Definition 2.7 (Value of Matrix Game). *The lower value of a matrix game A is defined by*

$$v^- := \max_{x \in X} \min_{y \in Y} x^T A y.$$

Player 1 is guaranteed to receive at least v^-. The upper value of the game is defined by

$$v^+ := \min_{y \in Y} \max_{x \in X} x^T A y.$$

Player 2 is guaranteed to lose at most v^+. The game has a value if $v^- = v^+$.

Proposition 2.8. $\min_{y \in Y} \max_{x \in X} x^T A y \geq \max_{x \in X} \min_{y \in Y} x^T A y$.

Proof. Similar to the case of pure strategies. □

2.2 Existence of saddle point equilibrium

In this section, we show that there always exists a mixed strategy saddle point equilibrium. To prove this, we need some tools from fixed point theory.

Definition 2.9. *A point $z \in \mathbb{R}^n$ is said to be a fixed point of a function $T : Z \to Z$, where $Z \subset \mathbb{R}^n$, if*

$$T(z) = z.$$

Theorem 2.10 (Brouwer's fixed point theorem). *Every continuous function $T : Z \to Z$ where $Z \subset \mathbb{R}^n$, has a fixed point if Z is a convex and compact set.*

Theorem 2.11 (Saddle point existence theorem). *There always exists a mixed strategy saddle point equilibrium for a matrix game A.*

Proof. Let $c_i(x, y)$ be the improvement to player 1 by switching from mixed strategy x to pure strategy i, and $d_j(x, y)$ be the improvement to player 2 by switching from mixed strategy y to pure strategy j. Therefore, for each $i \in I$ and $j \in J$, we have

$$c_i(x, y) = \begin{cases} \sum_{j \in J} a_{ij} y_j - x^T A y, \text{ if } \sum_{j \in J} a_{ij} y_j - x^T A y > 0 \\ 0, \text{ otherwise.} \end{cases} \tag{12}$$

$$d_j(x, y) = \begin{cases} x^T A y - \sum_{i \in I} a_{ij} x_i, \text{ if } x^T A y - \sum_{i \in I} a_{ij} x_i > 0 \\ 0, \text{ otherwise.} \end{cases} \tag{13}$$

Define a map $T : X \times Y \to X \times Y$ such that

$$T(x, y) = (x', y'), \tag{14}$$

where

$$x'_i = \frac{x_i + c_i(x, y)}{1 + \sum_{k \in I} c_k(x, y)}, \ \forall\, i \in I,$$

$$y'_j = \frac{y_j + d_j(x, y)}{1 + \sum_{l \in J} d_l(x, y)}, \ \forall\, j \in J.$$

It is clear that $x'_i \geq 0$ and $y'_j \geq 0$ for all $i \in I$ and $j \in J$, and

$$\sum_{i \in I} x'_i = \frac{\sum_{i \in I} x_i + \sum_{i \in I} c_i(x,y)}{1 + \sum_{k \in I} c_k(x,y)} = \frac{1 + \sum_{k \in I} c_k(x,y)}{1 + \sum_{k \in I} c_k(x,y)} = 1,$$

$$\sum_{j \in J} y'_j = \frac{\sum_{j \in J} y_j + \sum_{j \in J} d_j(x,y)}{1 + \sum_{l \in J} d_l(x,y)} = \frac{1 + \sum_{l \in J} d_l(x,y)}{1 + \sum_{l \in J} d_l(x,y)} = 1.$$

This implies, $x' \in X$ and $y' \in Y$. Therefore, the function $T(\cdot)$ is well defined and it is also a continuous function. The sets X and Y are convex and compact by definition. Then, the product set $X \times Y$ is also a convex and compact set. Therefore, from Brouwer's Theorem 2.10 there exists a point (x^*, y^*) which is a fixed point of $T(\cdot)$.

We show that (x^*, y^*) is a saddle point equilibrium. We first show that there exists an $i \in I$ such that $x^*_i > 0$ and $c_i(x^*, y^*) = 0$. From the definition of the payoff function we have

$$x^{*T} A y^* = \sum_{i \in I} x^*_i \sum_{j \in J} a_{ij} y^*_j. \tag{15}$$

Then, $\sum_{j \in J} a_{ij} y^*_j > x^{*T} A y^*$ cannot be true for all $i \in I$ for which $x^*_i > 0$. Because if $\sum_{j \in J} a_{ij} y^*_j > x^{*T} A y^*$ holds for all $i \in I$ for which $x^*_i > 0$. Then, from (15), we have

$$\begin{aligned} x^{*T} A y^* &= \sum_{i \in I} x^*_i \sum_{j \in J} a_{ij} y^*_j \\ &> \sum_{i \in I} x^*_i \left(x^{*T} A y^* \right) \\ &= x^{*T} A y^*. \end{aligned}$$

This gives a contradiction. Therefore, there exists at least an $\bar{i} \in I$ such that $x^*_{\bar{i}} > 0$ and $\sum_{j \in J} a_{\bar{i}j} y^*_j \leq x^{*T} A y^*$. Then, it follows from from (12) that $c_{\bar{i}}(x^*, y^*) = 0$. From the definition of fixed point, for $\bar{i} \in I$

$$x^*_{\bar{i}} = \frac{x^*_{\bar{i}} + 0}{1 + \sum_{i \in I} c_i(x^*, y^*)}.$$

This implies, $\sum_{i \in I} c_i(x^*, y^*) = 0$ because $x^*_{\bar{i}} > 0$. From the definition, $c_i(x^*, y^*) \geq 0$ for all $i \in I$. Therefore, $c_i(x^*, y^*) = 0$ for all $i \in I$. Then, from (12)

$$\sum_{j \in J} a_{ij} y^*_j \leq x^{*T} A y^*, \ \forall\, i \in I.$$

This, in turn, implies that

$$x^T A y^* \leq x^{*T} A y^*, \ \forall\, x \in X.$$

Similarly, we can show

$$x^{*T} A y^* \leq x^{*T} A y, \ \forall\, y \in Y.$$

That is,

$$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y, \ \forall\, x \in X, y \in Y.$$

Hence, (x^*, y^*) is an SPE. □

Theorem 2.12. *A saddle point equilibrium of a matrix game A exists if and only if the value of game exists, i.e.,*

$$\max_{x\in X}\min_{y\in Y} x^T Ay = \min_{y\in Y}\max_{x\in X} x^T Ay. \tag{16}$$

Proof. Let $(x^*, y^*) \in X \times Y$ be an SPE. Then,

$$x^T Ay^* \leq x^{*T} Ay^* \leq x^{*T} Ay, \ \forall\, x \in X, y \in Y.$$

This implies,

$$\max_{x\in X} x^T Ay^* = x^{*T} Ay^* = \min_{y\in Y} x^{*T} Ay.$$

Then,

$$\min_{y\in Y}\max_{x\in X} x^T Ay \leq \max_{x\in X} x^T Ay^* = x^{*T} Ay^* = \min_{y\in Y} x^{*T} Ay \leq \max_{x\in X}\min_{y\in Y} x^T Ay.$$

From, Proposition 2.8

$$\min_{y\in Y}\max_{x\in X} x^T Ay \geq \max_{x\in X}\min_{y\in Y} x^T Ay.$$

Hence,

$$\max_{x\in X}\min_{y\in Y} x^T Ay = \min_{y\in Y}\max_{x\in X} x^T Ay = x^{*T} Ay^*. \tag{17}$$

Let the value of game exists, i.e., condition (16) holds. Then,

$$\max_{x\in X} x^T Ay = \max_{x\in X} \sum_{i\in I} x_i (Ay)_i = \max_{i\in I}\ (Ay)_i.$$

The maximum of a finite number of continuous functions is a continuous function. Therefore, $\max_{x\in X} x^T Ay$ is a continuous function of y. The set Y is a compact set. Then, from Weierstrass theorem there exist $y^* \in Y$ which attains minimum at $\min_{y\in Y}\max_{x\in X} x^T Ay$. Similarly, there exists $x^* \in X$ which attains maximum at $\max_{x\in X}\min_{y\in Y} x^T Ay$. Then, from (16)

$$\min_{y\in Y} x^{*T} Ay = \max_{x\in X} x^T Ay^*.$$

Claim:

$$\min_{y\in Y} x^{*T} Ay = \max_{x\in X} x^T Ay^* = x^{*T} Ay^*. \tag{18}$$

The above claim is true because

$$\min_{y\in Y} x^{*T} Ay \leq x^{*T} Ay^*,$$

and

$$\min_{y\in Y} x^{*T} Ay = \max_{x\in X} x^T Ay^* \geq x^{*T} Ay^*.$$

Then,

$$\min_{y\in Y} x^{*T} Ay = x^{*T} Ay^* = \max_{x\in X} x^T Ay^*.$$

Hence, from (18)

$$x^T Ay^* \leq x^{*T} Ay^* \leq x^{*T} Ay, \ \forall\, x \in X, y \in Y.$$

That is, (x^*, y^*) is a saddle point equilibrium. $\square$

Remark 2.13. *We conclude from Theorems 2.11 and 2.12 that there exist both value and an SPE for a matrix game.*

Theorem 2.14. *A strategy pair (x^*, y^*) is an SPE if and only if*

$$x^* \in \operatorname*{argmax}_{x \in X} \min_{y \in Y} x^T Ay, \tag{19}$$

and

$$y^* \in \operatorname*{argmin}_{y \in Y} \max_{x \in X} x^T Ay. \tag{20}$$

Furthermore,

$$x^* Ay^* = \max_{x \in X} \min_{y \in Y} x^T Ay = \min_{y \in Y} \max_{x \in X} x^T Ay.$$

Proof. Let (x^*, y^*) be an SPE. Then,

$$x^T Ay^* \leq x^{*T} Ay^* \leq x^{*T} Ay, \ \forall \ x \in X, y \in Y.$$

This implies,

$$x^{*T} Ay^* = \max_{x \in X} x^T Ay^* = \min_{y \in Y} x^{*T} Ay, \tag{21}$$

and from (17)

$$x^{*T} Ay^* = \min_{y \in Y} \max_{x \in X} x^T Ay = \max_{x \in X} \min_{y \in Y} x^T Ay.$$

Then,

$$\min_{y \in Y} x^{*T} Ay = \max_{x \in X} \left(\min_{y \in Y} x^T Ay \right), \tag{22}$$

and

$$\max_{x \in X} x^T Ay^* = \min_{y \in Y} \left(\max_{x \in X} x^T Ay \right). \tag{23}$$

Therefore, from (22) and (23)

$$x^* \in \operatorname*{argmax}_{x \in X} \min_{y \in Y} x^T Ay,$$
$$y^* \in \operatorname*{argmin}_{y \in Y} \max_{x \in X} x^T Ay.$$

Let x^* and y^* be optimal strategies of player 1 and player 2 respectively. Then, from the proof of the Theorem 2.12 (x^*, y^*) is an SPE. □

Remark 2.15. *It follows from Theorem 2.14 that (x^*, y^*) is an SPE iff x^* is an optimal strategy of player 1 and y^* is an optimal strategy of player 2, where $x^{*T} Ay^*$ is the value of the game.*

Note: However, a strategy pair that gives the payoff same as value of the game need not be a saddle point equilibrium. This can be seen in the following example.

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 3 & 4 \\ 2 & 3 & 6 \end{pmatrix}.$$