2.6 Linear programming formulation

In this section, we propose equivalent linear programs to solve the zero sum game problem. We know that the optimal strategies of player 1 and player 2 are the solutions of $\max_{x \in X} \min_{y \in Y} x^T Ay$ and $\min_{y \in Y} \max_{x \in X} x^T Ay$ problems. We start with $\max_{x \in X} \min_{y \in Y} x^T Ay$ problem.

$$\begin{aligned} \max_{x \in X} \min_{y \in Y} x^T Ay &= \max_{x \in X} \min_{y \in Y} \sum_{j \in J} (x^T A)_j \cdot y_j \\ &= \max_{x \in X} \min_{j \in J} (x^T A)_j. \end{aligned}$$

Since, Y is a polytope, the second equality follows from the fact that an optimal solution of a linear program always exists on a extreme point. Let $v_1 = \min_{j \in J} (x^T A)_j$. Then, the optimal strategy of player 1 can be obtained by solving the following linear program

$$\begin{aligned} &\max_{v_1, x} v_1 \\ \text{s.t.} & \\ &\sum_{i \in I} a_{ij} x_i \geq v_1, \ \forall\, j \in J. \\ &\sum_{i \in I} x_i = 1, \\ &x_i \geq 0, \ \forall\, i \in I. \end{aligned} \tag{28}$$

Similarly

$$\min_{y \in Y} \max_{x \in X} x^T Ay = \min_{y \in Y} \max_{i \in I} (Ay)_i.$$

Let $v_2 = \max_{i \in I} (Ay)_i$. Then, the optimal strategy of player 2 can be obtained by solving the following linear program

$$\begin{aligned} &\min_{v_2, y} v_2 \\ \text{s.t.} & \\ &\sum_{j \in J} a_{ij} y_j \leq v_2, \ \forall\, i \in I. \\ &\sum_{j \in J} y_j = 1, \\ &y_j \geq 0, \ \forall\, j \in J. \end{aligned} \tag{29}$$

The linear programs (28) and (29) are dual of each other. Hence, a zero sum game problem can be solved using linear programming techniques.

Theorem 2.19. *(x^*, y^*) and v^* are saddle point equilibrium and value of the game if and only if (x^*, v^*) and (y^*, v^*) are optimal solutions of (28) and (29) respectively.*

Proof. Let (x^*, y^*) and v^* be saddle point equilibrium and value of the game. Then, $v^* = x^{*T}Ay^*$. From the definition of SPE

$$\sum_{j\in J} a_{ij}y_j^* \leq v^* \leq \sum_{i\in I} x_i^* a_{ij}, \quad \forall\, i \in I, j \in J.$$

This implies (x^*, v^*) and (y^*, v^*) are feasible points of (28) and (29) respectively, and hence optimal.

Let (x^*, v^*) and (y^*, v^*) be optimal solutions of (28) and (29) respectively. From the constraints of (28) and (29) we have

$$x^{*T}Ay^* \geq v^*,$$
$$x^{*T}Ay^* \leq v^*.$$

Therefore, $v^* = x^{*T}Ay^*$. Again by multiplying the constraints of (28) corresponding to each $j \in J$ by y_j and then sum over all $j \in J$, we have

$$x^{*T}Ay \geq v^* = x^{*T}Ay^*, \ \forall\, y \in Y.$$

Similarly,

$$x^TAy^* \leq x^{*T}Ay^*, \ \forall\, x \in X.$$

Hence,

$$x^TAy^* \leq x^{*T}Ay^* \leq x^{*T}Ay, \ \forall\, x \in X, y \in Y,$$

and $v^* = x^{*T}Ay^*$. This implies (x^*, y^*) is a saddle point equilibrium and v^* is the value of the game.

□

2.7 Summary

The theory of zero sum games can be summarized as follows:

1. An SPE and the value of the game need not exist in pure strategies.
2. A SPE and value of the game always exists in the space of mixed strategies.
3. A strategy pair is an SPE if and only if each component of the pair is an optimal strategy for the corresponding player.
4. In some cases it is possible to reduce the size of the game using iterated dominance procedure.
5. All 2×2 games can be solved graphically.
6. Some $2 \times n$ and $m \times 2$ games also can be solved graphically.
7. Finally any zero sum game can be solved using linear programming method.