4 Strategic games with continuous strategy sets

So far we have studied strategic games with finite number of pure strategies for each player. The existence of Nash equilibrium for these games is guaranteed in the space of mixed strategies. However, in more practical situations a player has continuum number of pure strategies. For example, a strategy of a player could be a real number $x \in [a, b]$, or it could be a vector $x \in \mathbb{R}^n$. We start with some practical examples of the games with continuous strategy sets.

4.1 Market Competition

Perfect Competition

Perfect competition is a market structure in which the following five criteria are met:

- All firms sell an identical product.
- All firms are price takers - they cannot control the market price of their product.
- All firms have a relatively small market share.
- Buyers have complete information about the product being sold and the prices charged by each firm.
- The industry is characterized by freedom of entry and exit.

The equilibrium condition in perfect competition is a point where a firm has no tendency to change its level of output. It needs neither expansion nor contraction. Let p be the market price and q be the quantity produced by a firm. The total revenue of the firm in this case is

$$TR = pq.$$

The marginal revenue is the additional revenue that will be generated by increasing product sales by one unit which is in this case is market price p. The marginal cost is the cost of producing one more unit of a good. For linear cost functions it is constant (say c). In this case, the profit of the firm is given by

$$\text{Profit} = (p - c)q.$$

Therefore, at the equilibrium the market price is same as marginal cost. Real markets are not perfectly competitive markets. Next, we discuss two simple example of imperfect competition.

Example 4.1 (Cournot duopoly). *Cournot competition is an economic model used to describe an industry structure in which companies compete on the amount of output they will produce. We consider the case of two firms which is called as Cournot duopoly. It has the following features*

- *Both the firms produce the homogeneous product.*
- *Both the firms do not cooperate.*

- *Firms have market power, i.e. each firm's output decision affects the good's price;*

- *The market price at which firms sell their product depends upon the total output of both the firms.*

- *Each firm is economically rational and is interested in maximizing its profit given the decisions of other firm.*

Let q_1, $q_2 \in [0, \infty)$ be the quantities of firm 1 and firm 2 which they produce simultaneously. The firms sell their product at market clearing price, i.e., the price at which the quantities produced are equal to the quantities demanded. The market clearing price depends upon the total output and it is denoted by $P(q_1 + q_2)$. The cost of producing q_i units of product by firm i, $i = 1, 2$, is given by $c_i(q_i)$. Hence, the profit of firm i is given by

$$u_i(q_1, q_2) = q_i \cdot P(q_1 + q_2) - c_i(q_i).$$

We consider linear market price $P(q_1 + q_2) = a - b(q_1 + q_2)$ and linear cost $c_i(q_i) = c \cdot q_i$ where $b > 0$ and c is a marginal cost.

Nash Equilibrium: A best response strategy of firm 1, for a fixed q_2, is given by solving

$$\frac{\partial u_1}{\partial q_1} = a - b(q_1 + q_2) - bq_1 - c = 0,$$

which gives

$$q_1 = \frac{a - bq_2 - c}{2b}.$$

The above q_1 is a maximizer because $\frac{\partial^2 u_1}{\partial q_1^2} = -2b < 0$. Hence, $r_1(q_2) = \frac{a - bq_2 - c}{2b}$ gives a best response strategy of firm 1 for a fixed strategy q_2 of firm 2. Similarly, $r_2(q_1) = \frac{a - bq_1 - c}{2b}$ gives a best response strategy of firm 2 for a fixed strategy q_1 of firm 1. A strategy pair (q_1^*, q_2^*) is said to be Nash equilibrium if and only if q_1^* and q_2^* are best response of each other. Therefore, it is a solution of

$$q_1^* = r_1(q_2^*) = \frac{a - bq_2^* - c}{2b},$$

and

$$q_2^* = r_2(q_1^*) = \frac{a - bq_1^* - c}{2b}.$$

This gives, $q_1^* = q_2^* = \frac{a-c}{3b}$. The total industry output is $\frac{2(a-c)}{3b}$. The equilibrium price is given by

$$P(q_1^* + q_2^*) = \frac{a + 2c}{3}.$$

The monopoly case:
For the case of only one firm the payoff function of the firm is given by

$$u(q) = q(a - bq) - cq.$$

Then, the optimal output of the firm is $q^* = \frac{a-c}{2b}$ and the price in this case is given by

$$P(q^*) = \frac{a+c}{2}.$$

Implications:

- The total output is greater with Cournot duopoly than monopoly.
- Price is lower with Cournot duopoly than monopoly, but not as low as with perfect competition.

Cournot competition with many firms:
Suppose there are N number of firms are involved in Cournot competing. The market price and cost functions can be derived in a similar manner as follows:

$$p(q) = a - b\sum_{i=1}^{N} q_i.$$

$$c_i(q_i) = cq_i,\ \forall\ i = 1, 2, \cdots, N.$$

The Nash equilibrium output of the firms are given by

$$q_i^* = \frac{a-c}{b(N+1)},\ \forall\ i = 1, 2, \cdots, N.$$

Therefore, total equilibrium output is given by

$$\sum_{i=1}^{N} q_i^* = \frac{N(a-c)}{b(N+1)},$$

and the equilibrium market price is given by

$$p(q^*) = \frac{a+Nc}{N+1}.$$

It is clear that

$$\lim_{N\to\infty} p(q^*) = c.$$

Hence, with many firms a Cournot market approximates a perfectly competitive market.

Example 4.2 (Bertrand duopoly). *In Bertrand competition the firms compete by setting prices simultaneously and consumers want to buy everything from a firm with a lower price. There are two firms producing a homogeneous product and can not cooperate in any way. If two firms charge the same price, consumers demand is split evenly between them. A crucial assumption about the technology is that both firms have the same constant unit cost of production.*

Payoff function: Let firm 1 and firm 2 set prices p_1 and p_2 respectively. The market demand is given by $q = D(p) = \frac{a}{b} - \frac{1}{b}p$, and the production cost $C(q) = cq$, where c the is unit cost of production. The payoff functions of both the firms are given by

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c) \cdot D(p_1), \text{ if } p_1 < p_2 \\ \frac{1}{2}(p_1 - c) \cdot D(p_1), \text{ if } p_1 = p_2 \\ 0, \text{ otherwise.} \end{cases} \tag{51}$$

$$u_2(p_1, p_2) = \begin{cases} (p_2 - c) \cdot D(p_2), \text{ if } p_2 < p_1 \\ \frac{1}{2}(p_2 - c) \cdot D(p_2), \text{ if } p_1 = p_2 \\ 0, \text{ otherwise.} \end{cases} \tag{52}$$

It is obvious that no firm will set its price below the unit cost c.

Nash equilibrium:

- $(p_1^*, p_2^*) = (c, c)$ is a Nash equilibrium of the game where no firm earns any thing. Because if a firm tries to set a price higher than c then it looses all the market, and if it tries to set a price lower than c then it gets negative payoff.
- There is no strategy pair other than (p_1^*, p_2^*) which is a Nash equilibrium. For example, if both firms set the same price above unit cost and share the market, then each firm has an incentive to undercut the other by an arbitrarily small amount and capture the whole market and almost double its profits. So there can be no equilibrium with both firms setting the same price above marginal cost. Also, there can be no equilibrium with firms setting different prices. The firms setting the higher price will earn nothing (the lower priced firm serves all of the customers). Hence the higher priced firm will want to lower its price to undercut the lower-priced firm. Hence the only equilibrium in the Bertrand model occurs when both firms set price equal to unit cost c.

Since the Bertrand model assumes that firms compete on price and not output quantity, it predicts that a duopoly is enough to push prices down to marginal cost level, meaning that a duopoly will result in perfect competition.