

3.2 n-player nonzero sum game

We can extend the two player game to the n -player case in similar way. It is defined by the following components:

- $\mathcal{N} = \{1, 2, \dots, n\}$ is the set of players.
- A_i is a finite action set of player $i \in \mathcal{N}$. A generic element of A_i is denoted by a_i .
- $A = \prod_{i \in \mathcal{N}} A_i$ denotes the set of all actions profiles of the players. A generic element of A is denoted by $a = (a_1, a_2, \dots, a_n)$.
- $(r_i(a))_{a \in A}$ is the payoff vector of player i where $r_i(a)$ is the payoff player i receives when the players choose actions a_1, a_2, \dots, a_n simultaneously.
- $\mathcal{T}_i = \{\tau_i \in \mathbb{R}^{|A_i|} \mid \sum_{a_i \in A_i} \tau_i(a_i) = 1, \tau_i(a_i) \geq 0, \forall a_i \in A_i\}$ denote the set of all mixed strategies of player i , $i \in \mathcal{N}$; $|A_i|$ is the cardinality of action set A_i .
- $\mathcal{T} = \prod_{i \in \mathcal{N}} \mathcal{T}_i$ is the set of mixed strategy profiles of the players, and $\mathcal{T}_{-i} = \prod_{j \in \mathcal{N}; j \neq i} \mathcal{T}_j$ is the set of mixed strategy profiles of all the players except player i . The generic elements of \mathcal{T} and \mathcal{T}_{-i} are denoted by $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ and $\tau_{-i} = (\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_n)$ respectively.

When each player i , $i \in \mathcal{N}$, chooses strategy τ_i simultaneously, the payoff of player i is given by

$$u_i(\tau) = \sum_{a \in A} \prod_{j \in \mathcal{N}} \tau_j(a_j) r_i(a). \quad (32)$$

For a given mixed strategy profile τ_{-i} of all the players except player i , the set of best response strategies of player i is given by

$$BR_i(\tau_{-i}) = \{\bar{\tau}_i \mid u_i(\bar{\tau}_i, \tau_{-i}) \geq u_i(\tau_i, \tau_{-i}), \forall \tau_i \in \mathcal{T}_i\}. \quad (33)$$

A strategy pair is said to be a Nash equilibrium iff for each $i \in \mathcal{N}$, the following inequality holds

$$u_i(\tau_i^*, \tau_{-i}^*) \geq u_i(\tau_i, \tau_{-i}^*), \forall \tau_i \in \mathcal{T}_i. \quad (34)$$

From (33) and (34), a strategy profile τ^* is a Nash equilibrium iff $\tau_i^* \in BR_i(\tau_{-i}^*)$ for all $i \in \mathcal{N}$.

3.2.1 Existence of Nash equilibrium

We show that there always exists a mixed strategy Nash equilibrium for a finite strategic game. We use Kakutani fixed point theorem to prove the result. Let $\mathcal{P}(\mathcal{T})$ be a power set of \mathcal{T} . Define a set valued map

$$G : \mathcal{T} \rightarrow \mathcal{P}(\mathcal{T})$$

such that

$$G(\tau) = \prod_{i \in \mathcal{N}} BR_i(\tau_{-i}).$$

A strategy profile τ is said to be a fixed point of $G(\cdot)$ if $\tau \in G(\tau)$, i.e., $\tau_i \in BR_i(\tau_{-i})$ for all $i \in \mathcal{N}$. This implies that a fixed point of $G(\cdot)$ is a Nash equilibrium of the game. Therefore, to show the existence of a Nash equilibrium it is enough to show that there exists a fixed point of the set valued map $G(\cdot)$.

Theorem 3.8. *There always exists a mixed strategy Nash equilibrium for an n -player finite strategic game.*

Proof. In order to show that G has a fixed point, we show that G satisfies all the following conditions of Kakutani fixed point theorem:

1. \mathcal{T} is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
2. $G(\tau)$ is non-empty and convex for all $\tau \in X$.
3. $G(\cdot)$ has closed graph: if $(\tau_n, \bar{\tau}_n) \rightarrow (\tau, \bar{\tau})$ with $\bar{\tau}_n \in G(\tau_n)$ for all n , then $\bar{\tau} \in G(\tau)$.

Condition 1 holds from the definition of \mathcal{T} . For fixed τ_{-i} , $u_i(\cdot, \tau_{-i})$ is a continuous function of τ_i . For each $i \in I$, $BR_i(\tau_{-i})$ is non-empty because a continuous function $u_i(\cdot, \tau_{-i})$ over a compact set \mathcal{T}_i always attains maxima. Hence, $G(\tau)$ is non-empty for all $\tau \in \mathcal{T}$. For each $i \in I$, $BR_i(\tau_{-i})$ is a convex set because $u_i(\cdot, \tau_{-i})$ is a linear function of τ_i . Hence, $G(\tau)$ is a convex set for all $\tau \in \mathcal{T}$. Now, we prove that $G(\cdot)$ is a closed graph. Assume that $G(\cdot)$ is not a closed graph, i.e., there is a sequence $(\tau^n, \bar{\tau}^n) \rightarrow (\tau, \bar{\tau})$ with $\bar{\tau}^n \in G(\tau^n)$ for all n , but $\bar{\tau} \notin G(\tau)$. In this case, $\bar{\tau}_i \notin BR_i(\tau_{-i})$ for some $i \in I$. Then, there is an $\epsilon > 0$ and a $\tilde{\tau}_i$ such that

$$u_i(\tilde{\tau}_i, \tau_{-i}) > u_i(\bar{\tau}_i, \tau_{-i}) + 3\epsilon. \quad (35)$$

Since, $u_i(\cdot)$ is a continuous function of τ , $u_i(\bar{\tau}_i^n, \tau_{-i}^n) \rightarrow u_i(\bar{\tau}_i, \tau_{-i})$. Then, there exists an integer N_1 such that

$$u_i(\bar{\tau}_i^n, \tau_{-i}^n) < u_i(\bar{\tau}_i, \tau_{-i}) + \epsilon, \quad \forall n \geq N_1. \quad (36)$$

From (35) and (36), we have

$$u_i(\bar{\tau}_i^n, \tau_{-i}^n) < u_i(\tilde{\tau}_i, \tau_{-i}) - 2\epsilon, \quad \forall n \geq N_1. \quad (37)$$

Similarly, $u_i(\tilde{\tau}_i, \tau_{-i}^n) \rightarrow u_i(\tilde{\tau}_i, \tau_{-i})$. Then, there exists an integer N_2 such that

$$u_i(\tilde{\tau}_i, \tau_{-i}^n) < u_i(\tilde{\tau}_i, \tau_{-i}) + \epsilon, \quad \forall n \geq N_2. \quad (38)$$

Let $N = \max\{N_1, N_2\}$. Then, from (37) and (38), we have

$$u_i(\tilde{\tau}_i, \tau_{-i}^n) > u_i(\bar{\tau}_i^n, \tau_{-i}^n) + \epsilon, \quad \forall n \geq N.$$

That is, $\tilde{\tau}_i$ performs better than $\bar{\tau}_i^n$ against τ_{-i}^n for all $n \geq N$ which contradicts $\bar{\tau}_i^n \in BR_i(\tau_{-i}^n)$ for all n . Hence, $G(\cdot)$ is a closed graph. That is, the set valued map $G(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem. Hence, $G(\cdot)$ has a fixed point τ^* . Such τ^* is a Nash equilibrium of the game. \square