4.2 Auctions

In an **auction**, a good is sold to the party who submits the highest bid. In a common form of auction, people sequentially submit increasing bids for an object. The auction stops when no one wishes to submit a bid higher than the current bid. In this case the person making the current bid obtains the object at the price he/she bids.

If every person is certain of his/her valuation of the object before the bidding begins, during the bidding no one can learn anything relevant to her actions. In that case, we can model the auction by assuming that each person decides about his/her maximal bid before bidding begins. When the players carry out their plans, the winner is the person whose maximal bid is the highest. Eventually only she and the person with the second highest maximal bid will be left competing against each other. Hence, in order to win, the person needs to bid slightly more than the second highest maximal bid. If the bidding increment is small, we can take the price the winner pays to be equal to the second highest maximal bid. This leads to the second-price auction.

Thus we can model such an auction as a strategic game in which each player chooses an amount of money, interpreted as the maximal amount he/she is willing to bid, and the player who chooses the highest amount obtains the object and pays a price equal to the second highest amount.

This game also models a situation in which the people simultaneously put bids in sealed envelopes, and the person who submits the highest bid wins and pays a price equal to the second highest bid. For this reason the game is called a second-price sealed-bid auction.

4.2.1 Second-price sealed-bid auction

Consider an auction with n number of bidders. Let v_i be the value player i attaches to the object and it is known to all the players. If he obtains the object at the price p his payoff is $v_i - p$. Assume that the players' valuations of the object are all different and all positive; number the players 1 through n in such a way that $v_1 > v_2 > \ldots > v_n > 0$. Each player i simultaneously submits a bid $b_i \in [0, \infty)$ in a sealed envelope. The player whose bid is the highest gets the object at the price of second highest bid. In case of tie, the player whose valuation is highest gets the object. The payoff of player i is given by

$$\text{Payoff}_i = \begin{cases} v_i - \max_{j \neq i} b_j, & \text{if } b_i > \max_{j \neq i} b_j, \\ v_i - \max_{j \neq i} b_j, & \text{if } b_i = \max_{j \neq i} b_j \text{ and } v_i > v_k \ \forall\ k \text{ such that } b_k = b_i, \\ 0, \text{otherwise}. \end{cases}$$

Claim: Bidding true valuation is a weakly dominant strategy.
Consider the bidder i. We show that $b_i = v_i$ is a weakly dominant strategy. Consider any strategy profile b_{-i} of the players but player i. Let $\bar{b} = \max_{j \neq i} b_j$. We need to show that, no matter what other players bid, $b_i = v_i$ will give player i at least as much as he gets for choosing other actions, and for some cases $b_i = v_i$ gives better payoff. This is summarized in the following tables:

Other players' bid / Player i's bid	$\bar{b} < v_i$		$\bar{b} \geq v_i$
	$b_i < \bar{b}$ or $\bar{b} = b_i$ & b_i losses	$b_i > \bar{b}$ or $\bar{b} = b_i$ & b_i wins	
$b_i < v_i$	0	$v_i - \bar{b}$	0
$b_i = v_i$	$v_i - \bar{b} > 0$		0

Other players' bid / Player i's bid	$\bar{b} \leq v_i$	$\bar{b} > v_i$	
		$b_i < \bar{b}$ or $\bar{b} = b_i$ & b_i loses	$b_i > \bar{b}$ or $\bar{b} = b_i$ & b_i wins
$b_i > v_i$	$v_i - \bar{b}$	0	$v_i - \bar{b} < 0$
$b_i = v_i$	$v_i - \bar{b}$	0	

From above tables it is clear that $b_i = v_i$ is a weakly dominant strategy.

Nash Equilibrium:
This game has many Nash equilibria. One equilibrium is $(b_1, b_2, \ldots, b_n) = (v_1, v_2, \ldots, v_n)$: each player's bid is equal to her valuation of the object. Because $v_1 > v_2 > \ldots > v_n$, the outcome is that player 1 obtains the object at the price b_2; his payoff is $v_1 - b_2$ and every other player's payoff is zero. This profile is a Nash equilibrium by the following argument.

- Fix $b_i = v_i$ for all $i = 2, \ldots, n$. The player 1 is not benefited by unilaterally deviating from $b_1 = v_1$. Because for $b_1 \geq v_2$ player 1's payoff is not changing, i.e., it's $v_1 - b_2$, and for $b_1 < v_2$ the player 2 gets the object and the payoff of player 1 is zero.

- The player i, $i \neq 1$, cannot be benefited by unilaterally changing his bid. Because for $b_i \leq v_1$ player i remains loser and payoff is zero, and for $b_i > v_1$ player i wins the object and gets the payoff $v_i - v_1 < 0$.

Another equilibrium is $(b_1, b_2, \ldots, b_n) = (v_1, 0, \ldots, 0)$. In this equilibrium, player 1 obtains the object and pays the price of zero. The profile is an equilibrium because

- if player 1 changes his bid unilaterally the outcome remains the same.

- for any other player i, $i \neq 1$, if $b_i \leq v_1$ the outcome remains the same. If $b_i > v_1$, player i wins the object and gets the payoff $v_i - v_1 < 0$.

In both of these equilibria, player 1 obtains the object. But there are also equilibria in which player 1 does not obtain the object. Consider, for example, the action profile

$(v_2, v_1, 0, \ldots, 0)$, in which player 2 obtains the object at the price v_2 and every player (including player 2) receives the payoff of zero. This action profile is a Nash equilibrium by the following argument.

- If player 1 changes his bid unilaterally to $b_1 \geq v_1$, he wins the object but his payoff remains zero. In other cases the outcome of the game remains same.
- If player 2 changes his bid unilaterally to $b_2 > v_2$, the outcome of the game does not change. For $b_2 \leq v_2$, player 2 loses and the payoff remains zero.
- If any other player i other than 1 and 2 changes his bid unilaterally to $b_i \leq v_1$, the outcome of the game remains same. For $b_i > v_1$, he wins the object and gets the payoff $v_i - v_1 < 0$.

4.2.2 First-price sealed-bid auctions

A first-price auction differs from a second-price auction only in that the winner pays the price she bids, not the second highest bid. The payoff of player i is given by

$$\text{Payoff}_i = \begin{cases} v_i - b_i, \text{ if } b_i > b_j, \ \forall\, j \neq i, \\ v_i - b_i, \text{ if } b_i = \max_{j \neq i} b_j \text{ and } v_i > v_k \ \forall\, k \text{ such that } b_k = b_i, \\ 0, \text{otherwise.} \end{cases}$$

Claims:

1. Bidding true valuation, i.e., $b_i = v_i$, is not a weakly dominant strategy for player i.

   Proof. Unlike in second-price auction, $b_i = v_i$ is not a weakly dominant strategy for player i. The reason is if player i bids v_i his payoff is 0 regardless of the other players' bids, whereas if he bids less than v_i his payoff is either 0 (if he loses) or positive (if he wins). For example, if the bids of other players are such that $\bar{b} < v_i$ where $\bar{b} = \max_{j \neq i} b_j$, player i can get positive payoff by bidding b_i such that $\bar{b} < b_i < v_i$. Therefore, it is possible to increase the payoff by bidding less than the true valuation. In fact, $b_i = v_i$ is weakly dominated by $b_i < v_i$. This can be shown by considering all possible cases of other players' bid. □

2. $b_i > v_i$ is weakly dominated by the bid $b_i = v_i$.

   Proof. The arguments are summarized in the following table:

<table>
<tr><td rowspan="2">Other players' bid
Player i's bid</td><td rowspan="2">$\bar{b} \leq v_i$</td><td colspan="2">$\bar{b} > v_i$</td></tr>
<tr><td>$b_i < \bar{b}$ or $\bar{b} = b_i$ & b_i loses</td><td>$b_i > \bar{b}$ or $\bar{b} = b_i$ & b_i wins</td></tr>
<tr><td>$b_i > v_i$</td><td>$v_i - b_i < 0$</td><td>0</td><td>$v_i - b_i < 0$</td></tr>
<tr><td>$b_i = v_i$</td><td>0</td><td colspan="2">0</td></tr>
</table>

□

Remark 4.3. *From claims 1 and 2 above, the bid $b_i < v_i$ weakly dominates bids $b_i \geq v_i$*

3. $b_i < v_i$ is not weakly dominated by any strategy.

Proof. This is true by the following arguments:

- It is not weakly dominated by a bid $b_i' < b_i$, because if the other players' highest bid is between b_i' and b_i then b_i' loses whereas b_i wins and yields player i a positive payoff.
- It is not weakly dominated by a bid $b_i' > b_i$, because if the other players' highest bid is less than b_i then both b_i and b_i' win and b_i yields a better payoff.

□

Remark 4.4. *From claims 1, 2, and 3, it is clear that there is no weakly dominant strategy.*

Nash equilibrium:
An action profile $(b_1, b_2, \cdots, b_n) = (v_2, v_2, v_3, \cdots, v_n)$ is a Nash equilibrium. The outcome of this action profile is that player 1 gets the object at the price v_2. The payoff of player 1 is $v_1 - v_2$ and the payoff of other players is 0. This action profile is a Nash equilibrium by the following arguments:

- Player 1 cannot improve his payoff by unilaterally changing his action. For example, if he bids $b_1 < v_2$, then he loses and gets zero payoff. If he bids $b_1 > v_2$, he gets the object at higher price.
- Other players cannot improve his payoff by unilaterally changing his action. Because by bidding more than v_2 a player wins the object but receives negative payoff, and by biding at most v_2 outcome remains same.

There are many other Nash equilibrium in first-price auction. Some important facts about the Nash equilibria are given below:

- At a Nash equilibrium profile the winner is the player who values the object most highly (player 1).

  Proof. Consider a Nash equilibrium profile $(b_1, b_2, \cdots, b_n)$ where player i, $i \neq 1$ wins the object. Then $b_i > b_1$. In that case the bid $b_i > 0$. If $b_i > v_2$, then the payoff of player i is $v_i - b_i < 0$ (because $v_i - b_i < v_i - v_2 \leq 0$), and player i can increase his payoff by reducing his bid to zero. If $b_i \leq v_2$, then player 1 can increase his payoff from zero to $v_1 - b_i$ by bidding b_i. Thus no such action profile is a Nash equilibrium. □

- In a Nash equilibrium of a first-price sealed-bid auction the two highest bids are the same, one of these bids is submitted by player 1, and the highest bid is at least v_2 and at most v_1.

  Proof. If the two highest bids in an action profile are not same, it cannot be a Nash equilibrium because the player with the highest can increase his payoff by winning the object with slightly lower bid.

  From the above argument in any equilibrium player 1 wins the object. Thus he submits one of the highest bids.

  If the highest bid is less than v_2, then player 2 can increase his bid to a value between the highest bid and v_2, win, and obtain a positive payoff. Thus in an equilibrium the highest bid is at least v_2. If the highest bid exceeds v_1, player 1's payoff is negative, and he can increase this payoff by reducing his bid. Thus in an equilibrium the highest bid is at most v_1. □