Noncommutative rings

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1 Preface

1.1 Structures of algebra

- groups
- rings
- fields
- vector spaces
- Homomorphisms
- Quotient structures

1.2 References

A survey of modern algebra ; Birkhoff and MacLane
Topics in algebra ; Herstein
Modern algebra ; van der Waerden (the chapter on field theory)

2 The Jacobson

2.1 Modules

R module :
Module over a ring R
vector space over a ring R

Definition 2.1. The additive abelian group M is said to be an R-module if there is a mapping from $M \times R$ to M (sending (m,r) to mr)

1. m(a+b)=ma + mb
2. $(m_1 + m_2)a = m_1 a + m_2 a$
3. (ma)b = m(ab)

for all $m \in M$ and all $a, b \in R$.

If R should have a unit element, **1**, and if $m1 = m$ for all $m \in M$ we then describe M to be a *unitary* R-module.

Note that the definition made above merely says that the ring elements induce endomorphisms on M considered merely as an additive abelian group an that furthermore these endomorphisms induced behave as they should with respect to the addition and multiplication of such endomorphisms.

thmet X be a complex Banach space and suppose that $T \in \mathcal{B}(X)$ is power bounded. Then

$$\lim_{n\to\infty} \|T^n(I-T)\| = 0 \tag{1}$$

if and only if $\sigma(T) \cup \mathbb{T} \subset \{1\}$.

Here $\mathcal{B}$ deontes the algebra of bounded linear operators on a complex Banach space X, $\sigma(T)$ deontes the spectrum of the operator $T \in \mathcal{B}(X)$, and an operator $T \in \mathcal{B}(X)$ is said to be power-bounded if $\sup_{n\geq 0} < \infty$. Moreover, $\mathbb{T}$ stands for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Limits of the type appearing in **??** play an important role for instance in the theory of iterative methods (see 16), so it is natural to ask at what *speed* convergence takes place. If $\sigma(T) \cup \mathbb{T} = \emptyset$ the decay is at least exponential, with the rate determined by the spectral radius of T, so the real interest is in the non-trivial case where $\sigma(T) \cap \mathbb{T} = \{1\}$.

Given a continuous non-increasing function $m : (0, \pi] \to [1, \infty)$ such that $\|R(e^{i\theta}, T)\| \leq m(|\theta\|)$ for $0 < |\theta| \leq \pi$, it is shown in 21 , Theorem 2.11, that , for any $c \in (0, 1)$, $\|(T^n(I - T))\| = O(m_{\log}^{-1}(cn)), n \to \infty$ where $m_{\log}^{-1}$ is the inverse function of the map $m_{\log}$ defined by

$$m_{\log}(\epsilon) = m(\epsilon)\log\left(1 + \frac{m(\epsilon)}{\epsilon}\right), 0 < \epsilon \leq \pi, \tag{2}$$

$$AX = \lambda X$$

$$v_1 \begin{bmatrix} 3 \\ 3 \\ 6 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 0$$

Lab activity 1.2.4

Find the difference quotient of $f(x)$ when $f(x) = x^3$.

We proceed as demonstrated in the lab manual; assuming that $h \neq 0$ we have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 2xh^2 + h^3}{h} \\
&= \frac{h(3x^2 + 2xh + h^2)}{h} \\
&= 3x^2 + 2xh + h^2
\end{aligned}$$

Lab activity 2.3.4

Use the definition of the derivative to find $f'(x)$ when $f(x) = x^{\frac{1}{4}}$.

Using the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h} \cdot \frac{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})}{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \lim_{h\to 0} \frac{1}{((x+h)^{1/4} + x^{1/4})((x+h)^{1/2} + x^{1/2})} \\
&= \frac{1}{(x^{1/4} + x^{1/4})(x^{1/2} + x^{1/2})} \\
&= \frac{1}{(2x^{1/4})(2x^{1/2})} \\
&= \frac{1}{4x^{3/4}} \\
&= \frac{1}{4}x^{-3/4}
\end{aligned}$$

Note: the key observation here is that

$$\begin{aligned}
a^4 - b^4 &= (a^2 - b^2)(a^2 + b^2) \\
&= (a - b)(a + b)(a^2 + b^2),
\end{aligned}$$

with

$$a = (x+h)^{1/4}, \qquad b = x^{1/4},$$

which allowed us to rationalize the denominator.