# A quantified Tauberian theorem for sequences

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**Abstract**

The main result of this paper is a quantified version of Ingham's Tauberian theorem for bounded vector-valued sequences rather than functions. It gives an estimate on the rate of decay of such a sequence in terms of the behaviour of a certain boundary function, with the quality of the estimate depending on the degree of smoothness this boundary function is assumed to possess. The result is then used to give a new proof of the quantified Katznelson-Tzafriri theorem recently obtained in 21.

## 1 Introduction

One of the cornerstones in the asymptotic theory of operators is the Katznelson-Tzafriri theorem [13], theorem 1 , which states the following .

**Theorem 1** *Let $X$ be a complex Banach space and suppose that $T \in \mathcal{B}(X)$ is power bounded. Then*

$$\lim_{n\to\infty} \|T^n(I-T)\| = 0 \tag{1}$$

*if and only if* $\sigma(T) \cup \mathbb{T} \subset \{1\}$.

Here $\mathcal{B}$ deontes the algebra of bounded linear operators on a complex Banach space X, $\sigma(T)$ deontes the spectrum of the operator $T \in \mathcal{B}(X)$ , and an operator $T \in \mathcal{B}(X)$ is said to be power-bounded if $\sup_{n\geq 0} < \infty$. Moreover, $\mathbb{T}$ stands for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Limits of the type appearing in 1 play an important role for instance in the theory of iterative methods (see 16), so it is natural to ask at what *speed* convergence takes place. If $\sigma(T) \cup \mathbb{T} = \emptyset$ the decay is at least exponential, with the rate determined by the spectral radius of $T$, so the real interest is in the non-trivial case where $\sigma(T) \cap \mathbb{T} = \{1\}$.

Given a continuous non-increasing function $m : (0,\pi] \to [1,\infty)$ such that $\left\|R(\mathrm{e}^{i\theta},T)\right\| \leq m(|\theta\|)$ for $0 < |\theta| \leq \pi$ , it is shown in 21 , Theorem 2.11, that , for any $c \in (0,1)$, $\|(T^n(I-T))\| = O(m_{\log}^{-1}(cn)), n \to \infty$ where $m_{\log}^{-1}$ is the inverse function of the map $m_{\log}$ defined by

$$m_{\log}(\epsilon) = m(\epsilon)\log\left(1 + \frac{m(\epsilon)}{\epsilon}\right), 0 < \epsilon \leq \pi, \tag{2}$$

$$AX = \lambda X$$
$$v_1 \begin{bmatrix} 3 \\ 3 \\ 6 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 0$$

### Lab activity 1.2.4

Find the difference quotient of $f(x)$ when $f(x) = x^3$.

We proceed as demonstrated in the lab manual; assuming that $h \neq 0$ we have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 2xh^2 + h^3}{h} \\
&= \frac{h(3x^2 + 2xh + h^2)}{h} \\
&= 3x^2 + 2xh + h^2
\end{aligned}$$

## Lab activity 2.3.4

Use the definition of the derivative to find $f'(x)$ when $f(x) = x^{\frac{1}{4}}$.

Using the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h} \cdot \frac{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})}{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \lim_{h\to 0} \frac{1}{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \frac{1}{(x^{1/4} + x^{1/4})(x^{1/2} + x^{1/2})} \\
&= \frac{1}{(2x^{1/4})(2x^{1/2})} \\
&= \frac{1}{4x^{3/4}} \\
&= \frac{1}{4}x^{-3/4}
\end{aligned}$$

Note: the key observation here is that

$$\begin{aligned}
a^4 - b^4 &= (a^2 - b^2)(a^2 + b^2) \\
&= (a - b)(a + b)(a^2 + b^2),
\end{aligned}$$

with

$$a = (x+h)^{1/4}, \qquad b = x^{1/4},$$

which allowed us to rationalize the denominator.