4.4 Inefficiency of Equilibria

We have seen numerous examples demonstrating that the outcome of rational behavior by self-interested players can be inferior to a centrally designed outcome. In this section, we discuss the question: by how much?

To begin, recall the Prisoner's Dilemma.

- Both players suffer a cost of 4 in the unique Nash equilibrium of this game, while both could incur a cost of 2 by coordinating.
- A qualitative observation is that the equilibrium is strictly Pareto inefficient, in the sense that there is another outcome in which all of the players achieve a smaller cost.

This qualitative perspective is particularly appropriate in applications where the "cost" or "payoff" to a player is an abstract quantity that only expresses the player's preferences between different outcomes. However, payoffs and costs have concrete interpretations in many applications, such as money or the delay incurred in a network. We can proceed more quantitatively in such applications by defining a specific objective function on the outcomes of the game, that numerically expresses the "social good" or "social cost" of an outcome. Two prominent objective functions are the utilitarian and egalitarian functions, defined as the sum of the players costs and the maximum player cost, respectively. Mathematically, these objective functions in the cost setting are defined by

$$\begin{aligned} C_u(x_1, x_2, \cdots, x_n) &= \sum_{i=1}^{n} C_i(x_1, x_2, \cdots, x_n) \\ C_e(x_1, x_2, \cdots, x_n) &= \max_{i=1,2,\cdots,n} \big\{ C_i(x_1, x_2, \cdots, x_n) \big\}. \end{aligned}$$

The Nash equilibrium in the Prisoner's Dilemma does not minimize either of these objective functions. The objective functions in the payoff setting are defined by

$$\begin{aligned} U_u(x_1, x_2, \cdots, x_n) &= \sum_{i=1}^{n} U_i(x_1, x_2, \cdots, x_n) \\ U_e(x_1, x_2, \cdots, x_n) &= \min_{i=1,2,\cdots,n} \big\{ U_i(x_1, x_2, \cdots, x_n) \big\}. \end{aligned}$$

Price of anarchy (PoA) and *price of stability* (PoS) are two popular measures of inefficiency of Nash equilibria.

Price of anarchy: It is defined as a ratio of the worst object function value of an equilibrium and optimal objective value. Let $C(\cdot)$ be the cost function which defines the objective function of the game. Then

$$\text{PoA} = \frac{\max \big\{ C(x) \mid x \text{ is a NE} \big\}}{\min \big\{ C(x) \mid \forall\ x \big\}}.$$

In this case, it is clear that $\text{PoA} \geq 1$.

Similarly, if the objective function of the game is defined by payoff function $U(\cdot)$,

$$\text{PoA} = \frac{\min\left\{U(x) \mid x \text{ is a NE}\right\}}{\max\left\{U(x) \mid \forall\ x\right\}}.$$

Hence, PoA ≤ 1 in the payoff setting. The games in which the price of anarchy is close to 1, all equilibria are good approximations of an optimal outcome. A game with multiple equilibria has a large price of anarchy even if only one of its equilibria is highly inefficient. The price of stability is a measure of inefficiency designed to differentiate between games in which all equilibria are inefficient and those in which some equilibrium is inefficient.

Price of stability: It is defined as ratio of the best objective function value of an equilibrium and optimal objective value. In the cost setting

$$\text{PoS} = \frac{\min\left\{C(x) \mid x \text{ is a NE}\right\}}{\min\left\{C(x) \mid \forall\ x\right\}}.$$

In this case, PoS ≥ 1. In the payoff setting

$$\text{PoS} = \frac{\max\left\{U(x) \mid x \text{ is a NE}\right\}}{\max\left\{U(x) \mid \forall\ x\right\}}.$$

In this case, PoS ≤ 1. For a game with multiple equilibria, its price of stability is at least as close to 1 as its price of anarchy, and it can be much closer. For Prisoner's Dilemma, PoA $=$ PoS $= 2$, because it has a unique Nash equilibrium.

Remark 4.10. *Note that the PoA and PoS of a game are defined with respect to a choice of objective function and a choice of equilibrium concept. It can be different for different choice of objective functions for the same equilibrium concept.*

5 Network Games

We will discuss two network games:

1. network routing with a continuum of users (selfish players drive on congested paths);
2. network formation with discrete users (players build paths & pay for their costs)

5.1 Selfish Routing Games

An instance of the routing game is specified by:

- A directed graph
- A designated source node (s) and destination node (t) in the graph
- A large population of users traveling from s to t define the congestion. We represent the large population by $[0, 1]$. A point $z \in [0, 1]$ denotes the fraction of users.
- There are k number of paths to reach from s to t.
- A vector $x = (x_{P_1}, x_{P_2}, \cdots, x_{P_k})$ defines a flow, where $x_{P_i} \geq 0$ represents the fraction of users using path P_i such that $\sum_{i=1}^{k} x_{P_i} = 1$.
- The cost of an edge e is a function of its congestion and it is defined by $c_e : [0, 1] \rightarrow R^+$.
- The cost of path P for a flow x is $C_P(x) = \sum_{e \in P} c_e(x)$.

Wardrop Equilibrium:
A flow x is in equilibrium if, for all paths P with $x_P > 0$ and all paths P', we have

$$C_P(x) \leq C_{P'}(x), \tag{62}$$

i.e., for each path used by a positive fraction of users, no other path is faster. In particular, all paths used by an equilibrium flow have equal cost.

Example 5.1 (Pigou's example)**.** *Consider the simple network shown in Figure 6. Two disjoint edges connect a source vertex s to a destination vertex t. Each edge is labeled with a cost function c(·), which describes the cost (e.g., travel time) incurred by users of the edge, as a function of the amount of traffic routed on the edge. The upper edge has the constant cost function $c(x) = 1$, and thus represents a route that is relatively long but immune to congestion. The cost of the lower edge, which is governed by the function $c(x) = x$, increases as the edge gets more congested. In particular, the lower edge is cheaper than the upper edge if and only if less than one unit of traffic uses it.*

Wardrop Equilibrium: For the above game it is clear that routing whole traffic on lower path is an equilibrium flow. The cost incurred by all the users in this case is 1. Because in this case a fraction of users have no incentive to change the path as they will incur the same cost. Let us consider the flow $(x, 1 - x)$, where fraction of traffic (users)

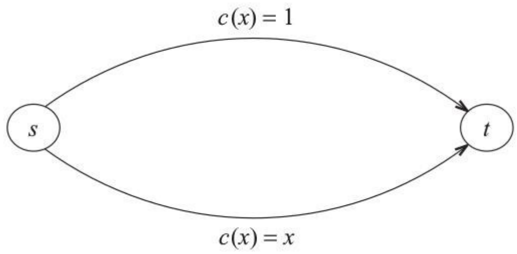

Figure 6: Pigous example

being routed on lower path is x and the traffic routed on the upper path is $1-x$. It is clear that in this case a fraction of users from the upper path has incentive to use the lower path. We can get the equilibrium flow in this case easily from (62). For a flow $(x, 1-x)$ to be an equilibrium flow, the cost of both the paths has to be same, which implies

$$\text{The cost of lower path} = x = \text{The cost of upper path} = 1.$$

This gives routing whole traffic on lower path ($x = 1$) as unique equilibrium.

Social-Cost: Equilibrium flows correspond to Wardrop equilibria when there are a continuum of users who all want to minimize their own delay. On the other hand, if a single designer were controlling all of the flow, they would want to minimize the average delay/social cost given by

$$C(x) = \sum_P x_P \cdot C_P(x).$$

For Pigou's example, consider an arbitrary flow $(x, 1-x)$. Then,

$$C(x) = x \cdot x + (1-x) \cdot 1 = x^2 - x + 1.$$

The minimum of $C(x)$ will occur at

$$2x^* - 1 = 0$$
$$\Rightarrow x^* = \frac{1}{2}.$$

The optimal social cost $C(x^*) = \frac{3}{4}$, and the cost at Wardrop equilibrium $C(1) = 1$. Since, the game has unique equilibrium,

$$\text{PoA} = \frac{4}{3} = \text{PoS}.$$

Exercise 5.2. *Consider the above Pigou's example with $c(x) = x^p$. What will be the PoA as $p \to \infty$?*

Example 5.3 (Braess's Paradox)**.** *Consider the four-node network shown in Figure 7(a). There are two disjoint routes from s to t, each with combined cost $1 + x$, where x is the amount of traffic that uses the route. Assume that there is one unit of traffic. What would be the equilibrium flow? Now suppose that, in an effort to decrease the cost encountered by the traffic, we build a zero-cost edge connecting the midpoints of the two existing routes. The new network is shown in Figure 7 (b). What is the new equilibrium flow?*

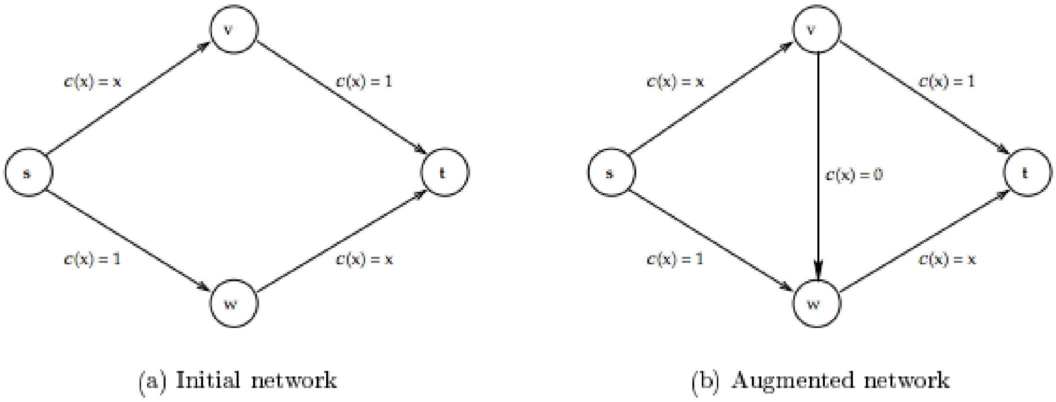

Figure 7: Braess's example

Wardrop Equilibrium & Price of Anarchy : (a) Let $(x, 1-x)$ be an arbitrary flow, where x is the flow on upper path and $1-x$ is the flow on lower path. At equilibrium flow the cost of both the paths has to be same. Then,

$$1 + x = 1 + 1 - x \Rightarrow x = \frac{1}{2}.$$

Therefore, routing half traffic on both path is a unique equilibrium flow. The social cost for a given flow $(x, 1-x)$ is

$$C(x) = x \cdot (1 + x) + (1 - x) \cdot (1 + 1 - x) = 2x^2 - 2x + 2.$$

Therefore, social cost will achieve minima at $x^* = \frac{1}{2}$. The optimal social cost in this case is $\frac{3}{2}$. The equilibrium flow is also optimal flow in this case. The social cost at equilibrium is $\frac{3}{2}$. Hence, the PoA of this game is 1.

(b) Now consider the game given in Figure 7 (b). Let $(x_1, x_2, 1-x_1-x_2)$ be an arbitrary flow, where x_1 is the traffic routed on $s \to v \to t$, x_2 is the traffic routed on $s \to w \to t$, and the remaining traffic $s \to v \to w \to t$. For the equilibrium flow

$$x_1 + 1 - x_1 - x_2 + 1 = x_2 + 1 - x_1 - x_2 + 1 = x_1 + 1 - x_1 - x_2 + x_2 + 1 - x_1 - x_2$$
$$\Rightarrow 2 - x_2 = 2 - x_1 = 2 - x_1 - x_2.$$

This gives $x_1 = 0$ and $x_2 = 0$. Therefore, at equilibrium flow whole traffic is routed on $s \to v \to w \to t$. The social cost at equilibrium is 2. The social cost at arbitrary flow $(x_1, x_2, 1-x_1-x_2)$ is given by

$$C(x_1, x_2) = 2 - x_1 - x_2 + x_1^2 + x_2^2. \text{ (Take this as Exercise)}$$

Then, the optimal cost is achieved at $x_1^* = x_2^* = \frac{1}{2}$ (Exercise). Hence, the optimal social cost is $\frac{3}{2}$. The PoA in this case is $\frac{4}{3}$.

Note: Braesss Paradox shows that the intuitively helpful action of adding a new zero-cost edge can increase the cost experienced by all of the traffic! This phenomenon has been reported, for example, in New York City: when a certain road was under repair, the traffic sped up.

5.2 Network design games

Network design games involves only finite number of players. We define a Shapley network design game as follows. Like selfish routing games, such a network design game occurs in a directed graph G. Each edge e of the graph has a fixed, nonnegative cost c_e . There are k players, and each player i is associated with a source vertex s_i and a destination vertex t_i. Player i wants to establish connectivity from its source to its destination, and its strategies are therefore the $s_i - t_i$ paths of G. Given a choice of a path P_i by each player i, we assume that the formed network is simply the union $\cup_i P_i$ of these. The cost of this network is the sum $\sum_{e \in \cup_i P_i} c_e$ of the costs of these edges, and we assume that the cost of each edge of the formed network is shared equally by the players who use it. More formally, each player i incurs cost $\frac{c_e}{f_e}$ for each edge e of its path P_i, where f_e denotes the number of players selecting paths that contain the edge e. The central objective of the game is to minimize the cost of the formed network.

Example 5.4. *Consider the network shown in Figure 8. There are k players, each with the same source s and destination t. The edge costs are k and $1 + \epsilon$, where $\epsilon > 0$ is arbitrarily small.*

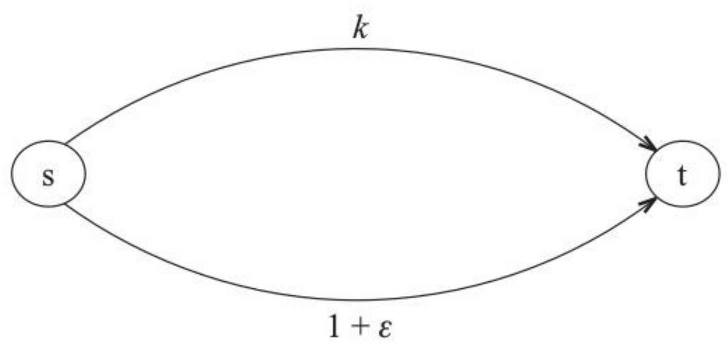

Figure 8: Shapley Network design game

Nash Equilibrium and Optimal outcome: When all the players choose the lower edge it will minimize the cost of network. This outcome is also a Nash equilibrium. On the other hand, suppose that all of the players choose the upper edge. Each player then incurs cost 1, and if a player deviates to the lower edge, it pays the larger cost of $1 + \epsilon$. This outcome is thus a second Nash equilibrium, and the cost of network is k in this case.

The price of anarchy of the game in Example 5.4 is roughly the number of players, and we view this as unacceptably large. However, the price of stability of the game is 1.

Example 5.5. *Consider the network shown in Figure 9. There are k players, all with the same target t, and $\epsilon > 0$ is arbitrarily small. For each $i \in \{1, 2, \cdots, k\}$, the edge (s_i, t) has cost $\frac{1}{i}$.*

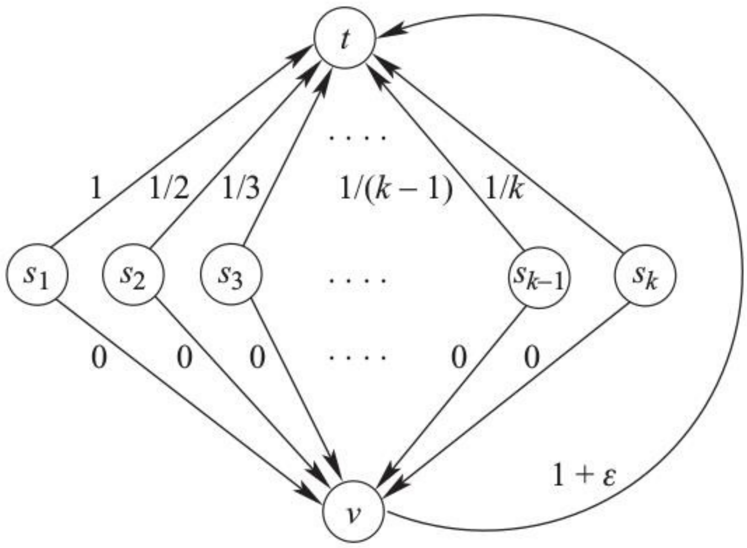

Figure 9: Shapley Network design game

Nash Equilibrium and Optimal outcome: In the optimal outcome, each player i chooses the path $s_i \to v \to t$ and the cost of the formed network is $1 + \epsilon$. Each player incurs cost $\frac{1+\epsilon}{k}$. This outcome is not a Nash equilibrium because player k can reduce its cost to $\frac{1}{k}$ by choosing the path $s_k \to t$. The outcome where each player i chooses direct path $s_i \to t$ is a Nash equilibrium because unilateral deviation by any player will cost $1 + \epsilon > \frac{1}{i}$ for all $i = 1, 2, \cdots, k$. The direct path $s_k \to t$ is a strongly dominant strategy for the kth player. Arguing inductively about the players $k-1, k-2, \cdots, 1$ shows that the above Nash equilibrium is unique. The cost of the network at this equilibrium is $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i}$. The PoA and PoS in this case is roughly $\mathcal{H}_k$.