Problem set 1
MTL 763 (Introduction to Game Theory)

1. Consider the game of **"Matching Pennies"**. Find a saddle point equilibrium and the value of the game.

2. Player 1 holds a black Ace and a red 8. Player 2 holds a red 2 and a black 7. The players simultaneously choose a card to play. If the chosen cards are of the same color, player 1 wins. Player 2 wins if the cards are of the different colors. The amount won is a number of dollars equal to the number on the winner's card (Ace counts as 1). Formulate this as a zero sum game by defining the payoff matrix. Find the value and saddle point equilibrium of the game.

3. Players 1 and 2 Simultaneously call out one of the numbers 1 or 2. Player 1 wins if the sum of the numbers is odd, and player 2 wins if the sum of the number is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. Find the saddle point equilibrium and value of the game.

4. Consider following matrix games

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{pmatrix}.$$

   Find the value and saddle point equilibrium in both the games A and B using graphical method.

5. Consider a matrix game

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

   - For a fixed strategy $x = (0, 1/2, 1/2)$ of player 1 what is the optimal strategy of player 2.
   - For a fixed strategy $y = (0, 1/2, 1/2)$ of player 2 what is the optimal strategy of player 1.
   - What can you say about the saddle point equilibrium of the game?

6. Consider a matrix game A given below

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ t & \frac{4}{3} & 1 \end{pmatrix},$$

   where $-\infty < t < +\infty$.

   (i) For a given strategy $x^* = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ of player 1, find the optimal strategies of player 2 for all $-\infty < t < +\infty$.

(*ii*) For a given strategy $y^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ of player 2, find the optimal strategies of player 1 for all $-\infty < t < +\infty$.

(*iii*) Using the information from (*i*) and (*ii*) can we say that (x^*, y^*) forms a saddle point equilibrium for a certain value of t.

7. Solve the game with matrix

$$\begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}$$

for an arbitrary real number t. Draw the graph of $v(t)$, the value of the game, as a function of t.

8. Solve the following matrix games:

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}.$$

9. Players 1 and 2 choose i and j simultaneously from the sets $\{1, 2, 3, 4, 5, 6, 7\}$. Player 1 wins 1 if $|i - j| = 1$, otherwise there is no payoff. Describe the payoff matrix of the game and find the value and saddle point equilibrium of the game.

10. Construct a 3×3 matrix game A which has a pure strategy saddle point equilibrium and each player has no dominant strategies.

11. Consider a diagonal game described by matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Find the value and saddle point equilibrium of the game.

12. Consider a rack-paper-scissors game described by the following payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Find the value and saddle point equilibrium of the game.

13. Colonel Blotto has 4 regiments with which to occupy two posts. The famous Lieutenant Kije has 3 regiments with which to occupy the same posts. The army sending the most units to either post captures it and all the regiments sent by the other side, scoring one point for the captured post and one for each captured regiment. If the players send the same number of regiments to a post, both forces withdraw and there is no payoff. Formulate this game as a zero-sum game. Find the saddle point equilibrium and value of the game.

14. Solve the above Blotto and Kije game under following situations:

   - Suppose Blotto has 2 units and Kije has just 1 unit, with 2 posts to capture.
   - Suppose Blotto has 3 units and Kije has 2 units, with 2 posts to capture.
   - Suppose Blotto has 4 units and Kije has 3 units, with 3 posts to capture.
   - Suppose Blotto has 4 units and Kije has 3 units, with 4 posts to capture.

15. Player 2 chooses a number $j \in \{1, 2, 3, 4\}$, and player 1 tries to guess what number player 2 has chosen. If he guesses correctly and the number was j, he wins 2^j dollars from player 2. Otherwise there is no payoff. Set up the matrix of this game and solve.

16. Player 2 chooses a number $j \in \{1, 2, 3, 4\}$, and player 1 tries to guess what it is. If he guesses correctly, he wins 1 from player 2. If he overestimates he wins $\frac{1}{2}$ from player 2. If he underestimates, there is no payoff. Set up the matrix of this game and solve.

17. Solve the games with the following matrices.

$$(a) \quad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad (b) \quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & \frac{3}{2} & 1 & 1 \\ 1 & 1 & \frac{4}{3} & 1 \\ 1 & 1 & 1 & \frac{5}{4} \end{pmatrix}, \quad (c) \quad \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

18. (a) Consider a matrix game A, where

$$A = \begin{pmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{pmatrix}.$$

   Show that (x, y), where $x = \left(\frac{6}{37}, \frac{20}{37}, 0, \frac{11}{37}\right)$ and $y = \left(\frac{14}{37}, \frac{4}{37}, 0, \frac{19}{37}, 0\right)$, is a saddle point equilibrium. What is the value of game?

   (b) Consider a matrix game

$$A = \begin{pmatrix} 1 & 2 & \alpha \\ \alpha & 2 & 3 \\ 2 & -1 & 3 \end{pmatrix}.$$

   Find value of the game and at least one saddle point equilibria for all values of α, where $\alpha \in (1, \infty)$.

19. Find a saddle point equilibrium and value of the following matrix game A

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 3 & 1 & 2 \\ 0 & \frac{3}{4} & \frac{1}{2} & 3 \end{pmatrix}.$$

20. Player 2 chooses a number $j \in \{1, 2, 3, 4\}$, and player 1 tries to guess what number player 2 has chosen. If he guesses correctly and the number was j, he wins 2^j dollars from player 2. Otherwise there is no payoff.

   - Set up the matrix A of this game.
   - Find the value of the game and saddle point equilibrium.
   - What will be the new saddle point equilibrium of A if we add a number 1 in all diagonal entries of A?
   - What will be the saddle point equilibrium of A if we add a number 1 in all the entries of A?