(\Gamma_i)$ and one in which $y \notin V(\Gamma_i)$, namely

sum1
$$\sum_{x,y \in V(\Gamma_i)} \sum_{\substack{l, w \in E(\Gamma) \\ \mathcal{C}(l) = \mathcal{C}(j), \mathcal{C}(w) \neq \mathcal{C}(l)}} (\alpha(l)^2 - \alpha(w)^2) c^l_{xj} c^w_{yl} u_w u_x u_y \tag{6.44}$$

sum2
$$+ \sum_{x \in V(\Gamma_i)} \sum_{y \notin V(\Gamma_i)} \sum_{\substack{l, w, \in E(\Gamma) \\ \mathcal{C}(w) \neq \mathcal{C}(l)}} (\alpha(l)^2 - \alpha(w)^2) c^l_{xj} c^w_{yl} u_w u_x u_y \tag{6.45}$$

Now suppose j is a root of Γ_i, say ρ^i_s. If we restrict ourselves to terms such that $\mathrm{depth}(x) > \mathrm{depth}(\rho^i_s) = s$, then the summation (6.45) reduces to

sum3
$$\sum_{\substack{x \in V(\Gamma_i) \\ \mathrm{depth}(x) > s}} \sum_{y \notin V(\Gamma_i)} \sum_{\substack{l, w \in E(\Gamma) \\ \mathcal{C}(w) \neq \mathcal{C}(l)}} -(\alpha(l)^2 - \alpha(w)^2) c^w_{y\overrightarrow{\rho^i_s x}} u_w u_x u_y$$

by (5.6) and (3.1). On the other hand, if x and y are permuted in (6.45), then $c^l_{y\rho^i_s} = 0$ by (4.1). It follows that $-(\alpha(\overrightarrow{\rho^i_s x})^2 - \alpha(w)^2) c^w_{y\overrightarrow{\rho^i_s x}}$ is the total coefficient of the monomial $u_w u_x u_y$ for $y \notin V(\Gamma_i)$, $x \in V(\Gamma_i)$, and $\mathrm{depth}(x) > s$ in the entry $\tilde{\mathfrak{b}}^2_{\rho^i_s}$. Using Corollary 6.33, we conclude that

diffy
$$c^w_{y\overrightarrow{\rho^i_s x}} = 0$$

for such w, x, y.

If $l \in E(\Gamma_i)$, then l is of the form $\overrightarrow{\rho^i_s x}$ for some root ρ^i_s of Γ_i and some vertex $x \in V(\Gamma_i)$ with $\mathrm{depth}(x) > s$, i.e. $X_l = [X_{\rho^i_s}, X_x]$ by (3.1). Thus we have

adyl
$$\mathrm{ad} X_y(X_l) = [\mathrm{ad} X_y(X_{\rho^i_s}), X_x] + [X_{\rho^i_s}, \mathrm{ad} X_y(X_x)]. \tag{6.46}$$

Suppose $\mathcal{C}(y) = t$. Since $\mathcal{C}(y) \neq \mathcal{C}(l)$ by assumption, Proposition 6.31 and (4.1) shows that $\mathrm{ad} X_y(X_{\rho^i_s})$ and $\mathrm{ad} X_y(X_x)$ must belong to $D_i + D_t$. This, together with Proposition 6.31 and (4.1) applied to the brackets on the right side of (6.46), implies that these brackets must belong to $D^2_i + D^2_t$.

Now we return our attention to (6.44), and take $j \in V(\Gamma_i)$ such that $\mathrm{depth}(j) > 0$. We will analyze the coefficients of two types of monomials in this entry, namely, $u^2_{\rho^i_0} u_w$ and $u_{\rho^i_0} u_v u_w$ where $v \in V(\Gamma_i)$ is a depth 0 vertex different from ρ^i_0. First, consider the monomial $u^2_{\rho^i_0} u_w$

(i.e., x and y are both equal to ρ_0^i in (6.44)) for a given $w \in E(\Gamma)$ with $\mathcal{C}(w) \neq i$. Clearly, this monomial appears in only one term of the summation and its coefficient is

$$(\alpha(\overrightarrow{\rho_0^i j})^2 - \alpha(w)^2) c^w_{\rho_0^i \overrightarrow{\rho_0^i j}}$$

(using the fact that $c^{\overrightarrow{\rho_0^i j}}_{\rho_0^i j} = 1$ by (5.6)). We conclude

cwterm
$$c^w_{\rho_0^i \overrightarrow{\rho_0^i j}} = 0 \tag{6.47}$$

by Corollary 6.33. On the other hand, for monomials of the form $u_{\rho_0^i} u_v u_w$, we see that the total coefficient in (6.44) is given by

$$\sum_{l \in E(\Gamma)} (\alpha(l)^2 - \alpha(w)^2) c^l_{vj} c^w_{\rho_0^i l} u_w u_{\rho_0^i} u_v + (\alpha(\overrightarrow{\rho_0^i j})^2 - \alpha(w)^2) c^{\overrightarrow{\rho_0^i j}}_{\rho_0^i j} c^w_{v \overrightarrow{\rho_0^i}} u_w u_{\rho_0^i} u_v$$

sum4
$$= \sum_{l \in E(\Gamma)} (\alpha(l)^2 - \alpha(w)^2) c^l_{vj} c^w_{\rho_0^i l} u_w u_{\rho_0^i} u_v + (\alpha(\overrightarrow{\rho_0^i j})^2 - \alpha(w)^2) c^w_{v \overrightarrow{\rho_0^i j}} u_w u_{\rho_0^i} u_v. \tag{6.48}$$

However, by (5.7) and (4.1), the summation over $l \in E(\Gamma)$ reduces to a summation over those $l \in E(\Gamma_i)$ such that $l = \overrightarrow{\rho_0^i x}$. Thus (6.48) becomes

sum5
$$\sum_{\substack{x \in V(\Gamma_i) \\ \mathrm{depth}(x) > 0}} (\alpha(\overrightarrow{\rho_0^i x})^2 - \alpha(w)^2) c^{\overrightarrow{\rho_0^i x}}_{vj} c^w_{\rho_0^i \overrightarrow{\rho_0^i x}} u_w u_{\rho_0^i} u_v + (\alpha(\overrightarrow{\rho_0^i j})^2 - \alpha(w)^2) c^w_{v \overrightarrow{\rho_0^i j}} u_w u_{\rho_0^i} u_v.$$

Now, using Corollary 6.33 and (6.47), it follows that

cw0
$$c^w_{v \overrightarrow{\rho_0^i j}} = 0, \quad v, j \in V(\Gamma_i), \mathrm{depth}(j) > 0, \mathrm{depth}(v) = 0$$

whenever $\mathcal{C}(w) \neq i$.

More generally, if we fix a root $\rho_s^i \in V(\Gamma_i)$ and $j \in V(\Gamma_i)$ such that $\mathrm{depth}(j) > s$, then we can follow the same arguments presented above for the monomial $u^2_{\rho_0^i} u_w$. We consider the monomials $u^2_{\rho_s^i} u_w$ and $u_j^2 u_w$ in the entries $\tilde{\mathfrak{b}}_j^2$ and $\tilde{\mathfrak{b}}^2_{\rho_s^i}$, respectively. By the same arguments as in the case of the monomial $u^2_{\rho_0^i} u_w$ above, we may conclude that the structure functions of the form $c^w_{\rho_s^i \overrightarrow{\rho_s^i j}}$ and $c^w_{j \overrightarrow{\rho_s^i j}}$ vanish for any $w \in E(\Gamma)$ such that $\mathcal{C}(w) \neq i$, $\mathrm{depth}(j) > s$.

Now let $\rho_s^i, \rho_t^i \in V(\Gamma_i)$ be roots of Γ_i such that $s < t$ and $j \in V(\Gamma_i)$ be a vertex such that $\mathrm{depth}(j) > t$. We will consider the coefficients of the monomials $u_{\rho_s^i} u_{\rho_t^i} u_w$, $u_j u_{\rho_s^i} u_w$, and $u_j u_{\rho_t^i} u_w$ in the entries $\tilde{\mathfrak{b}}_j^2$, $\tilde{\mathfrak{b}}^2_{\rho_t^i}$, and $\tilde{\mathfrak{b}}^2_{\rho_s^i}$, respectively. From (6.44) we calculate their respective coefficients to be

$$(\alpha(\overrightarrow{\rho_s^i j})^2 - \alpha(w)^2) c^w_{\rho_t^i \overrightarrow{\rho_s^i j}} + (\alpha(\overrightarrow{\rho_t^i j})^2 - \alpha(w)^2) c^w_{\rho_s^i \overrightarrow{\rho_t^i j}},$$

$$(\alpha(\overrightarrow{\rho_s^i \rho_t^i})^2 - \alpha(w)^2) c^w_{j \overrightarrow{\rho_s^i \rho_t^i}} - (\alpha(\overrightarrow{\rho_t^i j})^2 - \alpha(w)^2) c^w_{\rho_s^i \overrightarrow{\rho_t^i j}},$$

and

$$-(\alpha(\overrightarrow{\rho_s^i \rho_t^i})^2 - \alpha(w)^2) c^w_{j \overrightarrow{\rho_s^i \rho_t^i}} - (\alpha(\overrightarrow{\rho_s^i j})^2 - \alpha(w)^2) c^w_{\rho_t^i \overrightarrow{\rho_s^i j}}.$$

By Corollary 6.33, this set of coefficients defines a system of equations:

$$(\alpha(\overrightarrow{\rho_s^i j})^2 - \alpha(w)^2) c^w_{\rho_t^i \overrightarrow{\rho_s^i j}} + (\alpha(\overrightarrow{\rho_t^i j})^2 - \alpha(w)^2) c^w_{\rho_s^i \overrightarrow{\rho_t^i j}} = 0$$

$$(\alpha(\overrightarrow{\rho_s^i \rho_t^i})^2 - \alpha(w)^2) c^w_{j \overrightarrow{\rho_s^i \rho_t^i}} - (\alpha(\overrightarrow{\rho_t^i j})^2 - \alpha(w)^2) c^w_{\rho_s^i \overrightarrow{\rho_t^i j}} = 0$$

$$-(\alpha(\overrightarrow{\rho^i_s\rho^i_t})^2-\alpha(w)^2)c^w_{\overrightarrow{j\rho^i_s\rho^i_t}}-(\alpha(\overrightarrow{\rho^i_s j})^2-\alpha(w)^2)c^w_{\overrightarrow{\rho^i_t\rho^i_s j}}=0.$$

Since $w\notin E(\Gamma_i)$, the factors of α's are all non-zero. It follows that

c000
$$c^w_{\overrightarrow{\rho^i_t\rho^i_s j}}=c^w_{\overrightarrow{\rho^i_s\rho^i_t j}}=c^w_{\overrightarrow{j\rho^i_s\rho^i_t}}=0 \tag{6.49}$$

In particular, if we take $s=0$, then by Proposition 6.29 we have that

ad0
$$\mathrm{ad}X_{\rho^i_0}(D^2_i)\subset D^2_i. \tag{6.50}$$

The inclusion (6.50) implies that, if we consider a monomial of the form $u_v u_{\rho^i_0} u_w$ in (6.44) from the entry $\tilde{\mathfrak{b}}^2_j$ with $v,j\in V(\Gamma_i)$ such that $\mathrm{depth}(j)>0$, then the total coefficient of such a monomial is given by

sum6
$$-(\alpha(\overrightarrow{\rho^i_0 j})^2-\alpha(w)^2)c^w_{\overrightarrow{v\rho^i_0 j}}.$$

Thus, by Corollary 6.33,

c0000
$$c^w_{\overrightarrow{v\rho^i_0 j}}=0. \tag{6.51}$$

By taking $s=1$ in (6.49) and $v=\rho^i_1$ in (6.51), we similarly conclude that

ad1
$$\mathrm{ad}X_{\rho^i_1}(D^2_i)\subset D^2_i \tag{6.52}$$

Now we proceed in a similar manner to that which led to (6.51) by considering monomials of the form $u_v u_{\rho^i_1} u_w$ in the entry $\tilde{\mathfrak{b}}^2_j$ where $v\in V(\Gamma_i)$ and $\mathrm{depth}(j)>1$. From (6.44), using (6.52), we find that the total coefficient of such a monomial in $\tilde{\mathfrak{b}}^2_j$ is

$$-(\alpha(\overrightarrow{\rho^i_0 j})^2-\alpha(w)^2)c^w_{\overrightarrow{v\rho^i_1 j}}.$$

It follows from Corollary 6.33 that

$$c^w_{\overrightarrow{v\rho^i_1 j}}=0.$$

Proceeding inductively, we conclude that $c^w_{\overrightarrow{v\rho^i_s j}}=0$ for any $v,j\in V(\Gamma_i)$, $\mathrm{depth}(j)>0$, and $s\in[1:\omega_i]$. This, together with (4.1) and Propositions 6.29 and 6.30, implies that $\mathrm{ad}X_v(D^2_i)\subset D^2_i$ for $v\in V(\Gamma_i)$. □

From the combination of the previous propositions, we have the following

involutive **Corollary 6.35.** *The distributions D^2_i and $D^2_i\oplus D^2_j$ are involutive for $i,j\in[1:k]$.*

Proof. The Corollary follows from the Jacobi identity and Proposition 6.34. □

To finish the proof of the main theorem, we need to rotate the part of the frame belonging to D (i.e., $\{X_v\}_{v\in V(\Gamma)}$) to a suitable frame $\{\widetilde{X}_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ via a procedure described in [8]. We cite the results here without proof:

By Corollary 6.35, $\bigoplus_{s\in S} D^2_s$ is involutive for every subset S of $[1:k]$. We denote by $\mathcal{F}_S(q)$ the leaf of the foliation of $\bigoplus_{s\in S} D^2_s$ passing through the point $q\in M$. Now fix an $i\in[1:k]$ and let $S_i=[1:k]/\{i\}$. The distributions D^2_i and $\bigoplus_{s\in S_i} D^2_s$ are transversal, thus, for a sufficiently small neighborhood $\mathcal{U}\subset M$, the leaves $F_{\{i\}}(q_0)$ and $F_{S_i}(q_1)$ intersect at a single point whenever $q_0,q_1\in\mathcal{U}$. Therefore, we define the projection map $\pi^{q_0}_i$ by

$$\pi^{q_0}_i: U\to\mathcal{F}_{\{i\}}(q_0)$$
$$q\to F_{\{i\}}(q_0)\cap F_{S_i}(q_1).$$

Since these projections are diffeomorphisms of $\mathcal{F}_{\{i\}}(q)\cap\mathcal{U}$ to $\mathcal{F}_{\{i\}}(q_0)\cap\mathcal{U}$, the following can be shown:

zaithm **Theorem 6.36.** [8] *If g_1 and g_2 are affinely equivalent and not constantly proportional, then there exists a g_1 orthonormal frame $\{\widetilde{X}_v\}_{v\in V(\Gamma)}$ on D such that the following hold:*

(1) *$\widetilde{X}_v$ belongs to D_i for each $v \in V(\Gamma_i)$ and $i \in [1:k]$*
(2) *$d\pi_i^{q_0}(\widetilde{X}_v) = X_v$*
(3) *$[\widetilde{X}_v, \widetilde{X}_w] = 0$ for $\mathcal{C}(v) \neq \mathcal{C}(w)$.*

If, instead, g_1 and g_2 are projectively equivalent (and not affinely equivalent) and non-conformal, then the above holds with orthogonality replacing orthonormality.

The proof of the previous theorem is presented in Section 4.7 of [8] for the affine case. However, the proof presented there must be changed slightly for the case of projective equivalence. In particular, for a given D_i and $q \in M$, let $F_i(q)$ denote the matrix (with (v,w)-entry given by f^i_{vw}) such that

$$\widetilde{X}_v = \sum_{w\in V(\Gamma_i)} f^i_{vw} X_w$$

for $v \in V(\Gamma_i)$. The crucial step in the proof is the following equation (see equation (4.91) in [8]).

$$X_l(g_1(\widetilde{X}_v, \widetilde{X}_s)) = X_l(\sum_{w\in V(\Gamma_i)} f^i_{vw} f^i_{sw}) = -\sum_{w,j\in V(\Gamma_i)} (c^w_{lj} + c^j_{lw}) f^i_{vj} f^i_{sw}$$

for $l \in V(\Gamma_t)$ with $t \neq i$. Item (3) of Proposition 6.14 shows that $(c^w_{lj} + c^j_{lw}) = 0$ whenever $w \neq j$. However, if $w = j$, then $(c^w_{lw} + c^w_{lw}) = 0$ only in the affine case. Indeed, when two metrics are affinely equivalent, then the eigenvalues of the transition operator are constant, hence item (1) of Proposition 6.14 implies that $c^w_{lw} = 0$. Thus $g_1(\widetilde{X}_v, \widetilde{X}_s)$ is locally constant on leaves of $\mathcal{F}_{[1:k]\setminus\{i\}}$. It follows that orthonormality is preserved in a sufficiently small neighborhood by item (2) of Theorem 6.36 and the orthonormality of the original frame.

On the other hand, if we have projectively equivalent metrics which are not affinely equivalent, then the term $(c^w_{lw} + c^w_{lw})$ does not necessarily vanish. In that case, we have

diffq
$$X_l(g_1(\widetilde{X}_v, \widetilde{X}_s)) = X_l(\sum_{w\in V(\Gamma_i)} f^i_{vw} f^i_{sw}) = -(c^w_{lw} + c^w_{lw}) f^i_{vw} f^i_{sw}. \tag{6.53}$$

Now, by the Picard-Lindelöf Theorem, the solution to this ODE is unique. When $v \neq s$, the solution is identically 0 by item (2) of Theorem 6.36 and the orthonormality of the original frame. By the same arguments as in the affine case, orthogonality is preserved in a sufficiently small neighborhood. The distinction between the affine case and the projective case appears when $v = s$ since $g_1(d\pi_i^{q_0}\widetilde{X}_v, d\pi_i^{q_0}\widetilde{X}_v) = 1 \neq 0$. Thus, normality cannot be guaranteed from (6.53) in the projective case. Nevertheless, we can still prove a slightly weaker version of the Levi-Civita Theorem for the case of projective equivalence. The following lemmas are in order (see Lemmas 7 and 8 in [12] for their Riemannian variants).

Lemma 6.37. *For each $i \in [1:k]$ there exists a metric h_i on D_i and a function γ_i such that*

metric1
$$g_1 = \sum_{i=1}^k \gamma_i (\pi_i^{q_0})^* h_i$$

Proof. Notice that the conclusion of the lemma is equivalent to showing that, for each $i \in [1:k]$ and for any $X \in D_i(q)$, there is a metric h_i on $\mathcal{F}_{\{i\}}$ and function γ_i such that

metric2
$$g_1(X,X) = \gamma_i(q) h_i(d\pi_i^{q_0} X, d\pi_i^{q_0} X).$$

To show this, let h_i be the restriction of g_1 to the leaf $\mathcal{F}_{\{i\}}(q_0)$ passing through $q_0 \in \mathcal{U}$. Fix $u \in D_i(q_1)$ such that $g_1(u,u) = 1$. Since $\pi_i^{q_0}$ is a diffeomorphism from $\mathcal{F}_{\{i\}}(q) \cap \mathcal{U} \to \mathcal{F}_{\{i\}}(q_0) \cap \mathcal{U}$, there is a unique $X^u \in D_i(q)$ such that $d\pi_i^{q_0} X^u = u$. Define the function ϵ_u on $\mathcal{F}_{S_i}(q_1)$ by

$$\epsilon_u = g_1(X^u, X^u).$$

Let $Z \in D_t$ with $t \neq i$ and $g_1(Z, Z) = 1$. We may choose a frame adapted to the pair g_1 and g_2 such that

Need to explicitly define adapted frame above

$$X_v = \frac{X^u}{\sqrt{\epsilon_u}}, \quad \forall q \in \mathcal{F}_{S_i}(q_1)$$
$$X_w = Z.$$

We can choose local coordinates $\{x_s\}_{s\in V(\Gamma)\cup E(\Gamma)}$ on M such that $\mathcal{F}_{\{i\}}$ is parallel to the coordinate subspace spanned by $\{\partial_{x_s}\}_{s\in V(\Gamma_i)\cup E(\Gamma_i)}$. Thus we can write u in this basis as

$$u = \sum_{s\in V(\Gamma_i)\cup E(\Gamma_i)} u_s \partial_{x_s}$$

□

appendix

Appendix A. Proof of (5.13)

Lemma A.1. *The following equalities hold:*

$$R_j = \sum_{s,v\in V(\Gamma)\, s\neq v} \left(\alpha(j)^2 - \alpha(s)^2\right) c^s_{vj} u_v u_s + \alpha(j)^2 \sum_{s\in E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s.$$

Proof. From *need Rj ref*, we have

rj1

$$R_j = \vec{h}_1(\alpha(j)^2)u_j + \alpha(j)^2 \vec{h}_1(u_j) - \frac{1}{2}\alpha(j)^2 u_j \frac{\vec{h}_1(a^2)}{a^2} - \frac{1}{2}\sum X_j(\alpha(v)^2)u_v^2 - \sum \alpha(s)^2 c^s_{vj} u_v u_s. \tag{A.1}$$

Using (6.2), *need reference to h(P)/P=Q divisibility*, and the fact that α has no vertical dependence (i.e. no U-dependence), we can rewrite (A.1) as

rj2

$$\begin{aligned} R_j = &\sum_{v\in V(\Gamma)} X_v(\alpha(j)^2)u_v u_j + \alpha(j)^2 \sum_{s\in V(\Gamma)\cup E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s \\ &- \frac{1}{2}\alpha(j)^2 u_j \sum_{v\in V(\Gamma)} \frac{X_v(\alpha(v)^2)u_v}{\alpha(v)^2} - \frac{1}{2}\sum_{v\in V(\Gamma)} X_j(\alpha(v)^2)u_v^2 \\ &- \sum_{v,s\in V(\Gamma)} \alpha(s)^2 c^s_{vj} u_v u_s. \end{aligned} \tag{A.2}$$

In the above expression, we combine the second and last terms as follows:

$$\begin{aligned} &\alpha(j)^2 \sum_{s\in V(\Gamma)\cup E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s - \sum_{v,s\in V(\Gamma)} \alpha(s)^2 c^s_{vj} u_v u_s \\ &= \alpha(j)^2 \sum_{v,s\in V(\Gamma)} \left(1 - \frac{\alpha(s)^2}{\alpha(j)^2}\right) c^s_{vj} u_v u_s + \alpha(j)^2 \sum_{s\in E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s, \end{aligned}$$

which, by ***ref generalized divisibility property***, is equal to

rj3

$$\begin{aligned} &\alpha(j)^2 \left(\sum_{s,v\in V(\Gamma), s\neq v} \left(1 - \frac{\alpha(s)^2}{\alpha(j)^2}\right) c^s_{vj} u_v u_s - \frac{1}{2}\sum_{v\in V(\Gamma)} X_j\left(\frac{\alpha(v)^2}{\alpha(j)^2}\right) u_v^2 + \sum_{s\in E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s \right) \\ &= -\frac{\alpha(j)^2}{2} \sum_{v\in V(\Gamma)} X_j\left(\frac{\alpha(v)^2}{\alpha(j)^2}\right) u_v^2 + \sum_{s,v\in V(\Gamma)\, s\neq v} \left(\alpha(j)^2 - \alpha(s)^2\right) c^s_{vj} u_v u_s \\ &+ \alpha(j)^2 \sum_{s\in E(\Gamma)} \sum_{v\in V(\Gamma)} c^s_{vj} u_v u_s. \end{aligned} \tag{A.3}$$

Substituting (A.3) back into (A.2) for the first and last term, we obtain

rj4

$$\begin{aligned} R_j = &\sum_{v\in V(\Gamma)} X_v(\alpha(j)^2)u_v u_j - \frac{1}{2}\alpha(j)^2 u_j \sum_{v\in V(\Gamma)} \frac{X_v(\alpha(v)^2)u_v}{\alpha(v)^2} \\ &-\frac{1}{2}\sum_{v\in V(\Gamma)} X_j(\alpha(v)^2)u_v^2 - \frac{\alpha(j)^2}{2}\sum_{v\in V(\Gamma)} X_j\left(\frac{\alpha(v)^2}{\alpha(j)^2}\right)u_v^2 \\ &+\sum_{s,v\in V(\Gamma)\, s\neq v}\left(\alpha(j)^2-\alpha(s)^2\right)c_{vj}^s u_v u_s + \alpha(j)^2\sum_{s\in E(\Gamma)}\sum_{v\in V(\Gamma)} c_{vj}^s u_v u_s. \end{aligned} \tag{A.4}$$

Note that, for $v\neq j$, we have

$$\begin{aligned} X_j\left(\frac{\alpha(v)^2}{\alpha(j)^2}\right) &= \frac{1}{\alpha(j)}X_j\left(\frac{\alpha(v)^2}{\alpha(j)}\right) + \frac{\alpha(v)^2}{\alpha(j)}X_j\left(\frac{1}{\alpha(j)}\right) = \frac{\alpha(v)^2}{\alpha(j)}X_j\left(\frac{1}{\alpha(j)}\right) \\ &= -\frac{\alpha(v)^2}{\alpha(j)^3}X_j(\alpha(j)) \end{aligned}$$

by ***generalized divisibility***. Substituting this into the term

$$-\frac{\alpha(j)^2}{2}\sum_{v\in V(\Gamma)} X_j\left(\frac{\alpha(v)^2}{\alpha(j)^2}\right)u_v^2$$

in (A.4) and using ***P reference***, we obtain

rj5

$$\begin{aligned} R_j = &\sum_{v\in V(\Gamma)} X_v(\alpha(j)^2)u_v u_j - \frac{1}{2}\alpha(j)^2 u_j \sum_{v\in V(\Gamma)} \frac{X_v(\alpha(v)^2)u_v}{\alpha(v)^2} - \frac{1}{2}\sum_{v\in V(\Gamma)} X_j(\alpha(v)^2)u_v^2 \\ &+\frac{1}{2}\frac{X_j(\alpha(j))}{\alpha(j)}(P-\alpha(j)^2u_j^2) + \sum_{s,v\in V(\Gamma)\, s\neq v}\left(\alpha(j)^2-\alpha(s)^2\right)c_{vj}^s u_v u_s \\ &+\alpha(j)^2\sum_{s\in E(\Gamma)}\sum_{v\in V(\Gamma)} c_{vj}^s u_v u_s. \end{aligned} \tag{A.5}$$

Similarly, we may use ***general divisi*** to rewrite the term $-\frac{1}{2}\sum_{v\in V(\Gamma)} X_j(\alpha(v)^2)u_v^2$ as

$$-\frac{1}{2}\frac{X_j(\alpha(j))}{\alpha(j)}\sum_{v\in V(\Gamma), v\neq j}\alpha(v)^2u_v^2 - \frac{1}{2}X_j(\alpha(j)^2)u_j^2 = -\frac{1}{2}\frac{X_j(\alpha(j))}{\alpha(j)}(P-\alpha(j)^2u_j^2) - \frac{1}{2}X_j(\alpha(j)^2)u_j^2.$$

After substitution into (A.5) and effecting the cancellation, we are left with

rj6

$$\begin{aligned} R_j = &\sum_{v\in V(\Gamma)} X_v(\alpha(j)^2)u_v u_j - \frac{1}{2}\alpha(j)^2 u_j \sum_{v\in V(\Gamma)} \frac{X_v(\alpha(v)^2)u_v}{\alpha(v)^2} - \frac{1}{2}X_j(\alpha(j))^2u_j^2 \\ &+\sum_{s,v\in V(\Gamma)\, s\neq v}\left(\alpha(j)^2-\alpha(s)^2\right)c_{vj}^s u_v u_s + \alpha(j)^2\sum_{s\in E(\Gamma)}\sum_{v\in V(\Gamma)} c_{vj}^s u_v u_s. \end{aligned} \tag{A.6}$$

Notice that we may rewrite the first term $\sum_{v\in V(\Gamma)} X_v(\alpha(j)^2)u_v u_j$ (using ***generalized divisi***) as

rjj

$$\frac{\alpha(j)^2}{2}\sum_{v\in V(\Gamma), v\neq j}\frac{X_v(\alpha(v)^2)}{\alpha(v)^2}u_v u_j + X_j(\alpha(j)^2)u_j^2$$

which cancels with like terms in (A.6) leaving

rj7

$$R_j = \sum_{s,v\in V(\Gamma)\, s\neq v}\left(\alpha(j)^2-\alpha(s)^2\right)c_{vj}^s u_v u_s + \alpha(j)^2\sum_{s\in E(\Gamma)}\sum_{v\in V(\Gamma)} c_{vj}^s u_v u_s.$$

□

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