Problem set 3
MTL 763 (Introduction to Game Theory)

Q.1 Consider an n firm Cournot competition where the cost function for firm i is given by $C_i(x_i) = c_i x_i$ where $c_i \geq 0$. The inverse demand function is given $P(X) = a - bX$ where $a, b > 0$ and $X = x_1 + \cdots + x_n$. The payoff function for firm i is therefore $\pi_i(x_1, \cdots x_n) = \left(a - b\sum_j x_j\right) x_i - c_i x_i$.
(i) Compute a Nash equilibrium for the game. Compute also the equilibrium price.
(ii) What happens to the equilibrium quantity choice of firm j if there is an increase in firm i's cost; i.e., in c_i ? What happens to equilibrium price?
(iii) Take the cost function $C_i(x_i) = c_i x_i^2$. Compute the Nash equilibrium and equilibrium price of the game.

Q.2 Find a Nash equilibrium of a second-price sealed-bid auction in which player n obtains the object.

Q.3 Show that in a Nash equilibrium of a first-price sealed-bid auction the two highest bids are the same, one of these bids is submitted by player 1, and the highest bid is at least v_2 and at most v_1. Show also that any action profile satisfying these conditions is a Nash equilibrium.

Q.4 Let the cost function of player 1 and player 2 be given by

$$C_1(x, y) = x_1^2 + x_2^2 - 2x_1y_1 - 2x_2y_2 - x_1$$
$$C_2(x, y) = y_1^2 + 2y_2^2 - x_1y_1,$$

where $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$. Find a Nash equilibrium of this game.

Q.5 Consider a non-cooperative game $(u_1(\cdot), u_2(\cdot), X, Y)$ defined by:

$$X = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, x_1^2 + 2x_1^2 \leq 1\},$$

$$Y = \{y \in \mathbb{R}^3 | y \in M, y_1^2 + 2y_2^2 + 3y_3^2 \leq 4\},$$

where Y is non empty and $M \subset \mathbb{R}^3$ is a convex and compact set. The payoff functions $u_1(.)$ and $u_2(\cdot)$ are given by $u_1(x, y) = c_1||x||||y|| - \sum_{i=1}^{2} a_i x_i$ and $u_2(x, y) = c_2||x||||y|| - \sum_{j=1}^{3} y_j^2$, where $c_i < 0$, $i = 1, 2$ and $|| \cdot ||$ is the Euclidean norm. Show that there exists a Nash equilibrium.

Q.6 Consider a non-decreasing convex function $g : \mathbb{R} \to \mathbb{R}$. Let $f_1 : R \times \mathbb{R} \to \mathbb{R}$ and $f_2 : R \times \mathbb{R} \to \mathbb{R}$ be continuous and bi-convex functions, i.e., for a fixed value of one variable it is a convex function with respect to other variable. Define, the cost functions of player 1 and player 2 as $C_1(x, y) = (g \circ f_1)(x, y)$, and $C_2(x, y) = (g \circ f_2)(x, y)$ for all $x \in X \subset \mathbb{R}$, $y \in Y \subset \mathbb{R}$, where X and Y are non-empty convex and compact strategy sets of player 1 and player 2 respectively. Show that there exists a Nash equilibrium for two player non-cooperative game defined by cost functions $C_1(\cdot)$, $C_2(\cdot)$, and strategy sets X, Y.

Q.7 Consider the game $(N, (A_i), u_i)$ when $N = \{1, 2, \cdots, n\}$ and $A_i = \{1, 2, \cdots, n\}$ for all $i \in N$.

$$\begin{cases} u_1(a_1, a_2, \cdots, a_n) = b_{ik} > 0 \;\text{ if }\; a_1 = a_2 = \cdots = a_n = k \\ = 0 \;\text{ otherwise} \end{cases}$$

Show that the only pure strategy profiles which are not equilibrium points are those with exactly $(n-1)$ of the a_i equal.

Q.8 A two player game is called symmetric strategic form game in which strategy sets $S_1 = S_2$ and $u_1(s_1, s_2) = u_2(s_2, s_1)$ for all $s_1 \in S_1$ and $s_2 \in S_2$. Show that in such a game the strategy profile (s_1^*, s_2^*) is a pure strategy Nash equilibrium if and only if the profile (s_2^*, s_1^*) is also a pure strategy Nash equilibrium.

Q.9 Consider a Cournot competition among electricity firms over an electricity network comprises of a set of nodes. Let N denote the set of nodes and N_i denote the subset of nodes where firm i has installed its generation facilities. Let I_k denote the set of firms who owns generation facilities at node k. Let x_k^i be the generation quantity for firm i at node k and $(x_k^i)^2$ be its cost of generation. Denote a generation level vector of firm i by $x^i = (x_k^i)_{k \in N_i}$, and a generation level vector at node k by $\hat{x}_k = (x_k^i)_{i \in I_k}$. The price at each node k depends only on the generation quantities of the firms whose generation facilities are installed at node k, and it is determined by the sum of the generation quantities. The price at node $k \in N$ is given by $P_k(\hat{x}_k) = a_k - b_k \sum_{j \in I_k} x_k^j$, where $a_k \in \mathbb{R}$ and $b_k \geq 0$. We assume that the capacity of generation facility of firm i at node k is C_k^i. Formulate the above Cournot competition as a strategic game and show that there exists a Nash equilibrium for the game.

Q.10 An airline loses two identical suitcases that belong to two different travelers. The airline is liable for up to \$100 per suitcase. The airline manager, in order to obtain an honest estimate of each suitcase, separates each traveler i in a different room and proposes the following game: "Please write an integer $x_i \in [2, 100]$ in this piece of paper. As a manager, I will use the following reimbursement rule:

- If both of you write the same estimate, $x_1 = x_2 = x$, each traveler gets x.
- If one of you writes a larger estimate, i.e., $x_i > x_j$ where $i \neq j$, then:
  - The traveler who wrote the lowest estimate (traveler j) receives $x_j + k$, where $k > 1$; and
  - The traveler who wrote the largest estimate (traveler i) only receives $\max\{0, x_j - k\}$"

(i) Show that asymmetric strategy profiles, in which travelers submit different estimates cannot be a Nash equilibrium.

(ii) Show that symmetric strategy profiles, in which both travelers submit the same estimate and such estimate is larger than 2, cannot be a Nash equilibrium.

(iii) Show that the symmetric strategy profile in which both travelers submit the same estimate $(x_1, x_2) = (2, 2)$ is the unique pure strategy Nash equilibrium.

(iv) Does the above results still hold when the traveler writing the largest amount receives $x_j - k$ rather than $\max\{0, x_j - k\}$?

Q.11 Consider an industry with three identical firms each selling a homogeneous good and producing at a constant cost per unit c with $1 > c > 0$. The market price is given by $P(Q) = 1 - Q$, where Q is the sum of quantities produced by all three firms, i.e., $Q = q_1 + q_2 + q_3$. Competition in the marketplace is in quantities.
(i) Find the equilibrium quantities, price and profits. (ii) Consider now a merger between two of the three firms, resulting in duopolistic structure of the market. The merger might give rise to efficiency gains, in the sense that the firm resulting from the merger produces at a cost $e.c$, with $0 \leq e \leq 1$ (whereas the remaining firm still has a cost c)

a. Find the post-merger equilibrium quantities, price and profits.

b. For what values of c the merger reduce prices?

c. Under which conditions is the merger beneficial to the merging firms?

Q.12 Consider two candidates competing for office: Democrat (D) and Republican (R). Suppose that voters compare the two candidates only based on the budget share that each candidate promises to spend on education. Voters' ideal policies are uniformly distributed along the interval [0, 1], and each votes for the candidate with a policy promise closest to the voter's ideal. Candidates simultaneously and independently announce their policy positions. A candidate's payoff from winning is 1, and from losing is -1. If both candidates receive the same number of votes, then a coin toss determines the winner of the election.
(i) Formulate the above situation as a game problem between two candidates and find a Nash equilibrium of the game if it exists, otherwise give proper justifications that it does not exist, (ii) Does a Nash equilibrium exist, if a third candidate from independent party enters into the race? If yes, find it.

Q.13 Consider a communication channel shared by n users of maximum capacity 1. Each player may wish to send x_i units of flow, where $x_i \in [0, 1]$. The transition doesn't happen if total flow exceeds the maximum capacity of channel. The payoff function of player i is given by

$$u_i(x_i, x_{-i}) = \begin{cases} x_i(1 - \sum_{j=1}^{n} x_j) \text{ if } \sum_{i=1}^{n} x_i < 1 \\ 0 \text{ if } \sum_{i=1}^{n} x_i \geq 1. \end{cases}$$

Formulate the above problem as non-cooperative game and find its Nash equilibrium profile and Nash equilibrium payoffs. Is Nash equilibrium profile also a Pareto optimal profile?