

### 3.2 n-player nonzero sum game

We can extend the two player game to the  $n$ -player case in similar way. It is defined by the following components:

- $\mathcal{N} = \{1, 2, \dots, n\}$  is the set of players.
- $A_i$  is a finite action set of player  $i \in \mathcal{N}$ . A generic element of  $A_i$  is denoted by  $a_i$ .
- $A = \prod_{i \in \mathcal{N}} A_i$  denotes the set of all actions profiles of the players. A generic element of  $A$  is denoted by  $a = (a_1, a_2, \dots, a_n)$ .
- $(r_i(a))_{a \in A}$  is the payoff vector of player  $i$  where  $r_i(a)$  is the payoff player  $i$  receives when the players choose actions  $a_1, a_2, \dots, a_n$  simultaneously.
- $\mathcal{T}_i = \{\tau_i \in \mathbb{R}^{|A_i|} \mid \sum_{a_i \in A_i} \tau_i(a_i) = 1, \tau_i(a_i) \geq 0, \forall a_i \in A_i\}$  denote the set of all mixed strategies of player  $i$ ,  $i \in \mathcal{N}$ ;  $|A_i|$  is the cardinality of action set  $A_i$ .
- $\mathcal{T} = \prod_{i \in \mathcal{N}} \mathcal{T}_i$  is the set of of mixed strategy profiles of the players, and  $\mathcal{T}_{-i} = \prod_{j \in \mathcal{N}; j \neq i} \mathcal{T}_j$  is the set of mixed strategy profiles of all the players except player  $i$ . The generic elements of  $\mathcal{T}$  and  $\mathcal{T}_{-i}$  are denoted by  $\tau = (\tau_1, \tau_2, \dots, \tau_n)$  and  $\tau_{-i} = (\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_n)$  respectively.

When each player  $i$ ,  $i \in \mathcal{N}$ , chooses strategy  $\tau_i$  simultaneously, the payoff of player  $i$  is given by

$$u_i(\tau) = \sum_{a \in A} \prod_{j \in \mathcal{N}} \tau_j(a_j) r_i(a). \quad (32)$$

For a given mixed strategy profile  $\tau_{-i}$  of all the players except player  $i$ , the set of best response strategies of player  $i$  is given by

$$BR_i(\tau_{-i}) = \{\bar{\tau}_i \mid u_i(\bar{\tau}_i, \tau_{-i}) \geq u_i(\tau_i, \tau_{-i}), \forall \tau_i \in \mathcal{T}_i\}. \quad (33)$$

A strategy pair is said to be a Nash equilibrium iff for each  $i \in \mathcal{N}$ , the following inequality holds

$$u_i(\tau_i^*, \tau_{-i}^*) \geq u_i(\tau_i, \tau_{-i}^*), \forall \tau_i \in \mathcal{T}_i. \quad (34)$$

From (33) and (34), a strategy profile  $\tau^*$  is a Nash equilibrium iff  $\tau_i^* \in BR(\tau_{-i}^*)$  for all  $i \in \mathcal{N}$ .

#### 3.2.1 Existence of Nash equilibrium

We show that there always exists a mixed strategy Nash equilibrium for a finite strategic game. We use Kakutani fixed point theorem to prove the result. Let  $\mathcal{P}(\mathcal{T})$  be a power set of  $\mathcal{T}$ . Define a set valued map

$$G : \mathcal{T} \rightarrow \mathcal{P}(\mathcal{T})$$

such that

$$G(\tau) = \prod_{i \in \mathcal{N}} BR_i(\tau_{-i}).$$

A strategy profile  $\tau$  is said to be a fixed point of  $G(\cdot)$  if  $\tau \in G(\tau)$ , i.e.,  $\tau_i \in BR_i(\tau_{-i})$  for all  $i \in \mathcal{N}$ . This implies that a fixed point of  $G(\cdot)$  is a Nash equilibrium of the game. Therefore, to show the existence of a Nash equilibrium it is enough to show that there exists a fixed point of the set valued map  $G(\cdot)$ .

**Theorem 3.8.** *There always exists a mixed strategy Nash equilibrium for an  $n$ -player finite strategic game.*

*Proof.* In order to show that  $G$  has a fixed point, we show that  $G$  satisfies all the following conditions of Kakutani fixed point theorem:

1.  $\mathcal{T}$  is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
2.  $G(\tau)$  is non-empty and convex for all  $\tau \in X$ .
3.  $G(\cdot)$  has closed graph: if  $(\tau_n, \bar{\tau}_n) \rightarrow (\tau, \bar{\tau})$  with  $\bar{\tau}_n \in G(\tau_n)$  for all  $n$ , then  $\bar{\tau} \in G(\tau)$ .

Condition 1 holds from the definition of  $\mathcal{T}$ . For fixed  $\tau_{-i}$ ,  $u_i(\cdot, \tau_{-i})$  is a continuous function of  $\tau_i$ . For each  $i \in I$ ,  $BR_i(\tau_{-i})$ , is non-empty because a continuous function  $u_i(\cdot, \tau_{-i})$  over a compact set  $\mathcal{T}_i$  always attains maxima. Hence,  $G(\tau)$  is non-empty for all  $\tau \in \mathcal{T}$ . For each  $i \in I$ ,  $BR_i(\tau_{-i})$  is a convex set because  $u_i(\cdot, \tau_{-i})$  is a linear function of  $\tau_i$ . Hence,  $G(\tau)$  is a convex set for all  $\tau \in \mathcal{T}$ . Now, we prove that  $G(\cdot)$  is a closed graph. Assume that  $G(\cdot)$  is not a closed graph, i.e., there is a sequence  $(\tau^n, \bar{\tau}^n) \rightarrow (\tau, \bar{\tau})$  with  $\bar{\tau}^n \in G(\tau^n)$  for all  $n$ , but  $\bar{\tau} \notin G(\tau)$ . In this case,  $\bar{\tau}_i \notin BR_i(\tau_{-i})$  for some  $i \in I$ . Then, there is an  $\epsilon > 0$  and a  $\tilde{\tau}_i$  such that

$$u_i(\tilde{\tau}_i, \tau_{-i}) > u_i(\bar{\tau}_i, \tau_{-i}) + 3\epsilon. \quad (35)$$

Since,  $u_i(\cdot)$  is a continuous function of  $\tau$ ,  $u_i(\bar{\tau}_i^n, \tau_{-i}^n) \rightarrow u_i(\bar{\tau}_i, \tau_{-i})$ . Then, there exists an integer  $N_1$  such that

$$u_i(\bar{\tau}_i^n, \tau_{-i}^n) < u_i(\bar{\tau}_i, \tau_{-i}) + \epsilon, \quad \forall n \geq N_1. \quad (36)$$

From (35) and (36), we have

$$u_i(\bar{\tau}_i^n, \tau_{-i}^n) < u_i(\tilde{\tau}_i, \tau_{-i}) - 2\epsilon, \quad \forall n \geq N_1. \quad (37)$$

Similarly,  $u_i(\tilde{\tau}_i, \tau_{-i}^n) \rightarrow u_i(\tilde{\tau}_i, \tau_{-i})$ . Then, there exists an integer  $N_2$  such that

$$u_i(\tilde{\tau}_i, \tau_{-i}) < u_i(\tilde{\tau}_i, \tau_{-i}^n) + \epsilon, \quad \forall n \geq N_2. \quad (38)$$

Let  $N = \max\{N_1, N_2\}$ . Then, from (37) and (38), we have

$$u_i(\tilde{\tau}_i, \tau_{-i}^n) > u_i(\bar{\tau}_i^n, \tau_{-i}^n) + \epsilon, \quad \forall n \geq N.$$

That is,  $\tilde{\tau}_i$  performs better than  $\bar{\tau}_i^n$  against  $\tau_{-i}^n$  for all  $n \geq N$  which contradicts  $\bar{\tau}_i^n \in BR_i(\tau_{-i}^n)$  for all  $n$ . Hence,  $G(\cdot)$  is a closed graph. That is, the set valued map  $G(\cdot)$  satisfies all the conditions of Kakutani fixed point theorem. Hence,  $G(\cdot)$  has a fixed point  $\tau^*$ . Such  $\tau^*$  is a Nash equilibrium of the game.  $\square$