5.3 Network formation games

We consider the network games where the strategic relationships among players are represented by a graph. The nodes of a graph represent the players and edge between two nodes represent the relationship between two nodes. The graph can be both directed as well as undirected. A directed edge represent one-way relationship, i.e., when we don't need the consent of other player to create a link, which is generally true in citation networks or in links between web pages. An undirected edge represent two-way relationship, i.e., to create a link the consent of both the players are needed. For instance, in order to form a trading partnership, to maintain a friendship, maintaining a business relationship, alliance, etc.

5.3.1 Model

A network formation game is represented by a graph (N, g), where $N = \{1, 2, \cdots, n\}$ set of players (nodes) and $g = [g_{ij}]$, where $g_{ij} \in \{0, 1\}$ is an $n \times n$ adjacency matrix which defines the graph. For an undirected graph $g_{ij} = g_{ji}$, $g_{ij} = 1$ means i and j are connected with an edge, whereas $g_{ij} = 0$ means there is no edge between i and j. For an instance, if $N = \{1, 2, 3\}$, then

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is the undirected network given by Figure 10. For directed graph g_{ij} need not be

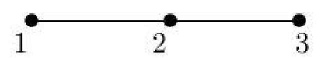

Figure 10: A network with two links

same as g_{ji}. Let the shorthand notation of $g + ij$ represent the network obtained by adding the link ij to an existing network g, and let $g - ij$ represent the network obtained by deleting the link ij from the network g. We can represent directed networks in an analogous manner, viewing ij as a directed link and distinguishing between ij and ji.

Network formation rules:

- In an undirected network, addition of a link require the consent of both the nodes while severing a link require the consent of only 1 player.
- In a directed network, addition as well as deletion of links can be done unilaterally. However, no self loops.

Let $G(N)$ be the set of all possible networks form on N. At each network, all the players receives certain payoffs. Let $u_i : G(N) \to \mathbb{R}$ defines the payoff function of player i.

The value of a network g is defined by

$$v(g) = \sum_{i \in N} u_i(g).$$

We assume that all the players are rational. Therefore, any of the players can sever the existing link unilaterally if it is beneficial for him or any two players can create a new link if both the players receive at least as much as they receive at old network and at least one player is strictly benefited. This leads to the definition of pairwise stable networks.

Definition 5.6 (Pairwise stability). *A network g is pairwise stable relative to payoff profile $(u_1, u_2, \cdots, u_n)$ if*

- *for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$, and $u_j(g) \geq u_j(g - ij)$, and*
- *for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.*

A network is pairwise stable if no player wants to sever a link and no two players both want to add a link.

Efficiency: A network g is efficient relative to the payoff profile $(u_1, u_2, \cdots, u_n)$ if it has highest value, i.e.,

$$\sum_{i \in N} u_i(g) \geq \sum_{i \in N} u_i(g'), \ \forall \ g' \in G(N).$$

Since, there are only finite number of networks, there always exists an efficient network.

Pareto Efficiency: A network g is Pareto efficient relative to the payoff profile $(u_1, u_2, \cdots, u_n)$ if there does not exist any $g' \in G(N)$ such that $u_i(g') \geq u_i(g)$ for all i with strict inequality for some i.

We say that one network Pareto dominates another if it leads to a weakly higher payoffs for all individuals, and a strictly higher payoffs for at least one. A network is then Pareto efficient if it is not Pareto dominated by any other network.

Claim: An efficient network g relative to the payoff profile $(u_1, u_2, \cdots, u_n)$ is also a Pareto efficient relative to $(u_1, u_2, \cdots, u_n)$.

Proof. Assume g is not Pareto efficient, then there exists a network g' such that $u_i(g') \geq u_i(g)$ for all i together with strict inequality for at least one i. This implies $\sum_{i \in N} u_i(g') > \sum_{i \in N} u_i(g)$. Then, g is also not efficient. This proves the Claim. □

Example 5.7 (Non-existence of a Pairwise Stable Network). *Consider the game with 4 players who obtain payoffs from trading with each other. The players have random endowments and the benefits from trading depend on the realization of these random endowments. The more players who are linked, the greater the gains from trade, but with diminishing marginal returns. In particular, there is a cost of a link of $c = 5$ to each player involved in the link. The utility of being alone is 0. Not accounting for the cost of links, the benefits to each player in a dyad is 12, the benefits for being connected (directly or indirectly) to two other players is 16, and of being connected to three other individuals is 18.*

It follows from the following arguments:

- Any network with more than three links is defeated by a network with fewer links, as some players will save the link cost by severing the link, and yet the full trading benefits are already realized with just three links.
- In one link networks, two isolated players can always form a link and earn 7.
- In two link networks there are two possibilities: *i*) three players are connected and one player is isolated, *ii*) there are two separate dyads. In first case, the player with two links can earn utility 1 by cutting one edge, and in second case two separate dyads gain by forming a link between them.
- in any network of four players that has just three links, one of the players who has more than one link will save 5 utility in cost by severing a link and only lose 2 utility in trading benefits.

The resulting payoffs for several of the key network configurations are pictured in Figure 11.

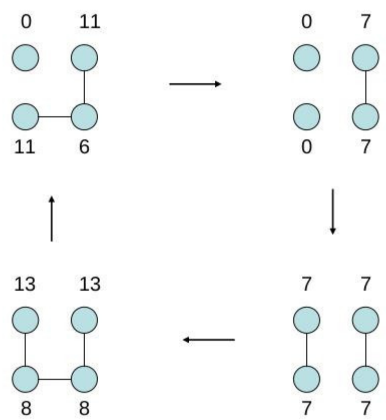

Figure 11: Payoffs pictured for one, two and three link networks and an improving cycle including the four networks.

Remark 5.8. *From example 5.7, it is clear that a pairwise stable network need not always exists.*

Example 5.9 (Unique Pairwise stable network)**.** *The network game with 3 players given in Figure 12 has unique pairwise stable network and it is also efficient network.*

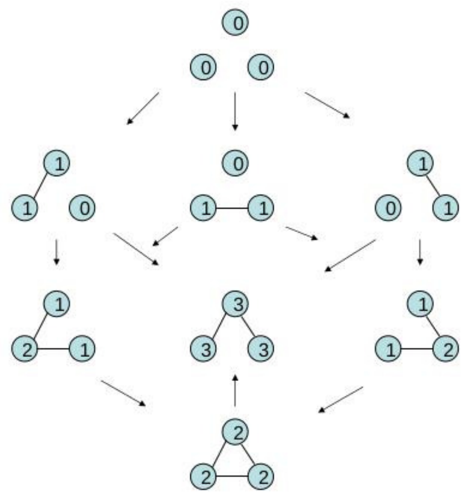

Figure 12: Improving Paths: The payoffs are listed in the nodes and the arrows point towards a network that defeats the one from which the arrow emanates. Following the arrows provides improving paths. There is a unique pairwise stable network.

Example 5.10 (Multiple pairwise stable networks)**.** *The network game with 3 players given in Figure 13 has two pairwise stable networks. One of them is also efficient network.*

In Examples 5.9 and 5.10, a pairwise stable network is also an efficient network. However, this is not always the case. We give an example where pairwise stable network and efficient network are not the same.

Example 5.11 (Pairwise stability vs Efficiency)**.** *Consider a 3 nodes network game, where*

- *Utility of each player is zero in an empty network.*
- *At one link networks utility of players who are connected with an edge have utility 1 and the isolated player has utility zero.*
- *The two links networks are star shaped network where the central player has utility -2 and other two players have utility 2.*
- *All the players at complete network have utility 1.*

In this example only one link networks are pairwise stable network. However, the complete network is the efficient network.
Connections Model: This example models social communication among individuals. Individuals directly communicate with those to whom they are linked. Through these links they also benefit from indirect communication from those to whom their adjacent nodes

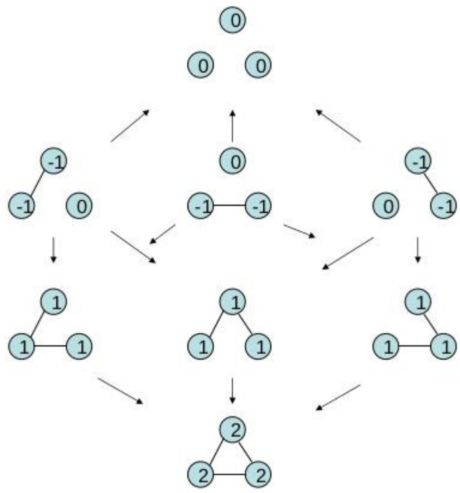

Figure 13: An Example with Two Pairwise Stable Networks, where Improving Paths can get Stuck at the Empty Network.

are linked, and so on. The value of communication obtained from other nodes depends on the distance to those nodes. Also, communication is costly so that individuals must weigh the benefits of a link against its cost. The utility of each player from the graph g is given by

$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - \sum_{j: ij \in g} c_{ij},$$

where c_{ij} denote the cost to i of maintaining the link ij, t_{ij} is the number of links in the shortest path between i and j (setting $t_{ij} = \infty$ if there is no path between i and j), and $0 < \delta < 1$ captures the idea that the value that i derives from being connected to j is proportional to the proximity of j to i. Less distant connections are more valuable than more distant ones, but direct connections are costly. For the symmetric version of this model $c_{ij} = c$ for all $ij \in g$. For simpler analysis we discuss the efficiency and pairwise stability for the case of only 3 players. Although, the results can be generalized for n-player case. We consider all possible cases of c.

Efficiency in symmetric connections model:
Case 1: $c < \delta - \delta^2$
In this case $\delta - c > \delta^2$. This implies the payoff earned by direct connections are more than the indirect connections. Hence, complete network is efficient network.

Case 2: $\delta - \delta^2 < c < \delta + \frac{\delta^2}{2}$
The value of all the networks are as follows:
v(empty)$= 0$,
v(star)$= 4(\delta - c) + 2\delta^2$,
v(complete)$= 6(\delta - c)$,
v(one link)$= 2(\delta - c)$.
It is easy show that v(star)$> v(g)$ for all networks g. Therefore, star network is efficient.

Case 3: $c > \delta + \frac{\delta^2}{2}$
In this case empty network is efficient.

Pairwise stability in symmetric connections model:
Case 1: $c < \delta - \delta^2$
In this case a complete network is unique pairwise stable network because

- Any two players at empty network has incentive to form a link because $\delta - c > \delta^2 > 0$,
- Any two players at one link networks have incentive to form a new link,
- At star network, two non-central nodes have incentive to form a new link and move to complete network,
- From complete network there is no incentive for any player to cut the link.

Case 2: $\delta - \delta^2 < c < \delta$
In this case star networks are pairwise stable because

- Any two players at empty network have incentive to form a link because $\delta - c > 0$,
- Any two players at 1-link networks have incentive to form a link because $\delta - c > 0$,
- Any player at complete network has incentive to cut the link as $\delta - c < \delta^2$.
- From star network, any pair of players have no incentive to change the structure of the network.

Case 3: $c > \delta$
In this case forming a direct link gives negative payoff. Therefore, at all the networks a player always has incentive to cut the link. Hence, empty network is the only pairwise stable network.

The Co-author Model: The nodes are interpreted as researchers who spend time writing papers. Each node's productivity is a function of its links. A link represents a collaboration between two researchers. The amount of time a researcher spends on any given project is inversely related to the number of projects that researcher is involved in. Thus, in contrast to the connections model, here indirect connections will enter the utility function in a negative way as they detract from one's co-author time. Let n_i and n_j be the number of projects researchers i and j are involved in. Then, the utility of player i, for $n_i > 0$, from the graph g is given by

$$u_i(g) = \sum_{j:ij\in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j}\right].$$

For $n_i = 0$, $u_i(g) = 0$. The above utility form assumes that each researcher has a unit of time which they allocate equally across their projects. The output of each project depends on the total time invested in it by the two collaborators, which is given by $\frac{1}{n_i} + \frac{1}{n_j}$, and on some synergy in the production process captured by the interactive term $\frac{1}{n_i n_j}$.

Efficiency and Pairwise stability:
For $n = 3$ a star network is an efficient network while complete network is pairwise stable. For $n = 4$ two separate dyad is an efficient network while the complete network is pairwise stable.
Pairwise Nash stable network: A network that will be stable over time in the sense that no players would like to delete any links (possibly multiple links at a time) is stable in Nash sense. We call such networks a Nash stable networks. The main drawback of this is that they are too many (see Example 5.12), including some which are easily seen to be unreasonable. It is not reasonable to consider Nash stability alone as a solution concept. Becasue, it does not capture the fact that it may be mutually advantageous for two players to form a new relationship. We need to move beyond Nash equilibrium to capture this. Although pairwise stability overcomes the difficulties, but it restricts attention to changes of one link at a time. This can lead to over-connected networks being pairwise stable (see example 5.13), even when some player would benefit from deleting multiple links at once. Therefore, the "reasonable" network is the one that is both Nash stable and pairwise stable. This has led to a concept of pairwise Nash stable networks. A network is pairwise Nash stable if it is Nash stable and pairwise stable.

Example 5.12 (Nash stable networks)**.** *Consider a 3 player network game given by Figure 14. All the networks except star networks are Nash stable.*

Example 5.13 (Pairwise Nash stable networks)**.** *Consider a 3 player network game given by Figure 15.*

This game has Nash stable, pairwise stable, and pairwise Nash-stable networks.

Link-Announcement Game: This is a strategic game which induces a network formation game. Consider an n-player game where each player simultaneously announces the set of players with whom he wants to form a link. Then, the strategy set of player i in

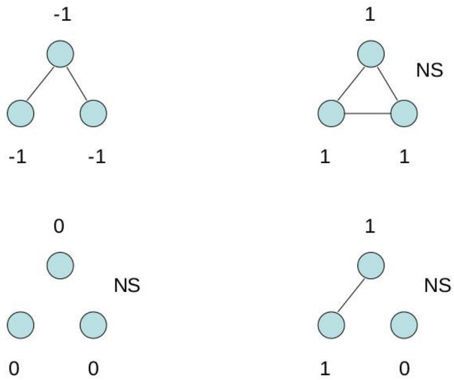

Figure 14: All Networks except Two-Link Networks are Nash Stable.

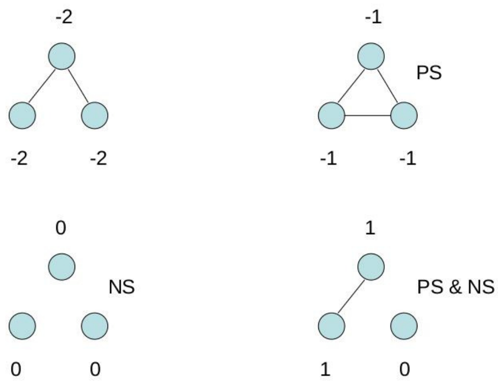

Figure 15: An Over-Connected Pairwise Stable Network.

this case is $S_i = 2^{N\setminus\{i\}}$, i.e., the set of subset of all players excluding the ith player. A link between two players is formed only if both the players simultaneously announce their interest to form a link with each other. Every strategy profile $s = (s_1, s_2, \cdots, s_n)$ induces a network $g(s)$. The payoff of player i at strategy profile s is given by $u_i(s) = u_i(g(s))$. It is possible that the different strategy profiles induces the same network. For example, consider the case of 3 players where strategy profiles $s = (s_1 = \{2,3\}, s_2 = \{1\}, s_3 = \{1,2\})$ and $s' = (s'_1 = \{2,3\}, s'_2 = \{1,3\}, s'_3 = \{1\})$ represent the star network with center node 1.

Remark 5.14. *A network g is Nash stable if it results from a pure strategy Nash equilibrium of the link-announcement game.*

Co-author example as a link announcement game: Let us consider the case of $n = 3$. In this case, the strategy sets of the players are defined as follows: $S_1 = \{\phi, \{2\}, \{3\}, \{2,3\}\}$, $S_2 = \{\phi, \{1\}, \{3\}, \{1,3\}\}$, $S_3 = \{\phi, \{1\}, \{2\}, \{1,2\}\}$. Multiple strategy profiles may correspond to the same network. For example, strategy profiles (ϕ, ϕ, ϕ), $(\{2\}, \{3\}, \{1\})$, $(\{3\}, \{1\}, \{2\})$ (possibly many more) correspond to the empty network, $\big(\{3\}, s_2 \in S_2, \{1\}\big)$, $\big(\{2,3\}, \{3\}, \{1\}\big)$, $\big(\{2,3\}, \phi, \{1,2\}\big)$ (possibly few more) correspond to one link network where the only link is between node 1 and node 3,

$\big(\{2,3\},\{1\},\{1\}\big)$, $\big(\{2,3\},\{1,3\},\{1\}\big)$, $\big(\{2,3\},\{1\},\{1,2\}\big)$ correspond to a star network with player 1 as center of the network, $\big(\{2,3\},\{1,3\},\{1,2\}\big)$ correspond to a complete network.

Nash equilibrium: The strategy profile $s^* = \big(\{2,3\},\{1,3\},\{1,2\}\big)$ is a Nash equilibrium of the game. It follows from the following arguments:

$$\begin{gathered} u_i(s^*) = 2.5,\ \forall\ i = 1,2,3, \\ u_1\big(\{2\},\{1,3\},\{1,2\}\big) = 2 \\ u_1\big(\{3\},\{1,3\},\{1,2\}\big) = 2 \\ u_1\big(\phi,\{1,3\},\{1,2\}\big) = 0. \end{gathered}$$

This implies

$$u_1(s_1^*, s_{-1}^*) \geq u_1(s_1, s_{-1}^*)\ \forall\ s_1.$$

Similarly, we can check that

$$u_2(s_2^*, s_{-2}^*) \geq u_2(s_2, s_{-2}^*)\ \forall\ s_2.$$

$$u_3(s_3^*, s_{-3}^*) \geq u_3(s_3, s_{-3}^*)\ \forall\ s_3.$$

This shows that s^* is a Nash equilibrium. The complete network is induced by s^* which makes it a Nash stable network. It is also pairwise stable network. Hence, complete network is pairwise Nash stable. In fact, all the networks in co-author example with $n = 3$ are Nash stable.

Note: While pairwise stability is natural and easy to work with, there are limitations to the concept that deserve discussion. First, pairwise stability is a weak notion in that it only considers deviations on a single link at a time. If other sorts of deviations are viable and attractive, then pairwise stability could be too weak concept. For instance, it could be that a player would not benefited from adding/severing any single link but would benefit from adding/severing several links simultaneously, and yet the network could still be pairwise stable. Second, pairwise stability considers only deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some more complicated reorganization of their links, which is not accounted for under pairwise stability. For example, the situation described in Figure 13 where it is always possible to move from empty network to complete network via the deviation by a grand coalition. To the extent that larger groups can coordinate their actions in making changes in a network, a stronger solution concept might be needed. This leads to the definition of strong stability of networks. Let $S \subset N$ be a coalition of players.

Definition 5.15. *A network $g' \in G(N)$ is obtainable from g via deviations by S as denoted by $g \rightarrow_S g'$, if*

(a) *$ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subset S$, and*

(b) *$ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \phi$.*

The above definition identifies changes in a network that can be made by a coalition S without the consent of any players outside of S. Part (a) requires that any new links only involve players in S, in line with the consent of both players being needed to add a link. Part (b) requires that at least one player of any deleted link be in S, in line with the idea that either player in a link can unilaterally sever the relationship.

Definition 5.16 (Improving deviation)**.** *A deviation by a coalition S from a network g to a network g' is said to be improving if*

(a) *$g \rightarrow_S g'$,*

(b) *$u_i(g') \geq u_i(g), \ \forall \ i \in S$ (with at least one strict inequality)*

Definition 5.17 (Strong stability)**.** *A network g is strongly stable if it is not possible for any coalition $S \subset N$ to make an improving deviation from g to some other network g'.*

Remark 5.18. *A strongly stable network is always a pairwise stable as well as Pareto efficient.*

Existence of strongly stable network: The set of strongly stable networks is a subset of the set of pairwise stable networks. This implies that a strongly stable network need not always exists, e.g., the Example 5.11 has three pairwise stable networks which are not strongly stable. Because the grand coalition can always make an improving deviation from one link networks to the complete network.

Example 5.19. *Consider a 3 nodes network game, where*

- *Utility of each player is zero in an empty network.*

- *At one link networks utility of players who are connected with an edge have utility 1 and the isolated player has utility zero.*
- *The utility of each player in the two links networks is -1.*
- *All the players at complete network have utility 1.*

In this game all the one link networks and complete network are pairwise stable. The complete network is also strongly stable and efficient. However, a strongly stable network and an efficient network need not be same. For example, in this game if we change the utility of central node in two link networks to 6 and other two node has same utility, then all two link networks are efficient but the unique strongly stable network is complete network.