3.2.2 Nash equilibrium computation for two-player games

In this section we give some algorithms to compute the Nash equilibria of a two player bimatrix game.

Definition 3.10. *The support of a mixed strategy is the set of pure strategies that have positive probability. Therefore,*

$$\mathcal{S}(x) = \{i \in I \mid x_i > 0\},$$

and

$$\mathcal{S}(y) = \{j \in J \mid y_j > 0\}.$$

For algorithmic purpose we redefine the action sets I and J to make them disjoint sets. Let $I = \{1, 2, \cdots, m\}$ and $J = \{m+1, m+2, \cdots, m+n\}$ be the action sets of player 1 and player 2 respectively. The following proposition gives the finite condition for the best response.

Proposition 3.11 (Best response condition)**.** *Let $\bar{x}$ and $\bar{y}$ be mixed strategies of player 1 and 2, respectively. Then $\bar{x}$ is a best response to $\bar{y}$ if and only if for all $i \in \mathcal{S}(\bar{x})$*

$$(A\bar{y})_i = \max\left\{ (A\bar{y})_k \mid k \in I \right\}, \tag{39}$$

and $\bar{y}$ is the best response to $\bar{x}$ if and only if for all $j \in \mathcal{S}(\bar{y})$

$$\left(B^T\bar{x}\right)_j = \max\left\{ \left(B^T\bar{x}\right)_l \mid l \in J \right\}. \tag{40}$$

Proof. We prove the if part by contraposition. Let the condition (39) does not hold for some $i \in \mathcal{S}(\bar{x})$. That is, $(A\bar{y})_i < \max_{k \in I}(A\bar{y})_k$. Then

$$
\begin{aligned}
\bar{x}^T A\bar{y} &= \sum_{i \in I} \bar{x}_i (A\bar{y})_i \\
&< \max_{k \in I} (A\bar{y})_{k \in I} \\
&= \max_{x \in X} x^T A\bar{y}
\end{aligned}
$$

Hence,

$$\bar{x}^T A\bar{y} < \max_{x \in X} x^T A\bar{y}.$$

This implies $\bar{x}$ is not best response to $\bar{y}$.

Consider a strategy $\bar{x}$ for which (39) is true for all the actions which are in its support. Then,

$$
\begin{aligned}
\bar{x}^T A\bar{y} &= \sum_{i \in I} \bar{x}_i (A\bar{y})_i \\
&= \max_{k \in I} (A\bar{y})_k \\
&= \max_{x \in X} x^T A\bar{y}.
\end{aligned}
$$

This implies $\bar{x}$ is a best response of $\bar{y}$. The result corresponding to player 2 can be proved using similar arguments. $\qquad\square$

Remark 3.12. *From the proof of proposition 3.11 a mixed strategy x, whose support satisfies condition* (39), *is a best response to* $\bar{y}$. *Similarly a mixed strategy y, whose support satisfies the condition* (40), *is a best response to* $\bar{x}$.

Note: Proposition 3.11 has the following intuition: Player 1's payoff $x^T A y$ is linear in x, so if it is maximized on a face of the simplex of mixed strategies of player 1, then it is also maximized on any vertex (i.e., pure strategy) of that face, and if it is maximized on a set of vertices then it is also maximized on any convex combination of them.

Proposition 3.11 can be used to compute the Nash equilibria of a bimatrix game by considering different sizes of support. Let us consider an example of a 3×2 bimatrix game (A, B) given by

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}.$$

It is easy to check that the action pair $(1, 4)$ is a pure strategy Nash equilibrium which can be represented by $((1, 0, 0), (1, 0))$ in mixed strategy setting. Let us consider the mixed strategies of support size 2. Let $\mathcal{S}(x) = \{1, 2\}$, $\mathcal{S}(y) = \{4, 5\}$. Then

$$3y_4^* + 3(1 - y_4^*) = 2y_4^* + 5(1 - y_4^*) \implies y_4^* = \frac{2}{3}, y_5^* = \frac{1}{3}.$$

The actions 1 and 2 of player 1 are best response to y^* because $Ay^* = (3, 3, 2)$. Therefore, from Proposition 3.11 any $x \in X$ such that $x_1 > 0$, $x_2 > 0$, $x_3 = 0$ is a best response of y^*. Again

$$3x_1^* + 2(1 - x_1^*) = 2x_1^* + 6(1 - x_1^*) \implies x_1^* = \frac{4}{5}, x_2^* = \frac{1}{5}, x_3^* = 0.$$

The actions 4 and 5 of player 2 are best response to x^* because $B^T x^* = (\frac{14}{5}, \frac{14}{5})$. Therefore, from Proposition 3.11 any $y \in Y$ such that $y_4 > 0$ and $y_5 > 0$ is best response to x^*. This implies, x^* and y^* are best response to each other. Therefore $(x^*, y^*) = \left(\left(\frac{4}{5}, \frac{1}{5}, 0\right), \left(\frac{2}{3}, \frac{1}{3}\right) \right)$ is a Nash equilibrium.

Similarly, by considering $\mathcal{S}(x) = \{2, 3\}$ and $\mathcal{S}(y) = \{4, 5\}$ we have a strategy pair $\left(\left(0, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right) \right)$ which is a Nash equilibrium. The support choice $\mathcal{S}(x) = \{1, 3\}$ and $\mathcal{S}(y) = \{4, 5\}$ doesn't give a Nash equilibrium. The first reason is

$$3 = 6y_5 \implies y_4 = \frac{1}{2} = y_5,$$

and $Ay = (3, 3.5, 3)$ shows that the actions 1 and 3 of player 1 are not best response to $y = (\frac{1}{2}, \frac{1}{2})$. The second reason is

$$3 = 2x_1 + 1 - x_1 \implies x_1 = 2, x_3 = -1,$$

which does not give a probability vector. In this game, it is not necessary to consider a mixed strategy x of player 1 where all three pure strategies have positive probability, because player 1 would then have to be indifferent between all these. However, a mixed strategy y of player 2 is already uniquely determined by equalizing the expected payoffs for two rows, and then the payoff for the remaining row is already different. This is the typical, "nondegenerate" case, according to the following definition.

Definition 3.13. *A two-player game is called nondegenerate if no mixed strategy of support size k has more than k pure best responses.*

In a degenerate game, Definition 3.13 is violated, for example, if there is a pure strategy that has two pure best responses. We consider an example of a degenerate game below.

Example 3.14. *Consider a bimatrix game*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

The pure strategy pairs $((0,1),(1,0,0))$ and $((1,0),(0,1,0))$ are Nash equilibria for this game. Let us consider the case of $\mathcal{S}(x) = \{1,2\}$ and $\mathcal{S}(y) = \{3,4,5\}$. Then,

$$y_3 + 2y_4 + 1 - y_3 - y_4 = 3y_3 + y_4 \Longrightarrow y_3 = \frac{1}{3}.$$

This implies player 1 is indifferent between his two pure actions for a $y^* \in Y$ such that $y^* = (\frac{1}{3}, y_4^*, y_5^*)$ where $y_4^* \geq 0$, $y_5^* \geq 0$ and $y_4^* + y_5^* = \frac{2}{3}$. Therefore, a $x \in X$ such that $x_1 \geq 0$, $x_2 \geq 0$ is a best response to y^*. Again from the indifference condition of player 2 we have

$$x_1^* + 2(1 - x_1^*) = 2x_1^* + 1 - x_1^* = 3x_1^* \Longrightarrow x_1^* = \frac{1}{2}, x_2^* = \frac{1}{2}.$$

Therefore, a mixed strategy y such that $y_3 \geq 0$, $y_4 \geq 0$, $y_5 \geq 0$ is a best response to $x^* = (\frac{1}{2}, \frac{1}{2})$. Hence, a strategy pair $\left((\frac{1}{2}, \frac{1}{2}), y^* \right)$ is a Nash equilibrium of the game. The case where the support size of y is 2 can be obtained by taking the respective component of y as 0. Therefore, the set of Nash equilibrium is given by

$$\mathcal{NE} = \left\{ (x^*, y^*) \mid x^* = \left(\frac{1}{2}, \frac{1}{2} \right), \ y^* = \left(\frac{1}{3}, y_4^*, y_5^* \right), \ \text{where } y_4^* \geq 0, \ y_5^* \geq 0, \ y_4^* + y_5^* = \frac{2}{3} \right\}.$$

The above example shows that there can be infinite number of Nash equilibrium. A non-degenerate game always has the finite number of Nash equilibria.

Proposition 3.15. *In any Nash equilibrium (x, y) of a nondegenerate bimatrix game, x and y have supports of equal size.*

Proof. Let (x, y) be a Nash equilibrium of nondegenerate game. Suppose $|\mathcal{S}(x)| < |\mathcal{S}(y)|$. Since, y is a best response of x, all the actions from $\mathcal{S}(y)$ will be the best response to x which contradicts the definition of a nondegenerate game. Similarly, we get the contradiction for $|\mathcal{S}(x)| > |\mathcal{S}(y)|$. Therefore, $|\mathcal{S}(x)| = |\mathcal{S}(y)|$. $\qquad\square$

The "support testing" algorithm for finding Nash equilibria of a nondegenerate bimatrix game then works as follows.

Algorithm 3.16. *(Equilibria by support enumeration) For each $k = 1, \cdots, \min\{m, n\}$ and each pair (M, N) of k-sized subsets of I and J, respectively, solve the equations $\sum_{i \in M} x_i b_{ij} = v$ for $j \in N$, $\sum_{i \in M} x_i = 1$, $\sum_{j \in N} a_{ij} y_j = u$ for $i \in M$, $\sum_{j \in N} y_j = 1$, and check that $x \geq 0$, $y \geq 0$, and that (39) holds for x and (40) holds for y.*

The linear equations considered in this algorithm may not have solutions, which then mean no equilibrium for that support pair. We only focus on nondegenerate games.

Equilibria via Labeled Polytopes

For a given strategy x of player 1 the maximum expected payoff of player 2 is given by

$$v = \max_{y \in Y} x^T B y$$
$$= \max_{j \in J} (B^T x)_j.$$

Therefore, the upper envelope of the payoffs of player 2 for a given strategy x of player 1 is given by

$$(B^T x)_j \leq v, \ \forall \, j \in J.$$

Similarly, the upper envelope of the payoffs of player 1 for a given strategy y of player 2 is given by

$$(Ay)_i \leq u, \ \forall \, i \in I.$$

The "best response polyhedron" of a player is the set of that player's mixed strategies together with the "upper envelope" of expected payoffs to the other player. Therefore, the best response polyhedron of player 1 and player 2 are given by

$$\bar{P} = \left\{ (x, v) \in \mathbb{R}^m \times \mathbb{R} \mid x \geq 0, \ \mathbf{1}^T x = 1, \ B^T x \leq v\mathbf{1} \right\},$$

$$\bar{Q} = \left\{ (y, u) \in \mathbb{R}^n \times \mathbb{R} \mid Ay \leq u\mathbf{1}, \ y \geq 0, \ \mathbf{1}^T y = 1 \right\}.$$

We say a point $(x, v) \in \bar{P}$ has a label $k \in I \cup J$ if the kth inequality in the definition of $\bar{P}$ is active. For example, for $k = j \in J$ the jth inequality $(B^T x)_j = v$, or for $k = i \in I$ the ith inequality $x_i = 0$. The labels of polyhedron $\bar{Q}$ are similarly defined. The pair $((x, v), (y, u))$ in $\bar{P} \times \bar{Q}$ is completely labeled, which means that every label $k \in I \cup J$ appears as a label either of (x, v) or of (y, u).

Claim: A mixed strategy pair (x, y) with the corresponding expected payoffs u and v is a Nash equilibrium if and only if the pair $((x, v), (y, u)) \in \bar{P} \times \bar{Q}$ is completely labeled.

Proof. Let (x, y) be a Nash equilibrium. Then, x and y are best response to each other and

$$u = x^T A y = \max_{\tilde{x} \in X} \tilde{x}^T A y = \max_{i \in I} (Ay)_i$$
$$v = x^T B y = \max_{\tilde{y} \in Y} x^T A \tilde{y} = \max_{j \in J} (B^T x)_j.$$

Therefore, from Proposition 3.11 for all $i \in I$ and $j \in J$

$$x_i > 0 \Longrightarrow (Ay)_i = u,$$

and

$$y_j > 0 \Longrightarrow (B^T x)_j = v.$$

This implies $((x, v), (y, u)) \in \bar{P} \times \bar{Q}$ is completely labeled.

Let the pair $((x, v), (y, u))$ be completely labeled in $\bar{P} \times \bar{Q}$. Then, $(Ay)_i = u$ for all $i \in I$ for which $x_i > 0$ and $(B^T x)_j = v$ for all $j \in J$ for which $y_j > 0$. Therefore, from Proposition 3.11 x and y are the best response of each other. Hence, (x, y) is a Nash equilibrium. $\qquad \square$

The best response polyhedron $\bar{P}$ and $\bar{Q}$ can be simplified by eliminating the payoff variables u and v, which works if these are always positive. If payoff matrices A and B are strictly positive, u and v will always be positive. The set of Nash equilibria of a game does not change if we add a same constant to each entries of the payoff matrices. Therefore, without loss of generality we can assume $A > 0$ and $B > 0$. We can also consider the matrices with zero entries and even negative entries as long as u and v remains positive. For $\bar{P}$, we divide each inequality $\sum_{i \in I} b_{ij} x_i \leq v$ by v, which gives $\sum_{i \in I} b_{ij} \left(\frac{x_i}{v} \right) \leq 1$ treat $\frac{x_i}{v}$ as a new variable that we call again x_i , and call the resulting polyhedron P. Similarly, $\bar{Q}$ is replaced by Q by dividing each inequality in $Ay \leq \mathbf{1}u$ by u. Then,

$$P = \left\{ x \in \mathbb{R}^m \mid x \geq 0,\ B^T x \leq \mathbf{1} \right\},$$

$$Q = \left\{ y \in \mathbb{R}^n \mid Ay \leq \mathbf{1},\ y \geq 0, \right\}.$$

Nonzero vectors $x \in P$ and $y \in Q$ are multiplied by $v = \frac{1}{\mathbf{1}^T x}$ and $u = \frac{1}{\mathbf{1}^T y}$ to turn them into probability vectors. The scaling factors v and u are the expected payoffs to the player 2 and player 1 respectively.

The set $\bar{P}$ is in one-to-one correspondence with $P - \{0\}$ with the map $(x, v) \to \frac{x}{v}$, and $\bar{Q}$ is in one-to-one correspondence with $Q - \{0\}$ with the map $(y, u) \to \frac{y}{u}$. The active inequalities in $\bar{P}$ and $\bar{Q}$ remains active in P and Q. Then, the points have the same labels. Hence, a completely labeled pair $(x, y) \in P \times Q$ gives a Nash equilibrium.

Nondegeneracy of a bimatrix game (A, B) can be stated in terms of the polytopes P and Q as follows: no point in P has more than m labels, and no point in Q has more than n labels. (If $x \in P$ and x has support of size k and L is the set of labels of x, then $|L \cap I| = m - k$, so $|L| > m$ implies x has more than k best responses in $L \cap J$, which contradicts the fact that bimatrix game (A, B) is nondegenerate). Furthermore only vertices of P can have m labels, and only vertices of Q can have n labels. Otherwise, a point of P with m labels that is not a vertex would be on a higher dimensional face, and a vertex of that face, which is a vertex of P , would have additional labels. In this case, the only possibility for $(x, y) \in P \times Q$ to be a completely labeled if x is a vertex of P and y is a vertex of Q. Therefore, only vertices of P and Q have to be inspected as possible Nash equilibrium strategies.

Algorithm 3.17 (Equilibria by vertex enumeration)**.** *Input: A nondegenerate bimatrix game. Output: All Nash equilibria of the game. Method: For each vertex x of $P - \{0\}$, and each vertex y of $Q - \{0\}$, if (x, y) is completely labeled, output the Nash equilibrium $\left(\frac{x}{\sum_{i \in I} x_i}, \frac{y}{\sum_{j \in J} y_j} \right)$.*

Consider the same nondegenerate game as before where

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix},\ B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}. \tag{41}$$

The best response polytopes P and Q for above game are given by

$$P = \{x \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 3x_1 + 2x_2 + 3x_3 \leq 1, 2x_1 + 6x_2 + x_3 \leq 1\},$$

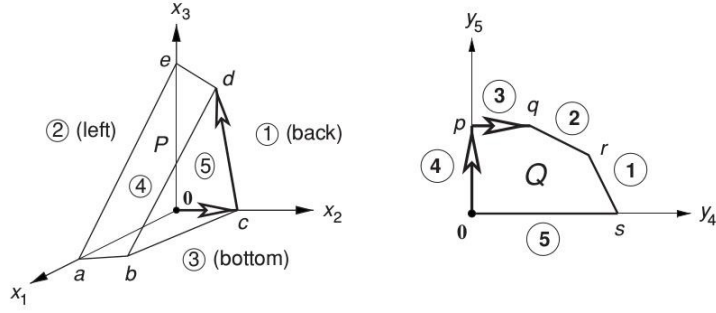$$Q = \{y \mid 3y_4 + 3y_5 \leq 1,\ 2y_4 + 5y_5 \leq 1,\ 6y_5 \leq 1,\ y_4 \geq 0,\ y_5 \geq 0\}.$$

Figure 5: The best response polytopes P and Q for the game in (41).

These polytopes are defined in Figure 5. The vertices of P and Q together with its labels are defined in table below:

No.	Vertex	Label
o	$(0, 0, 0)$	1, 2, 3
a	$\left(\frac{1}{3}, 0, 0\right)$	2, 3, 4
b	$\left(\frac{2}{7}, \frac{1}{14}, 0\right)$	3, 4, 5
c	$\left(0, \frac{1}{6}, 0\right)$	1, 3, 5
d	$\left(0, \frac{1}{8}, \frac{1}{4}\right)$	1, 4, 5
e	$\left(0, 0, \frac{1}{3}\right)$	1, 2, 4

No.	Vertex	Label
o	$(0, 0)$	4, 5
p	$\left(0, \frac{1}{6}\right)$	3, 4
q	$\left(\frac{1}{12}, \frac{1}{6}\right)$	2, 3
r	$\left(\frac{2}{9}, \frac{1}{9}\right)$	1, 2
s	$\left(\frac{1}{3}, 0\right)$	1, 5

From above table it is clear that (a, s), (b, r), and (d, q) are completely labeled vertex pairs. The vertex pair (a, s) corresponds to pure strategy Nash equilibrium $((1, 0, 0), (1, 0))$, and the vertex pair (b, r) corresponds to a mixed strategy Nash equilibrium $\left(\left(\frac{4}{5}, \frac{1}{5}, 0\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$, and the vertex pair (d, q) corresponds to a mixed strategy Nash equilibrium $\left(\left(0, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$.

Lemke-Howson Algorithm

In the previous section we have seen that a completely labeled vertex pair of $P \times Q$ gives a Nash equilibrium of a nondegenerate bimatrix game. The Lemke-Howson algorithm is based on moving from one vertex to another alternatively in polytopes P and Q. We start from vertex pair $(\mathbf{0}, \mathbf{0})$ which is a completely labeled pair and is also called as artificial equilibrium pair. Due to nondegenerate nature of the game at each time when we move from one vertex to another vertex in polytope P, we drop exactly one label at the current vertex and pick up a new label at new vertex. This new label which we picked up is a duplicate label as it is also present in polytope Q. Then, at the next step we drop the duplicate label at Q and pick up a new label which again needs to be dropped in polytope P to avoid duplicasy. The procedure stops when a completely labeled vertex pair is found. The Lemke-Howson algorithm follows a path of vertex pairs (x, y) of $P \times Q$ that starts at $(0, 0)$ and ends at a Nash equilibrium. Unlike the previous algorithms, the Lemke-Howson algorithm compute only one Nash equilibrium.

Lemke-Howson algorithm for Example 41:
For example 41 the polytopes P and Q are shown in Figure 5. Starting from o in P, suppose label 2 is dropped which we call as missing label, traversing the edge from o to vertex c, which is the set of points x of P that have labels 1 and 3, shown by an arrow in Figure 5. The endpoint c of that edge has label 5 which is picked up. At the vertex pair (c, o) of $P \times Q$, all labels except for the missing label 2 are present, so label 5 is now duplicate because it is both a label of c and of o. The next step is therefore to drop the duplicate label 5 in Q, traversing the edge from o to vertex p while keeping c in P fixed. The label that is picked up at vertex p is 3, which is now duplicate. Dropping label 3 in P defines the unique edge defined by labels 1 and 5, which joins vertex c to vertex d. At vertex d, label 4 is picked up. Dropping label 4 in Q means traversing the edge of Q from p to q. At vertex q, label 2 is picked up. Because 2 is the missing label, the current vertex pair (d, q) is completely labeled, and it is the Nash equilibrium found by the algorithm.

Lemke-Howson algorithm:
The Lemke-Howson algorithm follows the edges of a polyhedron, which is implemented algebraically by pivoting as used by the simplex algorithm for solving a linear program. In fact we can use integer pivoting in this case. The best response polytopes P and Q can be written using slack variables as follows:

$$B^T x + s = \mathbf{1},$$
$$x \geq 0, \ s \geq 0. \tag{42}$$

and

$$Ay + r = \mathbf{1},$$
$$r \geq 0, y \geq 0. \tag{43}$$

A basic solution to (42) is given by n basic (linearly independent) columns, where the nonbasic variables that correspond to the m (nonbasic) columns are set to be zero so that the basic variables are uniquely determined. Similarly a basic solution to (43) is given

by m basic columns, where n nonbasic variables are set to zero. A basic feasible solution also fulfills the nonnegative constraints and defines a vertex x of P and y of Q. The labels of such a vertex are given by the respective nonbasic columns. Pivoting is a change of the basis where a nonbasic variable enters and a basic variable leaves the set of basic variables, while preserving feasibility. In this case entering a nonbasic variable determine the label which is being dropped and the leaving a basic variable determine the new label which is being picked up.

We illustrate this procedure using Example 41. The best response polytopes P and Q for 41 can be written as

$$3x_1 + 2x_2 + 3x_3 + s_4 = 1,$$
$$2x_1 + 6x_2 + x_3 + s_5 = 1,$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_4 \geq 0, s_5 \geq 0,$$

and

$$3y_4 + 3y_5 + r_1 = 1$$
$$2y_4 + 5y_5 + r_2 = 1$$
$$6y_5 + r_3 = 1$$
$$r_1 \geq 0, r_2 \geq 0, r_3 \geq 0, y_4 \geq 0, y_5 \geq 0.$$

The initial basic matrices are identity matrices, and s_4, s_5 are initial basic variables for P and r_1, r_2, r_3 are initial basic variables for Q. Dropping the Label 2 means x_2 is an entering variable for polytope P. The leaving variable is decided using minimum ratio test. It will assign a new label which needs to be dropped in polytope Q by considering it as entering variable for Q. This will lead to a new basis in both P and Q. The change of basis is done by maintaining identity matrix corresponding to basic columns using elementary row operations which automatically gives the value of basic variables. We use integer pivoting and maintain the a constant multiple (say K) of identity matrix corresponding to basic columns. The values of corresponding basic variables can be obtained by dividing the corresponding right hand side values by the constant K. This procedure continues until we find a vertex pair which is completely labeled