Control Systems Engineering

Third Edition

S. K. Bhattacharya

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Control Systems Engineering

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Third Edition

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Delhi • Chennai

ISBN 9788131791653
eISBN 9789332517691

Head Office: A-8(A), Sector 62, Knowledge Boulevard, 7th Floor, NOIDA 201 309, India
Registered Office: 11 Local Shopping Centre, Panchsheel Park, New Delhi 110 017, India

To

my wife, Sumita;
son, Samrat; daughter-in-law, Anindita;
and grandson, Siddhant,
who inspired me to write.

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Preface to the Third Edition

It is indeed satisfying to note that this book on control systems engineering is becoming popular among students and teachers alike. Encouraged by this, I am happy to bring out its third edition in which I have included additional material to cover the syllabus prescribed by many Indian universities.

A complete chapter on MATLAB-based problems and solutions has been included. More solved numericals, particularly on Bode plot, Nyquist plot and root locus technique, have been incorporated. An introductory chapter on digital control has been added. The chapter on design and compensation has been enriched with additional material and insight.

The salient features of this book are highlighted below.

- This book covers the entire syllabus of linear control systems taught at the undergraduate level in all Indian universities.
- Plenty of examples have been used to clarify concepts, the explanations are simple and clear, and emphasis has been laid on developing higher order cognitive abilities.
- Adequate number of illustrations have been provided to help understand the concepts and procedures.
- Replete with MATLAB-based problems, multiple-choice questions, practice problems with answers and an exhaustive list of annexures, this edition provides abundant study material to hone the skills of the discerning reader.

I hope that students and teachers will find this edition of the book interesting and easy to read.

I would like to thank the publisher, Pearson Education, for having completed this work on time and bringing out an outstanding educational product.

S. K. BHATTACHARYA

Preface to the First Edition

Control systems engineering is an exciting and challenging field. It is a multidisciplinary subject. This textbook is meant for the basic course in control systems for the undergraduate students of electrical, electronics, instrumentation, chemical, mechanical, production, electronics and aerospace engineering. For this reason, no specific emphasis has been provided for any particular branch of engineering.

The purpose of writing this book has been to provide students with basics on control systems engineering which will help them understand the subject with ease, and also prepare them for the university examination.

The content of the book is based on the curriculum prescribed by different universities in India. The classical methods of control systems engineering have been thoroughly covered. Laplace transforms, transfer functions, root locus design, stability analysis, state variable method, frequency response methods, including Bode, Nyquist, steady-state error for standard test signals, and so on have been widely dealt with. A complete chapter on design of control systems with solved examples has been included. A chapter on control components has also been provided.

In addition to explaining the basic principles in a simplified manner, plenty of examples and exercises, design problems and MATLAB problems have been provided to help students understand the subject better. The MATLAB exercises will help students practice computer-aided design and re-work on some of the design examples.

Plenty of illustrated examples have been included to develop problem-solving skills. The review questions at the end of each chapter will help in retention of the concepts learnt.

Though an understanding of the subject requires a strong mathematical foundation, the approach followed has not been from the purely theoretical point of view, but to use mathematical fundamentals with the ultimate objective of understanding control systems design.

I would like to acknowledge that I have extensively consulted the established textbooks in this field. I would, therefore, like to convey my gratitude to all the authors who have contributed to the development of learning material as provided in the bibliography.

I am thankful to Ashis Kumar Das and Chinmoy Jana of NITTTR, Kolkata, for helping me prepare the manuscript.

I am also thankful to Pearson Education for bringing out the book on time and in a presentable manner.

I hope that students will find this book useful.

S. K. Bhattacharya

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Contents

1

INTRODUCTORY CONCEPTS

1.1 CONCEPTS OF PLANT, SYSTEM AND CONTROL SYSTEM

A system is a combination of devices and components connected together by some form of regular interactions to act together and perform a certain function. The system may be physical, biological, economic and the like.

The term control means to regulate, to direct or to command. Thus, a control system may be defined as *a combination of devices and components connected or related so as to command, direct or regulate itself or another system*.

The part of the system which is to be controlled is given different names, for example, plant, controlled system, process, etc. The control system designer develops a controller that will control the plant or the controlled system.

Control systems are used in many applications, for example, the control of temperature, liquid level, position, velocity, flow, pressure, acceleration, etc.

Let us consider a control system, e.g., the driving system of an automobile. Speed of the automobile is a function of the position of its accelerator. The desired speed can be maintained by controlling the pressure on the accelerator pedal. If the speed of the vehicle is to be increased, the driver has to increase the pressure on the pedal. Fig. 1.1(a) shows a typical driving system in a simple block diagram form.

The input to the system is force, called the command signal, on the accelerator pedal. The command signal is applied to the accelerator pedal which has various linkages and connections with the carburetter. When the fuel is burnt, the engine causes the vehicle to increase its speed. The output at variable speed is called controlled output because the signal has caused a control on the speed of the vehicle.

Thus, the controlled output of the above driving system can be depicted as in Fig. 1.1(b).

Increase or decrease of the output, that is, the speed as the output, is achieved by increase or decrease in the input, that is, the fuel input to the engine through the carburetter.

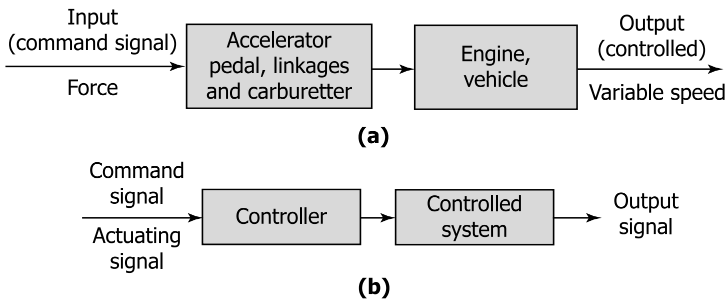


Fig. 1.1 *(a) Driving system of an automobile; (b) Control system of an automobile*

Change in the output signal requires a proportionate change in the input signal. This proportionate change is compared with reference input by a comparator, which is also called an error detector.

1.1.1 Examples of Control Systems

Some examples of control system are given below.

Control of Output Voltage of a DC Generator

Let us take an example of control of output voltage of a d.c. generator. The d.c. generator is rotated at constant speed with the help of a prime mover. Its output voltage can be controlled by changing the field current, I_f. The field current gets changed when a variable resistance connected in series with the field circuit is changed as shown in Fig. 1.2. In manual control system, the human operator will watch the output voltage using a voltmeter. He will compare the actual output voltage with the desired output voltage. Let us assume that the output voltage should remain constant at 220 V when the load on the generator changes. As the load changes, the output voltage changes. The operator has to keep a watch on the voltmeter reading. If the voltmeter reading is less than the desired value of 220 V, an error is created. Recognizing this error (negative error), the operator will have to adjust the field regulating resistance so as to increase the field current. If the field circuit resistance is decreased, field

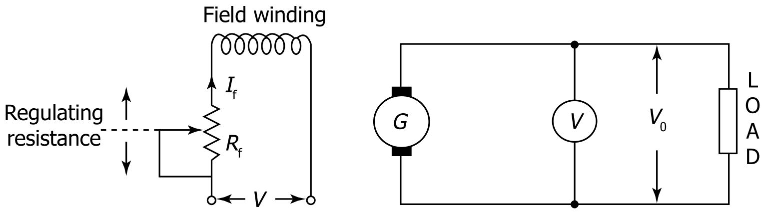


Fig. 1.2 *Control of output voltage of a d.c. generator*

current, I_f, will increase. If the field current is increased, the voltage output will increase. The field current should increase to such a value that the output equals the desired value thereby reducing the error to zero. *This is a manual control system*. This operation can be made automatic by using an automatic control system. A mechanism can be developed to compare the actual value with the desired value and the error signal is fed to a *servomotor* to control the field regulating resistance. The error voltage, also called signal, has to be sometimes amplified and fed to the servomotor. Such an automatic control system is also called *feedback control system*.

Control of Room Temperature

Let us take another example of temperature control system. The heating element of the heater can be kept ON for heating a room using a timer. An ordinary heater will remain ON for all the time irrespective of the temperature of the room. To make the heater automatic, a thermostat system has to be incorporated to measure the room temperature continuously. If the room temperature attains the desired temperature, the heater will be automatically switched off. This control system cannot be designed for fixed time basis, that is, switching on and off of the heater cannot be done for a fixed interval of time. This is because the room temperature will also depend upon the outside temperature.

1.1.2 Block Diagram Representation of Control Systems

A control system can be represented using block diagram. The simplest form of representing a system is as shown in Fig. 1.3(a). Here the input is fixed, once for all, for the desired output. When the output is controlled by measuring it and comparing with the desired output through a feedback system, the block diagram will be as shown in Fig. 1.3(b).

Now let us draw the block diagrams of a few control systems.

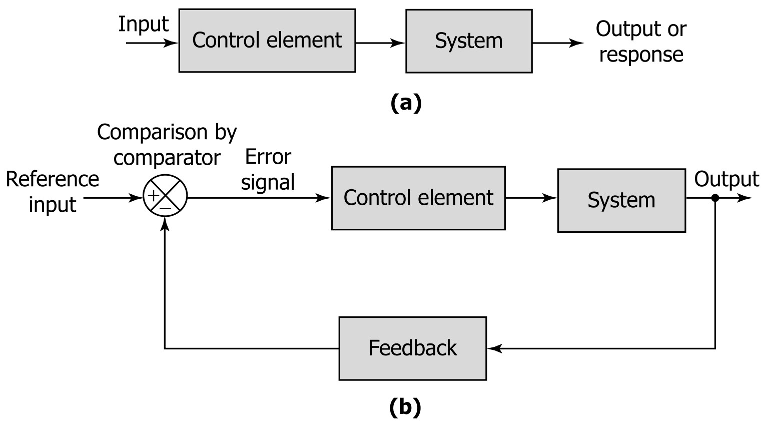


Fig. 1.3 *(a) A non-feedback control system; (b) A feedback control system*

Example 1.1 Represent a traffic control system in the form of a block diagram.

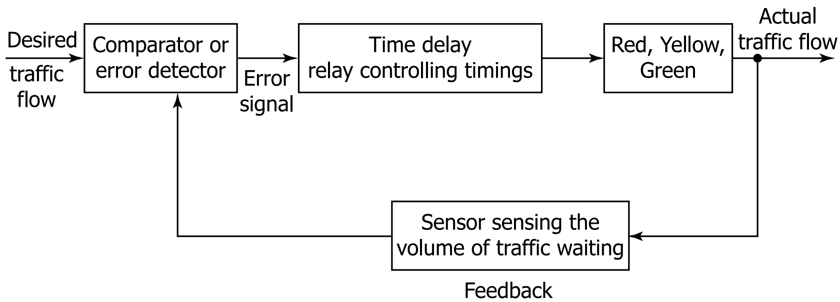


Fig. 1.4 Block diagram representation of a traffic control system

Solution

Fig. 1.4 shows the block diagram of a traffic light control system. In most of the places, the timings of the time delay relays or the electronic circuits controlling the time of operation of the traffic lights are fixed. This is irrespective of the volume of traffic on the road. Such a system would have to wait for green signal on one road even if there is no vehicle moving on the crossroad. The feedback sensor, if installed on the road crossings, will regulate the timings of the signal lamps according to volume of traffic at a particular time.

Example 1.2 Represent the manually operated air conditioning system in an electrical train.

Solution

The block diagram of a manually operated air conditioning system in an electric train has been shown in Fig. 1.5.

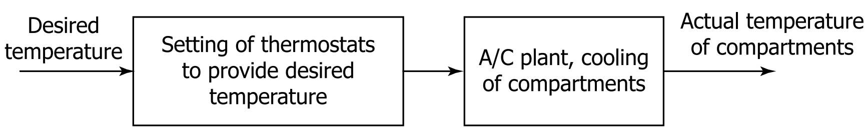


Fig. 1.5 Manually operated air conditioning system in a train

Once the thermostat setting is made for any desired temperature, say 24°C, the air conditioners will continue to work till the desired temperature is reached. During the time of setting, the actual temperature is made equal to the desired temperature. But during say night time, the outside temperature falls and hence less cooling time should be able to bring the temperature inside to the desired temperature. But here, the setting of the thermostat being unchanged, the air conditioner will over cool the compartments.

Example 1.3 Represent a room air conditioning system in the form of block diagram.

Solution

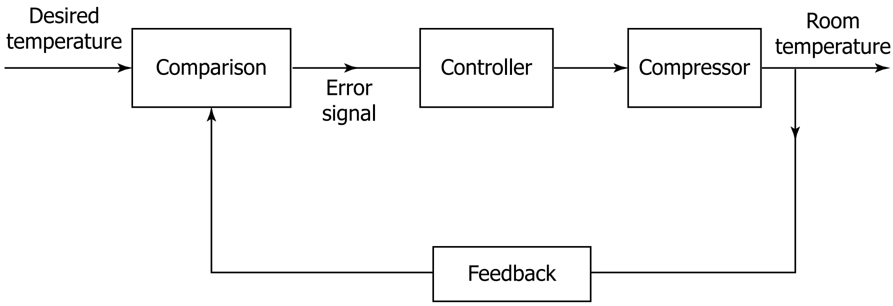


Fig. 1.6 *Room air conditioning system*

An air conditioner controls the temperature and humidity of a room to make it comfortable. An air conditioner has a compressor and a thermostat. By measuring the actual room temperature and comparing it with the desired temperature, say 24°C, an error signal is generated which either turns the compressor on, or switches it off. The simple block diagram for room air conditioning is shown in Fig. 1.6.

After examining the above examples, we can develop a generalized block diagram of a control system showing its basic components.

1.2 BASIC COMPONENTS OF A CONTROL SYSTEM

The basic components of a control system forming a control diagram are shown in Fig. 1.7.

For easy understanding we will consider room heating system using steam flowing through pipes fitted in the room. The flow of steam through the pipes is regulated automatically by a control valve. The amount of opening of the valve is regulated by a servomotor. The servomotor, controlling the opening or closing of the valve, works as a controller for amount of heating of the room. A temperature sensor will measure the room temperature and will provide a feedback signal for comparison in the comparator. A reference input is provided to the comparator for the desired temperature. In case the output temperature is different from the desired temperature, an error signal will be generated to control the hot steam flow through the valve.

The components shown in the diagram above are defined as follows:

1. **Reference Input:** This provides input signal for the desired output.
2. **Error Detector:** It is an element in which one system variable (feedback signal) is subtracted from another variable (reference signal) to obtain a third variable (error signal). It is also called *comparator*.
3. **Feedback Element:** Feedback signal is a function of the controlled output which is compared with the reference signal to obtain the error or the actuating signal. Feedback element measures the controlled output, converts or transforms to a suitable value for comparison with the reference input.

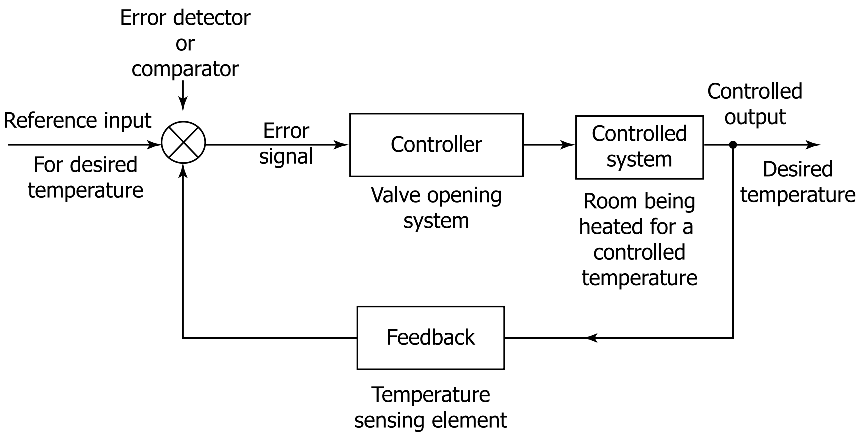


Fig. 1.7 *Representation of a control system*

4. **Error Signal:** It is an algebraic sum of the reference input and the feedback.
5. **Controller:** The controller is an element that is required to generate the appropriate control signal. The controller operates until the error between the controlled output and desired output is reduced to zero.
6. **Controlled System:** It is a body, a plant, a process or a machine of which a particular condition is to be controlled, for example, a room heating system, a spacecraft, reactor, boiler, CNC machine, etc.
7. **Controlled Output:** Controlled output is produced by the actuating signal available as input to the controller. Controlled output is made equal to the desired output with the help of the feedback system.

1.3 CLASSIFICATION OF CONTROL SYSTEMS

1.3.1 Open-loop and Closed-loop Control Systems

The systems in which output has no effect on the control action are called open-loop control systems. In other words, the output is neither measured nor fed back for comparison with reference input as shown in Fig. 1.8. For example, let us consider a washing machine in which soaking, washing and rinsing in the washer operates on the time basis.

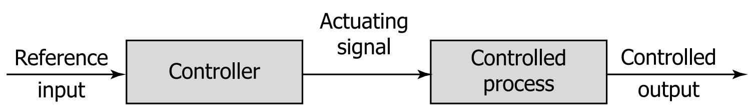
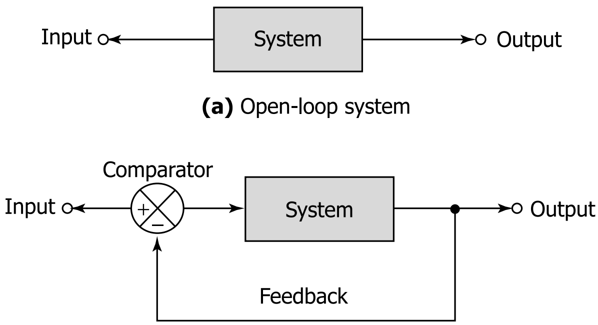


Fig. 1.8 *Elements of an open-loop control system*

The machine does not measure the output signal, that is, cleanliness of clothes. Such open-loop control systems can be used in practice, if the relationship between input and output is known and there are no external or internal disturbances. For the washing machine with open-loop system, the required time of its operation for the level of dirtiness of clothes is to be known.

Feedback control systems are also referred to as closed-loop control systems. In a closed-loop, the actuating error signal, which is the difference between the input signal and the feedback signal (output signal), is fed to the controller so as to reduce the error and bring the output of the system to the desired value. Such a system has already been shown in Fig. 1.7. It is desirable that a closed-loop system is not influenced by external disturbances and internal variations in system parameters.

The operation of a system may be controlled externally or automatically (by the system itself). When the control action of a system is independent of the output, the system is said to be an *open-loop* control system. However, if the control action is somehow dependent on the output, the system is called a *closed-loop* or *feedback* control system. If the feedback signal is equal to the controlled output, the feedback system is called unity feedback system.



(a) Open-loop system

(b) Closed-loop system

Fig. 1.9 *Open-loop and closed-loop systems*

Fig. 1.9 shows an open-loop system and a unity feedback closed-loop system. In a closed-loop system, the output is fed back and compared with the input so that any necessary corrective action is taken. The open-loop system, however, relies for its action on the settings of its components.

Some more examples of open-loop and closed-loop control systems are as follows:

a) Traffic-light systems vary in operation. Some systems are closed-loop and some are open-loop. In the open-loop type, there is a timing mechanism which is set to switch the lights at regular intervals, irrespective of the volume of traffic. In the closed-loop type, however, the amount of traffic passing through the junctions is measured (electronically or otherwise), and the duration of the "*red*" and "*green*" conditions are adjusted for each side of the roads of the junction accordingly.

b) An example of the action of an open-loop system is a car-wash machine in which all cars receive the same amount of washing irrespective of how dirty they are. Here, the output is the cleanliness of the cars, which corresponds to a given setting of the machine (amount of water and washing time). A human being, however, who washes cars, automatically makes sure that the dirtier cars receive more attention than others. Car washing by a responsible human being is, therefore, a closed-loop control system. An ordinary washing machine is an open-loop system.
c) A refrigerator is a closed-loop system. Its temperature is measured by a thermostat which turns the motor ON when the temperature rises above the desired value and turns the motor OFF when the temperature again reaches the desired value.
d) The lighting of a room is an open-loop system. Once the light is turned on, it will stay on (until it is switched off) irrespective of whether the room is dark or light.

The differences between an open-loop system and a closed-loop system are shown in Table 1.1.

Table 1.1 *Differences between Open-loop and Closed-loop Systems*

Open-loop System	Closed-loop System
a) No feedback used.	a) Feedback is there for comparison between desired output and reference input
b) Open-loop system is generally stable.	b) Closed-loop system can become unstable under certain conditions.
c) Their accuracy is determined by the calibration of their elements. Simple to develop and cheap.	c) They are more complex. Complicated to construct and costly.
d) Affected by non-linearities in the system.	d) Adjusts to the effects of non-linearities present in the system.
Examples: Washing machine, fixed time traffic control system, room heater, etc.	*Examples:* Servomotor control, generator output voltage control system, refrigerator, biological system, etc.

1.3.2 Linear and Non-linear Control Systems

A linear system is one which obeys the principle of superposition. The principle of superposition states that the response produced by simultaneous application of two different forcing functions is equal to the sum of individual responses.

Non-linear systems do not obey the principle of superposition. Almost all practical systems are non-linear to some extent. Non-linearities are introduced due to saturation effect of system components, frictional forces, play between gear trains, mechanical linkages, non-linearities of electronic components like power amplifiers, transistors, etc., used. Control of linear systems is easy as compared to the control of non-linear systems.

1.3.3 Time-invariant and Time-varying Control Systems

Time-invariant control system is one whose parameters do not vary with time. The response of such a system is independent of the time at which input is applied. For example, resistance, inductance and capacitance of an electrical network are independent of time.

A time-varying system is one in which one or more parameters vary with time. The response depends on the time at which input is applied. A space vehicle control system where mass decreases with time, as fuel it carries is consumed during flight, is an example of a time-varying system.

1.3.4 Continuous Time and Discrete Control Systems

In continuous control systems, all system parameters are function of continuous time, *t*. A discrete time control involves one or more variables that are known only at discrete instants of time.

A continuous time or continuous data control system is one in which the signals at various parts of the system are continuous functions of time. These signals are continuous time signals. For example, when we consider speed control of a d.c. motor, we know that the output, that is, the rotation in terms of radians per second is a function of voltage and current provided as input to it on a continuous time basis. Such a system takes a continuous time input and provides a continuous time output.

Sampled data control system is one in which the signals at one or more points of the system are in the form of pulses or in digital code. Sampled data control system will have discrete time input signal.

Discrete data control system is one in which a computer is used as one of the control circuit element. The input and output are in binary numbers. Digital (A to D) converters also form a part of the control system.

For example, in a generator excitation control system, the field current is adjusted continuously on the basis of feedback from the output voltage generated so as to keep the output voltage constant. Discrete signals are used in microprocessor and computer based control systems.

1.3.5 Single-Input–Single-Output (SISO) and Multi-Input–Multi-Output (MIMO) Control Systems

A system with one input and one output is called single-input–single-output control system. In other words, there is only one command and one controlled output.

A system with multiple inputs and multiple outputs is called multi-input–multi-output control system. For example, boiler drum level control, robot arm control, etc. The robot arm performs multiple functions with multiple inputs. These multiple functions are called degree of freedom.

1.3.6 Lumped Parameter and Distributed Parameter Control Systems

Control systems that can be described by ordinary differential equations are lumped parameter control systems whereas distributed parameter control systems are described by partial differential equations. The parameters of a long transmission line, that is, the resistance, inductance and capacitance, are distributed along the line but they may be considered as lumped parameters at certain points.

1.3.7 Deterministic and Stochastic Control Systems

A control system is deterministic if the response is predictable and repeatable. If not, the control system is a stochastic control system which involves random variable parameters.

1.3.8 Static and Dynamic Systems

A system is called dynamic or time dependent if its present output depends on past input, whereas, a static system is the one whose current output depends only on current input.

To use/understand a control system properly, we must learn how to:

i) Develop mathematical system descriptions and reduce them to block diagram forms: (Modelling/Mathematical Representation);
ii) Manipulate and solve the resulting system equations;
iii) Design system to satisfy general performance specifications and
iv) Evaluate results by analytical and simulation studies.

It is, thus, important that we become familiar with a variety of electrical and mechanical control systems.

1.4 SERVOMECHANISM, REGULATOR, PROCESS CONTROL AND DISTURBANCE SIGNAL

Servomechanism is an automatic control system in which the controlled variable value is forced to follow the variations of reference value, instead of regulating a variable value to "set point". For example, control of an industrial robot arm, a position control system, etc. It is also called tracking control system.

Regulator is a feedback control system in which controlled variable is maintained at a constant value inspite of external load on the plant. The reference input or command signal, although adjustable, is held constant for long period of time. The primary task is then to maintain the output at the desired value in the presence of disturbances. Examples are regulation of steam supply in steam engine by fly-ball governor, thermostat control of home heating systems, regulation of the voltage of an alternator, frequency controller and speed controller, and so on.

Process control refers to control of such parameters as level, flow, pressure, temperature and acidity of process variables. A particular parameter has only one desired value. The control system is required to ensure that the process output is maintained at the desired level inspite of external disturbances which affect the process.

Disturbance represents the undesired signals that tend to affect the controlled system. Disturbance may be due to changes made in set point, amplifier noise, variation in load, wind power disturbing outdoor installation, etc. Other disturbances that affect the performance of the control system may be changes in parameters due to wear, ageing, environmental effects, high frequency noise introduced by the measurement sensors, etc. Sometimes disturbance signals may be too fast for the control system to take care of. Low pass filters may be used to take care of high frequency disturbance signals thereby maintaining satisfactory performance of the control system.

1.5 ILLUSTRATIVE EXAMPLES OF CONTROL SYSTEMS

1. Temperature control system: The temperature of an electric furnace is measured by a thermometer which is an analog device. The analog data (temperature) is converted into digital data by A/D (Analog to Digital) converter. The digital equivalent data of temperature is fed to digital controller where it is compared with desired temperature and if there is any error, the controller sends out a signal to heater through an amplifier and relay to bring the temperature of the furnace to the desired value.

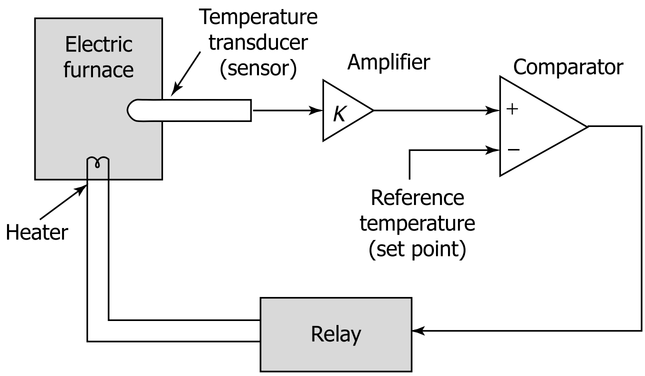


Fig. 1.10 *Temperature control system*

As shown in Fig. 1.10, the temperature of an electric furnace is measured by a temperature sensor (thermocouple) and the output of the transducer is amplified and applied to an error detector (comparator). The output of the comparator drives a relay which makes the heating system either ON or OFF. In this way, the temperature of the furnace is controlled as per desired value (set point).

2. Robot hand grasping force control system: Fig. 1.11 shows the schematic diagram of a grasping force control system using force sensing and slip sensing device.

If the grasping force is too weak, the robot hand will drop the mechanical object and if it is too strong, the hand may crush and damage the object. In the system shown in Fig. 1.11, the grasping force is preset at moderate level before the hand touches the mechanical object. The hand picks up and raises the object with preset grasping force. If there is slip in the raising motion, it will be observed by the slip sensing device and a signal will be sent back to the controller which will then increase the grasping force.

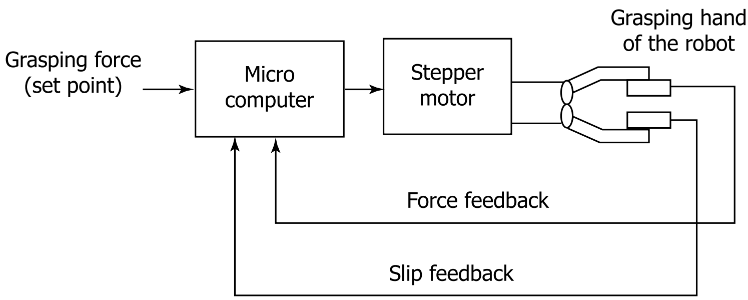


Fig. 1.11 *Robot hand control system*

In this way, a reasonable grasping force can be realized that can prevent slipping but will not damage the mechanical object.

3. Speed control system: James Watt developed and used his fly-ball generator for controlling the speed of a steam engine (see Fig. 1.12). It is recognized as the first automatic feedback controller used in an industrial process.

A centrifugal Watts' Governor uses the lift of the rotating ball as speedometer. The supply of steam is automatically controlled as speed tends to increase or decrease beyond the set point.

In Fig. 1.12, the position of the valve controls the opening of the steam supply to the engine, thus regulating the speed. For a desired speed position, the valve is fixed to a set point. An increase or decrease of speed of rotation will cause an increase or decrease in the opening of the valve, thereby causing the variation in steam supply resulting in automatic control of speed.

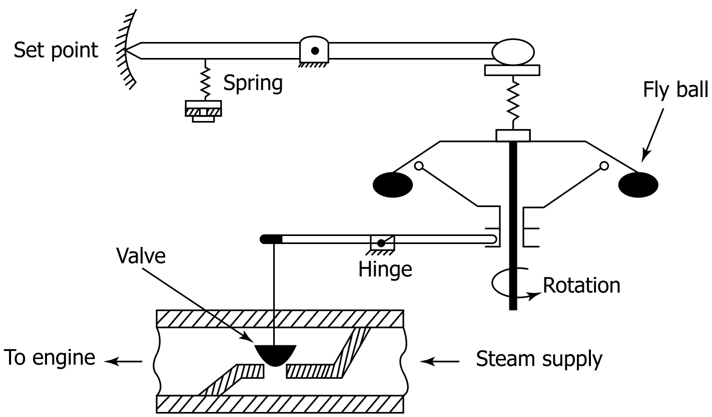


Fig. 1.12 *Control system for steam supply to the engine*

4. Pressure control system: Fig. 1.13 shows a pressure control system of a furnace. Here the pressure inside the furnace is automatically controlled by causing a change in the position of the damper, which moves up and down.

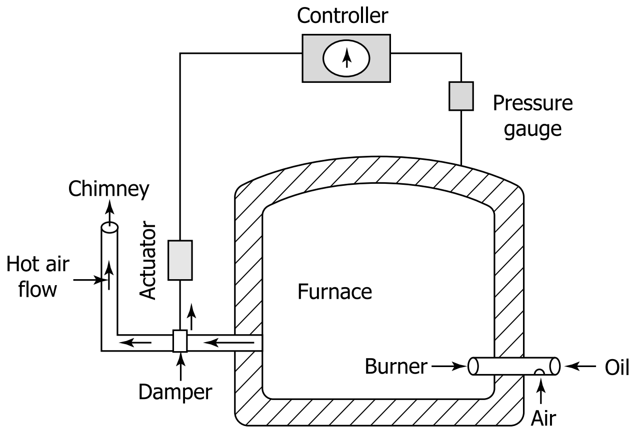


Fig. 1.13 *Pressure control system in a furnace*

The pressure inside the furnace is measured by pressure gauge. In case the pressure increases or decreases beyond a desired value, the controller and the actuator will cause a change in the position of the damper. The damper will increase or decrease the path of hot air to chimney to maintain the desired pressure.

1.6 FEEDBACK IN CONTROL SYSTEM AND EFFECT OF FEEDBACK

1.6.1 Importance of Feedback

Feedback control is now a basic feature of modern industry. In present-day technological society, in order to utilize natural resources optimally, some form of control is needed.

Control engineering is primarily concerned with controlling industrial processes and natural resources, and forces of nature purposefully and for the benefit of mankind.

Early machine and equipment used for control were primarily, manually operated type, requiring frequent adjustment so as to maintain and/or achieve the desired performance.

Advanced technology made revolutions in the procedure used for system analysis and design. However, the fundamental theory to study system performance has not changed over the years.

It is true that we design many complex systems using sophisticated computer control concepts for process control, space vehicle guidance and similar applications. But such complex system design requires routine design of control mechanism for items like temperature control, speed control, voltage regulation and so on.

Let us consider some more examples of control systems which we commonly encounter.

A forced air gas furnace is used for heating the interior of a building. Depending on the temperature level to be maintained in the building, the gas inlet to the furnace has to be turned on or off with the help of a temperature sensor placed for control purpose. A blower fan is also used to force the warm air to circulate inside the room.

When the room temperature drops below a preset reference level, measured by a temperature sensor like a thermostat, a relay is actuated to switch on the furnace fire. Subsequently, a blower fan is switched on to circulate the hot air. The blower fan runs till the heat is dissipated and then turns off automatically.

Next, consider the control problems associated with learning to drive a motor car. At the beginning, the driver must learn about the functions of various mechanisms provided for controlling the vehicle and the way the car responds under all conditions.

One would learn that frequent and abrupt application of the brake on slippery road should be avoided, since it results in the wheels sliding and loss of control of the vehicle. Control inputs are provided by the accelerator, steering wheel and the brake system.

Inputs are obtained through the driver's senses. The driver's brain, his feet and his arms are used to adjust the control mechanism by which these changes are made to influence the vehicle's motion. The fundamental controllers are the brakes, the steering mechanism and the accelerator.

The overall control system can become unstable as a result of skidding and hence total loss of control due to excessive speed on icy/muddy roads.

It can be seen that we are surrounded by control system applications. All large-scale chemical and industrial processes and automated production facilities depend heavily on feedback control technology.

1.6.2 Effects of Feedback

Use of feedback in control systems brings in significant changes in terms of improvement in overall gain, improvement in system stability, reduction of sensitivity of the system to variations in system parameters, and neutralizing or reducing the effect of disturbance signals. These will be discussed in details in chapters that will follow.

REVIEW QUESTIONS

1.1 Define a control system. Represent a control system by a block diagram.

1.2 Give examples of an open-loop control system and a closed-loop control system. State their differences.

1.3 Represent in block diagram form, the control system used in automatic control of speed of a steam turbine.

1.4 Discuss the advantages and disadvantages of an open-loop and closed-loop control systems. Give one example of each control system.

1.5 What is a feedback control system? Give two examples to show the use of control systems in our everyday life.

1.6 State, giving reasons, whether the following are open-loop or closed-loop control system:
   i) The room heater or a hot-air blower used to heat a room in winter.
   ii) The refrigerator.
   iii) The air conditioning system used in electric trains.
   iv) The traffic control system used in road crossing in typical Indian cities.
   v) Maintenance of normal body temperature of a human being.
   vi) Control of the speed of a motor vehicle by its driver depending upon road condition.
   vii) The room air conditioner.

1.7 Human being is said to be the best example of an automatic control system. Explain.

1.8 Traffic at the road crossings are controlled using signal lamps (red, yellow, green). Represent through block diagram a closed-loop traffic flow control system.

1.9 Define the following terms: System, control system, feedback, closed-loop system, open-loop system, error detector, comparator, linear and non-linear control systems, controller and servomechanism.

1.10 The following are examples of open-loop systems:
   i) The room heater.
   ii) A traffic light control system.
   iii) A domestic washing machine.
   iv) A street light system.
   v) A water pump lifting water to an overhead tank.

   How will you change each of them into closed-loop system?

1.11 Explain the function of all the basic components of a feedback control system.

1.12 Explain how an error signal is generated in a feedback control system. What is a unity feedback control system?

1.13 Distinguish between a linear control system and a non-linear control system.

1.14 A manually controlled street lighting system is an open-loop control system. How can you change it to a closed-loop control system?

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2

MODELLING A CONTROL SYSTEM—TRANSFER FUNCTION APPROACH

2.1 INTRODUCTION

We have discussed in the previous chapter that a control system consists of a number of sub-systems. All the sub-systems work in unison to achieve a desired output for a given input. Mathematical modelling of sub-systems and also of the whole system is required for carrying out performance studies. In this chapter, modelling of a control system using transfer function approach has been dealt with.

2.2 TRANSFER FUNCTION

In control theory, transfer functions are commonly used to characterize the input-output relationship of components or systems that can be described by linear, time-invariant differential equations. The relationship between input and output is represented by a diagram known as block as shown in Fig. 2.1.

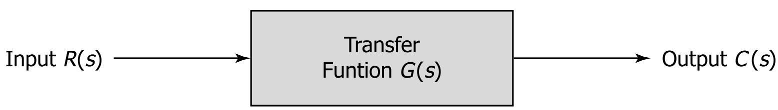

Fig. 2.1 *Block diagram representation of a system*

2.2.1 Definition of Transfer Function

The transfer function of a linear, time-invariant system is defined as the ratio of Laplace transform of the output to the Laplace transform of the input, under the assumption that all initial conditions are zero.

As shown in Fig. 2.1, $R(s)$ is the input, $C(s)$ is the output. The output changes according to the system transfer function, $G(s)$.

Therefore, $$G(s) = \frac{C(s)}{R(s)} \qquad \text{....(2.1)}$$

Now let us consider a linear, time-invariant system defined by the following differential equation:

$$a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots\dots\dots + a_{n-1} y + a_n$$

$$= (b_0 x^m + b_1 x^{m-1} + \dots\dots\dots + b_{m-1} x + b_m) \qquad \text{....(2.2)}$$

where $n \geq m$

y is the output of the system and

x is the input of the system.

Therefore, the transfer function of the system is obtained by taking Laplace transform of both the sides under the assumption that the initial conditions are zero.

$$(a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots + a_{n-1} s + a_n)\, Y(s)$$

$$= (b_0 s^m + b_1 s^{m-1} + \dots\dots\dots + b_{m-1} s + b_m)\, X(s)$$

$$\text{Transfer function } = G(s) = \left.\frac{L(\text{output})}{L(\text{input})}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} \qquad \text{....(2.3a)}$$

Using the concept of transfer function it is possible to represent system dynamics and at the same time, calculate the order of the system by knowing the highest power of s in the denominator. The order of the system is the same as the highest power of s in the denominator. Type of a system is defined as the number of open-loop poles at the origin. The order of a control system is always greater than or equal to its type.

2.2.2 Poles and Zeros of a Transfer Function

The expression for transfer function can be represented, after factorisation,

as
$$G(s)=\frac{k(s-z_1)(s-z_2)......(s-z_n)}{(s-p_1)(s-p_2)......(s-p_m)} \quad ....(2.3b)$$

where $K=\frac{a_o}{b_o}$, called the gain factor of the transfer function.

If we put s equal to $z_1, z_2, \ldots, z_n$, the transfer function becomes zero. Hence $z_1, z_2, z_3, \ldots$ are called the *zeros* of the transfer function.

On the other hand, if we put s equal to $p_1, p_2, \ldots, p_m$, the transfer function assumes infinite value. Hence $p_1, p_2, p_3, \ldots$ are called the *poles* of the transfer function. The positions of poles and zeros in the s-plane are important indicators of the performance of the system.

2.3 PROCEDURE FOR DETERMINING THE TRANSFER FUNCTION OF A CONTROL SYSTEM

Transfer function of any system can be determined through the following steps:

Step 1: Formulate the mathematical equation for the system.

Step 2: Take the Laplace transform of the system equation assuming all the initial conditions of the system as zero.

Step 3: Take the ratio of Laplace transform of the output to the Laplace transform of the input.

Let us take an example.

Let a system be described by the following differential equation:

Step 1: We write the equation representing the system

$$5\dddot{y}+3\ddot{y}+2\dot{y}+y=4\ddot{x}+2\dot{x}+x \quad ....(2.4)$$

where y is the system output and x is the system input.

Step 2: Take the Laplace transform of equation (2.4). By assuming all the initial conditions as zero, we get

$$(5s^3+3s^2+2s+1)\,Y(s)=(4s^2+2s+1)\,X(s)$$

Step 3: Take the ratio of Laplace transform of the output to the Laplace transform of the input,

$$\text{Transfer function, } G(s)=\frac{Y(s)}{X(s)}=\frac{4s^2+2s+1}{5s^3+3s^2+2s+1} \quad ....(2.5)$$

After factorization of the numerator and the denominator of the transfer function, the *poles* and *zeros* can be determined.

2.4 FORMULATION OF EQUATIONS OF PHYSICAL SYSTEMS AND THEIR TRANSFER FUNCTIONS

A physical system consists of a number of sub-systems connected together to serve a specific purpose. If we consider a motor car as a mechanical system, it has a number of sub-systems like ignition sub-system, pneumatic sub-system, power transmission sub-system, and so on. Similarly, we have electrical systems. In mechanical systems we have rotational systems, translational systems, and so on. To have an insight and better understanding about the performance of a control system it is convenient to develop mathematical models of such systems and study and modify them for giving better performance. It should be borne in mind that almost all physical systems are non-linear to some extent and as such it may be difficult to write exact mathematical equations for all systems. It may, therefore, be necessary to use the best possible linear approximation to analyse such systems.

There are a wide range of physical systems and therefore it may be difficult to use the same method of mathematical modelling and analysis in each case. In this section we will discuss briefly electrical systems and mechanical systems and their modelling using transfer function approach. We will assume that the systems are linear. The procedure for determining the transfer function will be the same as listed in the previous section.

2.4.1 Electrical Systems

A resistor, an inductor and a capacitor are the three basic elements of an electric circuit. The circuit is analysed by the application of Kirchhoff's voltage and current laws.

The relationship that exists between voltage and the current flowing through the circuit elements may be expressed as:

For a resistive circuit,

$$v = iR$$

Taking Laplace transform,

$$V(s) = RI(s)$$

For a capacitive circuit,

$$v = \frac{1}{C}\int_o^t i dt \quad \text{(Considering initial conditions to be zero)}$$

Taking Laplace transform,

$$V(s) = \frac{1}{Cs} I(s)$$

For an inductive circuit,

$$v = L\frac{di}{dt}$$

Taking Laplace transform,

$$V(s) = sLI(s)$$

When these basic elements form an electrical circuit, mathematical formulation is made by using Kirchhoff's laws.

The RLC circuit of Fig. 2.2 is analysed by Kirchhoff's voltage law applied to the closed loop.

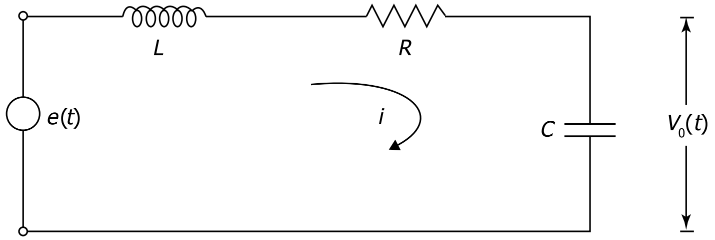

Fig. 2.2 *An RLC circuit*

The system equation is

$$e(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt \quad \text{....(2.6)}$$

Now taking Laplace transform on both sides, we get

$$E(s) = LsI(s) + RI(s) + \frac{1}{Cs}I(s)$$

(assuming all initial conditions to be zero)

$$E(s) = \left[Ls + R + \frac{1}{Cs}\right]I(s)$$

$$E(s) = \left[\frac{LCs^2 + RCs + 1}{Cs}\right]I(s)$$

Let the output voltage $v_o(t)$ be taken across the capacitor, C. Then,

$$v_o(t) = \frac{1}{C}\int idt \quad \text{....(2.7)}$$

Taking Laplace transform on both sides of equation (2.7), we get

$$V_0(s) = \frac{1}{Cs}I(s)$$

(assuming all initial conditions to be zero)

Therefore, the transfer function is given by

$$G(s) = \frac{V_o(s)}{E(s)}$$

$$= \frac{\frac{1}{Cs}}{(LCs^2 + RCs + 1)} . Cs \frac{I(s)}{I(s)}$$

$$\text{Transfer function, } G(s) = \frac{V_o(s)}{E(s)} = \frac{1}{(LCs^2 + RCs + 1)}$$

The block diagram representation shown in Fig. 2.3.

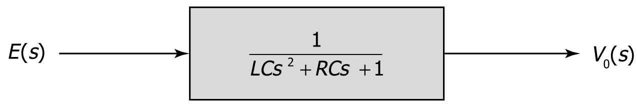

Fig. 2.3 *Block diagram of the RLC network*

This is a second order system since the highest power of s in the denominator is 2.

2.4.2 Mechanical Systems

In analysis of mechanical systems use is made of three idealized elements, namely, inertial elements such as mass or moment of inertia, spring and a damper. The inertial elements, i.e. mass and spring are capable of storing energy while the damper is capable of dissipating energy. A damper is often referred to as mechanical resistance. A mechanical system may have either purely translational motion or purely rotational motion.

The equations of motion are generally formulated using Newton's laws of motion. In pure translational systems, motion is considered to take place in straight lines. Therefore, the variables—force, displacement, velocity and acceleration—are aligned in a straight line. If a mass m moves in a straight line with x as its position at any time with reference to a fixed reference axis, then by applying Newton's laws of motion, the force F acting upon it may be expressed as

$$\text{Force = Mass} \times \text{Acceleration; i.e. } F = m\frac{d^2x}{dt^2} = m\ddot{x}$$

Again, if a mass m is moving with a velocity v, it will have a stored translational energy of $\frac{1}{2}mv^2$. This provides us with an idea that mass is an energy storing device.

A spring, when acted upon by a force, gets stressed. For a linear spring, the deformation produced is directly proportional to the magnitude of the applied force. The equation describing the relationship of F and x for a linear spring is $F = Kx$, where F is the force exerted on the spring, x is the deformation of the spring, and K is the stiffness coefficient of the spring. A spring is usually represented by a coil. A spring is compressed or elongated by an amount x due to a force F.

An ideal spring will return the same amount of energy (in the form of work) when allowed to come back to its original state.

Thus, we have seen that both mass and spring can store energy. Mass stores translational kinetic energy which is due to its velocity while the spring stores translation potential energy which is due to its position (elongated or compressed).

A damper is generally represented by a piston in a cylinder. A damper produces resistance velocity which is often termed as friction. In a fluid medium this friction is due to the viscosity. When the force due to friction is proportional to relative velocity, the friction is known as linear friction. The relationship of forces in a linear damper is given by

$$f(t) = B\frac{dx}{dt} = B\dot{x}$$

where $f(t)$ is the damping (resisting) force due to relative velocity $\dot{x}$ between two movable parts and B is the coefficient of viscous friction.

Rotation about a fixed axis takes place in a purely rotational system. The elements are moment of inertia, that is, rotational mass, torsional spring and damping. Moment of inertia is expressed as J. The governing equation relating angular velocity ω with applied torque T, according to Newton's second law of motion, is

$$\begin{aligned} T &= J\omega \\ &= J\frac{d^2\theta}{dt^2} = J\ddot{\theta} \end{aligned}$$

Mechanical elements which experience deformation due to applied torque may be considered rotational spring like a long shaft, helical spring, and so on. The relationship of torque T and angular displacement θ is expressed as

$$T = K\theta$$

Rotational damping is the resistance to the rate of relative angular velocity between parts of a rotating mechanical system. The relationship with torque, T is

$$T = B\frac{d\theta}{dt} = B\dot{\theta}$$

Mechanical systems and devices can be modelled by three ideal translatory elements or through three ideal rotational elements as shown in the following section.

2.4.2.1 Translational Mechanical System

The basic elements of any translatory mechanical system are the mass, the spring and the damper as shown in the Fig. 2.4.

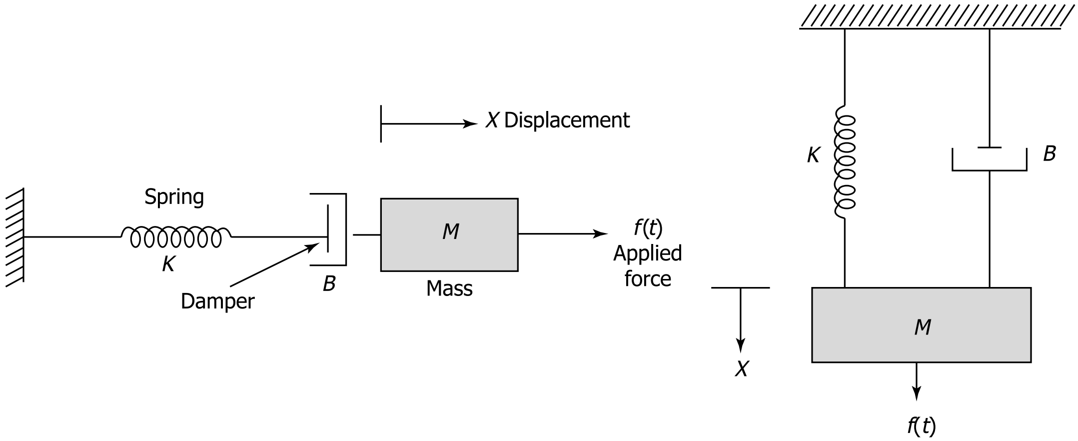

Fig. 2.4 *Basic elements (mass, spring, damper) of a mechanical system*

The mass of the system is M (the unit of M is kg). The displacement of mass due to applied force $f(t)$ (the unit of force is Newton) results in inertia force. This inertia force is the product of the mass and its acceleration. The spring deflection constant is K Newton/metre. The restoring force f_K for the spring is proportional to its displacement. The viscous damping in the system is offered by a dashpot damper. The damping force varies in direct proportion to the velocity. The coefficient of viscous damping is B Newton/rad/sec.

Application of force $f(t)$ to the mass results in a displacement of x metre. The equation of motion for the system is obtained by applying Newton's Second law of Motion.

Inertia force = mass × acceleration = $M\dfrac{d^2x}{dt^2}$

Applied force = Inertia force + Damping force + Restoring force by the spring.

Therefore,
$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad \text{....(2.8)}$$

Rearranging equation (2.8), the following equation is obtained

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t) \qquad \text{....(2.9)}$$

Taking Laplace transform on both sides and assuming all initial conditions to be zero,

$$Ms^2X(s) + Bs\,X(s) + KX(s) = F(s)$$

$$[Ms^2 + Bs + K]\,X(s) = F(s)$$

$$\text{Transfer function, } G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \qquad \text{....(2.10)}$$

This can be represented in block diagram form as shown in Fig. 2.5.

Fig. 2.5 *Block diagram representation of the mechanical system shown in Fig. 2.4*

Fig. 2.6 shows the equivalent circuit of the mechanical system of Fig. 2.4.

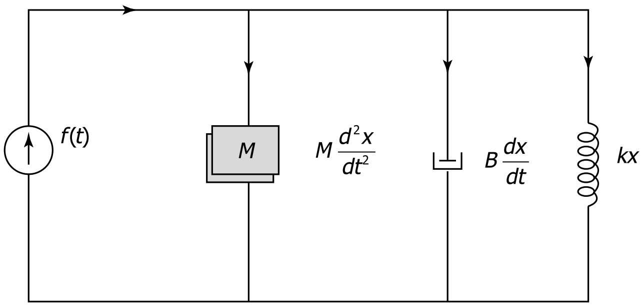

Fig. 2.6 *Equivalent circuit of the mechanical system*

How this circuit is analogous to the mechanical system will be understood from the Force-current analogy, which will be described a little later.

2.4.2.2 Rotational Mechanical System

Mechanical systems involving rotation around a fixed axis are often seen in machineries such as turbines, pumps, rotating discs, gears, generators, motors, and so on. Fig. 2.7 shows a rotational mechanical system that consists of rotating disc of moment of inertia J and a shaft of stiffness K. The disc rotates in a viscous medium with a viscous friction coefficient B.

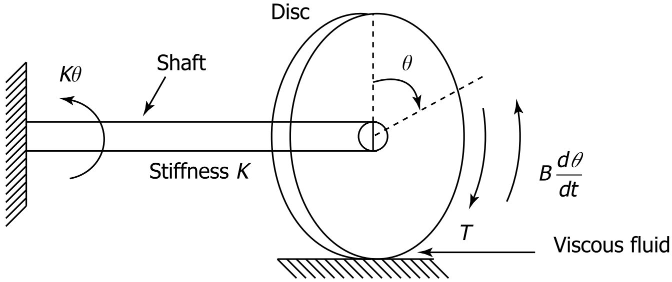

Fig. 2.7 *Rotational mechanical system*

Let T be the applied torque which tends to rotate the disc. The three basic components of the rotational system are moment of inertia, viscous friction, and spring stiffness (torsional).

The system equation can be written using the relation,

Applied Torque = Inertia torque + Damping torque + Angular displacement (torsional) torque ....(2.11)

Inertia Torque = Moment of Inertia × Angular acceleration $= J\dfrac{d^2\theta}{dt^2}$

Thus,
$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta \qquad \text{....(2.12)}$$

The equivalent circuit diagram of the mechanical system is shown in Fig. 2.8.

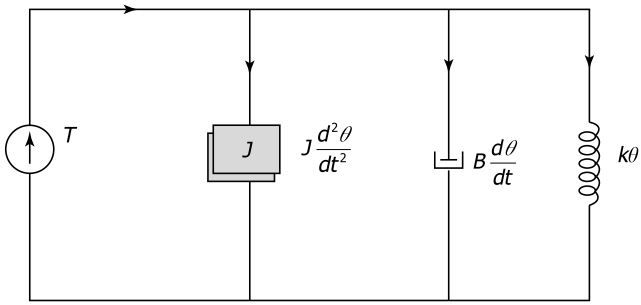

Fig. 2.8 *Mechanical equivalent circuit of Fig. 2.7*

Taking Laplace transform of equation (2.12) and assuming all initial conditions as zero,

$$T(s) = Js^2\theta(s) + Bs\theta(s) + K\theta(s)$$

$$T(s) = (Js^2 + Bs + K)\theta(s)$$

$$\text{Transfer function, } G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K} \qquad \text{....(2.13)}$$

This is represented in the form of a block diagram as shown in Fig. 2.9.

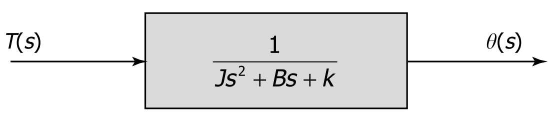

Fig. 2.9 *Block diagram of a rotational mechanical system*

2.4.3 Analogies of Mechanical and Electrical Systems

Sometimes mechanical and other systems are converted into electrical analogous systems for the ease of design, modification and analysis. Analogous systems have the same type of differential equations. We will discuss two types of analogies, namely, force-voltage analogy and force-current analogy.

2.4.3.1 Force–Voltage Analogy

Equation (2.9) which provides us with a relation representing a mechanical system is reproduced below as

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t) \qquad \text{Equation (2.9) reproduced.}$$

The analogy of this equation can be established by the voltage equation of a RLC electrical circuit. The voltage equation for the circuit as established, in equation (2.6) is reproduced as follows:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = e(t)$$

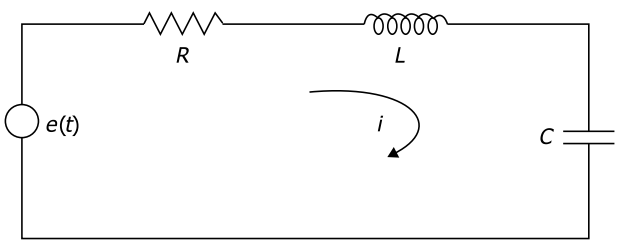

Fig. 2.10 Electrical circuit for force–voltage analogy

As the current is the rate of flow of electric charge, $i = \frac{dq}{dt}$. Thus, the equation becomes

$$L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad \ldots(2.14)$$

It is observed that equation (2.9) is analogous to equation (2.14).

The three equations, namely,

i) the equation for the mechanical translational system represented by mass, spring and damper;
ii) the mechanical rotational system represented by moment of inertia, viscous friction and stiffness; and
iii) the electrical system of inductance, resistance, and capacitance; are rewritten and placed together as

$$M\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Kx=f(t) \qquad \text{....(i)}$$

$$J\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}+K\theta=T \qquad \text{....(ii)}$$

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e(t) \qquad \text{....(iii)}$$

From the above three equations, the analogies between the mechanical translation, mechanical rotational, and electrical system are established and shown in Table 2.1.

In mechanical system, $f(t)$ is the force applied whereas in electrical system, $e(t)$ is the voltage applied. This is the reason for which the analogy is called **force-voltage analogy**. In rotational system, force is replaced by torque.

Table 2.1 *Analogous Quantities of Mechanical and Electrical Systems (Force–Voltage Analogy)*

Mechanical Translational Systems	**Mechanical Rotational Systems**	**Electrical Systems**
Force, f	Torque, T	Voltage, e
Mass, M	Moment of inertia, J	Inductance, L
Viscous friction coefficient, B	Viscous friction coefficient, B	Resistance, R
Spring Stiffness, K	Torsional spring stiffness, K	Reciprocal of capacitance, $1/C$
Displacement, x	Angular displacement, θ	Charge, q
Velocity, $\dot{x}$	Angular velocity, $\dot{\theta}$	Current, i

2.4.3.2 Force–Current Analogy

Applying Kirchhoff's current law in the L–R–C parallel circuit shown in Fig. 2.11, we can write

$$i_L+i_C+i_R=i(t) \qquad \text{....(2.15)}$$

we know that the emf induced in an inductor is

$$e=L\frac{di}{dt};\ \text{Current through the inductor is, } i_L=\frac{1}{L}\int e\,dt$$

and the charge on a capacitor is $\qquad q=\int i_C\,dt$

and $\qquad q=Ce$

$\therefore \qquad \int i_C\,dt=Ce$

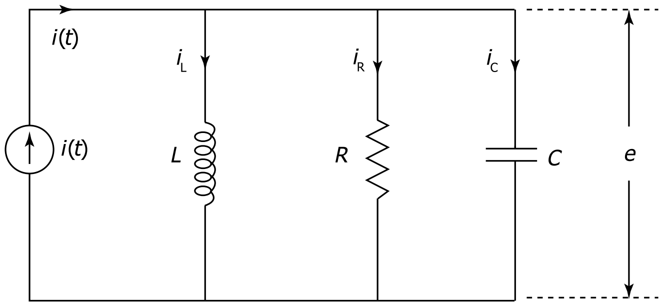

Fig. 2.11 *Electrical circuit for force–current analogy*

Current through the capacitor i_C is, therefore,

$$i_C = C\frac{de}{dt}$$

Also,
$$i_R R = e; i_R = \frac{1}{R}e$$

$$i_L = \frac{1}{L}\int e\,dt; i_R = \frac{1}{R}e; i_C = C\frac{de}{dt}$$

Substituting the values of i_L, i_R and i_C in equation (2.15), we get

$$\frac{1}{L}\int e\,dt + C\frac{de}{dt} + \frac{1}{R}e = i(t) \qquad ....(2.16)$$

In terms of flux linkage, ψ the above equation can be represented by putting,

$$e = \frac{d\psi}{dt} \text{ where } \psi = N\varphi, \text{ i.e. the flux linkage} \qquad ....(2.17)$$

Equation (2.16) becomes,

$$\frac{1}{L}\psi + \frac{1}{R}\frac{d\psi}{dt} + C\frac{d^2\psi}{dt^2} = i(t) \qquad \left[\because \int e\,dt = \psi\right]$$

Rearranging, we get

$$C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi = i(t) \qquad ....(2.18)$$

This equation can be compared with the equation relating mechanical translational system of equation (2.9) and equivalent rotational system as in Table 2.2.

Table 2.2 *Analogous Quantities of Mechanical and Electrical Systems (Force–Current Analogy)*

Mechanical Translational Systems	Mechanical Rotational Systems	Electrical Systems
Force, f	Torque, T	Current, i
Mass, M	Moment of inertia, J	Capacitance, C
Viscous friction coefficient, B	Viscous friction coefficient, B	Reciprocal of resistance, $1/R$
Spring stiffness, K	Torsional spring stiffness, K	Reciprocal of inductance, $1/L$
Displacement, x	Angular displacement, θ	Flux linkage, ψ
Velocity, $\dot{x}$	Angular velocity, $\dot{\theta}$	Voltage, e

On the basis of force-current analogy, we can derive the equations for the mechanical system by developing an equivalent circuit diagram, as has been done in the case of force-voltage analogy earlier. We will now consider a few systems and determine their transfer functions.

2.4.4 Hydraulic System

Let us determine the transfer function of a typical hydraulic system. A hydraulic system works due to liquid pressure difference.

A dashpot (or damper) is shown in Fig. 2.12. Whenever a step displacement, x is applied to the piston, a corresponding displacement y becomes momentarily equal to x. The applied force will make oil to flow through the restriction R and the cylinder will return to its original position.

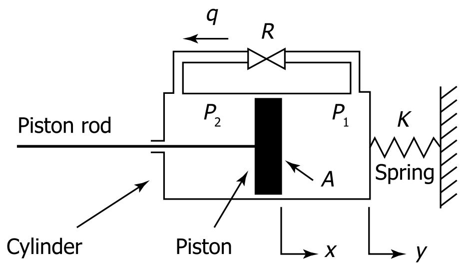

Fig. 2.12 *Damper of a hydraulic system*

Assuming inertia force to be a negligible, the forces balancing are represented as

$$A(p_1 - p_2) = ky \qquad \text{....(2.19)}$$

where A is the piston area, k is the spring constant and p_1, p_2 are the oil pressure existing on the right and left side of the piston.

The flow rate q through the restriction with resistance R is given by

$$q = \frac{p_1 - p_2}{R} \qquad \text{....(2.20)}$$

The oil is assumed to be incompressible (so oil density ρ = constant). As the mass of oil flow through the restriction in time dt must balance with the change in mass of the left side of the piston, we have

$$q\,dt = A\rho\,(dx - dy)$$

Rearranging and using equations (2.19) and (2.20), we obtain

$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{q}{A\rho} = \frac{p_1 - p_2}{RA\rho} = \frac{ky}{RA^2\rho}$$

or,

$$\frac{dx}{dt} = \frac{dy}{dt} + \frac{ky}{RA^2\rho}$$

Taking Laplace transform with zero initial conditions, we get

$$sX(s) = sY(s) + \frac{k}{RA^2\rho}Y(s)$$

So, the transfer function of the system becomes

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s}{s + \dfrac{k}{RA^2\rho}} = \frac{s}{s + \dfrac{1}{\tau}} = \frac{\tau s}{1 + \tau s}$$

where $\tau = \dfrac{RA^2\rho}{k}$, τ is called the time constant.

2.4.5 Pneumatic System

A pneumatic system works due to the pressure difference of air or any other gas. We shall consider a simple pneumatic system with an actuating valve as shown in Fig. 2.13.

As shown in Fig. 2.13, air at a pressure, p_i is injected through the input manifold. The plunger arrangement has a mass M, B and K being the coefficients of viscous friction and spring constant respectively. If A is the area of diaphragm, then the force exerted on the system will be Ap_i.

Force F = Pressure × Area

that is, $F = p_i A = Ap_i$

Again, $F = M\dfrac{d^2y}{dt^2} + B\dfrac{dy}{dt} + ky$

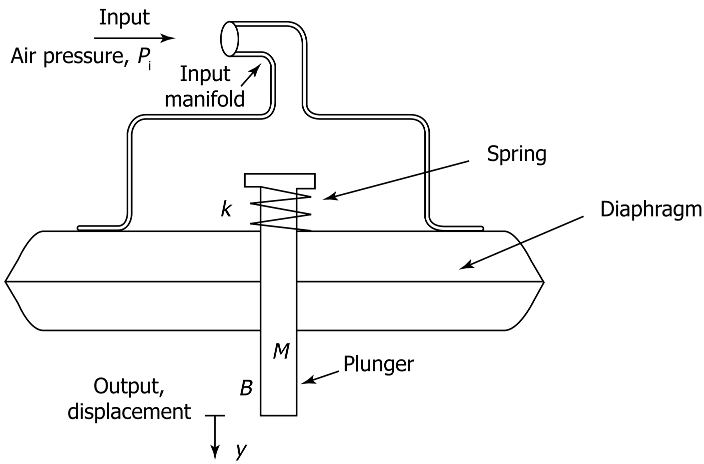

Fig. 2.13 *Schematic diagram of a simple pneumatic system*

$$\therefore \qquad M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = AP_i \qquad \text{....(2.21)}$$

Taking Laplace transform of both sides and assuming initial conditions to be zero, we get

$$Ms^2Y(s) + BsY(s) + KY(s) = AP_i(s)$$

The transfer function is

$$\text{Transfer function, } G(s) = \frac{Y(s)}{P_i(s)} = \frac{A}{Ms^2 + Bs + K} \qquad \text{....(2.22)}$$

2.4.6 Thermal System

Fig. 2.14 shows a water heating system (a thermal system) where input water temperature is θ_i and output water temperature is θ_0.

The following assumptions are made to analyse the thermal system and determine its transfer function.

i) The temperature of the medium is uniform
ii) The tank is insulated from the surrounding atmosphere.

Let the steady state temperature of inflowing water is θ_i and that of outflowing water is θ_0. The steady state heat input rate from the heater is Q. Let the water flow rate be constant. Any increase in heat input rate, ΔH will be balanced by increase in heat outflow rate, ΔH_1 and heat storage rate in the tank, ΔH_2.

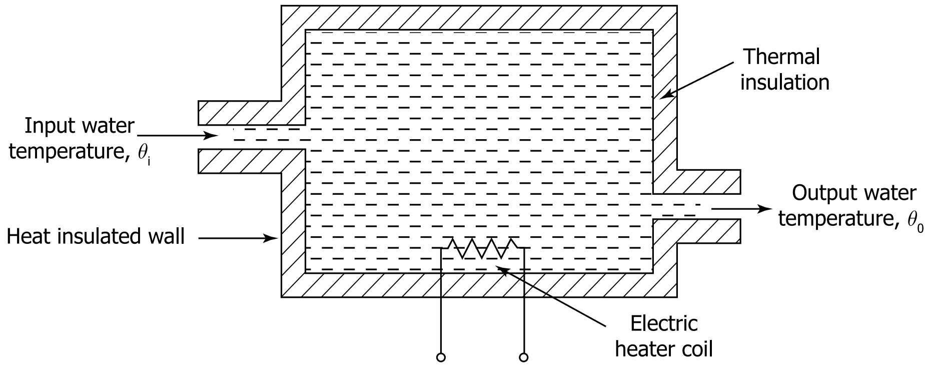

Fig. 2.14 *A simple water heating system*

Using heat balance equation, we can write

$$\Delta H = \Delta H_1 + \Delta H_2 \qquad \text{....(2.23)}$$

$$\Delta H_1 = QS\,\Delta\theta_0$$

where Q is water flow rate and S is the specific heat

$$\Delta H_1 = \frac{\Delta\theta_0}{R} \qquad \text{....(2.24)}$$

where $R = \dfrac{1}{QS}$, is called the thermal resistance

Heat storage rate in the tank,

$$\begin{aligned}\Delta H_2 &= MS\frac{d}{dt}(\Delta\theta_0)\\ &= C\frac{d}{dt}(\Delta\theta_0)\end{aligned} \qquad \text{....(2.25)}$$

where M is the mass of water in the tank; $C = MS$, is the thermal capacitance of water and $\dfrac{d(\Delta\theta_0)}{dt}$ is the rate of rise of temperature of water in the tank.

Now from equations (2.23), (2.24) and (2.25) we can write,

$$\Delta H = \frac{\Delta\theta_0}{R} + C\frac{d}{dt}(\Delta\theta_0)$$

In Laplace transform form, the quantities can be expressed as,

$$H(s) = \frac{1}{R}\theta_0(s) + Cs\,\theta_0(s)$$

$$H(s) = \left[\frac{RCs+1}{R}\right]\theta_0(s)$$

$$\text{Transfer function, } G(s) = \frac{\theta_0(s)}{H(s)} = \frac{R}{RCs+1}$$

Example 2.1 Write the equation relating the various forces acting on a mechanical system shown in Fig. 2.15(a). Show that the electrical circuit of Fig. 2.15(b) is its analogous system.

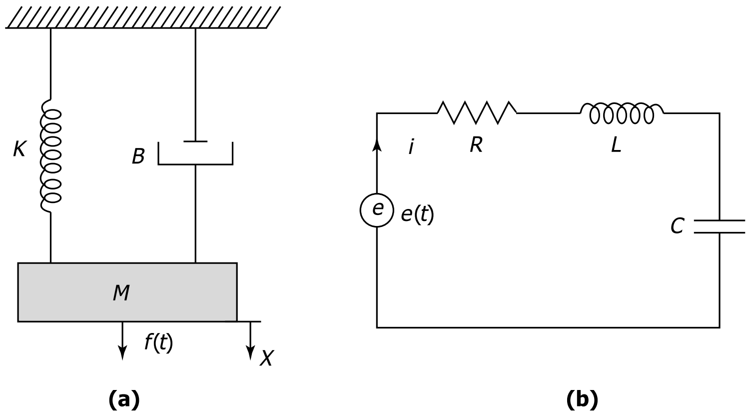

Fig. 2.15 *Mechanical and electrical analogous systems*

Solution

The forces acting on the system of Fig. 2.15(a) are external applied force $f(t)$ and the internal resisting forces, which are inertia force, $M\frac{d^2x}{dt^2}$; damping force, $B\frac{dx}{dt}$; and spring force, kx. Equating the applied force with the resisting forces, we get

$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad \text{....(2.26)}$$

The voltage equation for the R–L–C circuit shown in Fig. 2.15(b) is written as

$$e(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt \qquad \text{....(2.27)}$$

since $i = \frac{dq}{dt}$ and $q = \int idt$, the above equation is represented as

$$e(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \qquad \text{....(2.28)}$$

By observing equations (2.26) and (2.28) we conclude that the two systems shown in Fig. 2.15 are analogous (similar) systems.

Example 2.2 Show that circuit of Fig. 2.16(b) is the analogous electrical network for the system shown in Fig. 2.16(a) using force–voltage analogy.

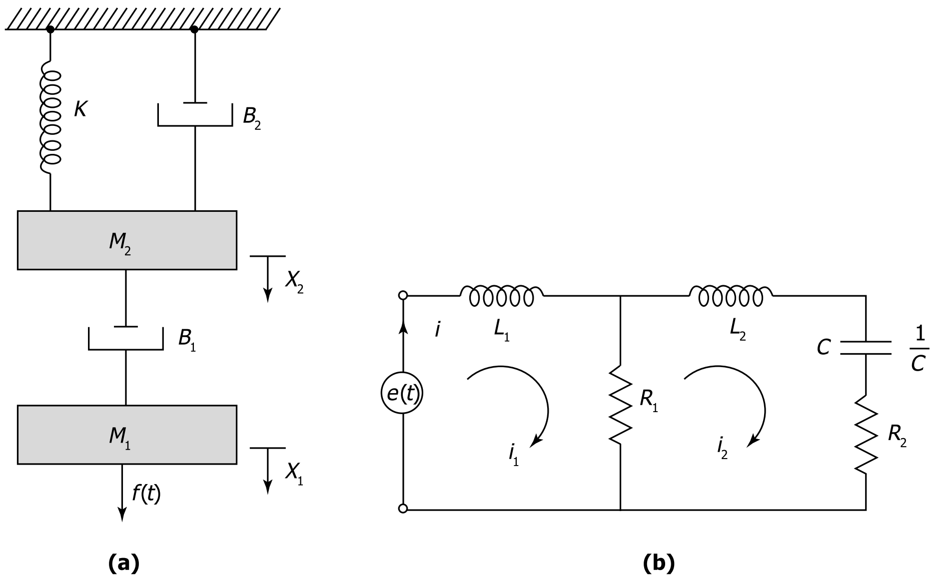

Fig. 2.16 *(a) A mass-spring damper system; (b) Analogous equivalent circuit*

Solution

The forces acting on mass M_1 and M_2 are represented using free body diagrams as shown in Fig. 2.17. The balancing equations are then written from Fig. 2.17(a) and (b) as

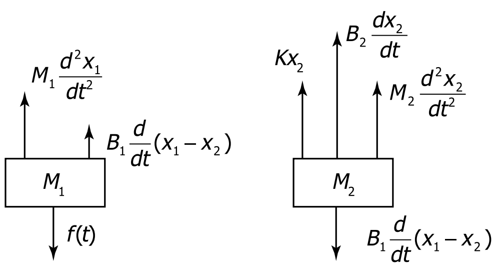

Fig. 2.17 *Free body diagrams of the system of Fig. 2.16(a)*

The balancing equations are:

$$f(t) = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{d}{dt}(x_1 - x_2) \qquad \text{....(2.29)}$$

$$B_1 \frac{d}{dt}(x_1 - x_2) = M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + kx_2 \qquad \text{....(2.30)}$$

We will apply force-voltage analogy for which we shall refer to Table 2.1.
Replacing $f(t)$ by $e(t)$, M_1 by L_1, M_2 by L_2, x_1 by q_1, x_2 by q_2, B_1 by R_1, B_2 by R_2 and K by $1/C$, these equations are written as

$$e(t) = L_1 \frac{d^2 q_1}{dt^2} + R_1 \frac{d}{dt}(q_1 - q_2)$$

and
$$R_1 \frac{d}{dt}(q_1 - q_2) = L_2 \frac{d^2 q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{1}{C} q_2$$

since, $q = \int i dt$ or, $\frac{dq}{dt} = i$, the equation can also be written as

$$e(t) = L_1 \frac{di_1}{dt} + R_1(i_1 - i_2)$$

and
$$R_1(i_1 - i_2) = L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C} \int i_2 dt$$

These equations are the voltage equations of the two loops for the analogous electrical circuit shown in Fig. 2.16(b).

Example 2.3 Obtain the transfer function of the mechanical system shown in Fig. 2.18 and draw its analogous circuit.

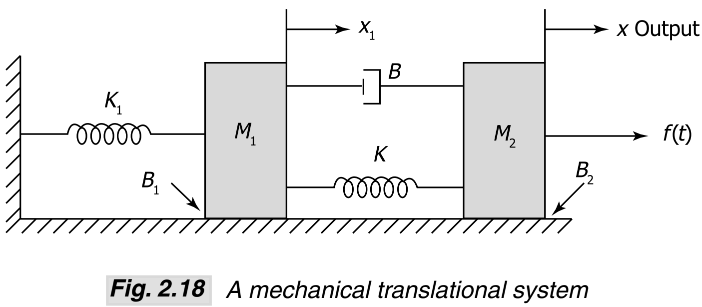

Fig. 2.18 *A mechanical translational system*

Solution

Fig. 2.19 shows the free body diagram of the various forces acting on mass M_1 and M_2.

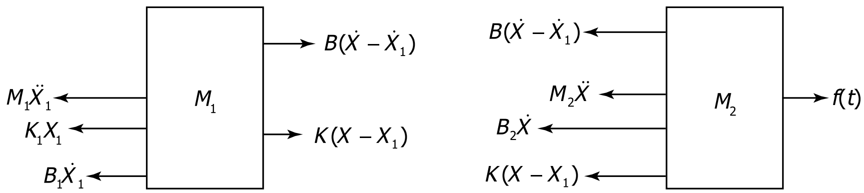

Fig. 2.19 *Free body diagrams of the system shown in Fig. 2.18*

The equations of the balancing forces on M_2 and M_1 respectively are

$$f(t) = M_2\ddot{x} + B_2\dot{x} + B(\dot{x} - \dot{x}_1) + K(x - x_1)$$

and
$$M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 = B(\dot{x} - \dot{x}_1) + K(x - x_1)$$

Using Laplace transform, these equations are written as

$$(M_2s^2 + B_2s + Bs + K)\,X(s) - (Bs + K)\,X_1(s) = F(s) \qquad \text{....(2.31)}$$

and
$$(M_1s^2 + B_1s + Bs + K + K_1)\,X_1(s) = (Bs + K)\,X(s) \qquad \text{....(2.32)}$$

Substituting the value of $X_1(s)$ from equation (2.32) in equation (2.31), we get

$$\left(M_2s^2 + B_2s + Bs + K\right)X\left(s\right) - \frac{\left(Bs+K\right)\left(Bs+K\right)X\left(s\right)}{\left(M_1s^2 + B_1s + Bs + K + K_1\right)} = F(s)$$

or
$$\{(M_2s^2 + B_2s + Bs + K)(M_1s^2 + B_1s + Bs + K + K_1) - (Bs + K)^2\}X(s)$$
$$= (M_1s^2 + B_1s + Bs + K + K_1)F(s)$$

Transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{M_1s^2 + B_1s + Bs + K + K_1}{\left(M_2s^2 + B_2s + Bs + K\right)\left(M_1s^2 + B_1s + Bs + K + K_1\right) - \left(Bs + K\right)^2}$$

Converting the nodal equations (2.31) and (2.32), into comparable electrical analogous equations we get

$$L_2\frac{di}{dt} + R_2i + R(i - i_1) + \frac{1}{C}\int(i - i_1)dt = e(t) \qquad \text{....(2.33)}$$

$$L_1\frac{di_1}{dt} + R_1i_1 + \frac{1}{C_1}\int i_1dt = R(i - i_1) + \frac{1}{C}\int(i - i_1)dt \qquad \text{....(2.34)}$$

Using equations (2.33) and (2.34) the electrical analogous circuit based on force–voltage analogy is drawn as shown in Fig. 2.20. Equation (2.33) is for loop ABCDA and equation (2.34) is for loop AEFDA.

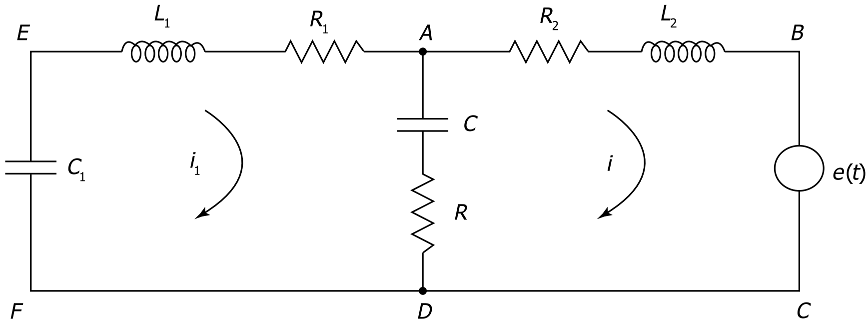

Fig. 2.20 *Electrical equivalent of the mechanical system shown in Fig. 2.18*

Here, $L_2 = M_2, R_2 = B_2, C = K, R_1 = B_1, L_1 = M_1, C_1 = K_1,$

$$i = x, i_1 = x_1, e(t) = f(t), R = B$$

Example 2.4 Obtain the transfer function of mechanical system shown in Fig. 2.21.

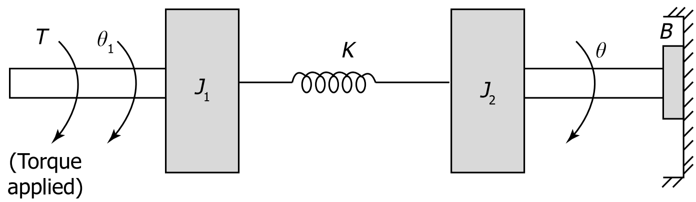

Fig. 2.21 *A mechanical rotational system*

Solution

The equivalent circuit diagram for the above system is shown in Fig. 2.22.

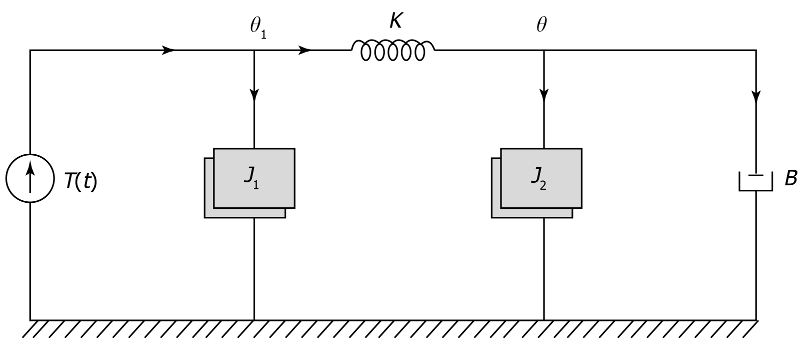

Fig. 2.22 *The mechanical system of Fig. 2.21 has been represented in the form of a network diagram*

The nodal equations at node θ_1 and θ are

At node θ_1, $$J_1\ddot{\theta}_1 + K(\theta_1 - \theta) = T(t)$$

At node θ, $$J_2\ddot{\theta} + B\dot{\theta} = K(\theta_1 - \theta)$$

or $$(J_1 s^2 + K)\theta_1(s) - K\theta(s) = T(s) \quad \text{....(2.35)}$$

and $$(J_2 s^2 + Bs + K)\theta(s) = K\theta_1(s) \quad \text{....(2.36)}$$

Substituting value of $\theta_1(s)$ from equation (2.36) in equation (2.35), we get

$$\left[\frac{\left(J_1 s^2 + K\right)\left(J_2 s^2 + Bs + K\right)}{K} - K\right]\theta(s) = T(s)$$

$\therefore$ Transfer function $$G(s) = \frac{\theta(s)}{T(s)}$$

$$= \frac{K}{\left(J_1 s^2 + K\right)\left(J_2 s^2 + Bs + K\right) - K^2}$$

$$= \frac{K}{J_1 J_2 s^4 + J_1 B s^3 + (KJ_1 + KJ_2)s^2 + KBs}$$

Example 2.5 Draw the network diagram for the system in Fig. 2.23 and also draw its analogous circuit.

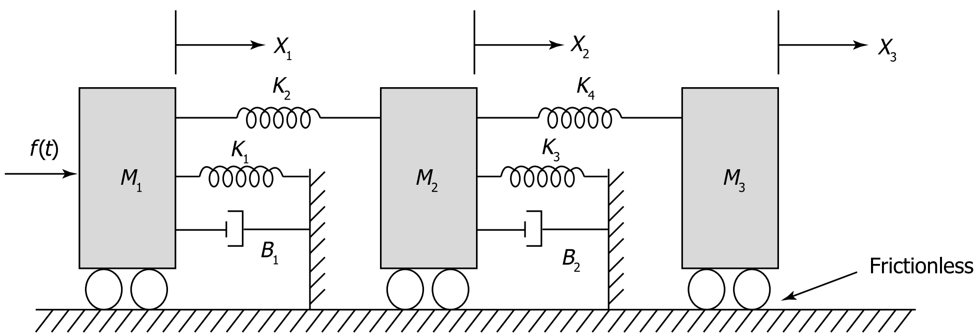

Fig. 2.23 *A mechanical translational system*

Solution

The network diagram for the system in Fig. 2.23 is shown in Fig. 2.24. This has been drawn by considering the forces acting on mass M_1, M_2, and M_3 respectively.

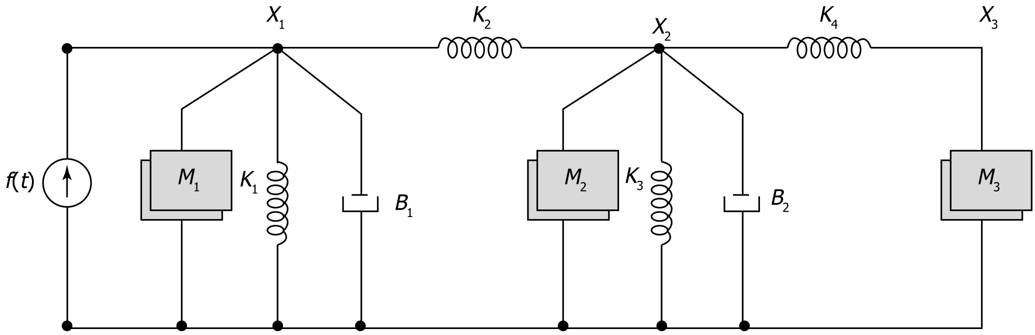

Fig. 2.24 *Circuit diagram for the system of example 2.5*

At Node x_1, $$f(t) = M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 + K_2(x_1 - x_2) \qquad \text{....(i)}$$

At Node x_2, $$K_2x_1 - K_2x_2 = M_2\ddot{x}_2 + B_2\dot{x}_2 + K_3x_2 + K_4x_2 - K_4x_3$$

or, $$M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_2 + K_3 + K_4)x_2 - K_3x_1 - K_4x_3 = 0 \qquad \text{....(ii)}$$

At Node x_3, $$M_3\ddot{x}_3 = K_4x_2 - K_4x_3$$

or, $$M_3\ddot{x}_3 - K_4x_2 + K_4x_3 = 0 \qquad \text{....(iii)}$$

Using force–current analogy (replacing force by current, mass by capacitance, velocity $\dot{x}$ by voltage v, damping coefficient B by $\frac{1}{R}$, stiffness k by $\frac{1}{L}$) the equations (i), (ii) and (iii) are expressed as (iv), (v) and (vi), respectively:

$$i(t) = C_1\frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{1}{L_1}\int v_1 dt + \frac{1}{L_3}\int v_2 dt - \frac{1}{L_3}\int v_2 dt \qquad \text{....(iv)}$$

and, $$C_2\frac{dv_2}{dt} + \frac{v_2}{R_2} + \left[\frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4}\right]\int v_2 dt - \frac{1}{L_4}\int v_3 dt - \frac{1}{L_2}\int v_1 dt = 0 \qquad \text{....(v)}$$

and, $$C_3\frac{dv_3}{dt} + \frac{1}{L_4}\int v_3 dt - \frac{1}{L_4}\int v_2 dt = 0 \qquad \text{....(vi)}$$

$$\left[\because \frac{dx}{dt} = v, x = \int v dt\right]$$

Using the above three equations the analogous electrical circuit is drawn as shown in Fig. 2.25.

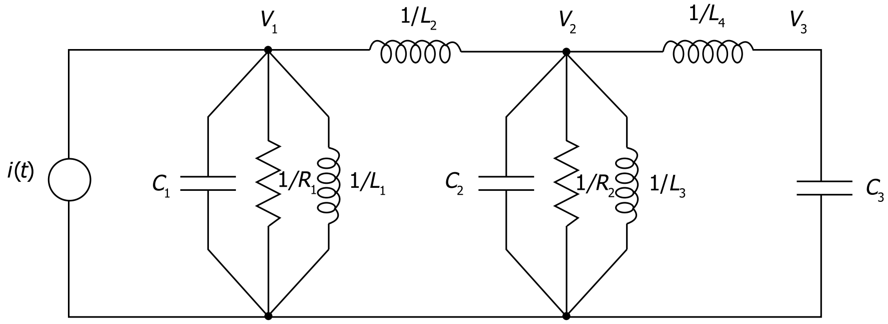

Fig. 2.25 *Electrical equivalent circuit of the mechanical system of Fig. 2.23 has been drawn using force–current analogy*

Example 2.6 Write the nodal equations for the system shown in Fig. 2.26 and draw its analogous electrical network.

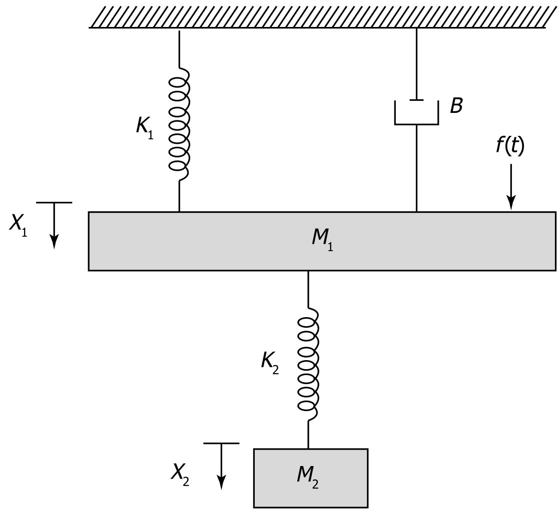

Fig. 2.26 *A mechanical system represented by mass-spring damper*

Solution

The mechanical network diagram for the given system has been drawn considering the forces acting on M_1 and M_2 respectively and is shown in Fig. 2.27.

The nodal equations are:

At Node x_1, $$f(t) = M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 + K_2(x_1 - x_2)$$

In Laplace transform form, $F(s) = (s^2M_1 + sB + K_1 + K_2)X_1(s) - K_2X_2(s)$ ….(i)

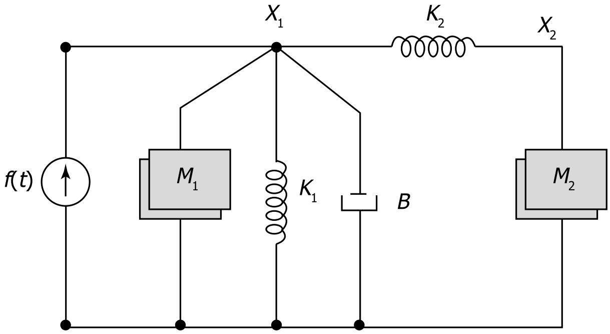

Fig. 2.27

At Node x_2, $$K_2(x_1 - x_2) = M_2\ddot{x}_2$$

Taking Laplace transform, $(s^2 M_2 + K_2)X_2(s) - K_2 X_1(s) = 0$ ....(ii)

Electrical analogous circuit based on force–voltage analogy is drawn in Fig. 2.28.

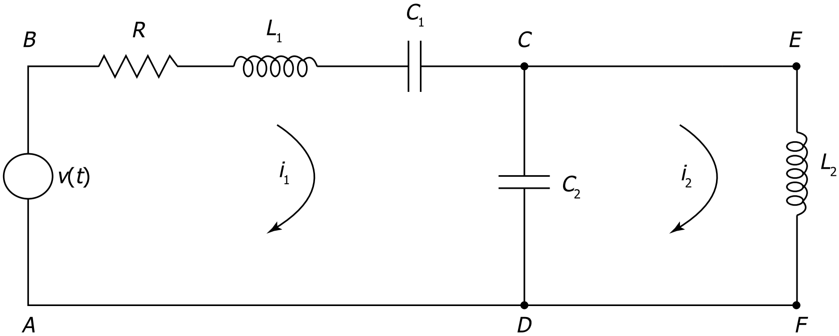

Fig. 2.28 Electrical equivalent circuit of the mechanical system represented as in Fig. 2.26

The differential equations for the electrical analog are

For loop ABCDA, $v(t) = Ri_1 + L_1\dfrac{di_1}{dt} + \dfrac{1}{C_1}\displaystyle\int i_1 dt + \dfrac{1}{C_2}\int i_1 dt - \dfrac{1}{C_2}\int i_2 dt$

or $$v(t) = R\frac{dq_1}{dt} + L_1\frac{d^2 q_1}{dt^2} + \frac{q_1}{C_1} + \frac{q_1}{C_2} - \frac{q_2}{C_2}$$

Taking Laplace transform

$$V(s) = \left(sR + s^2 L_1 + \frac{1}{C_1} + \frac{1}{C_2}\right)Q_1(s) - \frac{1}{C_2}.Q_2(s) \qquad \text{....(iii)}$$

or,
$$V(s)=\left(s^2L_1+sR+\frac{1}{C_1}+\frac{1}{C_2}\right)Q_1(s)-\frac{Q_2(s)}{C_2}$$

and for loop EFDCE,
$$L_2\frac{di_2}{dt}+\frac{1}{C_2}\int i_2dt-\frac{1}{C_2}\int i_1dt=0$$

or,
$$L_2\frac{d^2q_2}{dt^2}+\frac{q_2}{C_2}-\frac{q_1}{C_2}=0$$

$$\left[s^2L_2+\frac{1}{C_2}\right]Q_2(s)-\frac{Q_1(s)}{C_2}=0 \qquad \text{....(iv)}$$

Equations (iii) and (iv) are similar to the nodal equations (i) and (ii) formed earlier.

Example 2.7 Draw the network diagram for the system shown in Fig. 2.29 and write the equations of performance.

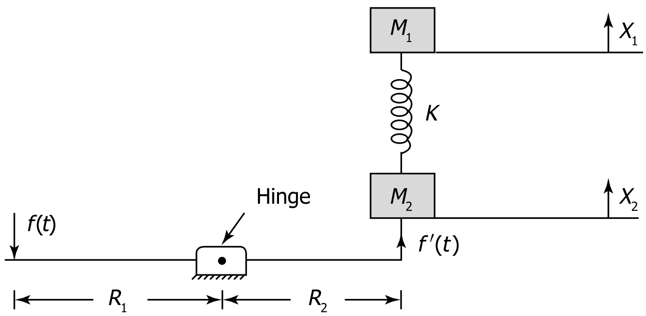

Fig. 2.29 *A mechanical system*

Solution

Equating (force × distance) on the both sides of the hinge,
$$f(t)\times R_1=f'(t)\times R_2 \qquad \text{....(2.37)}$$

or
$$f'(t)=f(t)\times\frac{R_1}{R_2}$$

Equating forces acting on mass M_2,
$$f'(t)=M_2\ddot{x}_2+K(x_2-x_1) \qquad \text{....(2.38)}$$

Equating forces acting on mass M_1,
$$K(x_2-x_1)=M_1\ddot{x}_1 \qquad \text{....(2.39)}$$

Using the above two equations, the network diagram is drawn as shown in Fig. 2.30.

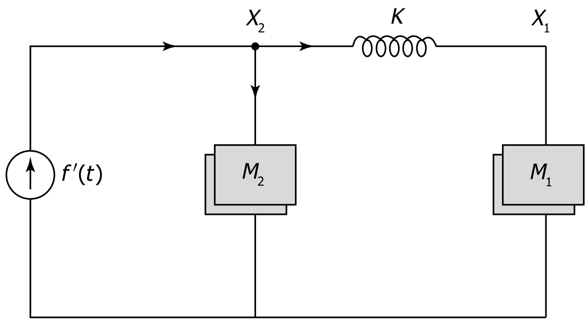

Fig. 2.30 *Network diagram for the mechanical system of Fig 2.29*

Example 2.8 A hydraulic actuator with incompressible oil inside the cylinder is shown in Fig. 2.31. The mass of piston and leakage are assumed negligible. Obtain the transfer function relating displacement x of the valve piston and displacement y of the main piston.

Solution

Fig. 2.31 shows partial diagram of a hydraulic system where linear motion of a small piston valve (not shown in the figure) in a cylinder can cause movement of large piston in the main cylinder.

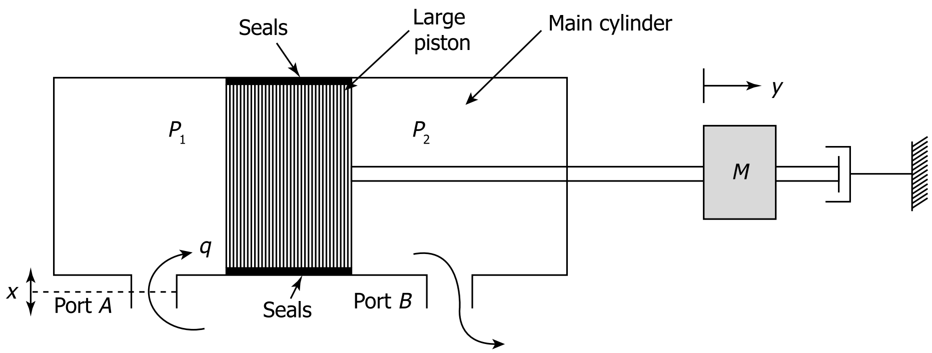

Fig. 2.31 *A simplified hydraulic pump*

On application of a small force piston in the main cylinder moves from left to right by a distance x. The two ports A and B of the main cylinder opens. Oil at high pressure (actuated by a pressure pump) enters through port A and develops a pressure $p_{1.}$ Oil at the other side of the piston will have comparatively lower pressure. The low pressure oil will return to pump through the outlet valve. Let us call the pressure difference on the two sides of the main piston as P_D.

The quantity of oil flow is q which is a function of displacement x and pressure difference, P_D.

Thus we can write,

Quantity of oil flow, $q = f(x, P_D)$ ....(2.40)

By partial differentiation, we get

$$\partial q = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial P_D} dP_D \qquad \text{....(2.41)}$$

with zero initial conditions for all the variables, on integration we get

$$q = \left(\frac{\partial q}{\partial x}\right) x + \frac{\partial q}{\partial P_D} P_D$$

or,
$$q = K_1 x - K_2 P_D \qquad \text{....(2.42)}$$

If the displacement of the main piston is dy in time dt (see Fig. 2.31) and assuming oil to be incompressible and neglecting leakage, the rate of flow of oil into the main cylinder with piston area A is given by

Quantity of oil flow,
$$q = A\frac{dy}{dt} \qquad \text{....(2.43)}$$

From equations (2.42) and (2.43), $A\dfrac{dy}{dt} = K_1 x - K_2 P_D$

or
$$P_D = \frac{1}{K_2}\left[K_1 x - A\frac{dy}{dt}\right] \qquad \text{....(2.44)}$$

Since pressure is force per unit area, the force (F) developed by the piston is AP_D.

$\therefore$
$$F = \frac{A}{K_2}\left[K_1 x - A\frac{dy}{dt}\right] \qquad \text{....(2.45)}$$

The equation relating force F applied to a mass M and having coefficient of viscous friction B at the load (see Fig. 2.31), is written as

$\therefore$
$$F = M\frac{d^2 y}{dt^2} + B\frac{dy}{dt} \qquad \text{....(2.46)}$$

Equating (2.45) and (2.46) we get,

$$M\frac{d^2 y}{dt^2} + B\frac{dy}{dt} = \frac{A}{K_2}\left[K_1 x - A\frac{dy}{dt}\right]$$

Taking Laplace transform of both sides and assuming initial conditions as zero, we obtain

$$Ms^2 Y(s) + BsY(s) = \frac{A}{K_2}[K_1 X(s) - AsY(s)]$$

or, $$Y(s)\left[Ms^2 + Bs + \frac{A^2 s}{K_2}\right] = \frac{AK_1}{K_2} X(s)$$

Transfer function, $$G(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{AK_1}{K_2}}{Ms^2 + Bs + \dfrac{A^2 s}{K_2}}$$

Multiplying numerator and denominator by $\dfrac{K_2}{BK_2 + A^2}$

$$G(s) = \frac{A\dfrac{K_1}{K_2} \times \dfrac{K_2}{BK_2 + A^2}}{s\left(\dfrac{BK_2 + A^2}{K_2} + Ms\right)\dfrac{K_2}{BK_2 + A^2}} = \frac{\dfrac{AK_1}{BK_2 + A^2}}{s\left(1 + s\dfrac{MK_2}{BK_2 + A^2}\right)}$$

$$= \frac{k}{s(1 + s\tau)}$$

Transfer function, $$G(s) = \frac{K}{s[1 + s\tau]} \qquad \text{....(2.47)}$$

where $$K = \frac{AK_1}{BK_2 + A^2} \text{ and } \tau = \frac{MK_2}{BK_2 + A^2}$$

Example 2.9 Derive the transfer function of mercury in a simple glass thermometer and draw its electrical analogous circuit.

Solution

The cross-section of mercury in a simple glass thermometer is shown in the Fig. 2.32. The thermometer is assumed to have thermal capacitance C which stores heat and thermal resistance $\Re$ which limits that flow.

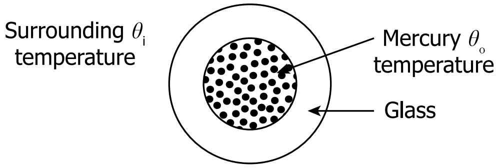

Fig. 2.32 *Cross-sectional view of a mercury glass thermometer*

The thermometer responds to temperature changes because of the heat absorbed or lost by the mercury. Heat input from the surrounding is $hA(\theta_i - \theta_0)$.

Heat stored $= C\dfrac{d\theta_0}{dt}$

Balancing heat absorbed with heat stored,

$$hA(\theta_i - \theta_0) = C\frac{d\theta_0}{dt}$$

or,
$$\tau\frac{d\theta_0}{dt} = \theta_i - \theta_0 \quad \text{....(2.48)}$$

$$\text{where } \tau = \frac{C}{hA}$$

τ is the time constant of the thermometer, C is the thermal capacitance, A is the surface area and h is the film coefficient of solid-liquid interface.

It may be noted that the heat capacity of glass has been neglected as it is insignificant as compared to that of mercury. The time constant can be written as $\tau = RC$, where $R = \dfrac{1}{hA}$ is the thermal resistance for convection.

Taking Laplace transform of equation (2.48) with zero initial conditions, we get

$$\text{Transfer function, } G(s) = \frac{\theta_0(s)}{\theta_i(s)}$$
$$= \frac{1}{\tau s + 1}$$

Referring to equation (2.48) an electrical analog for the thermometer system is given by the circuit shown in Fig. 2.33. The voltage equations are

$$e_i = Ri + \frac{1}{C}\int i dt \text{ and } e_0 = \frac{1}{C}\int i dt$$

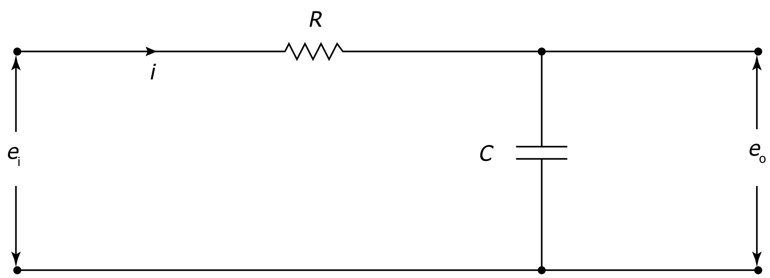

Fig. 2.33 *Electrical analogous circuit of a thermometer*

Taking Laplace transform, $\quad E_0(s) = RI(s) + \dfrac{1}{Cs}I(s)$

$$E_0(s) = \frac{1}{Cs} I(s)$$

$$G(s) = \frac{E_0(s)}{E_i(s)} = \frac{1}{1+RCs} = \frac{1}{1+\tau s}$$

The electrical circuit (Fig. 2.33) of the above equation is the electrical analogous circuit of thermometer.

Example 2.10 Find the transfer function of the following network

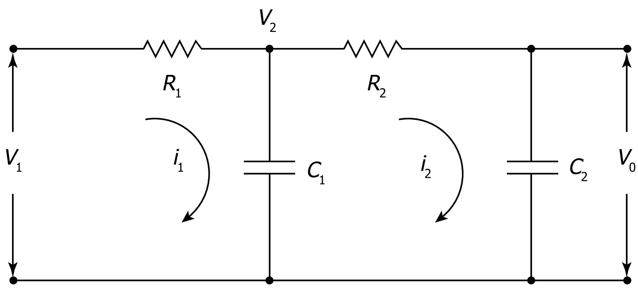

Fig. 2.34

Solution

Applying Kirchoff's voltage law (*KVL*) to loop 1

$$V_i = iR_1 + \frac{1}{C_1}\int (i_1 - i_2)dt$$

Taking Laplace transform,

$$V_i(s) = R_1 I_1(s) + \frac{I_1(s) - I_2(s)}{C_1 s} \quad \text{....(1)}$$

Applying *KVL* to loop 2

$$\frac{1}{C_1}\int (i_2 - i_1)dt + i_2 R_2 + \frac{1}{C_2}\int i_2 dt = 0$$

Taking Laplace transform,

$$\frac{I_2(s) - I_1(s)}{C_1 s} + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \quad \text{....(2)}$$

Also,

$$V_o = \frac{1}{C_2}\int i_2 dt$$

$$V_o(s) = \frac{I_2 s}{C_2 s} \quad \text{....(3)}$$

From Equation (2)

$$I_2(s)\left[\frac{1}{C_1 s}+R_2+\frac{1}{C_2 s}\right]=\frac{I_1(s)}{C_1 s}$$

Substituting this in Equation (1)

$$V_i(s)=R_1C_1sI_2(s)\left[\frac{1}{C_1 s}+R_2+\frac{1}{C_2 s}\right]-\frac{I_2(s)}{C_1 s}+I_2(s)\left[\frac{1}{C_1 s}+R_2+\frac{1}{C_2 s}\right]$$

$$\begin{aligned}V_i(s)&=\left[\frac{R_1C_1s}{C_1s}+R_2R_1C_1s+\frac{R_1C_1s}{C_2s}-\frac{1}{C_1s}+\frac{1}{C_1s}+R_2+\frac{1}{C_2s}\right]I_2(s)\\&=\left[R_1+R_1R_2C_1s+\frac{R_1C_1}{C_2}+R_2+\frac{1}{C_2s}\right]I_2(s)\end{aligned}\qquad \text{....(4)}$$

$\dfrac{V_o(s)}{V_i(s)}=$ Transfer function.

$$=\frac{I_2(s)/C_2s}{\left[R_1+R_1R_2C_1s+\dfrac{R_1C_1}{C_2}+R_2+\dfrac{1}{C_2s}\right]I_2(s)}$$

$$=\frac{1}{[C_2sR_1+R_1R_2C_1C_2s^2+R_1C_1s+R_2C_2s+1]}$$

REVIEW QUESTIONS

2.1 Draw the corresponding network diagram for the simple mass-spring-damper mechanical system shown in Fig. 2.35 and write its system equations.

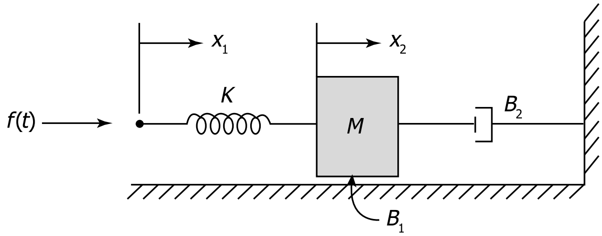

Fig. 2.35

2.2 Draw the corresponding network diagram for the damped mechanical system shown in Fig. 2.36 and write its system equations.

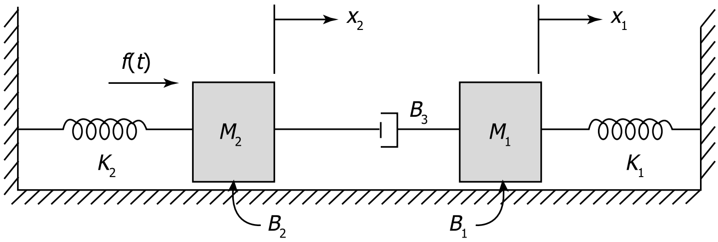

Fig. 2.36

2.3 Write the force or torque equations for the mechanical systems shown in Fig. 2.37. Draw their equivalent mechanical circuit.

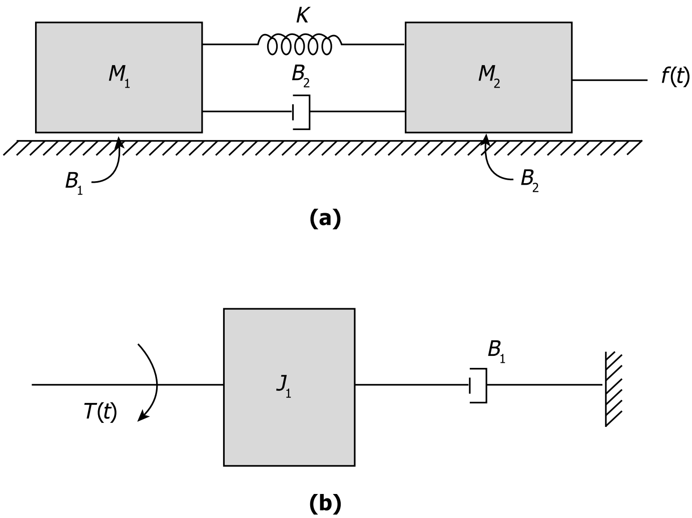

Fig. 2.37

2.4 Determine a mathematical model for the Operational Amplifier circuits shown in Fig. 2.38.

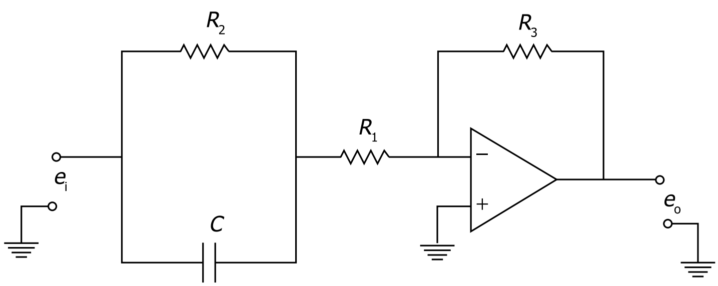

Fig. 2.38

2.5 Obtain the transfer function of the mechanical system shown in Fig. 2.39 and draw its analogous circuit.

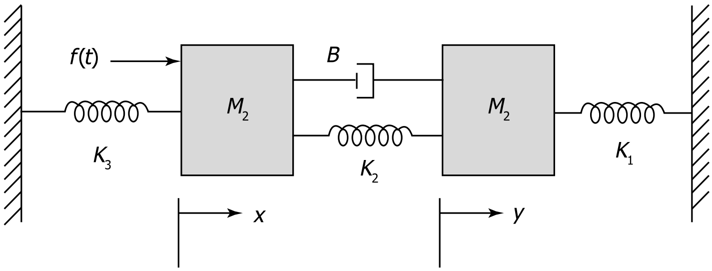

Fig. 2.39

2.6 A pneumatic bellow shown in Fig 2.40 made from corrugated copper, when employed with a variable resistance (turbulent) R_T, is useful in pneumatic controllers as a feedback element to sense and control the pressure p_i. The displacement x is transmitted by a linkage or beam balance to a pneumatic valve regulating the air supply. The elasticity of the bellow is modeled as a spring with constant K. Develop a model for the displacement x.

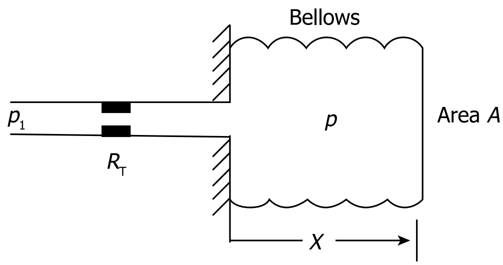

Fig. 2.40

2.7 Considering small deviations of variables from their steady-state values, obtain a model for outlet temperature θ_o of the air heating system shown in Fig. 2.41.

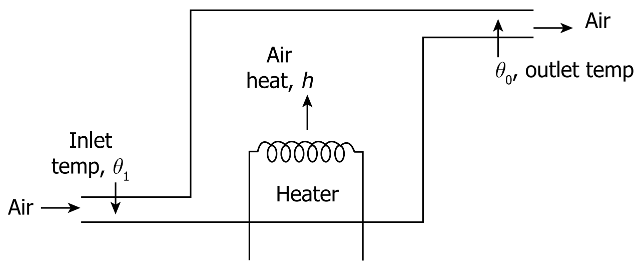

Fig. 2.41

θ_i is the temperature of the inlet air and h is the heat input.

2.8 Draw the mechanical network for the system shown in Fig 2.42 and draw its analogous circuit.

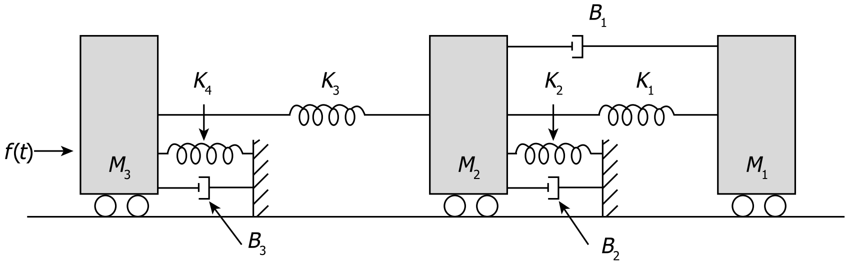

Fig. 2.42

2.9 Using force–voltage analogy show that the mechanical system shown in Fig. 2.43(a) below is analogous to the electrical system as in Fig. 2.43(b).

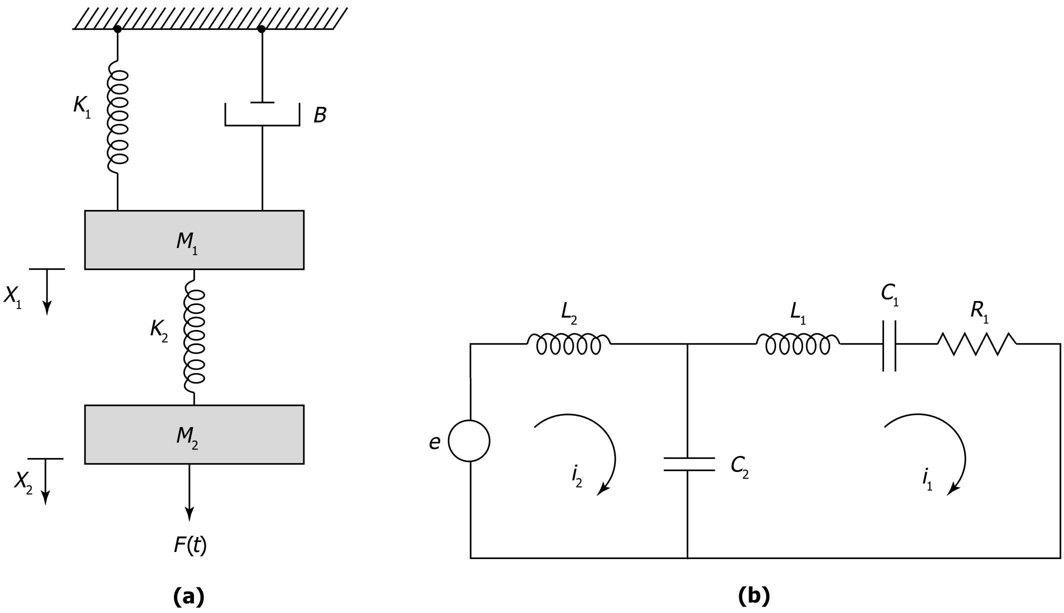

Fig. 2.43

2.10 With the help of suitable example of any one physical system illustrate the procedure for determining the transfer function.

2.11 Tabulate the analogous quantities of Mechanical and Electrical systems using (a) Force–voltage analogy; and (b) Force–current analogy.

2.12 Derive the system equations of the system shown in Fig. 2.44 and find the value of $x_2(s)/F(s)$.

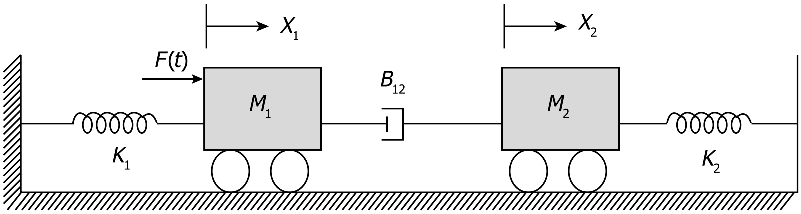

Fig. 2.44

2.13 Show that the systems shown in Fig. 2.45 are analogous to each other. In Fig. 2.45(a), X_1 represents the input displacement and X_0 represents the output displacement.

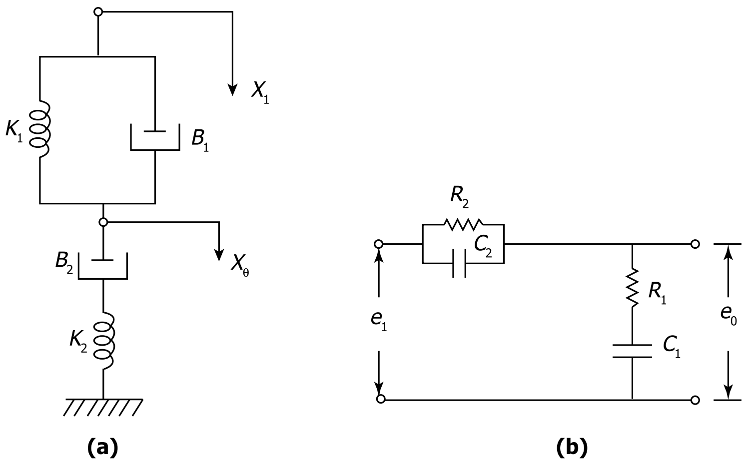

Fig. 2.45

2.14 Show that the two systems shown in Fig. 2.46 are analogous.

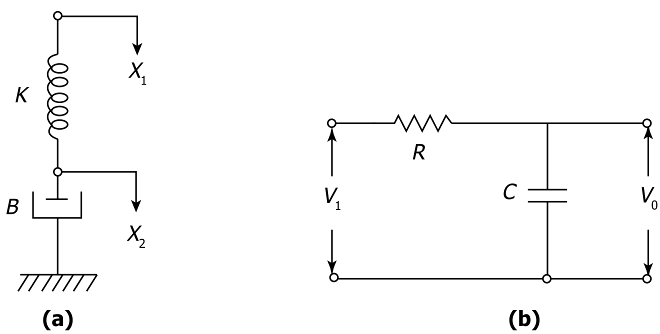

Fig. 2.46

2.15 Using force–voltage analogy of the mechanical system shown in Fig. 2.47, draw its analogous electrical circuit.

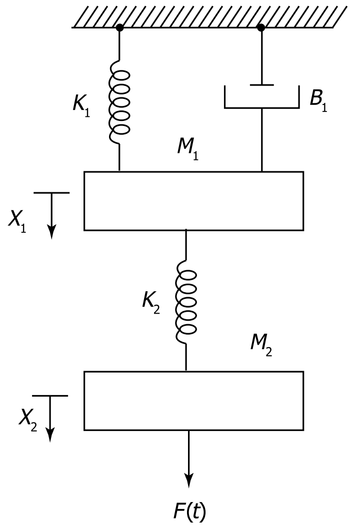

Fig. 2.47

2.16 Using force–voltage analogy draw the analogous electrical circuit of the system shown in Fig. 2.48.

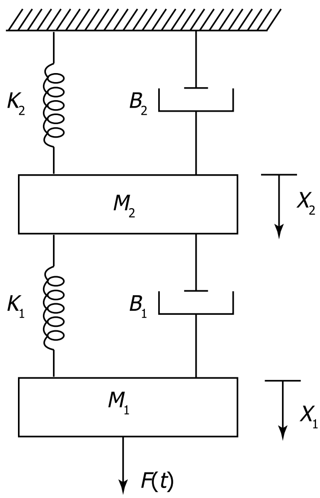

Fig. 2.48

2.17 Using force–voltage analogy and force–current analogy draw the equivalent analogous system for the system shown in Fig. 2.49.

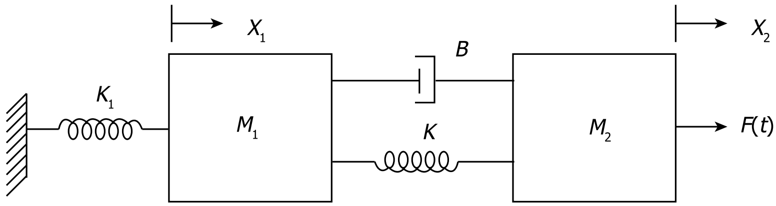

Fig. 2.49

2.18 For the mechanical system shown in Fig. 2.50 write the differential equations and draw the force-voltage analogous system.

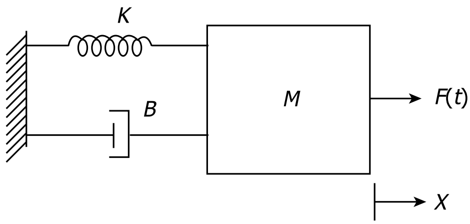

Fig. 2.50

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3

MODELLING A CONTROL SYSTEM—BLOCK DIAGRAM REPRESENTATION

3.1 INTRODUCTION

Any control system will have a number of control components. A control system can be represented in block diagram form. For making a block diagram, first the transfer function of the system components are determined. They are then showed in respective blocks. These blocks are connected by arrows indicating the direction of flow of signals in the control system whose block diagram is being represented. The signals can pass only in the direction of arrows represented in the diagram. The arrow head pointing towards a particular block indicates the input to the system component and the arrow head leading away from the block indicates the output. All the arrows in a block diagram are referred to as signals.

For complicated systems, the block diagram interconnecting all the sub-systems becomes a complex one. Reduction of such complex block diagram makes it easy to determine the overall transfer function of the whole system. Various techniques are then applied to analyse the performance of the system.

3.2 ADVANTAGES OF BLOCK DIAGRAM REPRESENTATION

The following are the main advantages of block diagram representation of control systems.

i) The overall block diagram of a system can be easily drawn by connecting the blocks according to the signal flow. It is also possible to evaluate the contribution of each of the components towards the overall performance of the control system.
ii) Block diagram helps in understanding the functional operation of the system more readily than examination of the actual control system physically. It may be noted that a

block diagram drawn for a system is not unique, that is, there may be alternative ways of representation of a system in block diagram form.

3.3 BLOCK DIAGRAM REPRESENTATION OF AN ERROR DETECTOR

In a feedback control system there is one reference input signal which corresponds to the desired output and one feedback signal. The feedback signal is received from the output (as a fraction of the output) if there is any deviation of the output from the desired output. Error detector is a summing device which produces a resultant signal. This signal is the difference between the reference input and the feedback signal of the control system. As shown in Fig. 3.1, $R(s)$ is the reference input corresponding to the desired output, $C(s)$ is the desired output, and $B(s)$ is the feedback signal.

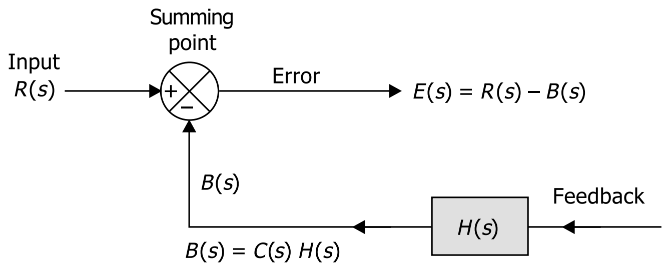


Fig. 3.1 Error detector

3.4 BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM AND ITS TRANSFER FUNCTION

The basic block diagram representation of a unity feedback control system has been shown in Fig. 3.2.

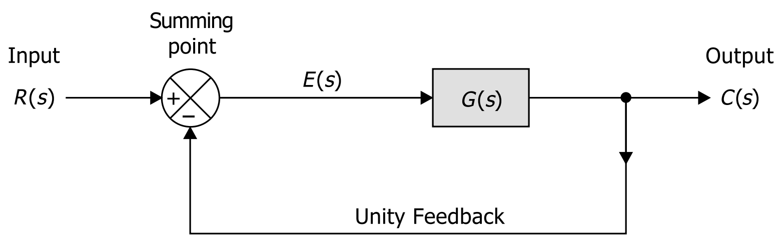


Fig. 3.2 A closed-loop system in block-diagram form

When the output is fed back to the summing point for comparison with the input to create an error signal, it is necessary to convert the form of the output signal to that of the input signal. The quantities being added or subtracted at the summing point should have the same dimensions and therefore, should have the same units. The conversion of a fraction of output signal on feedback path is done by the feedback element whose transfer function is $H(s)$. The feedback element modifies the output before it is compared with the input signal.

Fig. 3.3 shows the standard form of representation of a feedback control system in block diagram form. Here a fraction of output $B(s) = C(s)\,H(s)$ is brought to the summing point thereby producing an error signal. This block diagram is also called the canonical form of representation of a control system. $G(s)$ is the system transfer function.

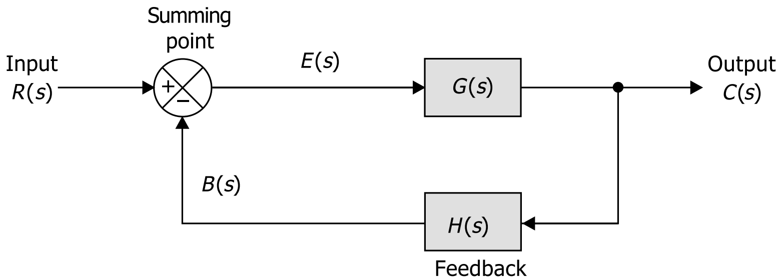


Fig. 3.3 *Standard way or canonical form of block diagram representation of a system with negative feedback*

Referring to Fig. 3.3, we can write,

$$E(s) = R(s) - B(s) \qquad \text{....(3.1)}$$

$$B(s) = H(s)\,C(s)$$

and
$$C(s) = E(s)\,G(s) \qquad \text{....(3.2)}$$

$\therefore$
$$E(s) = R(s) - H(s)\,C(s)$$

or,
$$R(s) = E(s) + H(s)\,C(s)$$

$$R(s) = E(s) + H(s)\,E(s)\,G(s) \qquad \text{....(3.3)}$$

Therefore, the closed loop transfer function is calculated by dividing equation (3.2) by equation (3.3),

$$\frac{C(s)}{R(s)} = \frac{E(s)G(s)}{E(s) + H(s)G(s)E(s)}$$

$$= \frac{G(s)}{1 + G(s)H(s)}$$

Transfer function,
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The above function is the closed loop transfer function for negative feedback. The transfer function for a closed loop system with positive feedback is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

This is because, the error signal for positive feedback will be, $E(s) = R(s) + B(s)$.

For a unity feedback control system, $H(s) = 1$. The transfer function for a unity feedback system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)} \qquad \text{....(3.4)}$$

where plus sign in the denominator stands for negative feedback.

The block diagram for a unity feedback control system is the same as shown in Fig. 3.2.

3.5 CHARACTERISTIC EQUATION OF A CONTROL SYSTEM

Consider the transfer function of the closed-loop system described earlier. The transfer function (T.F.) is written as

$$T.F. = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The denominator of the transfer function when equated to zero is called the characteristic equation of the control system. Accordingly, $1 + G(s)\, H(s) = 0$ is the characteristic equation of the system. For example, let the open-loop transfer function of a control system is given as $G(s) = \frac{7}{s(s+8)}$. Let the system is made a unity feedback system so that $H(s) = 1$. The characteristic equation is found from

$$T.F. = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Characteristic equation is

$$1 + G(s)\, H(s) = 0$$

or,
$$1 + \frac{7}{s(s+8)} \times 1 = 0$$

or,
$$s^2 + 8s + 7 = 0.$$

The roots of the characteristic equation are called the characteristic roots or closed-loop poles. The characteristic equation can be written as

$$(s + 1)(s + 7) = 0$$

The poles are at $s = -1$ and $s = -7$. These are the roots of the characteristic equation. The location of the poles in the s-plane indicates the condition of stability of the system. Use of the characteristic equation in the stability analysis will be dealt with in subsequent chapters.

3.6 RULES OF BLOCK DIAGRAM SIMPLIFICATION

Six rules are given so as to help simplify a block diagram of a control system. These rules are illustrated as under

a) When two or more blocks are connected in series, we are to multiply the transfer functions and put as one block as shown in Fig. 3.4.

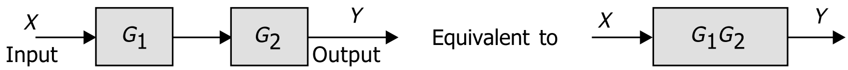


Fig. 3.4 *Equivalent of series connected blocks*

b) When two blocks are connected in parallel, the transfer functions are to be added as shown in Fig. 3.5.

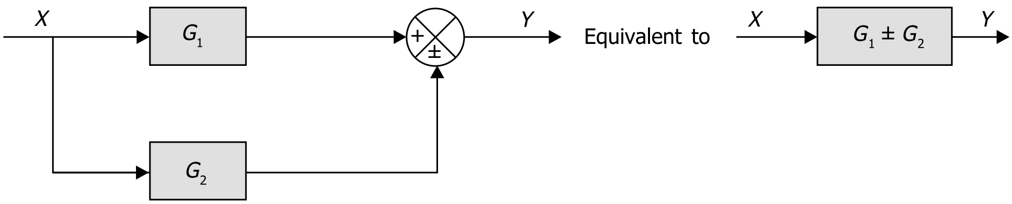


Fig. 3.5 *Equivalent of parallel connected blocks*

c) When shifting the summing point prior to, i.e. before a block,

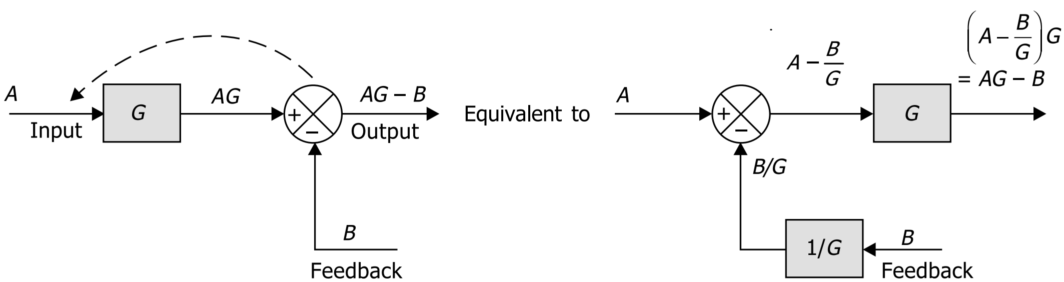


Fig. 3.6 *Shifting a summing point prior to a block*

Note that in this shifting the original input, output and feedback quantities, i.e. A, $AG - B$, and B respectively, must not change. Fig. 3.6 shows shifting of the summing point before block G with the help of a dotted line.

A block $1/G$ has to be introduced in the feedback path to maintain the input, output, and feedback signals unaltered.

d) Shifting the summing point beyond, i.e. after a block.

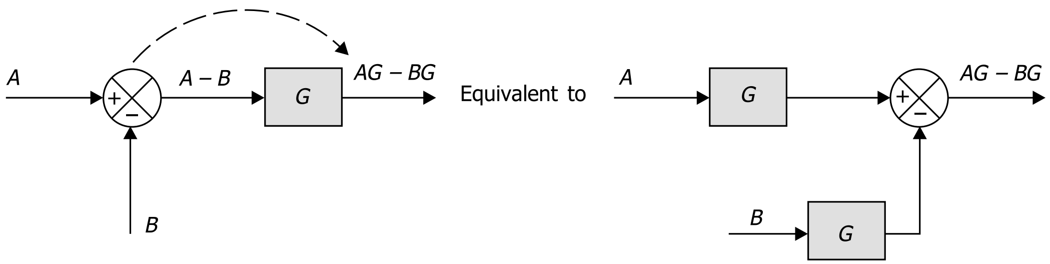


Fig. 3.7 *Shifting a summing point after a block*

As shown in Fig. 3.7, to keep the output signal unaltered, a block G has been placed in the feedback path.

e) Moving a take-off point from after a block to before a block.

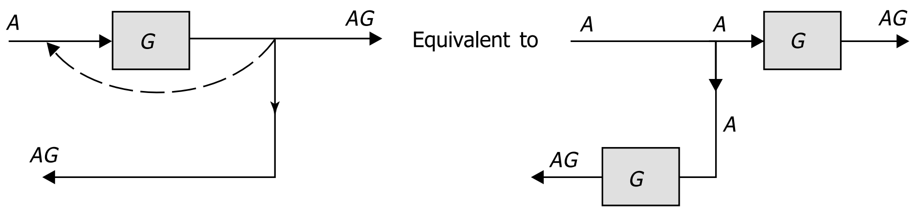


Fig. 3.8 *Moving a take-off point to before a block*

As shown in Fig. 3.8, a block G has to be introduced in the feedback path to keep feedback signal as AG unaltered.

f) Moving a take-off point beyond, i.e. after a block.

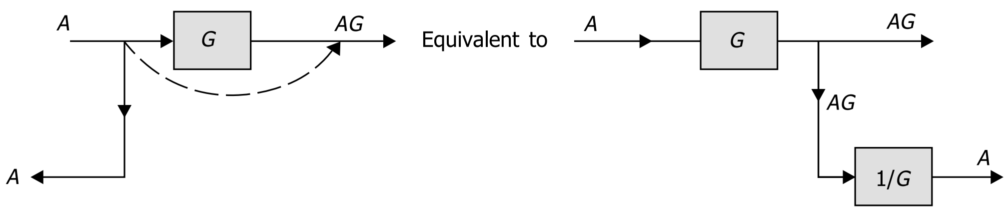


Fig. 3.9 *Moving a take-off point beyond a block*

A block of $1/G$ has to be introduced in the feedback path to maintain the equivalence as shown in Fig. 3.9.

Now let us take few examples of block diagram reduction.

Example 3.1 Reduce the block diagram as shown in Fig. 3.10 into canonical form and determine its transfer function.

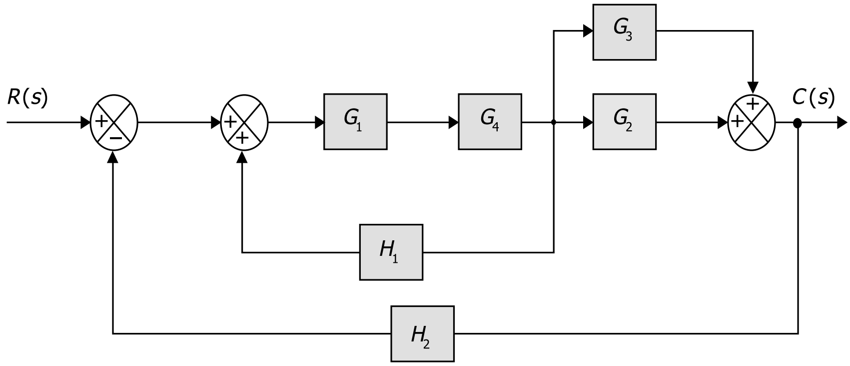
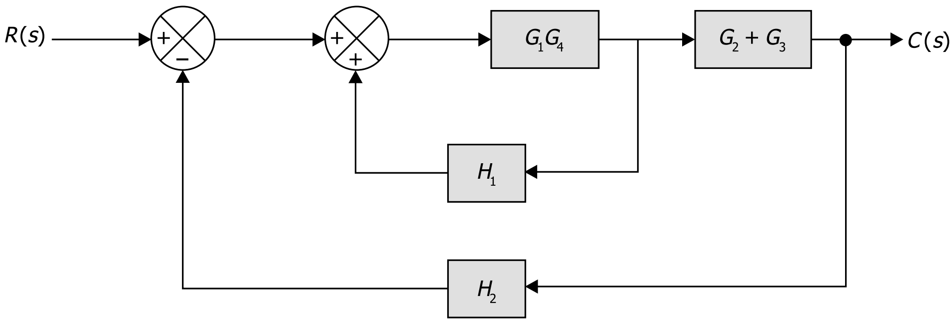


Fig. 3.10 Block diagram representation of a system

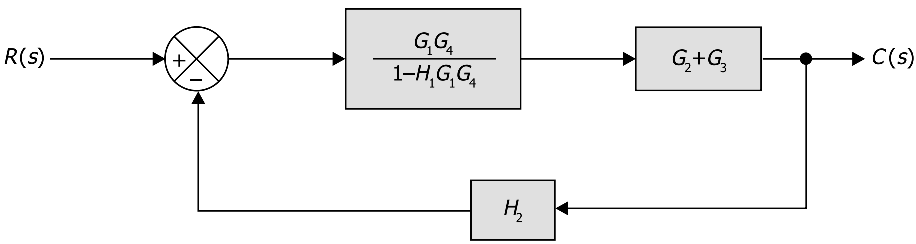
Solution

Combining elements G_1 and G_4 in series and elements G_2 and G_3 in parallel, we get (applying rule 1 and 2)



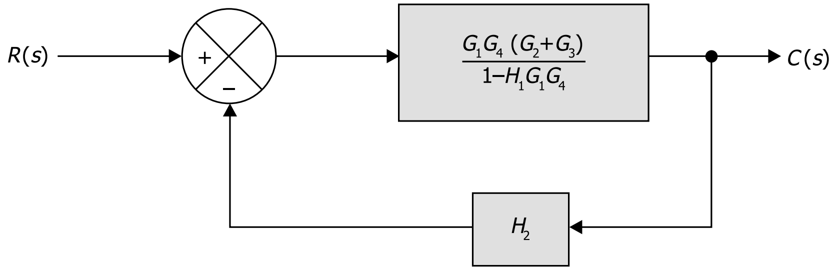
(a)

G_1, G_4 and H_1 form the feedback loop. Note that feedback is positive and therefore the Transfer Function (T.F.) will be of the form $\dfrac{G(s)}{1-G(s)H(s)}$.



(b)

$\dfrac{G_1G_4}{1-H_1G_1G_4}$, (G_2+G_3) are in series. They together form one block, and H_2 form the feedback loop.



(c)

Fig. 3.11 *Canonical form of representation*

$$T.F. = \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_4(G_2+G_3)}{1-H_1G_1G_4}}{1+\dfrac{G_1G_4(G_2+G_3)}{1-H_1G_1G_4}H_2}$$

$$= \frac{G_1G_4(G_2+G_3)}{1-H_1G_1G_4+H_2G_1G_4(G_2+G_3)}$$

Example 3.2 Reduce the block diagram of a system represented in Fig. 3.12 into canonical form.

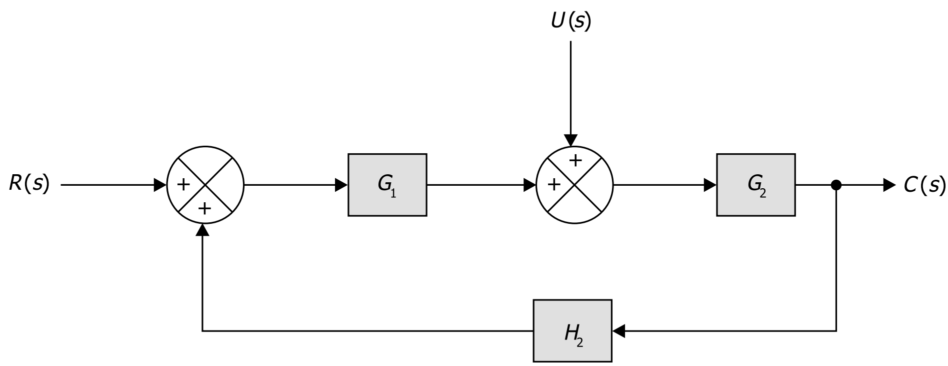


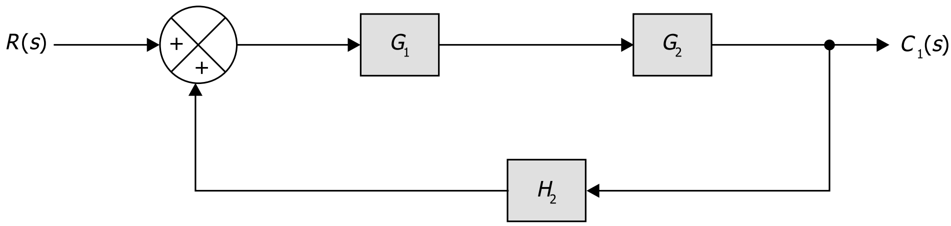
Fig. 3.12 *Block diagram representation of a system*

Solution

There are two input signals, $R(s)$ and $U(s)$, and one output signal, $C(s)$. We apply the superposition theorem to reduce the block diagram, considering the effect of one input at a time when other is set to zero.

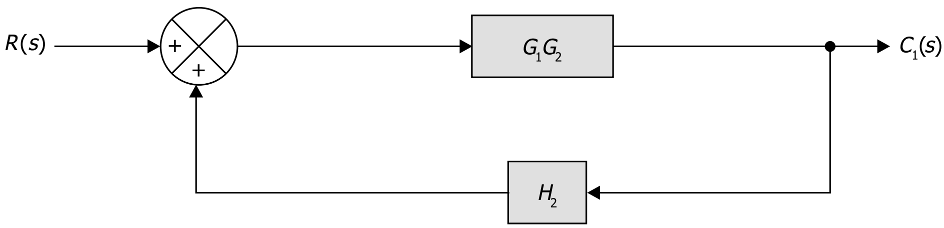
To determine the total response, the individual responses will be added.

i) Let us consider input $R(s)$ first, keeping $U(s) = 0$. The block diagram of Fig. 3.12 will be as shown in Fig. 3.13(a). Afterwards, we will consider the input $U(s)$, keeping $R(s) = 0$.



(a)

Combining G_1 and G_2 in series



(b)

Fig. 3.13 *Block diagram representation using superposition theorem with one input R(s) only*

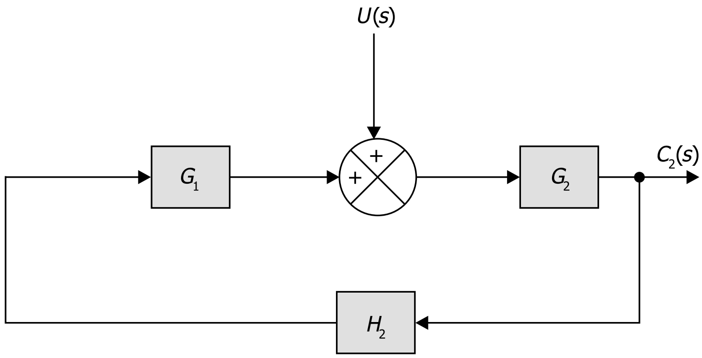
G_1G_2 and H_2 form the feedback loop.

Transfer Function, $$T.F_1 = \frac{C_1(S)}{R(S)} = \frac{G_1G_2}{1 - H_2G_2G_1}$$

(since the feedback is positive, we have used minus sign in the denominator)

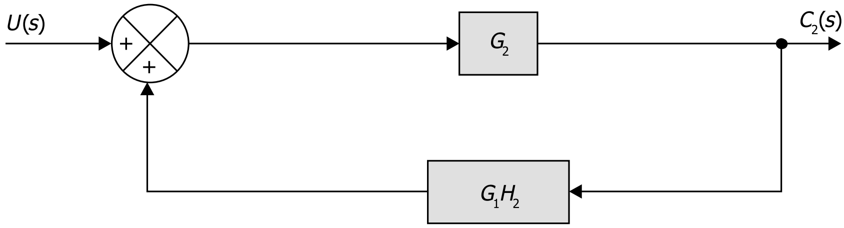
or Output, $$C_1(s) = \left(\frac{G_1G_2}{1 - G_1G_2H_2}\right)R(s)$$

ii) Considering second input $U(s)$ only, and keeping $R(s) = 0$, the network becomes as given below in Fig. 3.14(a).



(a)

Combining H_2, G_1 in series, and drawing the network in canonical form,



(b)

Fig. 3.14 *Canonical form of block diagram representation using superposition theorem with second input U(s) only*

G_2 and $G_1 H_2$ form the feedback loop.

Transfer Function, $$T.F_2 = \frac{C_2(S)}{U(S)} = \frac{G_2}{1 - G_1 G_2 H_2}$$

Therefore, $$C_2(s) = \left(\frac{G_2}{1 - G_1 G_2 H_2}\right) U(s).$$

Combining the responses in case (i) and case (ii), the net output response can be calculated.

Net output response $C(s) = C_1(s) + C_2(s) = \left(\frac{G_1 G_2}{1 - G_1 G_2 H_2}\right) R(s) + \left(\frac{G_2}{1 - G_1 G_2 H_2}\right) U(s).$

Example 3.3 Determine the net response of the system shown in Fig. 3.15 using block diagram reduction technique.

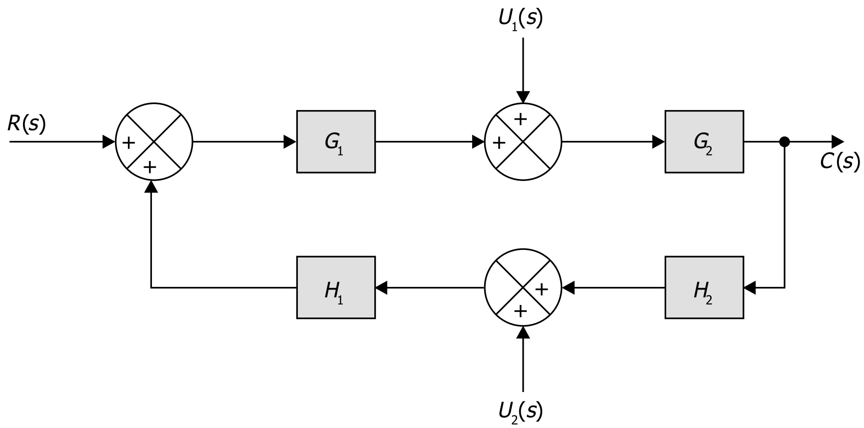
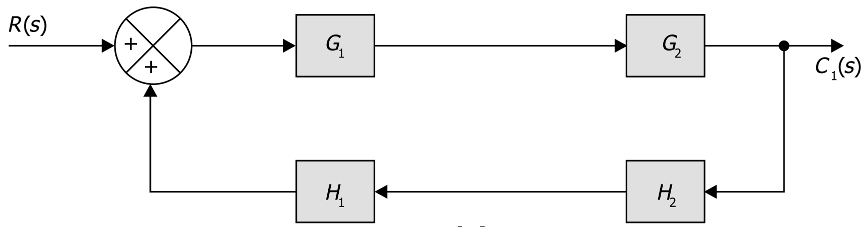


Fig. 3.15

Solution

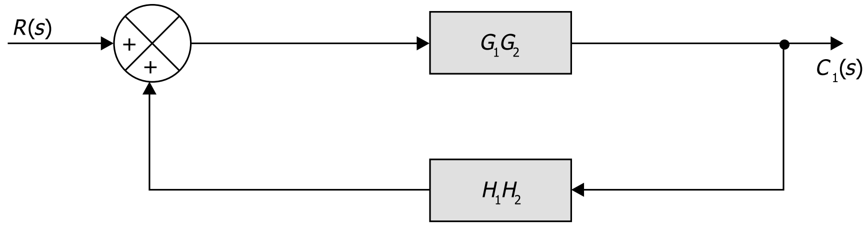
We will follow the same procedure as in the previous example. Here we have three inputs. We will take one input at a time, keeping other two inputs equal to zero.

i) Considering only input $R(s)$.
$U_1(s) = U_2(s) = 0$, Output $= C_1(s)$.



(a)

G_1, G_2 and H_1, H_2 are in series. There the circuit gets reduced to



(b)

Fig. 3.16

G_1, G_2 and H_1, H_2 from the feedback loop.

$$\frac{C_1(s)}{R(s)} = \frac{G_1G_2}{1 - G_1G_2H_1H_2} \quad \text{(minus sign in the denominator is due to positive feedback)}....(3.5)$$

ii) Considering input $U_1(s)$ only
$U_2(s) = R(s) = 0$, Output $= C_2(s)$

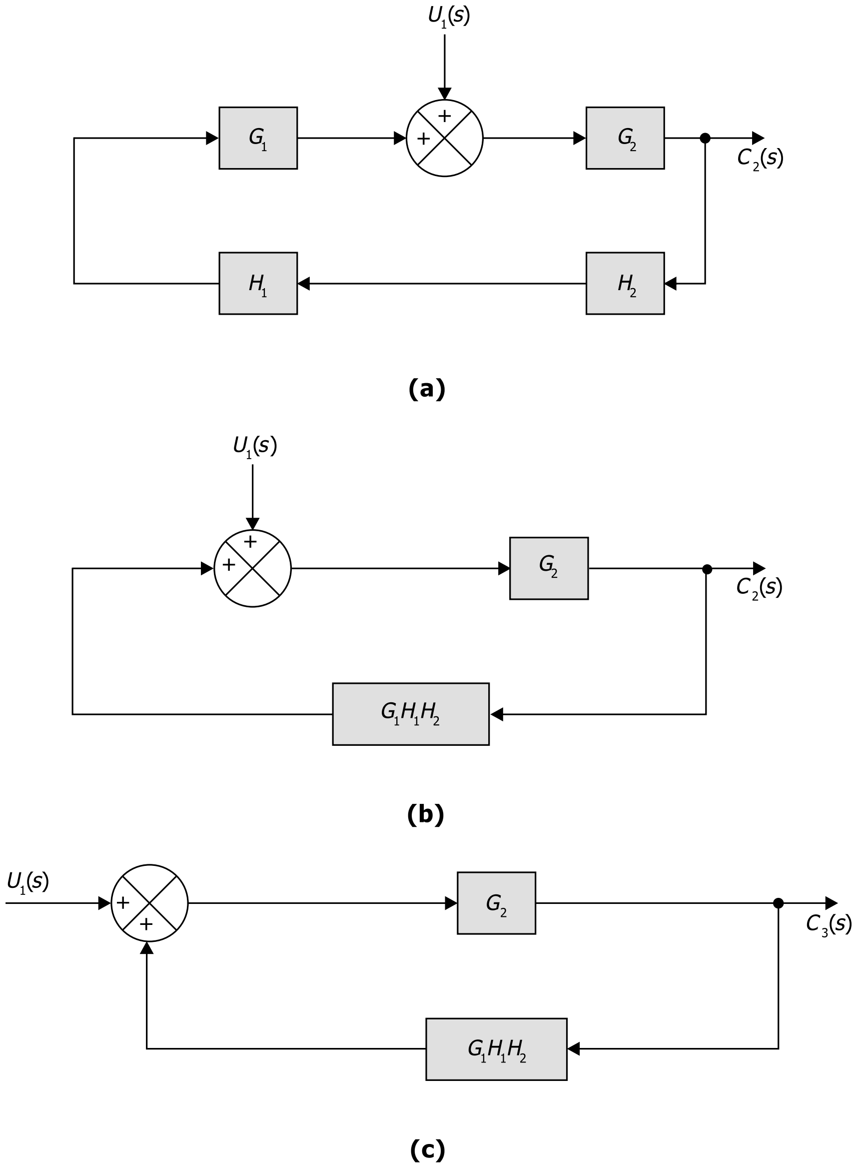
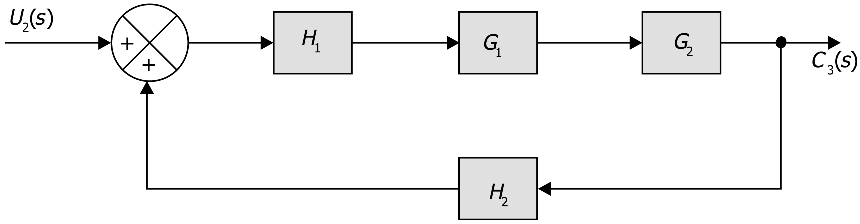


Fig. 3.17

$G_1H_1H_2$ and G_2 form the feedback loop.

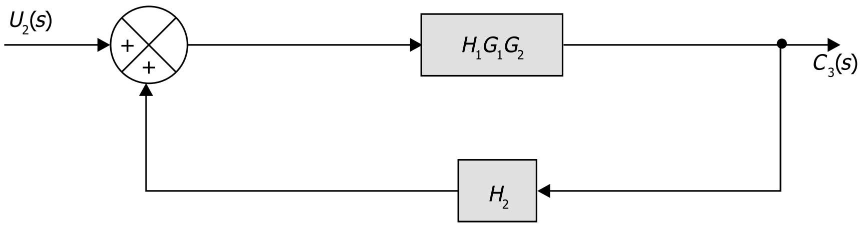
$$\frac{C_2(s)}{U_1} = \frac{G_2}{1 - G_1G_2H_1H_2} \qquad \text{....(3.6)}$$

iii) Considering input $U_2(s)$ only
$U_1(s) = R(s) = 0$, Output $= C_3(s)$.



(a)

H_1, G_1 and G_2 in the series



(b)

Fig. 3.18

H_1, G_1G_2 and H_2 form the feedback loop.

$$\frac{C_3(s)}{U_2(s)} = \frac{G_1G_2H_1}{1 - G_1G_2H_1H_2} \qquad \text{....(3.7)}$$

By adding equations (3.5), (3.6) and (3.7), we get the net response, i.e. the combined output as

$$C(s) = C_1(s) + C_2(s) + C_3(s)$$

or, $$C(s) = \left(\frac{G_1G_2}{1 - G_1G_2H_1H_2}\right)R(s) + \left(\frac{G_2}{1 - G_1G_2H_1H_2}\right)U_1(s) + \left(\frac{G_1G_2H_1}{1 - G_1G_2H_1H_2}\right)U_2(s)$$

Example 3.4 Reduce the block diagram of Fig. 3.19 to canonical form and derive the overall transfer function (Note that this problem has also been solved by Signal Flow Graph Method as in Example 4.3 in the following chapter, i.e. in Chapter 4)

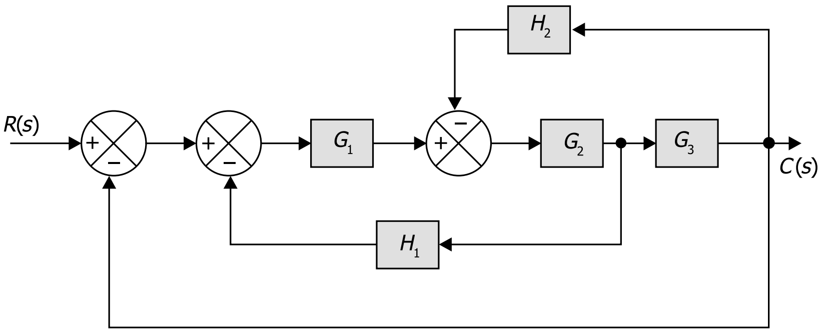
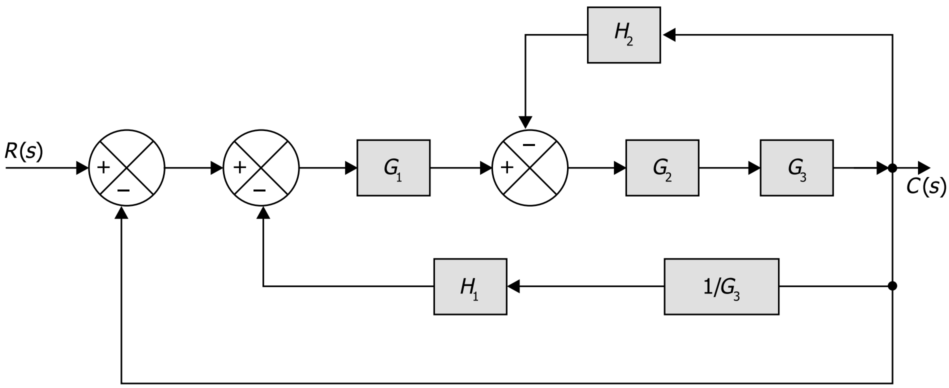


Fig. 3.19

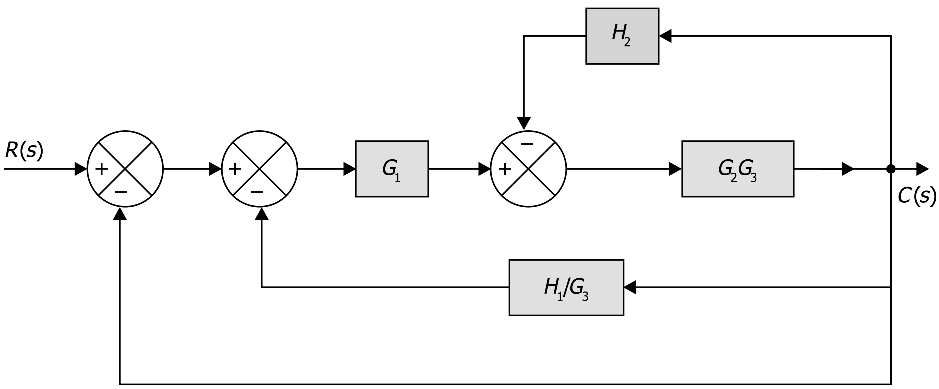
Solution

Shifting the take-off point after G_2, beyond the block G_3 as shown in Fig. 3.20.



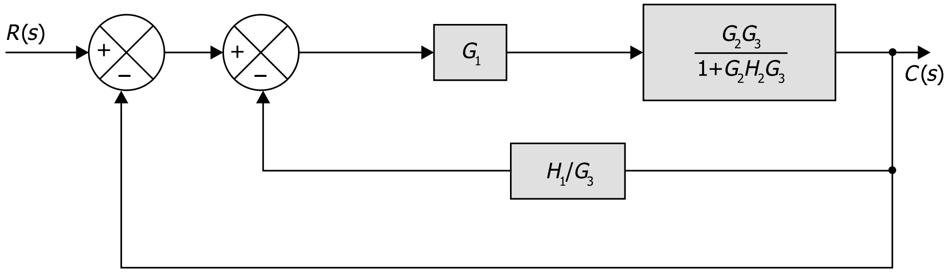
(a)

G_2, G_3 and H_1, $1/G_3$ are in series.



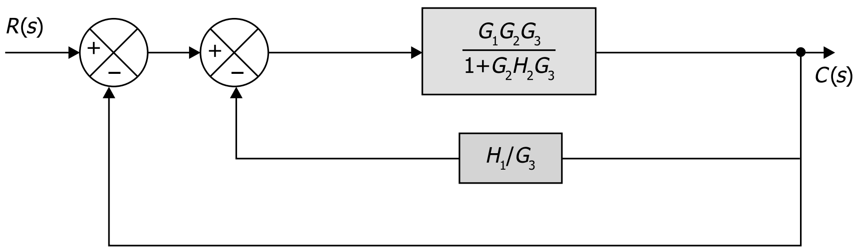
(b)

H_2 and G_2G_3 form a feedback loop.



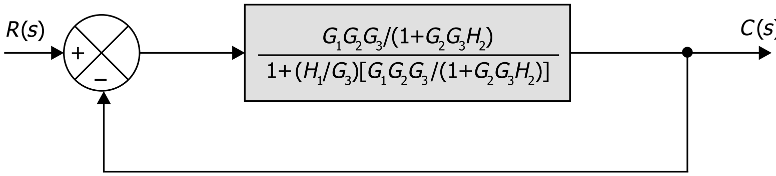
(c)

G_1 and $\dfrac{G_2G_3}{1-G_2G_3H_2}$ are in a series.



(d)

$\dfrac{H_1}{G_3}$ and $\dfrac{G_1G_2G_3}{1-G_2G_3H_2}$ form a feedback loop.



(e)

Fig. 3.20 *Canonical form of representation of the block diagram of Fig. 3.19*

This is a unity feedback system, where $H(s) = 1$.
So, the overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1}}{1+\dfrac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1}}$$

$$= \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}$$

Example 3.5 Reduce the block diagram of Fig. 3.21 to canonical form.

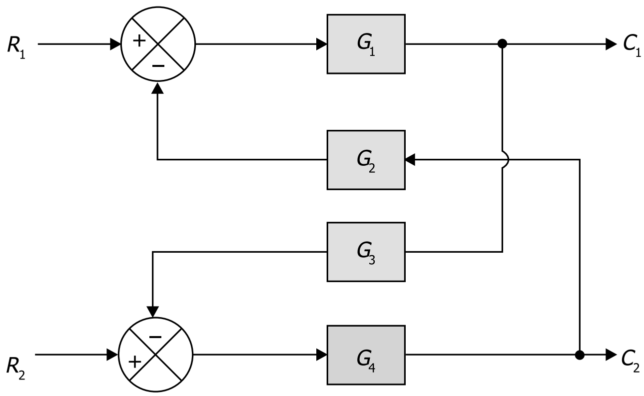


Fig. 3.21

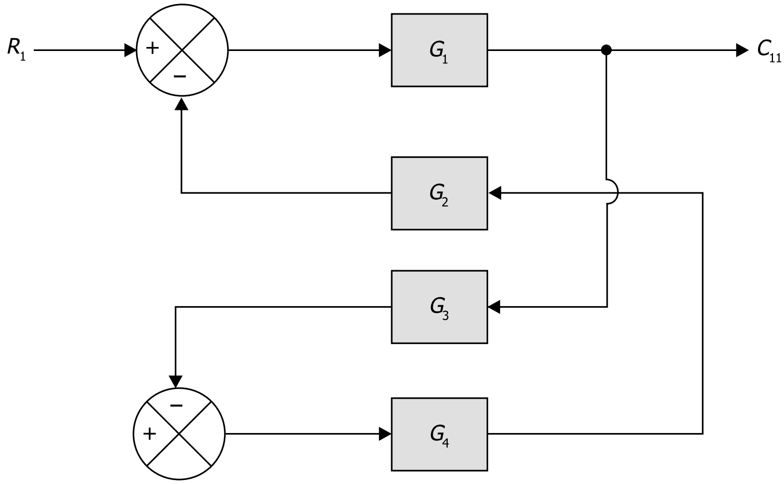
Solution

There are two inputs, R_1 and R_2, and there are two outputs, C_1 and C_2.

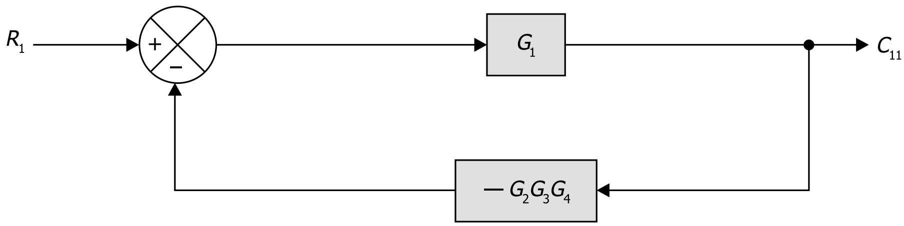
a) First of all, we shall find the output C_1, when $C_2 = 0$.
Now, we shall consider the inputs independently and then add the two responses C_{11} and C_{12} to find the overall response C_1.
C_{11} is the output with only R_1 as the input.
C_{12} is the output with only R_2 as the input.
$C_1 = C_{11} + C_{12}$. C_{11} and C_{12} will be calculated with one input.

1) To find C_{11}
Putting $R_2 = C_2 = 0$ in the network, $G_2\,G_3\,G_4$ are in series, $G_2\,G_3\,G_4$ and G_1 form the feedback loop and we get

$$\frac{C_{11}(s)}{r_1(S)} = \frac{G_1}{1-G_1G_2G_3G_4}$$



(a)



(b)

Fig. 3.22

2) To find C_{12},

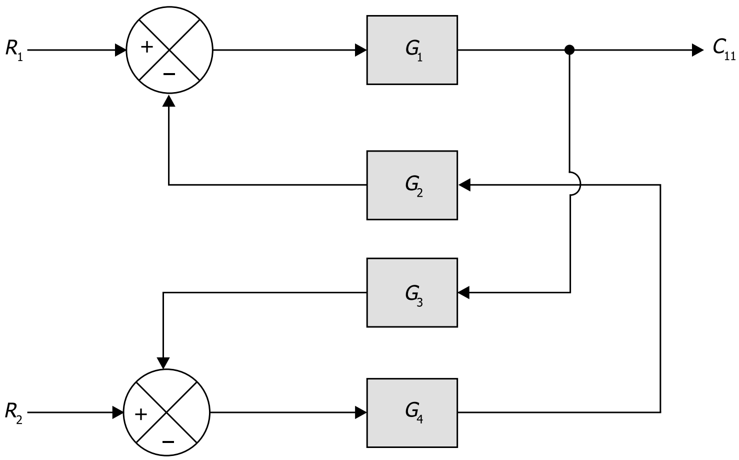
Putting $R_1 = C_2 = 0$ in the network, $G_1\ G_2\ G_4$ are in series, G_3 and $G_1\ G_2\ G_4$ form the feedback loop and we get

$$\frac{C_{12}(s)}{R_2(s)} = \frac{-G_1G_2G_4}{1 - G_1G_2G_3G_4}$$

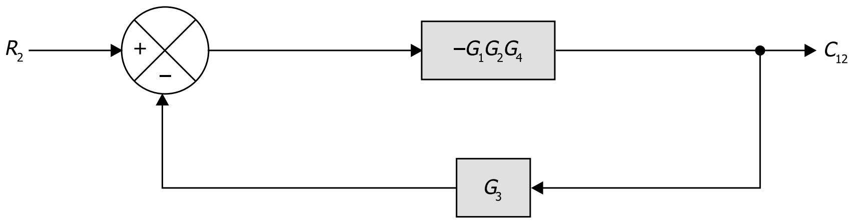
$\therefore$

$$C_1(s) = C_{11}(s) + C_{12}(s)$$

$$= \left(\frac{-G_1}{1 - G_1G_2G_3G_4}\right)R_1(s) + \left(\frac{-G_1G_2G_4}{1 - G_1G_2G_3G_4}\right)R_2(s)$$



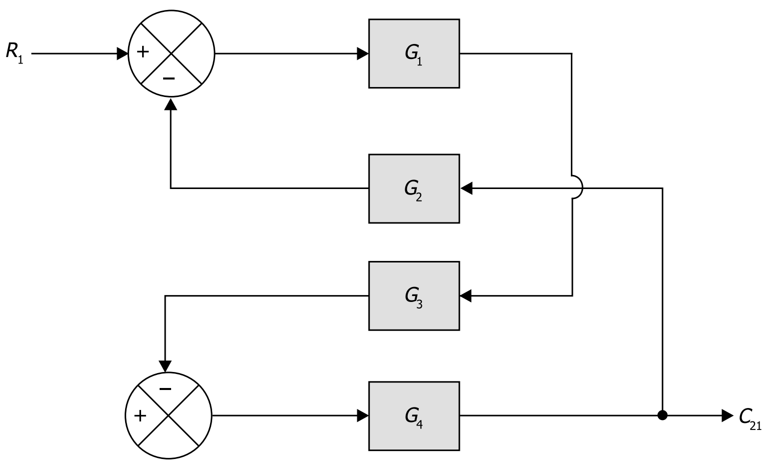
(a)



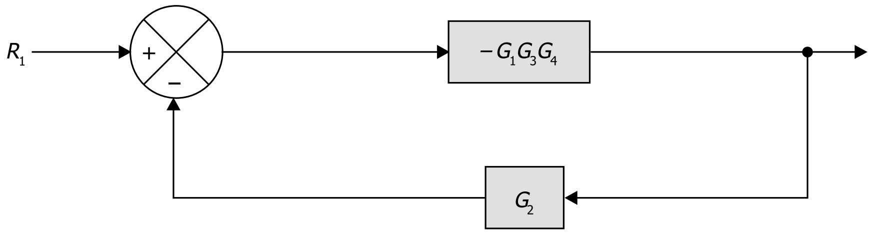
(b)

Fig. 3.23

3) Now, we shall find C_2, when $C_1 = 0$



(a)



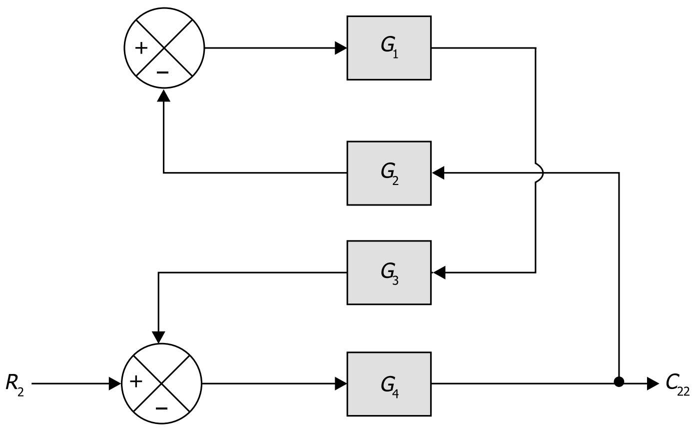
(b)

Fig. 3.24

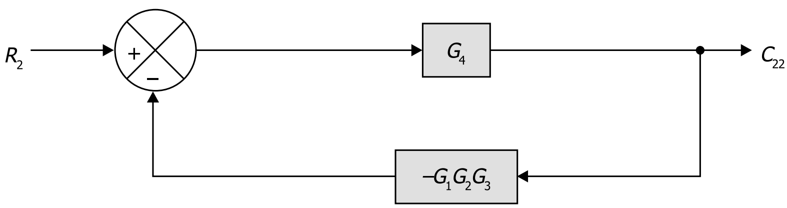
i) For finding C_{21}, consider input R_1 only $G_1\ G_3\ G_4$ are in series and $G_1\ G_3\ G_4$ and G_2 form the feedback loop.

Putting $R_2 = C_1 = 0$, we get $\dfrac{C_{21}(s)}{R_1(s)} = \dfrac{-G_1G_3G_4}{1 - G_1G_2G_3G_4}$.

ii) For finding C_{22}, consider input R_2 only and $R_1 = C_1 = 0$. In the network $G_1\ G_3\ G_4$ are in series and G_4 and $G_1\ G_3\ G_4$ form the feedback loop.



(a)



(b)

Fig. 3.25

$$\therefore \qquad \frac{C_{22}(s)}{R_2(s)} = \frac{G_4}{1 - G_1G_2G_3G_4}$$

$$\therefore \qquad C_2(s) = C_{21}(s) + C_{22}(s)$$

Thus $$C_2(s) = \left(\frac{G_4}{1 - G_1G_2G_3G_4}\right)R_2(s) + \left(\frac{-G_1G_3G_4}{1 - G_1G_2G_3G_4}\right)R_1(s)$$

Example 3.6 Obtain the transfer function for the block diagram shown in Fig. 3.26.

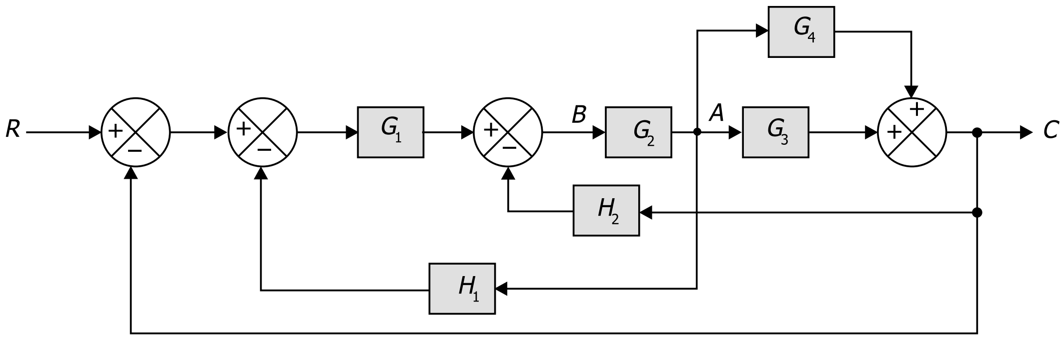


Fig. 3.26

Solution

Moving the take off point (Point A) between G_2 and G_3 ahead of G_2 (Point B) and inserting block G_2 with H_1 and G_4.

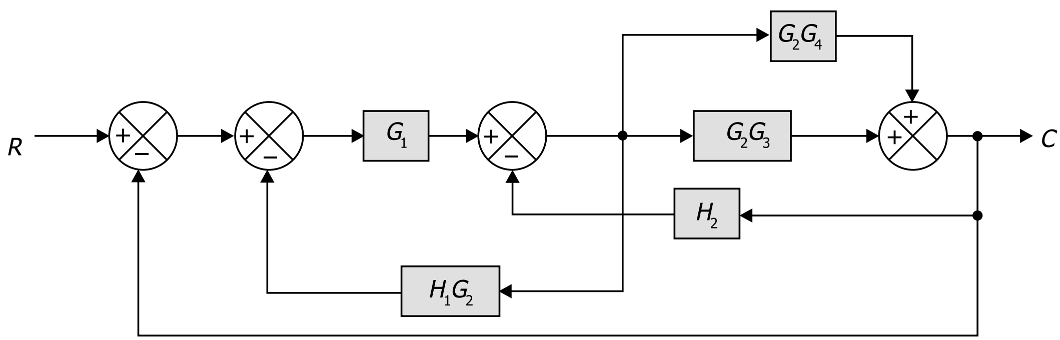


Fig. 3.27

$G_2\ G_4$ and $G_2\ G_3$ are in parallel,

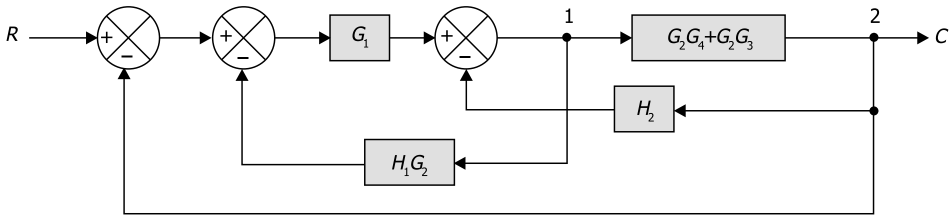


Fig. 3.28

Moving take off point 1 beyond block $(G_2\, G_4 + G_2\, G_3)$ at Point 2,

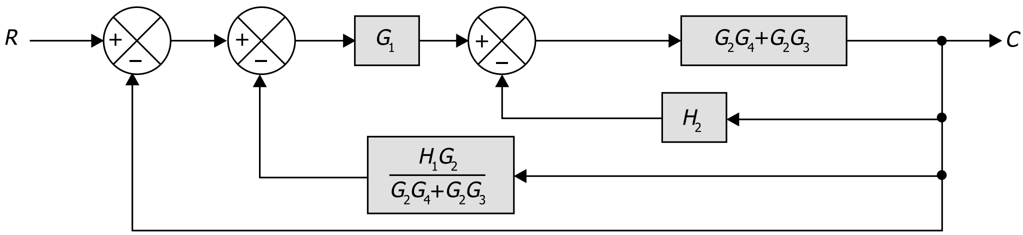


Fig. 3.29

Eliminating the feedback loops,

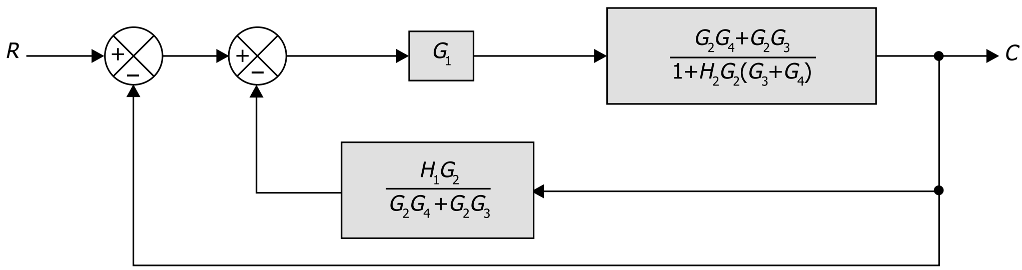


Fig. 3.30

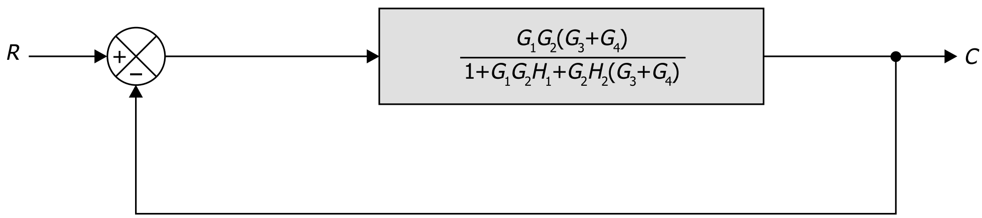


Fig. 3.31 *Block diagram of Fig. 3.27 reduced to canonical form through successive steps*

$$\frac{C}{R} = \frac{\dfrac{G_1G_2(G_4+G_3)}{1+G_2H_2(G_4+G_3)+H_1G_1G_2}}{1+\dfrac{G_1G_2(G_4+G_3)}{1+G_2H_2(G_4+G_3)+H_1G_1G_2}}$$

or,
$$\frac{C}{R} = \frac{G_1G_2(G_4+G_3)}{1+G_2H_2(G_4+G_3)+H_1G_1G_2+G_1(G_4+G_3)G_2}$$

$\therefore$
$$\frac{C}{R} = \frac{G_1G_2(G_4+G_3)}{1+G_2(G_3+G_4)(H_2+G_1)+H_1G_1G_2}$$

Example 3.7 Obtain the transfer function for the block diagram shown in Fig. 3.32.

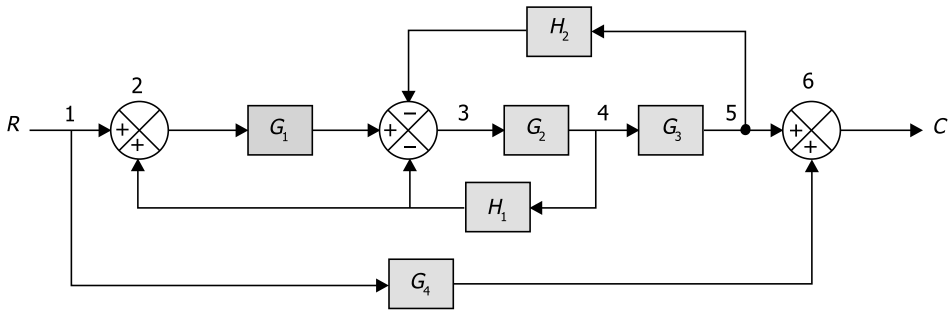


Fig. 3.32

Solution

Moving take off Point 4 beyond block G_3 (at Point 5).

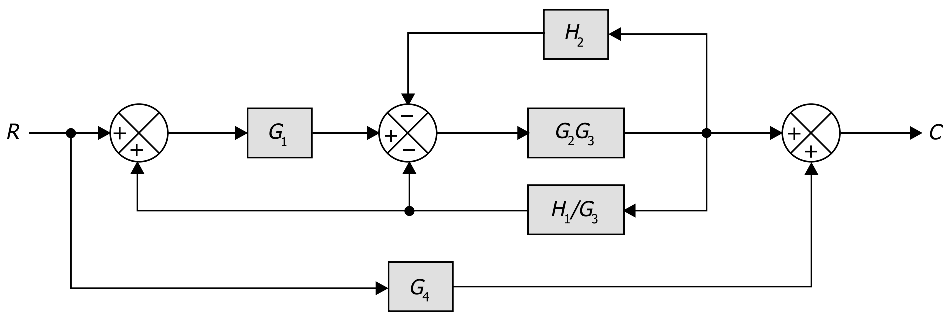


Fig. 3.33

Eliminating feedback with transfer function H_2,

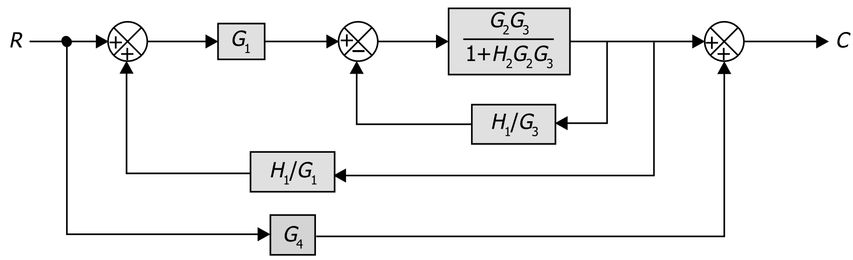


Fig. 3.34

Eliminating feedback loop,

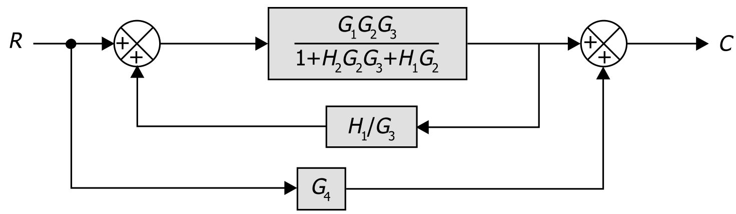


Fig. 3.35

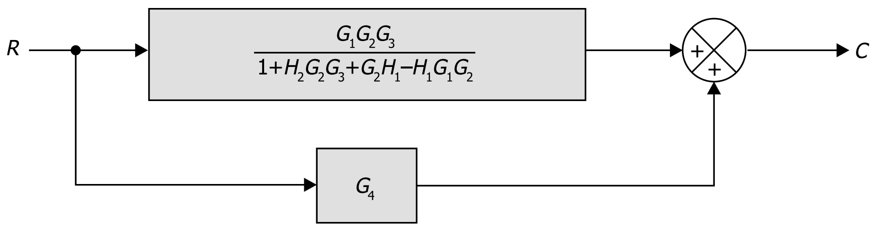


Fig. 3.36

The two blocks being parallel,

$$\frac{C}{R} = G_4 + \frac{G_1G_2G_3}{1 + H_2G_2G_3 + G_2H_1(1 - G_1)}$$

Example 3.8 Evaluate $\dfrac{C}{R_1}$ and $\dfrac{C}{R_2}$ for a system the block diagram representation of which is shown in Fig. 3.37. R_1 and R_2 are the inputs at the summing points.

Solution

First we will find $\dfrac{C}{R_1}$ with $R_2 = 0$ and then find $\dfrac{C}{R_1}$ with $R_1 = 0$.

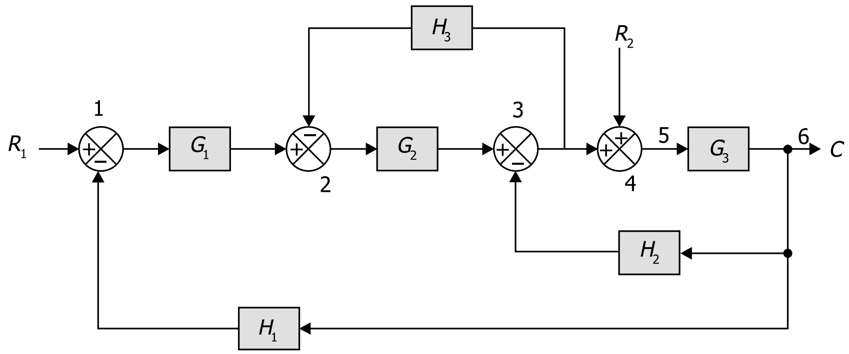


Fig. 3.37

i) Evaluation of C / R_1. Assume $R_2 = 0$. Therefore, summing Point 5 can be removed. Shift take off point 4 beyond block G_3.

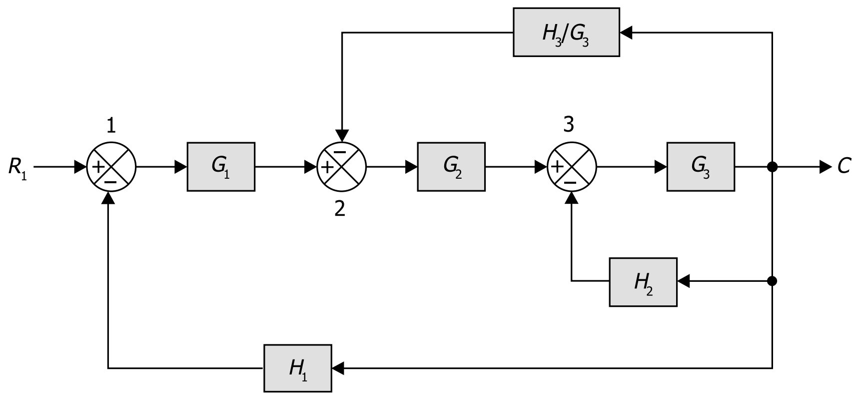


Fig. 3.38

Eliminate the feedback loop between Points 3 and 4 and then combining with G_2 in series.

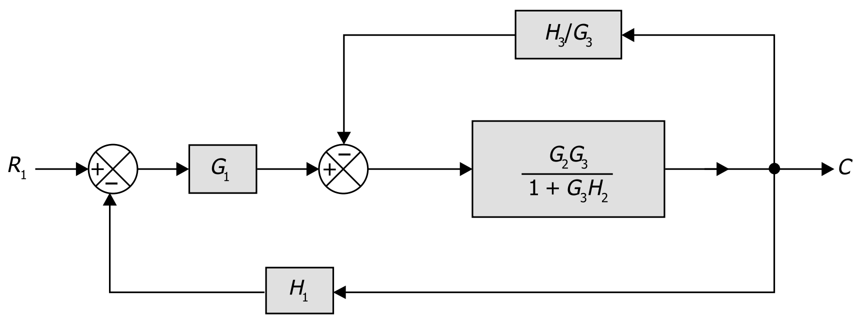


Fig. 3.39

Eliminating the feedback loop again, we get the block diagram as in Fig. 3.40.

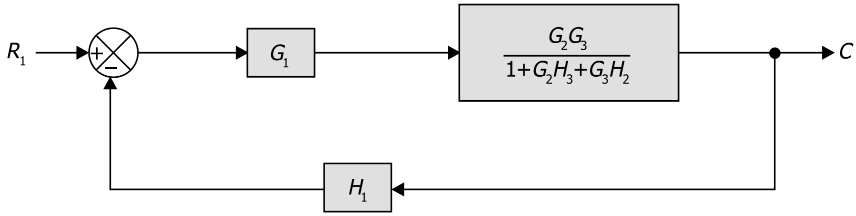


Fig. 3.40

With G_1 and $\dfrac{G_2G_3}{1+G_3H_2+G_2H_3}$ in series, we get

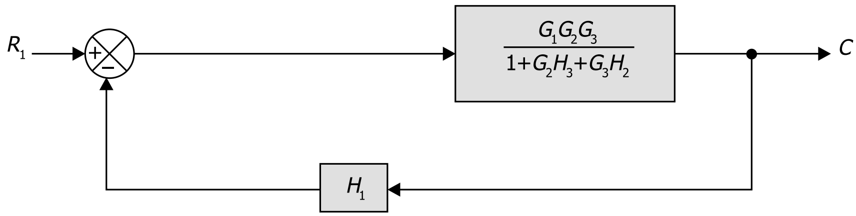


Fig. 3.41

$$\frac{C}{R_1} = \frac{G_1G_2G_3}{1+G_3H_2+H_3G_2+G_1G_2G_3H_1}$$

ii) Evaluation of C/R_2. Assume $R_1 = 0$. Thus summing point 1 in Fig. 3.64 is removed.

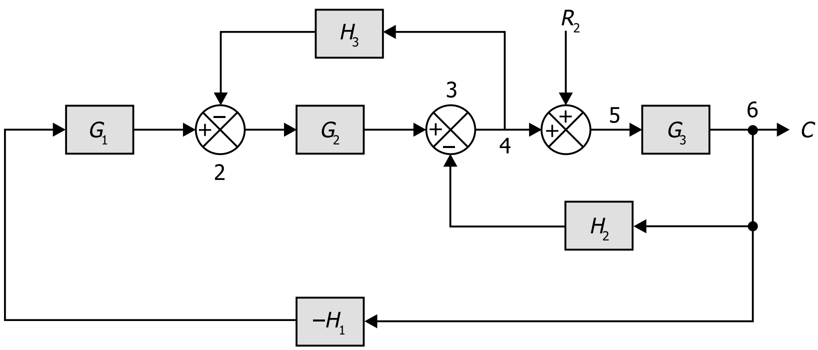


Fig. 3.42

Shifting the summing Point 2 beyond G_2 and rearranging,

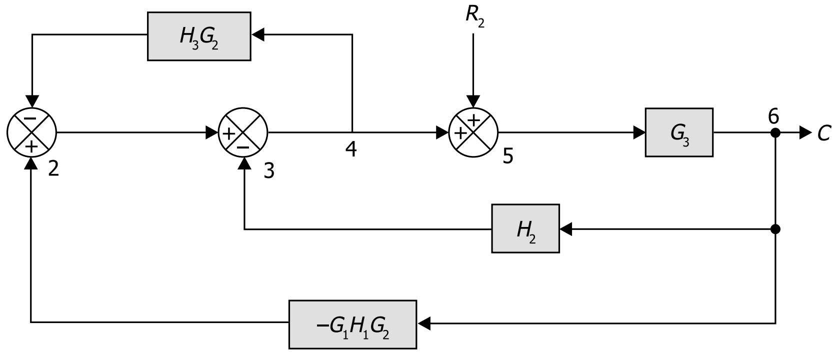


Fig. 3.43

Rearranging again, we get

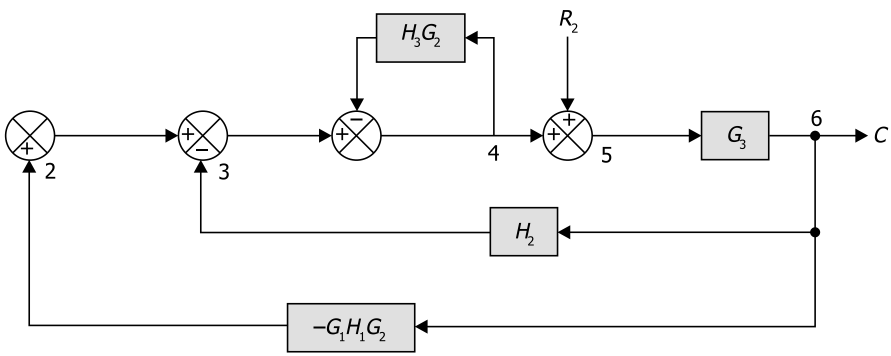


Fig. 3.44

Rearranging another time and eliminating the feedback loop,

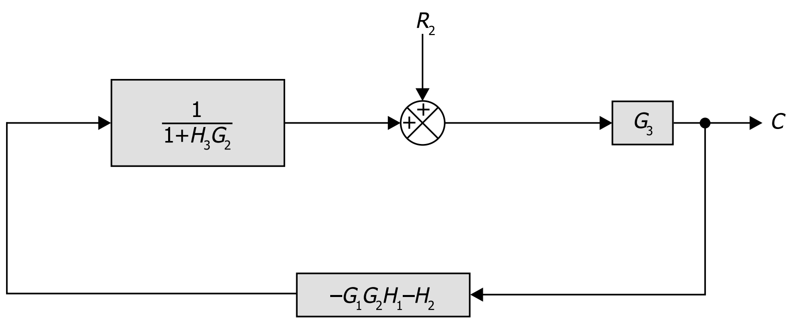


Fig. 3.45

Rearranging further,

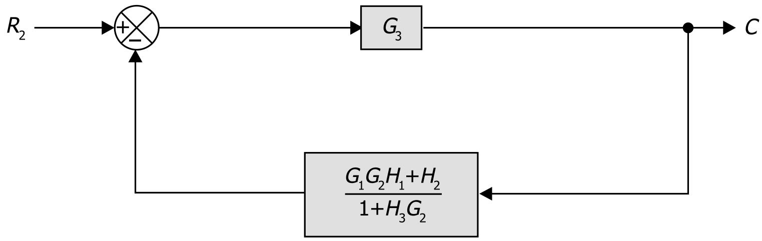


Fig. 3.46

So, we get
$$\frac{C}{R_2} = \frac{G_3(1+G_2G_3)}{1+G_2H_3+G_3G_2+G_1G_2G_3H_1}.$$

Example 3.9 Find the closed-loop transfer function of the system shown in Fig. 3.47.

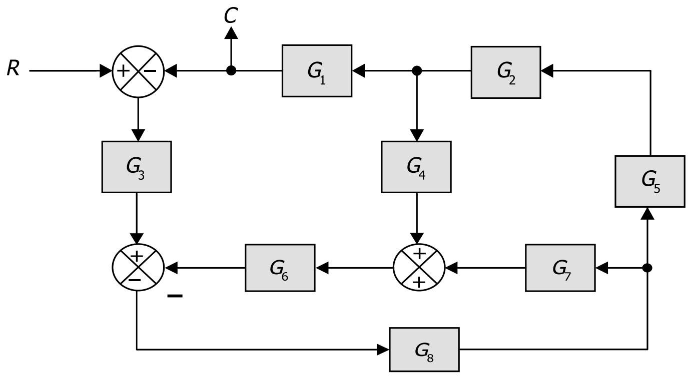


Fig. 3.47

Solution

Redrawing the diagram having input R to the extreme left and C taken to the extreme right as output and then joining each block step by step will help solve the problem with ease. Accordingly, the block diagram has been redrawn as shown in Fig. 3.48.

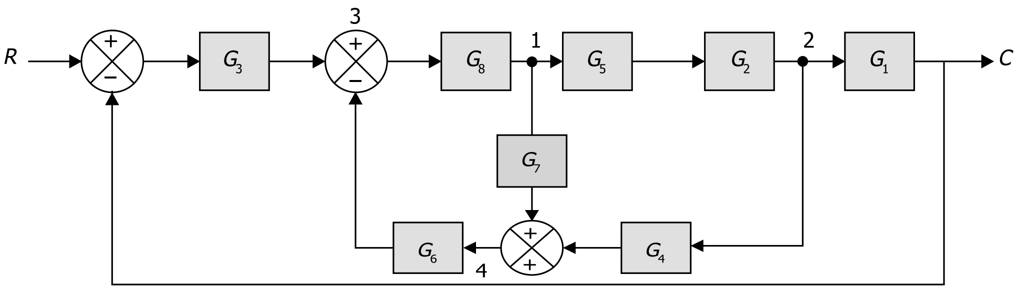


Fig. 3.48

Shifting the take off points at 1 and 2 beyond the block G_1,

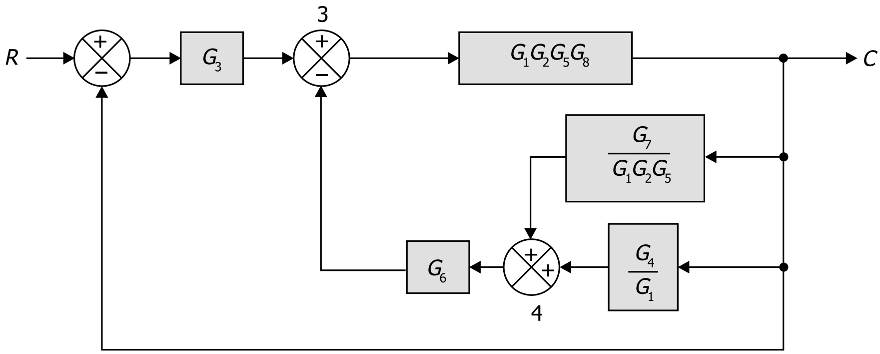


Fig. 3.49

Eliminating summing point at 4,

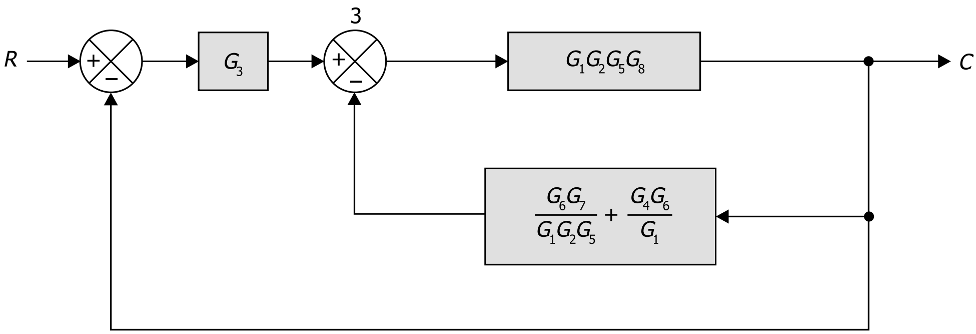


Fig. 3.50

Eliminating inner feedback loop and then combining G_3 in series we get,

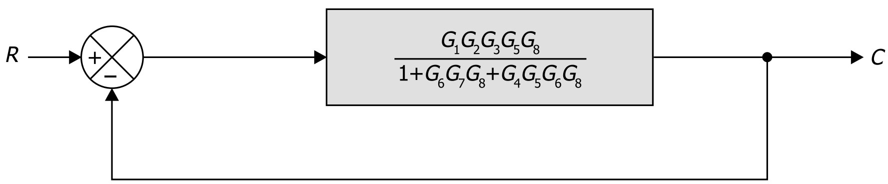


Fig. 3.51

Since this is unity feedback system, we calculate the transfer function as,

$$\frac{C}{R} = \frac{G_1G_2G_3G_5G_8}{1+G_6G_7G_8+G_2G_5G_8(G_1G_3+G_4G_6)}.$$

Example 3.10 Find the closed-loop transfer function of the system shown in Fig. 3.52.

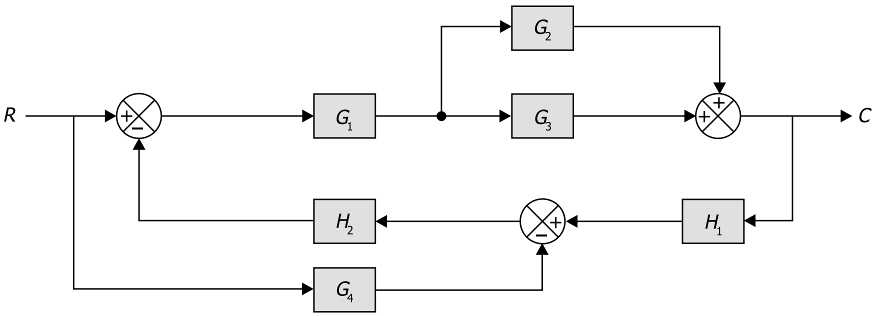


Fig. 3.52

Solution

Moving the summing point beyond the block H_2 and inserting G_4H_2 block with H_1, H_2 in series, we get

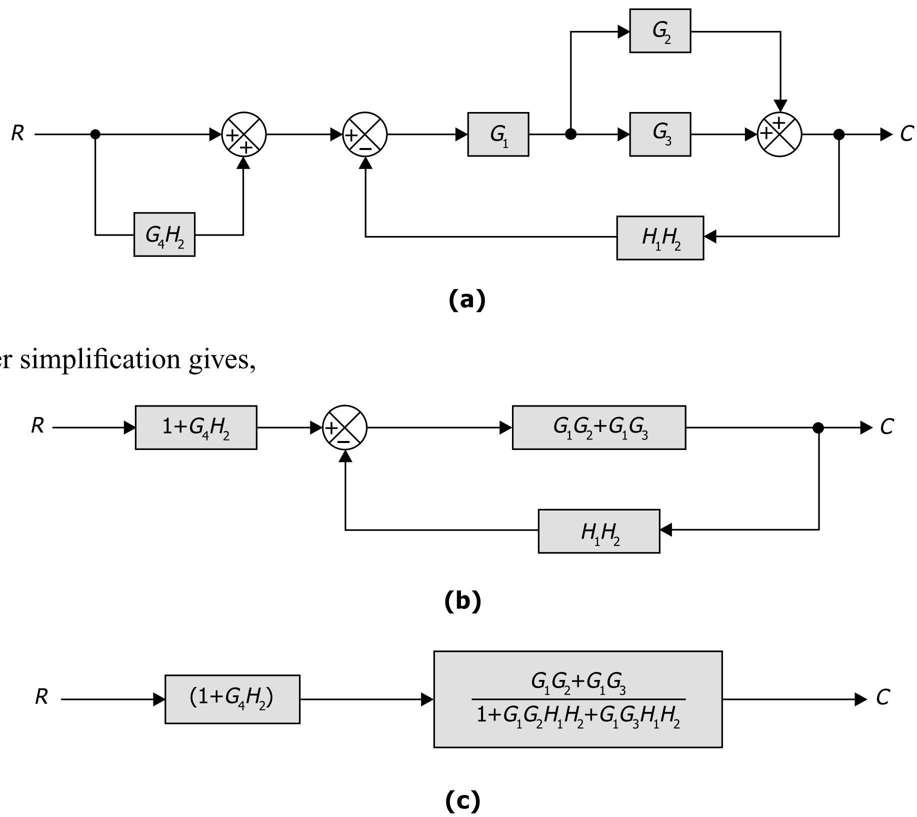


Fig. 3.53

- G_4H_2 and 1 are in parallel;
- G_2 and G_3 are in parallel;
- G_1 and $(G_2 + G_3)$ are in series.

Therefore, the transfer function is found as

$$\frac{C(s)}{R(s)} = \frac{(1+G_4H_2)(G_1G_2+G_1G_3)}{1+G_1G_2H_1H_2+G_1G_3H_1H_2}.$$

Example 3.11 Determine *C/R* of the system shown in Fig. 3.54 by the block diagram reduction technique.

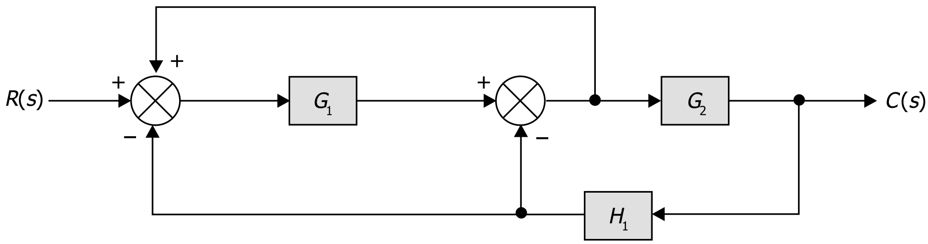


Fig. 3.54

Solution

Shifting the take off point ahead of G_2 to a position beyond G_2 and thereafter combining blocks G_2 and H_2 in feedback loop with G_1 in series,

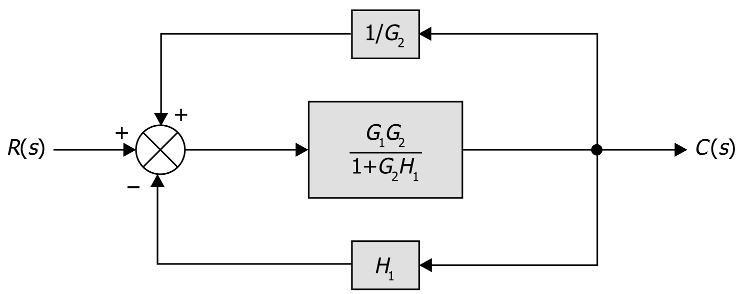


Fig. 3.55

Eliminating the negative feedback loop,

Eliminating the positive feedback loop we get the transfer function as,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2}{1+G_1G_2H_1+G_2H_1-G_1}.$$

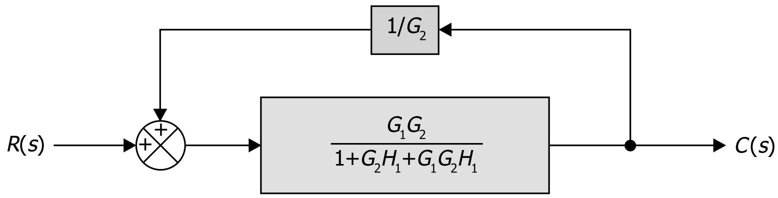


Fig. 3.56

Example 3.12 Using the block diagram reduction technique, reduce the following block diagram shown in Fig. 3.57 and determine the overall transfer function.

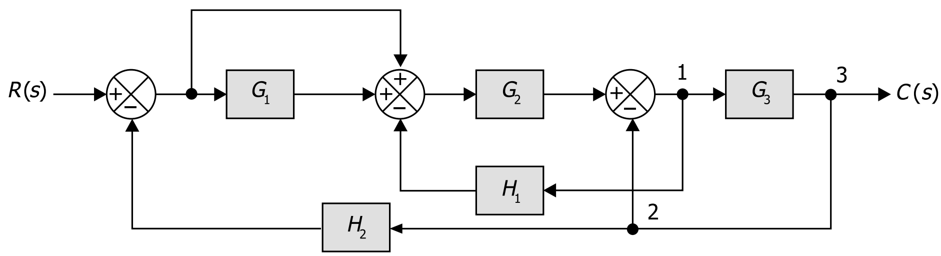


Fig. 3.57

Solution

Shift the take off point 1 to point 3, i.e. beyond G_3. The feedback element becomes $\dfrac{H_1}{G_3}$. Take point 2 to point 3. Transfer function of G_3 with unity feedback becomes $\dfrac{G_3}{1+G_3}$. This in series with G_2 becomes $\dfrac{G_2G_3}{1+G_3}$.

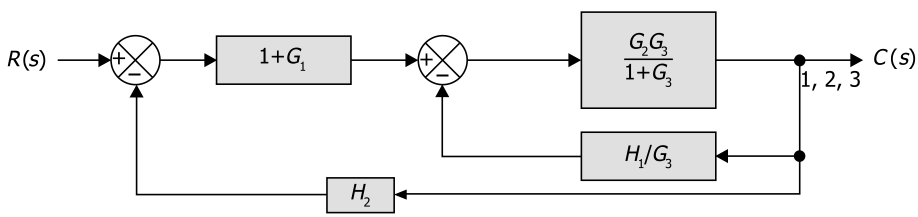


Fig. 3.58

Considering block $\dfrac{G_2G_3}{1+G_3}$ with feedback loop $\dfrac{H_1}{G_3}$ we get

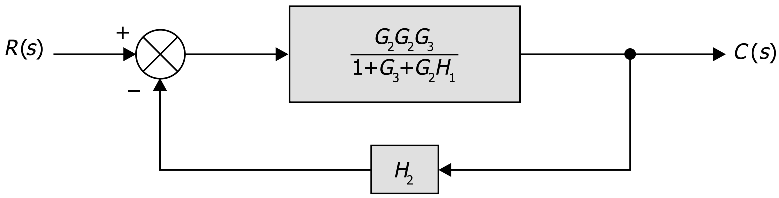


Fig. 3.59

With $\dfrac{G_1G_2G_3}{1+G_3+G_2H_1}$ and H_2 blocks in feedback loop, we get

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3}{1+G_3+G_2H_1+G_1G_2G_3H_2}$$

Example 3.13 Using the block diagram reduction technique, find the transfer function for the block diagram shown in Fig. 3.60.

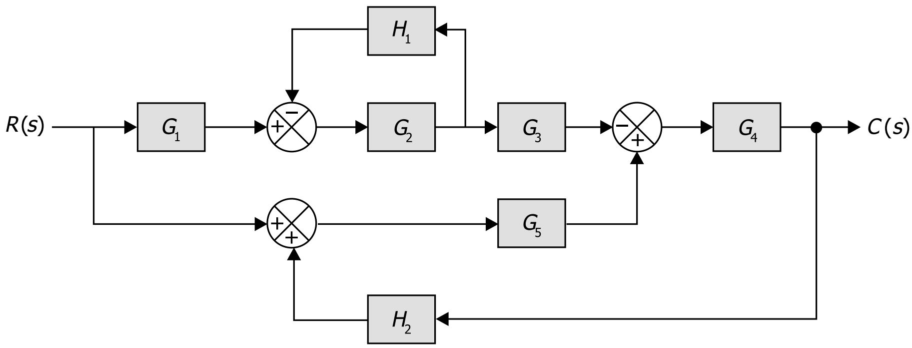
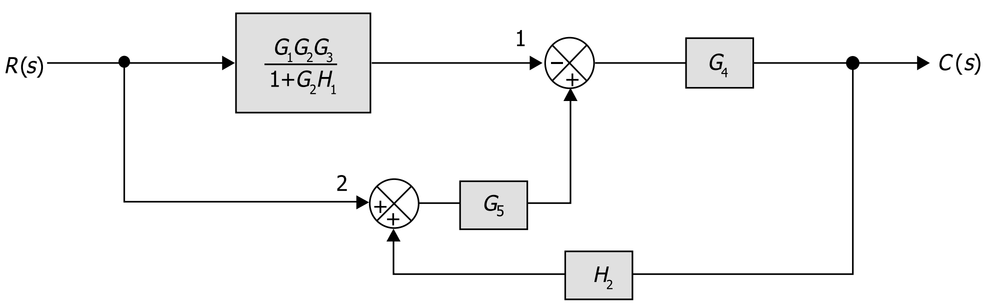


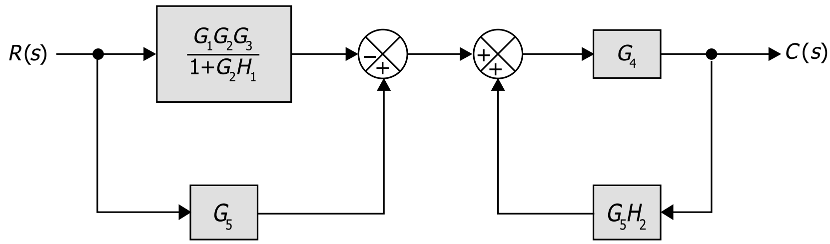
Fig. 3.60

Solution

Removing feedback loop with blocks G_2 and H_1 and then shifting the lower summing point beyond block G_5,



(a)



(b)

Fig. 3.61

The first loop is a parallel circuit with a negative sign on one element, and the second loop is a positive feedback loop. The diagram can further be reduced by calculating the transfer function of the two loops and then combining the two loops to get,

$$\frac{C(s)}{R(s)} = \left(\frac{-G_1G_2G_3}{1+G_2H_1} + G_5\right)\left(\frac{G_4}{1-G_4G_5H_2}\right)$$

$$= \frac{-G_1G_2G_3G_4 + G_4G_5 + G_2G_4G_5H_1}{1-G_4G_5H_2 - G_2G_4G_5H_1H_2 + G_2H_1}.$$

Example 3.14 Using the block diagram reduction technique, reduce the system shown in Fig 3.62 to the simplest possible form and find the transfer function.

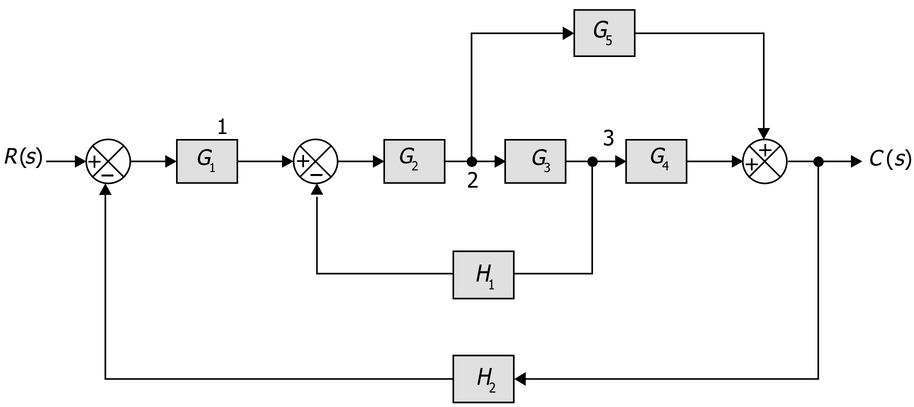


Fig. 3.62

Solution

Shifting take off point at 2 to point 3,

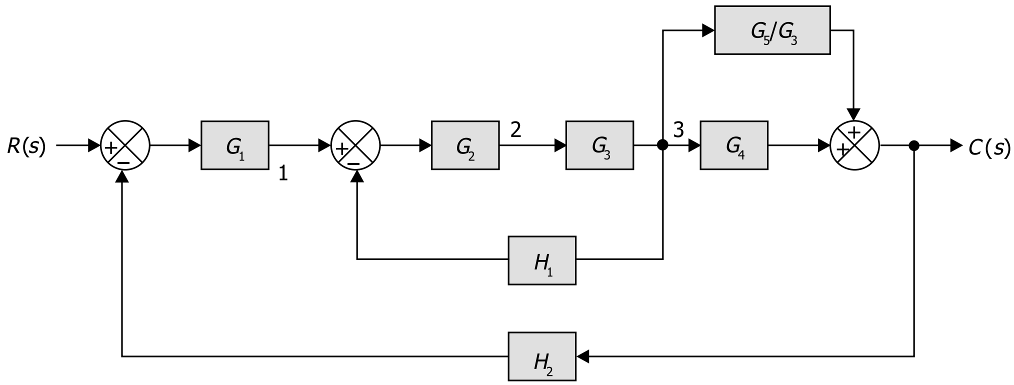


Fig. 3.63

Removing the feedback loop with G_2, G_3 and H_1 blocks,

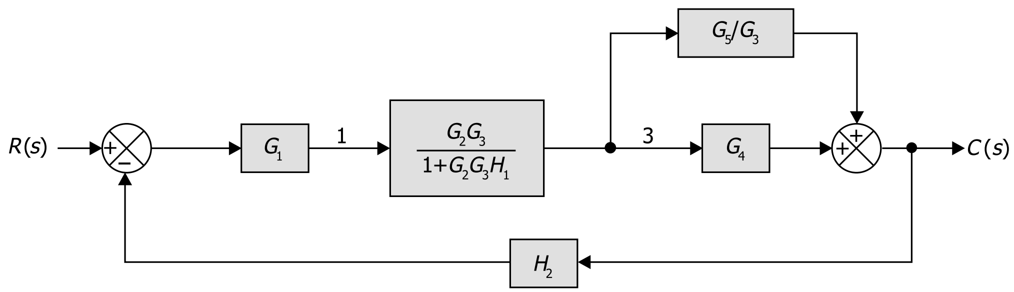


Fig. 3.64

Shifting take off Point at 3 to Point 1,

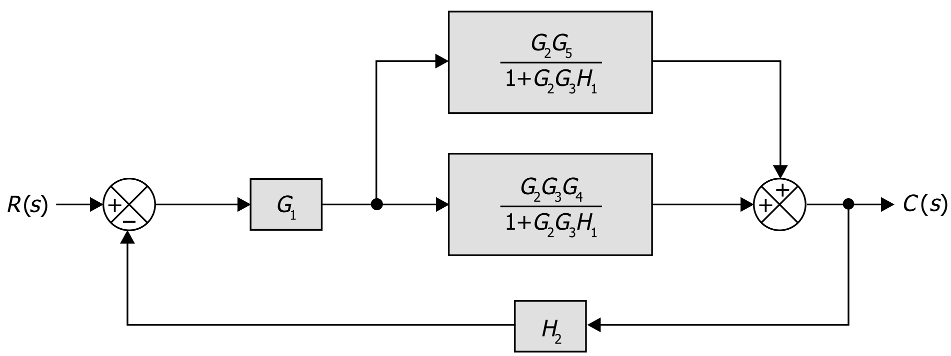


Fig. 3.65

Eliminating parallel blocks,

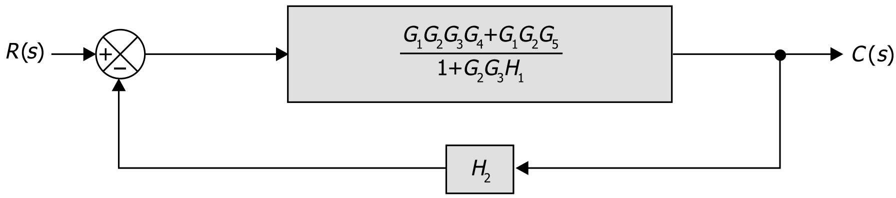


Fig. 3.66

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_1G_2G_5}{1 + G_2G_3H_1 + G_1G_2G_3G_4H_2 + G_1G_2G_5H_2}$$

Example 3.15 Determine the ratio $\frac{C(s)}{R(s)}$ for the multiple-loop system shown in Fig. 3.67.

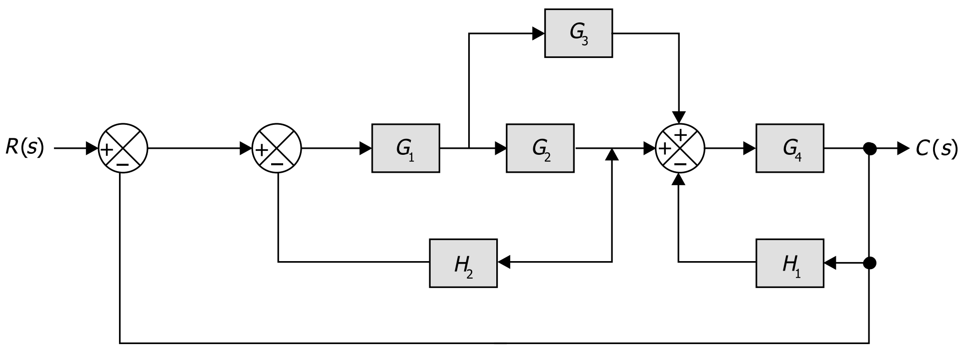


Fig. 3.67

Solution

Shifting take off point of feedback loop after G_2 to the point ahead of G_2 and eliminating feedback loop with G_4 and H_1, we get

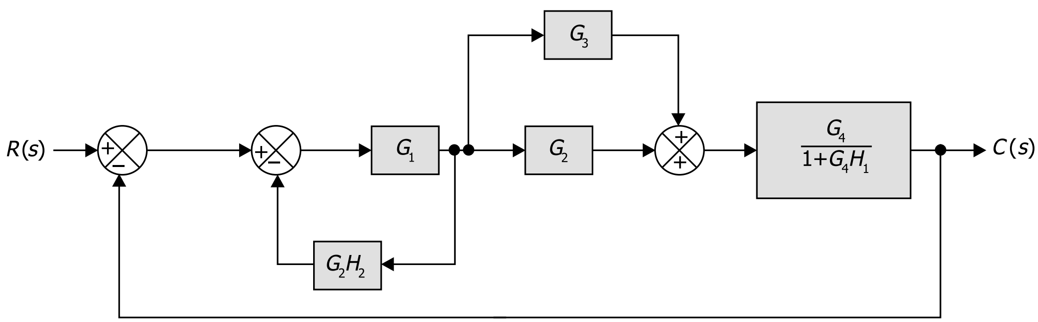


Fig. 3.68

Eliminating parallel blocks and the feedback loop, we get

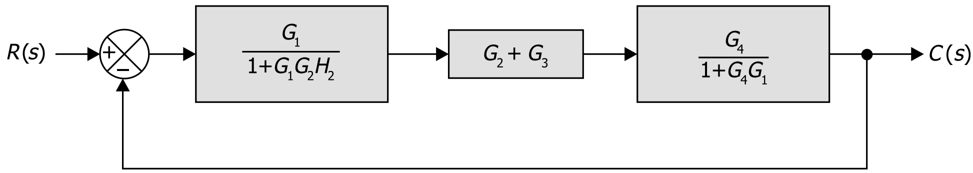


Fig. 3.69

Simplifying, the transfer function is determined as,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 + G_1G_2H_2 + G_4H_1 + G_1G_2G_4H_1H_2 + G_1G_2G_4 + G_1G_3G_4}$$

3.7 BLOCK DIAGRAM REPRESENTATION OF AN ELECTRICAL NETWORK

A system may be electrical, mechanical, or a combination of both having a number of elements or components interconnected to perform specific tasks. For drawing the block diagram of a system we have to first write the equations describing the dynamic behaviour of each component. Then, we have to take the Laplace transform of these equations, assuming zero initial conditions, and represent each Laplace transformed equation individually in block form. Finally, we have to assemble the elements into a complete block diagram. Let us consider a simple example of an electrical network.

Example 3.16 Draw the block diagram for the electrical network shown in Fig. 3.70. This an integrating circuit used in many practical applications.

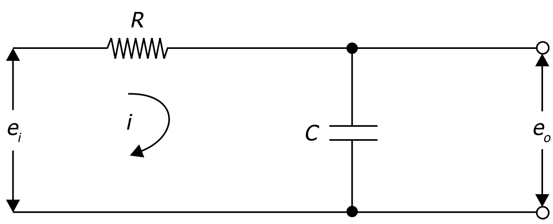


Fig. 3.70

Solution

i) The equations describing the system are

$$e_i = Ri + \frac{1}{C}\int idt$$

and $e_o = \dfrac{1}{C}\displaystyle\int idt.$

ii) Taking Laplace transform of above equations, we get

$$E_i(s) = RI(s) + \frac{1}{Cs} I(s) \qquad \text{....(3.8)}$$

and
$$E_o(s) = \frac{1}{Cs} I(s) \qquad \text{....(3.9)}$$

From equation (3.8),

$$E_i(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

or,
$$I(s) = \frac{E_i(s)}{R + \frac{1}{Cs}}$$

or,
$$I(s) = \frac{Cs\, E_i(s)}{1 + RCs}$$

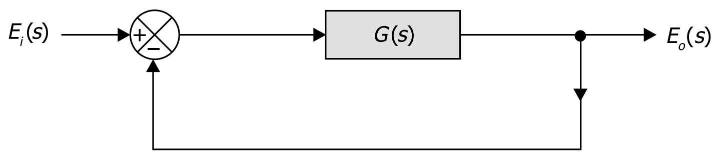


Fig. 3.71

Substituting the value of $I(s)$, in equation (3.9), we get

$$E_o(s) = \frac{1}{Cs} \frac{Cs\, E_i(s)}{(1 + RCs)}$$

Therefore,
$$\frac{E_o(s)}{E_i(s)} = \frac{1}{1 + RCs}$$

We have the standard form (Fig. 3.71) as

$$\frac{E_o(s)}{E_i(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

where $H(s) = 1$ (see Fig. 3.71). The transfer function is written as

$$T.F = \frac{G(s)}{1 + G(s)} = \frac{1}{1 + RCs}$$

or $$1 + G(s) = (1 + RCs)G(s)$$

or $$G(s) = \frac{1}{RCs}$$

Therefore the standard form is represented as shown in Fig. 3.72.

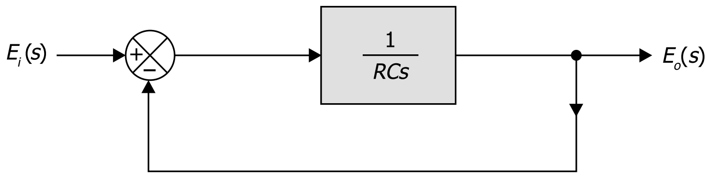


Fig. 3.72

From equations (3.8) and (3.9) we have

$$[E_i(s) - E_o(s)]\frac{1}{R} = I(s) \text{ and } E_o(s) = (1/Cs)I(s)$$

So Fig. 3.72 can be represented as shown in Fig. 3.73.

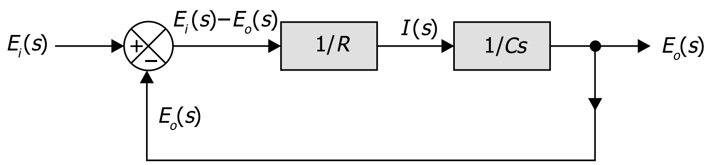


Fig. 3.73

3.8 BLOCK DIAGRAM REPRESENTATION OF COMPONENTS OF A SERVOMECHANISM

A servomechanism, also called a position control system, is a feedback control system and consists of a mechanism in which the output of the system may be some mechanical position, velocity or acceleration. Servomechanism and position control systems are synonymous.

In almost all servomechanism applications d.c. motor drives and gear mechanism are used. Block diagram representation and transfer function of these are discussed in the following sections.

3.8.1 Block Diagram of a DC Motor Drive

In servo applications, a DC motor is required to produce rapid acceleration from stand still (position of rest). Therefore, the physical requirements of such a motor are low inertia and high starting torque. Low inertia is attained with reduced armature diameter.

In control systems, DC motors are used in two different control modes:

a) Armature control mode with constant field current; and

b) Field control mode with fixed armature current.

These are discussed as follows.

1. Armature-controlled DC motor: In DC motors, we know that emf (E_b) is directly proportional to armature speed (N) and field flux (ϕ). Since, $E_b = \dfrac{\phi ZNP}{60A}$

or, $$E_b = k\phi N \qquad \text{where } k = \frac{ZP}{60A}$$

Also, $$E_b = V - I_a R_a$$

Where V = Terminal voltage

I_a = Armature current

R_a = Armature resistance

Thus, $$E_b = \frac{\phi ZNP}{60A} = K\phi N$$

Again, $$E_b = V - I_a R_a$$

Therefore, $$N = \frac{V - I_a R_a}{K\phi} \qquad \text{....(3.10)}$$

From equation (3.10), it can be seen that the speed of motor, N can be varied by changing V, i.e. the terminal voltage, or by changing the field flux, ϕ or by changing the armature circuit resistance

In armature control method, speed is varied by changing the armature voltage keeping the field flux constant. Fig. 3.74 shows an armature controlled DC motor. The armature resistance and inductance have been shown separately.

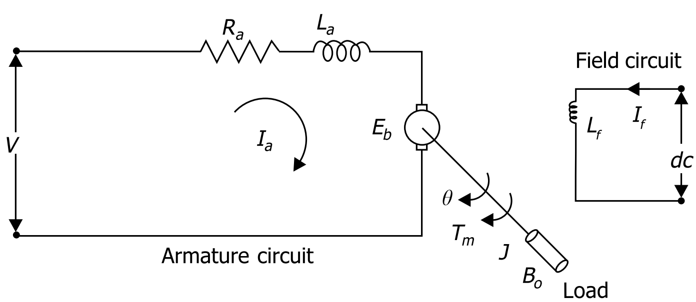


Fig. 3.74 *Armature controlled dc motor*

The motor has a rotating armature with load on its shaft having a moment of inertial J and a viscous friction coefficient B_0. Torque developed by the motor is T_m which causes rotation. The angular velocity is ω and angular acceleration is α. All these constants and variables of the motor drive are defined as

$V =$ applied armature voltage

$T_m =$ torque developed by the motor

$\theta =$ angular displacement of the motor shaft

$\omega = \dfrac{d\theta}{dt} =$ angular velocity of the motor shaft

$\omega = \dfrac{2\pi N}{60}$, where N is the number of revolution per minute

$\omega = 2\pi N$, if N is expressed in rps

$J =$ Equivalent moment of inertia of the rotor of the motor and that of the load referred to the motor shaft

$$\alpha = \frac{d^2\theta}{dt^2} = \text{angular acceleration}$$

Torque of the motor is due to armature current, I_a and the field flux ϕ. Again field flux is proportional to the field current, I_f.

Therefore,
$$T_m = K_1 \phi I_a$$
$$= K_1 K_f I_f I_a$$

If I_f is constant,

$$T_m = K_T I_a, \text{ where } K_T = K_1 K_f I_f \qquad \text{....(3.11)}$$

Back emf,
$$E_b = k\phi N = \frac{K\phi\omega}{2\pi} = \frac{KK_1 I_f}{2\pi}\frac{d\theta}{dt} = k_b \frac{d\theta}{dt} \qquad \text{....(3.12)}$$

Inertia torque $=$ Moment of inertia $\times$ Angular acceleration

Therefore,
$$T_i = J\frac{d^2\theta}{dt^2} \qquad \text{....(3.13)}$$

Torque due to bearing friction, windage friction and any other drag opposing rotation has been accounted for by an equivalent viscous friction coefficient B_0.

Friction torque $=$ Coefficient of viscous friction $\times$ Angular velocity

Therefore,
$$T_f = B_0\omega = B_0\frac{d\theta}{dt} \qquad \text{....(3.14)}$$

Torque developed by the motor is opposed by inertia torque and frictional torque. Thus, from Newton's second law of rotational motion,

$$T_i + T_f = T_m$$

or,
$$J\frac{d^2\theta}{dt^2} + B_0\frac{d\theta}{dt} = T_m = K_T I_a \quad \text{....(3.15)}$$

The differential equation involving quantities of the armature circuit can be written as

$$L_a\frac{dI_a}{dt} + R_a I_a + E_b = V \quad \text{....(3.16)}$$

Taking Laplace transform of equations (3.12), (3.15), and (3.16) respectively we get

$$E_b(s) = K_b s\,\theta(s) \quad \text{....(3.17)}$$

$$(Js^2 + B_0 s)\,\theta(s) = K_T I_a(s) = T_m(s) \quad \text{....(3.18)}$$

$$(L_a s + R_a)\,I_a(s) = V(s) - E_b(s) \quad \text{....(3.19)}$$

From equations (3.18) and (3.19),

$$[Js^2 + B_0 s]\theta(s) = K_T\frac{V(s) - E_b(s)}{L_a s + R_a}$$

Using equation (3.12) in the above,

or,
$$[Js^2 + B_0 s]\,[L_a s + R_a]\,\theta(s) = K_T V(s) - K_T K_b s\,\theta(s)$$

or,
$$[(Js^2 + B_0 s)\,(L_a s + R_a)]\,\theta(s) + K_T K_b s\,\theta(s) = K_T V(s)$$

$$\text{Transfer function} = \frac{\theta(s)}{V(s)} = \frac{K_T}{(Js^2 + B_0 s)(L_a s + R_a) + K_T K_b s}$$

$$= \frac{K_T}{s[(Js + B_0)(L_a s + R_a) + K_T K_b]}$$

Block diagram representation

From equation (3.19), we have
$$\frac{I_a(s)}{V(s) - E_b(s)} = \frac{1}{L_a s + R_a} \quad \text{....(3.20)}$$

From equation (3.18), we have
$$\frac{\theta(s)}{I_a(s)} = \frac{K_T}{s(Js + B_0)} \quad \text{....(3.21)}$$

Thus, from equation (3.21) we have the following block diagram

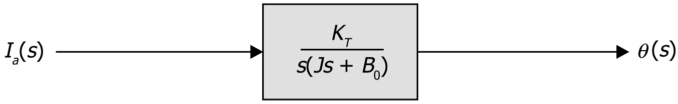


Fig. 3.75

and from equation (3.20), we have

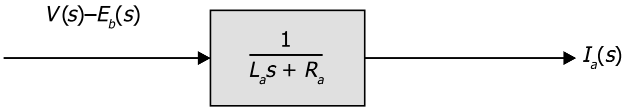


Fig. 3.76

With $V(s)$ as the input signal and $E_b(s)$ as the feedback signal, the comparator is represented as

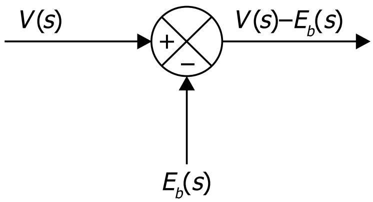
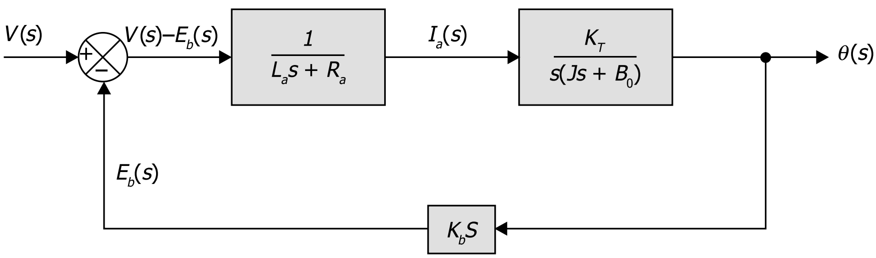


Fig. 3.77

Again from equation (3.17),

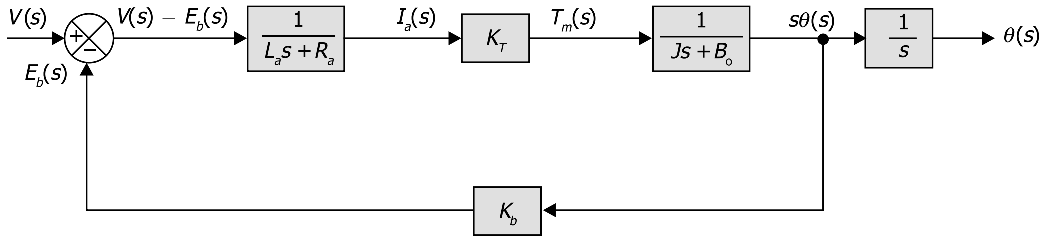
$$E_b(s) = K_b\, s\, \theta(s)$$

Thus, combining all the above blocks, the complete block diagram is represented as



(a)

Alternatively, the block diagram can be represented as



(b)

Fig. 3.78 *Block diagram representation of an armature controlled d.c. motor*

2. Field-controlled DC motor:

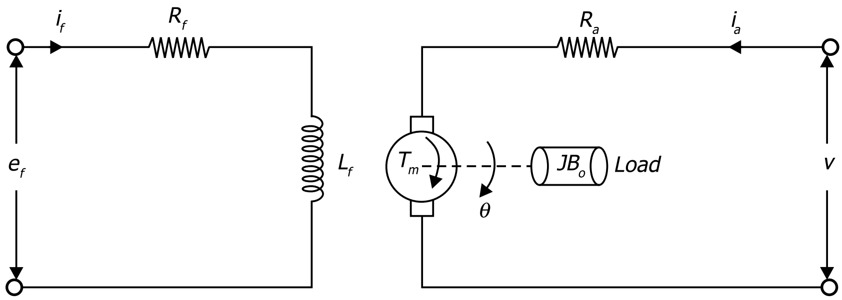


Fig. 3.79 *Field controlled dc motor*

Here

$$e_f = \text{Field voltage (control input)}$$

$$i_a = \text{Armature current (constant)}$$

$$T_m = K_1 \phi i_a = K_1 K_f i_f i_a = K i_f$$

where

$$K = K_1 K_f i_a$$

Also, the field circuit equation is,

$$e_f = L_f \frac{di_f}{dt} + R_f i_f \qquad \text{....(i)}$$

The torque equation is

$$J \frac{d^2\theta}{dt^2} + B_0 \frac{d\theta}{dt} = T_m = K i_f \qquad \text{....(ii)}$$

Taking Laplace transform of the above equations

$$(L_f s + R_f)\, I_f(s) = E_f(s) \qquad \ldots(3.22)$$

$$(Js^2 + B_0 s)\, \theta(s) = KI_f(s) \qquad \ldots(3.23)$$

or,
$$(Js^2 + B_0 s)\theta(s) = K \frac{E_f(s)}{L_f s + R_f} \qquad \text{[from equation (3.22)]}$$

$$T.F. = \frac{\theta(s)}{E_f(s)} = \frac{K}{s(Js + B_0)(L_f s + R_f)}$$

From equation (3.22),
$$\frac{I_f(s)}{E_f(s)} = \frac{1}{L_f s + R_f} \qquad \ldots(3.24)$$

From equation (3.23),
$$\frac{\theta(s)}{I_f(s)} = \frac{K}{Js^2 + B_0 s} \qquad \ldots(3.25)$$

Using equations (3.24) and (3.25), the block diagrams are developed as given in Figs. 3.80 and 3.81.

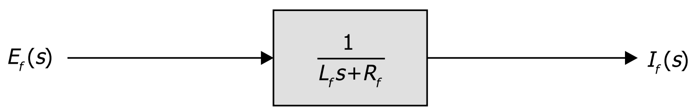


Fig. 3.80

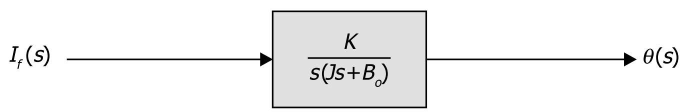


Fig. 3.81

Combining the above two blocks we get,

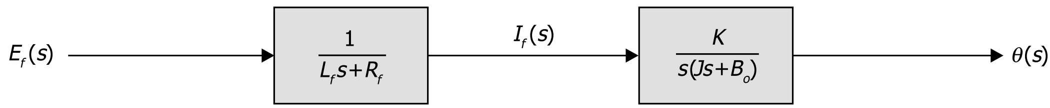


Fig. 3.82 Block diagram representation of a field-controlled d.c. motor

3.8.2 Block Diagram Representation of a Gear Train

Gear trains are made with a number of gears of different number of teeth connected together for power transmission. Power is a product of torque and speed. If speed is reduced, torque increases and vice-versa, which we experience while driving motor vehicles.

Gear trains are often used in servomechanism to reduce speed, to increase torque or to obtain the most efficient power transfer by matching the driving member to the driven load. Thus, in mechanical systems, gear trains act as matching devices like transformers in electrical systems. Fig. 3.83 shows a motor, driving a load through a gear train which consists of two gears meshed together. The gear with N_1 teeth is called the primary gear (analogous to primary winding of a transformer) and the gear with N_2 teeth is called the secondary gear (analogous to secondary winding of a transformer).

T_1 = Load torque on gear 1 due to the rest of the gear train; J_1 and J_2 are the moment of inertia; and B_1 and B_2 are viscous friction of the systems as shown in Fig. 3.83.

T_2 = Torque transmitted to gear 2.

The torque equations are

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + T_1 = T_m \qquad \text{....(3.26)}$$

$$J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + T_L = T_2 \qquad \text{....(3.27)}$$

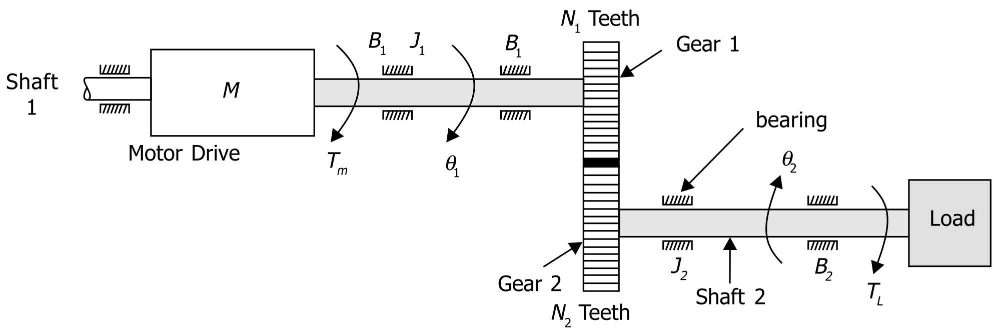


Fig. 3.83 *A motor driving a load through a gear train*

Since the linear distance traveled along the surface of each gear is the same,

$$\therefore \qquad \theta_1 r_1 = \theta_2 r_2$$

where r_1 and r_2 are the radius of gear 1 and gear 2 respectively

The number of teeth on the gear surface being proportional to the gear radius,

$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

In an ideal case, i.e. when there is no slip, work done by gear 1 = work done by gear 2 that is,

$$T_1\theta_1 = T_2\theta_2$$

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = n.$$

If the ratio of N_1 and N_2, i.e. n is less than 1, then the gear train reduces the speed but increases the torque.

$\therefore$
$$T_1 = T_2 \frac{N_1}{N_2}.$$

Substituting this value of T_1 in equation (3.26),

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + T_2 \frac{N_1}{N_2} = T_m \qquad \text{....(3.26A)}$$

Putting the value of T_2 from equation (3.27) in equation (3.26A),

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \frac{N_1}{N_2}(J_2\ddot{\theta}_2 + B_1\dot{\theta}_2 + T_L) = T_m \qquad \text{....(3.28)}$$

We had
$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

Differentiating we have,
$$\frac{\ddot{\theta}_2}{\ddot{\theta}_1} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{N_1}{N_2}$$

Therefore,

$$\left[J_1\ddot{\theta}_1 + J_2\left(\frac{N_1}{N_2}\right)^2 \ddot{\theta}_1\right] + \left[B_1\dot{\theta}_1 + B_2\left(\frac{N_1}{N_2}\right)^2 \dot{\theta}_1\right] + \left(\frac{N_1}{N_2}\right)T_L = T_m$$

or,
$$\ddot{\theta}_1\left[J_1 + J_2\left(\frac{N_1}{N_2}\right)^2\right] + \dot{\theta}_1\left[B_1 + B_2\left(\frac{N_1}{N_2}\right)^2\right] + \left(\frac{N_1}{N_2}\right)T_L = T_m \qquad \text{....(3.29)}$$

or,
$$J_{1eq}\ddot{\theta}_1 + B_{1eq}\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)T_L = T_m \qquad \text{....(3.30)}$$

where the equivalent moment of inertia and viscous friction of the gear train referred to shaft-1 are respectively,

$$J_{1eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2$$

and $$B_{1eq} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2$$

Similarly, we can express,

$$J_{2eq}\ddot{\theta}_2 + B_{2eq}\dot{\theta}_2 + T_L = T_m$$

where $$J_{2eq} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1$$

$$B_{2eq} = B_2 + \left(\frac{N_2}{N_1}\right)^2 B_1$$

Taking Laplace transform of equation (3.30)

$$J_{1eq}s^2\theta_1(s) + B_1 s\theta_1(s) = T_m(s) - \left(\frac{N_1}{N_2}\right)T_L(s)$$

or, $$J_{1eq}s^2\theta_1(s) + (B_{1eq}s)\theta_1(s) = T_m(s) - nT_L(s)$$

where, $$n = \frac{N_1}{N_2}$$

or, $$(s^2 J_{1eq} + B_{1eq}s)\theta_1(s) = T_m(s) - nT_L(s)$$

$\therefore$ $$\theta_1(s) = \frac{T_m(s) - nT_L(s)}{s(sJ_{1eq} + B_{1eq})}.$$

Block diagram representation is shown as

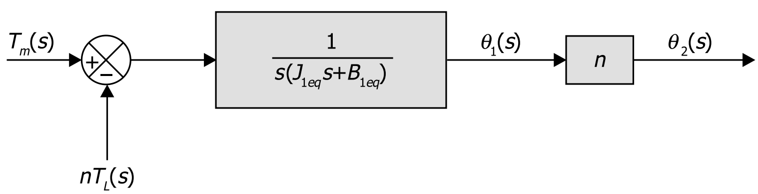


Fig. 3.84 *Block diagram representation of a gear train*

Example 3.17 A DC motor has been shown schematically in Fig. 3.85. L and R represent the inductance and resistance of the motor's armature circuit, and the voltage E_b represents the generated back emf which is proportional to the shaft velocity $\frac{d\theta}{dt}$. The torque T

developed by the motor is proportional to the armature current i. The inertia J represents the combined inertia of the motor armature and the load, and B is the total viscous friction acting on the output shaft. Determine the transfer function between the input voltage V and the angular position θ of the output shaft.

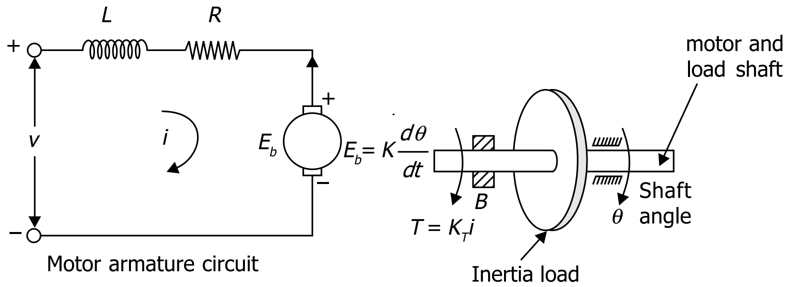


Fig. 3.85

Solution

The differential equations of the motor armature circuit and the inertia load are written as follows

$$V = iR + L\frac{di}{dt} + E_b$$

i.e. $$iR + L\frac{di}{dt} = V - E_b = V - K\frac{d\theta}{dt} \quad \ldots.(a)$$

and $$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_T i \quad \ldots.(b)$$

Taking the Laplace transform of the above equations,

$$(R + sL)I(s) = V(s) - Ks\theta(s)$$

and $$(Js^2 + Bs)\theta(s) = K_T I(s)$$

From the above two equations,

$$(JS^2 + BS)\theta(s) = K_T\frac{[V(s) - Ks\theta(s)]}{(Ls + R)}$$

$$\theta(s)\left[(Js^2 + Bs)(Ls + R) + KK_T s\right] = K_T V(s)$$

$$T.F = \frac{\theta(s)}{V(s)} = \frac{K_T}{(Js^2 + Bs)(Ls + R) + KK_T s}$$

3.8.3 Block Diagram of a Servomechanism or a Position Control System

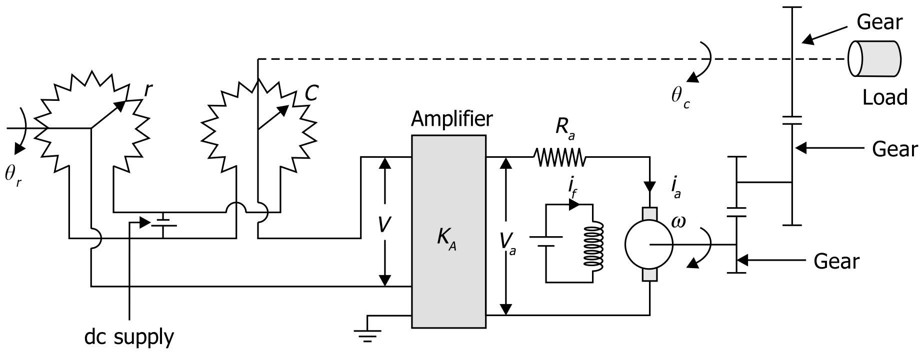


Fig. 3.86 *Positional servomechanism*

The servomechanism shown in Fig. 3.86 is used to control the position of the mechanical load in accordance with the reference position. A pair of potentiometers forms an error-measuring device which produces an error voltage of $V = k_p\,(r - c)$ with respect to input and output angular positions of r and c respectively of potentiometers' arms, where k_p is a constant of proportionality and $e = (r - c)$ is the error signal. An amplifier of gain k_A with high input impedance is used to feed an amplified voltage of $k_A V$ to the armature circuit of a DC servomotor, whose field is excited by a fixed voltage source. The armature inductance and low output impedance have been neglected as they are usually small. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero.

The complete block diagram for the system can be represented block by block using transfer functions for each block as has been shown in Fig. 3.87.

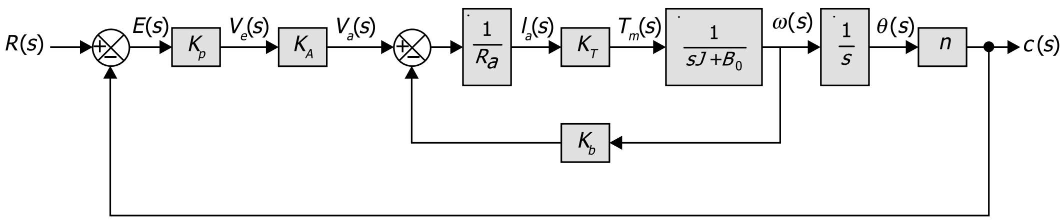


Fig. 3.87 *Block diagram of a position control system of a positional servomechanism*

The transfer function of the above block diagram is given by

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$

$$B = B_0 + \frac{K_b K_T}{R_a}$$

and $$K = \frac{K_p K_A K_T n}{R_a}$$

where K is a constant, J and B_0 are the moment of inertia and viscous friction coefficient respectively, referred to motor shaft for the combination of motor, load and gear train with gear ratio n. The quantities J, B and K when multiplied by $1/n^2$, produces inertia, viscous friction coefficient, and a constant referred to the output shaft.

$C(s)/R(s)$ can also be written as

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{Js^2 + Bs}}{1 + \dfrac{K}{Js^2 + Bs}} = \frac{G(s)}{1 + G(s)} \quad \text{as } H(s) = 1$$

The block diagram of diagram of Fig. 3.87 can, therefore, be simplified as shown in Fig. 3.88.

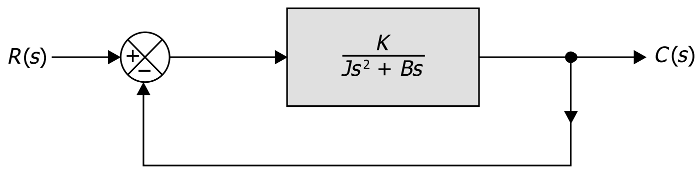


Fig. 3.88 Simplified block diagram of the system shown in Fig 3.87

Example 3.18 For the positional servomechanism shown in Fig. 3.89, the input to the system is the reference shaft position and the system output is the output shaft position. The numerical values for the system constants are as follows

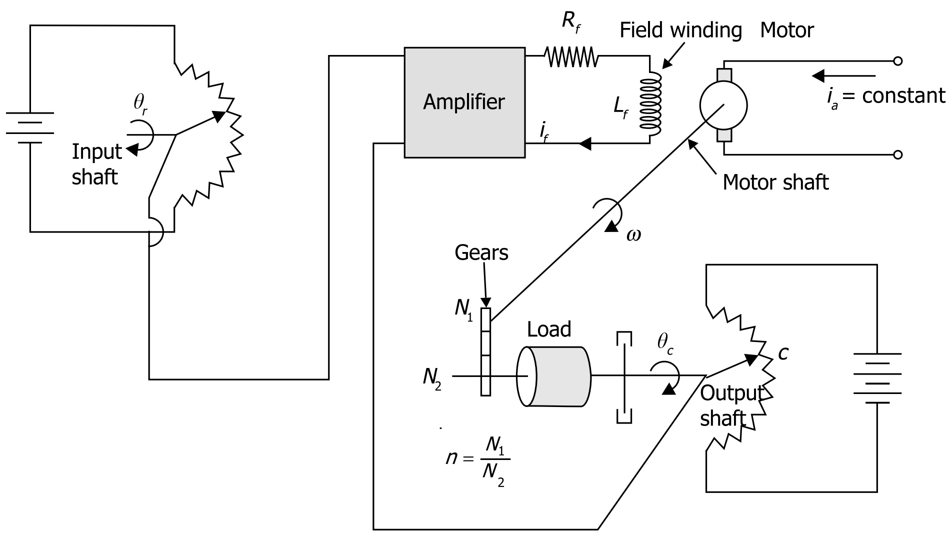


Fig. 3.89 Positional servomechanism

r = angular displacement of the reference input shaft, radians
c = angular displacement of the output shaft, radians
ω = angular displacement of the motor shaft, radians
K_1 = gain of the potentiometer error detector = $\frac{24}{\pi}$ volts/rad
K_p = amplifier gain = 10 V
R_f = field winding resistance = 2 ohms
L_f = field winding inductance = 0.1 henry
i_f = field winding current, amperes
e_f = applied field voltage, volts
K = motor torque constant = 0.05 Newton-m/amperes
n = gear ratio = 1/10
J = equivalent moment of inertia of the motor and load referred to the motor
B = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft = 0.02 Newton-m/rad/sec.

Draw a block diagram of the system showing the transfer function of each block. Finally, simplify the block diagram.

Solution

The equations describing the system dynamics are as follows.
For the potentiometric error detector

$$E(s) = K_1[R(s) - C(s)] = \frac{24}{\pi}[R(s) - C(s)] \qquad \text{....(3.31)}$$

For the amplifier

$$E_f(s) = K_p E(s) = 10E(s) \qquad \text{....(3.32)}$$

For field controlled d.c. motor (refer Fig. 3.82)

$$\frac{\theta(s)}{E_f(s)} = \frac{K}{s(L_f s + R_f)(Js + B)}$$

$$\frac{\theta(s)}{E_f(s)} = \frac{K_m}{s(T_f s + 1)(T_m s + 1)}$$

where,

$$K_m = \frac{K}{R_f B} = \frac{0.05}{(2)(0.02)}$$

$$= 1.25 \text{ rad/volt-sec.}$$

$$T_f = \frac{L_f}{R_f} = \frac{0.1}{2} = 0.05 \text{ sec.}$$

$$\mathrm{T}_m = \frac{J}{B} = \frac{0.02}{0.02} = 1 \text{ sec.}$$

$$\therefore \qquad \frac{\theta(s)}{E_f(s)} = \frac{1.25}{s(0.05s+1)(s+1)} \qquad \text{....(3.33)}$$

The block diagram of the system is drawn in Fig. 3.90 using equations (3.31), (3.32) and (3.33).

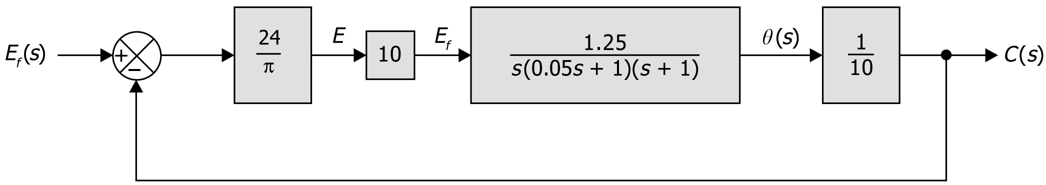


Fig. 3.90

The simplified block diagram of the above system is shown in Fig. 3.91 in general form as

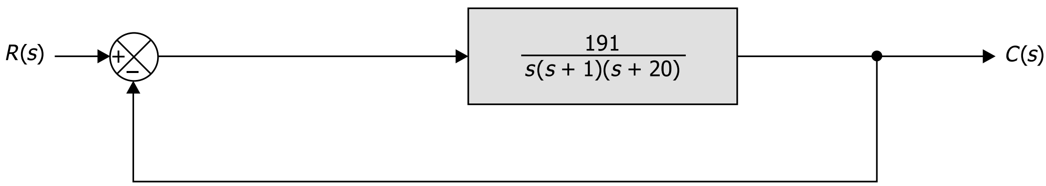


Fig. 3.91 *Block diagram of the system shown in Fig. 3.89*

REVIEW QUESTIONS

3.1 Determine the transfer function $C(s)/R(s)$ for the block diagram in Fig. 3.92

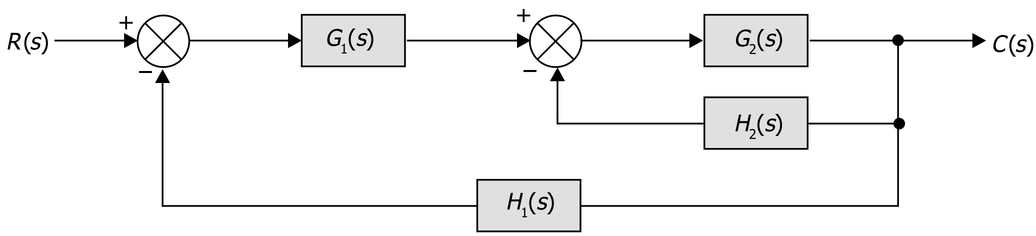


Fig. 3.92

3.2 Obtain the overall transfer function relating $C(s)$ and $R(s)$ for the block diagram shown in Fig. 3.93.

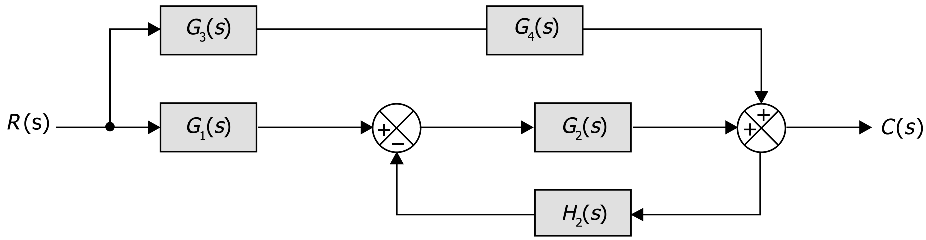


Fig. 3.93

3.3 Simplify the block diagram shown in Fig. 3.94 and find the transfer function $C(s)/R(s)$.

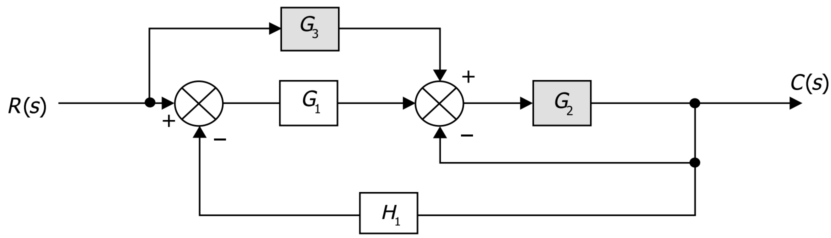


Fig. 3.94

3.4 Obtain the closed-loop transfer function $C(s)/R(s)$ after simplifying the block diagram shown in Fig. 3.95.

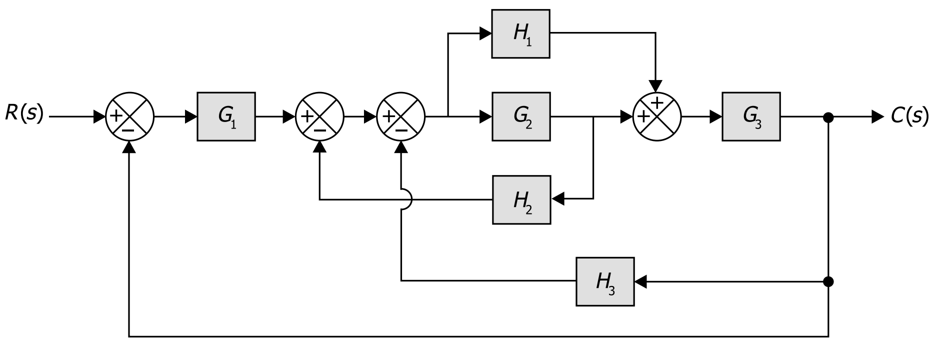


Fig. 3.95

3.5 Obtain the transfer function relation $V_2(s)$ and $V_1(s)$ and represent in the form of a block diagram for the differentiating circuit shown in Fig. 3.96.

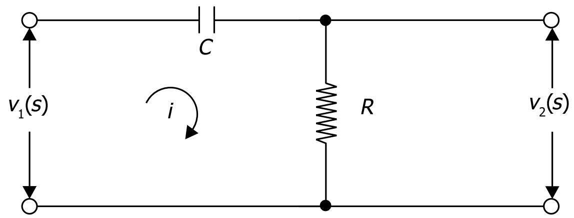


Fig. 3.96

3.6 A system block diagram is shown in Fig. 3.97. You are to do the following:

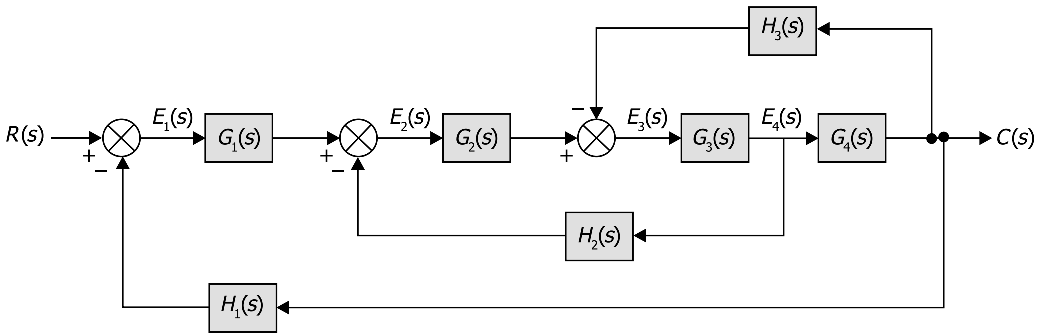


Fig. 3.97

i) represent the system, by block diagram reduction technique, in the form given below and find $G(s)$ and $H(s)$.

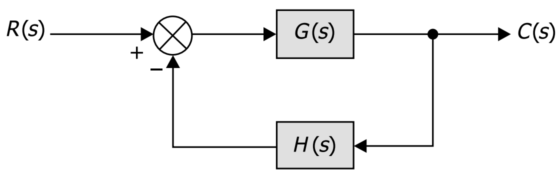


Fig. 3.98

ii) for the same system find $G_{eq}(s)$ for an equivalent unity-feedback system of the form shown below.

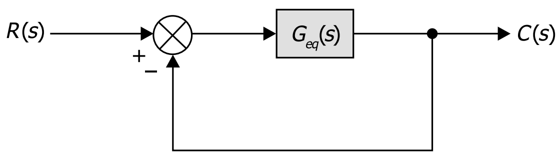


Fig. 3.99

3.7 Find the closed-loop transfer function of the system shown below by block diagram reduction technique.

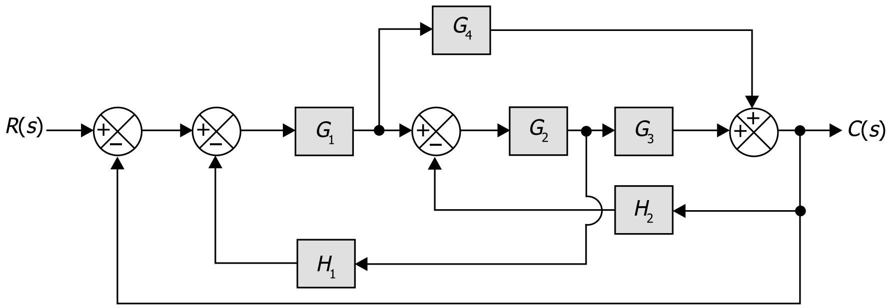


Fig. 3.100

3.8 Find the transfer function of the system represented by block diagram as shown below.

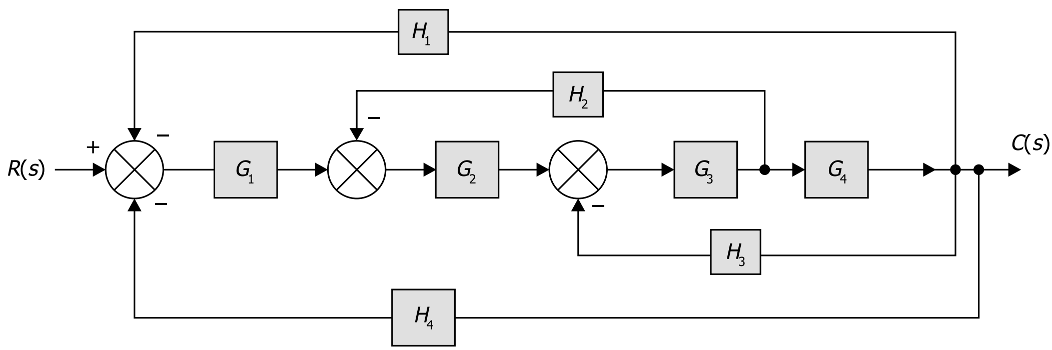


Fig. 3.101

3.9 Draw the block diagram and derive the transfer function of an armature controlled dc motor shown below.

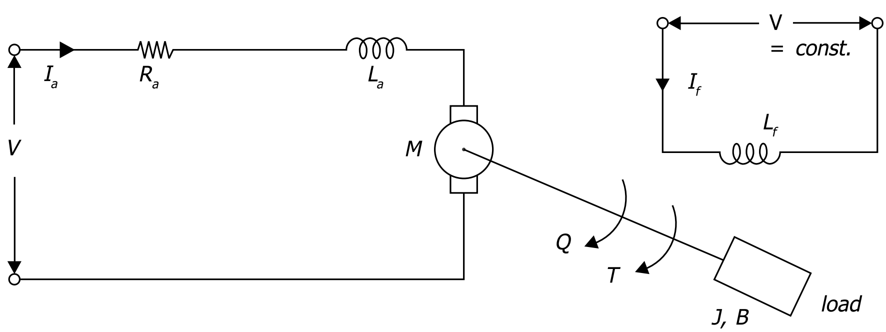


Fig. 3.102

3.10 Determine the transfer function $C(s)/R(s)$ for the system as shown below.

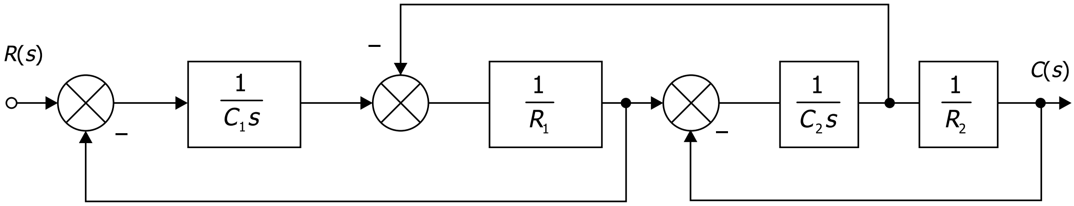


Fig. 3.103

3.11 Simplify the block diagram shown below and determine the transfer function.

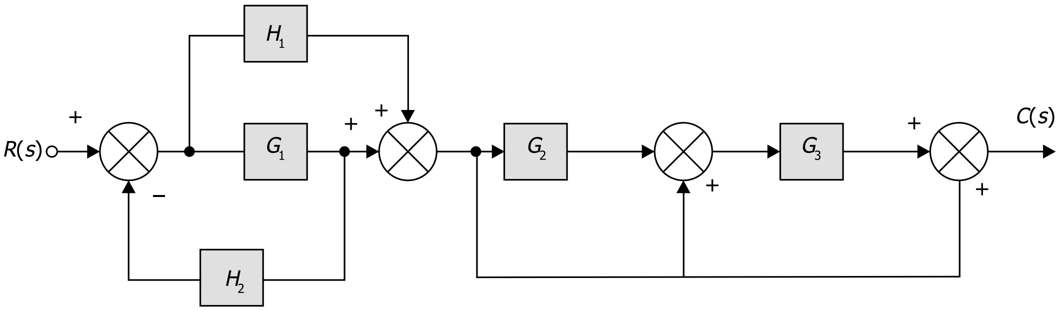


Fig. 3.104

3.12 Determine the transfer function of the system represented by the block diagram shown below.

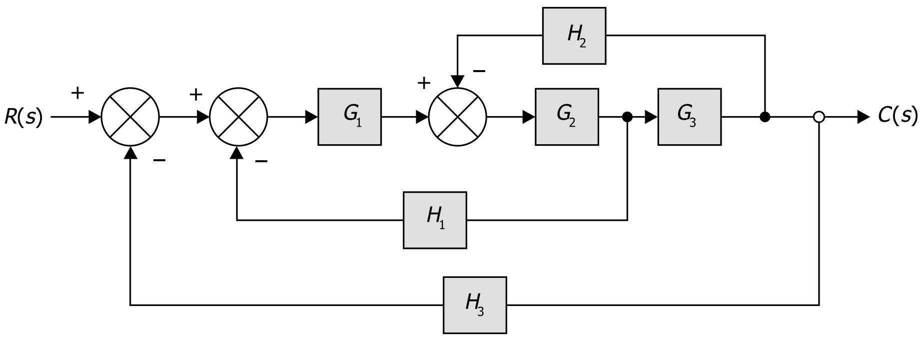


Fig. 3.105

4

MODELLING A CONTROL SYSTEM—SIGNAL FLOW GRAPH

4.1 INTRODUCTION

Block diagrams are very convenient in representing control systems. However, for complicated systems, the block diagram reduction approach for arriving at the transfer function relating the input and output variables is tedious and time consuming. An alternative approach is that of the signal flow graph (SFG) developed by S. J. Mason. A signal flow graph does not require any reduction process because of the availability of a flow graph gain formula which relates the input and output system variables.

Definition: A signal flow graph is a graphical representation of the relationship between the variables of a set of linear algebraic equations. It consists of a network in which nodes representing each of the system variables are connected by directed branches.

The meaning of the following terms need to be understood before proceeding further.

Node: It represents a system variable which is equal to the sum of all incoming signals at the node. Outgoing signals from the node do not affect the value of the node variable. Fig. 4.1(b) shows a signal flow graph. Points designated by R, E, and C are the nodes of this signal flow graph (SFG).

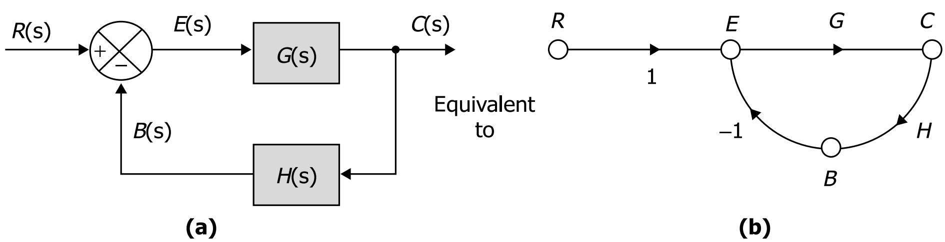


Fig. 4.1 *Representation of a control system: (a) Block diagram form and (b) equivalent signal flow graph*

Branch: A signal travels along a branch from one node to another in the direction indicated by the branch arrow and, in the process, gets multiplied by the gain or the transmittance. For example, the signal reaching the node *C* from the node *E* is given by *GE*, where *G* is the branch transmittance and the branch is directed form node *E* to node *C* (see Fig. 4.1).

Input Node or Source: It is a node with only outgoing branches; for example, *R* in Fig. 4.1 is an input node.

Output Node or Sink: It is a node with only incoming branches. However, this condition is not always met. An additional branch with unity gain may be introduced in order to meet the specified condition; for example, node *C* in Fig. 4.1 has one outgoing branch. However, after introducing an additional branch with unit transmittance as shown in Fig. 4.2, the node becomes an output node.

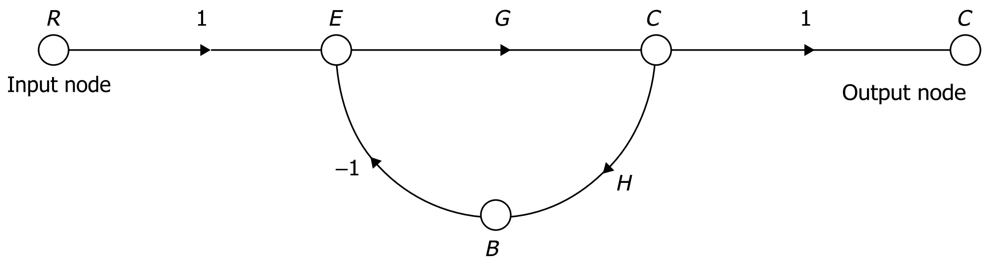


Fig. 4.2 *Input node and output node illustrated*

Path: It is the traversal of connected branches in the direction of the branch arrows such that no node is traversed more than once.

Forward Path: It is a path from the input node to the output node, when no node is encountered twice. *R*-*E*-*C* is a forward path.

Forward Path Gain: It is the product of branch gains in the forward path; for example, the forward path gain of the path *R*-*E*-*C* in Fig. 4.2 is *G*.

Loop: It is a path which originates and terminates at the same node; for example, *E*-*C*-*B*-*E* is a loop.

Loop Gain: It is the product of the branch gains encountered in traversing the loop, for example, the loop gain of the loop *E*-*C*-*B*-*E* in Fig. 4.2 is -*GH*.

Non-touching Loops: Loops are said to be non-touching if they do not possess any common node.

4.2 CONSTRUCTION OF SIGNAL FLOW GRAPH

4.2.1 Steps Followed in Drawing SFG

The equations, describing a system, are used to draw the signal flow graph. Consider a system described by the following set of equations. The input node is x_1 and the output node is x_2.

$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5 \qquad \text{....(4.1)}$$

$$x_3 = a_{23}x_2 \qquad \text{....(4.2)}$$

$$x_4 = a_{34}x_3 + a_{44}x_4 \qquad \text{....(4.3)}$$

$$x_5 = a_{35}x_3 + a_{45}x_4 + a_{55}x_5 \qquad \text{....(4.4)}$$

Here, the input variable is x_1 and output variable is x_5.

First, locate the nodes of the system, that is, x_1, x_2, x_3, x_4, x_5, as in (a) below and then draw SFG of the given equations as in (b), (c), (d) and (e).

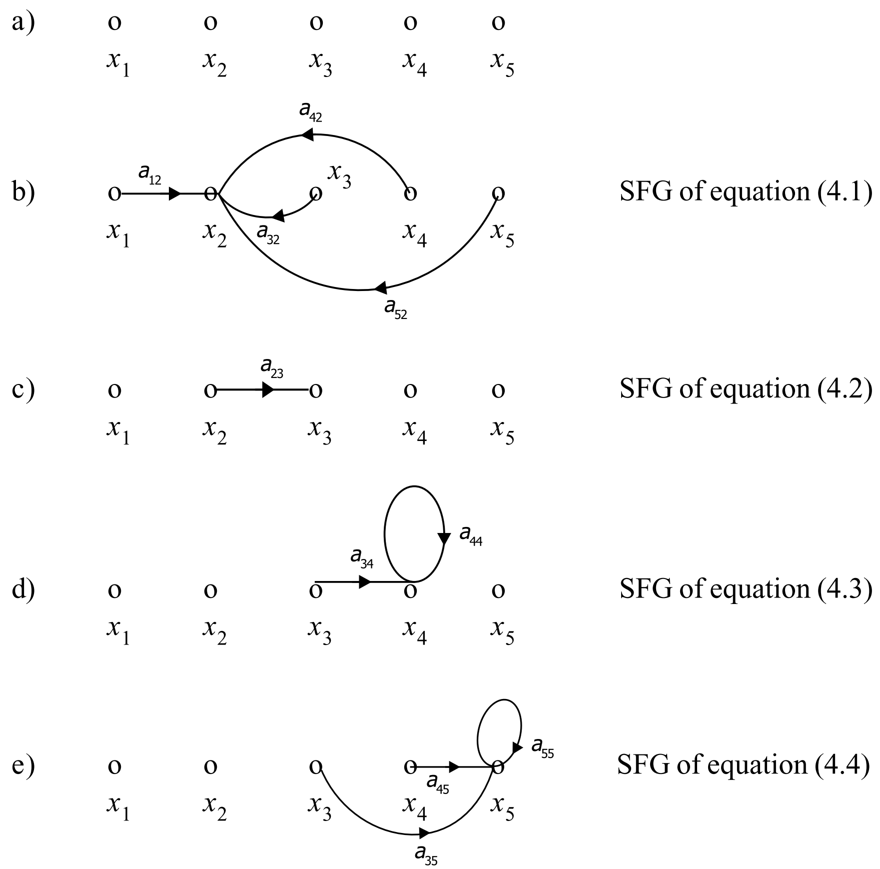


Fig. 4.3 *Development of signal flow graph for a set of equations describing a system*

The overall SFG can be drawn by adding the individual flow graphs of equations (4.1), (4.2), (4.3) and (4.4) as shown in Fig. 4.3. The overall SFG is shown in Fig. 4.4.

The overall gain from input to output may be obtained by using Mason's gain formula.

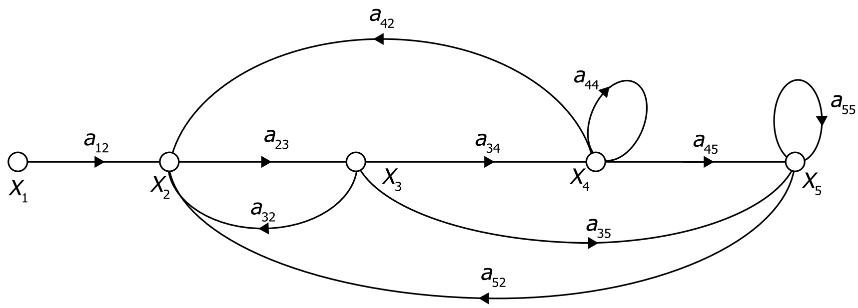


Fig. 4.4 *Complete signal flow graph of a system*

4.2.2 Mason's Gain Formula

According to Mason's gain formula, the overall gain T is expressed as

$$T = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

where P_k = path gain of the k^{th} forward path.

Δ = determinant of the path

= 1 – (sum of loop gains of all individual loops)

+ (sum of gain products of all possible combinations of two non-touching loops)

– (sum of gain products of all possible combinations of three non-touching loops)

+ ....

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + ...$$

Δ_k = the value of Δ for that part of the graph non touching the k^{th} forward path.

T = overall gain of the system.

P_{mr} = gain product of m^{th} possible combination of r non-touching loops.

Example 4.1

a) Obtain the transfer function of the system shown in Fig. 4.5 by block diagram reduction technique and also using signal flow graph

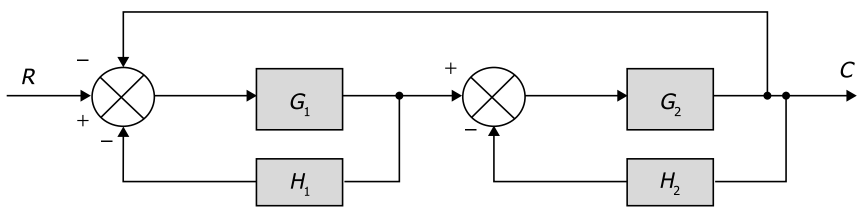


Fig. 4.5 *Block diagram of a control system*

b) Draw the signal flow graph for the following set of linear equations:

$$3y_1 + y_2 + 5y_3 = 0$$
$$y_1 + 2y_2 - 4y_3 = 0$$
$$-y_2 - y_3 = 0.$$

Solution

a) Using block diagram reduction technique the transfer function of system is determined as shown below.

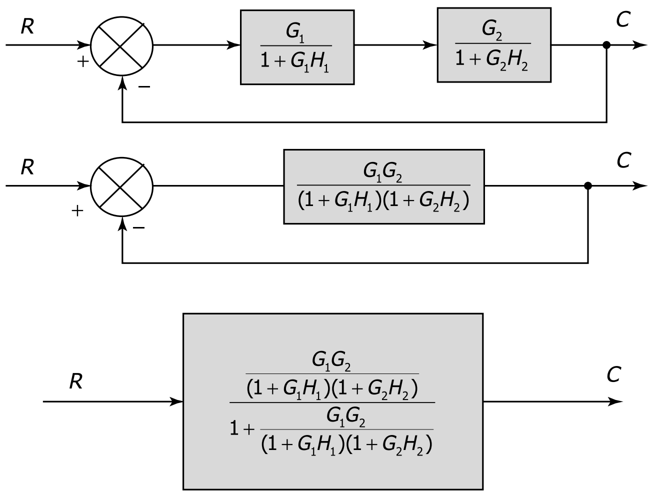


Fig. 4.6 *Determination of transfer function of a system using block diagram reduction technique*

$$\text{Transfer function} = \frac{G_1G_2}{(1+G_1H_1)(1+G_2H_2)+G_1G_2}$$

The signal flow graph of the system is drawn below.

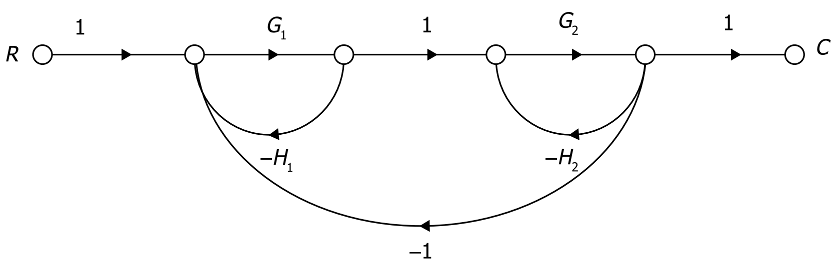
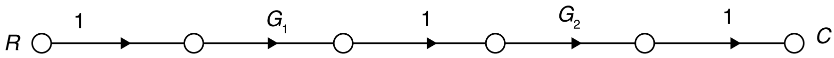


Fig. 4.7 *SFG of the system shown in Fig. 4.5*

1. The forward path is only one



The gain is, $P_1 = G_1G_2$

2. There are three individual loops; the loop gains are

$P_{11} = -G_1H_1; P_{12} = -G_2H_2; P_{13} = -G_1G_2$

3. There are two non-touching loops with gain

$P_{12} = (-G_1H_1)(-G_2H_2) = G_1G_2H_2H_2$

4. There is one forward path and all the loops touch the forward path

Therefore, $\Delta_1 = 1$

5. $\Delta = 1 -$ (sum of loop gains of all individual loops) + (sum of gain products of all possible combinations of two non-touching loops) $- \ldots$

$= 1 - (-G_1H_1 - G_2H_2 - G_1G_2) + G_1H_1G_2H_2$

$$\text{Transfer function} = \frac{P_1\Delta_1}{\Delta} = \frac{G_1G_2}{1+G_1H_1+G_2H_2+G_1G_2+G_1G_2H_1H_2} = \frac{G_1G_2}{(1+G_1H_1)(1+G_2H_2)+G_1G_2}.$$

b) To draw the signal flow graph for the given set of linear equations, first rearrange these equations as

$$3y_1 + y_2 + 5y_3 = 0 \quad \text{or,} \quad y_1 = -\frac{1}{3}y_2 - \frac{5}{3}y_3$$

$$y_1 + 2y_2 - 4y_3 = 0 \quad \text{or,} \quad y_2 = 2y_3 - 0.5y_1$$

$$-y_2 - y_3 = 0 \quad \text{or,} \quad y_3 = -y_2$$

Since there are three variables we consider three nodes of the signal flow graph as shown. The signal flow graph is drawn in such a way that the graphs satisfy the three equations as shown below.

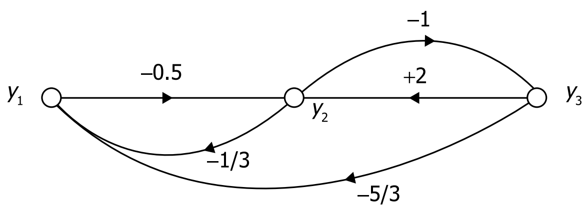


Fig. 4.8 SFG for the set of linear equation

Example 4.2 Find the gain of the system represented by the following equations.

$$x_2 = t_{12}x_1 + t_{32}x_3 \qquad \text{....(4.5)}$$

$$x_3 = t_{23}x_2 + t_{43}x_4 \qquad \text{....(4.6)}$$

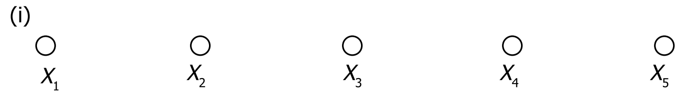
$$x_4 = t_{24}x_2 + t_{34}x_3 + t_{44}x_4 \qquad \text{....(4.7)}$$

$$x_5 = t_{25}x_2 + t_{45}x_4 \qquad \text{....(4.8)}$$

Here, the input node is x_1 and output node is x_5.

Solution

First of all, let us locate the nodes of the system.



Now, draw the signal flow graphs of the four equations as shown from (ii) to (v).

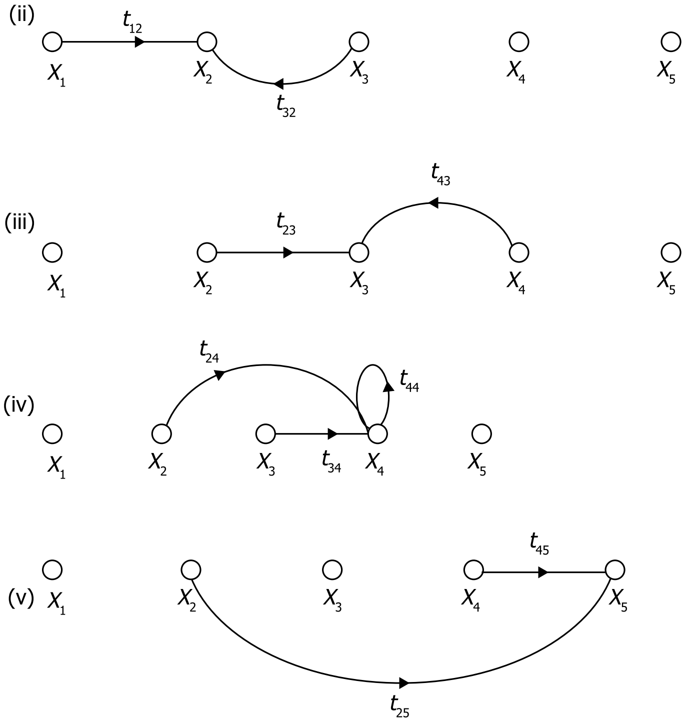


Fig. 4.9 *Development of SFG for a given set of four linear equations*

The overall signal flow graph is obtained by adding the graphs of the individual equations, as shown in Fig. 4.4.

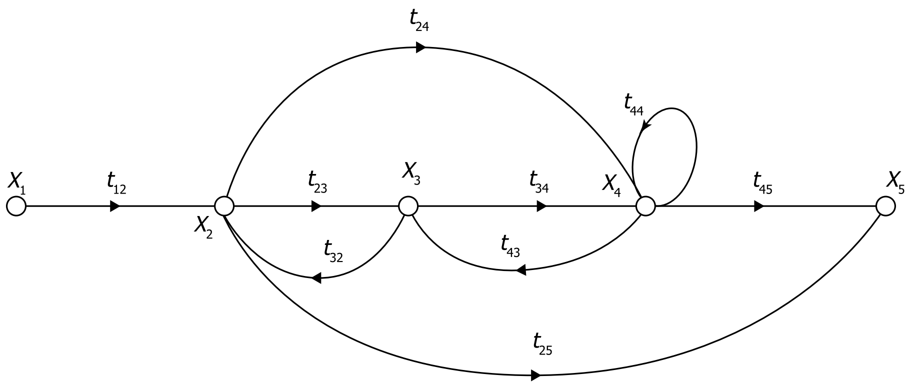
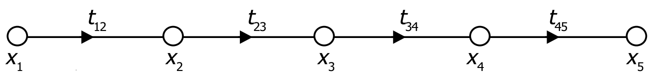


Fig. 4.10 *Complete SFG for the set of given equations*

Procedure for finding gain

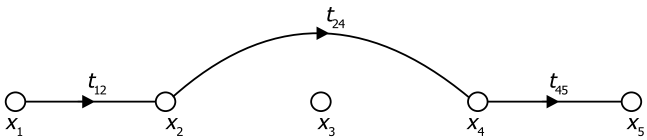
1. There are three forward paths as shown in (i), (ii) and (iii).

i)



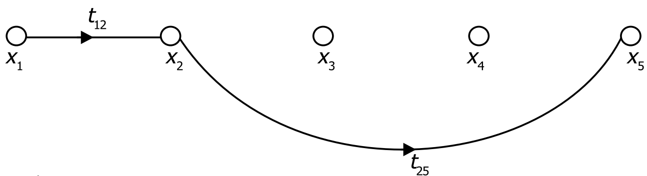
$\therefore$ Gain, $P_1 = t_{12}t_{23}t_{34}t_{45}$

ii)



$\therefore$ Gain, $P_2 = t_{12}t_{24}t_{45}$

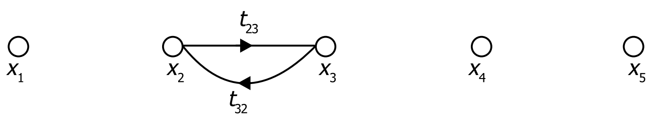
iii)



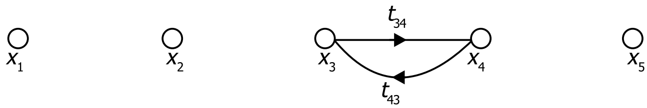
$\therefore$ Gain, $P_3 = t_{12}\, t_{25}$

2. There are four individual feedback loops as shown in (i), (ii), (iii) and (iv).

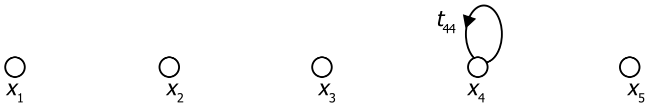
i)



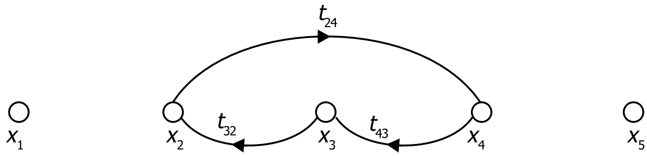
$\therefore$ Loop gain $P_{11} = t_{23}\, t_{32}$

ii) 

$\therefore$ Loop gain $P_{21} = t_{34}\, t_{43}$

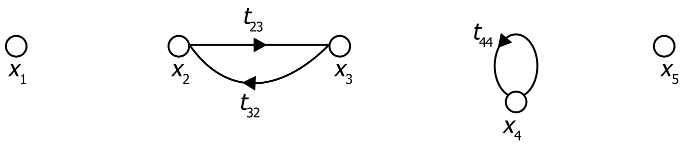
iii) 

$\therefore$ Loop gain $P_{31} = t_{44}$

iv) 

$\therefore$ Loop gain $P_{41} = t_{32}\, t_{43}\, t_{24}$

3. There is only one possible combination of two non-touching loops as shown.



$\therefore$ Gain, $P_{12} = t_{23}\, t_{32}\, t_{44}$

4.

i) The first forward path touches all the loops, so no individual loops are formed.

$$\therefore \Delta_1 = 1$$

ii) Also, when the second forward path is eliminated, no individual loop is formed.

$$\therefore \Delta_2 = 1$$

iii) When the third forward path is eliminated, two loops are formed.

$$\therefore \Delta_3 = 1 - t_{34}t_{43} - t_{44}$$

Now,
$$T.F. = \frac{x_5}{x_1} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

$$= \frac{(t_{12}t_{23}t_{34}t_{45}) + t_{12}t_{24}t_{45} + t_{12}t_{25}(1 - t_{34}t_{43} - t_{44})}{1 - (t_{23}t_{32} + t_{34}t_{43} + t_{44} + t_{32}t_{43}t_{24}) + t_{23}t_{32}t_{44})}$$

Example 4.3 Find the gain of the control system represented in block diagram form as in Fig. 4.5, using Mason's gain formula.

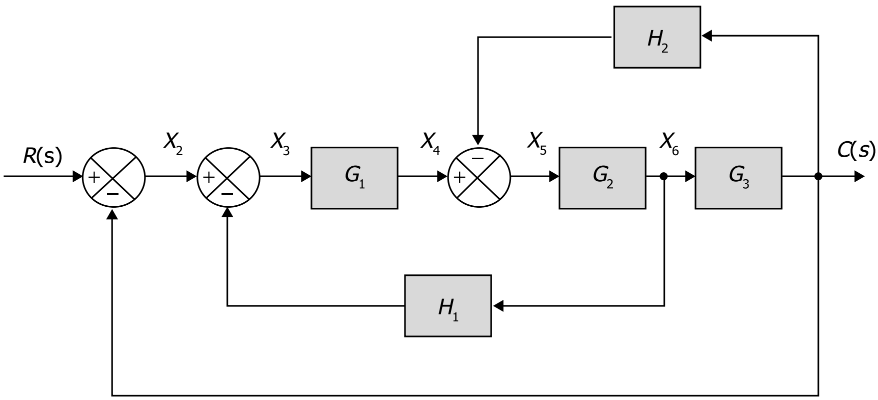


Fig. 4.11 Block diagram of a control system

Solution

It was mentioned at the beginning of the chapter that for complicated systems, the signal flow graph method is preferred over the block diagram reduction technique for finding the closed-loop transfer function. We have already solved this problem by the block diagram reduction technique in Example 3.4. Now, we shall solve it by the SFG method. The signal flow graph is shown in Fig. 4.12.

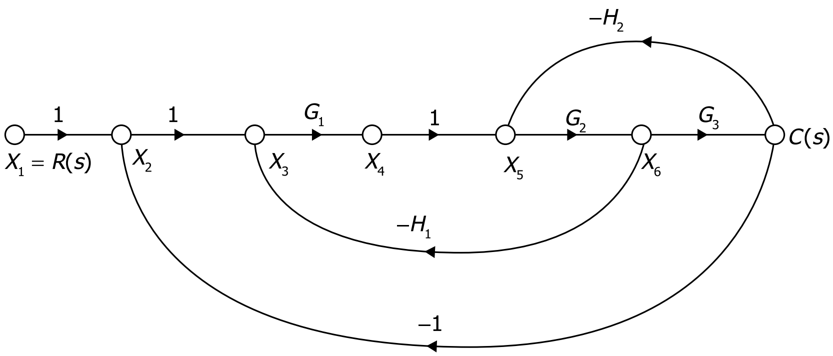
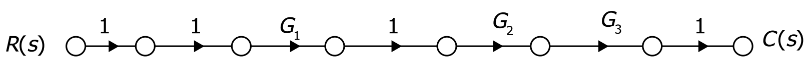


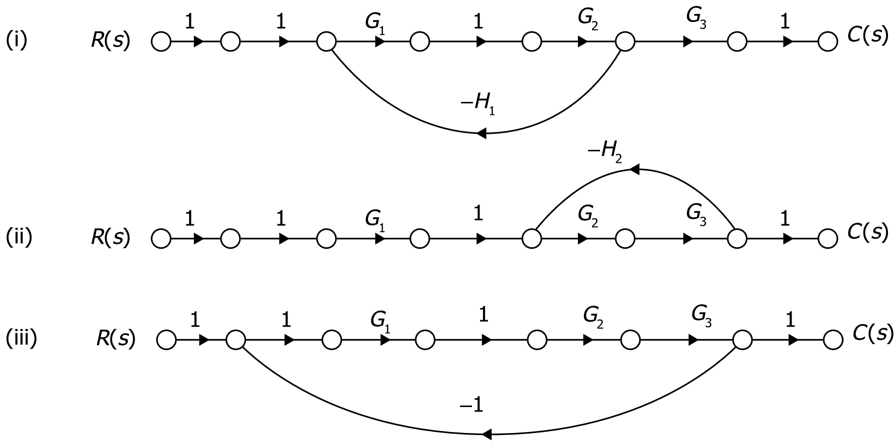
Fig. 4.12 Signal flow graph of the system shown in Fig. 4.11

1. There is only one forward path



$\therefore$ Gain, $P_1 = G_1\, G_2\, G_3$

2. There are three individual loops



$$\Delta = 1 - (-G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3)$$

or, $$\Delta = 1 + (G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3).$$

3. $\Delta_1 = 1$, as no loop is non-touching to the forward path

$\therefore$ $$T.F. = \frac{1}{\Delta}(P_1\Delta_1)$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3}$$

The result is the same as obtained in Example 3.4. This shows that we can use either block diagram reduction technique or signal flow graph for determining the transfer function.

Example 4.4 Find the outputs of the system, the SFG of which is as shown in Fig. 4.13.

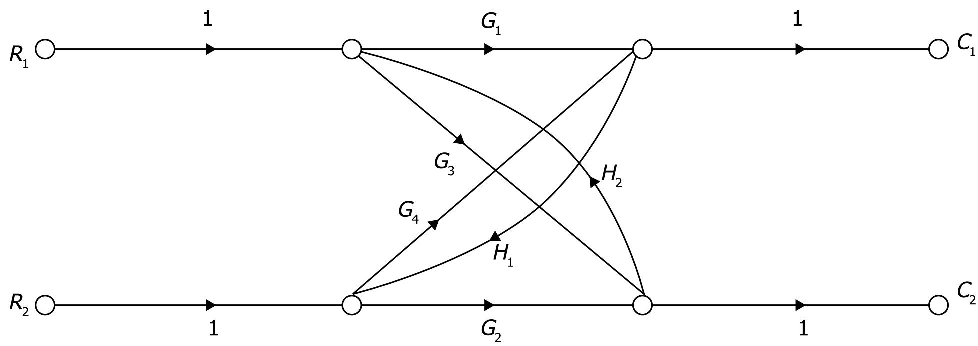
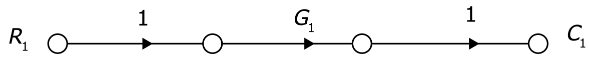


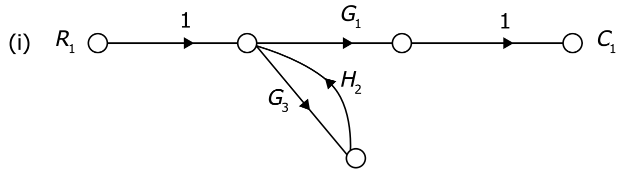
Fig. 4.13

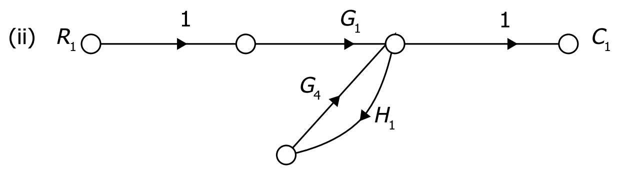
Solution

Consider output C_1 with one input R_1 only. The system has one forward path with path gain, $P_1 = G_1$.



It has two individual loops with loop gains as





$$P_{11} = G_3H_2 \text{ and } P_{21} = G_4H_1$$

The sum of the product of two non-touching loops will be

$$= G_3\, G_4\, H_1\, H_2$$

$\Delta_1 = 1$, as there is no non-touching loop with the forward path.

Hence, $$\frac{C_1}{R_2} = \frac{G_1}{1-(G_3H_2 + G_4H_1)+(G_3G_4H_1H_2)}.$$

Similarly, $$\frac{C_1}{R_2} = \frac{C_4}{1-G_4H_1}.$$

Therefore, the output C_1 is given by

$$C_1 = \frac{G_1R_1}{1-(G_3H_2 + G_4H_1)+(G_3H_1H_2G_4)} + \frac{G_4R_2}{1-G_4H_1}.$$

Similarly, the output C_2 can be calculated as

$$C_2 = \frac{G_2R_2}{1-(G_3H_2 + G_4H_1)+(G_3G_4H_1H_2)} + \frac{G_3R_1}{1-G_3H_2}.$$

Example 4.5 Find the transfer function of the system with SFG as shown in Fig. 4.14.

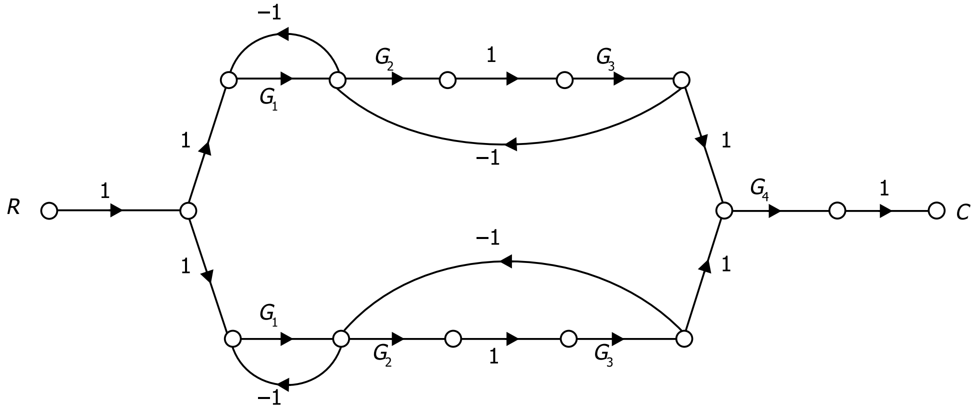


Fig. 4.14

Solution

It has two forward paths with path gains as shown in Fig. 4.15(i) and (ii) below.

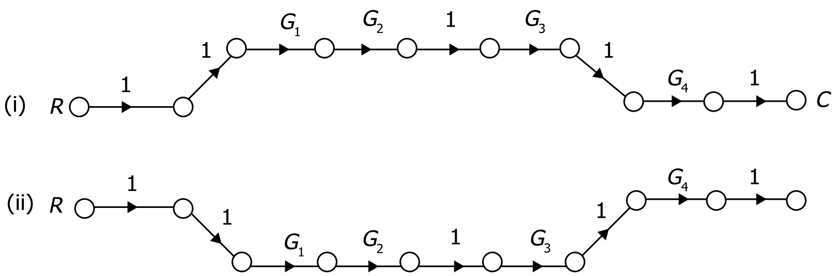


Fig. 4.15

$$P_1 = G_1\, G_2\, G_3\, G_4$$
$$P_2 = G_1\, G_2\, G_3\, G_4$$

There are four loops with gains

$$P_{11} = -\,G_1 \qquad P_{21} = -\,G_2\, G_3$$
$$P_{31} = -\,G_1 \qquad P_{41} = -\,G_2\, G_3$$

The sum of product of two non-touching loops will be

$$= (-G_1)(-G_1) + (-G_1)(-G_2G_3) + (-G_1)(-G_2G_3) + (-G_2G_3)(-G_2G_3)$$
$$= G_1^2 + 2G_1G_2G_3 \_ G_2^2G_3^2$$

Hence, $$\Delta = 1 + 2(G_1 + G_2G_3) + (G_1^2 + 2G_1G_2G_3 + G_2^2G_3^2)$$

$$\Delta_1 = 1 - (-G_1) - (-G_2G_3) = \Delta_2$$

∴ System transfer function by application of Mason's gain formula will be

$$\frac{C}{R} = \frac{2G_1G_2G_3G_4(1 + G_1 + G_2G_3)}{1 + 2(G_1 + G_2G_3) + (G_1^2 + 2G_1G_2G_3 + G_2^2G_3^2)}$$

Example 4.6 Consider a node having three input signals from source nodes R_1, R_2 and R_3. The transfer function from each source node is g_{10}, g_{20} and g_{30} respectively. There are two sink nodes C_1 and C_2 having transfer functions t_{01} and t_{02} through which signal is transmitted. This is shown in Fig. 4.16. You are to simplify the circuit to arrive at the transfer function.

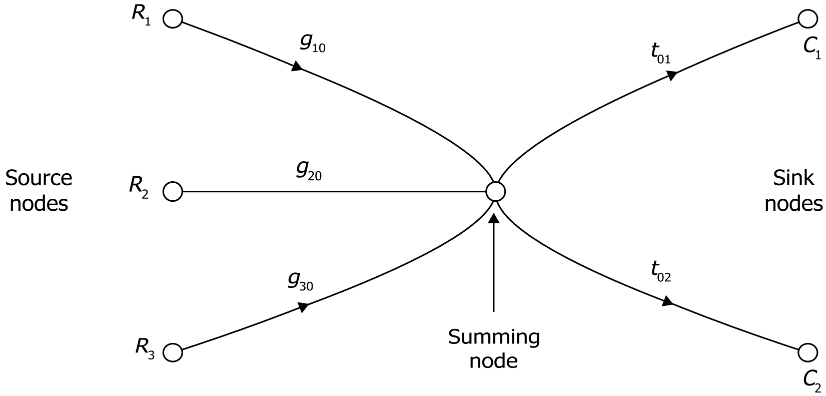


Fig. 4.16

Solution

From Fig. 4.16, the output at sink nodes will be

$$C_1 = g_{10}\, t_{01}\, R_1 + g_{20}\, t_{01}\, R_2 + g_{30}\, t_{01}\, R_3$$

$$C_2 = g_{10}\, t_{02}\, R_1 + g_{20}\, t_{02}\, R_2 + g_{30}\, t_{02}\, R_3$$

This state can also be shown by the SFG shown in Fig. 4.17.

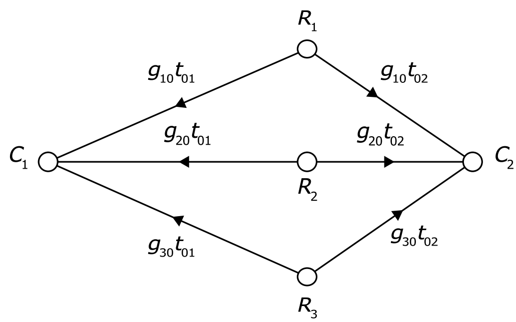


Fig. 4.17

This indicates that the series paths or cascaded nodes in any SFG can be replaced by a single path or pair of nodes having transfer function equal to the product of the individual paths. Similarly, the parallel paths can be combined by adding the transfer functions.

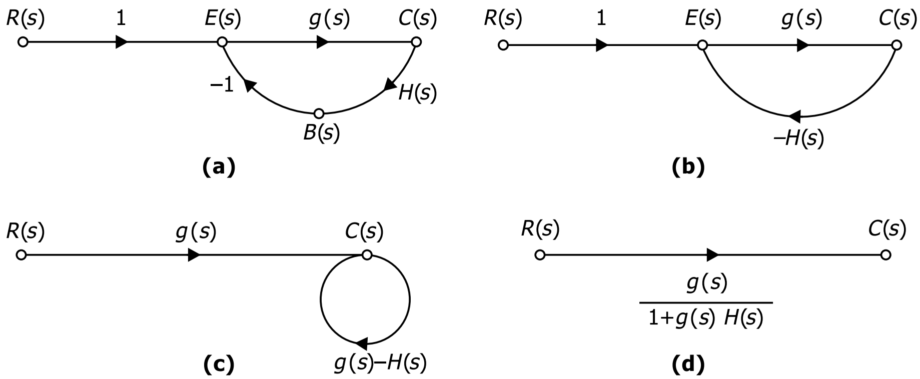


Fig. 4.18 *Node elimination method used*

The mixed nodes can be eliminated by using series, parallel and Mason's gain formula as shown in Fig. 4.18 for SFG of a closed-loop control system.

4.3 SFG FOR SOLUTION OF DIFFERENTIAL EQUATIONS

SFG technique can also be used to simplify the solution of differential equations. The procedure is explained with the help of the following examples.

Example 4.7 Draw a signal flow graph of a control system with differential equation for its response as $\ddot{x}+2\dot{x}+2x=y$. The system has all initial conditions as equal to zero.

Solution

Step 1: Write the equation for the highest order derivative as $\ddot{x}=y-2\dot{x}-2x$.

Step 2: Consider the highest derivative as a dependent variable and all the R.H.S. terms as independent variables. Prepare the SFG as shown in Fig. 4.19.

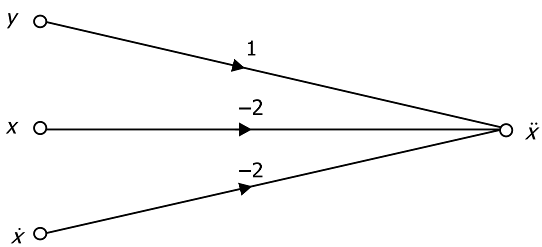


Fig. 4.19

Step 3: Insert branches which represent the auxiliary equations. Thereby connect the nodes of the highest order derivative with the one whose order is lower than this and the node with the lower order than the highest and so on. The flow of the signal must be from a higher order derivative to lower order derivative and insert the transfer function as $1/s$. The resultant graph is shown in Fig. 4.20.

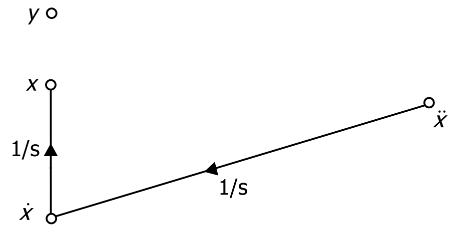


Fig. 4.20

Step 4: Reverse the signs of the branches which indicate integration. While applying this step observe the following rule.

Rule: To reverse the sign of a branch connecting the m^{th} node to n^{th} node (entering n^{th} node) of a SFG, without disturbing the *T.F.* of the graph, it is necessary to reverse the signs of all the remaining branches entering as well as leaving the n^{th} node as shown in Fig. 4.21.

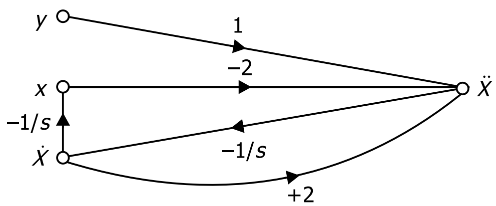


Fig. 4.21

It is to be noted that when the sign is reversed for $\ddot{x}$ to $\dot{x}$ branch, the sign reversal of $\dot{x}$ to x branch is already achieved and therefore, further reversal is not necessary.

Step 5: Redraw the SFG to indicate direct flow from independent variable to dependent variable as shown in Fig. 4.22.

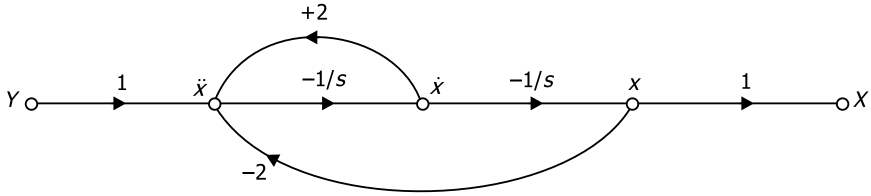


Fig. 4.22 *Resultant SFG*

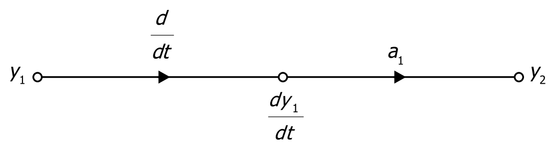
Example 4.8 Draw signal flow diagrams for the equations, as in (a), (b), and (c),

a) $y_2 = a_1 \dfrac{dy_1}{dt}$

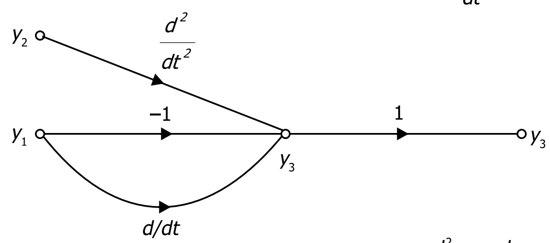
b) $y_3 = \dfrac{d^2 y_2}{dt^2} + \dfrac{dy_1}{dt} - y_1$

c) $\dfrac{d^2 y}{dx^2} + \dfrac{2}{3}\dfrac{dy}{dx} + \dfrac{11}{2} y = x$

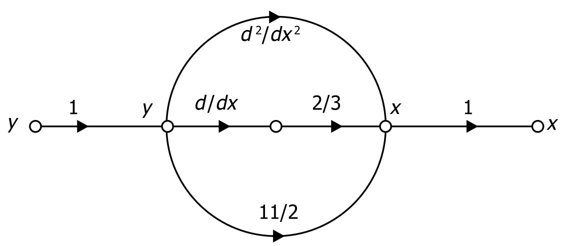
Solution



(a) SFG for $y_2 = a_1 \dfrac{dy_1}{dt}$



(b) SFG for $y_3 = \dfrac{d^2 y_2}{dt^2} + \dfrac{dy_1}{dt} - y_1$



(c) SFG for $\dfrac{d^2 y}{dx^2} + \dfrac{2}{3}\dfrac{dy}{dx} + \dfrac{11}{2} y = x$

Fig. 4.23

Alternatively, $\dfrac{d^2 y}{dx^2} + \dfrac{2}{3}\dfrac{dy}{dx} + \dfrac{11}{2} y = x$

or $$s^2Y(s)+\frac{2s}{3}Y(s)+\frac{11}{2}Y(s)=X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{1}{s^2+\frac{2}{3}s+\frac{11}{2}}=\frac{\frac{1}{s^2}}{1-\left(-\frac{2}{3s}-\frac{11}{2s^2}\right)}=\frac{\frac{1}{s^2}}{1-\left(-\frac{2}{3s}-\frac{11}{2s^2}\right)}$$

Comparing with Mason's gain formula, there is

a) One forward path with gain of $\frac{1}{s^2}$

b) Two feedback loops with gain of $\frac{-2}{3s}, \frac{-11}{3s^2}$

The signal flow graph is shown in Fig. 4.24.

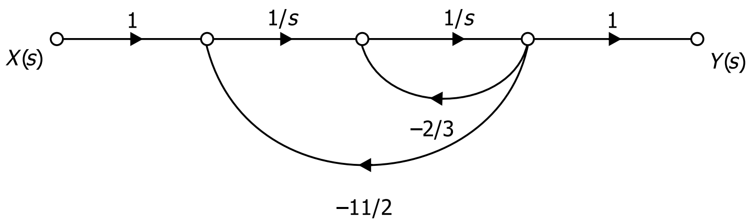


Fig. 4.24 *An alternative SFG for the differential equation*

Example 4.9 Find by signal flow graph technique the transfer function of the control system, the block diagram representation of which is shown in Fig. 4.25.

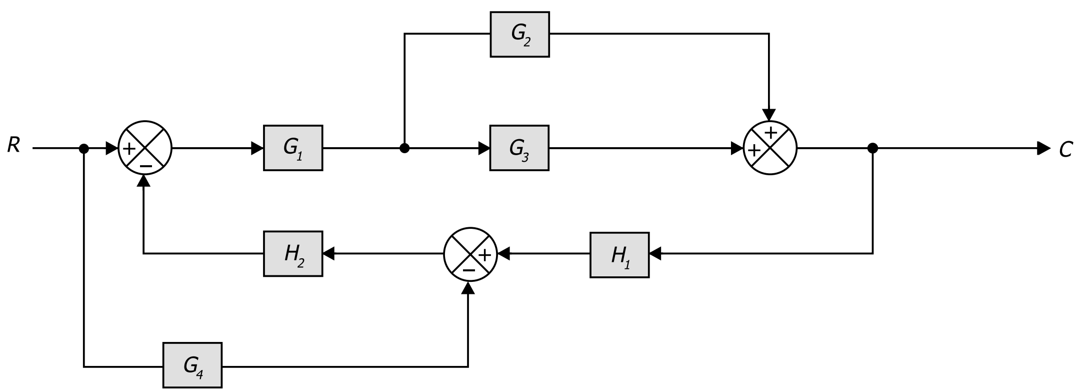


Fig. 4.25

Solution

The signal flow graph for the given block diagram representation is shown in Fig. 4.26.

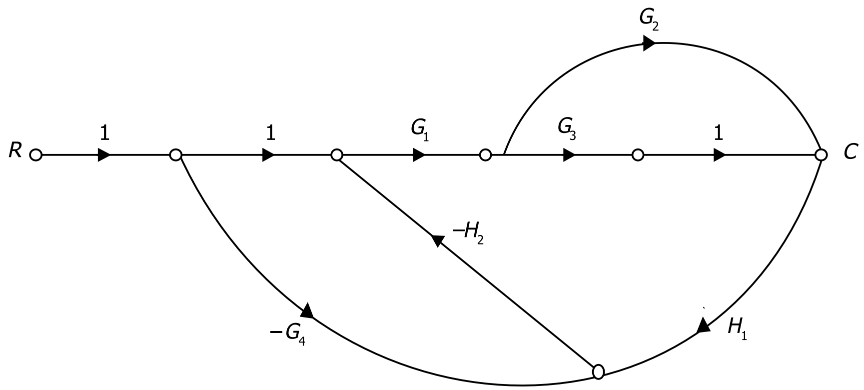


Fig. 4.26

Forward path gains are,

$P_1 = G_1 G_3$ $\quad\quad \therefore \Delta_1 = 1$

$P_2 = G_1 G_2$ $\quad\quad \therefore \Delta_2 = 1$

$P_3 = G_1 G_3 G_4 H_2$ $\quad\quad \therefore \Delta_3 = 1$

$P_4 = G_1 G_2 G_4 H_2$ $\quad\quad \therefore \Delta_4 = 1$

Loop gains, $L_1 = -G_1 G_3 H_1 H_2$

$L_2 = -G_1 G_2 H_1 H_2$

$\Delta = 1 - (L_1 + L_2) = 1 + G_1 G_2 H_1 H_2 + G_1 G_3 H_1 H_2$

$$T.F. = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$= \frac{G_1G_3 + G_1G_2 + G_1G_3G_4H_2 + G_1G_2G_4H_2}{1 + G_1G_3H_1H_2 + G_1G_2H_1H_2}$$

$$= \frac{G_1(G_3 + G_2)(1 + G_4H_2)}{1 + G_1H_1H_2(G_3 + G_2)}$$

Example 4.10 Find the transfer function of the system shown in Fig. 4.27.

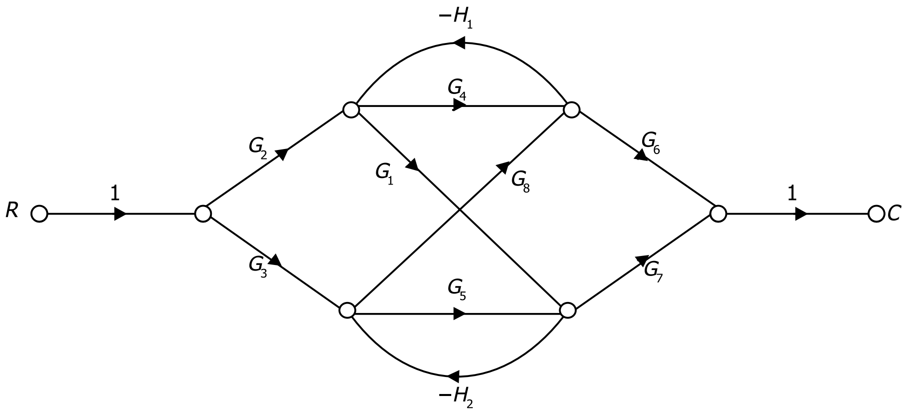


Fig. 4.27

Solution

Forward path gains are calculated as,

$$P_1 = G_2\, G_4\, G_6$$
$$P_2 = G_3\, G_5\, G_7$$
$$P_3 = G_2\, G_1\, G_7$$
$$P_4 = G_3 G_8\, G_6$$
$$P_5 = -\, G_2\, G_1\, H_2\, G_8\, G_6$$
$$P_6 = -\, G_3\, G_8\, H_1\, G_1\, G_7$$

Loop gains are, $L_1 = -G_4 H_1$

$$L_2 = -\, G_5 H_2$$
$$L_3 = G_1 H_2 G_8 H_1$$

Non-touching loops: There is one pair having gain product $= G_4 H_1\, G_5 H_2$

$$\Delta\ = 1 + G_4 H_1 + G_5 H_2 - G_1 H_2 G_8 H_1 + G_4 H_1 G_5 H_2$$
$$\Delta_1 = 1 + G_5 H_2$$
$$\Delta_2 = 1 + G_4 H_1$$
$$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$$

$$T.F. = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$$

$$= \frac{G_2G_4G_6(1+G_5H_2)+G_3G_5G_7(1+G_4H_1)+G_2G_1G_7+G_3G_8G_6-G_2G_6G_8G_1H_2-G_3G_7G_8G_1H_1}{1+G_4H_1+G_5H_2+G_4G_5H_1H_2+G_1G_8H_1H_2}$$

Example 4.11 Are the two systems shown in Fig. 4.28(a) and (b) equivalent? Justify your answer by determining their transfer functions.

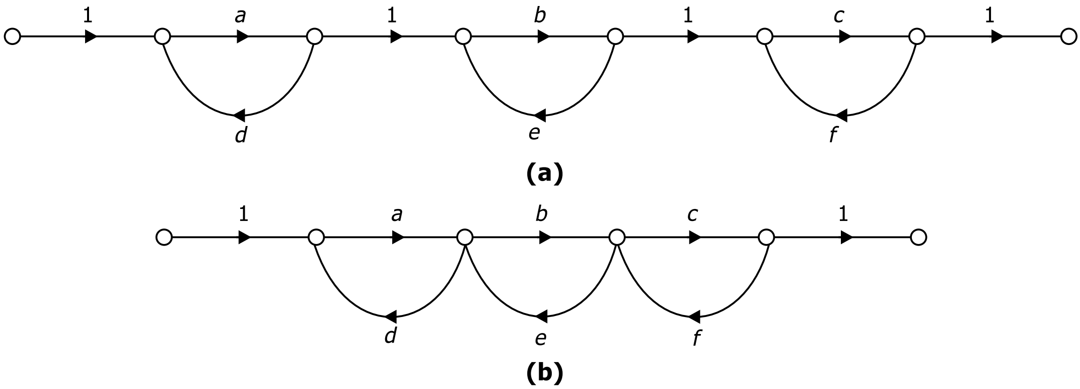


Fig. 4.28

Solution

Two systems can be considered equivalent to each other if their transfer functions are the same. The systems are not equivalent as the transfer function for Fig. 4.28(a) is

$$T.F. = \frac{abc}{1-(ad+be+cf)+(adbe+adcf+becf)-abcdef}$$

and the transfer function for Fig. 4.28(b) is

$$T.F. = \frac{abc}{1-(ad+be+cf)+(adcf)}.$$

Example 4.12 Find the transfer function for the system shown in Fig 4.29 by Mason's gain formula.

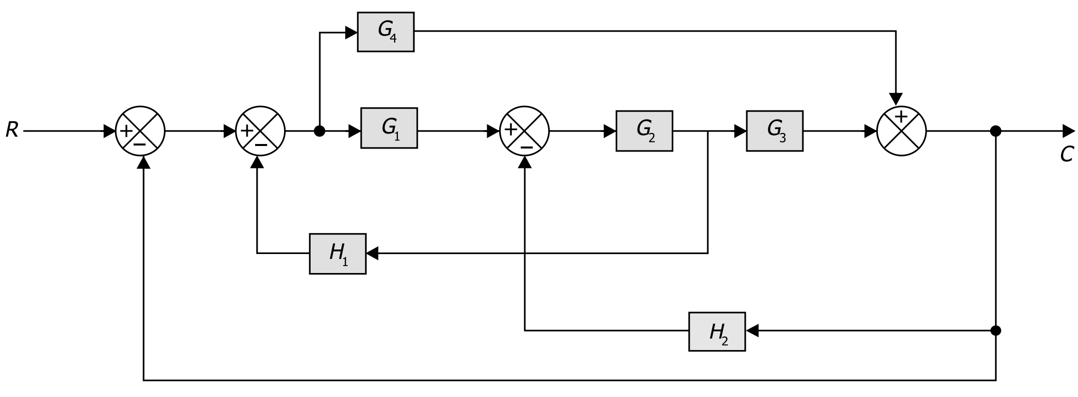


Fig. 4.29

Solution

The signal flow graph for the system is shown in Fig. 4.30.

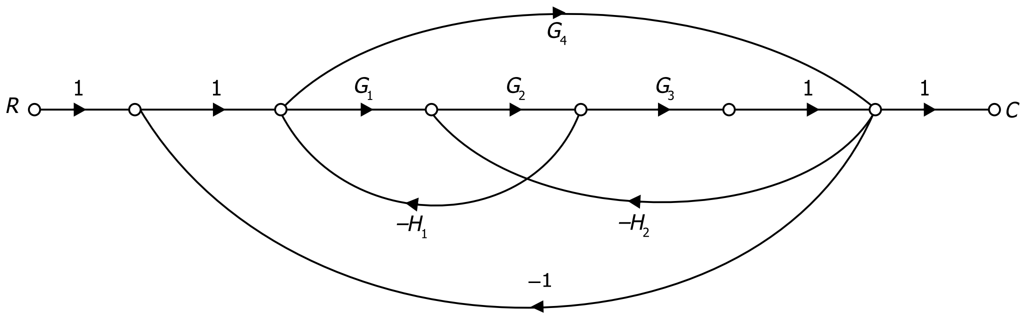


Fig. 4.30

Forward path gains
$$P_1 = G_1\, G_2\, G_3$$
$$P_2 = G_4$$

Loop gains
$$L_1 = -\, G_1\, G_2 H_1$$
$$L_2 = -G_1\, G_2\, G_3$$
$$L_3 = -\, G_2\, G_3 H_2$$
$$L_4 = -G_4$$
$$L_5 = G_4 H_2\, G_2 H_1$$
$$\Delta = 1 + G_1\, G_2 H_1 + G_1\, G_2\, G_3 + G_2\, G_3 H_2 + G_4 - G_2\, G_4 H_1 H_2$$
$$\Delta_1 = \Delta_2 = 1$$
$$T.F. = \frac{G_1 G_2 G_3 + G_4}{1 + G_1 G_2 H_1 + G_1 G_2 G_3 + G_2 G_3 H_2 + G_4 - G_1 G_8 H_1 H_2}$$

Example 4.13 Find $\dfrac{C(s)}{R(s)}$ for the control system shown in Fig. 4.31.

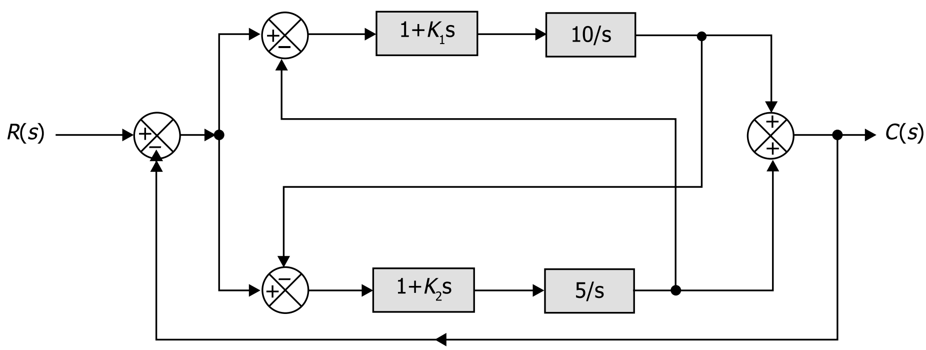


Fig. 4.31

Solution

The signal flow graph has been drawn as in Fig. 4.32.

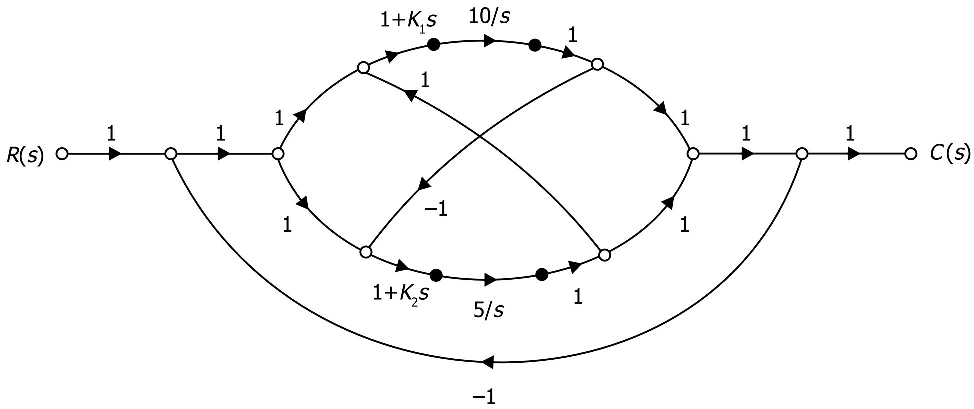


Fig. 4.32

Forward path gains are,

1. $P_1 = \dfrac{5(1+k_2 s)}{s}$ and $\Delta_1 = 1$

2. $P_2 = \dfrac{50(1+k_2 s)(1+k_1 s)}{s^2}$ and $\Delta_2 = 1$

3. $P_3 = \dfrac{10(1+k_1 s)}{s}$ and $\Delta_3 = 1$

4. $P_4 = \dfrac{-50(1+k_2 s)(1+k_1 s)}{s^2}$ and $\Delta_4 = 1$

Loop gains are,

1. $P_{11} = \dfrac{-5(1+k_2 s)}{s}$

2. $P_{21} = \dfrac{-10(1+k_1 s)}{s}$

3. $P_{31} = \dfrac{-50(1+k_2 s)(1+k_1 s)}{s^2}$

4. $P_{41} = \dfrac{50(1+k_1 s)(1+k_2 s)}{s^2}$

5. $P_{51} = \dfrac{-50(1+k_1 s)(1+k_2 s)}{s^2}$

$$\therefore \Delta = 1 - \sum_{m=1}^{5} P_{m1}$$

$$= 1 + \frac{5(1+k_2 s)}{s} + \frac{10(1+k_1 s)}{s} + \frac{50(1+k_1 s)(1+k_2 s)}{s^2}$$

$$T.F. = \frac{C(s)}{R(s)} = \frac{\left[\frac{5(1+k_2 s)}{s} + \frac{10(1+k_1 s)}{s} + \frac{50(1+k_2 s)(1+k_1 s)}{s^2} - \frac{50(1+k_1 s)(1+k_2 s)}{s^2}\right]}{\left[1 + \frac{5(1+k_2 s)}{s} + \frac{10(1+k_1 s)}{s} + \frac{50(1+k_2 s)(1+k_1 s)}{s^2}\right]}$$

$$\frac{C(s)}{R(s)} = \frac{5s(1+k_2 s) + 10s(1+k_1 s)}{s^2 + 5s(1+k_2 s) + 10s(1+k_1 s) + 50(1+k_2 s)(1+k_1 s)}$$

Example 4.14 Draw a block diagram of the signal flow graph shown in Fig. 4.27. Find the overall transfer function by block diagram reduction technique. Verify the result by applying Mason's gain formula.

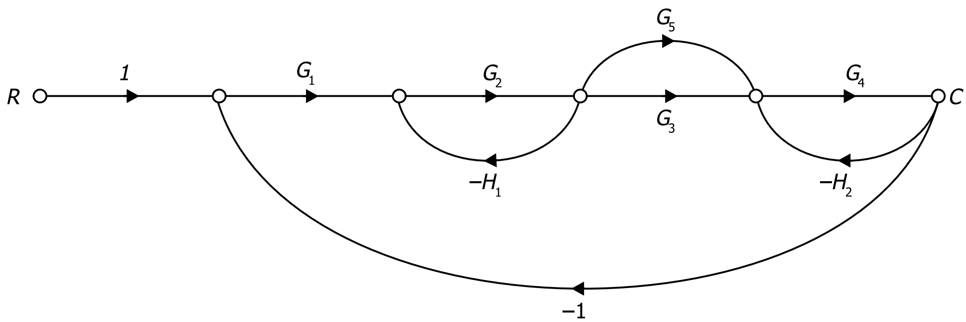


Fig. 4.33

Solution

Mason's gain formula

Forward paths:

1. $G_1\, G_2\, G_2\, G_4$
2. $G_1\, G_2\, G_5\, G_4$

Loops:

1. $-\, G_1\, G_2\, G_3\, G_4$
2. $-\, G_1\, G_2\, G_5\, G_4$
3. $-\, G_2 H_1$
4. $-\, G_4 H_2$

Non-touching loops $-\ G_2H_1$ and $-\ G_4H_2$

$$\frac{C}{R} = \frac{G_1G_2G_3G_4 + G_1G_2G_4G_5}{1 + G_1G_2G_3G_4 + G_1G_2G_4G_5 + G_2H_1 + G_4H_2 + G_2G_4H_1H_2}$$

The block diagram of the given signal flow graph is shown in Fig. 4.34.

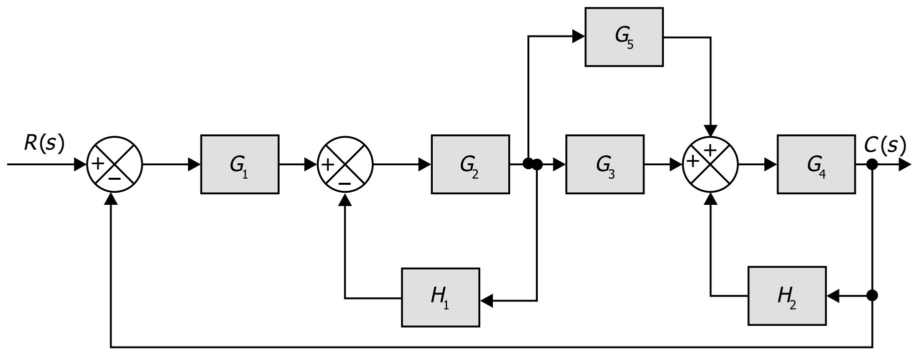


Fig. 4.34

By the block diagram reduction technique we get Fig. 4.35.

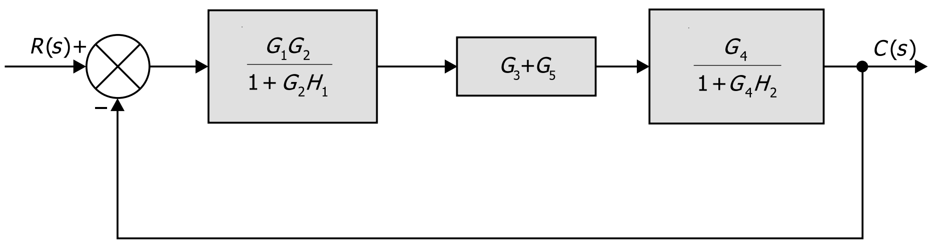


Fig. 4.35

$$\frac{C(s)}{R(s)} = \frac{\left[\dfrac{G_1G_2(G_3+G_5)G_4}{(1+G_2H_1)(1+G_4H_2)}\right]}{1 + \dfrac{G_1G_2(G_3+G_5)G_4}{(1+G_2H_1)(1+G_4H_2)}}$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_4G_5}{1 + G_1G_2G_3G_4 + G_1G_2G_4G_5 + G_2H_1 + G_4H_2 + G_2G_4H_1H_2}$$

This shows that the transfer function calculated by both the methods is the same.

Example 4.15 Using signal flow graph and Mason's gain formula, obtain the overall gain of the system depicted in Fig. 4.36. Also verify by block diagram reduction technique.

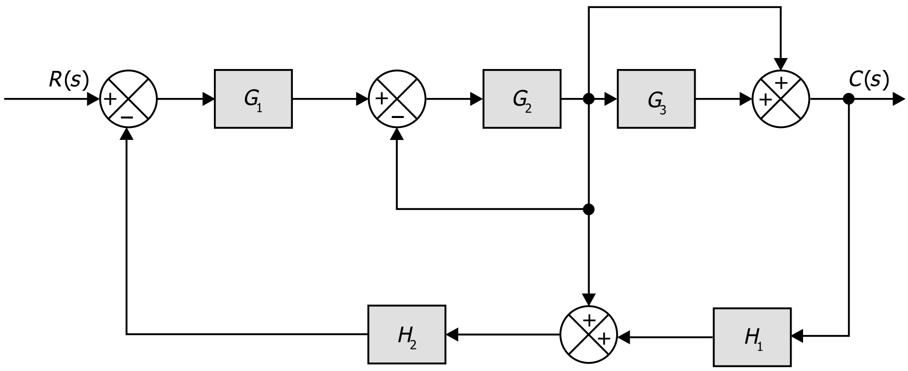


Fig. 4.36

Solution

The signal flow graph of the given block diagram is shown in Fig. 4.37.

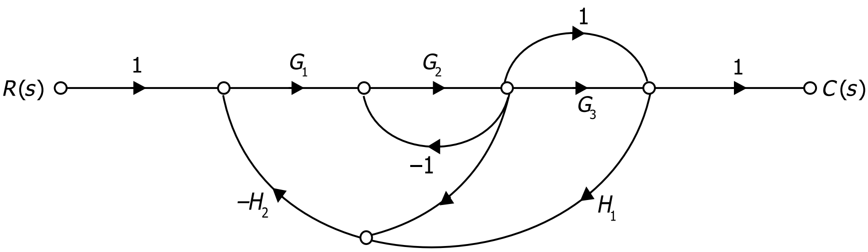


Fig. 4.37

Forward path gains are, $G_1\, G_2\, G_3$ and $G_1\, G_2$

Loop gains are:

(1) $-G_1\, G_2\, H_1\, H_2$; (2) $-G_1\, G_2 H_2$; (3) $-G_2$ and (4) $-G_1\, G_2\, G_3 H_1\, H_2$

Overall gain, i.e. the $T.F. = \dfrac{G_1G_2 + G_1G_2G_3}{1 + G_2 + G_1G_2H_2 + G_1G_2H_1H_2 + G_1G_2G_3H_1H_2}$

The following simplification have been made in Fig. 4.36 to get the circuit shown in Fig. 4.38.

i) reduction of unity feedback at G_2 as $\dfrac{G_2}{1+G_2}$

ii) reducing G_3 and its parallel path of unity as $(G_3 + 1)$.

iii) shifting the summing point beyond H_2 leaving H_1 and H_2 in series.

Block diagram reduction technique: Add block A and B, calculate *TF.* of $(A + B)$ with H_2 as the feedback loop, then add the *TF.* with block C, finally take *TF.* of the resultant block with H_1H_2 as the feedback loop.

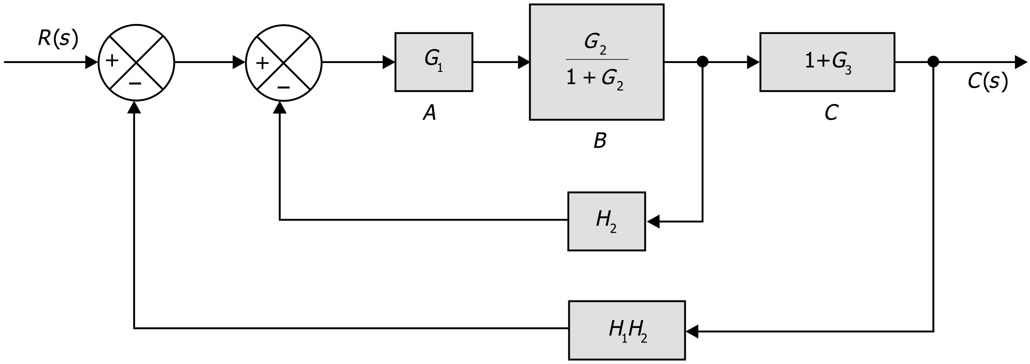


Fig. 4.38

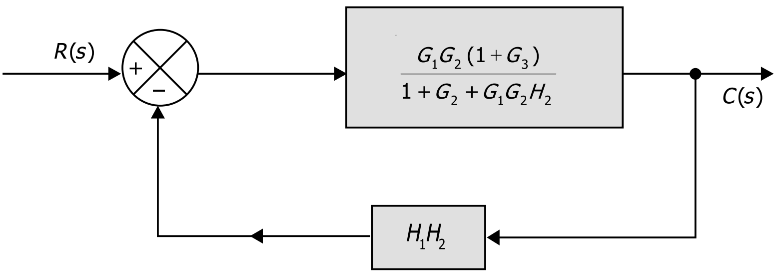


Fig. 4.39

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2(1+G_3)}{(1+G_2+G_1G_2H_2)}}{1+\dfrac{G_1G_2(1+G_3)}{1+G_2+G_1G_2H_2}H_1H_2}$$

$$= \frac{G_1G_2+G_1G_2G_3}{1+G_2+G_1G_2H_2+G_1G_2H_1H_2+G_1G_2G_3H_1H_2}$$

The results are the same in both the cases.

Example 4.16 Use Mason's gain formula for determining the overall transfer function of the system shown in Fig. 4.40.

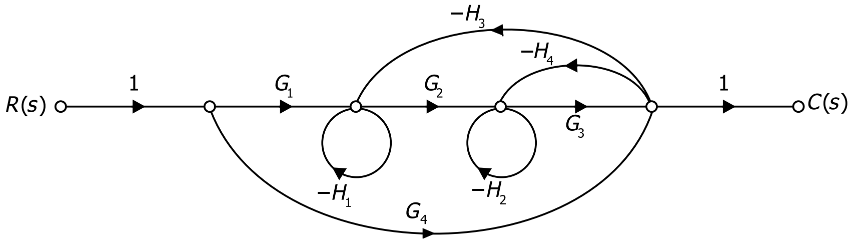


Fig. 4.40

Solution

Forward path gains, $P_1 = G_1 G_2 G_3$

$P_2 = G_4$

Loop gains, $P_{11} = -H_1$

$P_{21} = -H_2$

$P_{31} = -G_3 H_4$

$P_{41} = -G_2 G_3 H_3$

Non touching loops $P_{12} = H_1 H_2$

$P_{22} = H_1 H_4 G_3$

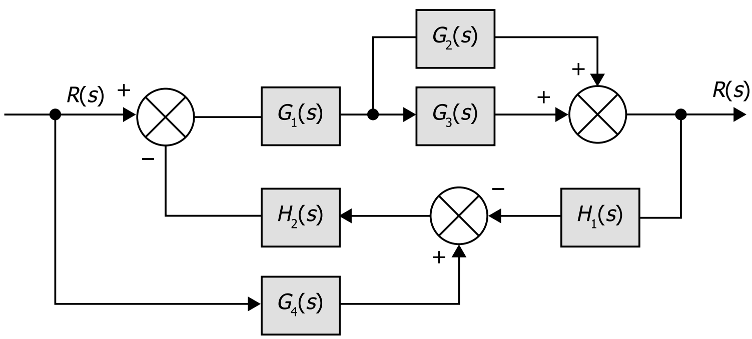
$\therefore$ $\Delta = 1 + H_1 + H_2 + G_3 G_4 + G_2 G_3 H_3 + H_1 H_2 + H_1 H_4 G_3$

$\Delta_1 = 1$ and

$\Delta_2 = 1 + H_1 + H_2 + H_1 H_2$

$$\therefore \qquad T.F. = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4(1 + H_1 + H_2 + H_1H_2)}{1 + H_1 + H_2 + G_3G_4 + G_2G_3H_3 + H_1H_2 + G_3H_1H_4}$$

Example 4.17 Find the overall transfer function of the system shown in Fig. 4.41 using signal flow graph.



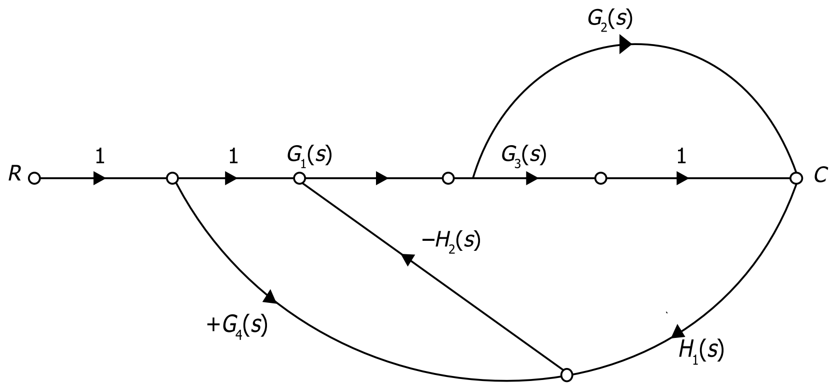


Fig. 4.41

Forward path gains are:

$$P_1 = G_1(s)\ G_3(s)\ \Delta_1 = 1$$

$$P_2 = G_1(s)\ G_2(s)\ \Delta_2 = 1$$

$$P_3 = G_1(s)\ G_3(s)\ G_4(s)\ H_2(s)\ \Delta_3 = 1$$

$$P_4 = G_1(s)\ G_4(s)\ G_2(s)\ H_2(s)\ \Delta_4 = 1$$

Loop gains,

$$L_1 = G_1(s)\ G_3(s)\ H_1(s)\ H_2(s)$$

$$L_2 = G_1(s)\ G_2(s)\ H_1(s)\ H_2(s)$$

$$\Delta = 1 - (L_1 + L_2)$$

$$= 1 + G_1(s)\ G_2(s)\ H_1(s)\ H_2(s) + G_1(s)\ G_3(s)\ H_1(s)\ H_2(s)$$

Transfer function,

$$TF = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$TF = \frac{G_1(s)G_3(s) + G_1(s)G_2(s) + G_1(s)G_3(s)G_4(s)H_2(s) + G_1(s)G_2(s)G_4(s)H_2(s)}{1 + G_1(s)G_3(s)H_1(s)H_2(s) + G_1(s)G_2(s)H_1(s)H_2(s)}$$

Thus,

$$\frac{C(s)}{R(s)} = \frac{G_1(s)[G_3(s) + G_2(s)][1 + G_4(s)H_2(s)]}{1 + G_1(s)H_1(s)H_2(s)[G_3(s) + G_2(s)]}$$

Example 4.18 Draw the signal flow graph for the following system and calculate the transfer function using Mason's gain formula.

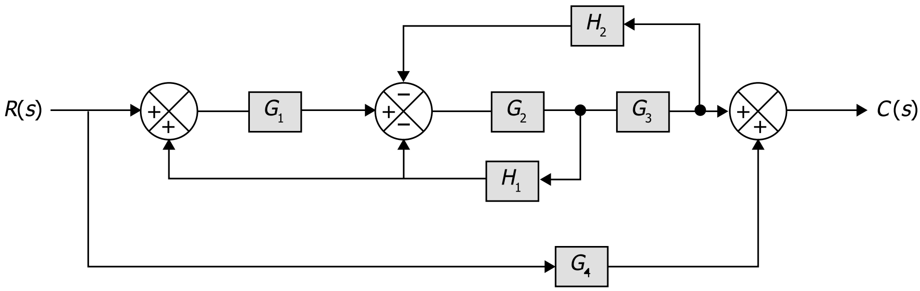


Fig. 4.42

Solution

The gain of the forward paths

$$g_1 = G_1G_2G_3$$
$$g_2 = G_4$$

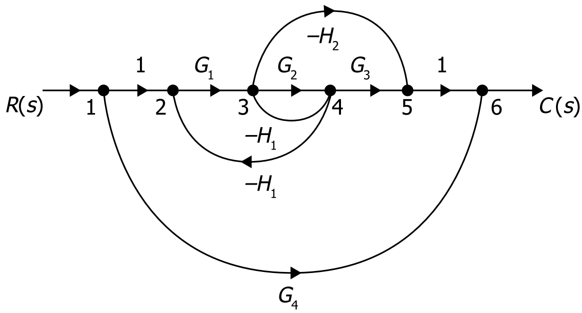


Fig. 4.43

Individual loops $L_1 = -G_2H_1$

$$L_2 = -G_1G_2H_1$$
$$L_3 = -G_2G_3H_2$$
$$\Delta_1 = 1$$
$$\Delta_2 = 1 - (-G_2H_1 - G_1G_2H_1 - G_2G_3H_2)$$
$$\Delta_2 = 1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2$$
$$\Delta = 1 - (L_1 + L_2 + L_3)$$
$$\Delta = 1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2$$
$$T.F. = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_4(1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2)}{1 + G_2H_1 + G_1G_2H_1 + G_2H_2G_3}$$

REVIEW QUESTIONS

4.1 Draw the signal flow graph for the system whose block diagram is shown in Fig. 4.44 and find its overall transfer function.

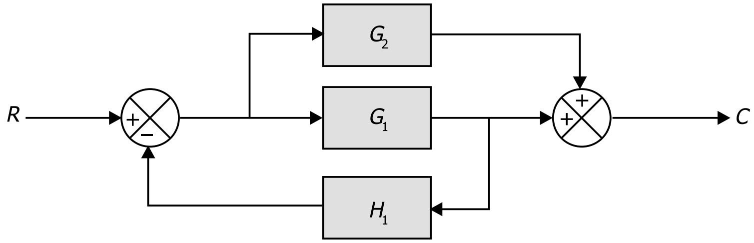


Fig. 4.44

4.2 Determine the transfer function relating to C and R for the system represented as

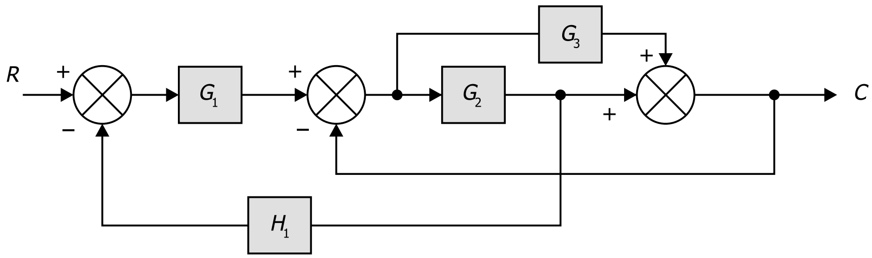


Fig. 4.45

4.3 Apply mason's gain formula to determine the overall transfer function of a control system represented through block diagram as shown below.

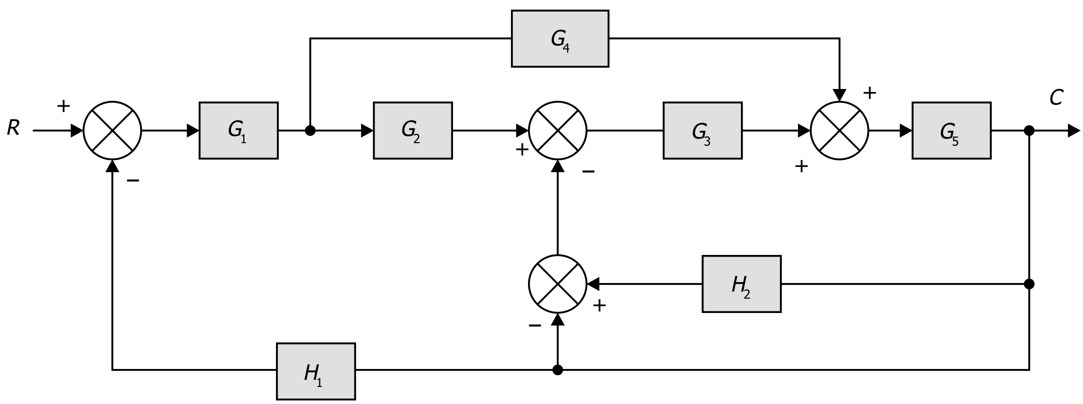


Fig. 4.46

4.4 Draw the signal flow graph for the system represented by block diagram as shown and determine the C/R ratio.

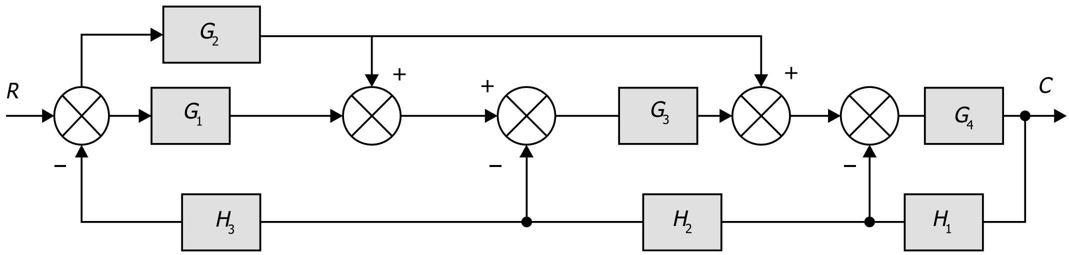


Fig. 4.47

4.5 Obtain the transfer functions G_6, G_7 and G_8 so that the three systems given below become equivalent. In the signal flow graph of Fig. 4.48(a), G_1, G_2, G_3, G_4, and G_5 are given.

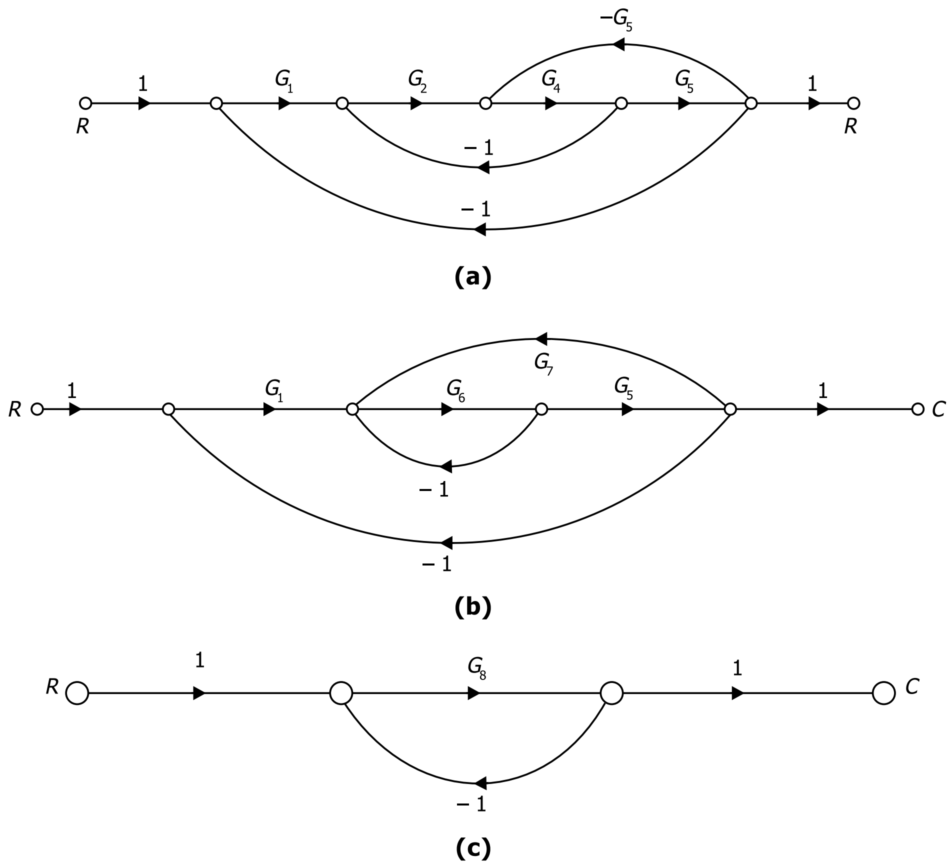


Fig. 4.48

4.6 Derive $C(s)/R(s)$ by the signal flow graph technique for the system the block diagram representation of which is shown in Fig. 4.49.

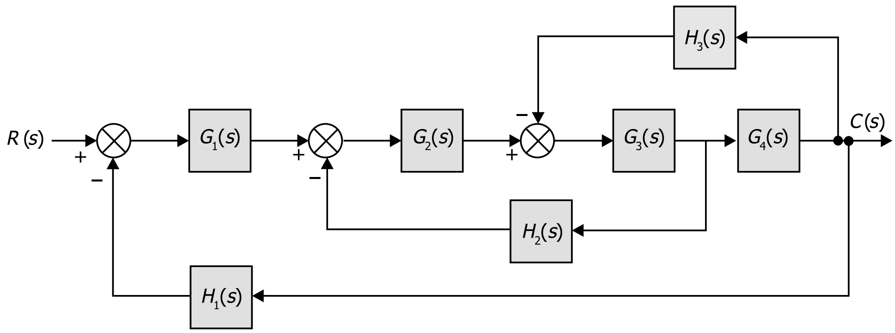


Fig. 4.49

4.7 Find the transfer function of the system shown in Fig. 4.50 by Mason's gain formula.

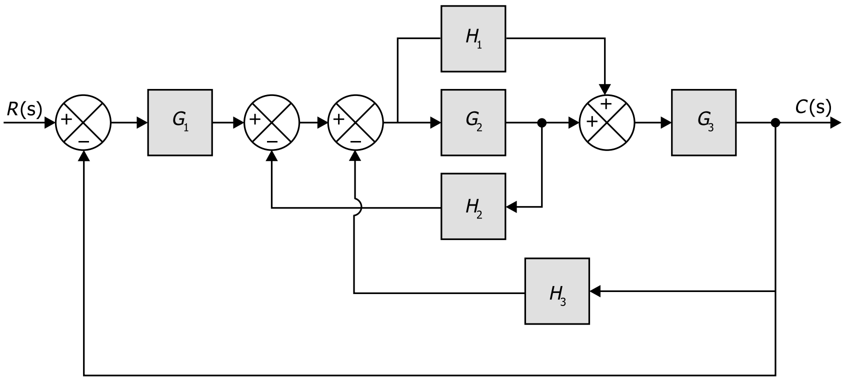


Fig. 4.50

4.8 Draw a block diagram of the signal flow graph given in Fig. 4.51. Find the overall transfer function by block diagram reduction and verify the result by Mason's gain formula.

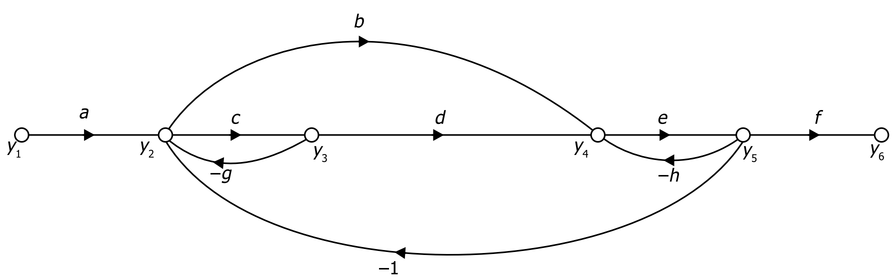


Fig. 4.51

4.9 The signal flow graph of a two input two output system is shown in Fig. 4.52. Find the output of the system.

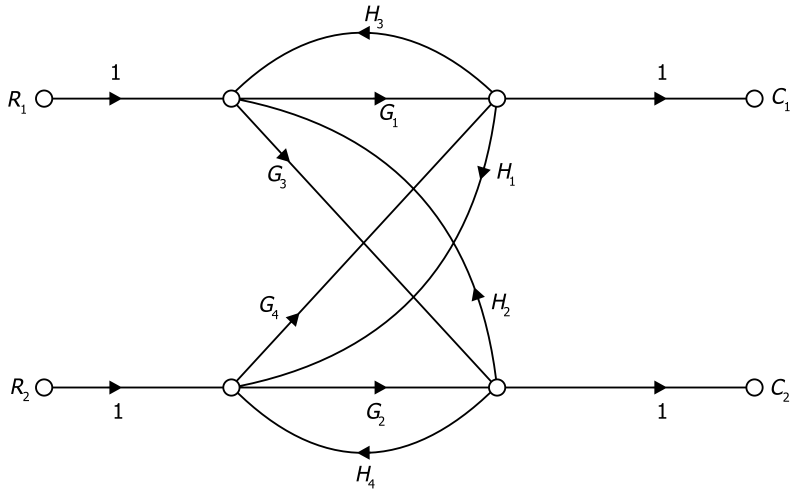


Fig. 4.52

4.10 Draw the signal flow graph of the system whose block diagram is given in Fig. 4.53. Find the outputs using Mason's gain formula.

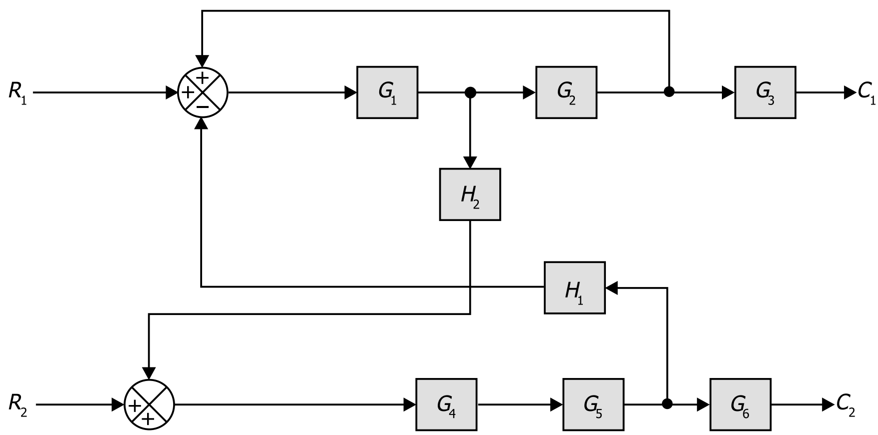


Fig. 4.53

5

FEEDBACK CONTROL SYSTEM AND EFFECT OF FEEDBACK ON SYSTEM PERFORMANCE

5.1 OPEN-LOOP AND CLOSED-LOOP CONTROL SYSTEM

Introduction to control system and the concept of feedback was introduced in Chapter 1. In this chapter we are, once again, discussing the same topic in some more details.

A control system is formed by interconnecting various components to give a desired system response. An open-loop or non-feedback system (Fig. 5.1), with the help of a controller or actuating device, directly generates the output in response to an input.

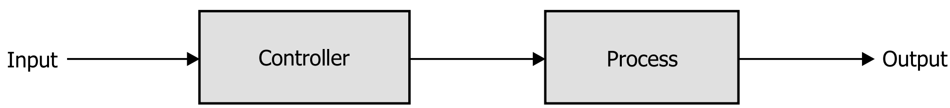


Fig. 5.1 *Open-loop or non-feedback system*

A closed-loop or feedback system (Fig. 5.2) generates the output in response to an error signal obtained by comparing the input with the feedback signal produced by measuring output so that the error is continually reduced and the process comes under control.

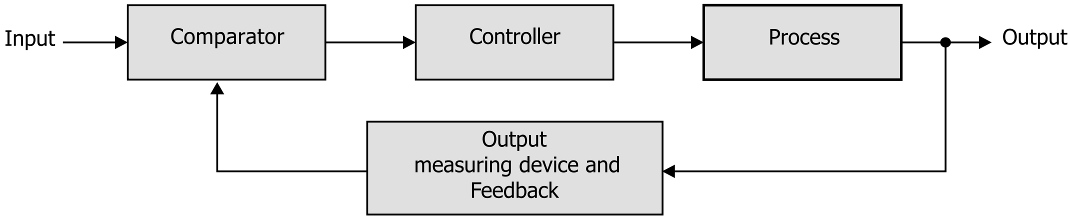


Fig. 5.2 *Closed-loop or feedback system*

A simplified block diagram and signal flow graph for the open-loop (Fig. 5.1) and closed-loop system (Fig. 5.2) are shown in Fig. 5.3(a) and (b) respectively.

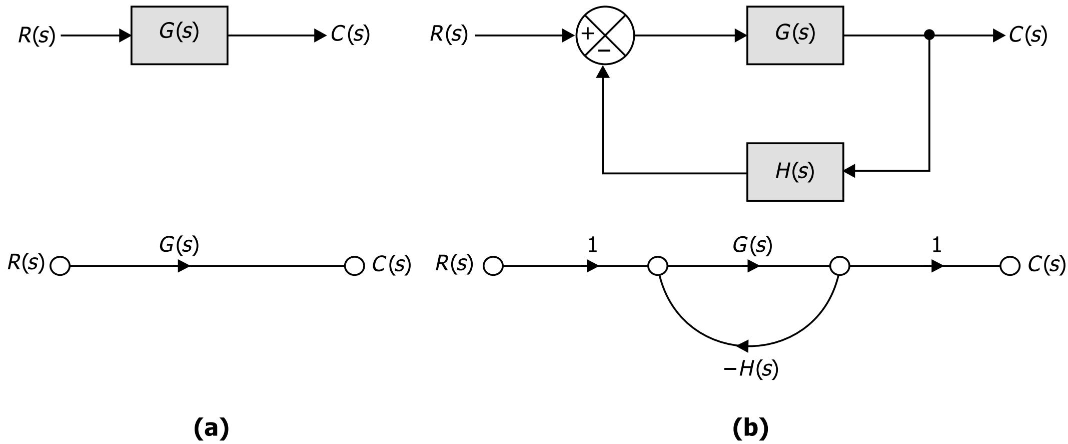


Fig. 5.3 *(a) Block diagram and signal flow graph of Fig. 5.1; (b) Block diagram and signal flow graph of Fig. 5.2*

Feedback is necessary to improve performance of control systems. Feedback is inherent in nature as in biological and physical systems. For example, the human heart rate control system is a feedback or closed-loop control system. A simple immersion rod type water heater is an example of an open-loop control system.

Feedback control systems are used in all applications like in production control, quality control, economy control, process control, etc.

Process control systems are feedback control systems where the output variables like temperature, pressure, humidity, etc. are regulated by feedback control mechanism.

5.2 FEEDBACK CONTROL SYSTEMS

In this section we will discuss some feedback control systems through examples of thermal, hydraulic, pneumatic and electrical systems and also draw their block diagrams.

5.2.1 Temperature Control System

A temperature control system is shown in Fig. 5.4.

The thermocouple at the outflowing liquid from a tank generates an output voltage e_t as a feedback signal proportional to the temperature of the outflowing liquid. The error voltage, $e = (e_r - e_t)$ produced by comparing the reference voltage (e_r) with the feedback voltage (e_t) from the thermocouple, actuates the controller to regulate the current i through the heating element.

To simplify the mathematical analysis, the following assumptions are made.

i) The flow rate (Q) at the inlet and outlet of the tank are same.
ii) The liquid in the tank is well-stirred so that its temperature is represented by that of the outflowing liquid.
iii) Heat loss and heat storage capacity of tank's wall are negligible.
iv) The electronic controller is linear that is, $i = k_1 e$, where k_1 is the gain of the controller in amps/volt.

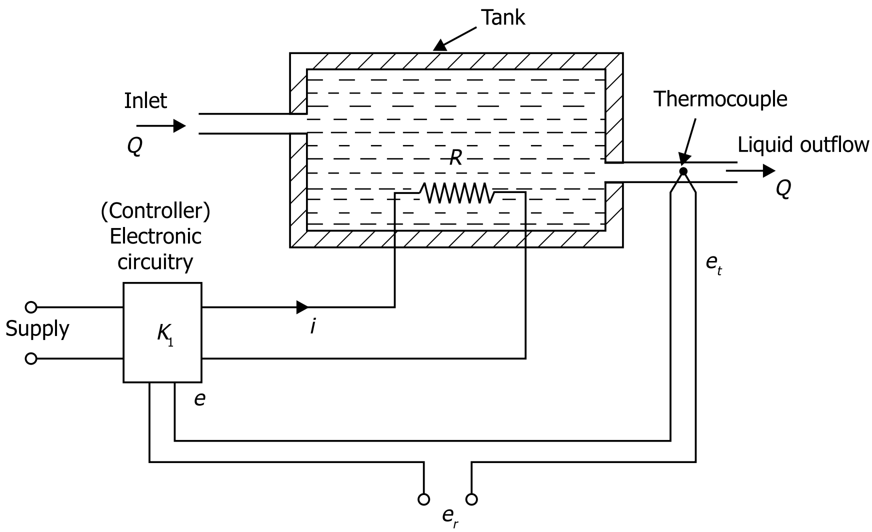


Fig. 5.4 *Temperature control system with feedback*

As the rate of heat generated in the heater is equal to the rate of heat storage in the tank plus the rate of heat removed by the out-flowing liquid, we can write

$$i^2 R = Mc\frac{d\theta}{dt} + Q\rho c(\theta - \theta_i)$$

or,

$$k_1^2 e^2 R = C\frac{d\theta}{dt} + \frac{1}{R_t}(\theta - \theta_i)$$

where $R_t = \dfrac{1}{Q\rho c} =$ thermal resistance

$C = Mc =$ thermal capacitance of the liquid in the tank

$M =$ mass of the liquid in the tank

$c =$ specific heat of the liquid

$\rho =$ density of the liquid

$Q =$ volume flow rate of the liquid

$R =$ resistance of the heating element

$\theta_i =$ temperature of liquid at inlet

$\theta =$ temperature of outflowing liquid

$K_1 =$ gain to the controller

$K_t =$ constant of the temperature sensor

Let the steady state and incremental values of the variables are $e = e_0 + \Delta e$, $d\theta = \Delta\theta$, $\theta = \theta_0 + \Delta\theta$, and $\theta_i = \theta_{io} + \Delta\theta_i$.

Replacing all the variables in the above equation by their steady plus incremental values and neglecting $(\Delta e)^2$, we get

$$k_1^2(e_0^2 + 2e_0\Delta e)R = C\frac{d(\Delta\theta)}{dt} + \frac{\theta_0 - \theta_{i0}}{R_t} + \frac{\Delta\theta - \Delta\theta_i}{R_t} \qquad \text{....(5.1)}$$

Putting incremental values to zero, we get the steady state equation as

$$Rk_1^2e_0^2 = \frac{\theta_0 - \theta_{i0}}{R_t} \qquad \text{....(5.2)}$$

Substituting the values of $Rk_1^2e_0^2$ into equation (5.1), we get

or,
$$K_1^2e_0^2R + 2K_1^2e_0R_t\Delta e = C\frac{d}{dt}(\Delta\theta) + \frac{\theta_0 - \theta_{i0}}{R_t} + \frac{\Delta\theta - \Delta\theta_i}{R_t}$$

$$2K_1^2e_0RR_t\Delta e = R_tC\frac{d(\Delta\theta)}{dt} + \Delta\theta - \Delta\theta_i \qquad \text{....(5.3)}$$

or,
$$K\Delta e = \tau\frac{d}{dt}(\Delta\theta) + \Delta\theta - \Delta\theta_i$$

Also we have,

$$\Delta e = \Delta e_r - \Delta e_t \qquad \text{....(5.4)}$$

and
$$\Delta e_t = K_t\Delta\theta \qquad \text{....(5.5)}$$

where K_t is the constant of thermocouple.

Taking Laplace transform of equations (5.3), (5.4) and (5.5), we get

$$K\Delta E(s) = (1+\tau s)\Delta\theta(s) - \Delta\theta_i(s)$$

$$\Delta\theta(s) = \frac{K\Delta E(s)}{\tau s+1} + \frac{\Delta\theta_i(s)}{\tau s+1} \qquad \text{....(5.6)}$$

$$\Delta E(s) = \Delta E_r(s) - \Delta E_t(s) \qquad \text{....(5.7)}$$

$$\Delta E_t(s) = K_t \Delta\theta(s) \qquad \text{....(5.8)}$$

where $K = 2K_1^2 e_0 R R_t$ and $\tau = R_t C$.

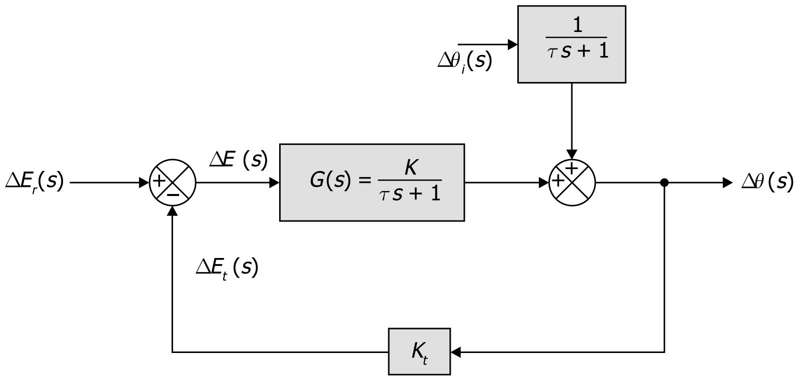


Fig. 5.5 *Block diagram of Fig. 5.4*

Using equations (5.6), (5.7) and (5.8) we draw the block diagram of the temperature control system as shown in Fig. 5.5.

5.2.2 Hydraulic System

In the hydraulic power steering mechanism (Fig. 5.6) of a car, when the steering wheel is rotated anticlockwise through an angle of θ_i, the input displacement x through the steering gear allows high pressure oil to move the power piston towards the right, giving a feedback displacement z of movable housing through a linkage ABC. So the error displacement, $e = (x - z)$, regulates the rate of flow of high pressure oil to give the power ram an output displacement of y which in turn rotates the car wheel to a desired angle of θ_0 through a drive linkage.

θ_i = angle of movement of steering wheel

x = movement of spool with the help of gear mechanism

θ_0 = desired displacement of the wheels

$e = x - z$ = net displacement of the spool = error signal

q = rate of flow of oil into power cylinder

p = pressure difference between high and low pressure sides

A = area of the power piston

M = mass of load on power piston

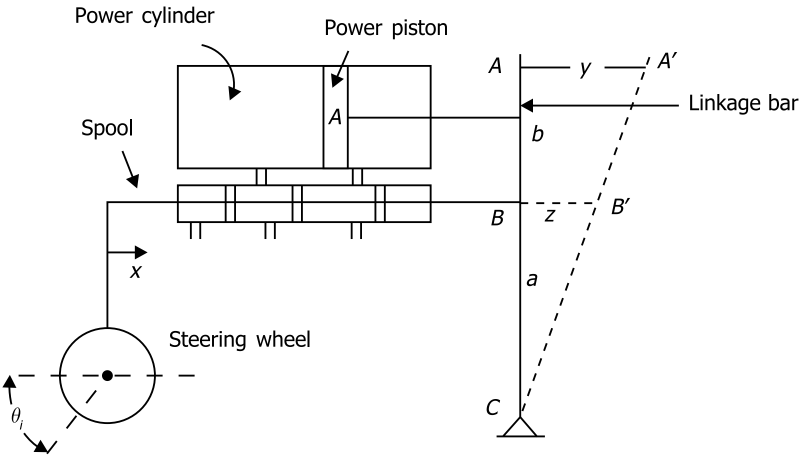


Fig. 5.6 *Hydraulic power steering mechanism of a car*

For a small movement (Fig. 5.6) of feedback linkage from its position ABC to position $A'B'C'$, we get

$$\frac{AA'}{AC} = \frac{BB'}{BC}$$

or
$$\frac{y}{a+b} = \frac{z}{a}. \qquad \text{....(5.9)}$$

Also, the error signal is given by

$$e = x - z. \qquad \text{....(5.10)}$$

If 'A' be the area of power piston then

$$q = A\frac{dy}{dt} \qquad \text{....(5.11)}$$

Neglecting leakage and compressibility of oil, the flow rate of oil may be written as

$$q = K_1 e - K_2 p \qquad \text{....(5.12)}$$

where $p = p_1 - p_2$ is the differential pressure of the power cylinder.

If M be the mass of load to the power piston and B be the viscous friction coefficient, then the force equation becomes

$$\text{Force} = Ap = M\frac{d^2y}{dt^2} + B\frac{dy}{dt} \qquad \text{....(5.13)}$$

Taking Laplace transform of equations (5.9), (5.10), (5.11), (5.12) and (5.13), with zero initial conditions, we get

$$\left.\begin{aligned} Z(s) &= \frac{a}{a+b}Y(s) \\ E(s) &= X(s) - Z(s) \\ Q(s) &= As\,Y(s) \\ Q(s) &= K_1 E(s) - \frac{K_2}{A}(Ms^2 + Bs)Y(s) \end{aligned}\right\} \qquad \text{....(5.14)}$$

Using equations (5.14), we now draw the block diagram of the hydraulic power steering system of a car as shown in Fig. 5.7.

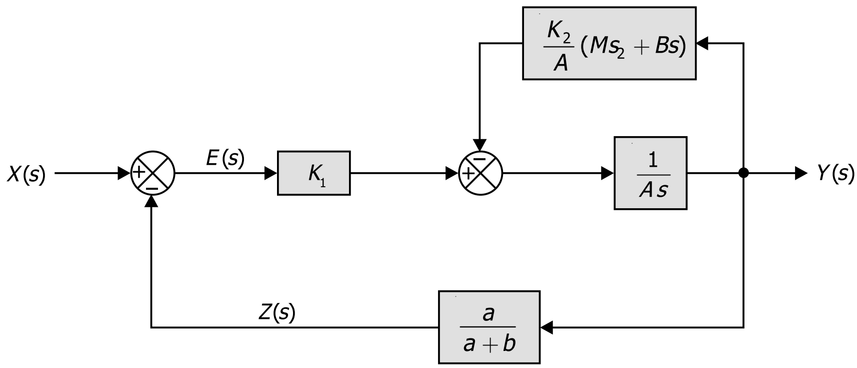


Fig. 5.7 *Block diagram of the hydraulic system shown in Fig. 5.6*

5.2.3 Pneumatic System

Fig. 5.8 shows schematic diagram of regulating the position control of a plug valve used to control the fluid flow through pneumatic control (air pressure control) mechanism. An input signal x is provided at A through a flapper that would correspond to the required opening of the fluid flow control valve. Air is supplied at a pressure P_s. The back pressure created at the nozzle is P_b. The nozzle back pressure operates the actuator which provides the required opening of the plug valve.

From triangle LMQ of Fig. 5.8,

$$\frac{LM}{LP} = \frac{ON}{NP}$$

or,
$$\frac{x+y}{a+b} = \frac{e+y}{a}$$

or,
$$\frac{e}{a} = \frac{x}{a+b} - \frac{b}{a(a+b)}y$$

or,
$$e = \frac{a}{a+b}x - \frac{b}{a+b}y$$

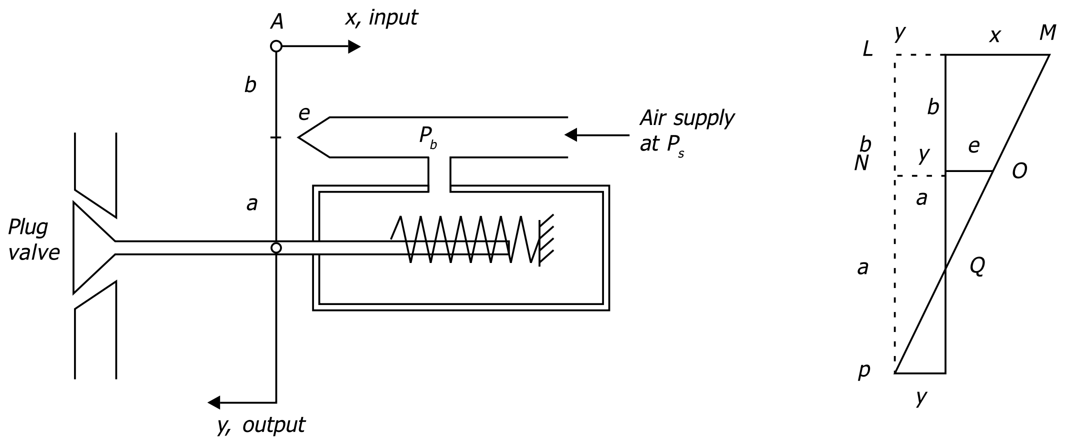


Fig. 5.8

Taking Laplace Transform,

$$E(s) = \frac{a}{a+b}x(s) - \frac{b}{a+b}y(s) \qquad \text{....(5.15)}$$

As the valve is operated in the linear zone of its characteristic, the nozzle back pressure (p_b) is given by, $P_b = K_e$. In Laplace transform form,

$$p_b(s) = KE(s) \qquad \text{....(5.16)}$$

Force $$F = AP_b = M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky$$

Taking Laplace transform,

$$Y(s) = \frac{AP_b(s)}{Ms^2 + Bs + K} \qquad \text{....(5.17)}$$

where K is the spring constant, M is the mass of plug valve and B is the coefficient of viscous friction.

Using equations (5.15), (5.16) and (5.17), the system represented in Fig. 5.8 can now be shown in the form of a block diagram as in Fig. 5.9.

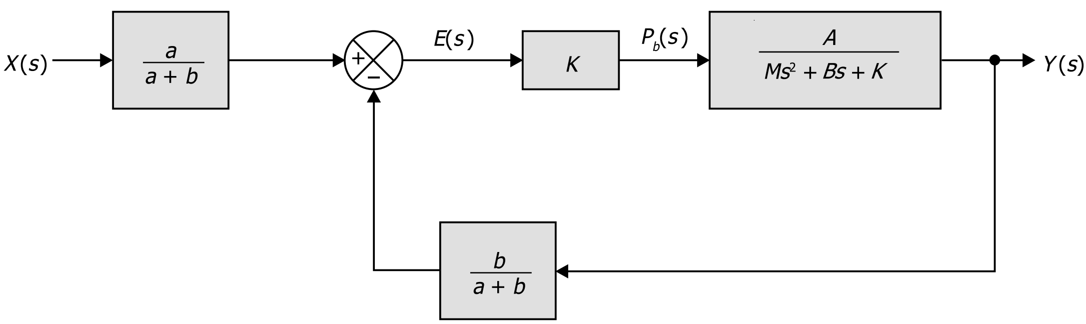


Fig. 5.9 *Block diagram of Fig. 5.8*

5.2.4 Speed Control System

In the speed control system (Fig. 5.10), a load is rotated at a desired speed with the help of an armature controlled DC motor. The field is fed by a constant voltage source and the armature is supplied with an amplified error voltage [$v_a = k_a(v_r - v_t)$] produced by comparing the reference voltage (v_r) with the feedback voltage (v_t). The feedback voltage ($v_t = k_t\omega$), proportional to the angular speed (ω) of the DC motor, is produced from the tachogenerator mounted on the shaft of the motor.

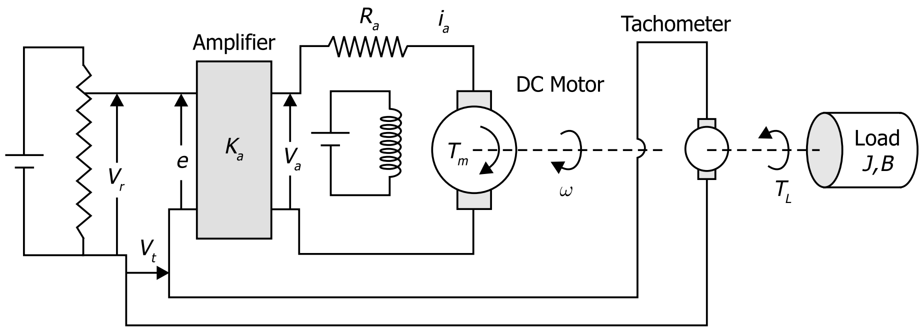


Fig. 5.10 *Speed control system*

The system equations and corresponding block diagrams are shown as follows:

Error signal, $$E(s) = V_r(s) - k_t\omega(s)$$

Signal Amplification, $$V_a(s) = k_a E(s)$$

For motor, $$V_a = E_b + I_a R_a$$

$$T_m = K_T I_a$$

Using Laplace transform,

$$V_a(s) = k_b\omega(s) + R_a I_a(s)$$

or, $$I_a(s) = \frac{V_a(s) - K_b\omega(s)}{R_a}$$

$$T_m(s) = K_T I_a(s)$$

and, $$= \frac{K_T}{R_a}[V_a(s) - K_b\omega(s)].$$

Again, $$T_m(s) = [Js + B]\,\omega(s) + T_L(s)$$

The complete block diagram is shown as in Fig. 5.11

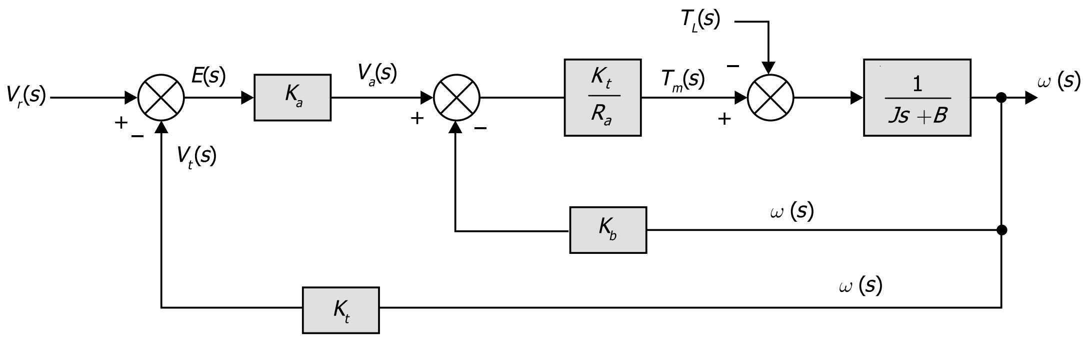


Fig. 5.11 *Block diagram of the system shown in Fig. 5.10*

5.3 EFFECT OF FEEDBACK

A feedback is provided to bring about improvement in the performance of a control system. The advantages of feedback in a control system are:

a) Feedback reduces the sensitivity of the system to its parameter variations. Parameters may vary due to ageing, environmental changes, etc. If feedback is introduced the system performance will not be adversely effected.
b) Feedback improves the sensitivity of a control system but there would be reduction in system gain.
c) Feedback improves the stability if properly designed.
d) Negative feedback reduces the overall gain of the system.
e) System response to disturbance signal can be reduced with feedback.

The above mentioned effects of feedback on the system performance are described in the following sections.

5.3.1 Effect of Feedback on Parameter Variations

The parameters of a system (see Fig. 5.3) with transfer function $G(s)$ may vary with changing environment, aging, ignorance of the exact values of the system parameters, and other natural factors that affect a control system. The primary advantage of using feedback in control system is to reduce the system's sensitivity to parameter variations.

Suppose due to system parameter changes, the transfer function $G(s)$ changes to $G(s) + \Delta G(s)$, the corresponding change in output, i.e. $\Delta C(s)$ with respect to open-loop and closed-loop system will be,

$$\Delta C(s) = \Delta G(s)\, R(s) \qquad \text{for open-loop system and}$$

$$= \frac{\Delta G(s) R(s)}{1 + G(s)H(s)} \qquad \text{for closed-loop system (ignoring change } \Delta G(s) \text{ in the denominator).}$$

So the change in output due to parameter variation of $G(s)$ in the closed-loop (feedback) system is reduced by a factor of $[1 + G(s)\, H(s)]$ which is usually much greater than unity.

For any given system we can define sensitivity to parameter variations in the following way.

Sensitivity is the ratio of relative variation of the overall transfer function of the system due to variation of $G(s)$. That is to say,

$$\text{Sensitivity} = \frac{\text{Percentage change in } C(s)/R(s)}{\text{Percentage change in } G(s)}$$

The effect of feedback in control system is to reduce the sensitivity to parameter variation on the system's output.

To have a highly accurate open-loop system, the components of $G(s)$ must be selected carefully to meet the exact specifications whereas the components of $G(s)$ in a closed-loop system can be less accurately specified because of reduced sensitivity to parameter variations.

Example 5.1 Let us consider the speed control system of Fig. 5.10 with zero load or disturbance to illustrate the reduction of sensitivity to parameter variation. From Fig. 5.11 with $T_L(s) = 0$, we get

$$\frac{\omega(s)}{V_a(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K_T}{R_a(Js+B)}}{1+\dfrac{K_T K_B}{R_a(Js+B)}} = \frac{\dfrac{K_T}{R_a(Js+B)}}{\dfrac{R_a(Js+B)+K_T K_b}{R_a(Js+B)}} \qquad \text{....(i)}$$

$$= \frac{K_T}{R_a Js + (R_a B + K_T K_b)} = \frac{\dfrac{K_T}{R_a B + K_T K_b}}{1+\dfrac{R_a J}{R_a B + K_T K_b}s} = \frac{K}{1+\tau s}$$

where $$K = \frac{K_T}{R_a B + K_T K_b}$$

and $$\tau = \frac{R_a J}{R_a B + K_T K_b}$$

From Fig. 5.11, the

$$\text{Overall Transfer function, } T.F. = \frac{\omega(s)}{V_r(s)} = \frac{\dfrac{K_a K}{1+\tau s}}{1+\dfrac{K_a K K_t}{1+\tau s}}$$

$$= \frac{K_a K}{1 + \tau s + K_a K K_t} \qquad \text{....(ii)}$$

The sensitivity of the system for open-loop operation for any variation of the value of K is unity whereas the sensitivity for the closed-loop system considering expression (ii) and assuming values of system parameters when calculated will be less than unity.

5.3.2 Effect of Feedback on Transient Response

Transient response is the response of a system with respect to time before steady-state is reached. To understand the effect of feedback on transient response, we may again consider the speed control system of Fig. 5.10.

Applying a step input of $V_r(s) = \frac{K_2}{s}$, we get the speed of the motor for a closed-loop operation as

$$\omega(s) = \frac{K V_r(s)}{\tau s + K K_t + 1} = \frac{K K_2 / \tau}{s(s + \frac{K K_t + 1}{\tau})}.$$

Taking inverse Laplace transform, we get

$$\omega(t) = \frac{K K_2}{1 + K K_t}\left[1 - e^{-\left(\frac{K K_t + 1}{\tau}\right)t}\right] \qquad \text{for a closed-loop system and}$$

$$= K K_2 (1 - e^{-\frac{-t}{\tau}}) \qquad \text{for an open-loop system,}$$

where $\qquad K = K_a K_1 = \frac{K_a K_T}{R_a B + K_T K_b};$

and $\qquad \tau = \frac{R_a J}{R_a B + K_T K_b}$

It is clear from the above expression of $\omega(t)$ that the open-loop system with a large time constant τ exhibits poor transient response. There is little choice but to choose another motor, if possible, with a lower time constant. As τ is dominated by the load inertia J, little hope remains for much alternation of transient response of an open-loop system. However in the closed-loop system, the time constant is reduced by a factor of $(1 + KK_t)$ and hence the transient response can be adjusted by varying amplifier gain K_a and tachometer gain K_t, if needed.

5.3.3 Effect of Feedback on Disturbance Signal

A disturbance signal is an unwanted input signal that affects the system's output. The block diagram of a closed-loop system with disturbance signal $T_d(s)$ is shown in Fig. 5.12.

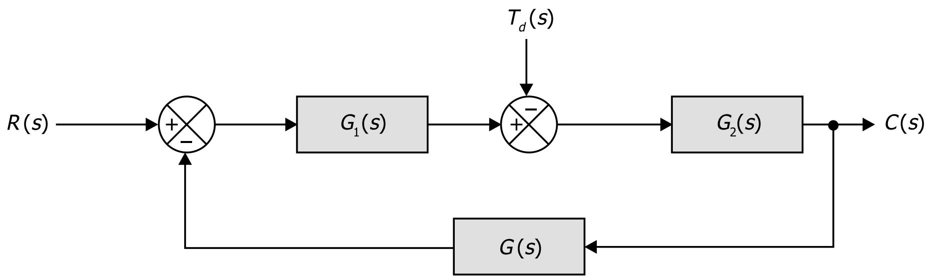


Fig. 5.12 *Block diagram of a closed-loop system with disturbance signal*

The ratio of output to disturbance signal is obtained by putting $R(s) = 0$ in Fig. 5.12.

$$\frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

If $|G_1(s)\ G_2(s)\ H(s)| >> 1$ in the working range of frequencies, then

$$\frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H(s)} = \frac{-1}{G_1(s)H(s)}$$

The effect of disturbance can be minimized through feedback if $G_1(s)$ is made sufficiently large. Again we may consider the example of the speed control system of Fig. 5.10.

Comparing Fig. 5.11 with Fig. 5.12, we get

$$R(s) = V_r(s);\ C(s) = \omega(s);$$

$$G_1(s) = \frac{K_a K_T}{R_a};\quad G_2(s) = \frac{1}{Js + B};$$

$$H(s) = K_t + \frac{K_b}{K_a}.$$

Thus the steady-state error for a closed-loop speed control system with $R(s) = V_r(s) = 0$ and $T_d(s) = A/s$ is given by

$$e_{ss}^{CL} = \lim_{s \to 0} s[R(s) - C(s)] = \lim_{s \to 0} s[-C(s)]$$

$$= \lim_{s \to 0} \frac{sG_2(s)T_d(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$= \lim_{s \to 0} \frac{\dfrac{A}{(Js + B)}}{1 + \dfrac{K_a K_T}{R_a}\left(\dfrac{1}{Js + B}\right)\left(K_t + \dfrac{K_b}{K_a}\right)}$$

$$= \frac{AR_a}{R_a B + K_T(K_b + K_a K_t)}$$

$K_t = 0$ gives the steady state error for an open-loop speed control system with $R(s) = V_r(s) = 0$ and $T_s(s) = A/s$.

$$e_{ss}^{OL} = \frac{AR_a}{R_a B + K_T K_b}$$

From the above two expressions of steady-state errors it is clears that by employing feedback the effect of load torque disturbance on system response can be considerably reduced by increasing the gains of amplifier and tachogenerator.

The primary reason for introducing feedback in a control system is to reduce the effects of disturbances and noise signal occurring within the feedback loop. A noise signal found in many systems is the noise generated from the measurement sensor $H(s)$. The effect of such noise on the output $C(s)$ is obtained from the signal flow graph of Fig. 5.13.

$$C(s) = \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s)} N(s)$$

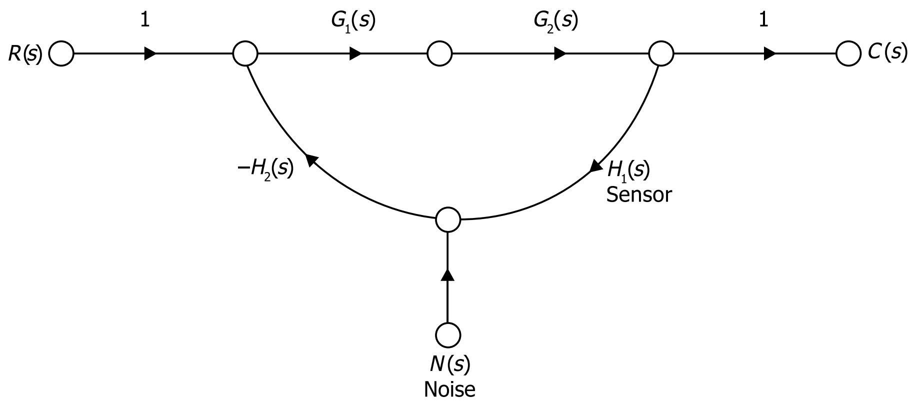


Fig. 5.13 *Closed-loop control system with measurement noise*

For large values of loop gain of $|G_1(s)G_2(s)H_1(s)H_2(s)| >> 1$, the system output becomes

$$C(s) = \frac{-1}{H_1(s)} N(s)$$

From the above expression it is clear that to maximise the signal-to-noise ratio, the output of measurement sensor $H_1(s)$ must be maximized. So the feedback element $H(s)$ must be well designed and operated with minimum noise, drift and parameter variation which is also supported by the sensitivity $S_H^M \cong -1$. The quality and constancy of feedback sensors can usually be achieved because the feedback elements operate at low power levels and can be well designed at reasonable cost.

5.3.4 Effect of Feedback on Steady-State Error

From Fig. 5.3(a) we may write the error $E(s)$ for open and closed-loop system as follows.

$$E(s) = R(s) - C(s)$$

$$= [1 - G(s)]R(s) \quad \text{for open-loop system and}$$

$$= \frac{R(s)}{1+G(s)} \quad \text{for closed-loop system with } H(s) = 1.$$

So, the steady-state error for open-loop system with step input is

$$\begin{aligned} e_{ss}^{OL} &= \lim_{s\to 0} SE(s) \\ &= \lim_{s\to 0} s[1-G(s)]\frac{1}{s} \\ &= 1-G(0) \end{aligned}$$

The steady-state error for closed-loop system with step input is

$$\begin{aligned} e_{ss}^{CL} &= \lim_{s\to 0} s[R(s)-C(s)] \\ &= \lim_{s\to 0} s\frac{1}{1+G(s)}\left(\frac{1}{s}\right) \\ &= \frac{1}{1+G(0)}. \end{aligned}$$

$G(0)$ is often called the DC gain and is normally greater than unity. So, the steady-state error for an open-loop system will be of significant magnitude as compared to that for a closed-loop system with a reasonably large DC loop gain $G(0)$.

For example, the steady-state error for open- and closed-loop speed control has been calculated in Section 5.5. There, it is evident that the steady-state error can be reduced by using feedback.

5.3.5 Effect of Feedback on Overall Gain

The overall transfer function of an open-loop system and a closed-loop system are respectively written as

For open-loop system,

$$\frac{C(s)}{R(s)} = G(s)$$

For closed-loop system,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Transfer function represents the gain.

Thus, the gain of an open-loop system is $G(s)$. When we use negative feedback, this gain gets reduced by a factor $1/[1 + G(s)H(s)]$.

5.3.6 Effect of Feedback on Stability

Use of feedback improves the stability of a system. From the transfer function, we can examine the location of the poles in the s-plane. If the poles get shifted more to the left-hand side of the imaginary axis, we can say that the system becomes more stable. For example, let us say that the span-loop transfer function $G(s)$ is $K/s + \tau$

The pole is located at $s = -\tau$.

The overall transfer function of the system with unity negative feedback will be

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s}+\tau}{1+\frac{K}{s+\tau}} = \frac{K}{s+(\tau+K)}$$

Now, the pole gets shifted to

$$s = -(\tau + K)$$

The pole gets shifted from $s = -\tau$ to $s = -(\tau + K)$.

Thus, we can see that feedback can make the system more stable.

Example 5.1 An open-loop control system is shown in the form of block diagram in Fig. 5.14. Examine the effect of incorporating a feedback in the control system.

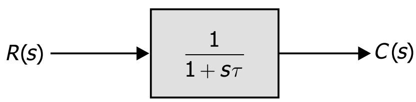


Fig. 5.14 An open-loop control system

Solution

From Fig. 5.14, the transfer function is expressed as

$$\frac{C(s)}{R(s)} = \frac{1}{1+s\tau}$$

where τ is the time constant.

The time response of the system for a step input is calculated by taking $R(s) = 1/s$.

Thus, $$C(s) = \frac{1}{(1+s\tau)} \times \frac{1}{s}$$

Taking inverse Laplace,

$$C(t) = 1 - e^{-t/\tau} \qquad \text{....(a)}$$

Now, let us use a feedback loop for the system as shown in Fig. 5.15.

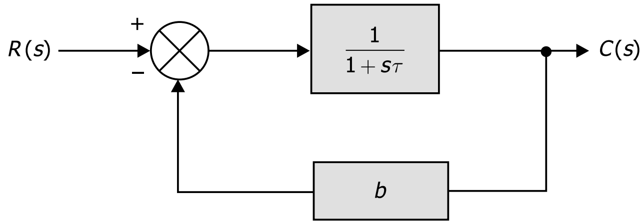


Fig. 5.15 *A feedback loop introduced in the open-loop system of Fig. 5.14*

We have taken feedback

$$H(s) = b$$

The transfer function of the system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\dfrac{1}{1+s\tau}}{1 + \dfrac{1}{1+s\tau} \times b}$$

$$= \frac{1}{1 + s\tau + b} = \frac{\dfrac{1}{\tau}}{s + \dfrac{1+b}{\tau}}$$

Taking $R(s) = 1/s$,

$$C(s) = \frac{\dfrac{1}{\tau}}{s + \left(\dfrac{1+b}{\tau}\right)} \times \frac{1}{s}$$

Taking Laplace inverse,

$$C(t) = \frac{1}{1+b}\left[b - e^{-\frac{t}{t/1+b}}\right] \qquad \text{....(b)}$$

$$= \frac{1}{1+b}\left[b - e^{-\frac{t}{\tau'}}\right]$$

where, τ' = new time constant = $\tau/1 + b$.

This shows that due to introduction of feedback, the time constant is reduced if 'b' is positive. A reduced time constant implies that the system will now be faster when there was no feedback.

Thus we can state that introduction of a negative feedback improves the response of a control system.

5.4 THE COST OF FEEDBACK

The use of feedback has several advantages as outlined in the previous sections. These advantages have an attended cost due to an increased number of components and complexity in the system.

In an open-loop system the transfer function is $G(s)$ and is reduced to $G(s)/[1 + G(s)\,H(s)]$ in a feedback (closed-loop) system. So, the loss of gain by the same factor of $1/[1 + G(s)\,H(s)]$ that reduces the sensitivity of the system to parameter variations is again an added cost of using feedback. However, it should be noted that although the gain of input-output transmittance is reduced through feedback, the system does retain the substantial power gain of a power amplifier and actuator that is fully utilized in the system.

Finally the addition of feedback may cause the system to be unstable. This possibility of instability is again another cost of using feedback.

However, in most cases, the advantages of using feedback outweigh the disadvantages and feedback is utilised in practice.

REVIEW QUESTIONS

5.1 A position control system is represented by the block diagram given below. Determine the sensitivity of closed-loop transfer function M with respect to G and H for $\omega = 1.5$ rad/sec.

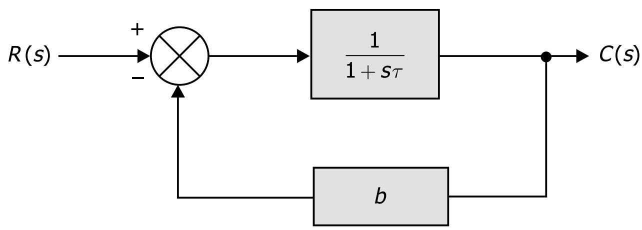


Fig. 5.16

5.2 For the audio system shown in Fig. 5.17 calculate

i) The sensitivity with respect to K_2.

ii) The effect of disturbance on the output $C(s)$.

iii) The value of amplifier gain K_1 to minimise the effect of disturbance.

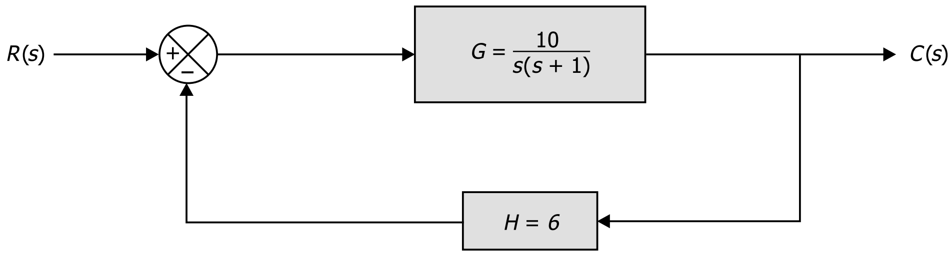


Fig. 5.17

5.3 For a speed control system shown below, the constants are given as follows.

- Motor back emf constant, $K_b = 5$ volt/rad/sec.
- Moment of inertia of motor and load, $J = 5$ kg m^2.
- Armature resistance of motor and generator, $R_a = 1$ ohm.
- Amplifier gain, $K_a = 5$ amps/volt.
- Generator gain constant, $K_g = 50$ volts/amp.
- Tachometer constant, $K_t = 0.5$ volt/rad/sec.

Assume that (a) the reference and feedback tachometer are identical, (b) friction of motor and load are negligible, and (c) generator field time constant is negligible and its generated voltage is K volt/amp.

i) For a sudden step input of 10 rad/sec, find the time variation of output speed ω. What is the steady state value of ω?

ii) If the same steady-state speed is achieved in the open-loop system by adjusting gain K_a, determine the speed variation with time and compare the speed of response for the tow cases.

iii) Obtain the sensitivity of ω to changes in amplifier gain K_a and generator speed ω_g, with and without feedback.

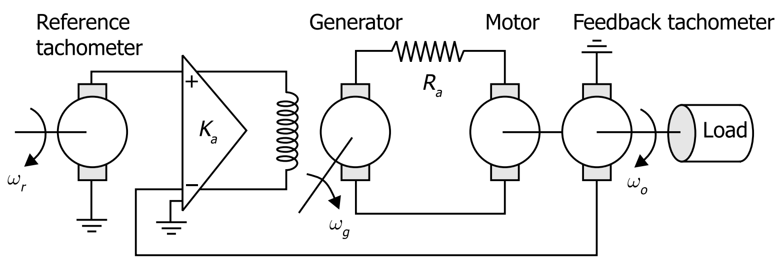


Fig. 5.18

5.4 In the paper-making process, uniform consistency of stock output has to be maintained as it progresses to drying and rolling. A thick stock consistency dilution control system and its signal-flow graph is shown in Fig. 5.19.

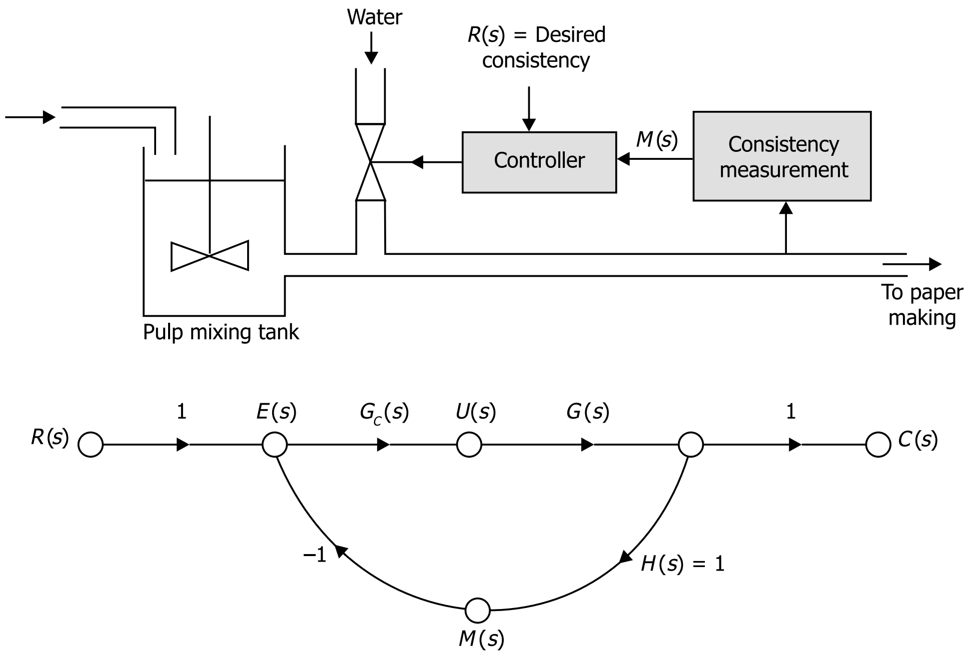


Fig. 5.19

If
$$G_c(s) = \frac{K}{8s+1} \text{ and}$$

$$G(s) = \frac{1}{4s+1},$$

determine

i) Closed-loop transfer function $T(s) = C(s)/R(s)$.

ii) The sensitivity S_K^T.

iii) Steady-state error for a step change of desired consistency $R(s) = A/s$.

iv) The value of K required to maintain a steady-state error of 1%.

5.5 The block diagram of a feedback control system with disturbance and noise is given in Fig. 5.14.

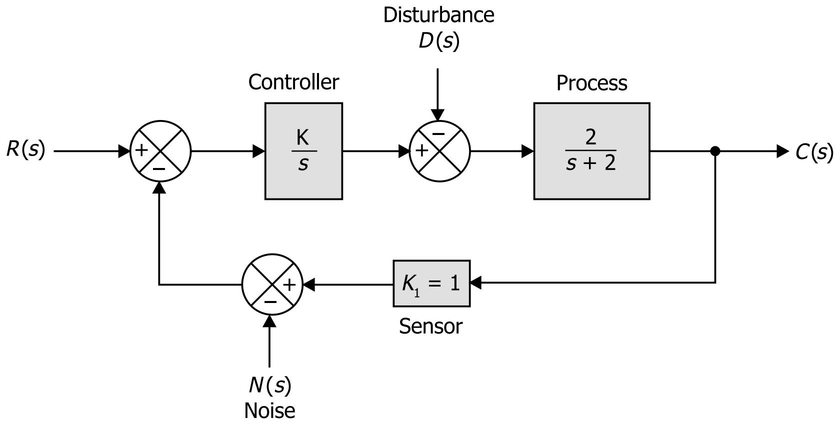


Fig. 5.20

Assuming $R(s) = 0$, determine

i) The effect of the disturbance on output $C(s)$.

ii) The effect of the noise on output $C(s)$.

iii) The best value of K when $1 \leq K \leq 100$ so that the effect of steady-state error due to disturbance and noise is minimised. Assume $D(s) = A/s$ and $N(s) = B/s$.

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6

ERROR ANALYSIS

6.1 INTRODUCTION

Errors in control systems can be attributed to many factors. Changes in the reference input will cause unavoidable error during transient period and may cause steady state error also. Imperfections in the system components such as static friction, backlash (a sudden strong response or reaction), amplifier drift, as well as aging or deterioration will cause errors at steady state.

6.2 TYPES OF INPUT SIGNALS

In analysing or designing a particular control system, we must have a basis of comparison of performance of various control systems. This basis may be set by specifying particular test input signals and by comparing the responses of various systems to these input signals. Usually the input signals to control systems are not fully known.

From experience it has been observed that the actual input signals which severely strain a control system are: *a sudden shock*, *a sudden change*, *a constantly increasing change* or *a constantly accelerating change*. Therefore, system dynamic behaviour for analysis and design can be studied and compared under application of *standard test signals such as an impulse signal (sudden shock)*, *a step signal (sudden change)*, *a ramp signal (constant velocity)* or *a parabolic signal (constant acceleration)*. *Sinusoidal signal is also another important test signal*. With these test signals, mathematical and experimental analysis of control systems can be carried out easily since these signals are very simple functions of time.

6.2.1 Standard Test Signals

1. **Step Signal:** A step signal gives an instantaneous change in the value of the reference $r(t)$ as shown in Fig. 6.1(a).

that is, $r(t) = 0$; $t < 0$ and $r(t) = A$; $t > 0$. The Laplace transform is, $R(s) = \dfrac{A}{s}$

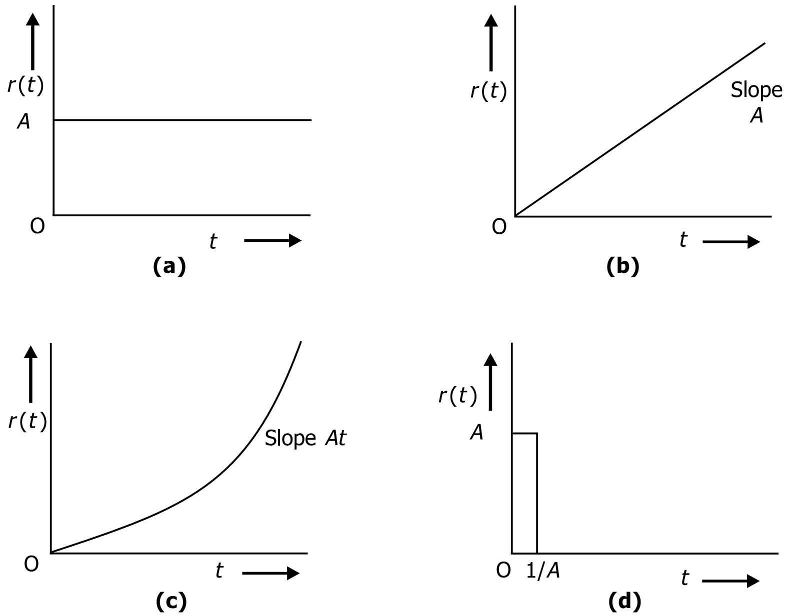


Fig. 6.1 *Standard test signals: (a) Step signal; (b) Ramp signal; (c) Parabolic signal; (d) Impulse signal*

2. **Ramp Signal:** A ramp signal gives a constant change in the value of the reference variable $r(t)$ with respect to time, as shown in Fig. 6.1(b). It is also the integral of a step signal.

   $r(t) = 0$; $t < 0$ and $r(t) = At$; $t > 0$. The Laplace transform is, $R(s) = \dfrac{A}{s^2}$.

3. **Parabolic Signal:** A parabolic signal gives an accelerating change in the value of the reference variable $r(t)$. This is the integral of ramp signal. It is shown in Fig. 6.1(c).

$$r(t) = 0;\ t < 0 \text{ and } r(t) = \frac{At^2}{2};\ t > 0$$

   The Laplace transform is, $R(s) = \dfrac{A}{S^3}$.

4. **Impulse Signal:** The unit-impulse signal gives an infinite magnitude to the value of the reference variable at $t = 0$ and a zero value everywhere except at $t = 0$. Since a perfect impulse cannot be achieved in practice it is approximated as a pulse of small width and of unit area shown in Fig. 6.1(d). Mathematically, the impulse function $r(t)$ can be

written as $r(t) = u(t)$, where $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$, that is, $u(t)$ is a unity step function.

$r(t) = 0$; $t < 0$ and $r(t) = 1$; $t = 0$.

The Laplace transform is, $R(s) = 1$.

Whether or not a given system will exhibit steady-state error for a given type of input depends upon the type of open-loop transfer function of the system.

6.3 CLASSIFICATION OF CONTROL SYSTEMS

Control systems may be of different orders viz. *zero order system*, *first order system*, *second order system*, and so on. The generalized relation between a particular input, say q_i and the corresponding output, say q_0, with proper simplified assumptions, can be written as

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-t} q_0}{dt^{n-1}} + \ldots + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \ldots + b_1 \frac{dq_i}{dt} + b_0 q_i \quad \text{....(6.1)}$$

where q_0 is the output quantity

q_i is the input quantity

t is the time

a's and b's are combination of system parameters which are assumed to be constants.

Taking Laplace transform and assuming all initial conditions to be zero, we get

$$\left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right] Q_0(s) = \left[b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right] Q_i(s)$$

$$\text{Transfer function, } G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right)}$$

The T.F. can also be written as,

$$= \frac{k(\tau_a s + 1)(\tau_b s + 1)\ldots}{s^N (\tau_1 s + 1)(\tau_2 s + 1)\ldots} \quad \text{....(6.1a)}$$

The power of s in the denominator determines the order of the system.

6.3.1 Zero Order System

In the generalized system equation, i.e. equation (6.1) if all the a's and b's are made zero except a_0 and b_0, we will get

$$q_0 = \frac{b_0}{a_0} q_i = kq_i \text{ where } k = \frac{b_0}{a_0} \quad \text{....(6.2)}$$

Therefore, $q_0 = kq_i$ represents a zero order system.

A simple example of a zero order system is a potentiometer where the output voltage is a fraction of the input voltage, i.e. $e_0 = ke_i$.

6.3.2 First Order System

If in the generalized system equation of (6.1), if all a's and b's other than a_1, a_0 and b_0 are taken as zero, we will get

$$a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

or,
$$\frac{a_1}{a_0} \frac{dq_0}{dt} + q_0 = \frac{b_0}{a_0} q_i$$

or,
$$\tau \frac{dq_0}{dt} + q_0 = kq_i \qquad \text{....(6.3)}$$

where, $\tau = \dfrac{a_1}{a_0}$ is called the *time constant*

$k = \dfrac{b_0}{a_0}$ is called the *static sensitivity*

Any system that follows the above relation of equation (6.3) is called a first order system. Taking Laplace transform,

$$\tau s Q_0(s) + Q_0(s) = kQ_i(s)$$

Transfer Function, $$G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{k}{\tau s + 1}$$

We have seen that a simple $R-C$ network having an input e_i and output across C as e_0 will have a similar transfer function and therefore can be called a first order system.

6.3.3 Second Order System

In the generalized equation if all a's and b's are made zero except a_2, a_1, a_0 and b_0 then we will get the equation for the second order system. Accordingly a second order system is one which follows the equation,

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

or,
$$\frac{a_2}{a_0} \frac{d^2 q_0}{dt^2} + \frac{a_1}{a_0} \frac{dq_0}{dt} + q_0 = \frac{b_0}{a_0} q_i$$

or,
$$\frac{1}{\omega_n^2} \frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_0}{dt} + q_0 = kq_i \qquad \text{....(6.4)}$$

where $\omega_n = \sqrt{\dfrac{a_0}{a_2}}$, is called the *undamped natural frequency*

$\zeta = \dfrac{a_1}{2\sqrt{a_0 a_2}}$, is called *the damping ratio.*

$$\left[\because \frac{a_1}{a_0} = \frac{2\zeta}{\omega_n}, \therefore \zeta = \frac{a_1\omega_n}{2a_0} = \frac{a_1\sqrt{a_0}}{2a_0\sqrt{a_2}} = \frac{a_1}{2\sqrt{a_0 a_2}}\right]$$

Taking Laplace transform of equation (6.4), we get

$$\left[\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n}\right] Q_0(s) = KQ_i(s)$$

$$\text{T.F.}, G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{K}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\zeta s}{\omega_n} + 1}$$

$$= \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

A mass spring damper system is an example of second order system. By referring to Fig. 2.4 and equation (2.9) in chapter 2, we can rewrite the equation of forces as

$$M\frac{d^2x}{dt} + B\frac{dx}{dt} + kx = f(t)$$

Taking Laplace transform,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} = \frac{1/K}{\dfrac{M}{K}s^2 + \dfrac{B}{K}s + 1} = \frac{K_1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $\omega_n = \sqrt{\dfrac{k}{M}}$, is the *undamped natural frequency in rad/sec*

and $\xi = \dfrac{B}{2\sqrt{KM}}$, is the *damping ratio*.

It will be seen later that as the system order number increases, the accuracy improves but the system tends to become unstable.

Before going into error and stability analysis, we will discuss steady state and dynamic error coefficients.

6.4 STEADY-STATE ERROR

The difference between the desired response and the actual response of a system is called the error. The error, if any, when the system settles down or stabilizes is called the steady-state error.

Transient state refers to the oscillatory condition of the system output, i.e. during the transient time before the system comes to final steady-state condition.

During design stage, a system is tested for its steady-state and transient state errors in simulated condition. Modifications are made in the system parameters including the amplifier gain setting so as to obtain the desired steady state and transient state performance of the system. The error is measured in a simulated condition by applying certain test signals described earlier.

Steady-state error is a measure of the accuracy of a control system. The steady-state error of a control system is judged by the steady-state error due to step, ramp or acceleration input. We shall investigate a type of steady-state error which is caused by the incapability of a system to follow particular types of inputs. Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input but the same system may exhibit non-zero, i.e. some steady-state error to a ramp input. We can, however, eliminate this error by modifying the system structure. Generally, steady state error should be as low as possible.

Fig. 6.2 shows the general block-diagram of a closed-loop system described earlier.

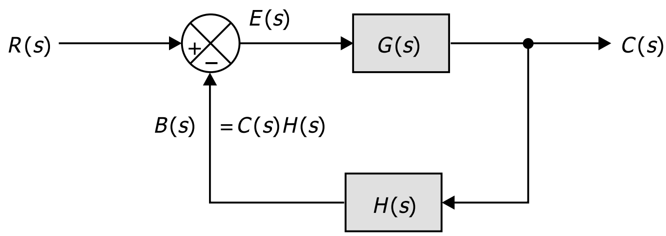


Fig. 6.2 *Block diagram of a closed-loop control system*

With reference to Fig. 6.2,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

and $$E(s) = R(s) - B(s)$$

or, $$E(s) = R(s) - C(s)\,H(s).$$

Dividing both sides by $R(s)$,

$$\frac{E(s)}{R(s)} = \frac{R(s)-C(s)H(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}H(s) = 1 - \frac{G(s)H(s)}{1+G(s)H(s)}$$

or, $$\frac{E(S)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$\therefore$ Error, $$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

The steady-state error is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ [from Final Value Theorem]

$$e_{ss} = \lim_{t\to\infty} e(t)$$

So, steady state error $$e_{ss} = \lim_{t\to\infty} e(t)$$
$$= \lim_{s\to o} sE(s)$$
$$= \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

6.4.1 Static Position Error Coefficient (K_p)

The steady-state error of the system for a unit step input is

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s} \qquad \left[R(s) = \frac{1}{s} \text{ for unit step input}\right]$$

or, $$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+K_p}$$

where, $K_p = \lim_{s\to 0} G(s)H(s)$; K_p is called the static position error coefficient.

Now, we shall find the value of K_p for different types of systems, that is, type 0, type 1, type 2.

Type is defined as the number of open-loop poles at the origin and is indicated by power of s, i.e. s^N in the denominator of the transfer function. For type 0 system, $s^0 = 1$; for type 1, $s^1 = s$; for type 2, $s^2 = s^2$, and so on.

i) For a type 0 system,

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)....}{s^0 (T_1 s+1)(T_2 s+1)....} \text{ (from equation (6.1a))}$$

$$K_p = \lim_{s\to 0} G(s)H(s) \frac{K(0+1)(0+1)....}{(0+1)(0+1)....}$$
$$= \frac{K}{s^0} = K$$

ii) For type 1 or higher system,

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)....}{s^N (T_1 s+1)(T_2 s+1)....}$$

$$K_p = \lim_{s\to 0} G(s)H(s) \frac{K(0+1)(0+1)....}{0(0+1)(0+1)....}$$

$$K_p = \infty \quad (\text{for } N \geq 1)$$

Now, we shall find the static position error for different systems at steady state.

i) For type 0 system,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K} \qquad [\because K_p = K]$$

ii) For type 1 or higher systems,

$$e_{ss} = \frac{1}{1+K_p}$$

$$= \frac{1}{1+\infty} = 0 \qquad [\because K_p = \infty]$$

Thus, for a unit step input, steady-state error for different types of systems is finite. Hence, every type of system is capable of following step input. Though type 0 system shows some error, higher type of systems can respond to step input very accurately.

6.4.2 Static Velocity Error Coefficient (K_v)

Static velocity error coefficient is associated with e_{ss} for unit ramp input. The steady-state actuating error of the system with a unit ramp input (unit velocity input) is given by

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s^2} \qquad \left[\because R(s) = \frac{1}{s^2}\right]$$

$$= \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} = \frac{1}{K_v}$$

where, $K_v = \lim_{s\to 0} sG(s)H(s)$; K_v is called the *static velocity error coefficient*.

Now, we shall find the static velocity error coefficient for different types of systems.

i) For a type 0 system,

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)....}{s^0 (T_1 s+1)(T_2 s+1)....}$$

$$\therefore \qquad = \lim_{s\to 0} sG(s)H(s)$$

$$= \lim_{s\to 0} \frac{sK(T_a s+1)(T_b s+1)....}{(T_1 s+1)(T_2 s+1)....} = 0$$

ii) For a type 1 system,

$$K_v = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} s\frac{K(T_a s+1)(T_b s+1)....}{s(T_1 s+1)(T_2 s+1)....} = K$$

iii) For a type 2 or higher system,

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\frac{K(T_a s+1)(T_b s+1)....}{s^{N-1}(T_1 s+1)(T_2 s+1)....}$$

$$= \lim_{s\to 0} s\frac{K(T_a s+1)(T_b s+1)....}{s^{N-1}(T_1 s+1)(T_2 s+1)....} = \infty \qquad [\because N \geq 2]$$

Now, we shall find static velocity error for different types of systems at steady state.

i) For type 0 system,

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty \qquad [\because K_v = 0]$$

ii) For type 1 system,

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K} \qquad [\because K_v = K]$$

iii) For a type 2 or higher systems,

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty = 0 \qquad [\because K_v = \infty]$$

For ramp input, steady-state error for type 0 system is infinite. Hence, a type 0 system is not capable of following ramp input. The static velocity error for type 1 system is finite. But static velocity error for type 2 system or higher system is zero. So type 2 or higher systems are capable of following a ramp input very accurately.

We see from this analysis that as we move from type 0 to type 2 or higher systems, the static velocity error goes on decreasing. But with increase in the type number, the number of poles on the $j\omega$-axis also goes on increasing, which will make the system response more oscillatory.

Hence, we have to make a compromise between stability and accuracy.

6.4.3 Static Acceleration Error Coefficient (K_a)

Static acceleration error coefficient is associated with e_{ss} for unit parabolic input. The steady-state actuating error of the system with unit-parabolic input (acceleration) is given by

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s^3} \qquad \left[\because R(s) = \frac{1}{s^3}\right]$$

$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)H(s)} = \frac{1}{K_a}$$

where $K_a = \lim_{s\to 0} s^2 G(s)H(s)$.

Now, we shall find the static acceleration error coefficient for different types of systems.

i) For type 0 system,

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)....}{s^0 (T_1 s+1)(T_2 s+1)....} = 0$$

ii) For a type 1 system,

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)....}{s(T_1 s+1)(T_2 s+1)....}$$

$$= \lim_{s\to 0} \frac{sK(T_a s+1)(T_b s+1)....}{(T_1 s+1)(T_2 s+1)....} = 0$$

iii) For a type 2 system

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)....}{s^2 (T_1 s+1)(T_2 s+1)....} = K$$

iv) For a type 3 or higher system,

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)....}{s^N (T_1 s+1)(T_2 s+1)....}$$

$$= \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)....}{s^{N-2} (T_1 s+1)(T_2 s+1)....} = \infty \qquad [\therefore N \geq 3]$$

Now we shall find the static acceleration error for different types of systems at steady state.

i) For a type 0 system,

$$e_{ss} = \frac{1}{K_a} = \infty \qquad [\therefore K_a = 0]$$

ii) For a type 1 system,

$$e_{ss} = \frac{1}{K_a} = \infty \qquad [\because K_a = 0]$$

iii) For a type 2 system,

$$e_{ss} = \frac{1}{K_a} = \frac{1}{K} \qquad [\because K_a = K]$$

iv) For a type 3 or higher system,

$$e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0 \qquad [\because K_a = \infty]$$

Thus type 0 and type 1 systems are capable of following a parabolic input. For a type 2 system, the error is finite, but for type 3 or higher systems the error is zero. Again, as we increase the type numbers, the error goes on reducing.

The steady-state errors in terms of forward path gain (K) of type 0, type 1 and type 2 closed-loop systems have been summarised in Table 6.1.

Table 6.1 *Steady-state Errors of Closed-loop Systems of Different Types*

System	Step Input $r(t)$ = 1	Ramp Input $r(t)$ = 1	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0	$\frac{1}{1+K}$	∞	∞
Type 1	0	$\frac{1}{K}$	∞
Type 2	0	0	$\frac{1}{K}$

The terms “position error”, “velocity error”, “acceleration error” mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out, the input and output move at the same velocity but have a finite position difference.

The error coefficient K_p, K_v and K_a, describe the ability of a system to reduce or eliminate steady-state error. It is desirable to increase the error coefficients while maintaining the transient response within an acceptable range. From Table 6.1, it can be seen that a type 0 system gives error for all the three types of inputs. A type 2 system gives error due to one type of input only, which is finite. So a type 2 system is better than a type 0 system or a type 1 system from the steady-state error point of view. Higher types of systems are better from the steady-state error point of view but are less stable.

6.5 DYNAMIC ERROR COEFFICIENTS

The static error coefficients suffer from the drawback of providing no information on the steady-state error when inputs are other than step, ramp, or parabolic inputs. The steady-state error obtained through the static error coefficients is either zero, a finite non-zero value, or infinity and do not provide any information on how the error varies with time.

The dynamic error coefficients, on the other hand, provide a simple way of estimating the error signal to arbitrary inputs and the steady-state error, without solving the system differential equation.

Now we shall introduce an error series for obtaining a more generalised steady-state error in terms of dynamic error coefficients, input function and its derivatives. Referring to Fig. 6.2 we have,

$$E(s) = \frac{R(s)}{1+G(s)H(s)} = W(s)R(s)$$

where
$$W(s) = \frac{1}{1+G(s)H(s)}$$

is known as the *error transfer function*.

Using the principle of convolution integral, the error signal $e(t)$ may be written as

$$e(t) = \int_0^t \omega(\tau)r(t-\tau)d\tau \qquad \text{....(6.5)}$$

The limit of the integral has been taken from 0 to t instead of $-\infty$ to t, as $r(t) = 0$ for $t < 0$. Assuming derivatives of $r(t)$ to exist, the function $r(t - \tau)$ can be expanded into the Taylor series as follows:

$$r(t-\tau) = r(t) - \tau\dot{r}(t) + \frac{\tau^2}{2!}\ddot{r}(t) - \frac{\tau^2}{3!}\dddot{r}(t) + \text{.........}$$

Substituting the value of $r(t - r)$ into equation (6.5) we get,

$$e(t) = r(t)\int_0^t \omega(\tau)d\tau - \dot{r}(t)\int_0^t \tau\omega(\tau)d\tau + \ddot{r}(t)\int_0^t \frac{\tau^2}{2!}\omega(\tau)d\tau - \text{.........} \qquad \text{....(6.6)}$$

Denoting $\lim_{t\to\infty} r(t) = r_s(s)$, that is, taking steady-state part of $r(t)$ as $r_s(t)$, a more generalised steady-state error (e_{ss}) is obtained from equation (6.6) as:

$$e_{ss} = \lim_{t\to\infty} e(t) = C_0 r_s(t) + C_1\dot{r}_s(t) + \frac{C_2}{2!}\ddot{r}_s(t) + \ldots + \frac{C_n}{n!}r_s^{(n)}(t) + \ldots \qquad \text{....(6.7)}$$

where
$$C_i = (-1)^i \int_0^\infty \tau^i \omega(\tau)d\tau$$

for $\quad i = 0, 1, 2, \ldots, n, \ldots.$

Equation (6.7) is called the *error series* and the coefficients C_i are called *generalised error coefficients* or *error coefficients*.

From the definition of Laplace transform, we have

$$W(s) = \int_0^{\infty} \omega(\tau) e^{-\tau s} d\tau \qquad \text{....(6.8)}$$

So the error coefficients can be readily obtained from equation (6.8) as follows.

$$\left.\begin{aligned} C_0 &= \lim_{s \to 0} \omega(s) \\ C_i &= \lim_{s \to 0} \frac{d^i \omega(s)}{ds^i} \end{aligned}\right\} \qquad \text{....(6.9)}$$

for $\quad i = 1, 2, \ldots, n, \ldots$

We may now define the *dynamic error coefficients* K_i as

$$K_i = \frac{(i-1)!}{C_{i-1}} \qquad \text{....(6.10)}$$

for $\quad i = 1, 2, 3, \ldots$

The dynamic error in terms of dynamic error coefficients can be obtained by removing the limit from equation (6.7) as follows.

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!} \ddot{r}(t) + \ldots\ldots\ldots$$

or,
$$e(t) = \frac{1}{K_1} r(t) + \frac{1}{K_2} \dot{r}(t) + \frac{1}{K_3} \ddot{r}(t) + \ldots\ldots\ldots \qquad \text{....(6.11)}$$

Taking Laplace transform on both sides of equation (6.11) with zero initial conditions, we get

$$E(s) = \frac{1}{K_1} R(s) + \frac{1}{K_2} sR(s) + \frac{1}{K_3} s^2 R(s) + \ldots\ldots\ldots$$

or,
$$\frac{E(s)}{R(s)} = \frac{1}{K_1} + \frac{1}{K_2} s + \frac{1}{K_3} s^2 + \ldots\ldots\ldots \qquad \text{....(6.12)}$$

So, in other words, dynamic error coefficients may be obtained directly from $E(s)/R(s)$ when expanded into a series in ascending powers of s.

K_1 = dynamic position error coefficient;

K_2 = dynamic velocity error coefficient;

K_3 = dynamic acceleration error coefficient.

6.6 INTEGRAL SQUARE ERROR (ISE) AND ITS MINIMISATION

In the design of a control system, the performance specifications to be satisfied may be given in terms of transient response specifications to a specific input or in terms of a performance index which is a number that indicates the "goodness" of system performance. The performance indices which are generally used are as follows.

i) Integral square error

$$\text{ISE} = \int_0^{\infty} e^2(t)dt \qquad \text{....(6.13)}$$

ii) Integral of time multiplied square error

$$\text{ITSE} = \int_0^{\infty} te^2(t)dt \qquad \text{....(6.14)}$$

iii) Integral absolute error

$$\text{IAE} = \int_0^{\infty} |e(t)|\,dt \qquad \text{....(6.15)}$$

iv) Integral of time multiplied absolute error

$$\text{ITE} = \int_0^{\infty} t|e(t)|\,dt \qquad \text{....(6.16)}$$

To be useful, a performance index must have the following properties.

i) It must be a function of system parameters.
ii) It must offer selectivity, that is, an optimal adjustment of parameters must clearly distinguish non-optimal adjustment of parameters.
iii) It must exhibit an extremum that is, a maximum or a minimum.
iv) It must be easily computed, analytically or experimentally.

Although ITAE offers the best selectivity, ISE is extensively used for both deterministic and statistical inputs because of the ease of computing the integral both analytically and experimentally. The upper limit in ISE may be replaced by T which is chosen to be sufficiently large so that $e(t)$ $t > T$ is negligible. Usually T is estimated to be settling time (t_s) or, a multiple of settling time.

that is, $$\text{ISE} = \int_0^{T} e^2(t)dt$$

where $T = t_s$ or nt_s.

For a control system with reference input $r(t)$ and actual output $c(t)$, error $e(t)$ is defined by $e(t) = r(t) - c(t)$.

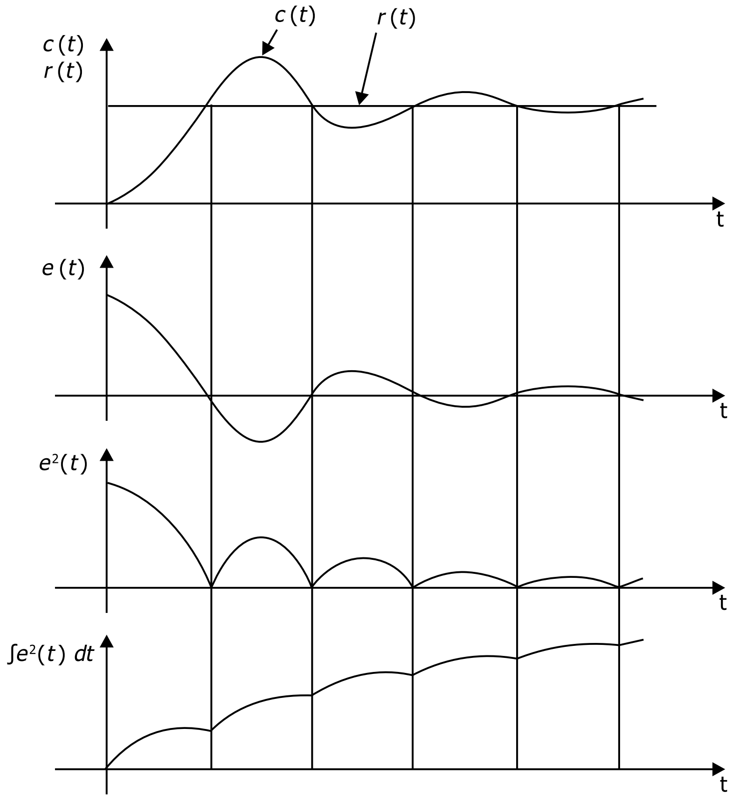


Fig. 6.3 *Performance indices of a control system*

Fig. 6.3 shows the curves of $r(t)$, $c(t)$, $e(t)$, $e^2(t)$ and $\int e^2(t)dt$ when the reference input is a unit step. ISE is the total area under the curve $e^2(t)$.

The main drawback of ISE is that a system designed by this criterion exhibits poor relative stability due to fast and oscillatory response caused by a rapid decrease in a large initial error. However, ISE is often of practical significance because its minimisation results in the minimisation of power consumption for some systems such as spacecraft.

An optimal control system results from the minimisation of a selected performance index. To define minimisation of performance index, let us consider the case of parameter optimisation which is solved as stated below.

If $J = J(K_1, K_2, \ldots, K_n)$ be the performance index expressed as a function of free parameters K_i (for $i = 1, 2, \ldots, n$) of the system with fixed configuration, then

i) The necessary conditions for J to be minimum are

$$\frac{\partial J}{\partial K_i} = 0 \quad (i = 1, 2, \ldots, n)$$

for a set of parameters $K_i (i = 1, 2, \ldots, n)$ that satisfy the above equations.

ii) The sufficient condition for J to be minimum is that for the same set of parameters K_i, $(i = 1, 2, \ldots, n)$ satisfying the necessary conditions, the Hessian matrix (H) given below must be positive which requires that all the principal minors (or eigen values) of H must be positive.

$$H = \begin{bmatrix} \dfrac{\partial^2 J}{\partial K_1^2} & \dfrac{\partial^2 J}{\partial K_1 \partial K_2} & \cdots\cdots & \dfrac{\partial^2 J}{\partial K_1 \partial K_n} \\ \dfrac{\partial^2 J}{\partial K_2 \partial K_1} & \dfrac{\partial^2 J}{\partial K_2^2} & \cdots\cdots & \dfrac{\partial^2 J}{\partial K_2 \partial K_n} \\ \cdots & \cdots & & \cdots \\ \dfrac{\partial^2 J}{\partial K_n \partial K_1} & \dfrac{\partial^2 J}{\partial K_n \partial K_2} & \cdots\cdots & \dfrac{\partial^2 J}{\partial K_n^2} \end{bmatrix}$$

The matrix H is always symmetric as $\dfrac{\partial^2 J}{\partial K_i \partial K_j} = \dfrac{\partial^2 J}{\partial K_j \partial K_i}$

iii) With more than one set of parameters (say, K_i', K_i'', and so on, for $i = 1, 2, \ldots, n$) satisfying the necessary as well as sufficient conditions for J to be minimum, the set of optimal parameters is the one that gives the smallest value of J.

Example 6.1 Consider the system shown in Fig. 6.4(a). The steady-state error to a unit ramp input is $e_{ss} = \dfrac{2\zeta}{\omega_n}$. Show that the steady-state error for ramp input may be eliminated if the input is introduced to the system through a proportional plus derivative element in Fig. 6.4(b).

Solution

From Fig. 6.4(b), we get

$$\frac{C(s)}{R(s)} = \frac{(1+Ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Again,

$$E(s) = R(s) - C(s)$$

$$= R(s) - \frac{(1+Ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$$

$$\therefore \qquad E(s) = \frac{s^2 + 2\zeta\omega_n s - \omega_n^2 Ks}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s).$$

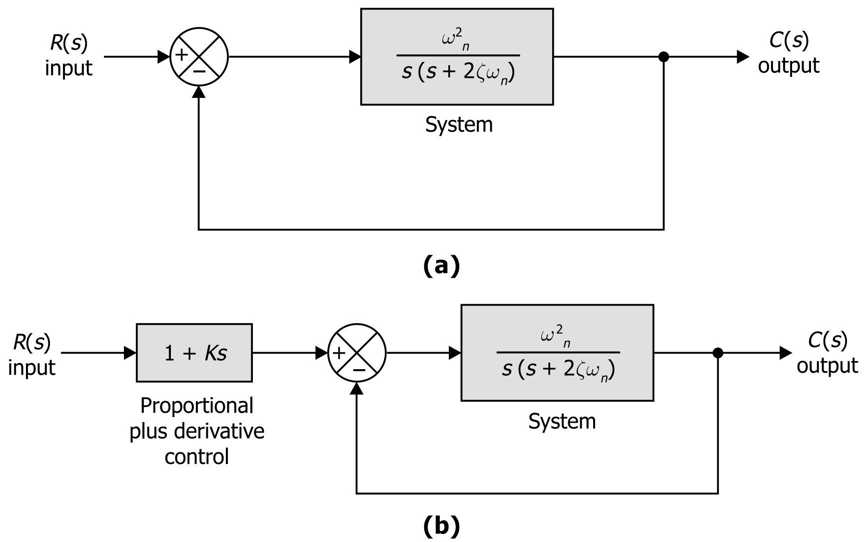


Fig. 6.4

So the steady-state error (e_{ss}) is given by

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} s \frac{s^2 + 2\zeta\omega_n s - \omega_n^2 Ks}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s^2} \qquad \left[\text{For unit ramp input } R(s) = \frac{1}{s^2}\right] \\ &= \lim_{s \to 0} \frac{s + 2\zeta\omega_n - \omega_n^2 K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{2\zeta\omega_n - \omega_n^2 K}{\omega_n^2} = \frac{2\zeta}{\omega_n} - K. \end{aligned}$$

Thus the value of K, for which steady-state error for the ramp input will be eliminated, is given by

$$e_{ss} = 0$$

or

$$\frac{2\zeta}{\omega_n} - K = 0$$

$$\therefore \qquad K = \frac{2\zeta}{\omega_n}$$

Example 6.2 Consider a unity feedback control system whose open-loop $T.F.$ is $G(s) = \dfrac{K}{s(Js+F)}$.

Discuss the effect of variation of gain K and viscous friction coefficient F on the steady-state error in unit ramp response. Sketch typical unit ramp response curves for a small value, medium value and large value of K.

Solution

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \qquad [\because H(s) = 1 \text{ for unity feedback system}]$$

or
$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Fs + K}$$

Again,
$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{K}{Js^2 + Fs + K}$$

or
$$E(s) = \frac{Js^2 + Fs}{Js^2 + Fs + K} R(s)$$

$$\therefore \qquad e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \left(\frac{Js^2 + Fs}{Js^2 + Fs + K} \right) \frac{1}{s^2} \qquad \left[\because R(s) = \frac{1}{s^2} \right]$$

$$\therefore \qquad e_{ss} = \frac{F}{K}.$$

From the result we see that steady-state error can be reduced by either increasing the gain K or reducing the viscous friction coefficient F. However, this causes damping to decrease and the transient response becomes more oscillatory. See Fig. 6.5.

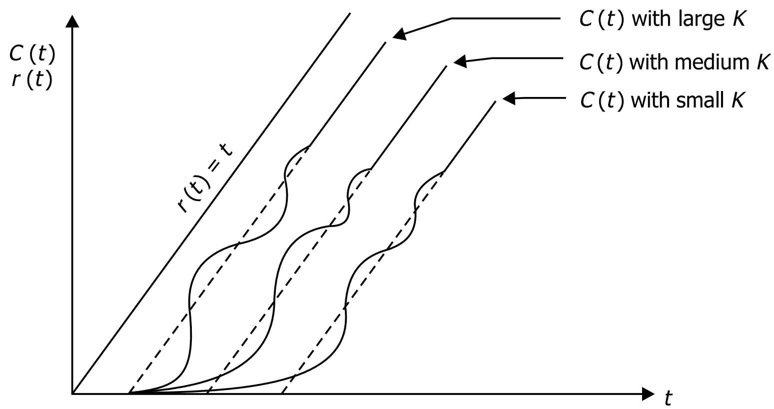


Fig. 6.5

Example 6.3 For a unity feedback system having an open-loop transfer function as $G(s) = \dfrac{K(s+2)}{s(s^3 + 7s^2 + 12s)}$, determine the i) type of system, ii) error constant K_p, K_v and K_a and iii) steady-state error for parabolic input.

Solution

$$G(s) = \frac{K(s+2)}{s^2(s^2 + 7s + 12)}$$

i) The open-loop transfer function $G(s)$ has poles of order two at the origin. So the type of system is two.

ii)

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{K(s+2)}{s^2(s^2 + 7s + 12)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{K(s+2)}{s(s^2 + 7s + 12)} = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{K(s+2)}{s^2 + 7s + 12} = \frac{K}{6}$$

iii) Let the parabolic input be

$$r(t) = t^2$$

$$R(s) = \frac{2}{s^3}$$

Steady-state error,

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} \qquad [\because H(s) = 1]$$

$$= \lim_{s \to 0} \frac{s \dfrac{2}{s^3}}{1 + \dfrac{K(s+2)}{s^2(s^2 + 7s + 12)}}$$

$$= \lim_{s \to 0} \frac{2(s^2 + 7s + 12)}{s^2(s^2 + 7s + 12) + K(d + 2)} = \frac{24}{2K} = \frac{12}{K}$$

Example 6.4 A unity feedback control system has forward loop transfer function as $\dfrac{K}{s}$. Determine the value of K so as to have minimum integral square error.

Solution

Given $$G(s) = \frac{K}{s} \text{ and } H(s) = 1.$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\frac{K}{s}}{1 + \frac{K}{s}} = \frac{K}{K + s}$$

$$\therefore \quad E(s) = R(s) - C(s) = R(s) - \frac{K}{K+s}R(s)$$

$$= \frac{s}{K+s}R(s)$$

For unit step function $$R(s) = \frac{1}{s}$$

$$\therefore \quad E(s) = \frac{1}{s+K}$$

or $$e(t) = e^{-kt}$$

$$\therefore \quad \text{ISE} = \int_0^\infty e^2(t)dt = \int_0^\infty e^{-2kt}dt = -\frac{1}{2K}\left[e^{-2kt}\right]_0^\infty = \frac{1}{2K}$$

So, to have minimum integral square error (ISE),

$$\frac{d}{dK}(\text{ISE}) = 0$$

or $$-\frac{1}{2K^2} = 0$$

$$\therefore \quad K = \infty$$

Example 6.5 Find the dynamic error coefficients of the unity feedback control system whose transfer function is given by

$$G(s) = \frac{10}{s(1+s)}.$$

Also obtain the steady-state error to the input defined by the relation,

$$r(t) = a_o + a_1 t + a_2 t^2$$

Solution

For the given system,

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s+s^2}{10+s+s^2}$$

Dividing $(s + s^2)$ by $(10 + s + s^2)$ we may write

$$\frac{E(s)}{R(s)} = 0.1s + 0.09s^2 - 0.019s^3 + \text{..........}$$

So the dynamic error coefficients are $\left[\text{by comparing with} \frac{E(s)}{R(s)} = \frac{1}{K_1} + \frac{1}{K_2}s + \frac{1}{K_3}s^2 + \cdots\cdots\right]$:

$$K_1 = \infty$$

$$K_2 = \frac{1}{0.1} = 10$$

$$K_3 = \frac{1}{0.09} = 11.1$$

As the input is given by

$r(t) = a_0 + a_1 t + a_2 t_2$

$\therefore$

$$\dot{r}(t) = a_1 + 2a_2 t$$

$$\ddot{r}(t) = 2a_2$$

$$\dddot{r}(t) = 0$$

From the above series in s-domain the steady-state error is

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{t\to\infty}\left[0.1\dot{r}(t) + 0.09\ddot{r}(t) - 0.019\dddot{r}(t) + \text{........}\right] \\ &= \lim_{t\to\infty}\left[0.1(a_1 + 2a_2 t) + 0.09(2a_2)\right] = \lim_{t\to\infty}\left[0.1a_1 + 0.18a_2 + 0.2a_2 t\right] \end{aligned}$$

Thus, the steady-state error becomes infinity, unless $a_1, a_2 = 0$.

REVIEW QUESTIONS

6.1 The closed-loop transfer function of a unity feedback control system is given by

$$\frac{C(s)}{R(s)} = \frac{Ks+a}{s^2+bs+a}$$

Obtain the open-loop transfer function $G(s)$ and hence show that for unity ramp response

$$e_{ss} = \frac{1}{K_v} = \frac{b-K}{a}$$

6.2 The forward and feedback path transfer functions of a control system are as follows.

$$G(s) = \frac{K}{s(s+1)} \text{ and } H(s) = \frac{1}{s+5}$$

The permissible error is 0.5 for a steady-state velocity of 0.1 π rad/sec. Determine the value of K.

6.3 Determine the position, velocity and acceleration error constants for the unity feedback control systems whose open-loop transfer functions are given as follows.

i) $\dfrac{10}{(1+0.4s)(1+0.5s)}$

ii) $\dfrac{K}{(s1+0.1s)(1+0.5s)}$

iii) $\dfrac{11(s+30)}{s^3(1+s)(1+0.2s)(s^2+5s+15)}$

iv) $\dfrac{K(1+s)(1+4s)}{s^2(s^2+2s+10}$

6.4 Use error coefficients to find $e(\infty)$ for the systems of Problem 6.4 when the inputs are (a) $r(t) = 4$, (b) $r(t) = 3t$ and (c) $r(t) = t^2$.

6.5 A non-unity feedback control system has the following transfer functions.

$$G(s) = \frac{5}{s(s+5)(s+10)} \text{ and } H(s) = 0.6\left[\frac{s+4}{s+2.4}\right]$$

a) Find G_{eq} and its system type.
b) Find steady-state output $C_{ss}(t)$ and steady-state error $e_{ss}(t)$ for a unity step input.

6.6 Determine $e_{ss}(t)$ for a system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{s(s+4)}{s(s+1)(s+5)}$$

The input to the system is defined by
$r(t) = 0$ for $t < 0$;
$r(t) = 2(1 + t)$ for $r > 0$.

6.7 A unity feedback control system has an open-loop transfer function given by

$$G(s) = \frac{400}{s(1+0.1s)}$$

Determine the steady-state error of the system for the following inputs.

i) $r(t) = t^2\, u(t)/2$;

ii) $r(t) = (1 + t)^2\, u(t)$.

6.8 The two systems shown in Fig. 6.6(a) and (b) below are equivalent.

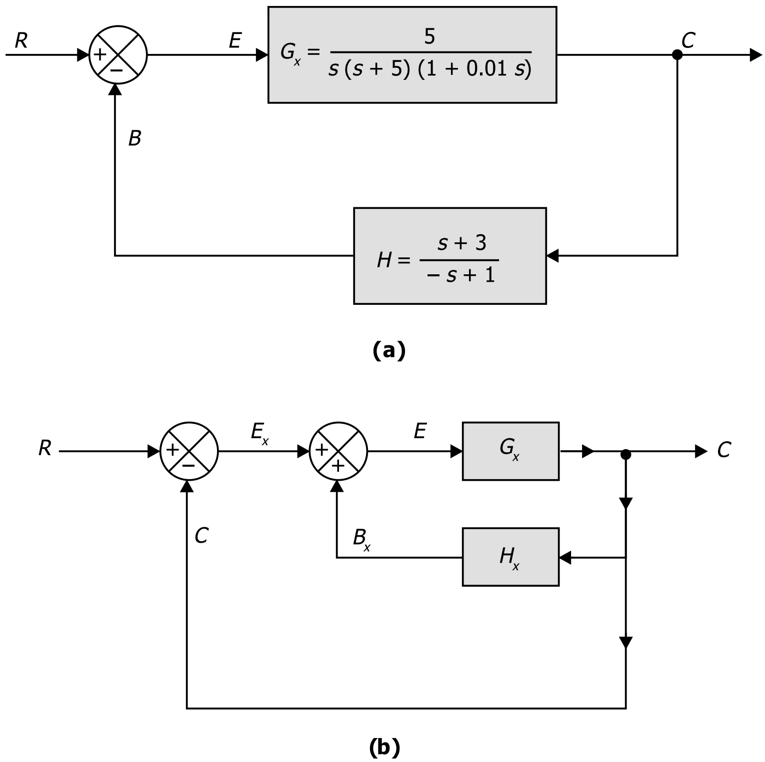


Fig. 6.6

i) Find the transfer function $H_x(s)$.

ii) What is the type of the system in Fig. 6.6(b)?

iii) Determine the error coefficients of the system in Fig. 6.6(b).

iv) Determine the final value of $c(t)$ if $r(t) = u(t)$.

v) What are the values of $e_x(t)_{ss}$ and $e(t)_{ss}$?

6.9 The block diagram of a system is shown in Fig. 6.7.
Determine the steady state error for unit ramp input taking the value of K equal to 300. Also determine the value of K for which the state steady state error for unit ramp input will be 0.025.

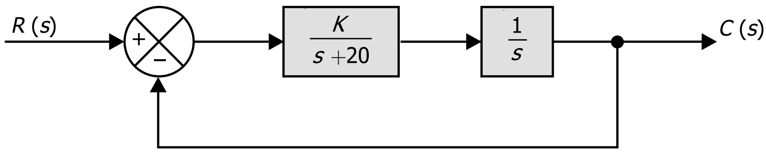


Fig. 6.7

6.10 For the system shown below calculate the steady state error for (i) unit step input; (ii) unit ramp input; (iii) unit acceleration input.

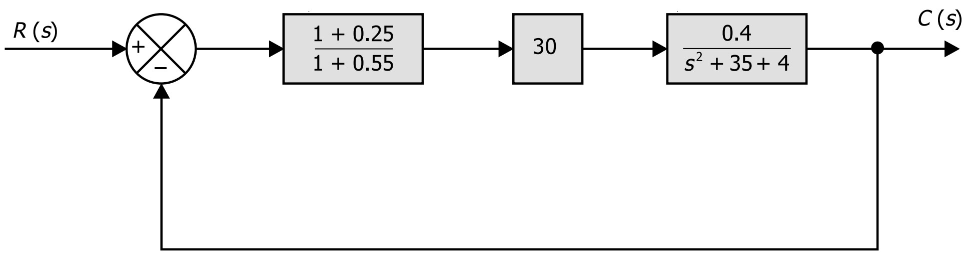


Fig. 6.8

6.11 Calculate the error constants and steady state errors for three basic types of inputs for the following system.

$$G(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)} \text{ and } H(s) = 1$$

6.12 The open-loop transfer function servo system with unity feedback is represented as

$$G(s) = \frac{10}{s(0.1s+1)}$$

Calculate the error constants K_p, K_v, K_a, for the system.

7

TIME RESPONSE ANALYSIS

7.1 INTRODUCTION

Time response analysis is also called time domain analysis. Here, we study the response, i.e. the output as a function of time.

Total time response $c(t)$ of a control system consists of transient response (dynamic response $c_t(t)$ and steady state response $c_{ss}(t)$.

$$c(t) = c_t(t) + c_{ss}(t)$$

where $c(t)$ = total time response

$c_t(t)$ = transient response

$c_{ss}(t)$ = steady-state response.

The transient state of the system remains for a very short time while steady-state is that stage of the system as time t approaches infinity. *A feedback control system has the inherent capabilities that its parameters can be adjusted to alter both its transient and steady-state behaviour*. In order to analyse the transient and steady-state behaviour of control systems, we obtain a mathematical model of the system. For any specific input signal, a complete time response expression can then be obtained through the Laplace transform inversion. This expression yields the steady-state behaviour of the system with time tending to infinity. In case of simple deterministic signals, steady-state response expression can be calculated by the use of the final value theorem.

Before proceeding with the time response analysis of a control system, it is necessary to test stability of the system through indirect tests without actually obtaining the transient response. In case a system happens to be unstable, hence of no practical use, we need not proceed with its transient response analysis.

Typical test signals that is, signals which can be generated in the laboratory are:

i) Unit step
ii) Unit ramp
iii) Parabolic
iv) Sinusoidal

The nature of the transient response of a system is dependent upon system poles only and not on the type of input. Therefore, we shall analyse the transient response to one of the standard test signals. A step signal is generally used for this purpose. The time consuming transient analysis need not be carried out for all the test signals by the final value theorem.

7.2 TIME RESPONSE OF FIRST ORDER SYSTEM TO STEP INPUT

Let us consider a simple *RC* circuit as shown in Fig. 7.1.

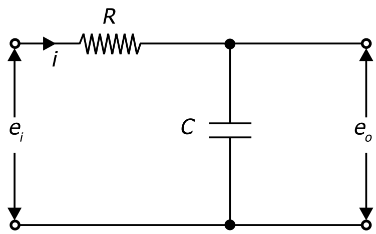


Fig. 7.1 *An RC circuit as a first order system*

We write the circuit equation as

$$e_i = Ri + \frac{1}{C}\int idt$$

and

$$e_o = \frac{1}{C}\int idt.$$

Taking Laplace transform of these two equations

$$E_i(s) = RI(s)\frac{1}{Cs}I(s)$$

or,

$$E_i(s) = \left(R + \frac{1}{Cs}\right)I(s)$$

and

$$E_o(s) = \frac{1}{Cs}I(s)$$

$$\text{Transfer function} = \frac{\text{output}}{\text{input}} = \frac{E_o(s)}{E_i(s)} = \left(\frac{1}{1+RCs}\right)$$

where $\tau = RC = \textit{Time constant}$ of RC circuit

$$\therefore \qquad T.F. = \left(\frac{1}{1+\tau s}\right)$$

The block diagram of the system is shown in Fig. 7.2.

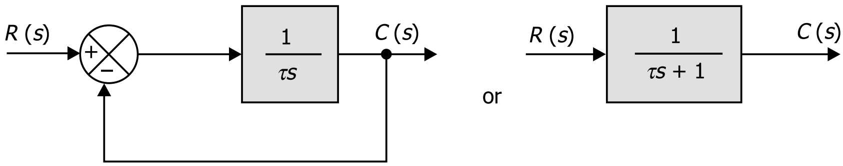


Fig. 7.2 *Block diagram representation of a RC circuit*

Here
$$G(s) = \frac{1}{\tau s}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$H(s) = 1$ for unity feedback

So,
$$\frac{C(s)}{R(s)} = \frac{\frac{1}{\tau s}}{1+\frac{1}{\tau s}} = \frac{1}{\tau s+1}$$

So,
$$C(s) = R(s)\frac{1}{(s\tau+1)}.$$

Time response:

Time response to the unit step input, $R(s) = \frac{1}{s}$ will now be calculated:

Putting $R(s) = \frac{1}{s}$,

$$C(s) = \frac{1}{s(\tau s+1)}$$

or,
$$C(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$$

Taking inverse Laplace transform,

Therefore,
$$C(t) = \left[1 - e^{-t/\tau}\right]$$

The unit step response of the above system is shown in Fig. 7.3. It is seen that as time tends to infinity, the error $e(t)$ goes on reducing and finally becoming zero. The steady state error becomes zero.

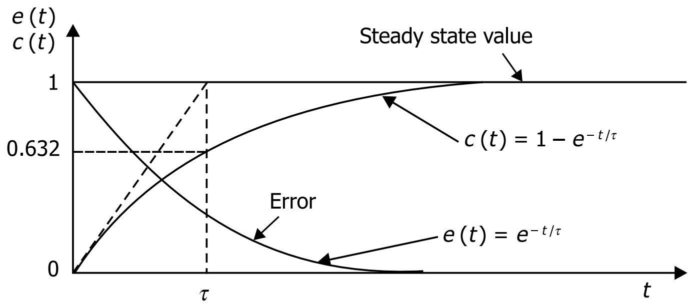


Fig. 7.3 *Unit step response of a first order system*

The time constant is indicative of how fast the system tends to reach the final value. The speed of the response can be quantitatively defined as the time for the output to become a particular percentage of its final value. A large time constant corresponds to a sluggish response and a small time constant corresponds to a fast response as shown in Fig. 7.4. As shown in Fig. 7.4, time constant τ_1 is greater than τ_2 and hence the response is as shown, i.e. it will take more time to reach the final value.

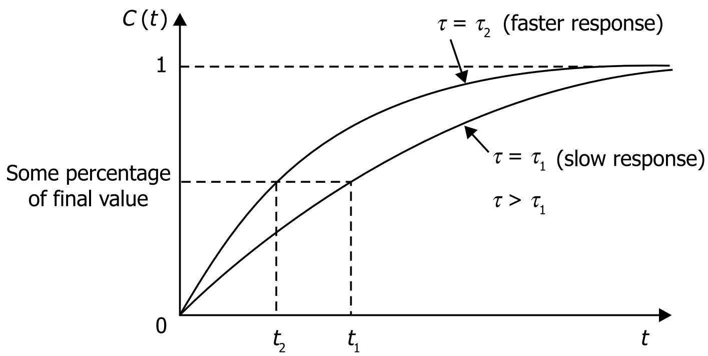


Fig. 7.4 *Effect of time constant on system response*

Error is

$$e(t) = r(t) - c(t)$$

or, $$e(t) = 1 - (1 - e^{-t/\tau})$$

or, $$e(t) = e^{-t/\tau}$$

Steady-state error is given by

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{-t/\tau} = 0.$$

The error response has been shown in Fig. 7.3.

7.3 RESPONSE OF FIRST ORDER SYSTEM TO RAMP INPUT

The transfer function of a first order system is written as

$$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

$$R(s) = \frac{1}{s^2} \text{ (for ramp input)}$$

or,
$$C(s) = \frac{R(s)}{\tau s + 1}$$

or,
$$C(s) = \frac{1}{s^2(\tau s + 1)}$$

or,
$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

Taking inverse Laplace transform,

$$C(t) = t - \tau\left(1 - e^{-t/\tau}\right)$$

$$R(s) = \frac{1}{s^2}$$

$$r(t) = t$$

Error is given by

$$e(t) = r(t) - c(t)$$

or
$$e(t) = t - [t - \tau(1 - e^{-t/\tau})]$$

$$e(t) = \tau(1 - e^{-t/\tau})$$

Steady-state error is given by

$$e_{ss} = \lim_{t\to\infty} e(t) = \tau$$

Therefore, for a ramp input reducing the system time constant improves the speed of response of the system as well as reduces its steady-state error to a ramp input. We, therefore, need to examine only the steady state error to ramp input which can also be obtained by applying the final value theorem as follows.

$$
\begin{aligned}
e_{ss} &= \lim_{t\to\infty} e(t) \\
&= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s[R(s) - C(s)] \\
&= \lim_{s\to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(\tau s+1)}\right] \\
&= \lim_{s\to 0} \frac{s}{s^2}\left[1 - \frac{1}{(\tau s+1)}\right] \\
&= \lim_{s\to 0} \frac{s}{s^2}\left[1 - \frac{1}{(\tau s+1)}\right] \\
&= \lim_{s\to 0} \frac{1}{s}\left[\frac{\tau s+1-1}{\tau s+1}\right] \\
&= \lim_{s\to 0} \frac{1}{s}\left[\frac{\tau s}{\tau s+1}\right] \\
&= \lim_{s\to 0} \left[\frac{\tau}{\tau s+1}\right]
\end{aligned}
$$

$\therefore \qquad e_{ss} = \tau.$

Thus there is no need of taking inverse Laplace transform, if we use the final value theorem. The response of a first order system to unit ramp input has been shown in Fig. 7.5(a).

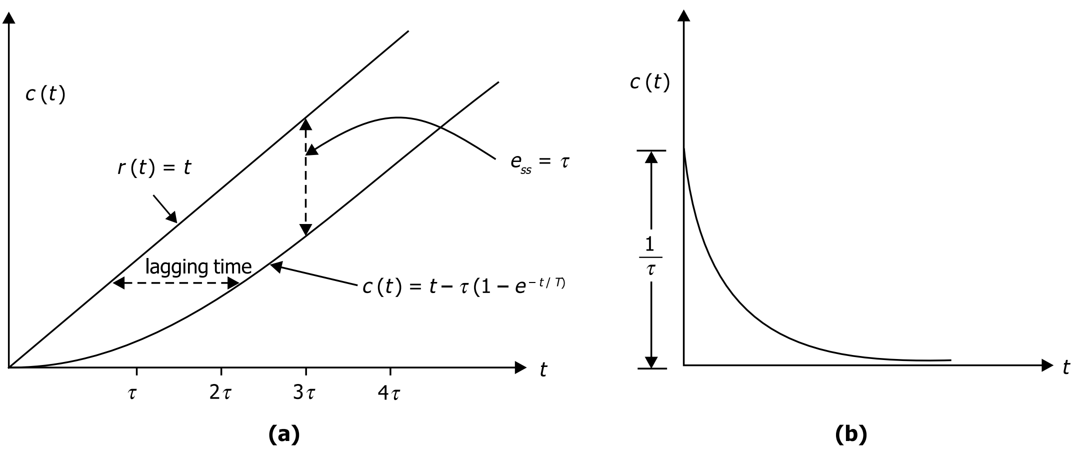


Fig. 7.5 *(a) Unit ramp response of a first order system; (b) Impulse response of first order system*

7.4 RESPONSE OF FIRST ORDER SYSTEM TO IMPULSE INPUT

For unit impulse input $r(t) = u(t)$ where $u(t)$ is an unit step function. So, $R(s) = 1$.

But, $$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

$\therefore$ $$C(s) = \frac{1}{\tau s + 1}$$

or, $$c(t) = \frac{1}{\tau} e^{-t/\tau}$$

Steady-state error

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} s[R(s) - C(s)] \\ &= \lim_{s \to 0} s\left[1 - \frac{1}{(\tau s + 1)}\right] \\ &= \lim_{s \to 0} s\left[\frac{\tau s + 1 - 1}{\tau s + 1}\right] \\ &= \lim_{s \to 0} s\left[\frac{\tau s}{\tau s + 1}\right] \\ e_{ss} &= 0. \end{aligned}$$

To summarise, the response of a first order system having transfer function $\left(\frac{1}{\tau s + 1}\right)$ to different input signals has been tabulated in Table 7.1.

Table 7.1 *Response of a First Order System to Different Types of Input Signals*

Signal	**Input** $r(t)$	**Output** $c(t)$	**Steady state error,** $e_{ss}(t)$
(i) Ramp	t	$t - \tau(1 - e^{-t/\tau})$	τ
(ii) Unit step	1	$1 - e^{-t/\tau}$	0
(iii) Impulse	$u(t) = \delta(t)$	$\frac{1}{\tau} e^{-t/\tau}$	0

It may be seen from Table 7.1 that input at (ii) is derivative of input at (i); also output at (ii) is derivative of output at (i). This rule can be generalized for higher order systems also.

We have found the step response of the second order system. We can find the impulse response of the second order system by differentiating the step response found earlier. Similarly, the ramp response can be found by integrating the step response. Since initial conditions are already zero, we can find the constants of integration easily.

7.5 TIME RESPONSE OF SECOND ORDER SYSTEMS

The block diagram representation of the second order system with unity feedback has been shown in Fig. 7.6.

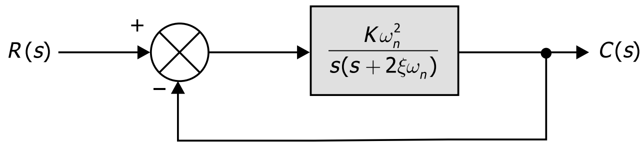


Fig. 7.6 *Block diagram of a second order system*

A second order system is represented by the equation,

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

The transfer function of a second order system is given as,

$$T.F. = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $$\omega_n = \sqrt{\frac{a_0}{a_2}} \text{ and } \xi = \frac{a_1}{2\sqrt{a_0 a_2}}$$

ω_n is *the undamped natural frequency* and ξ is *the damping ratio*. The characteristic equation is written by equating the denominator of the transfer function to zero. The characteristic equation of the second order system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of the characteristic equation are

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$
$$= \sigma \pm j\omega_d$$

Here s_1, s_2 are the complex frequencies, $\sigma = \xi\omega_n$, is *called the attenuation*, and

$\omega_d = \omega_n\sqrt{1-\xi^2}$, is called the *frequency of damped oscillation.*

The time domain response of the system will depend upon the roots of the characteristic equation. *For overdamped system, $\xi > 1$ and for an underdamped system, $\xi < 1$.* For a critically damped system $\xi = 1$.

7.5.1 Positional Servo System as a Second Order System and Its Analysis

As an example, let us consider a second order system. A positional servomechanism system has been shown.

Let us examine the positional servo system as in Fig. 7.7.

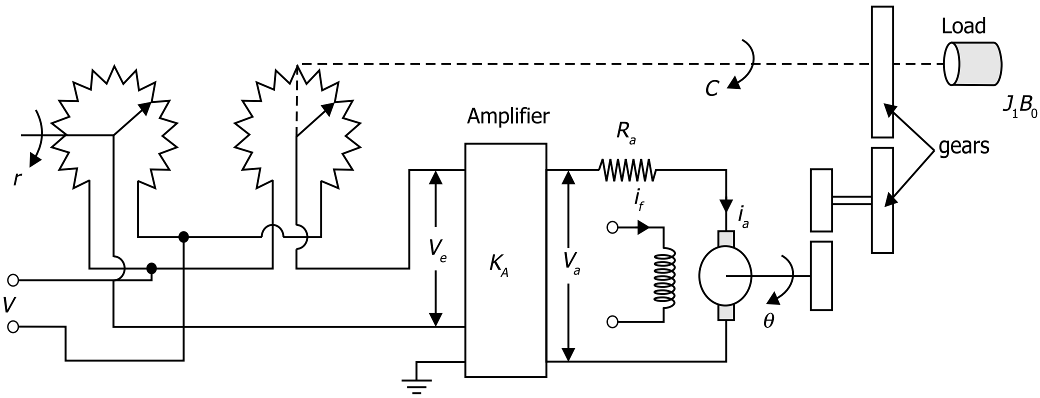


Fig. 7.7 *Positional servo mechanism*

Let us take,

R_a = resistance of armature
i_a = armature current
i_f = field current
v_a = applied armature voltage
e_b = back emf
T_m = torque developed by motor
θ = angular displacement of motor shaft
J = equivalent moment of inertia of motor and load referred to motor shaft
B_o = equivalent viscous friction coefficient of motor and load referred to motor shaft
ϕ = air-gap flux
K_A = amplifier gain
K_p = potentiometer sensitivity
v_e = voltage across the amplifier
r = reference shaft position
c = output shaft position
n = gear train ratio of load shaft speed $(\dot{c})$ to motor shaft speed $(\dot{\theta})$

The servo mechanism controls the position of the mechanical load in accordance with the position of the reference shaft. The error signal (v_e) appears at the leads of the potentiometer wiper arms. From Fig 7.7 we may write the following.

$$v_e = K_p(r - c) \quad \text{....(7.1)}$$

$$v_a = K_A v_e = K_A K_p(r - c) \quad \text{....(7.2)}$$

$$e_b = k_b \frac{d\theta}{dt} \quad \text{....(7.3)}$$

where k_b = back emf constant. Neglecting the small armature inductance, the equation of armature circuit is

$$R_a i_a + e_b = v_a$$

or,
$$R_a i_a + k_b \frac{d\theta}{dt} = v_a \quad \text{....(7.4)}$$

In servo application, a DC motor is used in the linear zone of magnetization curve and i_f = constant, for armature-controlled DC motor.

$$\therefore \quad \phi \propto i_f \ \text{ and } \ T_m \propto \phi i_a$$

$$\therefore \quad T_m = K_T i_a \quad \text{....(7.5)}$$

where K_T = motor torque constant.

The torque equation is given by

$$J \frac{d^2\theta}{dt^2} + B_o \frac{d\theta}{dt} = T_m$$

or,
$$J \frac{d\dot{\theta}}{dt} + B_o \dot{\theta} = K_T i_a \quad \text{....(7.6)}$$

For the gear train we have

$$\dot{C} = n\dot{\theta} \quad \text{....(7.7)}$$

Taking Laplace transforms of equations (7.1) through (7.7) we have

$$V_e(s) = K_P \, [R(s) - C(s)] \quad \text{....(7.8)}$$

$$V_a(s) = K_A V_e(s) = K_A K_p [R(s) - C(s)] \quad \text{....(7.9)}$$

$$E_b(s) = K_b \dot{\theta}(s) \quad \text{....(7.10)}$$

$$R_a I_a(s) + K_b \dot{\theta}(s) = v_a(s) \quad \text{....(7.11)}$$

$$T_m(s) = K_T I_a(s) \quad \text{....(7.12)}$$

$$Js\dot{\theta}(s) + B_o \dot{\theta}(s) = K_T I_a(s)$$

or, $$(Js + B_o)\dot{\theta}(s) = K_T I_a(s) \qquad \text{....(7.13)}$$

$$sC(s) = ns\theta\,(s)$$

or, $$C(s) = n\theta\,(s) \qquad \text{....(7.14)}$$

Also $$\frac{d\theta}{dt} = \dot{\theta} \text{ and its Laplace Transform is } s\theta(s) = \dot{\theta}(s) \qquad \text{....(7.15)}$$

Hence by using equations (7.8) through (7.15), we can draw the block diagram as shown in Fig. 7.8.

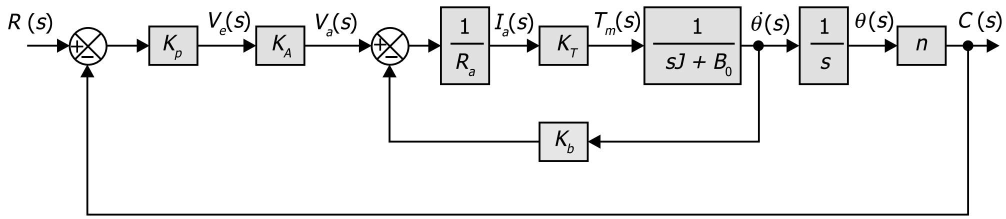


Fig. 7.8 *Block diagram of servo mechanism system*

Using equations (7.9), (7.11), (7.13) to (7.15) we get

$$K_P K_A[R(s) - C(s)] = R_a I_a(s) + K_b\dot{\theta}(s) = R_a I_a(s) + K_b s\theta(s)$$

$$= \frac{R_a}{K_T}(Js + B_o)\dot{\theta}(s) + K_b s\theta(s)$$

$$= \frac{R_a}{K_T}(Js + B_o)s\theta(s) + K_b s\theta(s)$$

$$= \left[\frac{R_a}{K_T}(Js + B_o)s + K_b s\right]\frac{C(s)}{n}$$

or, $$K_P K_A R(s) = \left[K_P K_A + \frac{R_a s}{nK_T}(Js + B_o) + \frac{K_b s}{n}\right]C(s)$$

or, $$\frac{C(s)}{R(s)} = \frac{K_P K_A}{K_P K_A + \frac{R_a s}{nK_T}(Js + B_o) + \frac{K_b s}{n}}$$

$$= \frac{K_P K_A n K_T / R_a J}{\frac{K_P K_A K_T n}{R_a J} + \left(\frac{B_o}{J} + \frac{K_b K_T}{R_a J}\right)s + s^2}$$

Putting $$\frac{K_P K_A K_T n}{R_a} = K$$

and $B_o + \dfrac{K_b K_T}{R_a} = B,$ we get

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{J}}{\dfrac{K}{J} + \dfrac{B}{J}s + s^2}$$

$$= \frac{\omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2}$$

$$= \frac{\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}{1 + \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}$$

$$= \frac{G(s)}{1 + G(s)}$$

where, $$\omega_n^2 = \frac{K}{J}$$

or, $$\omega_n = \sqrt{\frac{K}{J}}$$

and $$2\zeta\omega_n = \frac{B}{J}$$

or, $$\zeta = \frac{B}{J}\frac{\sqrt{J}}{2\sqrt{K}} = \frac{B}{2\sqrt{JK}}$$

Here ω_n is the undamped natural frequency; ζ is the damping ratio and B is the viscous friction coefficient referred to motor shaft. B is the actual damping and for $\zeta = 1$, the critical damping is $B_c = 2\sqrt{JK}$.

$$\therefore \quad \zeta = \frac{B}{B_c}$$

$$= \frac{\text{Actual damping value}}{\text{Critical damping value}}.$$

The damping ratio actually refers to the damping of oscillations.

Now the simplified block diagram of the servomechanism is shown in Fig. 7.9.

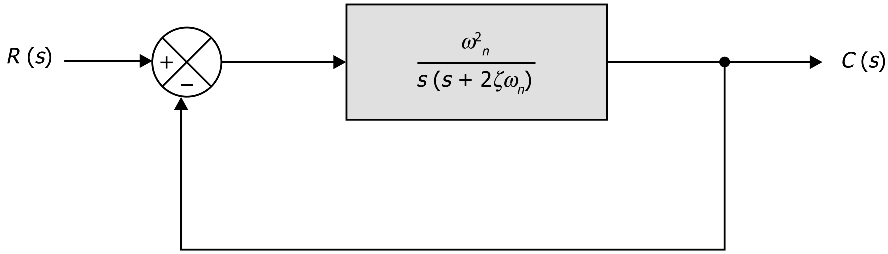


Fig. 7.9 *Simplified block diagram of Fig. 7.8 which is a second order system*

Characteristic Equation and Position of Roots/Poles

As mentioned earlier characteristic equation is the equation obtained by equating the denominator polynomial of the overall closed-loop transfer function to zero.

$$T.F. = \frac{G(s)}{1+G(s)} \text{ for } H(s) = 1$$

$$= \frac{\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1+\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}}$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

So, the characteristic equation of the closed-loop second order system is given by

$$s^2+2\zeta\omega_n s+\omega_n^2 = 0 \qquad \text{....(7.16)}$$

If $\zeta \geq 1$, the roots of the equation are real and the system response will be over damped. On the other hand, if $\zeta < 1$, the roots of the equation are complex conjugate and the system response will be oscillatory in nature. The condition of stability of the system is that the value of ζ should be positive. So for $\zeta \geq 1$, the real roots the of equation are

$$r_1, r_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

For $\zeta < 1$, the complex conjugate roots of the equation are

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\sigma \pm j\omega_d$$

where ζ is the damping ratio, ω_n is the undamped natural frequency. $\sigma = \zeta\omega_n$ is called the attenuation and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the frequency of damped oscillation. Characteristic roots are also the poles of the system. So the poles of second order for different values of damping ratio ζ are depicted in Fig. 7.10. Poles are at $+j\omega_n$ and $-j\omega_n$ for $\zeta = 0$. For $\zeta = 1$, double poles are at $-\omega_n$.

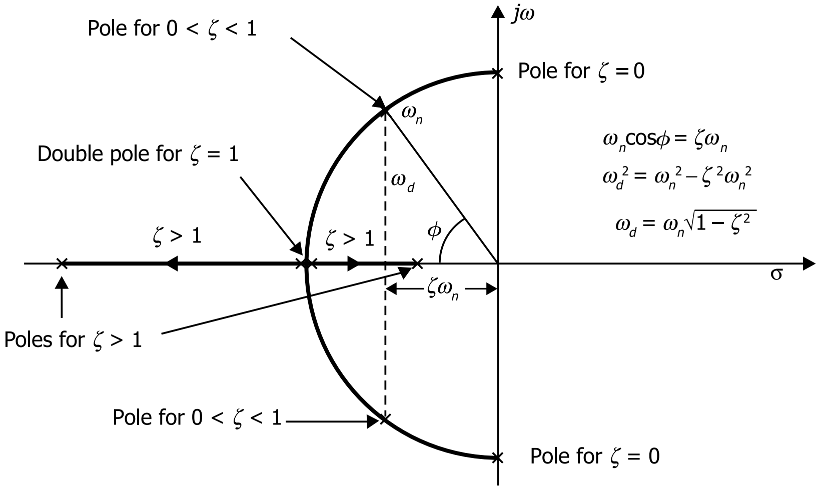


Fig. 7.10 *Location of poles of second order system for different values of damping ratio ζ in the s-plane*

Classification of Time Response for Second Order Control System

According to different values of ζ, the time response of a second order control system can be classified as follows.

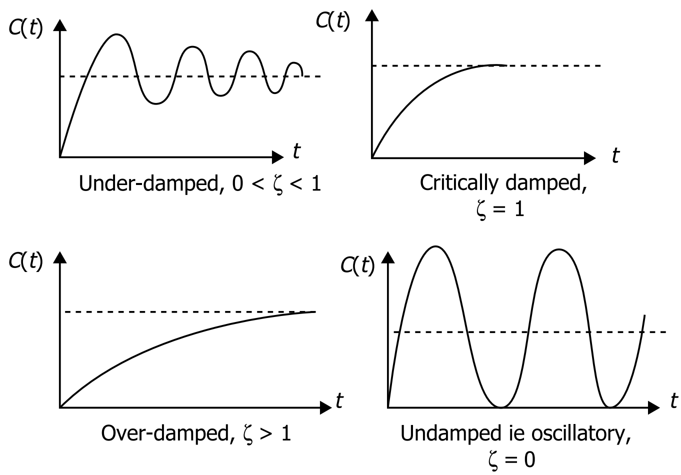


Fig. 7.11 *Time response of a second order system for different values of damping ratio*

a) *Underdamped:* For $0 < \zeta < 1$, the transient response is oscillatory in nature, but decays exponentially to give a stable response.

b) *Critically damped:* For $\zeta = 1$, the response just becomes non-oscillatory and gives a stable response after transient disappears.

c) *Overdamped:* For $\zeta > 1$, the response is non-oscillatory and gives a somewhat delayed stable response after transient disappears.

d) *Undamped:* For $\zeta = 0$, the transient does not disappear and the response gives a sustained oscillation.

The time response $c(t)$ of a second-order control system for different values of damping ratio ζ are shown in Fig. 7.11.

7.5.2 Time Response of Second Order Control System Subjected to Unit Step Input

a) *Under-damped case (when $0 < \zeta < 1$)*

Here $R(s) = \dfrac{1}{s}$ and so $C(s)$ can be written as

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \frac{\omega_n}{(s + \zeta\omega_n)^2 \omega_d^2}$$

[where $\omega_d = \omega_n\sqrt{1-\zeta^2} =$ damped natural frequency]

Taking inverse Laplace transform, we get

$$C(t) = 1 - e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t) \quad [\because \omega_n \sin\phi = \omega_d = \omega_n\sqrt{1-}\zeta^2]$$

[where $\cos\phi = \zeta$]

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

Thus, time response $C(t) = 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[(\omega_n\sqrt{1-\zeta^2})t + \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right]$. ....(7.17)

As $\lim_{t\to\infty} c(t) = 1$, the time response reaches its steady-state value of unity and hence the steady-state error (e_{ss}) becomes zero as deduced below.

$$
\begin{aligned}
e_{ss} &= \lim_{t\to\infty}[r(t) - c(t)] \\
&= \lim_{s\to 0} s[R(s) - C(s)] \\
&= \lim_{s\to 0} s\left[\frac{1}{s} - \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \\
&= \lim_{s\to 0}\left[1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \\
&= 0
\end{aligned}
$$

The underdamped response for a second order system is shown in Fig. 7.12.

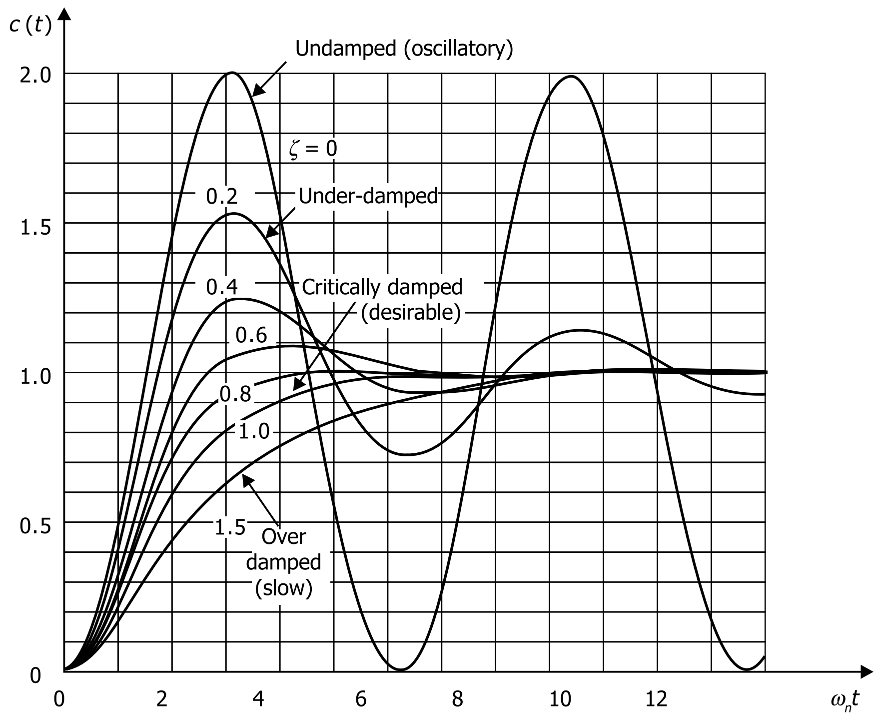


Fig. 7.12 *Comparison of unit step input response of a second order system for different values of damping ratio, ζ*

b) *Critically damped case (when $\zeta = 1$)*

As $R(s) = \dfrac{1}{s}$, and so $C(s)$ can be written as

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} \qquad [\text{Putting } \zeta = 1]$$

$$= \frac{(s+\omega_n)^2 - s(s+\omega_n) - s\omega_n}{s(s+\omega_n)^2}$$

$$= \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}.$$

Taking inverse Laplace transform on both sides, we get

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}(1 + \omega_n t)$$

The unit step response for this case also approaches unity, as $\lim\limits_{t\to\infty} c(t) = 1$ and also steady-state error $e_{ss} = 0$ as before. The response curve is shown in Fig. 7.12.

c) *Overdamped case (when $\zeta > 1$)*

Here also $R(s) = \dfrac{1}{s}$, and $C(s)$ may be written as $C(s) = \dfrac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$

$$\text{or,} \quad C(s) = \frac{\omega_n^2}{s\left[(s+\zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)\right]} = \frac{\omega_n^2}{s\left[s + \omega_n(\zeta + \sqrt{\zeta^2 - 1})\right]\left[s + \omega_n(\zeta - \sqrt{\zeta^2 - 1})\right]}$$

$$= \frac{1}{s} + \frac{1}{2\sqrt{\zeta^2-1}(\zeta + \sqrt{\zeta^2-1})[s + \omega_n(\zeta + \sqrt{\zeta^2-1})]} \qquad \text{(obtaining partial fraction)}$$

$$- \frac{1}{2\sqrt{\zeta^2-1}(\zeta - \sqrt{\zeta^2-1})[s + \omega_n(\zeta + \sqrt{\zeta^2-1})]}$$

Taking inverse Laplace transform on both sides, we have

$$c(t) = 1 + \frac{e^{-(\zeta + \sqrt{\zeta^2-1})\omega_n t}}{2\sqrt{\zeta^2-1}(\zeta + \sqrt{\zeta^2-1})} - \frac{e^{-(\zeta - \sqrt{\zeta^2-1})\omega_n t}}{2\sqrt{\zeta^2-1}(\zeta - \sqrt{\zeta^2-1})}$$

The time response curve is shown in Fig. 7.12.

If ζ becomes comparatively larger than 1, then the second term having a smaller time constant $\left[1/(\zeta+\sqrt{\zeta^2-1})\omega_n\right]$ decays more quickly than the third term with the larger time constant $\left[1/(\zeta-\sqrt{\zeta^2-1})\omega_n)\right]$. So after the time of the smaller time constant has elapsed, the response is similar to that of a first order system and $C(s)$ may be approximated as

$$\frac{C(s)}{R(s)}=\frac{1}{1+\tau s} \quad \text{or,} \quad C(s)=\frac{1/s}{1+\tau s}$$

Substituting value of τ, we get $C(s)=\dfrac{1/s}{1+\left[1/(\zeta-\sqrt{\zeta^2-1})\omega_n\right]s}$

$$=\frac{(\zeta-\sqrt{\zeta^2-1})\omega_n}{s\left[s+(\zeta-\sqrt{\zeta^2-1})\omega_n\right]}$$

Taking inverse Laplace transform, we get

$$C(t)=1-e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} \quad \text{for } \zeta>>1.$$

Here also $\lim_{t\to\infty} C(t)=1$ and $e_{ss}=0$

d) *Undamped case (when $\zeta = 0$)*

As and $R(s)=\dfrac{1}{s}, \zeta=0,$ we may write $C(s)$ as given below.

$$C(s)=\frac{\omega_n^2}{s(s^2+\omega_n^2)}=\frac{(s^2+\omega_n^2)-s^2}{s(s^2+\omega_n^2)}$$

$$=\frac{1}{s}-\frac{s}{s^2+\omega_n^2}.$$

Taking inverse Laplace transform on both sides, we have

$$C(t) = 1 - \cos \omega_n t$$

In this case the time response oscillates with the undamped natural frequency of ω_n and so it does not follow the step input. The response curve is shown in Fig. 7.12.

From the plot of $c(t)$ against ω_n for all values of damping it is observed that the way $c(t)$ reaches its final value from initial zero value depends on the damping condition, i.e. on the value of damping ratio, ζ. For an undamped system, however, $\zeta = 0$ and the system output is $c(t) = (1-\cos \omega_n t)$. The system will continue to oscillate around its final value of 1, with pick values changing from 0 to 2. As damping is increased, ζ increases towards unity and system oscillation gets reduced.

Example 7.1 Derive the time response of a second order system subjected to impulse input function.

Solution

For unit impulse $R(s) = 1$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Breaking into partial fraction

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2}$$

With characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Roots of this equation are

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$
$$s_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

Substituting $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$s_1 = -\zeta\omega_n + j\omega_d$$
$$s_2 = -\zeta\omega_n - j\omega_d$$

$$K_1 = \left.\frac{\omega_n^2}{s - s_2}\right|_{s=s_1}$$

$$= \frac{\omega_n^2}{-\zeta\omega_n + j\omega_d + \zeta\omega_n + j\omega_d}$$

$$= \frac{\omega_n^2}{2j\omega_d}$$

$$K_2 = \frac{\omega_n^2}{-j2\omega_d}$$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{2j\omega_d}\frac{1}{s - s_1} - \frac{\omega_n^2}{2j\omega_d}\frac{1}{s - s_2}$$

Taking inverse Laplace

$$C(t) = \frac{\omega_n^2}{2j\omega_d}[e^{s_1 t} - e^{s_2 t}] = \frac{\omega_n^2}{2j\omega_d}[e^{(-\zeta\omega_n + j\omega_d t)} - e^{(-\zeta\omega_n + j\omega_d t)}]$$

$$= \frac{\omega_n^2 e^{-\zeta\omega_n t}}{2j\omega_d} \times [2j \sin \omega_d t] = \frac{\omega_n^2 e^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t$$

Substituting the value of ω_d in the above equation,

$$C(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\right)t$$

which is the required response.

7.5.3 Transient Response Specifications

The unit step response is easy to generate and mathematically the response to any input can be derived if the response to a step input is known. Therefore, the performance characteristics of a control system are described in terms of transient response to a unit step input; with standard initial conditions of output and all time derivatives being zero when the system is at rest. The time response of second and higher order control systems to a unit step input is generally damped oscillatory in nature before reaching steady state. The following are the transient response specifications (as shown in Fig. 7.13) of a control system to a unit step input.

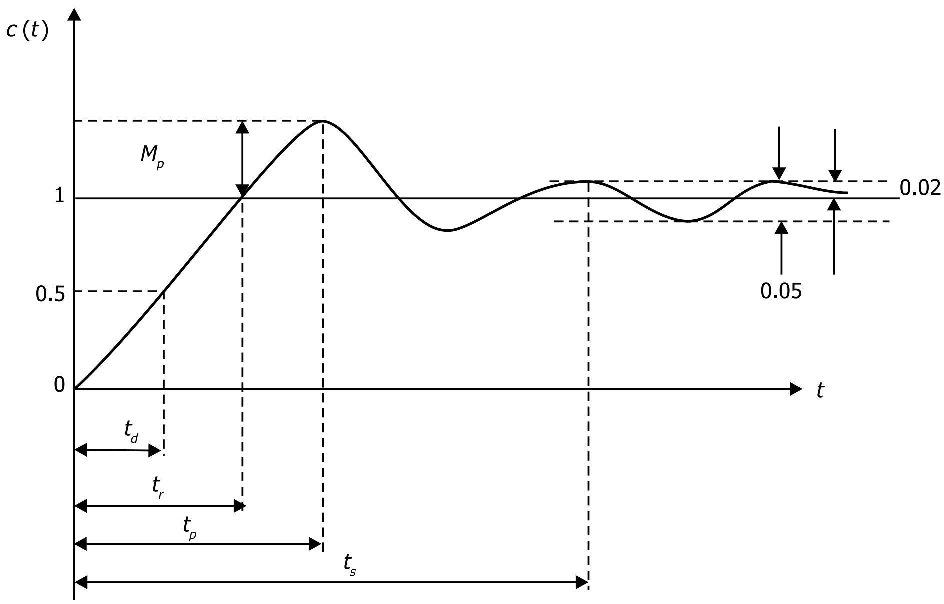


Fig. 7.13 Transient time response specifications to unit step input

Transient Response Specifications:

a) **Rise time (t_r):** It is the time required for the response to rise from 10 per cent to 90 per cent (for overdamped or critically damped systems) and 0 per cent to 100 per cent (for underdamped systems) of its final value. The 10 per cent to 90 per cent and 0 per cent to 100 per cent rise time are commonly used for overdamped and underdamped second order systems respectively.

b) **Peak time (t_p):** The peak time is the time required for the response to reach the first peak of the overshoot. See Fig. 7.13.

c) **Maximum overshoot and maximum percentage overshoot:** The maximum overshoot (M_p) is the maximum peak value of the response measured from unity. So M_p is given by $M_p = c(t_p) - 1$.
If the steady-state value is not unity, then the maximum per cent overshoot as defined below is used.
Maximum percent overshoot

$$= \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100.$$

d) **Setting time (t_s):** It is the time required for the response curve to reach and stay within a specified tolerance band (either 2 per cent or 5 per cent) of final value.

e) **Delay time (t_d):** It is the time required by the response to reach half of its final value at the first attempt.

7.5.4 Determination of Transient Response Specifications of the Second Order System

Assuming the system to be underdamped second order, we shall obtain the rise time (t_r), peak time (t_p), maximum overshoot (M_p) and settling time (t_s) in terms of ζ and ω_n. We have deduced (see equation (7.17)) earlier that time response $c(t)$ under unit step input is given by:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left((\sqrt{1-\zeta^2})\omega_n t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

a) **Rise time (t_r):** Rise time is obtained from $c(t_r) = 1$
Putting $c(t) = 1$ in the above equation we get

$$1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left((\omega_n \sqrt{1-\zeta^2})t_r + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right) = 1$$

or, $$\sin\left((\omega_n \sqrt{1-\zeta^2})t_r + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right) = 0 = \sin\pi$$

Now, as $0 < \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta} < \dfrac{\pi}{2}$ for $0 < \zeta < 1$

$$\therefore (\omega_n\sqrt{1-\zeta^2})t_r + \tan^{-1}\frac{1-\zeta^2}{\zeta} = \pi$$

or,
$$t_r = \frac{\pi - \tan^{-1}\dfrac{1-\zeta^2}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} = \text{The response time} \qquad \text{....(7.18)}$$

b) **Peak time (t_p):** Peak time is obtained by differentiating $c(t)$ with respect to t and equating to zero. At maxima the slope is zero. Therefore peak time is obtained from $\left.\dfrac{dc(t)}{dt}\right|_{t=t_p} = 0$

or,
$$\frac{\zeta\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin\left[(\omega_n\sqrt{1-\zeta^2})t_p + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

$$-\omega_n e^{-\zeta\omega_n t_p}\cos\left((\omega_n\sqrt{1-\zeta^2})t_p + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

or,
$$\tan\left(\omega_n\sqrt{1-\zeta^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

The general solution of the above equation is

$$(\omega_n\sqrt{1-\zeta^2})t_p = n\pi \qquad \text{....(7.19)}$$

where $n = 0, 1, 2, 3, \ldots$ and so on.

As t_p is the time required for the response to reach the first peak of the overshoot, so t_p is obtained for $n = 1$. (By putting $n = 2$ we can get t_p for the second peak, and so on.)

For the peak overshoot, $(\omega_n\sqrt{1-\zeta^2})t_p = \pi$

or, $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ Peak for the first maxima.

Here $n = 1$ first undershoot
$n = 2$ second overshoot etc.

c) **Maximum overshoot (M_p):**
Maximum overshoot M_p is found by substituting the value of t_p in the expression for $c(t)$ and subtracting the steady state response value from it.

Thus,

$$M_P = c(t_p) - 1 = c\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right) - 1$$

$$= 1 - \frac{e^{\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}.\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) - 1$$

$$= -\frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left(\pi + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$= +\frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Let $$\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \theta$$

$\therefore$ $$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\sin\theta = \frac{1}{\text{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}}$$

$\therefore$ $$= \frac{1}{\sqrt{1+\frac{\zeta^2}{1-\zeta^2}}} = \sqrt{1-\zeta^2}$$

$\therefore$ $$M_P = -\frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}.\sqrt{1-\zeta^2}$$

$\therefore$ $$M_P = e^{-\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \quad \text{....(7.20)}$$

Maximum per cent overshoot $$= \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

$$= \frac{1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100 \quad \text{....(7.21)}$$

$$= 100e^{-\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}.$$

d) **Settling time (t_s):** The transient response curve $c(t)$ always lies in between its two envelop curves $1 \pm (e^{-\zeta\omega_n t} / \sqrt{1-\zeta^2})$ whose time constant is $T = (1/\zeta\omega_n)$. So, the settling time (t_s) for the tolerance band of x per cent may be obtained from the given equation as follows.

$$\left|1 - \left\{1 \pm \left(e^{-\zeta\omega_n t} / \sqrt{1-\zeta^2}\right)\right\}\right| = \frac{x}{100}$$

or,
$$e^{-\zeta\omega_n t} = \frac{x}{100}\sqrt{1-\zeta^2}$$

For $0 < \zeta < 0.9$ and for convenience in comparing the responses of systems, we commonly define the settling time to be

$$t_s = 4T = \frac{4}{\zeta\omega_n} \quad \text{(for 2 per cent tolerance band)} \qquad \text{....(7.22)}$$

or,
$$t_s = 3T = \frac{3}{\zeta\omega_n} \quad \text{(for 5 per cent tolerance band).}$$

The setting time reaches a minimum value of ζ around 0.76 (for 2 per cent tolerance band) or $\zeta = 0.68$ (for 5 per cent tolerance band) and then increases almost linearly for large values of ζ.

7.6 DOMINANT CLOSED-LOOP POLES OF HIGHER ORDER SYSTEMS

The closed-loop poles that have dominant effect on the transient response behaviour are called dominant closed-loop poles. The dominant closed-loop poles often occur in the form of a complex-conjugate pair. The results, which were obtained in the second order system could easily be extended to higher order systems with a dominant pair of complex poles. That is, as long as the real part of these poles is much less than those of the other poles, the time response of higher order systems can be approximated by their having only dominant poles.

For example, let us take a third order system with transfer function,

$$E_a(s) = E(s) + K_d sE(s) + \frac{K_i}{s}E(s)$$

where $p = \dfrac{1}{\tau}$.

The unit step response by Laplace transform can be obtained as

$$c(t) = 1 - K_1 e^{-pt} + K_2 e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\right)t$$

where, $K_1 = \dfrac{\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2}$

$$K_2 = \frac{1}{\sqrt{\left[1-\zeta^2\left(1-2\zeta\omega_n / p + \omega_n^2 / p^2\right)\right]}}$$

and $$\phi = \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta}$$

Now two cases arise.

Case I: If $p << \zeta\omega_n$ then it can be shown that $\lim_{p\to 0} K_1 = 1$ and $\lim_{p\to 0} K_2 = 0$ so the pole at $s = -p$ being closer to the origin dominates in the response than the two conjugate poles at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Case II: If $p >> \zeta\omega_n$ then it can be shown that $\lim_{p\to\infty} K_1 = 0$ and $\lim_{p\to 0} K_2 = \dfrac{1}{\sqrt{1-\zeta^2}}$ so the pole at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ being closer to the origin dominates in the response than the poles at $s = -p$. It is observed that for $p = 6\zeta\omega_n$, the response of the third order system can be approximated as the response of the second order system which has transfer function:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The relative locations of the dominant poles of the third order system are shown in Fig. 7.14.

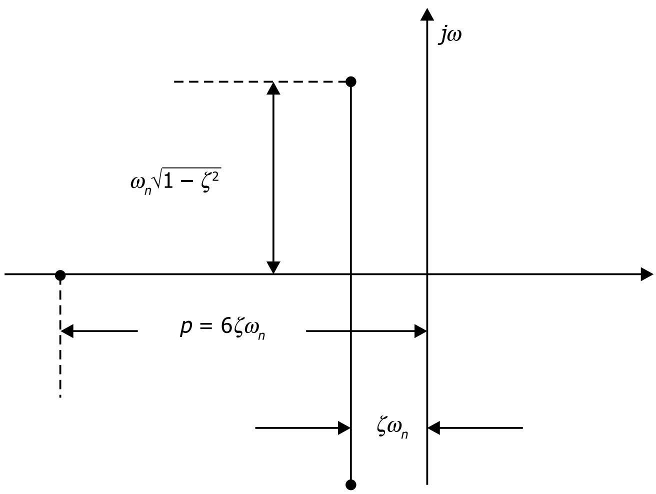


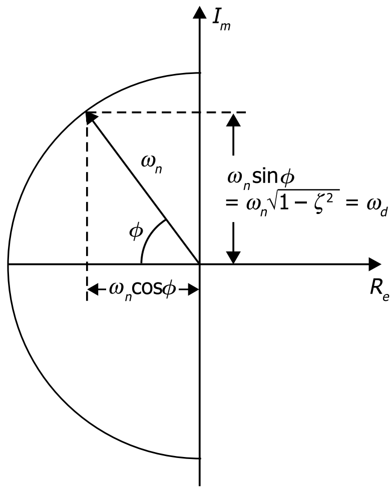
Fig. 7.14 *Location of dominant poles of a third order system*

Comments on Transient Response Specifications

It is always desirable that the transient response should be sufficiently fast and well damped. The output should reach the final value in a small interval of time, that is, the system response is quick. We have already seen from the response curves of the second order system that for values of ζ (0.5 to 0.8), steady-state conditions are obtained earlier but for such an underdamped system (0.5 to 0.8 $= \zeta$), there are oscillations at the equilibrium position. For large values of ζ, the maximum overshoot is more but settling time $\left(T_s = \dfrac{1}{\zeta\omega_n}\right)$ for higher values of ζ is less. *Thus, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other unavoidably becomes larger.*

Example 7.2 For a second order system $\zeta = 0.6$, $\omega_n = 5$ rad/sec. Find the values of t_r, t_p, M_p, t_s.

Solution



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= 5\sqrt{1-(.6)^2}$$

$$\omega_d = 4 \text{ rad/sec}$$

$$\sigma = \zeta\omega_n$$

$$= 0.6 \times 5$$

$$\sigma_s = 3$$

i) $$t_r = \frac{\pi - \beta}{\omega_d}$$

$$t_r = \frac{3.14 - \beta}{4}, \qquad \beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

$$t_r = \frac{3.14 - 0.93}{4}$$

$$t_r = 0.55 \text{ sec}$$

ii) $$t_p = \frac{\pi}{\omega_d}$$

$$= \frac{3.14}{4}$$

$$t_p = .785 \text{ sec.}$$

iii) $$M_p = e^{-\left(\frac{\sigma}{\omega_d}\right)\pi}$$

$$= e^{-\left(\frac{3}{4}\right)3.14}$$

$$M_P = 0.095$$

iv) $$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33\,\text{sec.} \quad \text{(for 2\% criterion)}$$

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1\,\text{sec.} \quad \text{(for 5\% criterion)}$$

Example 7.3 Consider the servomechanism shown in Fig. 7.15. Determine the values of k and k_1 so that the maximum overshoot in unit step response is 25 per cent and the peak time is 2 seconds.

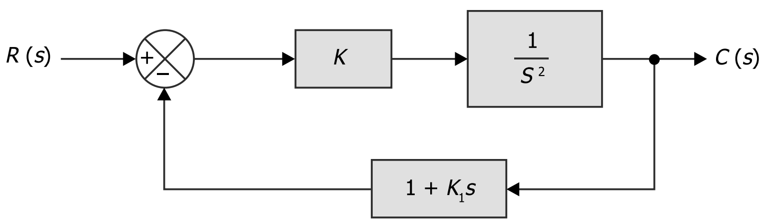


Fig. 7.15

Solution

$$M_P = e^{(-\zeta/\sqrt{1-\zeta^2})\pi} 0.25$$

and $$t_p = \frac{\pi}{\omega_d} = 2 \text{ given.}$$

$\therefore$ $$\frac{\zeta}{\sqrt{1-\zeta^2}}\pi = 1.39 \quad \text{and} \quad \omega_d = 1.57$$

$\therefore$ $$\zeta = 0.4$$

and $$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$$

$$= \frac{1.57}{\sqrt{1-(0.4)^2}}$$

$$= 1.71$$

$$\frac{C(s)}{R(s)} = \frac{k}{s^2 + kk_1 s + k} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Comparing, $k = \omega_n^2 = (1.71)^2 = 2.93$ and $kk_1 = 2\zeta\omega_n$

$\therefore$ $$K_1 = \frac{2(0.4)(1.71)}{2.93} = 0.47.$$

Example 7.4 The block diagram of a servo system is shown in Fig. 7.16. Determine the characteristic equation of the system. Also, calculate the following when the unit step input is given:

a) Undamped frequency of oscillation;
b) Damped frequency of oscillation;
c) Damping ratio and damping factor;
d) Maximum overshoot;
e) Time interval after which maximum and minimum output will occur;
f) Settling time;
g) Number of cycles completed before the output is settled within 2 per cent, 5 per cent of final value.

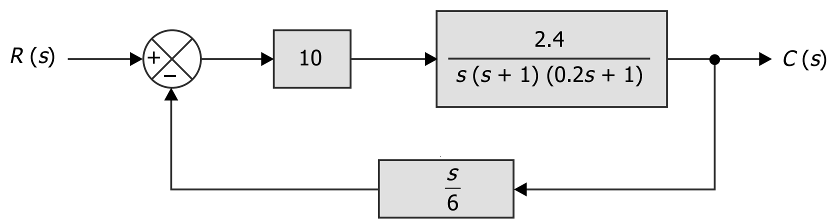


Fig. 7.16

Solution

The problem indicates a second order controlled system of the type having output function for unit step input as

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \cdot \frac{1}{s}$$

$$c(t) = 1 - \frac{\omega_n}{\omega_d} e^{0\sigma t} \sin\left[\omega_d t + \tan^{-1}\frac{\omega_d}{\sigma}\right]$$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\sigma = \omega_n \zeta$$

Now, for the given problem the open-loop transfer function is $G(s) = \dfrac{10 \times 2.4}{s(s+1)(0.2s+1)}$

$$G(s) = \frac{24}{s(s+1)(0.2s+1)}$$

and feedback is,

$$H(s) = \frac{s}{6}.$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{24}{s(s+1)(.2s+1)}}{\left[1+\dfrac{24}{s(s+1)(0.2s+1)}\right]\dfrac{s}{6}}$$

$$= \frac{24}{s(0.2s^2+1.2s+1)+4s} = \frac{24}{s(0.2s^2+1.2s+5)}$$

$$= \frac{120}{s(s^2+6s+25)}$$

This gives the characteristic equation as $(s^2 + 6s + 25) = 0$. Comparing it with the generalised equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, we get

a) $\omega_n^2 = 25$ or $\omega_n = 5$ rad/sec.

b) $2\zeta\omega_n = 6$ or $\zeta = 0.6$

$\therefore$ Damping factor = damping ratio × natural frequency

$$= \zeta\omega_n = 0.6 \times 5 = 3.$$

c) Damping frequency $= \omega_n \sqrt{1-\zeta^2}$

$$= 5\sqrt{1-(0.6)^2}$$
$$= 4 \text{ rad / sec}.$$

d) Maximum overshoot $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$$= e^{-0.6\pi/\sqrt{1-(0.6)^2}}$$
$$= 0.095.$$

e) Time interval after which maximum and minimum will occur is given by

$$2t_p = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{4} = 1.57 \text{ sec}.$$

f) Settling time (2 per cent), $t_s = \dfrac{4}{\zeta\omega_n} = 1.33\,\text{sec}.$

Settling time (5 per cent), $t_s = \dfrac{3}{\zeta\omega_n} = 1\,\text{sec}.$

g) Number of cycles completed before settling within 2 per cent is

$$\frac{2\sqrt{1-\zeta^2}}{\pi\zeta} = \frac{2\times 0.8}{\pi\times 0.6} = 0.85 \text{ cycle.}$$

Setting time changes from 1.33 sec to 1 sec, to for a change of tolerance from 2 per cent to 5 per cent.

Hence, the number of cycles completed before reaching 5 per cent tolerance will be reduced linearly to

$$\frac{0.85}{1.33} = 0.64$$

Example 7.5 A flywheel is driven by a motor which develops a torque of 1600 Nm/rad. The moment of inertia of the system is 100 kg m^2. The flywheel is controlled by a handwheel. If the handwheel is suddenly given a movement through $\pi/4$ rad from rest, calculate the response or output of the flywheel. The coefficient of viscous friction is 400 Nm/rad/sec.

Solution

Let θ_i is the reference position of the handwheel,
θ_0 is the output of the flywheel.

Torque developed = 1600 $[\theta_i(s) - \theta_0(s)]$

Also, torque developed is equated to $J\dfrac{d^2\theta_0}{dt^2} + B\dfrac{d\theta_0}{dt}$

Taking Laplace transform, equating and putting values,

$$1600[\theta_i(s) - \theta_0(s)] = [100s^2 + 400s]\,\theta_0(s)$$

or
$$\frac{\theta_0(s)}{\theta_i(s)} = \frac{1600}{100s^2 + 400s + 1600} = \frac{16}{s^2 + 4s + 16}$$

When a unit step input of $\pi/4$ rad is applied, we express $\theta_i(s) = \dfrac{\pi}{4}\dfrac{1}{s}$

Thus,
$$\theta_0(s) = \frac{\pi}{4s}\frac{16}{(s^2 + 4s + 16)}$$

$$= \frac{\pi}{4s}\frac{16}{s(s^2 + 4s + 16)} = \frac{\pi}{4}\left[\frac{1}{s} - \frac{s+4}{(s+2)^2 + 12}\right]$$

The output can be determined by taking inverse Laplace transform as,

$$\theta_0(t) = \frac{\pi}{4}\left[1 - e^{-2t}(\cos 3.46t + 0.578 \sin 3.46t)\right]$$

Example 7.6 A unity feedback system has an open-loop transfer function $G(s) = \dfrac{25}{s(s+8)}$. Determine its damping ratio, peak overshoot and time required to reach the peak output. Now a derivative component having transfer function of $s/5$ is introduced in the system. Discuss its effect on the values obtained above.

Solution

Without derivative feedback,

$$G(s) = \frac{25}{s(s+8)},$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{25/s(s+8)}{1 + \dfrac{25}{s(s+8)} \cdot 1}$$

$$= \frac{25}{s^2 + 8s + 25}$$

Its characteristic polynomial can be written as

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 8s + 25.$$

Comparing, $\omega_n = 5$ rad/s

and $$2\zeta\omega_n = 8$$

$\therefore$ Damping ratio $$\zeta = \frac{8}{2 \times 5} = 0.8.$$

Percentage peak overshoot

$$M_p = 100\, e^{-\left(\pi\zeta/\sqrt{1-\zeta^2}\right)}$$

$$= 100 e^{-\pi \times 0.8/\sqrt{1-(.8)^2}}$$

$$= 100 \times 0.0152$$

$$= 1.52\%$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{\pi}{5\sqrt{1-0.8^2}}$$

$$t_p = 1.047 \text{ second.}$$

When the derivative control is introduced in the feedback,

$$H(s) = 1 + \frac{s}{5}$$

$$M(s) = \frac{C(s)}{R(s)}$$

$$= \frac{\dfrac{25}{s(s+8)}}{1 + \dfrac{25}{s(s+8)} \cdot \dfrac{(s+5)}{5}}$$

$$= \frac{25}{s^2 + 8s + 5s + 25}$$

$$= \frac{25}{s^2 + 13s + 25}.$$

So the characteristic polynomial can be written as

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 13s + 25.$$

Comparing $\qquad \omega_n^2 = 25, \quad \omega_n = 5$

$$\zeta = \frac{13}{2 \times 5}$$

$$\zeta = 1.3.$$

There is no change in ω_n where as the value of ζ is now more than unity. The value of ζ has changed from 0.8 to 1.3. An underdamped system has now become an overdamped one.

Since the system, overdamped, the peak overshoot will not be attained.

Thus, the effect of derivative control in a unity feedback system is summarized as:

i) There is no change in natural frequency, but the damped frequency will change as per the damping ratio.

ii) There is an increase in the value of the damping ratio.

iii) The time to reach the peak overshoot is increased.

iv) The value of percentage peak overshoot is reduced.

Example 7.7 A feedback system having derivative feedback control is shown in Fig. 7.17.

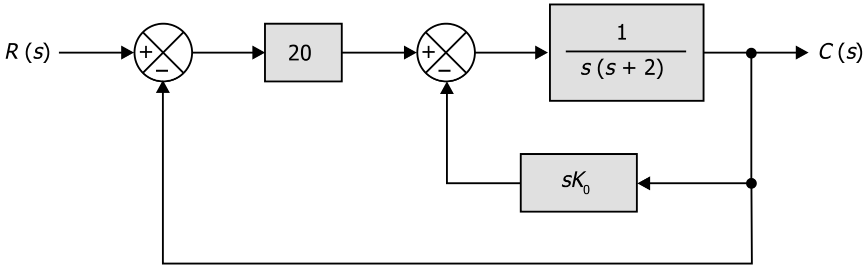


Fig. 7.17

i) If $k_0 = 0$, determine the damping ratio and natural frequency of the system. What is the steady-state error to a unit ramp input?

ii) Determine the value of k_0 for which the damping ratio of the system will be 0.6. Find the steady state error to a unit ramp input for this value of k_0.

iii) How can the steady-state error and damping ratio of the system with derivative feedback be kept to 0.2 and 0.6 respectively by applying unit ramp input?

Solution

i) When $k_0 = 0$, then the transfer function of the system is

$$M(s) = \frac{C(s)}{R(s)} = \frac{20}{s^2 + 2s + 20}$$

Its characteristic equation will be

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 20.$$

Comparing to get natural frequency,

$$\omega_n = \sqrt{20} = 4.48 \text{ rad/sec}, \; 2\zeta\omega_n = 2$$

Damping ratio

$$\zeta = \frac{2}{2\omega_n} = \frac{2}{2\sqrt{20}} = \frac{1}{\sqrt{20}} = 0.223.$$

Steady-state error

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}.$$

Now for unit ramp input,

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} \frac{s / s^2}{1 + \dfrac{20}{s(s+2)}}$$

$$= \lim_{s \to 0} \frac{(s+2)}{s^2 + 2s + 20}$$

$$= 0.1.$$

ii) With derivative feedback, the forward path transfer function of the system will be

$$G(s) = 20 \left[\frac{\dfrac{1}{s(s+2)}}{1 + \dfrac{sk_0}{(s+2)}} \right]$$

$$= \frac{20}{s^2 + s(2 + k_0)}.$$

So the characteristic polynomial of the system can be written as

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + (2 + K_0)s + 20.$$

Comparing the coefficients we get, $\omega_n = \sqrt{20}$ and $2\zeta\omega_n = 2 + K_0$

or,
$$\zeta = \frac{2 + K_0}{2\omega_n}$$

$$\zeta = \frac{(2 + K_0)}{2\sqrt{20}}.$$

To have its value as 0.6,

$$2 + K_0 = \zeta + 2\sqrt{20}$$

or,
$$K_0 = (0.6 \times 2 \times \sqrt{20}) - 2$$

$$= 5.4 - 2 = 3.4$$

Steady-state error

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

$$= \lim_{s\to 0} \frac{s/s^2}{1 + 20/(s^2 + 2s + K_0 s)}$$

$$= \frac{s + 2 + K_0}{s^2 + 2s + K_0 s + 20}$$

$$= \lim_{s\to 0} \frac{s + 2 + K_0}{s^2 + (2 + K_0)s + 20} = \frac{2 + K_0}{20}$$

Putting the value of $K_0 = 3.4$,

$$e_{ss} = \frac{2 + 3.4}{20} = 0.27$$

iii) To keep $e_{ss} = 0.2$ and $\zeta = 0.6$ let the amplifier gain be adjusted to a higher value of K. Hence the characteristic polynomial can be written as:

$$s^2 + (2 + K_0)s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Comparing, we get

$$2\zeta\omega_n = 2 + K_0 \text{ and } \omega_n = \sqrt{K}$$

$\therefore$ $$2 + K_0 = 2(0.6)\sqrt{K}$$

or, $$2 + K_0 = 1.2\sqrt{K} \qquad \text{....(7.23)}$$

Again, steady-state error $$e_{ss} = \frac{2 + K_0}{K} = 0.2$$

or, $$K = 5(2 + K_0)$$

or, $$K = 5(1.3)\sqrt{K} \qquad \text{[from equation (7.23)]}$$

or, $$\sqrt{K} = 6.1$$

$\therefore$ $$K = 36.0$$

Putting the value of K in equation (7.23),

$$2 + K_0 = (1.2)\,(6.1)$$

$\therefore$ $$K_0 = 5.93.$$

Example 7.8 Block diagram representation of a control system has been shown in Fig 7.18. Determine the value of gain K and velocity feedback constant K_h, so that the maximum overshoot in unit step response is 25 per cent and the peak time is 2 seconds.

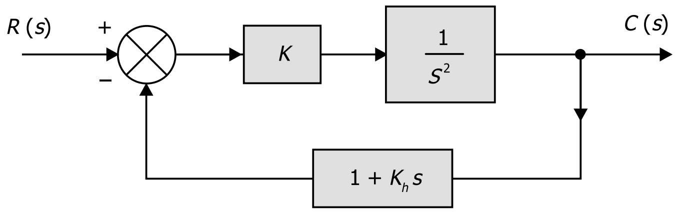


Fig. 7.18

Solution

The system transfer function $C(s)/R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s^2}}{1+\frac{K}{s^2}(1+K_h s)}$$

The characteristic equation is obtained by putting the denominator of the transfer function to zero.

Thus, $$1+\frac{K}{s^2}(1+K_h s) = 0$$

or, $$s^2 + K\,K_h s + K = 0$$

By comparing this equation with the basic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

We find, $$\omega_n^2 = K \text{ or } \omega_n = \sqrt{K}$$

$$2\zeta\omega_n = KK_h$$

∴ $$\zeta = \frac{KK_h}{2\omega_n} = \frac{KK_h}{2\sqrt{K}}$$

Maximum overshoot $$= 0.25 = e^{-\frac{2\pi}{\sqrt{1-\zeta^2}}}$$

$$\zeta = 0.404$$

Peak time, $$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2 = \frac{\pi}{\omega_n\sqrt{1-(0.404)^2}}$$

from which, $$\omega_n = 1.72$$

$$K = \omega n^2 = 2.96 \quad \text{and} \quad K_h = \frac{2\zeta\omega_n}{K} = \frac{2\zeta\sqrt{K}}{K} = \frac{2\zeta}{K} = \frac{2\times 0.404}{\sqrt{1.72}} = 0.47$$

7.7 SENSITIVITY OF A CONTROL SYSTEM

Consider the case of an open-loop system where, the overall transfer function is

$$M(s) = \frac{C(s)}{R(s)} = G(s).$$

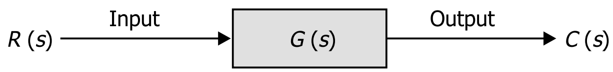


Fig. 7.19 *Block diagram of an open-loop control system*

A very small change $\Delta G(s)$ in the transfer function $G(s)$ changes it to $G(s) + \Delta G(s)$. The changed output of the system is

$$C(s) + \Delta C(s) = [G(s) + \Delta G(s)]\ R(s)$$

or, $$C(s) + \Delta C(s) = G(s)\ R(s) + \Delta G(s)\ R(s)$$

or, $$C(s) + \Delta C(s) = C(s) + [\Delta G(s)]\ R(s)$$

or, $$\Delta C(s) = [\Delta G(s)]\ R(s) \qquad \text{....(i)}$$

Now take the case of a closed-loop system the diagram of which has been shown in Fig. 7.20.

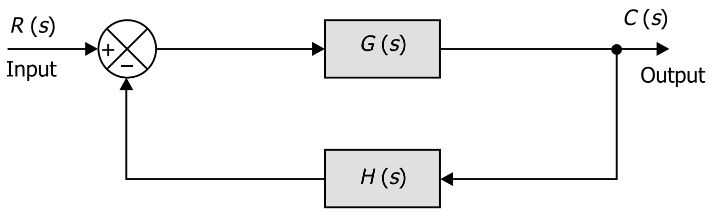


Fig. 7.20 *Block diagram of a closed-loop control system*

The overall transfer function, $$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

A very small change $\Delta G(s)$ in the forward path transfer function changes $G(s)$ to $G(s) + \Delta G(s)$.

$$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + [G(s) + \Delta G(s)]H(s)}.R(s)$$

or, $$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)}.R(s)$$

or, $$C(s)+\Delta C(s)=\frac{G(s)}{1+G(s)H(s)}.R(s)+\frac{\Delta G(s)}{1+G(s)H(s)}R(s) \quad \text{[neglecting } \Delta G(s)H(s)]$$

or, $$C(s)+\Delta C(s)=C(s)+\frac{\Delta G(s)}{1+G(s)H(s)}.R(s)$$

or, $$\Delta C(s)\cong\frac{\Delta G(s)}{1+G(s)H(s)}.R(s) \qquad \text{....(ii)}$$

From (i) and (ii) it is observed that feedback reduces the variation of output by a factor [1 + *G*(*s*) *H*(*s*)]. Since [1 + *G*(*s*) *H*(*s*)] >> 1 output variation is more sensitive (to variation of *G*(*s*)) in case of open-loop systems as compared to closed-loop systems.

Sensitivity of a control system is expressed as

$$\text{Sensitivity}=\frac{\text{Percentage change in overall transfer function, i.e.}\dfrac{C(s)}{R(s)}}{\text{Percentage change in } G(s)}$$

The sensitivity function for the overall transfer function *M*(*s*) with respect to variation in *G*(*s*) is written as

$$s_G^M=\frac{\partial M(s)/M(s)}{\partial G(s)/G(s)};\ \text{or,}\ s_G^M=\frac{G(s)}{M(s)}.\frac{\partial M(s)}{\partial G(s)}$$

For the closed-loop control system of Fig. 7.18 the sensitivity S_G^M is given by

$$S_G^M=\frac{G(s)}{\dfrac{G(s)}{1+G(s)H(s)}}.\frac{[1+G(s)H(s)]-G(s)H(s)}{[1+G(s)H(s)]^2}$$

$$=\frac{1}{1+G(s)H(s)}$$

From the above analysis the following are concluded:

i) Use of feedback reduces sensitivity of the system to parameter variations;

ii) Open-loop systems to be more accurate, components of *G*(*s*) must be selected to strictly meet the desired specifications;

iii) For closed-loop systems the components of *G*(*s*) may not be so tightly specified

iv) Improvement of sensitivity by use of feedback is off-set by the need for reduction of system gain.

7.8 CONTROL ACTIONS FOR DESIRED OUTPUT

Control actions involve steps to be taken to maintain the system output within the desired limit. For automatic control the error detector detects any variation in the output and generates an actuating signal to be fed to a controller. Various types of control actions, viz Proportional Control, Derivative Control, Integral Control, etc. are discussed as follows.

7.8.1 Proportional Control

Here, after obtaining the error signal, a proportional actuating signal is obtained by introducing a proportional gain controller having a gain K_p and its output is fed to the system as shown in Fig. 7.21. A proportional controller is basically an op-amp amplifier with controllable gain. A proportional controller is simply an amplifier of gain K, which amplifies the error signal and provides input to the actuator.

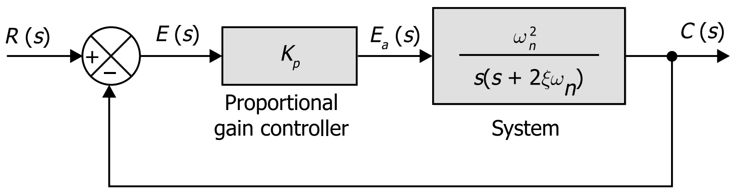


Fig. 7.21

The actuating signal is given by

$$E_a(t) = K_p E(t)$$

and its Laplace transform is given by

$$E_a(s) = K_p E(s)$$

For the system shown in Fig. 7.19, $E(s) = R(s) - C(s) = R(s) - E(s)\dfrac{K_p\omega_n^2}{s(s+2\zeta\omega_n)}$

or,
$$\frac{E(s)}{R(s)} = \frac{s(s+2\zeta\omega_n)}{s^2+2\zeta\cdot\omega_n s + K_p\omega_n^2}$$

For a unit ramp input, the steady state error is given by

$$e_{ss} = \lim_{s\to 0} SE(s)$$

$$= \lim_{s\to 0} s\frac{1}{s^2}\frac{s(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s + K_p\omega_n^2}$$

$$= \frac{2\zeta}{K_p\omega_n}$$

As e_{ss} is inversely proportional to K_p, steady state-error will decrease as the proportional gain K_p is increased.

To study the effect of change of gain K_p on the transient response of the system, we write the closed-loop transfer function, which is given as

$$\frac{C(s)}{R(s)} = \frac{K_p\omega_n^2}{s^2 + 2\zeta\omega_n s + K_p\omega_n^2}$$

The roots of the characteristic equation,

$$s^2 + 2\zeta\omega_n s + K_p\omega_n^2 = 0$$

is given as, $-\zeta\omega_n \pm j\omega_n\sqrt{K_p - \zeta^2}$

The damped natural frequency, $\omega_d = \omega_n\sqrt{K_p - \zeta^2}$ will thus increase as K_p is increased, indicating that the system output will oscillate at a higher frequency. The system however remains stable as the two roots lie in the left half of s-plane. Due to increase in ω_d, the rise time will decrease, making the system respond faster. However, minimum overshoot will increase. This follows from the expression for rise time and M_p as given below.

$$t_r = \frac{\pi - \beta}{\omega_d} \text{ where } \beta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n}$$

and

$$M_p = e^{-\left(\frac{\zeta\omega_n}{\omega_d}\right)\pi}$$

Thus, a proportional controller reduces the steady state error and makes the system respond faster but at the expense of increased maximum overshoot and oscillatory response.

For third and higher order systems, the system becomes unstable beyond a particular value of K_p as one or more roots shifts to the right hand side of the s-plane as the value of K_p increases.

7.8.2 Proportional Plus Derivative Control

Basically derivative control involves using an actuating signal which is proportional to the rate of change of the error signal. However, for achieving proportional plus derivative control action, we should have an actuating signal consisting of proportional error signal plus the derivative of the error signal. The actuating signal thus will be

$$e_a(t) = K_p e(t) + K_d\frac{d(Ke(t))}{dt} = e(t) + K_d\frac{de(t)}{dt}.$$

where K_d is a constant, and $K_p = 1$.

Taking Laplace transform, we get

$$E_a(S) = E(s) + sK_d E(s)$$

or,

$$E_a(s) = (1 + sK_d)E(s).$$

Derivative control action when added to a proportional controller increases the sensitivity of the controller and tends to increase the stability of the system by producing early corrective action before the magnitude of the error signal $e(t)$ becomes too large.

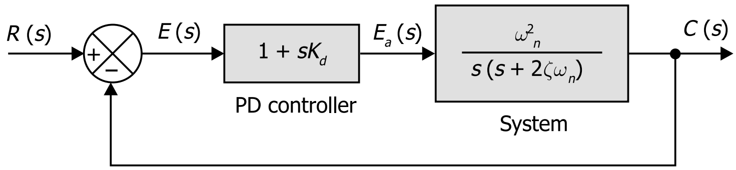


Fig. 7.22 *Second order unity feedback control system using derivative control*

The overall transfer function of the closed-loop second order control system using derivative control (Fig. 7.22) is given by

$$\frac{C(s)}{R(s)} = \frac{(1+sK_d)\omega_n^2}{s^2+(2\zeta\omega_n+\omega_n^2K_d)s+\omega_n^2}.$$

So the characteristic equation is

$$s^2+(2\zeta\omega_n+\omega_n^2K_d)s+\omega_n^2=0.$$

The standard from is, $s^2+2\zeta\omega_n s+\omega_n^2=0.$

From the co-efficient of s, the effective damping with derivative control and considering ζ as ζ_n is given by

New damping ratio, $\zeta_n = \dfrac{2\zeta\omega_n+\omega_n^2K_d}{2\omega_n} = \zeta + \dfrac{\omega_nK_d}{2}.$

Thus derivative control increases the damping of the second order system. Since natural frequency ω_n remains unchanged and the maximum overshoot, rise time and settling time are reduced.

For a closed loop system we can write

$$\frac{C(s)}{R(s)} = 1-\frac{E(s)}{R(s)}.$$

or,
$$\frac{E(s)}{R(s)} = 1-\frac{C(s)}{R(s)}.$$

From Fig. 7.20, we get

$$\frac{E(s)}{R(s)} = \frac{s(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n+\omega_n^2T_{kd})s+\omega_n^2}.$$

For unit ramp input, the steady-state error is given by

$$e_{ss} = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{s(s+2\zeta\omega_n)}{s^2 + 2(\zeta\omega_n + \omega_n^2 K_d)s + \omega_n^2} = \frac{2\zeta}{\omega_n}.$$

$K_d = 0$ will also give the same result of $e_{ss} = \frac{2\zeta}{\omega_n}$. *Thus steady-state error is not affected by derivative control*. While derivative control action has an advantage of being anticipatory thereby improving transient response, it has the disadvantages that it amplifies noise signals and may cause a saturation effect in the actuator. Derivative control action can never be used alone because this control action is effective only during transient period.

7.8.3 Proportional Plus Integral Control (PI Control)

In integral control use is made of an actuating signal which is proportional to the integral of error signal. In proportional plus integral control we use actuating signal which is proportional to error signal plus the integral of the error signal. The actuating signal for integral control action, can therefore be written as,

$$e_a(t) = e(t) + k_i \int e(t)\,dt, \text{ where } k_i \text{ is a constant and assuming } K_p = 1.$$

Taking Laplace transform, we get

$$E_a(s) = E(s) + k_i \frac{E(s)}{s}.$$

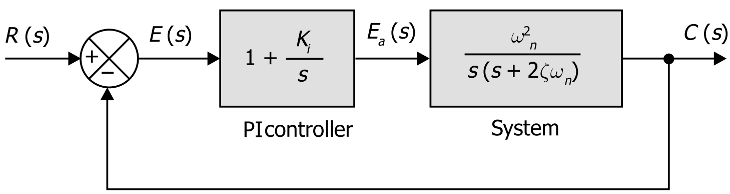


Fig. 7.23 *Second order unit feedback system using proportional plus integral control*

The overall transfer function of the closed-loop second order control system (Fig. 7.23) is given by

$$\frac{C(s)}{R(s)} = \frac{(s + K_i)\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}.$$

So the characteristic equation is obtained by equating denominator of the transfer function to zero, so that

$$s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2 = 0.$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{(s+K_i)\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$$

or,
$$\frac{E(s)}{R(s)} = \frac{s^2(s+2\zeta\omega_n)}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}.$$

or,
$$E(s) = R(s)\frac{s^2(s+2\zeta\omega_n)}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$$

For unit ramp input where $R(s) = \frac{1}{s^2}$, steady-state error is given by

$$e_{ss} = \lim_{s\to 0} sE(s)$$

or,
$$e_{ss} \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{s^2(s+2\zeta\omega_n)}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2} = 0$$

$K_i = 0$ will give the steady-state error $e_{ss} = \frac{2\zeta}{\omega_n}$. So without integral control action, a steady-state error or offset $e_{ss} = \frac{2\zeta}{\omega_n}$ exists in a second order closed-loop control system with unit ramp input. This offset or steady-state error can be reduced to zero by using integral control action. *The integral control action, while improving steady-state accuracy by removing offset, may lead to oscillatory response of a slowly decreasing amplitude or, even increasing amplitude, both of which are usually undesirable.*

7.8.4 Proportional Plus Integral Plus Derivative Control (PID Control)

PI control was introduced to meet the high accuracy requirement and improve steady state performance. Introduction of PI control changed a second order system into a third order system. Third order system introduces a distinct possibility of system instability. In PID control we take advantage of PI control with derivative control. PID controller is therefore a three action controller which combines proportional, integral and derivative control. The actuating signal is given by

$$e_a(t) = e(t) + K_d\frac{d}{dt}e(t) + K_i\int e(t)dt.$$

In Laplace transform form,

$$E_a(s) = E(s) + K_d sE(s) + \frac{K_i}{s}E(s)$$

or,
$$E_a(s) = E(s)\left[1 + sK_d + \frac{K_i}{s}\right]$$

PID controller is represented in the block diagram as shown in Fig. 7.24.

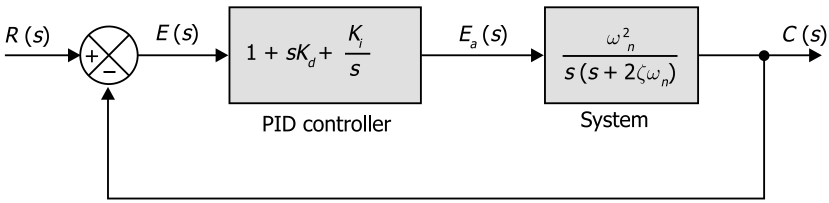


Fig. 7.24 *PID control contains the advantages of proportional, derivative, and integral control actions*

A PID control contains all the advantages of P, I, and D controllers. The disadvantages get almost neutralized to a large extent. PID controllers are widely used in industrial control applications.

7.8.5 Derivative Feedback Control

For achieving derivative feedback control action the actuating signal is obtained as the difference between the proportional error signal and derivative of the output signal. The actuating signal can, therefore, be expressed as

$$e_a(t) = e(t) - K_t \frac{dc(t)}{dt},$$

where K_t is a constant.

Taking Laplace transform, we get

$$E_a(s) = E(s) - sK_t C(s)$$

The overall transfer function for the unity feedback second order control system (Fig. 7.25) is given by

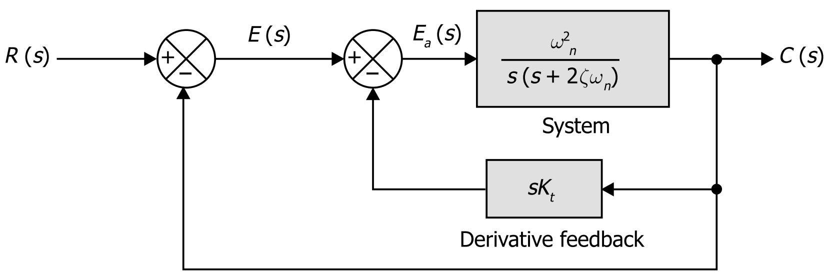


Fig. 7.25 *Derivative (or rate) feedback control*

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s + \omega_n^2}.$$

The characteristic equation is

$$s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s + \omega_n^2 = 0.$$

So the effective damping is given by,

$$\zeta' = \frac{2\zeta\omega_n + \omega_n^2 K_t}{2\omega_n}$$

$$= \zeta + \frac{\omega_n K_t}{2}.$$

Therefore, like derivative control the damping ratio is increased by using derivative feedback control. The maximum overshoot is thereby decreased and rise time however is increased. Since natural frequency ω_n is unchanged, this controller decreases the response settling time.

From Fig. 7.23 we get,

$$\frac{E(s)}{R(s)} = \frac{s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s}{s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

For unit ramp input, steady-state error is given by

$$e_{ss} = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s}{s^2 + (2\zeta\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

$$= \frac{2\zeta}{\omega_n} + K_t$$

So for a ramp input, the steady-state error is increased by using derivative feedback control action.

Example 7.9 The control valve of a hydraulic servomotor is shown in Fig. 7.24(a). Assume that the hydraulic fluid is incompressible and the oil flow rate is proportional to the control valve displacement (x). The power piston and load are of negligibly small mass.

Show that

a) The servomotor of Fig. 7.26(a) is an integral controller
b) The servomotor of Fig. 7.26(a) can be modified to a proportional controller by means of a floating feedback link ABC as shown in Fig. 7.26(b).

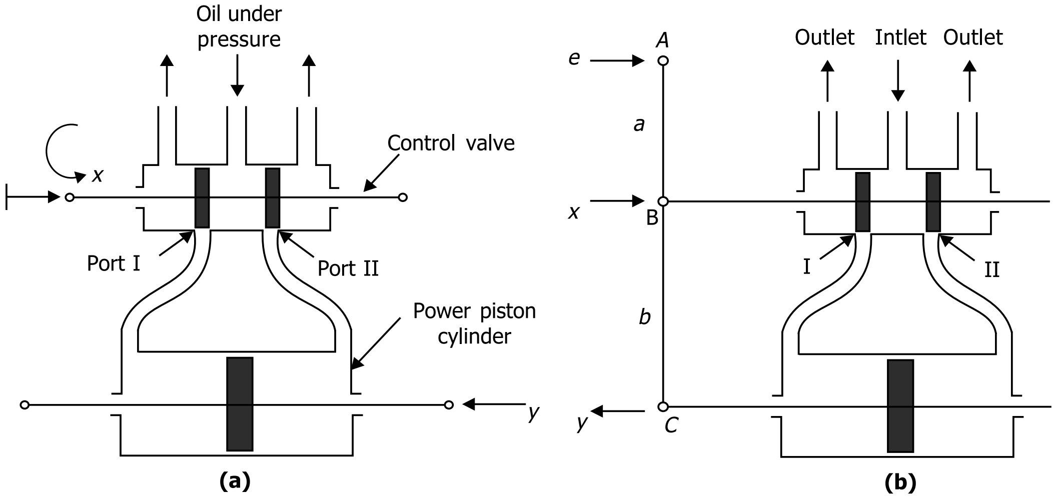


Fig. 7.26 *(a) Hydraulic servomotor; (b) Hydraulic proportional controller*

Solution

a) If the input x moves the pilot valve to the right, high pressure oil enters the right side of the power piston through port II and low pressure oil in the left side of the power piston returns to the drain through port I.
The rate (q) of flow of oil times dt is equal to the power piston displacement (dy) times the piston area (A) times the density (ρ) of oil.
That is, $A\rho\, dy = q\, dt$.
Again from given assumption,

$$q = K_1 x$$

where K_1 is a positive constant.
Combination the above two equations, we get

$$A\rho \frac{dy}{dt} = K_1 x.$$

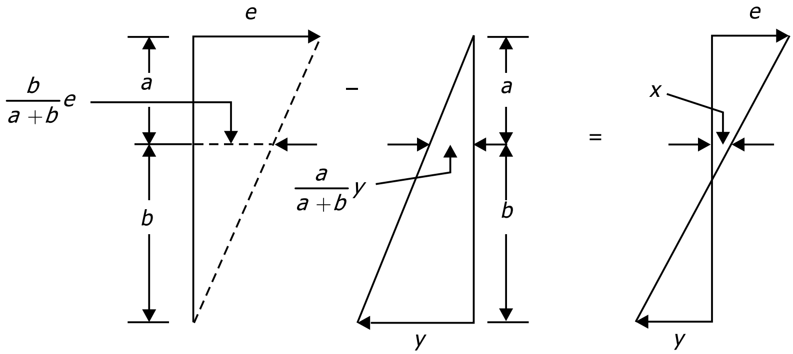
Taking Laplace transform with zero initial conditions,

$$\frac{Y(s)}{X(s)} = \frac{K_1}{A\rho s} = \frac{K}{s} \qquad \text{....(7.24)}$$

So the hydraulic servomotor shown in Fig. 7.26(a) acts as an integral controller.

b) In the Fig. 7.26(b), if input e moves the control valve to the right, the power piston will move to the left with feedback link ABC, thereby moving the pilot valve to the left until it returns to its original position.
The displacement x is given by

$$x = \frac{be}{a+b} - \frac{ay}{a+b}$$



Taking Laplace transform with zero initial condition,

$$X(s) = \frac{b}{a+b}E(s) - \frac{a}{a+b}Y(s) \qquad \text{....(7.25)}$$

Again from equation (7.24),

$$Y(s) = \frac{K}{s}X(s) \qquad \text{....(7.26)}$$

Using equations (7.25) and (7.26), the block diagram of Fig. 7.26(b) is given below.

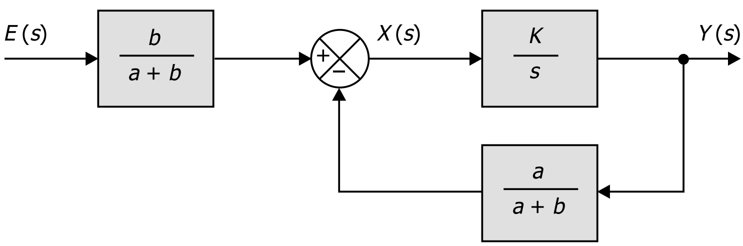


Fig. 7.27 *Block diagram of Fig. 7.26(b)*

From Fig.7.27, we get

$$\frac{Y(s)}{E(s)} = \frac{\left(\frac{b}{a+b}\right)\left(\frac{K}{s}\right)}{1+\left(\frac{K}{s}\right)\left(\frac{a}{a+b}\right)} = \frac{\frac{bK}{s(a+b)}}{1+\frac{Ka}{s(a+b)}} = \frac{bK}{s(a+b)} \times \frac{s(a+b)}{Ka} = \frac{b}{a}$$

Since under normal operation, we have $Ka|[s(a+b)]| >> 1$, the above transfer function is simplified as

$$\frac{Y(s)}{E(s)} = \frac{b}{a} = K_p$$

Thus the hydraulic servomotor with a feedback link works as a proportional controller of gain K_p.

Example 7.10 Show that the electronic controller shown in Fig. 7.28 works like a PD controller.

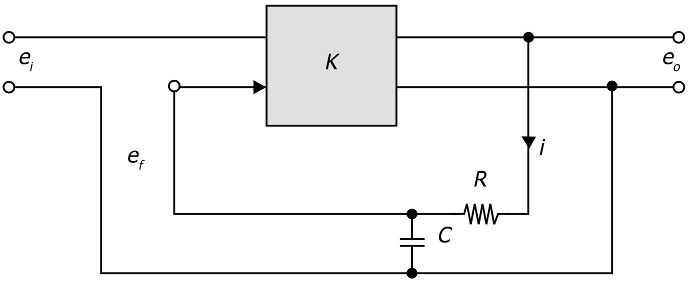


Fig. 7.28 *Electronic controller*

Solution

Output, $e_0 = (e_i - e_f)K.$

In Laplace transform form,

$$E_0(s) = K[E_i(s) - E_f(s)]$$

Again, $$e_0 = R_i + \frac{1}{C}\int idt,\ E_0(s) = RI(s) + \frac{1}{Cs}I(s)$$

and $$E_f = \frac{1}{C}\int idt,\ E_f(s) = \frac{1}{Cs}I(s)$$

$$\frac{E_f(s)}{E_0(s)} = \frac{\frac{1}{Cs}I(s)}{\left(R + \frac{1}{Cs}\right)I(s)} = \frac{1}{1+RCs}$$

$\therefore$ $$E_f(s) = \frac{1}{1+RCs}E_0(s)$$

We had, $$E_0(s) = K[E_i(s) - E_f(s)]$$

Putting the value of $E_f(s)$ we have

$$E_0(s) = KE_i(s) - \frac{K}{1+RCs}E_0(s)$$

or,
$$E_0(s)\left[1+\frac{K}{1+RCs}\right]=KE_i(s)$$

or,
$$\frac{E_0(s)}{E_i(s)}=\frac{K}{1+\dfrac{K}{1+RCs}}$$

Assuming

$$\frac{K}{1+RCs}\gg 1,\quad \frac{E_0(s)}{E_i(s)}=\frac{K}{\dfrac{K}{1+RCs}}=1+RCs$$
$$=(1+K_d s)$$

where $K_d = Rc$

This $(1 + K_d s)$ is equivalent to a PD controller and is represented as in Fig. 7.29.

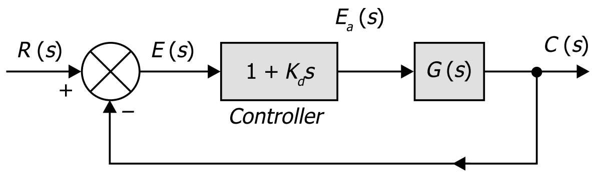


Fig. 7.29 PD controller

Example 7.11 A unity feedback control system with proportional plus derivative control is shown. Calculate the value of K_d so that the system is critically damped. Also for a ramp input calculate the values of steady state error, maximum overshoot, and settling time for the system without and with control actions.

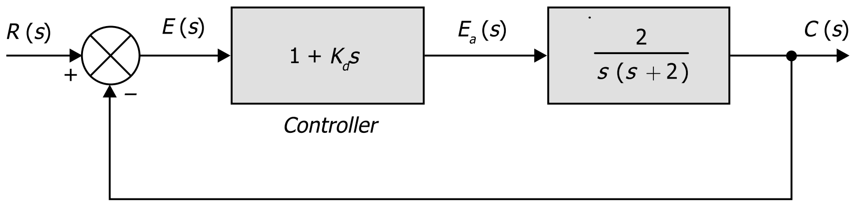


Fig. 7.30 Use of PD controller

Solution

With control action,

$$\frac{C(s)}{R(s)}=\frac{\dfrac{2(1+K_d s)}{s(s+2)}}{1+\dfrac{2(1+K_d s)}{s(s+2)}}=\frac{2(1+K_d s)}{s(s+2)+2(1+K_d s)}$$

The characteristic equation is

$$s(s+2)+2(1+K_d s)=0$$

or,
$$s^2+(2+2K_d)\,s+2=0.$$

Putting in the standard form and comparing,

$$s^2+2\zeta\omega_n s+\omega_n^2=0$$

$$2\zeta\omega_n = 2+2K_d$$

$$= 2(1+K_d)$$

$$\zeta\omega_n = 1+K_d$$

and $\omega_n^2 = 2$, i.e. $\omega_n = \sqrt{2}$

Putting the value of ω_n and taking $\zeta = 1$ for critically damped system,

$$1\times\sqrt{2} = 1+K_d$$

$$K_d = 0.414.$$

Without control action,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{2}{s(s+2)}}{1+\dfrac{2}{s(s+2)}}$$

$$= \frac{2}{s^2+2s+2}$$

as $H(s) = 1$

The characteristic equation is

$$s^2+2s+2=0$$

Comparing with the standard equation $s^2+2\zeta\omega_n s+\omega_n^2=0$,

$$\omega_n^2 = 2 \text{ or, } \omega_n = \sqrt{2} = 1.414$$

$$2\zeta\omega_n = 2 \text{ or, } \zeta = \frac{2}{2\omega_n} = \frac{2}{2\sqrt{2}} = 0.707$$

Steady state error,
$$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2\times 0.707}{1.414} = 1 \text{ rad.}$$

Setting time,
$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\frac{1}{\sqrt{2}}\sqrt{2}} = 4 \text{ seconds.}$$

Maximum overshoot, M_p is calculated as

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{\frac{-3.14\times 0.707}{\sqrt{1-(.707)^2}}} = e^{-3.11} = 0.12 = 12 \text{ percent}$$

With control action:

The characteristic equation is

$$s(s + 2) + 2(1 + K_d s) = 0$$

K_d was calculated as 0.414.

Therefore, the equation becomes

$$s^2 + 2s + 2 + 2 \times 0.414s = 0$$

or,
$$s^2 + 2.828s + 2 = 0$$

Comparing with the standard form $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$,

$$\omega_n^2 = 2 \text{ or } \omega_n = \sqrt{2} = 1.414 \ \zeta = 1$$

$$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2\times 1}{1.414} = 1.414 \text{ rad}$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{1\times\sqrt{2}} = 2\sqrt{2} = 2.828 \text{ seconds}$$

Maximum overshoot,
$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{\frac{-\pi\times 1}{\sqrt{1-1}}} = e^{-\infty} = 0$$

Example 7.12 A feedback control system is shown in Fig. 7.31. Determine the value of K so that the system will have a damping ratio of 0.6. Also for this value of K calculate maximum overshoot, setting time, and steady state error for a unit ramp input.

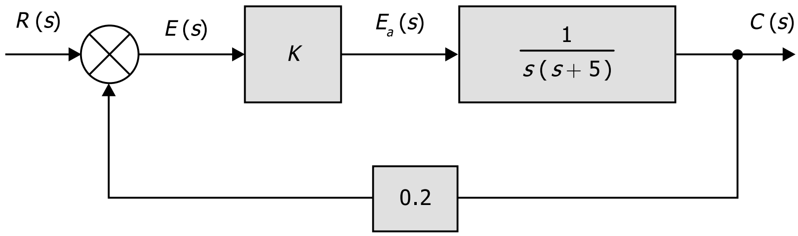


Fig. 7.31 *Proportional control action*

Solution

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s(s+5)}}{1+\frac{K\times 0.2}{s(s+5)}} = \frac{K}{s(s+5)+0.2K}$$

The characteristic equation is

$$s^2 + 5s + 0.2K$$

Comparing with the standard equation $s^2 + 2\zeta\omega_n s + \omega_n^2$,

$$\omega_n^2 = 0.2K \text{ or } \omega_n = \sqrt{0.2K}$$

$$2\zeta\omega_n = 5 \text{ or, } 2\times 0.5\,\omega_n = 5, \text{ or } \omega_n = 5$$

$$5 = \omega_n = \sqrt{0.2K}$$

or, $$0.2K = 5^2 = 25$$

or, $$K = \frac{25}{0.2} = 125$$

The characteristic equation with this value of K is

$$s^2 + 5s + 25 = 0$$

$$\omega_n^2 = 25 \text{ or } \omega_n = 5 \text{ and } \zeta = 0.6 \quad \text{(given)}$$

Settling time, $$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6\times 5} = 1.33 \text{ seconds.}$$

Steady state error, $$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2\times 0.6}{4} = 0.3 \text{ rad.}$$

Maximum overshoot, $$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{\frac{-3.14\times 0.6}{\sqrt{1-0.6^2}}} = e^{-2.35} = 0.15$$

Example 7.13 A feedback control system with proportional gain controller and having derivative feedback is shown in Fig. 7.32. Determine the damping ratio and steady state error for ramp input without the derivative feedback. Consider the gain of the amplifier $K_A = 9$. Now calculate with derivative control the values of K_A and K_t so that the damping ratio is increased to 0.6 without any change in the steady state error.

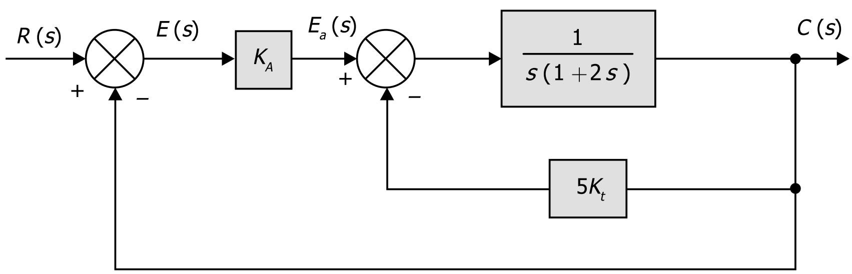


Fig. 7.32 Feedback control system

Solution

When derivative feedback loop is absent the system is represented as in Fig. 7.33.

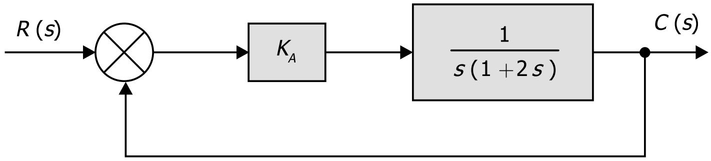


Fig. 7.33

With amplifier gain $K_A = 9$, the overall transfer function is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K_A}{s(1+2s)}}{1+\dfrac{K_A}{s(1+2s)}} = \frac{K_A}{s(1+2s)+K_A} = \frac{9}{2s^2+s+9} = \frac{4.5}{s^2+0.5s+4.5}$$

The characteristic equation is,

$$s^2 + 0.5s + 4.5 = 0$$

Comparing with equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

$$\omega_n^2 = 4.5 \text{ or } \omega_n = \sqrt{4.5} = 2.14 \text{ rad/sec}$$

and $$2\zeta\omega_n = 0.5, \text{ or } \zeta = \frac{0.5}{2\omega_n} = \frac{0.5}{2\times 2.14} = 0.117$$

Steady state error, $$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{2\times 0.117}{2.14} = 0.1$$

When the derivative feedback loop is present,

$$\frac{C(s)}{E_a(s)} = \frac{\dfrac{1}{s(1+2s)}}{1+\dfrac{sK_t}{s(1+2s)}} = \frac{1}{s(1+2s)+sK_t} = \frac{1}{2s^2+s(1+K_t)}$$

$$\frac{C(s)}{E(s)} = \frac{K_A}{2s^2+s(1+K_t)}$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K_A}{2s^2+s(1+K_t)}}{1+\dfrac{K_A}{2s^2+s(1+K_t)}} = \frac{K_A}{2s^2+s(1+K_t)+K_A} = \frac{0.5K_A}{s^2+\dfrac{(1+K_t)s}{2}+\dfrac{K_A}{2}}$$

The characteristic equation is,

$$s^2 + \frac{1+K_t}{2}s + \frac{K_A}{2} = 0$$

Comparing with $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$,

$$\omega_n^2 = \frac{K_A}{2} \text{ or } \omega_n = \frac{\sqrt{K_A}}{\sqrt{2}}$$

$$2\zeta\omega_n = 0.5(1+K_t)$$

$$2\times 0.6\times\frac{\sqrt{KA}}{2} = 0.5(1+K_t) \qquad \text{....(i)}$$

$$1.7\sqrt{K_A} = 1+K_t$$

$$E(s) = R(s) - C(s)$$

or,

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}$$

$$E(s) = R(s)\left[1-\frac{C(s)}{R(s)}\right] = R(S)\left[1-\frac{K_A}{2s^2+s(1+K_t)+K_A}\right]$$

$$= R(s)\frac{s\left[2s+(1+K_t)\right]}{2s^2+s(1+K_t)+K_A}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = s\frac{1}{s^2}\frac{s\left[2s+(1+K_t)\right]}{2s^2+s(1+K_t)+K_A}$$

$$= \frac{\left[2s+(1+K_t)\right]}{2s^2+s(1+K_t)+K_A}$$

$$= \frac{1+K_t}{K_A}$$

This new e_{ss} must be equal to the earlier calculate e_{ss} as 0.1.

Therefore, $0.1 = \dfrac{1+K_t}{K_A}$ ....(ii)

From (i) and (ii),

$$0.1K_A = 1+K_t = 1.7\sqrt{K_A}$$

or, $\sqrt{K_A} = 17$, i.e. $K_A = 289$

and $K_t = 0.1\,K_A - 1 = 28.9 - 1 = 27.9.$

Example 7.14 Determine the resistance values to obtain the following electronic PI controller with proportional and integral gain of 2 and 0.04 respectively. Use a 1 μF capacitor.

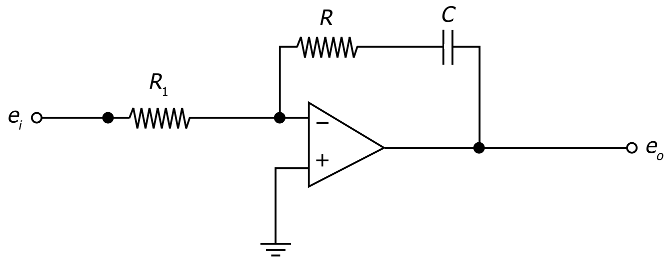


Fig. 7.34 Electronic PI controller

Solution

The transfer function of the above controller with zero initial conditions is given by

$$\frac{E_0(s)}{E_i(s)} = -\frac{RCs+1}{R_1Cs}$$

$$= -\left(K_p + \frac{K_1}{s}\right)$$

where, $K_p = \dfrac{R}{R_1}$ = proportional gain

and $K_I = \dfrac{1}{R_1C}$ = integral gain.

Given, $K_p = 2$ and $K_I = 0.04$

$$\therefore \quad \frac{R}{R_1} = 2$$

or, $$R = 2R_1 \qquad \text{....(7.27)}$$

and $$\frac{1}{R_1C} = 0.04$$

or, $$R_1C = 25 \qquad \text{....(7.28)}$$

Given that $C = 1\ \mu F$.
From equation (7.28)

$$R_1 = \frac{25}{1 \times 10^{-6}}\Omega = 25\ \text{M}\Omega.$$

From equation (7.27)

$$R = (2 \times 25)\ \text{M}\Omega = 50\ \text{M}\Omega.$$

Example 7.15 Determine the resistance values to obtain the following electronic PID controller with proportional, integral and derivative gain of 5, 0.7 and 2 respectively. The circuit should limit frequencies above 250 rad/sec. Take one capacitance to be 1 μF.

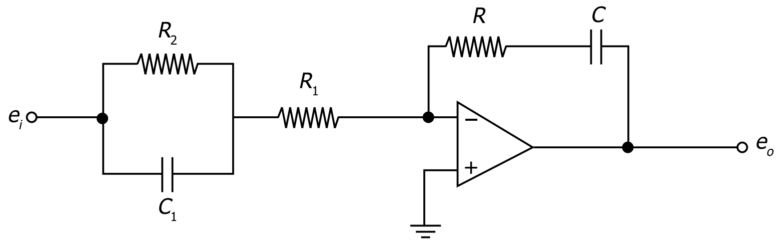


Fig. 7.35 *Electronic PID controller*

Solution

The transfer function of the above controller with zero initial conditions is given by

$$\frac{E_0(s)}{E_i(s)} = \frac{(RCs+1)(R_2C_1s+1)}{Cs(R_1+R_2+R_1R_2C_1s)}$$

$$= -\left(\frac{RC+R_2C_1}{R_2C} + \frac{1}{R_2Cs} + RC_1s\right)\left(\frac{\beta}{\beta R_1C_1s+1}\right)$$

$$= -\left(K_p + \frac{K_I}{s} + K_Ds\right)\frac{1}{(\beta R_1C_1s+1)}$$

where,
$$\beta = \frac{R_2}{R_1+R_2} \qquad \text{....(7.29)}$$

$$K_p = \beta\frac{RC+R_2C_1}{R_2C} \qquad \text{....(7.30)}$$

$$K_I = \frac{\beta}{R_2C} \qquad \text{....(7.31)}$$

$$K_D = \beta RC_1. \qquad \text{....(7.32)}$$

The denominator term $(\beta R_1C_1s + 1)$ limits the effect of frequencies above $\omega = \dfrac{1}{\beta R_1C_1}$.
As $\omega = 250$ rad/sec (given),

$$\therefore \qquad \frac{1}{\beta R_1C_1} = 250. \qquad \text{....(7.33)}$$

Eliminating β from equations (7.32) and (7.33)

$$\frac{R}{R_1} = 250K_D = 500 \quad [K_D = 2 \text{ given}]$$

or, $R = 500\ R_1$. ....(7.34)

Eliminating β from equations (7.30) and (7.31)

$$RC + R_2C_1 = \frac{K_p}{K_I} = \frac{5}{0.7} \ [K_p = 5 \text{ and } K_I = 0.7 \text{ given}]$$

or, $$RC + R_2C_1 = \frac{50}{7}$$

or, $$500R_1C + R_2C_1 = \frac{50}{7} \qquad ....(7.35)$$

Putting the value of β from equation (7.29) into equation (7.31), we get

$$(R_1 + R_2)C = \frac{10}{7}$$

or, $$C = \frac{10}{7(R_1 + R_2)}. \qquad ....(7.36)$$

Putting the value of β from equation (7.29) into equation (7.33), we get

$$C_1 = \frac{R_1 + R_2}{250R_1R_2}. \qquad ....(7.37)$$

Putting the values of C and C_1 into equation (7.35), we get

$$\frac{5000R_1}{7(R_1 + R_2)} + \frac{R_1 + R_2}{250R_1} = \frac{50}{7}$$

or, $$7\left(\frac{R_1 + R_2}{R_1}\right)^2 - 12500\left(\frac{R_1 + R_2}{R_1}\right) + 1250000 = 0$$

$$\therefore \quad \frac{R_1 + R_2}{R_1} = 106.33 \text{(chosen)}$$

or, $R_2 = 105.33\ R_1$ ....(7.38)

From equations (7.36) and (7.38) we get

$$106.33R_1 = \frac{10}{7C}$$

Taking $C = 1\ \mu$F, we have

$$R_1 = \frac{10^7}{106.33 \times 7} \Omega = 13.435 \text{ K}\Omega$$

$$R_2 = 105.33 R_1 = 1.415 \text{ M}\Omega$$

$$R = 500 R_1 = 6.717 \text{ M}\Omega$$

and
$$C_1 = \frac{R_1 + R_2}{250 R_1 R_2} = 0.3 \mu\text{F}.$$

7.9 TRANSIENT RESPONSE ANALYSIS USING MATLAB

MATLAB (the abbreviated form of MATrix LABoratory) is a matrix-based system for mathematical and engineering calculations. It has many commands and predefined matrix functions that can be called by the user through MATLAB programs to solve different types of problems on control systems. In this section, transient responses will be illustrated by solving a few problems through MATLAB.

Example 7.16 Using MATLAB obtain the unit impulse response of the system the transfer function of which is as follows.

$$G(s) = \frac{C(s)}{R(s)}$$
$$= \frac{1}{s^2 + 0.2s + 1}.$$

Assume zero initial conditions and use "step (num, den)" command in place of "impulse (num, den)".

Solution

For unit-impulse input, $R(s) = 1$. Hence

$$\frac{C(s)}{R(s)} = C(s) = G(s)$$
$$= \frac{s}{s^2 + 0.2s + 1} . \frac{1}{s}$$

Thus with zero initial conditions the unit-step response of $sG(s)$ is the same as the unit-impulse response of $G(s)$. Here we shall use arrays for numerator and denominator polynomials with coefficients in descending powers of s as follows.

```
num = [0   1    0]
den = [1   0.2  1]
```

The MATLAB program for unit-step response of $sG(s)$ is given below.

MATLAB PROGRAM 7.1

```
% - - - - Unit-step response of sG(s) – – – – –
% *** Enter the numerator and denominator of the transfer function***
num  =  [0   1   0];
den  =  [1   0.2   1];
% *** Enter the following step-response command***
step (num, den);
% *** Enter grid and title of the plot***
grid;
title (‘Unit-step Response of sG(s) = s/(s^2 + 0.2s + 1)’)
```

Program lines beginning with ‘%’are comments or remarks which are not executed. A semicolon at the last character of a statement will display no result, though the command is still executed. The last two statements put the grid lines and title to the graph. The ‘step (num, den)’ command will generate the unit-step response (Fig. 7.36) with automatic labels of ‘Time (sec)’ and ‘Amplitude’ on x-axis and y-axis respectively.

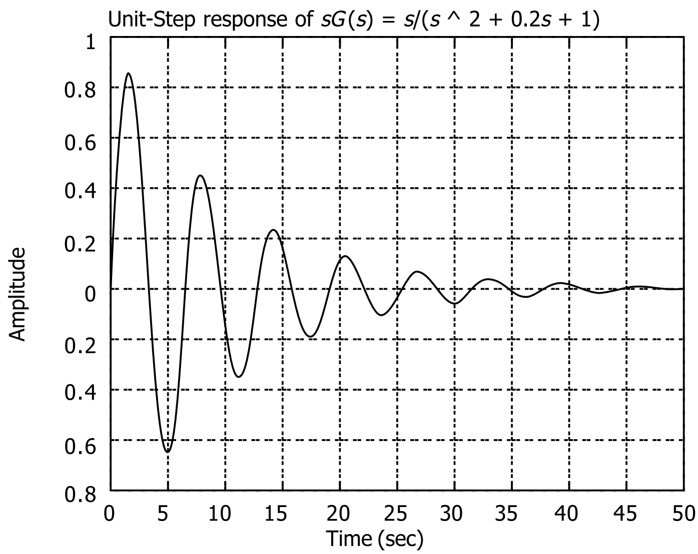


Fig. 7.36 *Unit-impulse response curve*

Example 7.17 Using MATLAB, obtain the unit-ramp response of the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + s + 1}.$$

Assume zero initial conditions.

Solution

For a unit-ramp input, $$R(s) = \frac{1}{s^2}$$

Hence, $$C(s) = \frac{1}{(s^2 + s + 1)s} \cdot \frac{1}{s}.$$

As there is no ramp command in MATLAB, we have to use step command on the transfer function $G(s)/s$ to obtain unit-ramp response through the following program.

MATLAB PROGRAM 7.2

```
% - - - - Unit-ramp response of – – – – –
% *** Enter the numerator and denominator of G(s)/s ***
num = [0   0   0   1];
den = [1   1   1   0];
% *** Assume the computing time points to be t = 0:0.1:7 and then
% enter step-response command as stated below***
t = 0:0.1:7;
c = step (num, den, t);
% *** The left-hand argument in the step command will not show plot on
% the screen. The following plot command will show input curve of solid(-)
% line type and output curve of circular (0) point type ***
plot (t, c, '0', t, t, '-')
% *** Add grid, title of response, x-axis label and y-axis label***
grid;
title ('Unit-Ramp Response of G(s) = 1/s^2 + s + 1)')
x label ('t sec')
y label ('Input and Output')
```

Note that the step command with left-hand argument is used here to place user-defined labels on the x-axis and y-axis. The above program, when executed, gives us the following response curve Unit-Ramp Response for $G(s) = 1/(s^2 + s + 1)$

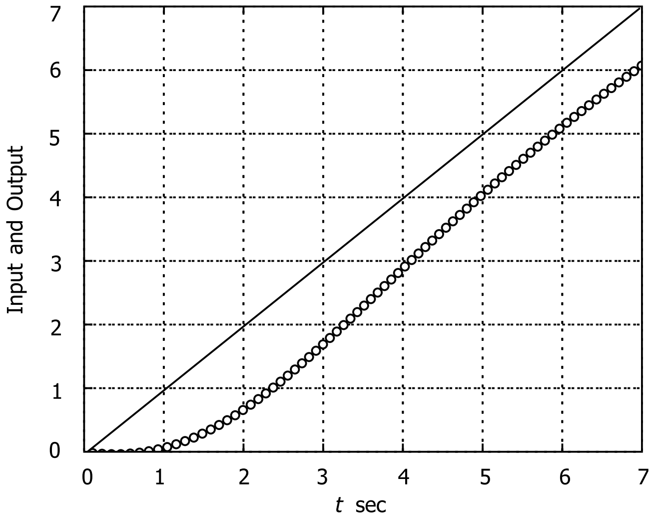


Fig. 7.37 Unit-ramp response

Example 7.18 A second order system is given by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}.$$

Using MATLAB, obtain the unit-impulse response curves for each zeta given as $\zeta = 0.1$, 0.3, 0.5, 0.7 and 1.0. Assume zero initial conditions.

Solution

For a unit-impulse input $R(s) = 1$, the MATLAB program for drawing all the response curves in one diagram is

MATLAB PROGRAM 7.3

```
% - - - - Unit-impulse response – – – – –
% *** This is obtained as step response of sG(s) ***
% *** Enter numerator and denominator of sG(s) for zeta = 0.1 ***
num = [0   1   0];
den1 = [1   0.2   1];
% *** Let computing time point be t = 0:0.1:10. The following step and
% text commands given below***
t = 0:0.1:10;
step (num, den1, t);
text (2.2, 0.89, 'zeta = 0.1'
% ***Hold this plot to add other response curves***
hold
current plot held
% ***Enter denominators of sG(s) for other zeta values ***
```

MATLAB PROGRAM 7.3 *(Contd)*

```
den2 = [1   0.6   1];
den3 = [1   1   1];
den4 = [1   1.4   1];
den5 = [1   2   1];
% ***Superimpose other response curves to the first curve by giving
% step and text command as before***
step (num, den2, t);
text (1.32, 0.73, '0.3')
step (num, den3, t);
text (1.14, 0.59, '0.5')
step (num, den4, t);
text (1.1, 0.47, '0.7')
step (num, den5, t);
text (0.8, 0.29, '1.0')
% ***Add grid and title of the plot grid***
title ('Impulse-Response Curves for G(s) = 1[s^2 + 2(zeta)s + 1]')
% ***Clear hold command on graphics***
hold
current plot released
```

After execution of the above program, the following unit-impulse curves for different values of zeta are obtained in a single diagram (Fig. 7.38).

Impulse-response curves for $G(s) = 1/[s\wedge 2 + 2(\text{zeta})s + 1]$

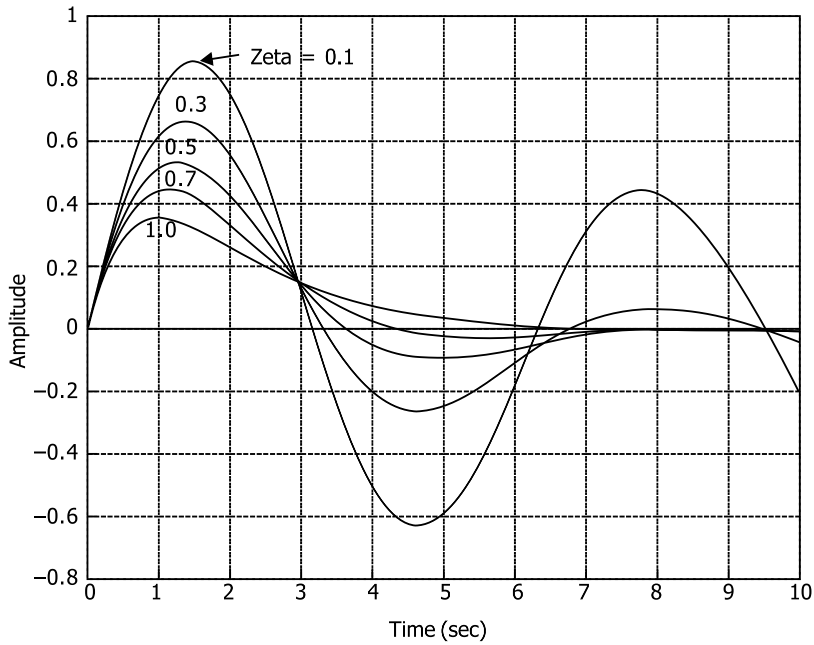


Fig. 7.38 *Response curves to unit-impulse input*

REVIEW QUESTIONS

7.1 The block diagram of a system is shown below. Find the value of k such that the damping ratio is 0.5. For the same-value of k obtain rise time (t_r), peak time (t_p), maximum overshoot (M_p) and settling time (t_s) in the unit-step response.

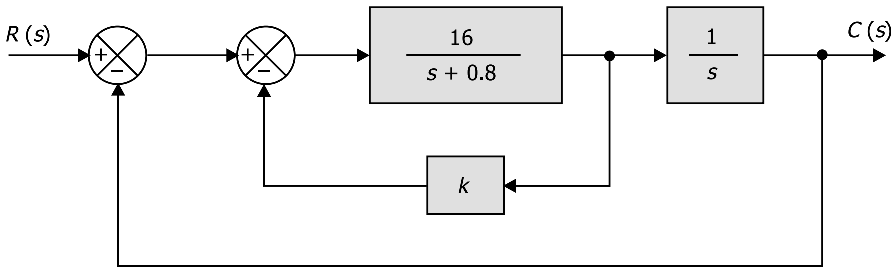


Fig. 7.39

7.2 The mass of a mechanical system oscillates as per Fig. 7.40(a) and (b) when it is subjected to a force of 10 N. Determine M, B and K of the system.

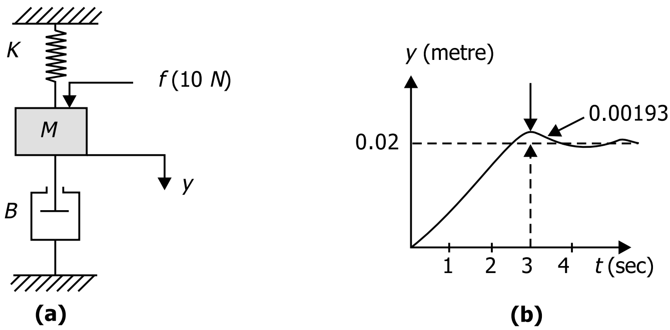


Fig. 7.40

7.3 The transfer function of a second order control system is given by

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

A step input of 10 Nm is applied to the system and the test results are found to be $M_p = 6$ per cent, $t_p = 1$ sec. and steady-state output of 0.5 radian. Determine the value of J, B and K.

7.4 For the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

i) Find the values of ζ and ω_n if the system responds to a step input with 5 per cent overshoot and with a settling time of 2 secs. (use 2 per cent criterion).

ii) Find the value of ζ is the system exhibits a damped oscillation as recorded in Fig. 7.41.

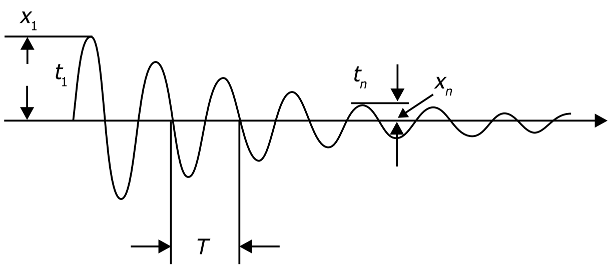
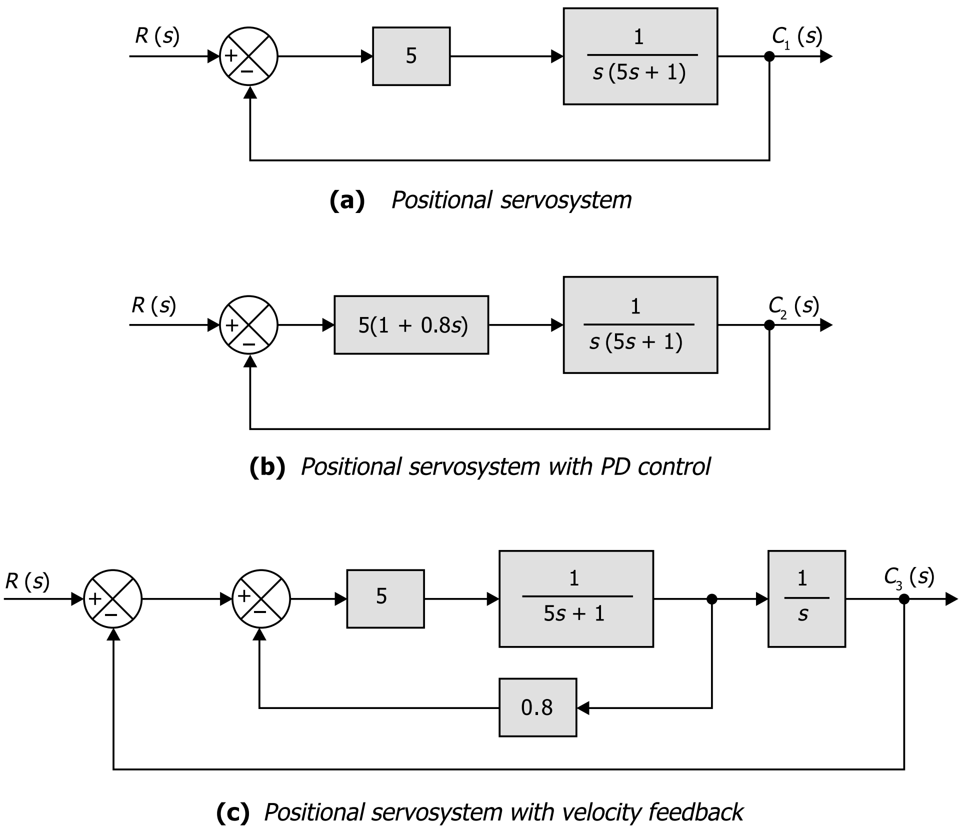


Fig. 7.41

7.5 Compare the unit-step, unit-impulse and unit-ramp response of the three systems shown below. Identify the best system with respect to the speed of response and maximum overshoot in the step response.



(a) *Positional servosystem*

(b) *Positional servosystem with PD control*

(c) *Positional servosystem with velocity feedback*

Fig. 7.42

7.6 The open-loop transfer function of a unity feedback control system is given by $G(s) = \dfrac{K(s+2)}{s^3 + \beta s^2 + 4s + 1}$.

Determine the value of K and β if

i) The closed-loop unit step response has $\omega_n = 3$ rad/sec. and $\zeta = 0.2$.

ii) The system exhibits sustained oscillations with $\omega_n = 4$ rad/sec.

7.7 The angular position of a flywheel is controlled by an error-actuated closed-loop automatic control system to follow the motion of input lever. The lever maintains sinusoidal oscillations through $\pm$ 60° with an angular frequency of 2 rad/sec. The inclusive moment of inertia of flywheel is 150 kg m^2 and the stiffness of the control is 2400 Nm per radian of misalignment. Find the viscous frictional torque required to produce critical damping and hence calculate the amplitude of the swing of the flywheel and the time lag between the flywheel and the control lever.

7.8 The maximum overshoot for a unity feedback control system with forward path transfer function of $K/s\,(sT + 1)$ is reduced from 60 per cent to 20 per cent. Determine the factor by which the gain should be reduced to achieve this. Assume that K and T are positive constants.

7.9 The system parameters of a servomechanism as per the figure below are as follows:

Sensitivity of synchro, $K_s = 1$ volt/rad

Amplifier gain, $K_a = 30$ volt/volt

Torque constant of motor, $K_T = 1 \times 10^{-5}$ Nm/volt

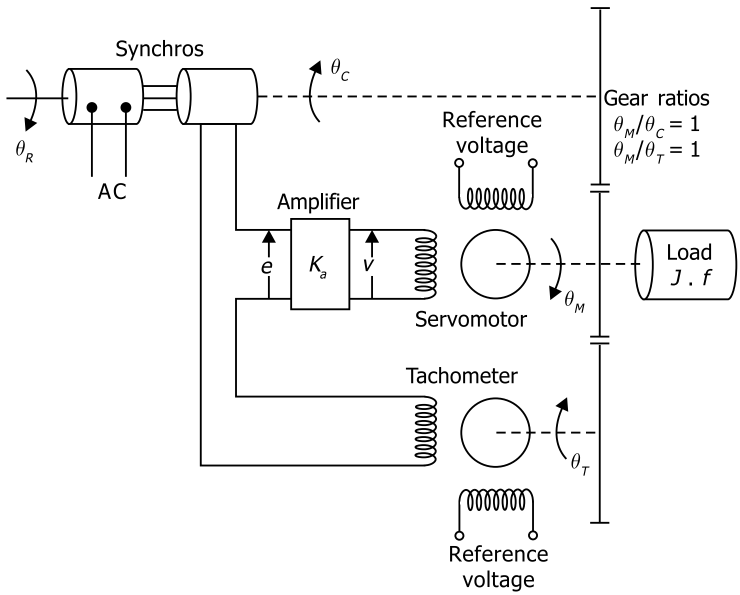


Fig. 7.43

Load inertia, $J = 1.5 \times 10^{-5}$ Kgm^2
Viscous friction, $f = 1 \times 10^{-5}$ Nm/rad/sec
Tachometer constants, $K = 0.2$ volt/rad/sec
Neglecting motor inertia and friction,

i) Obtain the value of damping ratio, when tachometer is disconnected. Also find the steady-state error for an input velocity of 1 rad/sec.
ii) Obtain the value of damping ratio when tachometer is a part of the system.
iii) Compare the steady-state behaviour of the system with that of part (i) if the tachometer is removed and the amplifier output is given by $v = K_a\left(e \int e\,dt\right)$

7.10 The system shown below employs proportional plus error-rate control. Find the value of error-rate constant K_e so that the damping ratio is 0.6. Also determine the values of settling time, maximum overshoot and steady-state error (for unit-ramp input) with and without error-rate control.

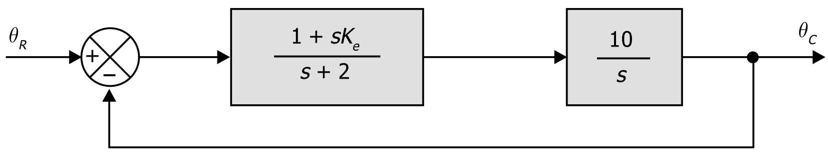


Fig. 7.44

7.11 Evaluate proportional gain, integral time and derivative time for the following PID controller.

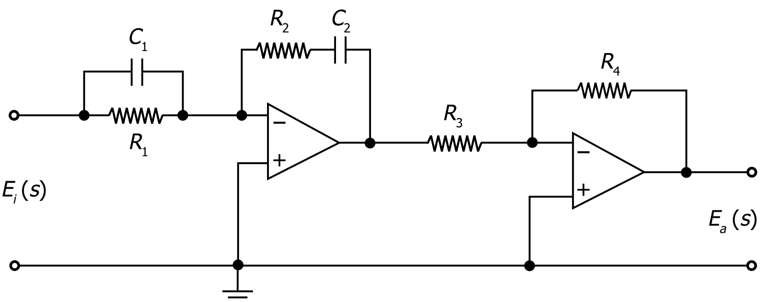


Fig. 7.45

7.12 Determine the sensitivity of the overall gain for the system shown below and find the value of K if the sensitivity is 0.2 under steady-state condition.

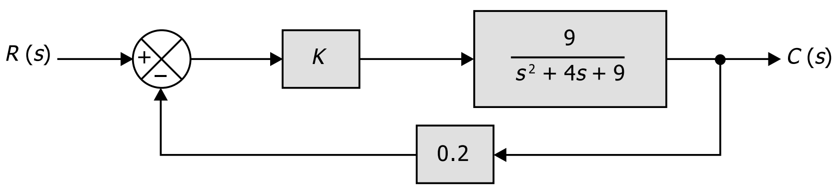


Fig. 7.46

7.13 A feedback system employing output rate damping is shown in Fig. 7.47. Find the value of K_1 and K_2 so that the closed-loop system resembles a second order system with damping ratio equal to 0.5 and frequency of damped oscillation 9.5 rad/sec.

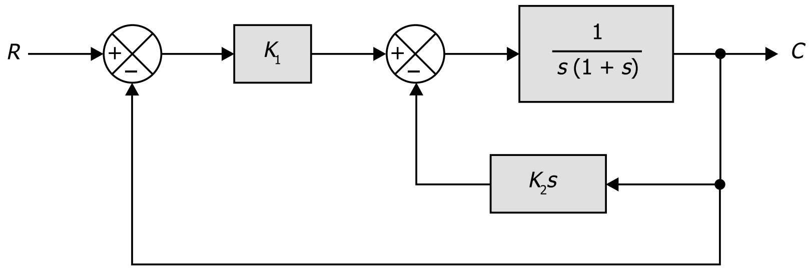


Fig. 7.47

7.14 Integral control of the plant $G_p(s) = \dfrac{K}{\tau s + 1}$ results in a system that is too oscillatory. Will derivative action improve this situation?

7.15 Using MATLAB, determine the unit-step, unit-ramp and unit-impulse response of the system the closed-loop transfer function of which is given by

$$\frac{C(s)}{R(s)} = \frac{8}{s^2 + 2s + 10}.$$

7.16 Determine the position, velocity and acceleration error constants for the unity feedback control systems whose open-loop transfer functions are given as follows.

i) $\dfrac{10}{(1+0.4s)(1+0.5s)}$

ii) $\dfrac{K}{(s^1+0.1s)(1+0.5s)}$

iii) $\dfrac{11(s+30)}{s^3(1+s)(1+0.2s)(s^2+5s+15)}$

iv) $\dfrac{K(1+2s)(1+4s)}{s^2(s^2+2s+10)}$

7.17 Determine $e_{ss}(t)$ for a system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{s(s+4)}{s(s+1)(s+5)}$$

The input to the system is defined by

$r(t) = 0$ for $t < 0$;

$r(t) = 2(1 + t)$ for $t \geq 0$

7.18 A unity feedback control system has an open-loop transfer function given by

$$G(s) = \frac{400}{s(1+0.1s)}$$

Determine the steady-state error of the system for the following inputs.

i) $r(t) = t^2u(t)/2$;

ii) $r(t) = (1 + t)^2u(t)$

7.19 Fig. 7.48 shows the block diagram of a system.
Determine the steady-state error for unit ramp input taking the value of K equal to 300. Also determine the value of K for which the state steady-state error for unit ramp input will be 0.025.

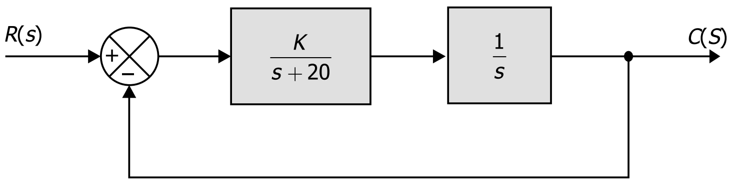


Fig. 7.48

7.20 For the system shown in Fig. 7.49 calculate the steady-state error for (i) unit step input; (ii) unit ramp input and (iii) unit acceleration input.

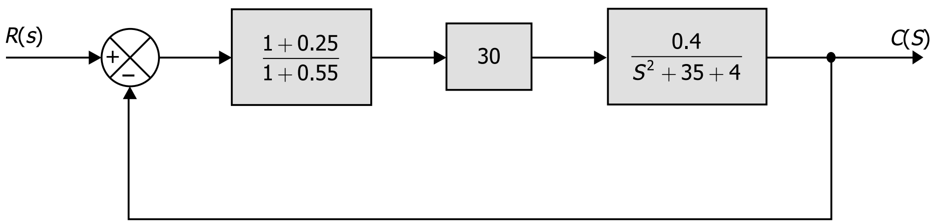


Fig. 7.49

7.21 (a) Calculate the error constants and steady-state errors for three basic types of inputs for the following system.

$$G(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)} \text{ and } H(s) = 1$$

(b) The open-loop transfer function servosystem with unity feedback is represented as

$$G(s) = \frac{10}{s(0.1s+1)}$$

Calculate the error constants K_p, K_v, K_a, for the system.

8

CONCEPT OF STABILITY AND ROUTH-HURWITZ CRITERION

8.1 CONCEPT OF STABILITY

System stability is one of the most important performance specification of a control system. A system is considered unstable if it does not return to its initial position but continues to oscillate after it is subjected to any change in input or is subjected to undesirable disturbance. For any time invariant control system to be stable the following two conditions need to be satisfied.

These are:

i) The system will produce a bounded output for every bounded input;
ii) If there is no input, the output should tend to be zero, irrespective of any initial conditions.

Stability of a system may be referred to as absolute stability or in terms of relative stability. The term relative stability is used in relation to comparative analysis of stability of systems and their operating conditions.

8.2 POLE-ZERO LOCATION AND CONDITIONS FOR STABILITY

The transfer function of a single-input, single-output system can be written as

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0s^m+b_1s^{m-1}+\ldots+b_m}{a_0s^n+a_1s^{n-1}+\ldots+a_n};\quad m<n$$

The denominator of the transfer function is called characteristic polynomial and its roots are the poles of the transfer function. The characteristic equation is formed by equating the characteristic polynomial to zero. Thus, for the above case, the characteristic equation is

$$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\ldots+a_n=0;\ \ a_0>0$$

The stability of the closed-loop system can be determined by examining the poles of the closed-loop system, that is, by the roots of the characteristic equation. As we know, the nature of time response of a system is related to the location of the roots of characteristic equation in s-plane. For the system to be stable, the roots should have negative real parts. A system will be stable, unstable, or oscillatory depending upon the positions of the roots of the characteristic equation as shown below. In Fig. 8.1, the position of a pole is indicated by a cross.

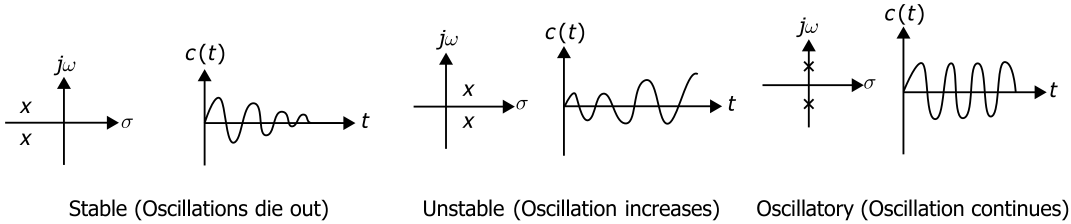


Fig. 8.1 *Positions of poles and the response of a control system*

The presence of negativeness of any of the coefficients of the characteristic equation implies that the system is either unstable or at the most has limited stability.
Therefore, *the necessary conditions for stability are as follows*.

a) The positiveness of the coefficients of characteristic equation is necessary. This is the condition for stability of the systems of the first and second order. The characteristic equation for a first order system is

$$a_0 s + a_1 = 0$$

which has a single root $s_1 = - a_1/a_0$. The system is stable provided a_0 and a_1 are both positive. The characteristic equation for a second order system is $a_0 s^2 + a_1 s + a_2 = 0$ which has two roots $s_1, s_2 = [-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}\,]/2a_0$. Again if a_0, a_1 and a_2 are positive the system will be stable.

b) For third and higher order systems, the positiveness of all the coefficients of characteristic equation does not ensure the negativeness of the real parts of complex roots. Therefore, it is a necessary but not sufficient condition for the systems of third and higher order.

If all the coefficients are positive, the possibility of stability of the system exists. To examine the sufficient condition for stability, there is a criterion known as the Routh-Hurwitz's stability criterion, explained in this chapter.

8.3 ROUTH'S STABILITY CRITERION AND ITS APPLICATION

The Routh-Hurwitz criterion for stability does not require calculation of the actual values of the roots of the characteristic equation. This criterion tells us about the number of roots on the right side of the imaginary axis. Moreover, this criterion gives just a qualitative result. It is the quickest method if we just want to know whether the system is stable or unstable.

The general characteristic equation is given by,

$$a_0 s^n + a_1 s^{n-1} + \dots\dots\dots + a_{n-1} s + a_n = 0.$$

Let a_0 be positive. If it is negative, multiply both sides of the equation above by -1.

The following are the steps for applying Routh's stability criterion:

Step 1: If any of the coefficients, $a_1, a_2, \dots\dots a_{n-1}, a_n$ is negative or zero, there is at least one root of the characteristic equation which has positive real part and the corresponding system is unstable. No further analysis is needed.

Step 2: If all the coefficients are positive, then from the first step, we cannot conclude anything about the location of the roots. Then we have to form the following array

$$\begin{array}{llllll}
s^{n} & a_0 & a_2 & a_4 & A_6 & \dots\dots\dots \\
s^{n-1} & a_1 & a_3 & a_5 & A_7 & \dots\dots\dots \\
s^{n-2} & b_1 & b_2 & b_3 & B_4 & \dots\dots\dots \\
s^{n-3} & c_1 & c_2 & c_3 & C_4 & \dots\dots\dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s^2 & f_1 & & & & \\
s^1 & g_1 & & & & \\
s^0 & a_n & & & &
\end{array}$$

where, $$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \dots\dots\dots\dots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \dots\dots\dots\dots$$

Now, examine the elements of the first column of the array.

$$a_0, a_1, b_1, c_1, \dots\dots\dots\dots, f_1, g_1, a_n$$

The following conclusions are derived from the Routh-Hurwitz criterion:

a) If any of these elements (a_0, a_1, b_1, c_1, ...) is negative, we have at least one root on the right of the imaginary axis and the system is unstable.

b) The number of sign changes in the elements of the first column is equal to the number of roots located at the right of the imaginary axis.

Therefore, a system is stable if all the elements in the first column of the array are positive, i.e. there is no change in the sign.

Depending on the coefficients of the equation, the following difficulties may arise.

i) The first element in any of the rows of the array is zero, but the others are not.

ii) The elements in one row of the array are all zero.

In the first case, replace the zero element in the first column by an arbitrary small positive number ε, and then proceed with array formation and ultimately let ε tend to zero.

The second case of problem indicates that there are symmetrically located roots in the s-plane. *The polynomial whose coefficients are just above the row of zeros in the array is called an auxiliary polynomial*. The auxiliary polynomial is always an even polynomial; that is, only even powers of s appear. The roots of the auxiliary equation also satisfy the original equation. To continue with the array, the following steps should be adopted.

a) Form the auxiliary equation, $A(s) = 0$;

b) Take derivative of the auxiliary equation with respect to s and equate to zero.

i.e.,
$$\frac{dA(s)}{ds} = 0;$$

c) Replace the row of zeros with the coefficients of
$$\frac{dA(s)}{ds} = 0;$$

d) Continue the array in the usual manner with replaced coefficients.

Example 8.1 Check the system represented by the following characteristic equation for absolute stability

$$s^3 + 5s^2 + 25s + 10 = 0$$

Solution

The Routh's array is

$$\begin{array}{llll}
s^3 & 1 & 25 & b_1 = \dfrac{5\times 25 - 10}{5} = 23 \\
s^2 & 5 & 10 & b_2 = 0 \\
s^1 & 23 & 0 & c_1 = \dfrac{23\times 10 - 5\times 0}{23} = 10 \\
s^0 & 10 & 0 & c_2 = 0
\end{array}$$

All the elements in the first column are positive. There is no root in the right half of the s-plane. The system is therefore absolutely stable.

Example 8.2 Check the system represented by the following equation for absolute stability.

$$s^5 + s^4 + 3s^3 + 4s^2 + 16s + 10 = 0.$$

Solution

The array is

s^5	1	3	16
s^4	1	4	10
s^3	-1	6	0
s^2	10	10	0
s^1	7	0	0
s^0	10		

In the first column there are two sign changes. From $+1$ to -1 and then to 10. Thus, there are two roots in the right half of the s-plane. The system is unstable.

Example 8.3 Consider the characteristic equation of a system

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

Find the status of its stability.

Solution

Array

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	
s^1	0	0	

The auxiliary equation is formed by using the coefficients just above the row of zeros, i.e.

$$A(s) = 4s^2 + 4 = 0.$$

The derivative of $A(s)$ with respect to s is

$$\frac{dA(s)}{ds} = 8s. \text{Therefore, we consider } 8\,s = 0$$

Replace zeros of the s^1 row with the coefficients of the above equation.

s^1	8	0
s^0	4	

Since there is no sign change, no root is lying in the positive half of s-plane. Therefore, the system is stable.

Example 8.4 The characteristic equation of a feedback control system is found as

$$s^4 + 9s^3 + 11s^2 + 6s + K = 0.$$

Determine the value of K for which the system is absolutely stable and marginally stable. Also determine the frequency of sustained oscillation.

Solution

The characteristic equation is $s^4 + 9s^3 + 11s^2 + 6s + K = 0$.

The array is

s^4	1	11	K
s^3	9	6	0
s^2	10.3	K	0
s^1	$\dfrac{61.8-9K}{10.3}$	0	0
s^0	K	0	0

For stability $K > 0$

$61.8 - 9K > 0$ or $K < 6.8$

Hence for absolute stability range q gain K is

$0 < K < 6.8$

The system will be marginally stable when $K = 6.8$.

The auxiliary equation being an even polynomial, is expressed with $K = 6.8$ as

$$10.3\, s^2 + 6.8 = 0$$

or, $$s^2 = -\frac{6.8}{10.3} = 0.66$$

Therefore, $s = \pm j\, 0.82 = \pm j\, \omega$

Frequency of sustained oscillation = 0.82 rad/second.

Example 8.5 Using the Routh-Hurwitz stability criterion, ascertain stability for each of the following three cases.

a) $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$.

Solution

The array is prepared using the characteristic equation as shown.

The first column of the array shows no sign change. Therefore, no roots are on the right-hand side of the s-plane. However, the occurrence of a zero row indicates the presence of symmetrically located roots in the s-plane. The roots of the auxiliary equation are given by

$$(s^4 + 6s^2 + 8) = 0$$

or, $(s^2 + 4)(s^2 + 2) = 0$

$s = \pm 2j$ and $s = \pm j\sqrt{2}$.

					Remarks
s^6	1	8	20	16	Auxiliary equation is
s^5	2	12	16		$s^4 - 6s^2 + 8 = 0$
s^5	1	6	8		Its derivative is
s^4	2	12	16		$4s^3 + 12s = 0$
s^4	1	6	8		or,
s^3	0	0			$s^3 + 3s = 0$
s^3	1	3			The s^3 row is now completed using the
s^2	3	8			coefficients of the above equation and
s^1	1/3	0			the array is completed
s^0	8	0			

As the coefficients of s^3 row are all zeros, the auxiliary $A(s)$ and its first derivative are formed as shown under 'Remarks'. The s^3 row was then replaced by the coefficients of $dA(s)/ds = 0$. As there are non-repeated imaginary roots on the imaginary axis, the system is limitedly stable. Here the roots of the auxiliary equation are also the roots of the characteristic equation as stated in the problem.

b) $s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$.

Solution

					Remarks
s^6	1	−2	−7	−4	Auxiliary
s^5	1	−3	−4		equation is
s^4	1	−3	−4		$s^4 - 3s^2 - 4 = 0$
s^3	0	0	0		Its derivative is
s^3	2	−3			$4s^3 - 6s = 0$
s^2	−1.5	−4			or,
s^1	−25/3	0			$2s^3 - 3s = 0$
s^0	−4				

There is one sign change in the first column of the array. Therefore, one root is lying on the right-hand side of the s-plane. The system is unstable. Without going for the formation of array it may be said, in other words, that the system is not stable as the necessary condition of positiveness of all the coefficients does not hold good.

c) $s^6 + s^5 + 4s^4 + 2s^3 + 5s^2 + s + 2 = 0.$

Solution

The array is

s^6	1	4	5	2	**Remarks**
s^5	1	2	1		$A_1(s) = s^4 + 2s^2 + 1 = 0$
s^4	1	2	1		$\frac{dA_1(s)}{ds} = 4s^3 + 4s = 0$
s^3	0	0			or, $s^3 + s = 0$
s^3	1	1			$A_2(s) = s^2 + 1 = 0$
s^2	1	1			
s^1	0				$\frac{dA_2(s)}{ds} = 2s = 0$
	2				
s^0	1				

There are two rows which become zero and there is no sign change in the first column of the array. The roots of $A_1(s) = 0$ are also the roots of the original characteristic equation; so the roots of $A_1(s) = 0$ are given by

$$(s^4 + 2s^2 + 1) = 0$$

or,
$$(s^2 + 1)^2 = 0$$

$$\therefore \quad s = \pm j \quad \text{and} \quad s = \pm j.$$

The multiplicity of the pair of roots on $j\omega$ -axis makes the system unstable.

Example 8.6 The characteristic equation of a servosystem is given by:

$$b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 = 0.$$

Determine the conditions for stability using Routh-Hurwitz stability criterion.

Solution

The array has been shown. From the array we find the conditions for stability.
The conditions for stability are stated as follows.

$b_1 > 0, b_0 > 0, (b_1 b_2 - b_0 b_3) > 0,$

$(b_1 b_2 b_3 - b_0 b_3^2 - b_1^2 b_4) > 0$ and $b_4 > 0$.

$$\begin{array}{c|ccc}
s^4 & b_0 & b_2 & b_4 \\
s^3 & b_1 & b_3 & \\
s^2 & \dfrac{b_1 b_2 - b_0 b_3}{b_1} & \dfrac{b_1 b_4}{b_1} & \\
s^1 & \dfrac{\dfrac{b_3(b_1 b_2 - b_0 b_3) - b_1^2 b^4}{b_1}}{\dfrac{b_1 b_2 - b_0 b_3}{b_1}} & & \\
s^0 & b_4 & &
\end{array}$$

Example 8.7 Determine whether the largest time constant of the characteristic equation given below is less than, greater than or equal to 1.0 sec.

$$s^3 + 5s^2 + 8s + 6 = 0.$$

Solution

$$s^3 + 5s^2 + 8s + 6 = 0.$$

Its roots are calculated after writing the characteristic equation as

$$(s + 3)\,(s^2 + 2s + 2) = 0$$

The roots are

$$s = -3 \text{ and } s = -1 \pm j$$

for $$s = -3$$

$$\tau = 1/3 = 0.33 \text{ sec.}$$

for $$s = -1 \pm 1j$$

$$\tau = ?$$

Refer to Fig. 8.2 where the complex roots have been plotted.

$$\zeta = \cos\phi = \frac{|OB|}{|OA|}$$

$$= \frac{1}{\sqrt{2}}.$$

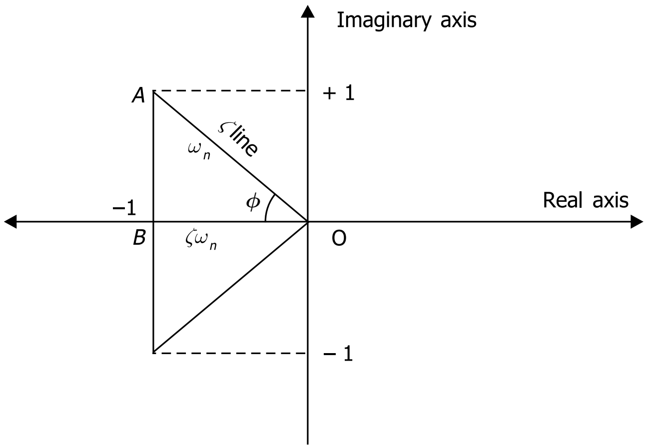


Fig. 8.2

Also,

$$\omega_n = |OA| = \sqrt{2}$$

$$\therefore \qquad \zeta\omega_n = \frac{1}{\sqrt{2}} \times \sqrt{2} = 1.$$

$$\therefore \qquad \tau = \frac{1}{\zeta\omega_n} = 1\,\text{sec}.$$

So the largest time constant is equal to 1 sec.

Example 8.8 A unity feedback control system has an open-loop transfer function $G(s) = \dfrac{K(s+13)}{s(s+3)(s+7)}$. Using the Routh-Hurwitz stability criterion, find the range of K for the system to be stable. If $K = 1$, check all the poles of the closed-loop transfer function having damping factor greater than 0.85. Assume unity feedback system.

Solution

The characteristic equation of the system is

$$1 + G(s) = 1 + \frac{K(s+13)}{s(s+3)(s+7)} = 0$$

or $$s^3 + 10s^2 + (21 + K)\,s + 13K = 0.$$

The array is prepared as follows.

$$\begin{array}{c|cc|}
s^3 & 1 & (21+K) \\
s^2 & 10 & 13K \\
s^1 & \dfrac{210-3K}{10} & - \\
s^0 & 13K & -
\end{array}$$

Condition

a) $13K > 0$ or $K > 0$

b) $\dfrac{210-3K}{10} > 0$ $K < 70$

For stability the range is $0 < K < 70$. When $K = 1$, the characteristic equation is

$$s^3 + 10s^2 + 22s + 13 = 0$$

or $$(s + 1)\,(s^2 + 9s + 13) = 0.$$

The roots are

$$s = -1, \frac{1}{2}(-9 \pm \sqrt{29}).$$

All roots are negative and real and hence lie on the negative real axis.

Now, damping ratio $\zeta = \cos \phi$.

Since, all roots are negative and real, therefore, damping ratio $\zeta = 1$. Hence, for all root's damping ratio is greater than 0.5.

Example 8.9 The characteristic equation for a feedback control system is given by $s^4 + a_1 s^3 + a_2 s^2 + a_3 s + K = 0$. If the numerical values of $a_1 = 22$, $a_2 = 10$ and $a_3 = 2$, find the value K which will keep the system stable.

Solution

The characteristic equation is given as

$$s^4 + 22s^3 + 10s^2 + 2s + K = 0$$

The array is

$$\begin{array}{c|ccc}
s^4 & 1 & 10 & K \\
s^3 & 22 & 2 & \\
s^2 & \dfrac{218}{22} & K & \\
s^1 & 2-\dfrac{484K}{218} & & \\
s^0 & K & &
\end{array}$$

Applying the condition of stability, we have to check for all the coefficients in the first column to be positive. Thus,

$$2-\frac{484K}{218}>0$$

or
$$\frac{484K}{218}<2$$

or
$$K<\frac{2\times 218}{484}$$

or
$$K < 0.9$$

and
$$K > 0.$$

$\therefore$ The range of K for stability is therefore determined as:

$$0 < K < 0.9.$$

Example 8.10 The open-loop transfer function of a unity feedback system is given by

$$\frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying the Routh-Hurwitz criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillation frequencies?

Solution

We have,

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)} \quad (\because H(s)=1)$$

Characteristic equation is

$$1 + G(s) = 0$$

or
$$1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}=0.$$

or,
$$s^4 + 12s^3 + 69s^2 + 198s + (200 + K) = 0.$$

The array is prepared as shown.

The system will be stable, if

$$200 + K > 0 \quad \text{or} \quad K > -200$$

and

$$198 - \frac{12(200+K)}{52.5} > 0 \quad \text{or} \quad K < 666.25.$$

$$\begin{array}{c|ccc}
s^4 & 1 & 69 & (200+K) \\
s^3 & 12 & 198 & \\
s^2 & 52.5 & (200+K) & \\
s^1 & 198 - \dfrac{12(200+K)}{52.5} & & \\
s^0 & (200+K) & &
\end{array}$$

Oscillations will occur when $K = 666.25$ and in this case the auxiliary equation is

$$52.5s^2 + (200 + K) = 0$$

or
$$52.5s^2 + (200 + 666.25) = 0$$

or
$$s = \pm\,(4.062)\,j$$

The frequency of sustained oscillation is therefore 4.062 rad/sec.

Example 8.11 The open-loop transfer function of a closed-loop system with unity feedback is $\dfrac{K(s+2)(s+1)}{(s+0.1)(s-1)}$. Using Routh-Hurwitz criterion show whether the system is stable or unstable.

Solution

The characteristic equation is

$$1 + G(s) = \frac{K(s+2)(s+1)}{(s+0.1)(s-1)} + 1 = 0$$

or,
$$K(s + 2)\,(s + 1) + (s + 0.1)\,(s - 1) = 0$$

or
$$(K + 1)s^2 + (3K - 0.9)s + (2K - 0.1) = 0$$

The array is

$$\begin{array}{ccc}
s^2 & (K+1) & (2K-0.1) \\
s^1 & (3K-0.9) & - \\
s^0 & (2K-0.1) & -
\end{array}$$

Comments:
For stability, the coefficients of the first column should be positive

That is, a) $K > -1$

b) $K > 0.9/3$ that is, $K > 0.3$

c) $K > 0.1/2$ that is, $K > 0.05$.

So the system will be stable if $K > 0.3$.

Example 8.12 Check the following six systems for their absolute stability.

a) $s^3 - 3s^2 + s + 6 = 0$

Solution

$$s^3 - 3s^2 + s + 6 = 0$$

Array is shown below.

s^3	$+1$	1	
s^2	-3	6	In the first column sign changes from positive value
s^1	$+3$	0	to negative value and then again to positive value
s^0	$+6$		

Let us inspect the first column. There are two changes in sign (positive value to negative and then to positive value). It implies that there are two roots on the right half of s-plane. So the system is unstable.

b) $2s^4 + s^3 + 3s^2 + 5s + 10 = 0$

Solution
Array is shown below.

$$\begin{array}{llll}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & \dfrac{3-10}{1} = -7 & 10 & 0 \\
s^1 & \dfrac{-35-10}{-7} = \dfrac{45}{7} & 0 & 0 \\
s^0 & 10 & 0 & 0
\end{array}$$

In the first column, there are two sign changes which implies that there are two roots on the right half of s-plane. So the system is not stable.

c) $(s-1)^2(s+2)=0$

Solution

$$s^3+3s+2=0$$

or,
$$s^3+0s^2-3s+2=0$$

Array is shown below.

$$\begin{array}{lcc} s^3 & 1 & -3 \\ s^2 & \varepsilon & 2 \\ s^1 & \dfrac{-3\varepsilon-2}{\varepsilon} & 0 \\ s^0 & 2 & 0 \end{array}$$

Since there is one entry in the first column as zero, we have replaced that zero by a very small positive number ε. There are two sign changes when ε tends to zero. Thus the system is unstable since two roots are there on the right half of s-plane.

d) $s^4+s^3-3s^2-s+2=0$

Solution

Array is shown below.

$$\begin{array}{lccc} s^4 & 1 & -3 & 2 \\ s^3 & 1 & -1 & 0 \\ s^2 & -2 & 2 & 0 \\ s^1 & 0 & 0 & 0 \\ s^1 & -4 & 0 & 0 \\ s^0 & 2 & & \end{array}$$

It is observed that in the above problem, all the entries in s^1 row are zero. So, for finding the coefficient of the next row, we write the auxiliary equation and differentiate it. The equation obtained after differentiation gives the elements of the row which were zero.

In the above problem, the auxiliary equation is

$$-2s^2+2=0$$

Differentiating this we have $-4s=0$ which will give the coefficients of the s^1 row. Then, we are to complete the array by the usual method.

As there are two sign changes in the first column of the array, the system is not stable because of two roots being on the right half of the s-plane.

e) $s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0.$

Solution

The array is presented as follows

s^6	1	−2	−7	−4
s^5	1	−3	−4	0
s^4	1	−3	−4	0
s^3	0	0	0	0
s^3	4	−6	0	0
s^2	$\frac{-3}{2}$	−4	0	0
s^1	$\frac{-50}{3}$	0	0	0
s^0	−4	0	0	0

We see that all the coefficients of the s^3 row are zero. They have been replaced by the coefficient of $dA(s)/ds = 0$ as follows.

Auxiliary equation $A(s)$:

$$A(s) = s^4 - 3s^2 - 4 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 - 6s = 0$$

Since there is one sign change in the first column of the array, we conclude that the system is unstable because of existence of one root on the right half of s-plane.

f) $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0.$

Solution

Array is shown

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	−6	0	0
s^0	5	0	0

There are two sign changes in the first column of the array. Thus, there are two poles in the right half of the s-plane. The system is, therefore, unstable.

g) $s^3 + 2s^2 + s + 2 = 0$

Solution
Array

			Remarks
s^3	1	1	Auxiliary equation
s^2	2	2	
s^1	0	0	$2s^2 + 2 = 0$
s^1	4		Its derivative
s^0	2	0	$\frac{d}{ds}(2s^2+2) = 4s$

There is no change in the sign of the entries in the first column. So, there is no root in the right half of the s-plane. Hence, the system is stable.

Example 8.13 Find the range of values of k for which the system, represented by the following expression, is stable.

$$s^3 + 30s^2 + 600s + 600k = 0.$$

Solution

s^3	1	600
s^2	30	$600k$
s^1	$20(30 - k)$	0
s^0	$600\,k$	

For the system to be stable, all the values in the first column should be positive.

$$20(30 - k) > 0$$

or, $$30 - k > 0$$

or, $$k < 30$$

and $$600\,k > 0,\ k > 0. \text{ Thus } 0 < k < 30.$$

The range of values of k for which the system will be stable is $0 < k < 30$.

Example 8.14 Determine the range of k for stability of the following system.

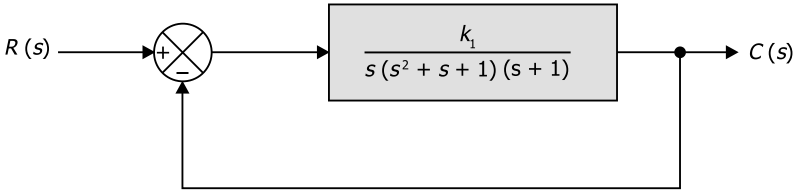


Fig. 8.3

Solution

Closed-loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{\dfrac{k_1}{s(s^2+s+1)(s+1)}}{1+\dfrac{k_1}{s(s^2+s+1)(s+1)}}$$

$$= \frac{k_1}{s^4+2s^3+2s^2+s+k_1}.$$

Characteristic equation is

$$s^4 + 2s^3 + 2s^2 + s + k_1 = 0.$$

s^4	1	2	K_1
s^3	2	1	0
s^2	3/2	K_1	0
s^1	$\frac{3-4K_1}{3}$	0	0
s^0	K_1	0	0

For the system to be stable

$$\frac{3-4K_1}{3} > 0$$

or, $$K_1 < 3/4$$

and $$K_1 > 0$$

For the system to be stable the range of k_1 is $0 < k_1 < 3/4$.

Example 8.15 Predict whether the following two given systems are stable or not.

i) $G(s) = \dfrac{12(s+1)}{s(s-1)(s+5)}, \quad H(s) = 1$

ii) $G(s) = \dfrac{12}{s(s-1)(2s+3)}, \quad H(s) = 1$

Solution

i) Closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\dfrac{12(s+1)}{s(s-1)(s+5)}}{1+\dfrac{12(s+1)}{s(s-1)(s+5)}}$$

$$= \frac{12(s+1)}{s(s-1)(s+5)+12(s+1)}$$

$$= \frac{12(s+1)}{s^3+4s^2+5s+12}.$$

Characteristic equation $s^3 + 4s^2 + 5s + 12 = 0$.

Array

s^3	1	5
s^2	4	12
s^1	2	0
s^0	12	0

The system is stable because all the coefficients in the first column of the array are positive.

ii) Closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\dfrac{12}{s(s-1)(2s+3)}}{1+\dfrac{12}{s(s-1)(2s+3)}}$$

$$= \frac{12}{s(s-1)(2s+3)+10}$$

$$= \frac{12}{2s^3+s^2-3s+12}.$$

Characteristic equation $2s^3 + s^2 - 3s + 12 = 0$.
Array

s^3	2	−3
s^2	1	12
s^1	−27	0
s^0	12	

The system is unstable because there are two changes in sign (positive values to negative value and then to positive value), there are two roots in the right half of the s-plane.

Example 8.16 Check the following system for absolute stability

$$s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10 = 0$$

Solution
Array

s^5	1	3	16
s^4	1	9	10
s^3	−6	6	0
s^2	10	10	0
s^1	12	0	0
s^0	10		

There are two sign changes in the first column. Thus, there are two roots in the right half of the s-plane. The system is unstable.

Example 8.17 Check the system represented by the following characteristic equation for absolute stability $s^3 + 5s^2 + 25s + 10 = 0$.

Solution
Array

s^3	1	25
s^2	5	10
s^1	23	0
s^0	10	0

All the elements in the first column are positive. There is no root in the right half of the s-plane. The system is, therefore, absolutely stable.

Example 8.18 The characteristic equation of a third order system is given by

$$s^3 + 7s^2 + 25s + 39 = 0.$$

Check whether the roots of the characteristic equation are more negative than -1.

Solution

Let us shift the origin to $s = -1$ by substituting $s = z - 1$ (as shown in Fig. 8.4) in the given characteristic equation.

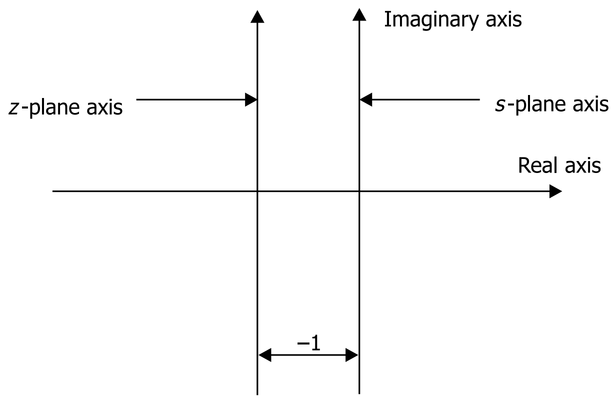


Fig. 8.4 Relation between s-axis and z-axis

The characteristic equation in terms of z-axis now becomes

$$(z - 1)^3 + 7(z - 1)^2 + 25(z - 1) + 39 = 0$$

or,
$$z^3 + 4z^2 + 14z + 20 = 0.$$

Forming the array, we get

z^3	1	14
z^2	4	20
z^1	9	
z^0	20	

Thus all the roots of the characteristic equation in terms of z lie to the left half of the z-plane, because the signs of all the elements in the first column of the array are positive. This implies that all the roots of the given characteristic equation lie to the left of $s = -1$ in the s-plane.

Example 8.19 Determine the stability of a system whose characteristic equation is

$$s^6 + s^5 + 5s^4 + 3s^3 + 2s^2 - 4s - 8 = 0$$

Solution

Consider the equation $s^6 + s^5 + 5s^4 + 3s^3 + 2\,s^2 - 4s - 8 = 0$

Constructing the Routh array

$$\begin{array}{lcccc}
s^6 & 1 & 5 & 2 & -8 \\
s^5 & 1 & 3 & -4 & \\
s^4 & 2 & 6 & -8 & \\
s^3 & 0 & 0 & & \\
s^2 & & & & \\
s^1 & & & & \\
s^0 & & & &
\end{array}$$

Since a row becomes zero, we formulate auxiliary equation as,

$$2s^4 + 6s^2 - 8 = 0$$

Differentiating the auxiliary equation

$$8s^3 + 12s = 0$$

Using the coefficient and reconstructing array

$$\begin{array}{lccc}
s^4 & 2 & 6 & -8 \\
s^3 & 8 & 12 & \\
s^2 & 3 & -8 & \\
s^1 & \dfrac{100}{3} & & \\
s^0 & -8 & &
\end{array}$$

Since the last element is negative, there is change of sign in the first column, and hence the system is unstable.

REVIEW QUESTIONS

8.1 Use the Routh-Hurwitz criterion to determine the stability of the systems having characteristic equations given below. Also find the number of roots of each equation that may lie in the right half of the s-plane.

i) $s^3 + 20s^2 + 9s + 200 = 0.$

ii) $3s^4 + 10s^3 + 5s^2 + s + 2 = 0.$

iii) $s^5 + 2s^4 + 2s^3 + 4s^2 + s + 1 = 0.$

iv) $s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0.$

8.2 Determine the values of K so that the systems with the following characteristic equations will be stable.

i) $s^3 + as^2 + Ks + b = 0$, where a, b are constants.

ii) $s^4 + 20Ks^3 + 5s^2 + (10 + K)s + 15 = 0.$

iii) $s(2s + 1)(3s + 1) + K(s + 1) = 0$

iv) $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

8.3 Derive the stability conditions which must be satisfied by the coefficients of the following characteristic equation.

$a_3s^3 + a_2s^2 + a_1s + a_0 = 0,$ where $a_3 > 0$

8.4 A unity feedback control system has the forward transfer function

$$G(s) = \frac{K(2s+1)}{s(4s+1)(s+1)^2}$$

Determine the stability of the system for the minimum value of K obtained from $e_{ss} \leq 0.1$ and $r(t) = 1 + t$.

8.5 Determine the stability for each of the systems with loop transfer functions as given below.

i) $G(s)H(s) = \dfrac{1}{(s+2)(s+4)}$

ii) $G(s)H(s) = \dfrac{Ks}{(s+1)^2}$

iii) $G(s)H(s) = \dfrac{K(s-1)}{s+(s^2+4s+4)}$

iv) $G(s)H(s) = \dfrac{K(s+2)}{s(s+3)(s^2+2s+3)}$

8.6 The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K(s+5)(s+40)}{s^3(s+200)(s+1000)}.$$

Discuss the stability of the closed-loop system in terms of K. What value of K will cause sustained oscillation? Find the frequencies of the oscillations.

8.7 A certain system has the characteristic equation $s(T_s + 1) + K = 0$. It is desired that all the roots lie to the left of the line $s = -a$ to guarantee a time constant not larger than

$1/a$. Use the Routh-Hurwitz criterion to find the values of K and T so that both roots meet this requirement.

8.8 Given a system with the characteristic equation $s^3 + 9s^2 + 26s + K = 0$, determine the value of K that will give a dominant time constant not larger than 0.5. The design of the system should be slightly underdamped. Is it possible? If so, find the value of K needed and the resulting damping ratio.

8.9 Find the value of K for which the system represented is stable

$$G(s) = \frac{K}{(s+1)(s+2)}$$

$$H(s) = 1$$

8.10 A control system as given below has the transfer function

$$G(s) = \frac{K}{s(1+0.1s)(1+0.2s)}$$

Determine the maximum value of K that makes the overall system absolutely stable.

8.11 Determine the range of k such that the characteristic equation $s^3 + 3(k+1)s^2 + (7k+5)s + (4k+7) = 0$ will have roots greater than $s = -1$.
(**Hint:** Put $s = z - 1$ in the given characteristic equation so as to shift the imaginary axis towards left by -1.)

8.12 Using Routh's stability criterion determine the relation between K and T so that the unity feedback control system whose open loop transfer function given below is stable.

$$G(s) = \frac{K}{s[s(5+10)+T]}$$

8.13 Using Routh's stability criterion determine the values of 'K' and 'a' so that the system represented below oscillates at a frequency of 2 rad/sec.

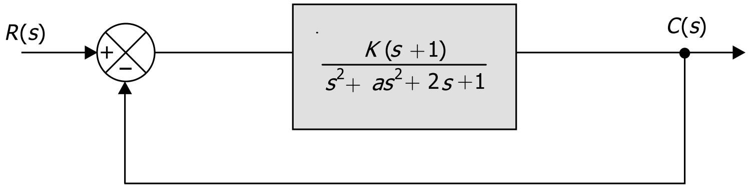


Fig. 8.5

8.14 State the necessary and sufficient conditions of stability for first and second order control systems. Explain why these conditions are necessary but not sufficient for stability of higher order systems.

8.15 A feedback control system whose $G(s)$ and $H(s)$ are

$$G(s) = \frac{K(s+40)}{s(s+10)} \text{ and } H(s) = \frac{1}{s+20}$$

Find the value of gain K which gives marginal stability. Determine the oscillation frequency. Also determine the range of K for which the system is absolutely stable.

9

THE ROOT LOCUS TECHNIQUE

9.1 INTRODUCTION

We have observed in the previous chapter that there are two related points with respect to study of stability of a control system. First, all the poles of the closed loop system must lie in the left hand side of the imaginary axis in the s-plane. The second, the examination that has to be made as to see how close the poles are with the $j\omega$-axis of the s-plane. The second aspect provides us with information regarding the relative stability of the control system.

We have also seen that the poles are the roots of the characteristic equation. The characteristic equation is obtained by putting the denominator of the closed loop transfer function to zero. The characteristic equation is useful in the study of dynamic characteristics of a control system.

From the design point of view, by writing the differential equation of the system and solving the differential equation with respect to a controlled variable, the time response can be found out. Thus, we will know the accurate solution of the equation and hence the performance of the system. But this approach of solution may be difficult for even a slightly complex system. Efforts involved in determining the roots of the characteristic equation has been avoided by applying Routh-Hurwitz criterion as described in the previous chapter. But this criterion only tells the designer as to whether a system is stable or unstable. The designer of a control system cannot remain satisfied with this information alone, because he is unable to indicate the degree of stability of the system. The degree of stability will tell about the amount of overshoot, settling time, etc. for an input, say a step or a ramp input.

Root locus technique, described in this chapter, is a powerful graphical method used for the analysis and design of a control system. This method of analysis not only indicates whether a system is stable or unstable but also shows the degree of stability of a stable system.

Root locus is a plot of the roots of the characteristic equation of the closed loop system as a function of gain. The effect of adjusting the closed loop gain of the system on its stability can be studied by root locus method of stability analysis.

9.2 THE ROOT LOCUS CONCEPT

Consider a second order system shown below in Fig. 9.1.

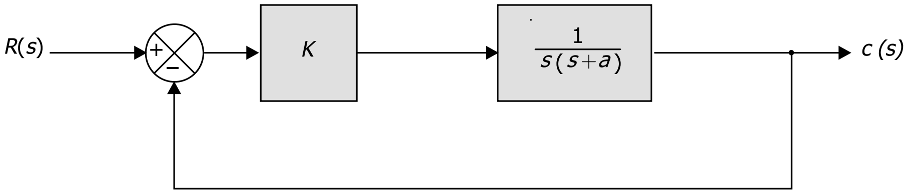


Fig. 9.1 *Block diagram of a second order system*

The open-loop transfer function of the system is given as,

$$G(s) = \frac{K}{s(s+a)}$$

The characteristic equation is, $s(s + a) = 0$.

It has two poles at $s = 0$ and $s = -a$

The closed-loop transfer function of the system is $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{K}{s^2+as+K}$.

The characteristic equation of the system is,

$$s^2 + as + K = 0 \qquad \text{....(9.1)}$$

The second order system as above will be stable for positive values of a and K. Its dynamic behaviour will be controlled by the roots of the characteristic equation (9.1).

Roots of the characteristic equation are

$$s_1, s_2 = \frac{-a \pm \sqrt{a^2 - 4K}}{2}$$

or,

$$s_1, s_2 = \frac{-a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - K}.$$

If we vary K from zero to infinity, the two roots (s_1, s_2) will describe a loci in the s-plane. The root locations for different values of K will change.

By examining the values of roots, i.e. s_1 and s_2 as above we observe that

1) When $0 \le K < \dfrac{a^2}{4}$, the roots are real and distinct.

   When $K = 0$ the roots are $s_1 = 0$ and $s_2 = -a$ which are open-loop poles.

2) When $K = \dfrac{a^2}{4}$, the roots are real and equal, that is, $s_1 = s_2 = \dfrac{-a}{2}$.

3) When $\dfrac{a^2}{4} < K < \infty$, the roots are complex conjugate with constant real part equal to $-a/2$.

The root loci drawn for changing values of K has been shown in Fig. 9.2. The root locus indicates the following system behaviour.

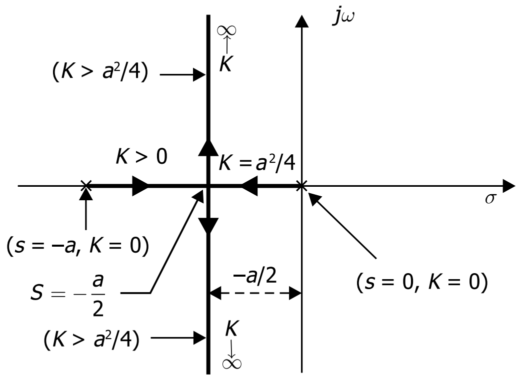


Fig. 9.2 *Root loci of a second order closed loop system with characteristic equation, $s^2 + as + K = 0$ for changing values of gain K*

1) The root locus plot has two branches starting at the two open-loop poles ($s = 0$ and $s = -a$) when the value of gain K is 0.

2) As K is increased from 0 to $a^2/4$, the root loci move along the real axis towards the point $\left(\dfrac{-a}{2}, 0\right)$ from opposite directions. The system will behave as an overdamped system. At $K = \dfrac{a^2}{4}$, the roots are $\dfrac{-a}{2}, \dfrac{-a}{2}$. At this value of K the system will behave as critically damped system.

3) At $K > \dfrac{a^2}{4}$, the roots become complex with real part equal to $\dfrac{-a}{2}$, that is, the roots break away from the real axis, become complex conjugate and move towards infinity along the vertical line at $\sigma = \dfrac{-a}{2}$. Since the loci move away from the real axis, the

system becomes underdamped. For this case, settling time is nearly constant as the real part is constant.

We have drawn the root locus by the direct solution of the characteristic equation. For higher order systems, this procedure will become complicated and time consuming. *Evans* developed a simplified graphical technique for root locus plot which is described below. The characteristic equation of the closed-loop system is

$$1 + G(s)\,H(s) = 0$$

or, $\quad G(s)\,H(s) = -1 \qquad \text{....(9.2)}$

a) Magnitude criterion

From equation (9.2), we see that the magnitude of the open-loop transfer function is equal to unity for all the roots of the characteristic equation $|G(s)H(s) = 1|$.

b) Angle criterion

The angle of the open-loop transfer function is an odd integral multiple of π.

$$\angle G(s)H(s) = \pm 180^\circ(2q+1);$$

where, q = 0, 1, 2...

The gain factor K does not affect the angle criterion.

For any point to be on the root locus in the s-plane, it has to satisfy both angle criterion and magnitude criterion. The magnitude criterion is checked after confirming the existence of the point on the root locus by applying the angle criterion.

To understand this, let us consider an example where

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Let us examine whether $s = -0.5$ lies on the root locus or not.

First we apply the angle criterion as

$\angle G(s)\,H(s)$ at $s = -0.5 = \pm\ 180^\circ\ (2q + 1)$ where $q = 0, 1, 2, \ldots$

$$\angle G(s)H(s) = \frac{K}{(-0.5)(-0.5+1)(-0.5+2)}$$

$$= \frac{K}{(-0.5+j0)(0.5+j0)(1.5+j0)}$$

$$= \frac{K\underline{|0^\circ}}{180^\circ \quad 0^\circ \quad 0^\circ} = -180^\circ$$

Since the angle criterion is satisfied, the point $s = -0.5$ lies on the root locus. Now we will also check by applying the magnitude criterion to find the value of K for which the point $s = -0.5$ lies on the root locus.

Using magnitude criterion

$$|G(s)H(s) = 1| \text{ at } s = -0.5$$

Here, $$\frac{K}{|-0.5||0.5||1.5|} = 1 \quad \text{or} \quad K = 0.375$$

Thus, for $K = 0.375$ point $s = -0.5$ lies on root locus.

9.3 ROOT LOCUS CONSTRUCTION PROCEDURE

As we have seen, root locus is the graphical plot of the poles of a closed loop system with respect to change in the gain parameter K of the system from 0 to ∞. The knowledge of open loop poles and zeros are important here as the root locus always starts from open loop poles and terminate on an open-loop zero or infinity. We will take up an example of plotting the root locus and along with write the general rules or guidelines.

Example 9.1 Sketch the root locus of a control system whose transfer function is $G(s) = \dfrac{K}{s(s+1)}$ with unity feedback.

Solution

We have,

$$G(s) = \frac{K}{s(s+1)} \text{ and } H(s) = 1$$

$$G(s)H(s) = \frac{K}{s(s+1)}$$

By examining the denominator of $G(s)$, we find that the number of open loop poles, $n = 2$. They are $s = 0$ and $s = -1$. As evident from the numerator, the number of open loop zeros, $m = 0$.

The closed loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{k/s(s+1)}{1+\dfrac{K}{s(s+1)}} = \frac{K}{s^2+s+K}$$

The characteristic equation is

$$s^2 + s + K = 0$$

$$s_1, s_2 = \frac{-1 \pm \sqrt{1^2 - 4K}}{2}$$

$$= -0.5 \pm \sqrt{0.25 - K}$$

For $K > 0.25$, the roots are complex conjugates.

The root loci starts at 0 and -1, i.e. at the open loop poles.
The number of branches or root loci = Number of open-loop poles
Number of asymptotes = Number of open loop poles – Number of open-loop zeros
In this case, asymptotes = 2 – 0 = 2.
Determination of Breakaway Points:
Characteristic equation is

$$1 + G(s)\,H(s) = 0$$

Substituting, $$\frac{K}{s(s+1)} = -1$$

or, $$K = -s(s+1)$$

or, $$K = -s^2 - s.$$

We put $$\frac{dK}{ds} = 0$$

$\therefore$ $$\frac{dK}{ds} = -2s - 1 = 0$$

or, $$s = \frac{1}{2} = -0.5$$

The two root loci, starting at 0 and -1 respectively approach each other and breakaway asymptotically at -0.5.
The value of K at the breakaway point on the real axis is calculated as

$$\begin{aligned} K &= -s(s+1) \\ &= -\left[-\frac{1}{2}\left(-\frac{1}{2}+1\right)\right] \\ &= -\left[-\frac{1}{2}\times\frac{1}{2}\right] \\ &= 0.25 \end{aligned}$$

For a value of $K = 0.25$, the two root loci meet at the real axis at $s = -0.5$ and breakaway at $\phi_A = 90°$ and $270°$ asymptotically as K increases beyond 0.25 towards infinity, where
Angle of asymptotes, ϕ_A is,

$$\phi_A = \frac{(2q+1)180°}{n-m}$$

where n = no. of open loop poles m = no. of open loop zeros. $q = 0, 1, 2, \ldots (n-m) - 1$

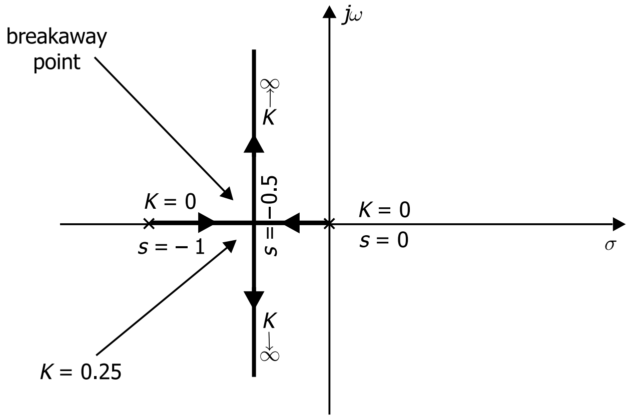


Fig. 9.3 *Root loci of $s^2 + s + K = 0$*

and $$\phi_A = \frac{(2\times1+1)180^\circ}{2-1} = 270^\circ$$

The root locus has been plotted as in Fig. 9.3. It may be observed that the root locus is symmetrical about the real axis, i.e. σ axis. There is no existence of the root locus to the extreme left pole at $s = -1$ on the real axis.

9.4 ROOT LOCUS CONSTRUCTION RULES

As mentioned earlier, root locus is the path of the roots of the characteristic equation, $1 + G(s)H(s) = 0$ traced out in s-plane as the system parameter (gain K) is changed.

The root locus diagram or plot can be completed using the following procedure. The procedure is presented in the form of certain rules.

a) **Starting and termination of root locus**—From the open-loop transfer function, locate the poles and zeros. Each branch of the root locus originates from an open-loop pole with $K = 0$ and terminates either on an open-loop zero or at infinity as the value of K increases from 0 to ∞. In most cases, we will have more poles than zeros. If we have n poles and m zeros, and $n > m$. then $n - m$ branches of the root locus will reach infinity. Because the root loci originate at the poles, the number of root loci is equal to number of poles.

b) **Root locus on the real axis**—The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.
Let the open-loop transfer function of a control system be $G(s) = K(s + 1)/(s + 2)$. The pole is at $s = -2$ and the zero is at $s = -1$ as shown in Fig. 9.4(a). The root locus will start at $s = -2$ and terminate at zero at $s = -1$. There is existence of root locus to the left of Z and no existence to the left of P on the real axis (root locus on real axis exists to the left of odd number of poles and zeros).

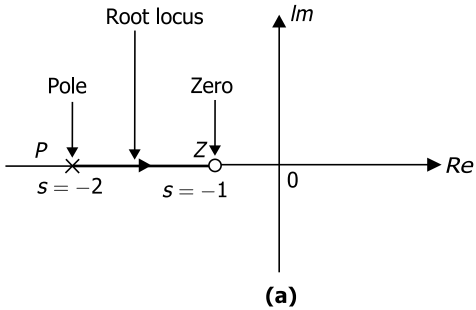


Fig. 9.4 (a) Location of poles and zeros and the root locus on the real axis

c) **Symmetry of the root locus**—The root loci must be symmetrical about the real axis because the complex roots appear in pairs.

d) **The member of asymptotes and their angles with the real axis**—The $(n - m)$ branches of root loci move towards infinity. They do so along straight line asymptotes. The angle of asymptotes with respect to the real axis is given by

$$\phi_A = \frac{(2q+1)}{n-m}180°, \quad q = 0, 1, 2, \ldots$$

where n is the number of poles and m is the number of zeros.

e) **Centroid of the asymptotes**—The linear asymptotes are centred at a point on the real axis. This is called the centroid which is given by the relation

$$\sigma_A = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n-m}$$

f) **Breakaway points**—The root locus breakaway from the real axis where a number of roots are available, normally, where two roots exist. The method of determining the breakaway point is to rearrange the characteristic equation in terms of K. We then evaluate $dK/ds = 0$ in order to find the breakaway point. Since the characteristic equation can have real as well as complex multiple roots, its root locus can have real as well as complex breakaway points. However, because of conjugate symmetry of root loci, the breakaway point must either be on the real axis or must occur in complex conjugate pairs.

g) **Intersection of the root locus with the imaginary axis**—The point at which the locus crosses the imaginary axis, in case it does, is determined by applying Routh-Hurwitz criterion. The value of K for which the locus crosses the imaginary axis is calculated by equating the terms in the first column of the Routh array of s^1 and s^0 to zero.

h) **Angle of departure of the root locus**—The angle of departure of the locus from a complex pole is calculated as

$\phi_d = 180°$ – sum of angles made by vectors drawn from the other poles to this pole + sum of angles made by vectors drawn from the zeros to this pole.

Let us consider an example. Let

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+2s)}$$

The poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$, $s_4 = 3 - j4$. There are no zeros. The positions of poles are shown in Fig. 9.4(b). The angle of departure of the root locus from the complex pole at P_3 is calculated as

$\phi_d = 180° - (127° + 104° + 90°) + 0 = 180° - 321° = -141°$

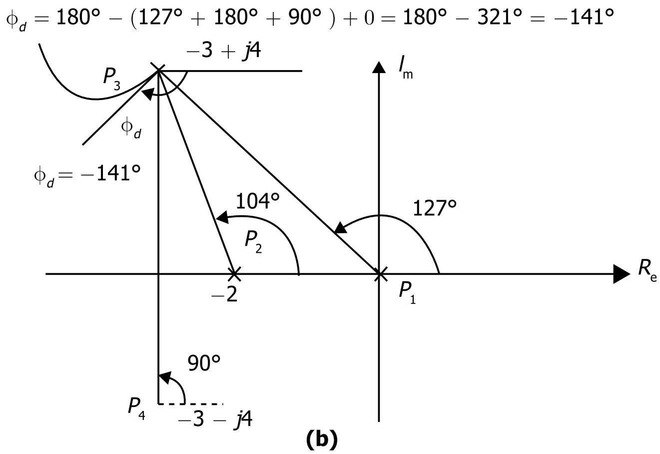


Fig. 9.4 (*b*) *Calculation of angle of departure of the root locus from a complex pole*

9.5 ROOT LOCUS CONSTRUCTION RULES—ILLUSTRATED THROUGH EXAMPLES

The root locus construction rules have been explained in the previous section. However, for the sake of better understanding the rules are once again illustrated through examples.

The following rules are applicable in sketching the root locus plot.

Rule 1: Symmetry of root locus—Any root locus must be symmetrical about the real axis, that is, the upper half of the root locus diagram is exactly the same as the lower half about the real axis. This can be seen from any root locus diagram.

Rules 2 and 3: Starting and termination of root loci—Root locus will start from an open-loop pole with gain $K = 0$ and terminate either on an open-loop zero or to infinity with $K = \infty$.

Let us illustrate these rules with an example. Let open-loop transfer functions of control systems are

1) $G(s)H(s) = \dfrac{K(s+1)}{(s+2)}$

2) $G(s)H(s) = \dfrac{K}{s(s+1)}$

For (1), the root locus will start at the pole at $s = -2$ and terminate at zero at $s = -1$ as shown in Fig. 9.5(a).

For (2) the poles are at $s = 0$ and $s = -1$. As there is no zero, the locus will originate at $s = 0$ and $s = -1$ and approach towards each other and then break away to infinity as the value of gain K is increased continuously as shown in Fig. 9.5(b)

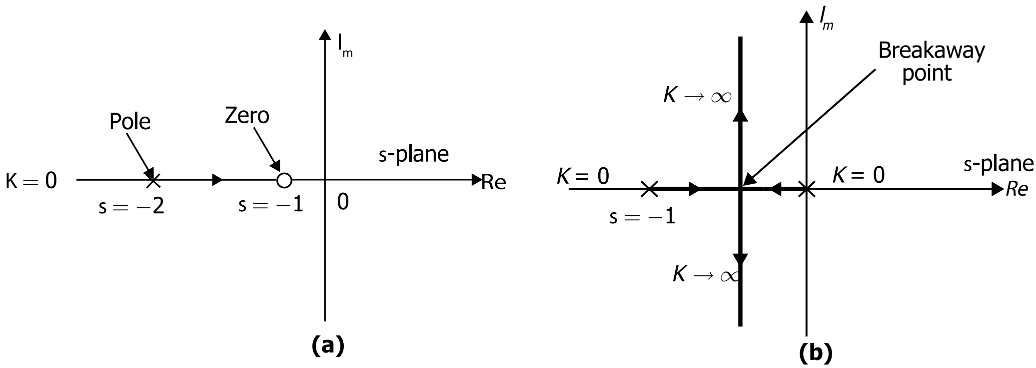


Fig. 9.5 *(a) Starting and termination of root locus where number of poles are equal to number of zeros, i.e. n = m;* ***(b)*** *Starting and termination of root locus when n > m*

for $G(s)H(s) = \dfrac{K(s+1)}{(s+2)}$ and $G(s)H(s) = \dfrac{K}{s(s+1)}$ respectively.

Rule 4: Number of root loci—If P is the number or poles and Z is the number of zeros in the transfer function $G(s)\ H(s)$, the number of root loci N will be as follows:

$N = P$ if $P > Z$

$N = P = Z$ if $P = Z$

For example, in the root locus shown under Rules 2 and 3, in Fig. 9.5(a), $N = P$, $Z = 1$. Therefore, the number of root loci, $N = P = Z = 1$. And in Fig. 9.5(b), $P = 2$, $Z = 0$. Therefore, $N = 2$. The two root loci originating from origin and -1 respectively are approaching to ∞ in two directions.

Rule 5: Root loci on the real axis—The root locus on the real axis will lie in a section of the real axis to the left of an odd number of poles and zeros.

This rule is illustrated through the following examples:

1) $$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

Its root locus is shown below.

There is no root locus on the real axis between P_2, and Z_1, because the number of poles and zeros to its right is even, that is 2. See Fig. 9.6.

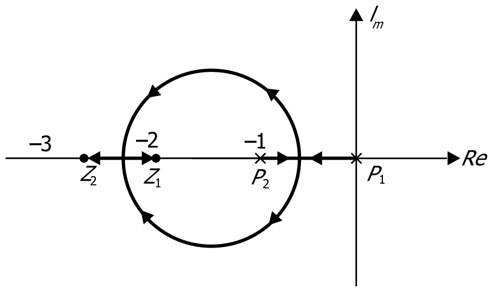


Fig. 9.6 *Location of root locus on the real axis*

Similarly, beyond, z_2 to the left on the real axis there cannot be any root locus as the number of poles and zeros is even, that is 4 in this case.

Rule 6: The number of asymptotes and their angles with the real axis—As the value of K is increased to ∞, some branches of root locus from the real axis approach infinity along some asymptotic lines. These asymptotic lines are straight lines originating from the real axis making certain angles with the real axis. The total number of asymptotic lines and the angles they would make are calculated as follows:

Number of asymptotic lines or asymptotes

$$= P - Z$$

where P is the number of poles and Z is the number of zeros of the open-loop transfer function, $G(s)H(s)$.

The angle of asymptotes with the real axis is

$$\phi_A = \frac{(2q+1)180^\circ}{P-Z} \text{ where } q = 0,1,2... \qquad \text{....(i)}$$

Let us consider,

$$G(s)H(s) = \frac{K}{s(s+2)}$$

Here, the number of poles $P = 2$; they are at $s = 0$ and $s = -2$ and number of zeros $Z = 0$. Number of asymptotes $= P - Z = 2 - 0 = 2$

Let the angle of two asymptotes be ϕ_A and ϕ_A' respectively. Then, by using equation (i),

$$\phi_A = \frac{(2\times 0+1)180^\circ}{2-0} = 90^\circ \quad \text{for } q=0$$

$$\phi_Z' = \frac{(2\times 1+1)180^\circ}{2-0} = 270^\circ \quad \text{for } q=1$$

The root locus with the asymptotes are shown in Fig. 9.7. One asymptote, *LM* is making 90° with the real axis and another asymptote *LN* is making an angle of 270° with the real axis. There are two root loci originating at the poles at $s = 0$ and $s = -2$. They approach each other and break away at *L* and approach towards infinity along the asymptotic lines as the value of *K* increases from 0 to ∞.

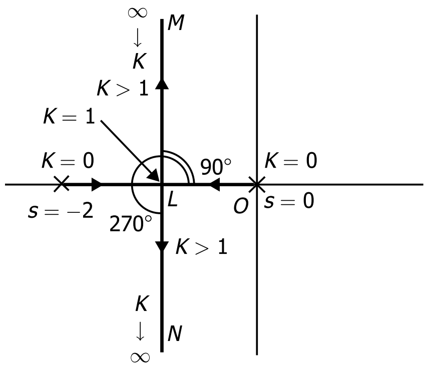


Fig. 9.7 *The root locus with the asymptotes*

Rule 7: Centroid of the asymptotes—The point of intersection of the asymptotes with the real axis is called the centroid σ_A which is calculated as

$$-\sigma_A = \frac{\sum \text{Real parts of poles } - \sum \text{Real parts of zeros}}{P-Z}$$

Let us consider the example of Rule 6 where $G(s) = K/s(s + 2)$

$$-\sigma_A = \frac{[0-2]-[0]}{2-0} = -\frac{2}{2} = -1$$

Therefore, the point *L* where the two asymptotes start, is at a distance of -1 from the origin. Let us consider another example,

$$G(s)H(s) = \frac{k}{s(s+4)(s+5)}$$

The poles are at $s = 0$, $s = -4$ and $s = -5$, which are shown in Fig. 9.8. There will be no part of the root locus to the left of *B* up to point *A* as there are even number of poles to the right of B. There will be part of root locus to the left of point A as there are odd number of poles to the right of A. There will be three root loci originating at the poles at C, B, and A. They will terminate at zeros. If there is not zero present in $G(s)H(s)$, they will approach toward infinity as the value of K is increased.

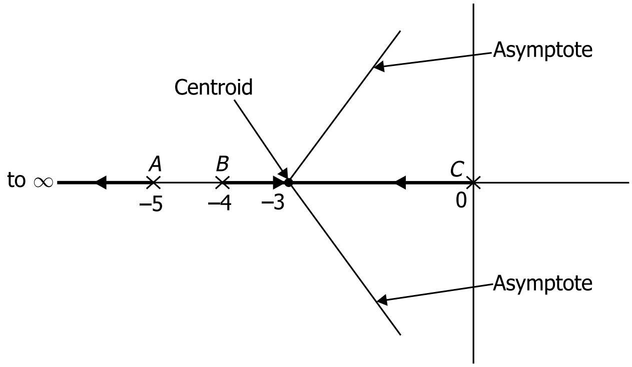


Fig. 9.8 *Centroid, i.e. the point of intersection of the asymptotes*

The root loci from B and C will approach each other along the real axis and break away towards infinity at a point on the real axis in between points B and C which is the centroid. The centroid of the asymptotes is calculated as

$$-\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{P - Z}$$

$$-\sigma_A = \frac{-5-4+0-0}{3-0} = \frac{-9}{3} = -3$$

Rule 8: Breakaway points—The root locus between two adjacent poles approaching each other break away on the real axis at a point on the real axis and move towards infinity as the value of K increases. For determining the breakaway point, we write the characteristic equation in terms of K and evaluate $dK/ds = 0$. The breakaway points will be either on the real axis or must occur as complex conjugate pairs. This is illustrated through examples as follows:

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

The characteristic equations is $1 + G(s)H(s) = 0$,

or $$s(s + 1)(s + 2) + K = 0$$

or $$K = -s^3 - 3s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

that is,

$$3s^2 + 6s + 2 = 0$$

The roots are

$$s_1, s_2 = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$
$$= -0.43, -1.57$$

The root locus sketch is shown in the following:

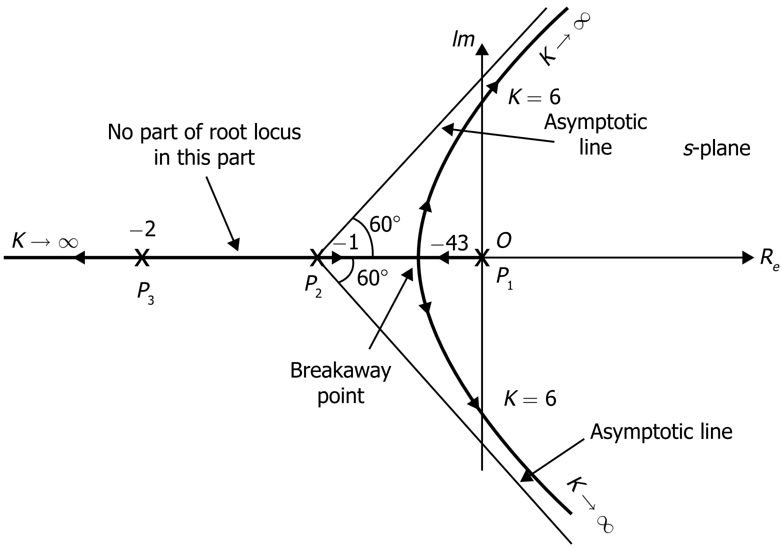


Fig. 9.9 *Determination of breakaway point of the root locus on the real axis*

P_1, P_2 and P_3 are the positions of the poles at $s = 0$, $s = -1$ and $s = -2$, respectively. There is no zero in the numerator of the transfer function. The presence of root locus on the real axis will be to the left of odd number of poles and zeros. So there will be no part of the root locus between point -1 and -2 as has been shown in Fig. 9.9. So, the breakaway point of root locus will be at -0.43 and not at -1.57.

The intersection of the root locus on the imaginary axis also has to be found out since the characteristic equation can have real as well as complex multiple roots.

The intersection of the root locus on the imaginary axis is calculated using the characteristic equation and forming the Routh's array as

$$s(s + 1)(s + 2) + K = 0$$

or $$s^3 + 3s^2 + 2s + K = 0$$

The Routh array is represented as

$$\begin{array}{lcc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & 0 \\ s^0 & K & 0 \end{array}$$

The value of K for which the root locus crosses the imaginary axis is calculated by equating the terms of the first column of Routh array of s^1 and s^0 to zero.

So we write,

$$\frac{6-K}{3} = 0 \text{ and } K = 0$$

Thus, $K = 6$

The auxiliary equation is

$$3s^2 + K - 0$$

The roots of the auxiliary equation are dominant roots which are close to the imaginary axis or on the imaginary axis.

Thus,

$$3s^2 = -K = -6$$

or $$s^2 = -2 \text{ or } s = j\sqrt{2}$$

Number of asymptotes = Number of open-loop poles = 3.

The angles are

$$\phi_A = \frac{(2q+1)180^\circ}{P-z} \quad q = 0,1,2$$

$$= \frac{180}{3} = 60^\circ \text{ for } q = 0$$

$$= \frac{(2+1)180^\circ}{3} = 180^\circ \text{ for } q = 1$$

$$= \frac{(2\times2+1)180^\circ}{3} = 300^\circ \text{ for } q = 22$$

Centroid
$$-\sigma_A = \frac{\sum \text{Real parts of poles} - \sum \text{ Real parts of zeros}}{P-z}$$

$$= \frac{(-1-2)-0}{3-0} = -1^\circ$$

Thus, the complete root locus is as shown in Fig. 9.10.

Rule 9: Angle of departure of the root locus from a complex pole and the angle of arrival at a zero—Angle of departure of the root locus from a complex pole is given as

$\phi_d = 180°$ – sum of angles of vectors drawn from other poles to this pole + sum of angles of vectors drawn to this pole from other zeros.

Let us illustrate this with an example.

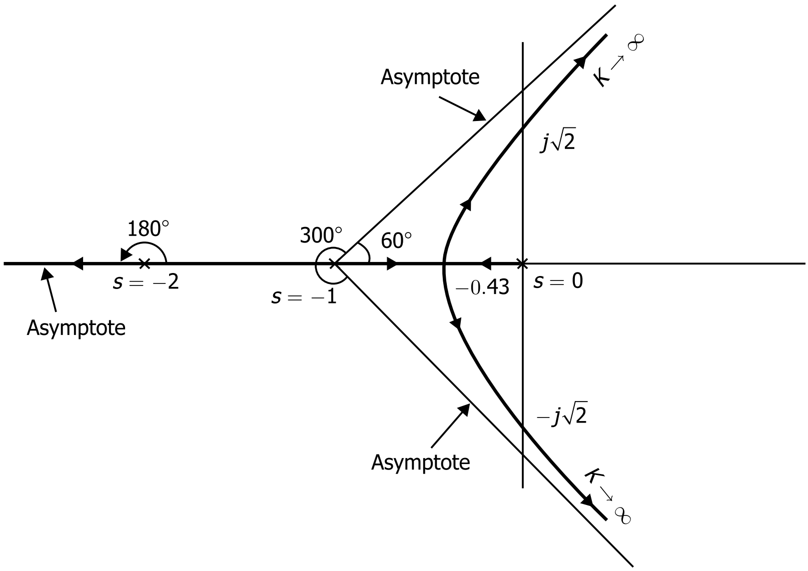


Fig. 9.10 *Determination of angle of departure of the root locus from a complex pole*

Let

$$G(s)H(s) = \frac{K}{s(s+2)s^2 + 62 + 2s)}$$

The poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$ and $s_4 = -3 - j4$.

The positions of poles are shown in Fig. 9.11. The angle of departure ϕ_A is calculated as,

$$\phi_A = 180° - (127° + 104° + 90°)$$
$$= 180° - 321° = -141°$$

9.5.1 Additional Techniques

a) Determination of *K* on root loci

For determining the value of *K* at any point on the root locus, we can use the following:

$$K = \frac{\text{Product of all vector lengths drawn from the poles of } G(s)H(s) \text{to that point}}{\text{Product of all vector lengths drawn from the zeros of } G(s)H(s) \text{ to that point}}$$

b) Ascertainment of any point to be on root locus

For any point to be on the root locus in the *s*-plane, it has to satisfy the angle criterion and magnitude criterion. First, we apply the angle criterion to check that any point in *s*-plane which satisfied the angle condition has to be on the root locus. Magnitude condition is used only after confirming the existence of point on the root locus by angle condition.

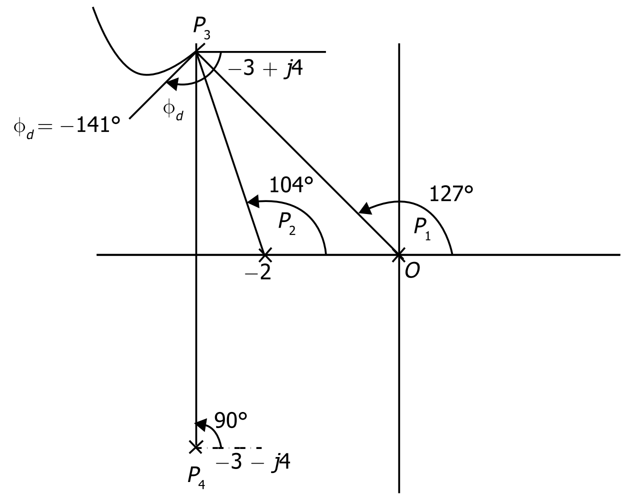


Fig. 9.11 *Method of determination of any point to be on the root locus*

Let us consider, for example, where

$$G(s)H(s)=\frac{K}{s(s+1(s+2)}$$

Let us examine whether $s = -0.5$ lies on the root locus or not.

First we apply angle criterion as

$$\angle G(s)H(s) \text{ at } s = -0.5 = \pm 180°(2q + 1) \text{ where } q = 0, 1, 2, \qquad \text{....(i)}$$

$$\angle G(s)H(s)=\frac{K}{(-0.5+j0)(-0.5+1)(-0.5+2)}=\frac{K}{(-0.5+j0)(0.5+j0)(1.5+j0)}$$

$$=\frac{K\angle 0°}{180°0°0°}=180°$$

Thus, the angle condition as in (i) above is satisfied and the point $s = -0.5$ lies on the root locus. Now we will use magnitude condition as

$|G(s)H(s)| = 1$ at $s = -0.5$

$$\frac{K}{|-0.5||0.5||1.5|}=1$$

or $$K = 0.375$$
for this value of K, point $s = -0.5$ lies on the root locus.

Example 9.2 A block diagram representation of a unity feedback control system is shown below.

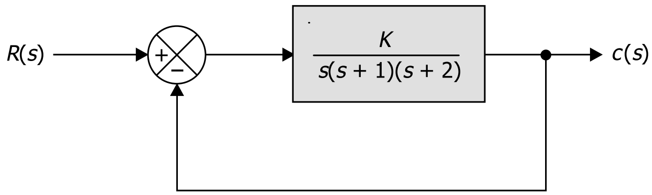


Fig. 9.12

For this system sketch the root locus. Also determine the value of K so that the damping ratio, ξ of a pair of complex conjugate closed loop poles is 0.5.

Solution

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

For determining the open loop poles, we equate the denominator of G(s) to 0.

$\therefore$ $$s(s+1)(s+2) = 0$$

a) There are three open loop poles at $s = 0$, $s = -1$, and $s = 2$.
b) We know that the number of root locus asymptotes will be equal to number of open loop poles minus the number of open loop zeros. Here there is no open loop zero.
   There will be three branches of the root locus originating respectively at $s = 0$, $s = -1$ and $s = -2$.
c) The three branches of the root locus will move towards infinity, as k changes, along the asymptotic lines whose angles with the real axis are

$$\phi_A = \frac{(2q+1)180°}{n-m}; q = 0, 1, 2$$
$$= (2q+1)180° = 60°, 180°, 300°$$

d) The root locus exist on the real axis between $s = 0$, and $s = -1$; and $s = -2$ moving toward ∞.
e) The centroid, $-\sigma_A$ is calculated as

$$-\sigma_A = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{\text{number of poles} - \text{number of zeros}}$$
$$= \frac{(-1-2)-0}{3-0} = -1$$

f) The break away points on the real axis is found by putting $\frac{dK}{ds} = 0.$
The characteristic equation is

$$s(s + 1)(s + 2) + K = 0$$

or,
$$K = -s^3 - 3s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

i.e., $3s^2 + 6s + 2 = 0$

$$s_1, s_2 = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$
$$= -0.43, -1.57$$

g) Intersection of the root locus on the imaginary axis is determined as follows.
The characteristic equation of the system is

$$s(s + 1)(s + 2) + K = 0$$

or,
$$s^3 + 3s^2 + 2s + K = 0$$

The Routh Array is

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & 0 \\ s^0 & K & 0 \end{array}$$

We know that the occurrence of a zero row in the Routh array indicates the presence of symmetrically located roots in the s-plane.

For this,
$$\frac{6 - K}{3} = 0$$

or,
$$K = 6$$

The auxiliary equation is

$$3s^2 + K = 0$$

or,
$$3s^2 = -K = -6$$

or,
$$s = \pm j\sqrt{2}.$$

The position of poles, the asymptotes, and the root locus plot have been shown in Fig. 9.13

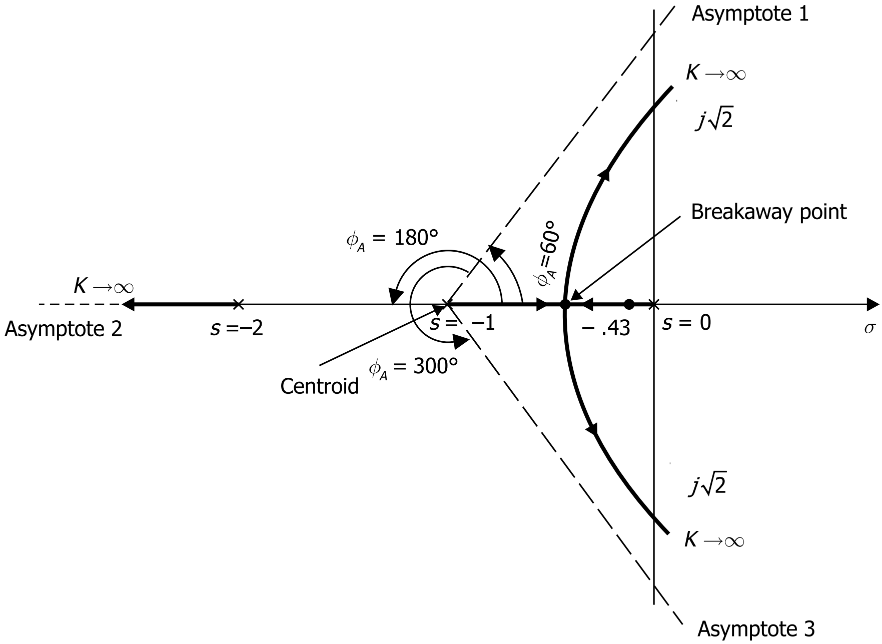


Fig. 9.13 *Root locus plot for $s^3 + 3s^2 + 2s + K = 0$*

Note that for breakaway point at $s = -1.57$, the angle criterion is not satisfied and hence can not be considered.

Example 9.3 The block diagram of a unity feedback control system is shown below.

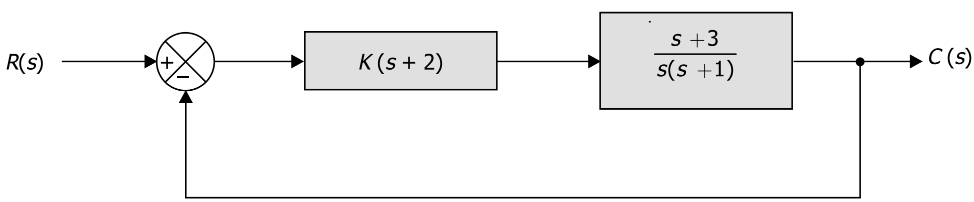


Fig. 9.14

Draw the root locus diagram for the above represented control system.

Solution

The open loop transfer function,

$$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

The number of open loop poles is 2 at $s = 0$ and $s = -1$.
The number of open loop zeros is 2 at $s = -2$ and $s = -3$.
Therefore, the number of root locus asymptotes $= 2 - 2 = 0$.
The number of branches will be 2 originating at $s = 0$ and $s = -1$.
The root loci will terminate at the zeros at $s = -2$ and $s = -3$.
Now let us calculate the breakaway points. The characteristic equation is

$$1 + G(s)\,H(s) = 0.$$

Putting actual values we get

$$\frac{k(s+2)(s+3)}{s(s+1)} = -1$$

or,
$$K = -\frac{s(s+1)}{(s+2)(s+3)} = \frac{-(s^2+s)}{(s^2+5s+6)}$$

We have to make
$$\frac{dK}{ds} = 0.$$

Thus,
$$\frac{dK}{ds} = \left[\frac{-(s^2+5s+6)(2s+1)+(s^2+5)}{(s^2+5s+6)^2}\right] = 0$$

or,
$$2s^2 + 6s + 3 = 0$$

or,
$$s_1 s_2 = -0.63,\ -2.36.$$

The root loci branches originate at $s = 0$ and $s = -1$ and get terminated at zeros at $s = -2$ and $s = -3$.

Thus root locus exists between 0 and -1 and between -2 and -3. Thus both the breakaway points $s = -0.63$ and $s = -2.36$ are valid (breakaway and break-in points).

The values of K at $s = -0.634$ and $s = -2.366$ are calculated as

$$K_1 = \left|\frac{(0.634)(0.366)}{(1.366)(2.366)}\right| = 0.07$$

and
$$K_2 = \left|\frac{(-2.366)(-1.366)}{(-0.366)(0.634)}\right| = 13.93$$

The root locus plot has been shown is Fig. 9.15. The two root loci originates at $s = 0$ and $s = -1$ on the real axis. As the value of K is increased they approach each other and breakaway at $s = -0.634$ with value of $K = 0.07$. As the value of K increases the two root loci makes semicircles and break in at $s = -2.366$ with value of $K = 13.93$. They get terminated at the two zeros at $s = -3$ and $s = -2$ as shown.

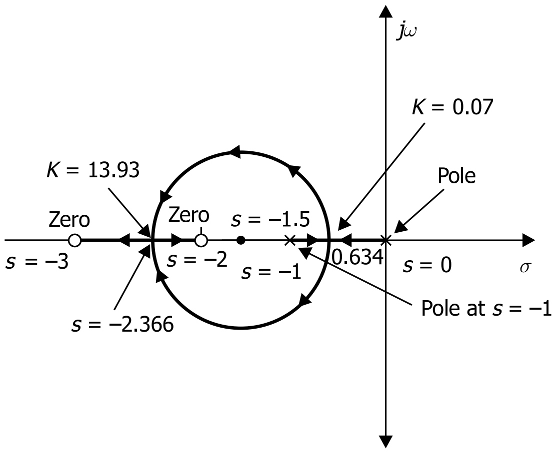


Fig. 9.15 *Root locus plot of* $G(s)H(s) = \dfrac{k(s+2)(s+3)}{s(s+1)}$

The root locus is a circle with its centre at -1.5 (i.e. $\dfrac{2.366-0.634}{2}+0.634$).

Example 9.4 The open loop transfer function of a system with unity feedback has been shown below.

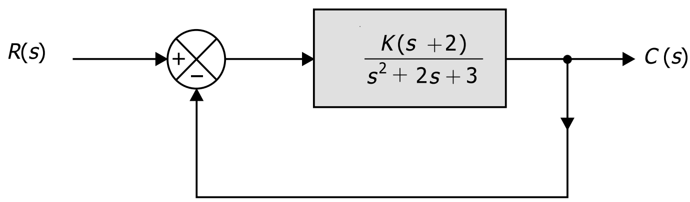


Fig. 9.16

Draw the root locus diagram for the above system.

Solution

The open loop poles are calculated from the equations, $s^2 + 2s + 3 = 0$.

$$s_1, s_2 = \frac{-2 \pm \sqrt{2^2 - 4\times 1\times 3}}{2}$$
$$= -1 \pm j\sqrt{2}$$

So, there are two open loop poles. The open loop zero is at $s = -2$.

The number of branches of the root locus is equal to the number of open loop poles, i.e. 2.

Number of asymptotes = Number of poles − Number of zeros
$= 2 - 1$
$= 1$

Angle of the asymptote

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; q = 0$$
$$= \frac{180^\circ}{1} = 180^\circ$$

Now, let us determine the break in point. For this we take the characteristic equation of the closed loop system as

$$1 + G(s)\, H(s) = 0.$$

or,
$$G(s)\, H(s) = -1.$$

Putting the values,

$$\frac{K(s+2)}{s^2+2s+3} = -1$$

or
$$K = -\frac{(s^2+2s+3)}{(s+2)}$$

Now let us calculate dK/ds and equate it to zero.

$$\frac{dK}{ds} = -\left[\frac{(s+2)(2s+2)-(s^2+2s+3)}{(s+2)^2}\right] = 0$$
$$= -\left[\frac{2s^2+6s+4-s^2-25-3}{(s+2)^2}\right] = 0$$

or,
$$s^2 + 4s + 1 = 0$$

$$s_1, s_2 = -2 \pm \sqrt{3} = -3.732, -0.268$$

The value of K at $s = -3.732$ is

$$K = \left|\frac{s^2+2s+3}{s+2}\right| = \frac{(3.732)^2 + 2\times 3.732 + 3}{3.732+2}$$
$$= 5.46.$$

The break in point at $s = -3.732$ lies between the position of zero at $s = -2$ and the infinity. At $s = -2$, the vale of $K = \infty$.

Thus the two root locus branches originate respectively from $-1+j\sqrt{2}$ and $-1-j\sqrt{2}$ and break in at the real axis at $s = -3.732$ as K increases to a value of 5. 46 as shown in Fig. 9.17.

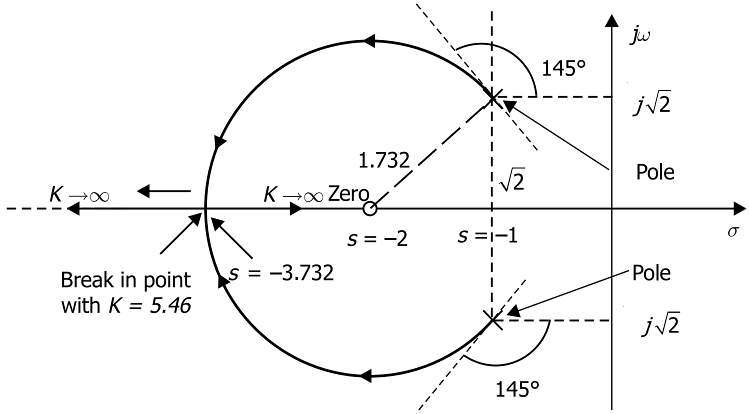


Fig. 9.17 *The root locus plot of* $G(s)H(s) = \dfrac{k(s+2)}{s^2+2s+2}$

The radius of the circle is $\sqrt{3}$, i.e. 1.732. The angle of take off can be calculated 145° as shown. Thus the two branches starting from the open loop zeros follow a circular path with the increase of gain factor K. As the value of K becomes 5.46, they meet at the real axis at $s = -3.732$ and then breakaway towards $s = -2$ and infinity respectively with increasing values of K.

Example 9.5 A unity feedback control system has an open loop transfer function.

$$G(s) = \frac{K}{s(s^2+4s+13)}$$

Draw the root locus as the value of K changes from 0 to ∞. Also find the value of K and the frequency at which the root loci crosses the $j\omega$ axis.

Solution

We have

$$G(s) = \frac{K}{s(s^2+4s+13)}$$

Number of open loop poles = 3
Number of open loop zeros = 0
Poles are at $s_1 = 0$ and

$$s_2, s_3 = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j\sqrt{3}$$

The root locus has three branches.
The centroid of the asymptotes

$$\sigma_A = \frac{(-2 - j\sqrt{3} - 2 + j\sqrt{3})}{3}$$
$$= -\frac{4}{3}$$
$$= -1.33$$

The angles the asymptotes make with the real axis

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2$$
$$= \frac{(2\times 0+1)180^\circ}{3-0} = 60^\circ$$

and $$\phi_A = \frac{(2\times 1+1)180^\circ}{3-0} = 180^\circ$$

and $$\phi_A = \frac{(2\times 2+1)180^\circ}{3-0} = 300^\circ$$

Let us examine if there is any breakaway point or not.
The closed loop characteristic equation is

$$1 + G(s)\, H(s) = 0.$$

or, $$\frac{K}{s(s^2+4s+13)} = -1$$

or, $$K = -s^3 - 4s^2 - 13s.$$

$$\frac{dK}{ds} = -3s^2 - 8s - 13 = 0$$

or, $$3s^2 + 8s + 13 = 0$$

$$s_1, s_2 = \frac{-8 \pm \sqrt{64-156}}{6} = -\frac{4}{3} \pm j\sqrt{92}$$

This is on imaginary axis and hence there does not exist any breakaway point on the real axis.
As shown in Fig. 9.19, the angle of departure from a pole is calculated as

$$\theta_p = \pm\,(2q + 1)180^\circ - \phi, \qquad q = 0, 1, 2, \ldots$$

where ϕ is the angle contribution at a pole of all other poles and zeros.
The root locus plot has been shown in Fig. 9.18.

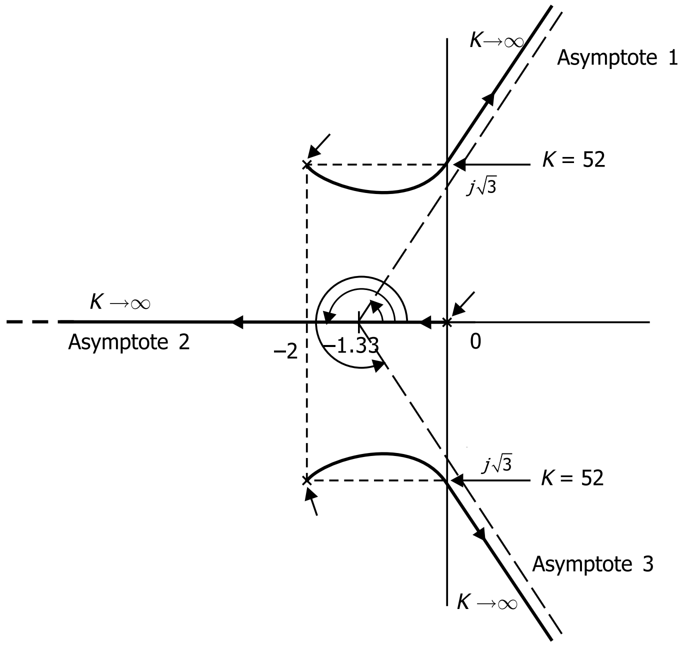


Fig. 9.18 Root locus plot of $G(s) = \dfrac{K}{s(s^2 + 4s + 13)}$

Angle of departure, $\phi_p = 180° - \phi$

$\phi_p = 180° - 213.7° = -33.7°$

Since $\phi = (90° + 123.7°) = 213.7°$

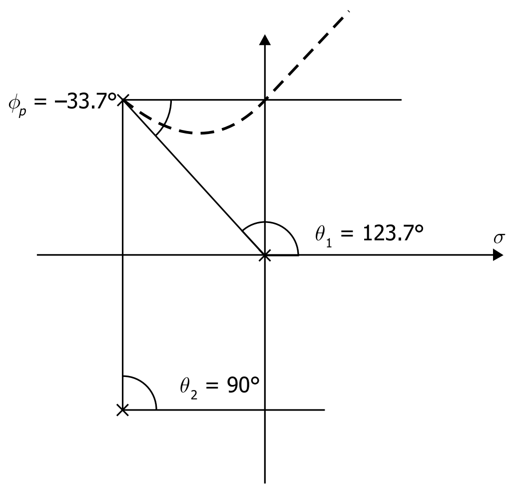


Fig. 9.19

To calculate the value of K

The characteristic equation is $s^3 + 4s^2 + 13s + K = 0$

Routh Array is

$$\begin{array}{ccc} s^3 & 1 & 13 \\ s^2 & 4 & K \\ s^1 & \dfrac{52-K}{4} & 0 \\ s^0 & K & 0 \end{array}$$

Applying condition for stability

$$\frac{52-K}{4} > 0$$

and $$K > 0$$

$\therefore$ $$0 < K < 52$$

For a maximum value of $K = 52$, the root locus will cut the imaginary axis. If the value of K equals 52, the system will be oscillatory. For values of K more than 52, the system will be unstable. When $K = 52$, the frequency of oscillation is calculated as

$$4s^2 + K = 0$$

or, $$s^2 = -\frac{52}{4} = -13$$

$$s = \pm j\sqrt{13} = \pm j\omega$$

Therefore, $$\omega = \sqrt{13} = 3.6 \text{ rad/sec}.$$

Example 9.6 The open loop transfer function of a control system is given as

$$G(s)H(s) = \frac{K}{(s+1)(s+10)(s+30)}$$

Draw the root locus. Determine the value of K for which the system is critically damped and also the value of K for which the system becomes unstable.

Solution

From $G(s)\,H(s)$ we find, $P = 3$ and $Z = 0$. The poles are at $s = -1$, $s = -10$, and $s = -30$. Since there are three poles there will be three root locus branches.

$$\text{Centroid } \sigma_A = \frac{(-1-10-30)-0}{3-0} = -\frac{41}{3} = -13.67$$

The asymptotes originate at $s = -13.67$ and make angles of

$$\phi_A = \frac{(2q+1)180°}{n-m} = 60°, 180° \text{ and } 300°.$$

The breakaway point is calculated using the characteristic equation as

$$(s + 1)(s + 10)(s + 30) + K = 0$$

\or, $$K = s^3 + 41s^2 - 340s - 300$$

$$\frac{dk}{ds} = -3s^2 - 82s - 340 = 0$$

or, $$3s^2 - 82s + 340 = 0$$

$$s_1, s_2 = \frac{-82 \pm \sqrt{(82)^2 - (4 \times 3 \times 340)}}{2 \times 3} = -5.1 \text{ and } -22.2$$

The poles are at $s = -1$, $s = -10$, and $s = -30$. the asymptotes are at angles of 60°, 180°, and 300°. They originate from $s = -13.67$. Two root loci originating from $s = -1$ with $K = 0$ and $s = -10$ with $K = 0$ approach each other and breakaway at $s = -5.1$. The breakaway point at $s = -22.2$ is discarded because this is not possible.
The root locus plot has been drawn as shown in Fig. 9.20.

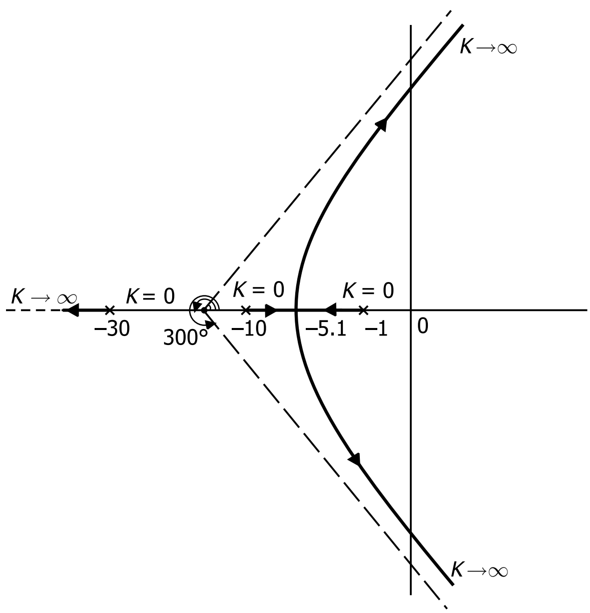


Fig. 9.20 *Root locus for* $G(s)H(s) = \dfrac{K}{(s+1)(s+10)(s+30)}$

We write the Routh Array for the equation $s^3 + 41s^2 + 340s + 300 + K = 0$

$$
\begin{array}{lcc}
s_3 & 1 & 340 \\
s_2 & 41 & 300+K \\
s_1 & \dfrac{340\times 41-(300+K)}{41} & \\
s_0 & 300+K &
\end{array}
$$

Condition for stability is that

$$\frac{13640-300-K}{41} > 0$$

or, $$K > 13640$$

Thus the value of K for which the system will be unstable is 13640 and above.

At $s = -5.1$, the loci leave the real axis at the breakaway point. The value of K corresponding to this point of the root locus is the one at which the system is critically damped. Putting $s = -5.1$ in the expression for K,

$$
\begin{aligned}
K &= -s^3 - 41s^2 - 340s + 300 \\
&= -(-5.1)^3 - [41(-5.1)^2] - 340(-5.1) - 300 \\
&= 500.2
\end{aligned}
$$

Thus at $K = 500.2$, the system is critically damped.

Example 9.7 The open-loop transfer function of a system is given as

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)}.$$

Draw the root locus diagram and establish the condition for stability with respect to the value of gain K. Calculate the phase margin and gain margin when the value of gain is reduced, say equal to 6.

Solution

Open loop poles are at $s = 0$, $s = -1$, and $s = -3$. There are no zeros. The number of root locus branches are $(P - Z) = 3 - 0 = 3$.

The centroid, σ_A is calculated as

$$\sigma_A = \frac{\Sigma(-1-3)-\Sigma 0}{3-0} = -\frac{4}{3} = -1.33$$

Angle of asymptotes ϕ_A are calculated as

$$
\begin{aligned}
\phi_A &= \frac{(2q+1)}{n-m} \text{ at } q = 0, 1, 2 \\
&= 60°, 180°, 300°
\end{aligned}
$$

The characteristic equation is $s(s + 1)(s + 3) + K = 0$

$$s^3 + 4s^2 + 3s + K = 0$$

$$K = -[s^3 + 4s^2 + 3s]$$

$$\frac{dK}{ds} = -[3s^2 + 8s + 3] = 0$$

or,
$$3s^2 + 8s + 3 = 0$$

$$s = \frac{-8 \pm \sqrt{8^2 - 4 \times 3 \times 3}}{2 \times 3}$$

$$s_{1,2} = -1.33 \pm \sqrt{0.77}$$

$$= -1.33 \pm 0.86$$

$$= -2.19, -0.47$$

Breakaway at $s = -2.19$ is not possible. The breakaway will be at $s = -0.47$ Routh Array using the characteristic equation is written as

$$s^3 + 4s^2 + 3s + K = 0$$

$$\begin{array}{ccc} s_3 & 1 & 3 \\ s_2 & 4 & K \\ s_1 & \dfrac{12-K}{4} & 0 \\ s_0 & K & 0 \end{array}$$

The condition for stability is, $K > 0$

$$\frac{12-K}{4} > 0$$

or,
$$K < 12$$

Therefore
$$0 < k < 12$$

At $k = 12$, the system is oscillating in nature. The value of s at $K = 12$ is calculated using the Routh Array as

$$4s^2 + K = 0; \text{ or } 4s^2 + 12 = 0$$

or,
$$s^2 + 3 = 0$$

or,
$$s = \pm j\sqrt{3} = \pm j1.732$$

using the above calculated values, the root locus plot is made as shown in Fig 9.21

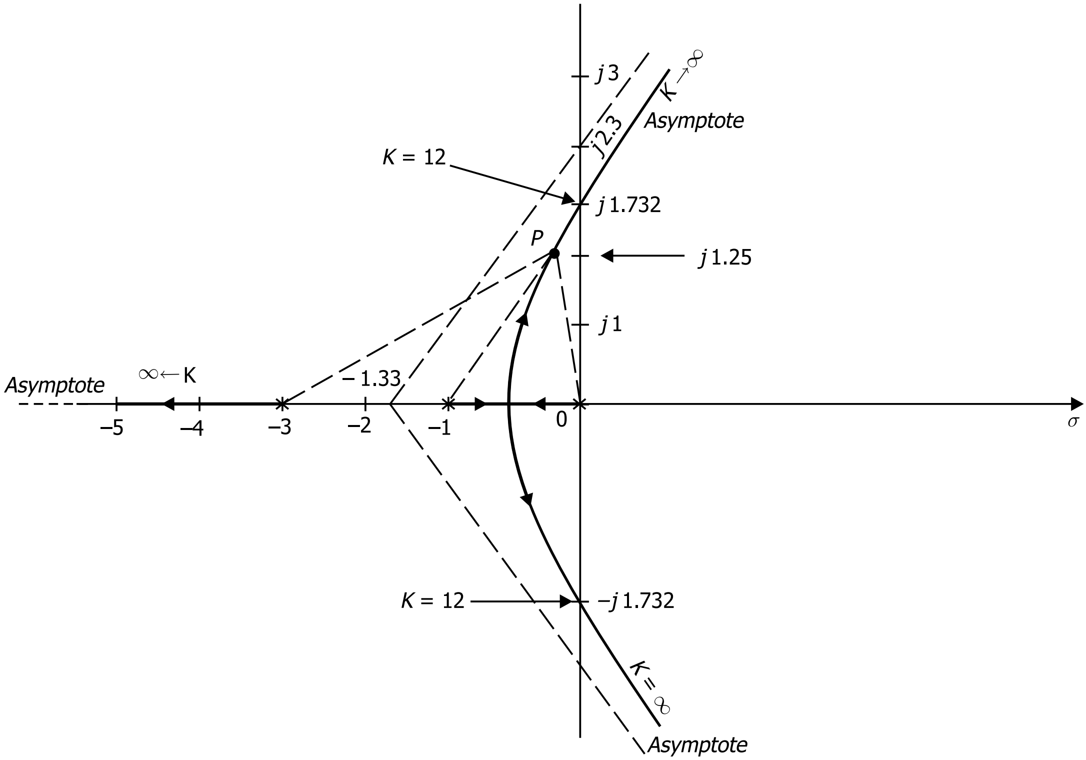


Fig. 9.21 *Root locus of* $G(s)H(s) = \dfrac{K}{s(s+1)(s+3)}$

The root loci crosses the $j\omega$ axis at 1.732 when the value of $K = 12$ and the system becomes oscillatory. If the value of K is reduced to 6, the system will be conditionally stable and we have to locate a point on the root loci somewhat below j 1.732. Let this point is at P. By trial and error, the point P on the root loci has to be fixed such that its distance from all the poles when multiplied equals gain $K = 6$. The distances of the three poles from the point P are 1.2, 1.6 and 3.15 respectively.

$$K = 6 = 1.2 \times 1.6 \times 3.15$$

$$\text{Gain margin} = \frac{\text{Value of limiting } K}{\text{Value of derived } K}$$

$$= \frac{12}{6} = 2$$

$$\text{Gain margin in dB} = 20\log_{10}(2)$$

$$= 6 \text{ dB.}$$

To determine the phase margin for k = 6 we have to locate a point on the imaginary axis of the root locus plot such that the magnitude of the transfer function for a frequency corresponding to that point is 1.

That is,

$$|G(j\omega_1)\ H(j\omega_1)| = 1 \quad \text{for } s = j\omega_1$$

By trial and error, i.e. by repeated calculation we will find that frequency, ω_1
Let $\omega_1 = 2$,

$$\left|G(j2)H(j2)\right| = \frac{6}{|j2|\left|\sqrt{2+1}\right||j2+3|}$$

$$= \frac{6}{1.4 \times 2.2\ 5 \times 3.8}$$

$$= < 1$$

Let $\omega = 1$,

$$\left|G(j1)H(j1)\right| = \frac{6}{|j1|\left|\sqrt{j1+1}\right||j1+3|}$$

$$= \frac{6}{1 \times \sqrt{2} \times \sqrt{10}}$$

$$= \frac{6}{1 \times 1.41 \times 3.1}$$

$$= 1.35 => 1$$

Let $\omega = 1.25$,

$$\left|G(j1.25)H(j1.25)\right| = \frac{6}{|j1.25||j1.25+1||j1.25+3|}$$

$$= \frac{6}{1.25 \times \sqrt{2.26} \times \sqrt{10.56}}$$

$$\simeq 1.$$

Therefore, $\omega_1 = 1.25$, is calculated as the gain cross-over frequency.

Phase margin = 180 $\quad$ $G(j1.25)\ H(j1.25)$

$=180° +$ (angle contributed by the three poles at a point on root having $\omega_1 = 1.25$)
Let θp_1, θp_2, θp_3 be the angle contributed by the poles respectively. If angles contributed by the poles are taken as negative,

$$\begin{aligned}\text{Phase margin} &= 180° + (-\theta p_1 - \theta p_2 - \theta p_3)\\ &= 180° + \left(-90° - \tan^{-1}\frac{1.25}{1} - \tan^{-1}\frac{1.25}{3}\right)\\ &= 180° - (90 + 50° + 22°)\\ &= 18°\end{aligned}$$

Example 9.8 The transfer function of a unity feedback system is $G(s) = \dfrac{K}{s(s+5)}$. Draw the root locus plot and determine the value of K for a damping ratio of 0.707.

Solution

The two poles are at $s = 0$ and $s = -5$

There is no zero. Number of root locus branches are $P - Z = 2 - 0 = 2$. Since there is no open loop zero, both the root loci will terminate at ∞. We are to calculate the angle of the asymptotes, ϕ_A

$$\begin{aligned}\phi_A &= \frac{(2q+1)\times 180°}{0-2} \quad q = 0, 1\\ &= 90°, 270°\end{aligned}$$

The asymptotes intersect at the negative real axis as the controid, σ_A.

$$\begin{aligned}\sigma_A &= \frac{\Sigma\,\text{real parts of poles} - \Sigma\,\text{real parts of zeros}}{P - z}\\ &= \frac{(0-5)-0}{2-1} = -2.5\end{aligned}$$

To calculate the breakaway point, i.e. the point on the negative real axis where from the root loci depart the real axis and become asymptotic and approach ∞, we write the characteristic equation which as

$$s(s + 5) + K = 0.$$

$$s^2 + 5s + K = 0$$

or,

$$K = -(s^2 + 5s)$$

$$\frac{dK}{ds} = -[2s + 5] = 0$$

$$\begin{aligned}s &= -\frac{5}{2}\\ &= -2.5\end{aligned}$$

This shows that the breakaway point and the point of intersection of asymptotes on real axis is the same.
We have,

$$\omega_n \cos\theta = \xi\omega_n \qquad \textit{(see Fig. 9.15)}$$

$$\cos\theta = \xi = 0.707$$

$$\theta = \cos^{-1} 0.707 = 45°$$

We will draw the ξ-line at 45° with the negative real axis as shown in Fig. 9.22. The ξ-line intersects the root locus at R.

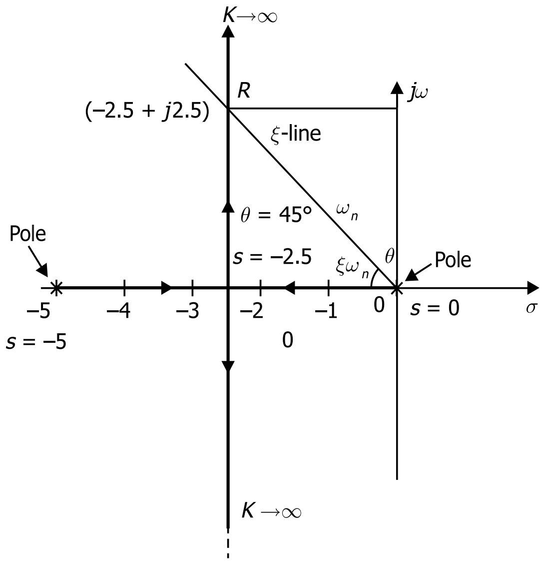


Fig. 9.22 *Root locus plot of* $G(s) = \dfrac{k}{s(s+5)}$ *and determination of gain K for a damping ratio of 0.707*

At point R, the value of $s = -2.5 + j2.5$
Since R is on the root locus, the magnitude criterion of $|G(s)\,H(s)| = 1$ will be satisfied.
Therefore,

$$|G(s)\,H(s)| = 1$$

or,
$$\left|\frac{K}{s(s+5)}\right| = 1$$

or,
$$\left|\frac{K}{(-2.5+j2.5)[(-2.5+j2.5)+5]}\right| = 1$$

or, $$\left|\frac{K}{(-2.5+j2.5)(2.5+j2.5)}\right|=1$$

or, $$\left|\frac{K}{[(6.25+6.25)(6.25+6.25)]^{1/2}}\right|=1$$

or, $$\frac{K}{12.5}=1$$

i.e. $$K = 12.5.$$

Example 9.9 The transfer function of a unity feedback control system is given as $G(s)=\dfrac{K}{s(s+2)(s+3)}$. Calculate the value of K for a damping ratio of 0.707. Also calculate the value of K for which the system will become oscillatory. Calculate the value of K at $s = -4$ and -5.

Solution

The poles are at $s = 0, -2$, and -3. There is no zero. Number of branches of the root locus is, $P - Z = 3 - 0 = 3$. We know that the poles terminate at zeros or infinity. As there is no zero, all the three root loci originating from the poles will terminate at ∞.

The characteristic equation is

$$1 + G(s) = 0$$

or, $$1+\frac{K}{s(s+2)(s+3)}=0$$

or, $$s\,(s + 2)\,(s + 3) + K = 0$$

$$K = -\,[s\,(s + 2)\,(s + 3)]$$

$$= -\,[s^3 + 5s^2 + 6s]$$

Putting $$\frac{dK}{ds}=0,$$

$$\frac{dK}{ds}=-[3s^2+10s+6]=0$$

$$3s^2+10s+6=0$$

$$s=\frac{-10\pm\sqrt{10^2-4\times3\times6}}{2\times3}$$

$$s=-\frac{10}{6}\pm\frac{5.3}{6}=-0.8,-2.54$$

The angle of asymptotes, ϕ_A is calculated as

$$\phi_A = \frac{(2q+1)\times 180^\circ}{p-2} \; q = 0, 1, 2$$
$$= \frac{(2\times 0+1)180^\circ}{3} = 60^\circ \text{ for } q = 0$$

For $q = 1$ & 2, $\phi_A = 180^\circ$, and 300°

The point of intersection of the asymptotes with the real axis, i.e. the centroid, σ_A is calculated as

$$\sigma_A = \frac{\Sigma \text{ poles} - \Sigma \text{ zeros}}{P-Z} = \frac{(0-2-3)-0}{3-0}$$
$$= -\frac{5}{3} = -1.67.$$

The value of K for which the system is stable is calculated by applying Routh-Hurwitz criterion using the characteristic equation.

We have the characteristic equation,

$$s(s+2)(s+3) + K = 0$$

or,
$$s^3 + 5s^2 + 6s + K = 0$$

Routh Array is,

$$\begin{array}{lcc} s^3 & 1 & 6 \\ s^2 & 5 & K \\ s^1 & \dfrac{30-K}{5} & 0 \\ s^0 & K & 0 \end{array}$$

For stability all the terms of the first column must be positive.

Hence, $K > 0$

and $\dfrac{30-k}{5} > 0$, i.e. $K < 30$.

Therefore, the system is stable if $0 < K < 30$.

At $K = 30$, the system is oscillatory. For $K < 30$ all the roots will lie on the left hand side of s-plane. If K exceeds 30, the roots will lie on the right hand side of the s-plane. At $K = 30$, the locus will intersect the imaginary axis. The value of s at which the locus cuts the imaginary axis is calculated from the Routh Array's second row as

$$5s^2 + K = 0$$

or,
$$5s^2 + 30 = 0$$

or,
$$s^2 = 6$$

or, $$s = \pm j\sqrt{6} = \pm j2.4 \;= \pm j\omega_n$$

The frequency of oscillation is equal to the distance from origin to the point of intersection of the root locus with the $j\omega$ axis.

The frequency of oscillation, $\omega_n = 2.45$ rad/sec.

The root locus plot has been shown in Fig. 9.23.

Now we have to calculate the value of K for damping ratio of 0.707.

We have $$\omega_n \cos\theta = \xi\omega_n$$

i.e., $$\cos\theta = \xi = 0.707$$

$$\theta = \cos^{-1} 0.707 = 45°.$$

We draw the ξ line with an angle of 45° with the negative real axis. The ξ line cuts the root locus at a point R. The value of s for that point is $-0.6 + j0.8$.

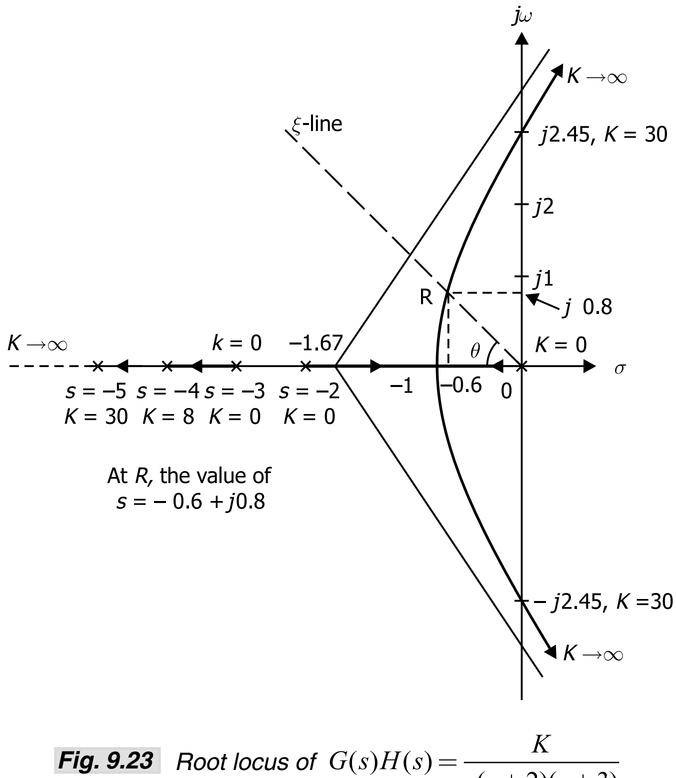


Fig. 9.23 *Root locus of* $G(s)H(s) = \dfrac{K}{s(s+2)(s+3)}$

Since point R for which $s = -0.6 + j0.8$ lies on the root locus, the magnitude condition must be satisfied which is

$$|G(s)H(s)| = 1$$

Substituting,

$$\left|\frac{K}{s(s+2)(s+3)}\right| = 1$$

$$\left|\frac{K}{(-0.6+j0.8)(-0.6+j0.8+2)(-0.6+j0.8+3)}\right| = 1$$

or, $$\left|\frac{K}{\sqrt{(-0.6)^2+(0.8)^2}\left(\sqrt{(1.4)^2+(0.8)^2}\right)\left(\sqrt{(2.4)^2+(0.8)^2}\right)}\right| = 1$$

or, $$\frac{k}{1\times 1.6\times 2.5} = 1$$

or, $$K = 4$$

The value of K at $s = -4$ and -5 can be calculated by putting the value of s in the transfer function as

$$\left|\frac{K}{s(s+2)(s+3)}\right| = 1$$

Put $$s = -4$$

$$\left|\frac{K}{-4(-4+2)(-4+3)}\right| = 1$$

or, $$K = 8$$

Put $$s = -5$$

$$\left|\frac{K}{-5(-5+2)(-5+3)}\right| = 1$$

or $$K = 30$$

We note that at $s = -3$, K is 0 as the root loci originating from the pole at $s = -3$ approaches infinity along the asymptote at 180°, the value of K increases, becoming $K = 8$ at $s = -4$ and $K = 30$ at s $= -5$.

Example 9.10 The open-loop transfer function of a control system is given as $G(s)H(s) = \dfrac{k}{s(s+4)(s^2+4s+15)}$. Draw the root locus plot and calculate the value of k for which the system becomes oscillatory.

Solution

There are 4 poles at $s = 0$, $s = -4$ and

$$s = \frac{-4 \pm \sqrt{4^2 = 4 \times 15 \times 1}}{2} = -2 \pm j3.3$$

There is no zero. Hence the number of branches of the root locus is 4. There are four asymptotes whose angles are

$$\phi_A = \frac{(2q+1)180°}{P-Z}, q = 0,1,2,3$$
$$= 45°, 135°, 225°, 315°.$$

The intersection of the asymptotes on the negative real axis is calculated as

$$\sigma_A = \frac{\sum \text{poles} - \sum \text{zeros}}{P-Z} = \frac{(0-4-2-2)-0}{4} = -2$$

We have now to calculate the breakaway point and the intersection of the root locus with the imaginary axis and the corresponding value of K.

The characteristic equation is

$$s(s+4)(s^4 + 4s + 15) + K = 0$$

or, $$s^4 + 8s^3 + 31s^2 + 60s + K = 0$$

First we will find the value of k when the root locus crosses the imaginary axis by applying Routh's criterion.

$$s^4 + 8s^3 + 31s^2 + 60s + K = 0$$

s^4	1	31	K
s^3	8	60	
s^2	23.5	K	
s^1	$\dfrac{23.5 \times 60 - 8K}{23.5}$		
s^0	K		

For stability, all the terms of the first column must be positive.

Hence $K > 0$

and $$\frac{(23.5 \times 60 - 8\,K)}{23.5} > 0$$

or, $$60 - 0.33\,K > 0$$

or, $$K > 180.$$

At $K = 180$, the system will be oscillatory.
For $K = 180$, the value of s is calculated form the third row of the Routh array as

$$23.5\, s^2 + K = 0$$

or, $$23.5\, s^2 + 180 = 0$$

or, $$s = \pm j\sqrt{7.7} = \pm j2.7$$

To locate the breakaway point we have to start with the characteristic equation as

$$K = -\left[s^4 + 8s^3 + 31s^2 + 60s\right]$$

$$\frac{dK}{ds} = -\left[4s^3 + 24s^2 + 62s + 60\right]$$

Putting $$\frac{dK}{ds} = 0,$$

$$4s^3 + 24s^2 + 62s + 60 = 0$$

From this equation we have to calculate the three values of s. From the root locus plot shown in Fig. 9.25, we notice that one of the breakaway points is at $s = -2$. To determine the other two breakaway points we divide the above equation by $(s + 2)$.

We get, $$\frac{4s^3 + 24s^2 + 62s + 60}{(s+2)} = 0$$

or, $$4s^2 + 16s + 30 = 0$$

$$s = \frac{-16 \pm \sqrt{16^2 - 4\times 4\times 30}}{2\times 4} = -2 \pm j\sqrt{3.5}$$

$$= -2 \pm j1.87$$

With the above calculated values, the complete root locus plot has been drawn as shown in Fig. 9.25. We have, however, also to calculate the angle of departure of the root locus from the complex poles at $(-2 + j3.3)$ and $(-2 - j3.3)$. For this we have to refer to the location of the poles and zeros. Since there is no zero in this case, the locations of the poles are shown as in Fig. 9.25.

$$\theta p + (\theta p_1 + \theta_2 + \theta p_3) = 180°$$

Here $$\theta p_1 = \tan^{-1}\frac{3.3}{2} = 59°$$

$$\theta p_2 = 180° - \tan^{-1}\frac{3.3}{2} = 180° - 59° = 121°$$

$$\theta p_3 = 90°$$

Therefore, from Fig. 9.24,

$$\begin{aligned}\theta p &= 180° - (59° + 121° + 90°)\\ &= 180° - 270°\\ &= -90°\end{aligned}$$

The angle of departure of the root loci from the other conjugate pole will be 90°.

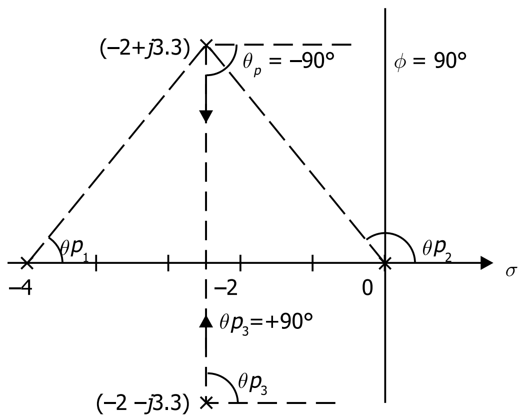


Fig. 9.24

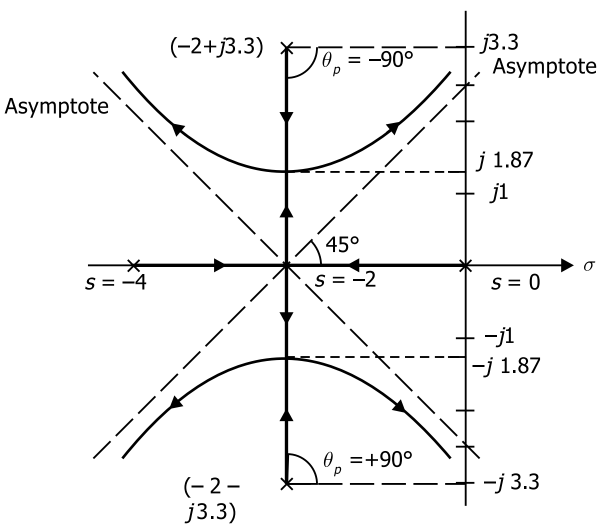


Fig. 9.25 *Root locus of* $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+15)}$

Example 9.11 The open-loop transfer function of a control system is given as

$$G(s)H(s)=\frac{K}{s^2(s+4)}$$

Draw the root locus plot for the system.
Determine the value of K for which the system is stable.

Solution

Given, $$G(s)H(s)=\frac{K}{s^2(s+4)}$$

Number of zeros $= 0$
Number of poles $= 3$ at $s = 0$ and $s = -4$
There would be three asymptotes for three root locus branches.

$$\text{Centroid } \sigma_A=\frac{(-4)}{3}=-1.33.$$

$$\phi_A=\frac{(2q+1)180}{n-m} \quad q=0,1,2$$
$$=60°, 180°\ 300°.$$

Closed loop characteristic equation is

$$s^2(s+4)+K=0$$

or, $$s^3+4s^2+K=0$$

or, $$K=-s^3-4s^2$$

$$\frac{dK}{ds}=-3s^2-8s=0$$

$$s_1, s_2=0, -\frac{8}{3}, \text{ i.e. } -2.67.$$

The two roots originating at $s = 0$, breakaway at $s = 0$ and move toward ∞ as $K \to \infty$. The other at $s = -4$ approaches ∞ as K tends to infinity. The root locus plot has been shown in Fig. 9.26. Using the characteristic equation the Routh Array is

written as,

s^3	1	0
s^2	4	K
s^1	$-\frac{K}{4}$	0
s^1	K	

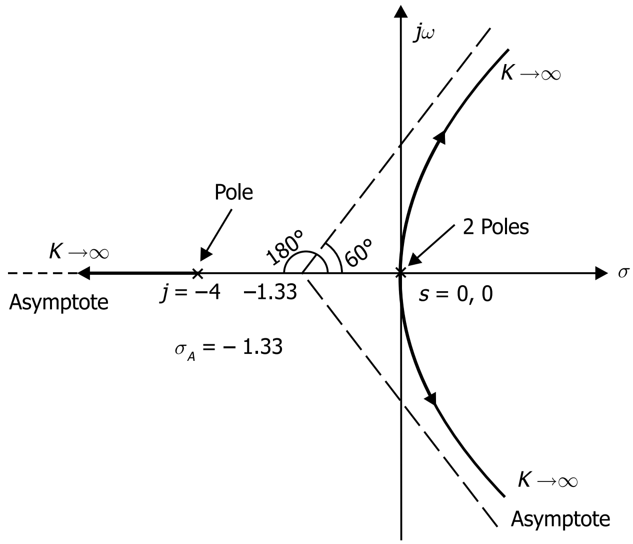


Fig. 9.26 *Root locus plot for* $G(s)H(s) = \dfrac{K}{s^2(s+4)}$

From the above, it is seen that the system is unstable for every value of K.

Example 9.12 Plot the root locus for a system having open loop transfer function of

$$G(s) = \frac{K}{s(s+1)^2}$$

Determine the value of K for which the system response will become oscillatory. Also calculate the frequency of oscillation at that value of K.

Solution

Given

$$G(s) = \frac{K}{s(s+1)^2}$$

Number of zeros = Nil
Number of poles = 3 at $s_1 = 0$, $s_{2,3} = -1, -1$
There are three branches of the root locus.

The centroid,
$$\sigma_A = \frac{(-1-1)}{3} = -0.67$$

$$\phi_A = \frac{(2q+1)180°}{3} \quad q = 0, 1, 2$$
$$= 60°, 180°\ 300°.$$

Breakaway point of the root locus is calculated by using the characteristic equation which is

$$s(s+1)^2 + K = 0$$

or,
$$s^3 + 2s^2 + s + K = 0$$

or,
$$K = -s^3 - 2s^2 - s$$

$$\frac{dK}{ds} = -3s^2 - 4s - 1 = 0$$

or,
$$3s^2 + 4s + 1 = 0$$

$$s_1, s_2 = \frac{-4 \pm \sqrt{4^2 - 4\times 3}}{2\times 3} = \frac{-4 \pm 2}{6}$$
$$= -1, -0.33$$

Breakaway point of -0.33 is to be considered.
The root locus plot has been shown in Fig. 9.27.
One root locus starts from $s = -1$ and goes towards ∞ as k increases. The other two branches, one from $s = -1$ and $s = 0$ approach each other on the real axis and breakaway at $s = -0.33$, crosses the $j\omega$ axis and moves towards infinity with increase in value of K.

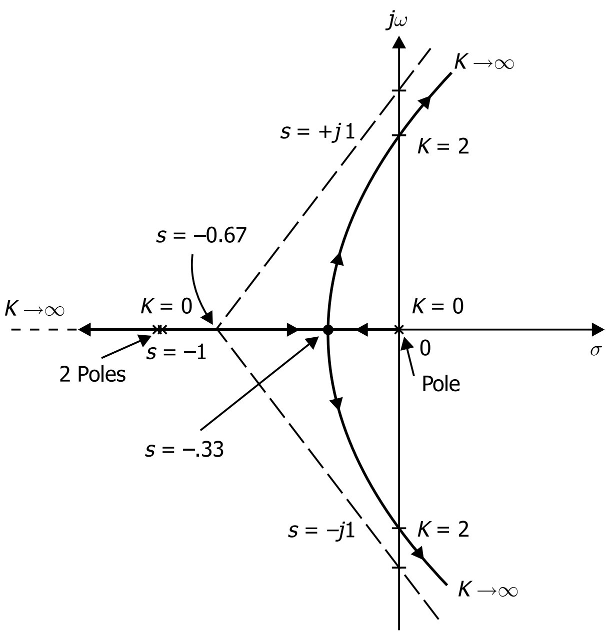


Fig. 9.27 *Root locus plot for* $G(s) = \dfrac{K}{s(s+1)^2}$

To determine the value of K for stable operation we write the Routh Array using the characteristic equation as:
The characteristic equation is

$$s^3 + 2s^2 + s + K = 0$$

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & K \\ s^1 & \dfrac{2-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For stable operation,

$$K > 0$$

and $$\frac{2-K}{2} > 0$$

or, $$k < 2$$

Thus the condition for stability is $0 < K < 2$.
at $K = 2$ the root locus crosses the $j\omega$ axis where the system is oscillatory.

We write the auxiliary equation from the Routh Array

$$2s^2 + K = 0$$

or, $$2s^2 + 2 = 0$$

$$s^2 = -1$$

or, s $= \pm j1$
The root locus cuts the $j\omega$ axis at $+j1$ and at $-j1$.
Frequency of oscillation, $s = \pm j\omega$. Therefore, $\omega = 1$ rad/sec.

Example 9.13 A feedback control system has an open-loop transfer function

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+4)}$$

Draw the root locus as K varies from zero to infinity.

Solution

Given $$G(s)H(s) = \frac{K(s+1)}{s^2(s+4)}$$

The poles are at $s = 0, 0, -4$
The zero is at $s = -1$
There are three branches of the root locus.

Centroid, $$\sigma_A = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{\text{number of poles} - \text{number of zeros}}$$

$$= \frac{-4-(-1)}{3-1} = -1.5$$

The angle ϕ_A of the asymptotes are

$$\phi_A = \frac{(2q+1)180°}{n-m} \text{ at } q = 0,1$$
$$= 90° \text{ and } 270°$$

The breakaway point of the root loci is calculated using the characteristic equation which is

$$s^2(s+4) + K(s+1) = 0$$

or $$s^3 + 4s^2 + Ks + K = 0$$

$$K = -\frac{(s^3 + 4s^2)}{s+1}$$

$$\frac{dK}{ds} = -\left[\frac{(s+1)(3s^2+8s) - (s^3+4s^2)\times 1}{(s+1)^2}\right] = 0$$

or, $$2s^3 + 7s^2 + 8s = 0$$

or, $$s(2s^2 + 7s + 8) = 0$$

Therefore, $$s_1 = 0$$

and, $$s_2 = \frac{-7 \pm \sqrt{49-64}}{4} = -\frac{7}{4} \pm \sqrt{-15}$$

Breakaway at $s_2 = -\frac{7}{4} \pm \sqrt{-15}$ is not applicable.

The two root loci breakaway at $s = 0$ and become asymptotic as the value of K increases as shown in Fig. 9.28. The value of K for stability is found using Routh Array as

$$s^3 + 4s^2 + Ks + K = 0$$

$$\begin{array}{ccc} s^3 & 1 & K \\ s^2 & 4 & K \\ s^1 & \frac{4-K}{4} & 0 \\ s^0 & K & 0 \end{array}$$

For stability $K > 0$

and $$\frac{4K-K}{4} > 0, \text{ i.e. } \frac{3K}{4} > 0 \text{ or } K > 0$$

The system is therefore stable for all values of K.

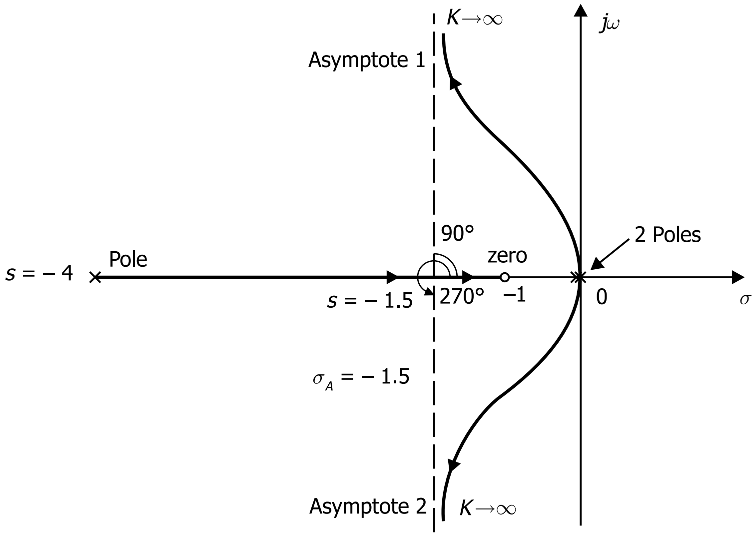


Fig. 9.28 *Root locus plot for* $G(s)H(s) = \dfrac{K(s+1)}{s^2(s+4)}$

Example 9.14 A feedback control system has an open loop transfer function

$$G(s)H(s) = \frac{K(s+4)}{s^2(s+1)}$$

Draw the root locus as K varies from zero to infinity and comment on its stability (Note the difference in T.F. of this problem with respect to the T.F. of the previous problem).

Solution

Given, $$G(s)H(s) = \frac{K(s+4)}{s^2(s+1)}$$

The number of zeros, $z = 1$ at $s = -4$; the number of poles, $p = 3$ at $s = 0, 0, -1$
There will be three root locus branches originating at 0, 0; and -1 in the s-plane. One root locus will terminate at the zero at $s = -4$. The other two will terminate at infinity.

The centroid, $$\sigma_A = \frac{-1-(-4)}{3-1} = +1.5$$

$$\phi_A = \frac{(2q+1)\times 180^\circ}{n-m} \text{ at } q = 0, 1$$
$$= 90^\circ, \text{ and } 270^\circ$$

Characteristic equation is

$$s^2 (s + 1) + K(s + 4) = 0$$

or,
$$s^3 + s^2 + Ks + 4 = 0$$

or,
$$K = -\frac{s^2(s+1)}{s+4} = -\frac{(s^3+s^2)}{(s+4)}$$

Breakaway point,

$$\frac{dK}{ds} = -\left[\frac{(s+4)(3s^2+2s)-(s^3+s^2)}{(s+4)^4}\right] = 0$$

or,
$$2s^3 + 13s^2 + 8s = 0$$

$\therefore$
$$s = 0$$

Routh Array:

$$s^3 + s^2 + Ks + 4K = 0$$

s^3	1	K
s^2	1	$4K$
s^1	$-3K$	0
s^0	$4K$	0

This shows that the system is unstable. The root locus plot has been shown in Fig. 9.29.

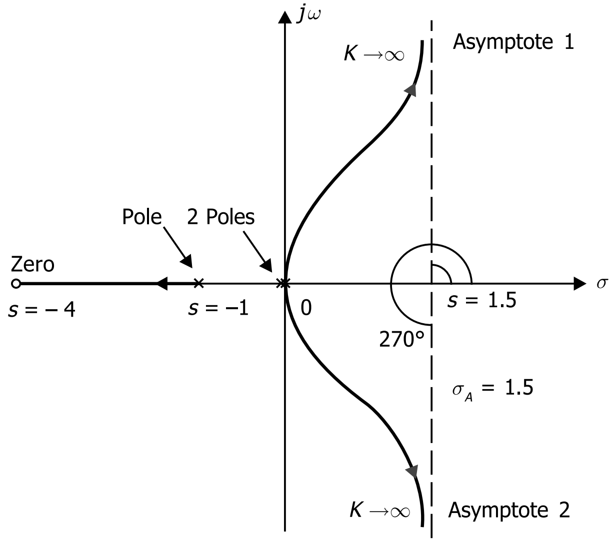


Fig. 9.29 *Root locus plot for* $G(s)H(s) = \dfrac{K(s+4)}{s^2(s+1)}$

Example 9.15 A feedback control system has open-loop transfer function

$$G(s)H(s) = \frac{K(s+30)}{(s+1)(s+10)(s+20)}$$

Draw the root locus as K varies from 0 to ∞.

Solution

On examining $G(s)H(s)$ we find

$$P = 3 \text{ at } s = -1. -10, \text{ and } -20$$

$$Z = 1 \text{ at s} = -30.$$

Number of root locus branches $= P = 3$.

Centroid, σ_A is calculated as

$$\sigma_A = \frac{\sum(-1-10-20)-\sum(-30)}{3-1}$$

$$= -0.5$$

$$\phi_A = \frac{(2q+1)180°}{n-m} \quad \text{at } q = 0,1,2$$

$$= 90°, 270°$$

Characteristic equation is

$$(s + 1)(s + 10)(s + 20) + K(s + 30) = 0$$

or, $$s^3 + 31s^2 + (230 + K)s + (200 + 30K) = 0.$$

or, $$K = -\frac{(s^3+31s^2+230s+200}{(s+30)}$$

Making $\dfrac{dK}{ds} = 0,$ breakaway point is calculated as $s = -5.2$.

Routh Array using the characteristic equation,
$s^3 + 31s^2 + (230 + K)s + (200 + 30K) = 0.$

$$\begin{array}{lll}
s^3 & 1 & 230 + K \\
s^2 & 31 & 200 + 30K \\
s^1 & \dfrac{6930+K}{31} & 0 \\
s^0 & 200 + 30K & 0
\end{array}$$

This shows that system is inherently stable for all values of K. The root locus is shown in Fig. 9.30.

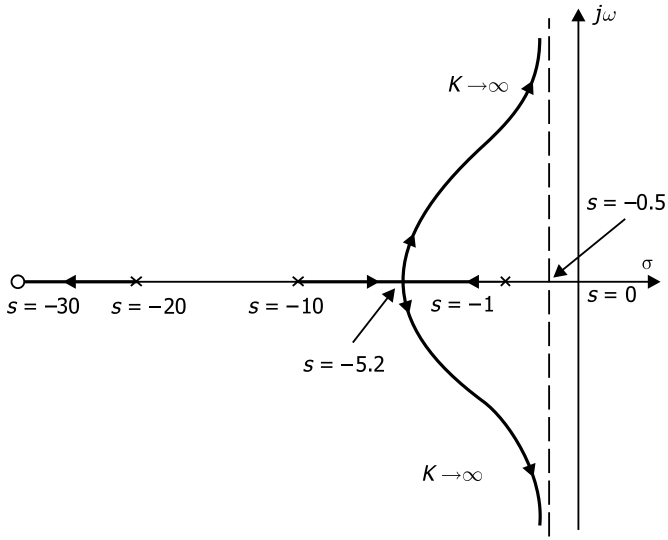


Fig. 9.30 Root locus of $G(s)H(s) = \dfrac{K(s+30)}{(s+1)(s+10)(s+20)}$

Example 9.16 Draw a complete root locus plot for a system having open-loop transfer function

$$G(s) = \frac{K}{s(s^2+4s+13)}$$

and calculate the values of the following:

i) location of poles and zeros
ii) centroid and angle of asymptotes
iii) angle of departure at poles
iv) angle of arrivals at zeros
v) breakaway points
vi) values of K and the frequency at which root-locus crosses the $j\omega$ axis.

Solution

$$G(s) = \frac{K}{s(s^2+4s+13)}$$

Step 1: Plot the poles and zeroes

$$s = 0$$

$$s = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm j3$$

Step 2: From $s = 0$ to the left is the part of the root locus.

Step 3: Number of root loci $N = P = 3$

Step 4: Centroid of the asymptotes

$$
\begin{aligned}
6A &= \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z} \\
&= \frac{0 - 2 + j3 - 2 - j3 - 0}{3 - 0} \\
&= -\frac{4}{3} = -1.33
\end{aligned}
$$

Step 5: Angle of asymptotes

$$\phi = \frac{2k+1}{P-Z} 180^\circ$$

$K = 0 \quad \phi_1 = 60^\circ$
$K = 1 \quad \phi_2 = 180^\circ$
$K = 2 \quad \phi_3 = 300^\circ$

Step 6: Breakaway point
The characteristic equation is

$$1 + G(s) + H(s) = 0$$

$$1 + \frac{K}{s(s^2 + 4s + 13)} = 0$$

$$K = -[s^3 + 4s^2 + 13s]$$

$$\frac{dk}{ds} = -[3s^2 + 8s + 13] = 0$$

$$\therefore \qquad s = \frac{-8 + \sqrt{64 - 156}}{6} = \frac{-8 \pm j9.59}{6} = -1.33 \pm j1.5$$

Since it is complex root, there will be no breakaway point on real axis.

Step 7: Point of intersection of root loci on imaginary axis of characteristic equation $s^3 + 4s^2 + 13s + K = 0$

$$
\begin{array}{lll}
s^3 & 1 & 13 \\
s^2 & 4 & K \\
s^1 & \dfrac{52 - K}{4} & \\
s^0 & K &
\end{array}
$$

For sustained oscillation,
$52 - K = 0$ or $K = 52$

Auxiliary equation

$$A(s) = 4s^2 + K$$
$$4s^2 + 52 = 0$$
$$s^2 = -13$$
$$s = \pm j3.6$$

$\therefore$ The value of K at the point of intersection on imaginary axis = 52.
The frequency at this point = 3.6 rad/sec.

Step 8: Angle of departure at the upper complex pole
$\phi_d = 180° - (124° + 90°) = -34°$

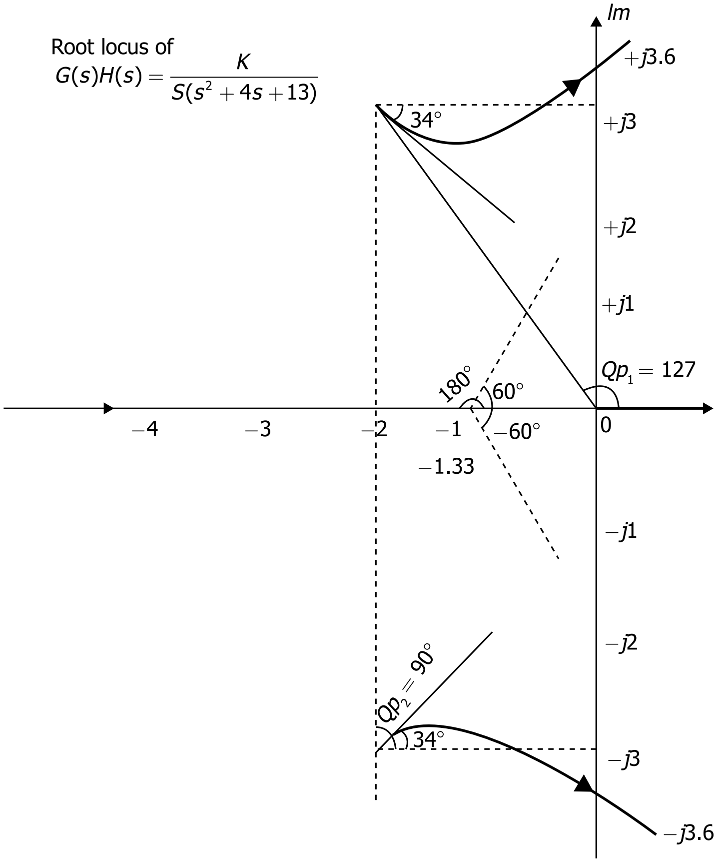


Fig. 9.31 Root locus of $G(s)H(s) = \dfrac{K}{s(s^2+4s+13)}$

Example 9.17 The open-loop transfer function of a control system is given by $G(s)\,H(s) = K/s(s+6)\,(s^2+4s+13)$

Sketch the root-locus and determine

a) the break away points
b) angle of departure from complex poles
c) the stability condition.

Solution

The root-locus is drawn following the steps given below.

Step 1: Plot poles and zeros. By examining the transfer function we all that the poles are at $0, -6, -2 \pm j3$ and zeros approach infinity.

Step 2: The segment between 0 and -6 lies on the root locus.

Step 3: Number of root loci $N = P = 4$.

Step 4: Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P - Z}$$

$$= \frac{0 - 6 - 2 + j3 - 2 + j3 - 0}{4} = -2.5.$$

Step 5: Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z} 180°$$

$K = 0 \quad \phi_1 = 45°$
$K = 1 \quad \phi_2 = 135°$
$K = 2 \quad \phi_3 = 225°$
$K = 3 \quad \phi_4 = 315°$

Step 6: Breakaway point

The characteristic equation $1 + G(s)\,H(s) = 0$

$$1 + \frac{K}{S(s+6)(s^2+4s+13)}$$

or $$s(s + 6)\,(s^2 + 4s + 13) + K = 0$$

or $$K = -(s^4 + 10s^3 + 37s^2 + 78s) = 0$$

$$\frac{dK}{dS} = -(4s^3 + 30s^2 + 74s + 78) = 0$$

$$4s^3 + 30s^2 + 74s + 78 = 0$$

By trial-and-error method $\quad s = -4.2$

1. Breakaway point $\quad = -4.2$

Step 7: Point of intersection of root loci on imaginary axis.

Characteristic equation is

$$s^4 + 10s^3 + 37s^2 + 78s + K = 0$$

Routh array

s^4	1	37	K
s^3	10	78	
s^2	29.2	K	
s^1	78	$-0.342K$	
s^0	K		

For sustained oscillations

$$78 - 0.34K = 0$$
$$K = 229.41$$

The auxillary equation

$$A(s) = 29.2s^2 + k$$
$$29.2s^2 + 229.41 = 0$$
$$s = \pm j2.8$$

Step 8: From Routh table for stability

$$K > 0$$
$$78 - 0.342K > 0 \text{ or } K < 229.41$$

Step 9: The angle of departure from upper complex pole is

$$\phi_d = 180° - (124° + 90° + 37°)$$
$$\phi_d = -70°$$

Similarly, the angle of departure from lower complex pole is $\phi_d = +71°$.
The complete root locus is shown in Fig. 9.32.

Example 9.18 The open-loop transfer function of a feedback control system is given by

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}.$$

Draw the root locus as K varies from 0 to ∞. Also calculate the value of K for which the system becomes oscillatory.

Solution

Number of open loop poles = 4. They are at $s = 0, -3, -1 \pm j\,1$. There is no open loop zero. There are four root locus branches originating at the four poles with value of $K = 0$. These branches will terminate at ∞ (i.e. open loop zeros). This is because there are no finite zeros. The four branches will tend to reach infinity with the value of K increasing towards an infinite value along asymptotic path (see Fig. 9.32).

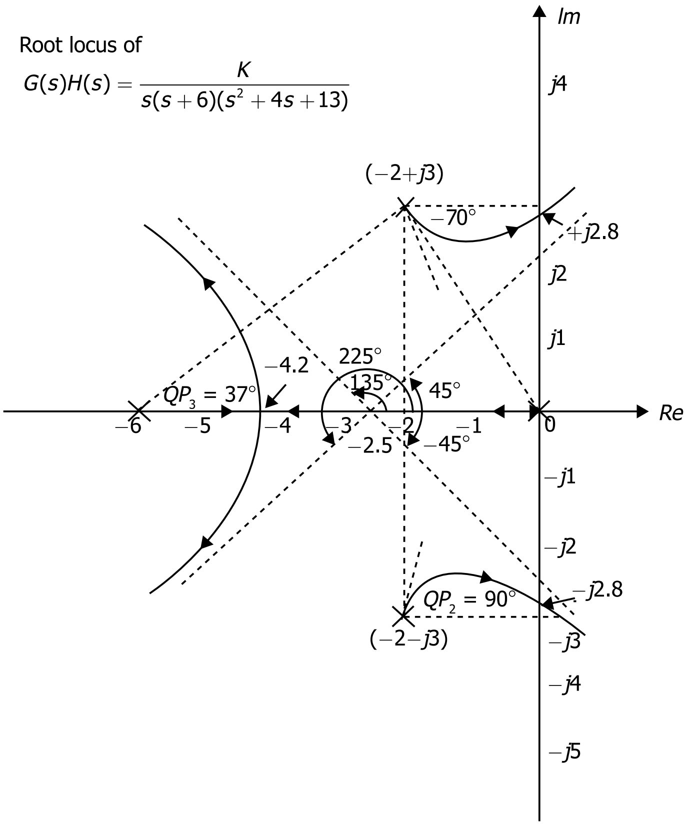


Fig. 9.32 Root locus of $G(s)H(s) = \dfrac{K}{s(s+6)(s^2+4s+13)}$

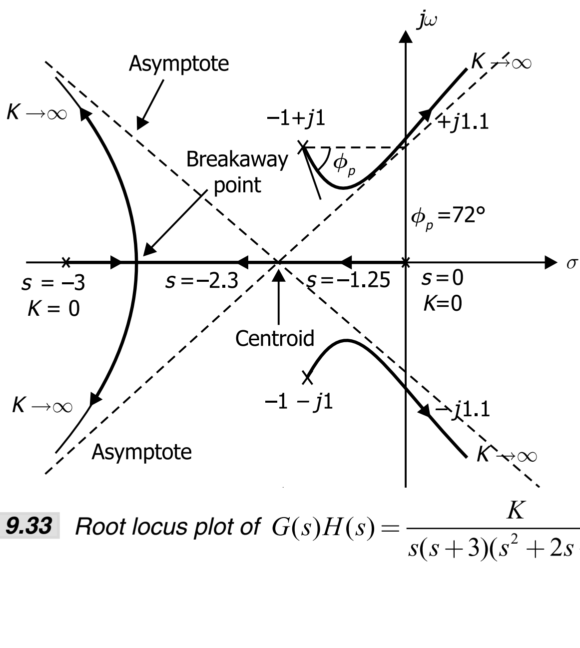


Fig. 9.33 Root locus plot of $G(s)H(s) = \dfrac{K}{s(s+3)(s^2+2s+2)}$

The angles the asymptotes with the real axis are calculated as

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \text{ at } q = 0,1,2,3.$$
$$= \frac{(2q+1)180^\circ}{4-0} = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$$

The centroid of the asymptotes is calculated as

$$\sigma_A = \frac{\sum(-3-1-1) - \sum(0)}{4-0}$$
$$= -\frac{5}{4} = -1.25$$

Therefore, the asymptotes will be originating at $s = -1.25$ on the real axis.
The breakaway point or points are calculated using the characteristic equation, which is $1 + G(s)\,H(s) = 0$

or,
$$\frac{K}{s(2+3)(s^2+2s+2)} = -1$$

or,
$$K = -s\,(s+3)\,(s^2+2s+2)$$
$$= -\,(s^4 + 5s^3 + 8s^2 + 6s)$$

To calculate s, we have to make $\frac{dK}{ds} = 0$

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 16s + 6) = 0$$
$$s = -2.3, -0.725 \pm j0.365$$

The breakaway point must be at $s = -2.3$ as it lies on the real axis. The two root locus branches originating at $s = 0$ and $s = -3$ approach each other on the real axis and breakaway at $s = -2.3$. The root locus plot is shown in Fig. 9.33.

For calculating the value of K at which the root locus intersects the imaginary axis and the system becomes oscillatory, we start with the characteristic equation and the Routh Array as: Characteristic equation is

$$s(s+2)\,(s^2+2s+2) + K = 0$$

or,
$$s^4 + 5s^3 + 8s^2 + 6s + K = 0$$

Applying Routh criterion,

s^4	1	8	K
s^3	5	6	0
s^2	34/5	K	0
s^1	$\dfrac{\left(\dfrac{204/5-5K}{5}\right)}{34/5}$	0	0
s^0	K	0	0

For stability, $0 < K < 8.16$.

At a value of $K = 8.16$, the two root loci intersects the $j\omega$ axis.

The point of intersection is calculated from the auxiliary equation formed from the coefficients of s^2 row when $K = 8.16$ as

$$\frac{34}{5}s^2 + K = 0$$

or,
$$s^2 = -\frac{K}{34/5} = -\frac{8.16\times 5}{34} = -\frac{40.8}{34} = -1.21$$

$\therefore$
$$s = \pm j\ 1.1$$

The angle of departure of the two root loci originating at $s = -1 \pm j\ 1$ is calculated as (see Fig. 9.34):

$$\theta_p = 180° - \phi$$

where ϕ is the angle contribution by other poles to this pole.

$$\theta_p = 180° - (90° + 135° + 27°) = -72°.$$

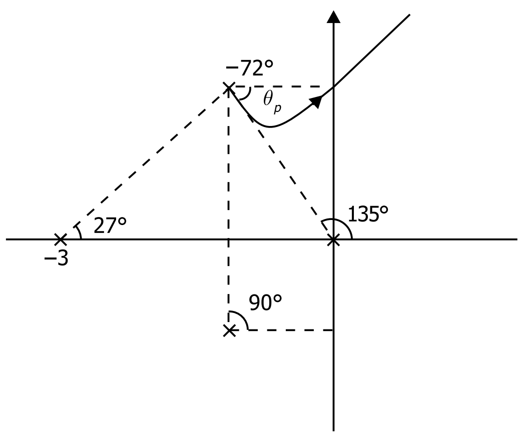


Fig. 9.34

Similarly the angle of departure of the root locus from the other pole at $(-1\ -j1)$ can be calculated as to be $+70°$.

Example 9.19 Find the roots of the following polynomial by root locus method.

$$3s^4 + 10s^3 + 21s^2 + 24s - 16 = 0.$$

Solution

We will rearrange the polynomial into $\dfrac{P(s)}{Q(s)}$ form in the following way.

$$3s^4 + 10s^3 + 21s^2 = -24s + 16 = -(24s - 16)$$

or,
$$\frac{24s-16}{3s^4+10s^3+21s^2} = -1$$

or
$$\frac{8\left(s-\dfrac{2}{3}\right)}{s^2(3s^2+10s+21)\times\dfrac{1}{3}} = -1$$

Let us assume $K = 8$

Therefore,
$$\frac{K(s-2/3)}{s^2\left(s^2+\dfrac{10}{3}s+7\right)} = -1$$

This is of the form, $G(s)\,H(s) = -1$

Number of open-loop poles $= 4$ at $s = 0, 0, -1.67 \pm j2.06$

Number of zeros is equal to 1 at $s = 0.67$

Number of branches of the root locus $= 4$

Number of asymptotes $= P - Z = 4 - 1 = 3$

Centroid,
$$\sigma_A = \frac{\Sigma(-1.67-1.67)-\Sigma(-0.67)}{3}$$
$$= -1.33$$

Angle of asymptotes,
$$\phi_A = \frac{(2q+1)180°}{3} = 60°, 180°, 300°.$$

Angle of departure of the root locus from $-1.67 \pm j\,2.06$ is calculated as

$$\theta_p = \pm\,[180° - \Sigma \text{ angles from other poles} + \Sigma \text{ angles from zeros}]$$
$$= \pm\,[180° - (90° + 128° + 128°) + 138°] = \pm\,30°$$

To calculate the value of k when the root locus intersects the imaginary axis and the points of intersection is calculated using the characteristic equation as

$$s^4 + \frac{10}{3}s^3 + 7s^2 + Ks - \frac{2}{3}K = 0$$

The Routh Array is shown below.

$$
\begin{array}{llll}
s^4 & 1 & 7 & -2/3K \\
s^3 & 10/3 & K & 0 \\
s^2 & \left(7-\dfrac{3}{10}K\right) & -\dfrac{2}{3}K & 0 \\
s^1 & \dfrac{\left(-\dfrac{3}{10}K^2+\dfrac{83}{9}K\right)}{\left(7-\dfrac{3}{10}K\right)} & 0 & 0 \\
s^0 & -\dfrac{2}{3}K & 0 & 0
\end{array}
$$

The value of K is calculated from s^1 term as

$$-\frac{3}{10}K^2+\frac{83}{9}K=0$$

or,
$$K=30.7$$

Condition for stability $0<K<30.7$

The value of s for $K=30.7$ is calculated from the auxiliary equation as

$$\left(7-\frac{3}{10}K\right)s^2-\frac{2}{3}K=0$$

or
$$s=\pm j\,3.04.$$

The root locus has been drawn using the data obtained as shown in Fig. 9.35.

The above solution has been done assuming $K=8$. We have to by trial and error locate the points on the root loci where $K=8$ and from there the roots can be located. From the root locus the roots are found as $-\ 2.25$, $-0.75\pm j\,2.15$ for value of $K=8$ as shown by using dots on the root locus.

Example 9.20 The simplified form of open-loop transfer function of an aeroplane with an auto-pilot is represented as

$$G(s)H(s)=\frac{K(s+a)}{s(s-b)(s^2+2\xi\omega_n s+\omega_n^2)}$$

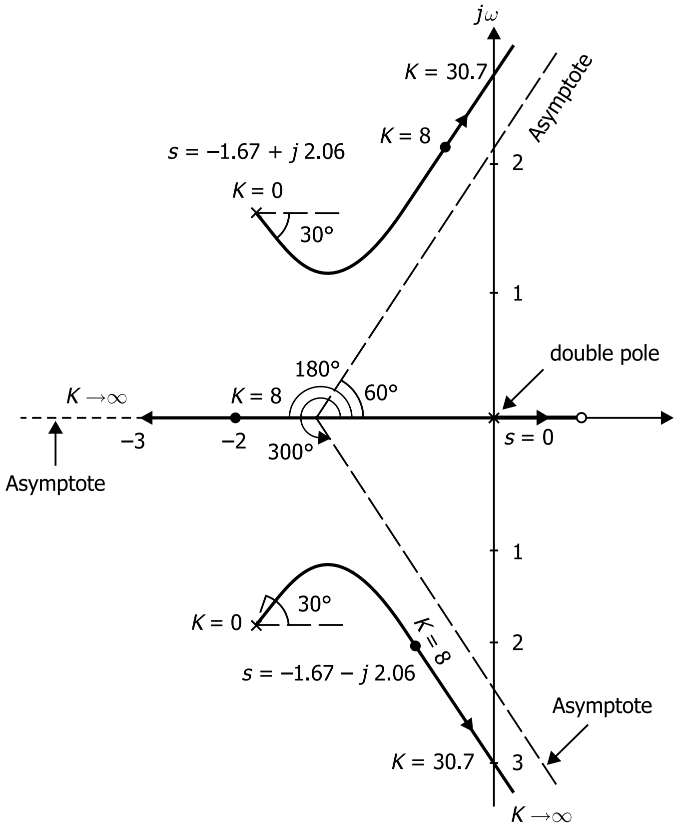


Fig. 9.35 *Root locus plot of a system whose characteristic equation is $3s^4 + 10s^3 + 21s^2 + 24s - 16 = 0$*

Sketch the root locus plot when $a = b = 1$, $\xi = 0.5$ and $\omega_n = 4$. For what value of K the system will remain stable?

Solution

The open loop transfer function after putting the values of a, b, ξ and ω_n is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

Number of open loop poles $= 4$ at $s = 0, +1, -2 \pm j3.5$, Number of zeros $= 1$ at $s = -1$

Number of root locus branches = No. of poles = 4

Number of asymptotes = No. of poles − No. of zeros

$= 4 - 1 = 3.$

Angles of asymptotes with the real axis,

$$\phi_A = \frac{(2q+1)180°}{3-1} q = 0,1,2$$

$$= 60°, 180°, 300°.$$

The asymptotes intersect on the real axis at the centroid, σ_A

$$\sigma_A = \frac{\sum(0+1-2-2)-(-1)}{4-1}$$

$$= -\frac{2}{3} = -0.67$$

The root locus diagram with the position of poles and zeros and the asymptotes are shown in Fig. 9.36.

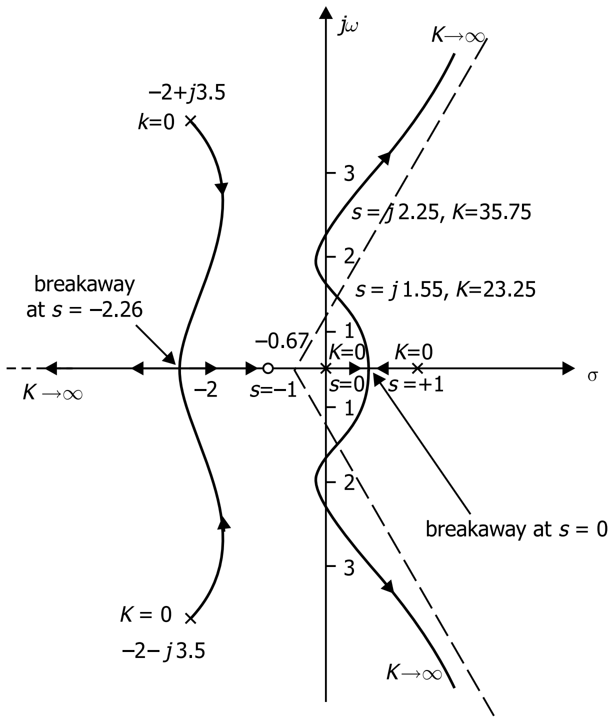


Fig. 9.36 *Root locus plot of* $G(s)H(s) = \dfrac{K(s+a)}{s(s-6)(s^2+2\zeta w_n s + w_n^2)}$

We will now find breakaway points using the characteristic equation which is $s\,(s-1)\,(s^2+4s+16)+k\,(s+1)=0$

or
$$K = -\frac{s(s-1)(s^2+4s+16)}{(s+1)}$$

$$\frac{dK}{ds} = -\frac{(3s^4+10s^3+21s^2+16)}{(s+1)^2} = 0$$

From which
s_1, s_2, s_3, s_4 are calculated as equal to 0.45, – 2.26, – 0.76 + j2.16 and – 0.76 – j2 – 16 respectively.
Breakaway points are on the real axis and are at 0.45 and – 2.26.
Now we locate the breakaway points on Fig. 9.35.
Now let us determine the value of K for which the root locus intersects the imaginary axis.
The characteristic equation is

$$s^4 + 3s^3 + 12s^2 + (K - 16)\,s + K = 0$$

Routh Array is presented as

$$\begin{array}{llll} s^4 & 1 & 12 & K \\ s^3 & 3 & K-16 & 0 \\ s^2 & \dfrac{52-K}{3} & K & 0 \\ s^1 & \dfrac{(-K^2+59K-832)}{(52-K)} & 0 & 0 \\ s^0 & k & 0 & 0 \end{array}$$

The value of K at the intersection on the imaginary axis by the root locus is calculated as $\dfrac{(-K^2+59K-832)}{52-K} = 0$

or $$K^2 - 59K + 832 = 0$$

$$K = \frac{59 \pm \sqrt{(59)^2 - 4\times 432}}{2}$$

$$= \frac{59 \pm \sqrt{153}}{2} = 35.75, 23.25.$$

The shows that the system is stable for

$$23.25 < K < 35.75$$

For calculating the crossing points on the imaginary axis, we will use the auxiliary equation from Routh array of s^2 as

$$\frac{(52-K)s^2}{3} + K = 0.$$

or $$s = \pm\, j\,2.25 \text{ for } K = 35.75$$

and $$s = \pm\, j\,1.55 \text{ for } K = 23.25$$

The complete root locus diagram can now be plotted as shown in Fig. 9.36.

9.6 EFFECTS OF ADDING POLES AND ZEROS TO G(S) H(S)

Often the desired performance specifications of a closed-loop system is not met by merely adjusting the gain parameter. Once the effects on the root locus of addition of poles and/or zeros are fully understood, we can readily determine the poles and zeros of a compensator that will reshape the root locus to meet the desired performance specifications.

The addition of a pole to the open loop transfer function has the effect of shifting the root locus to the right, thereby decreasing relative stability and increasing settling time (slow). This effect may be compared with that of *integral control* which makes the system less stable by adding a pole at the origin.

Fig. 9.37(b) and (c) shows respectively the effects of adding one pole and two poles to the single-pole system of Fig. 9.37(a). The root locus gradually bends towards right.

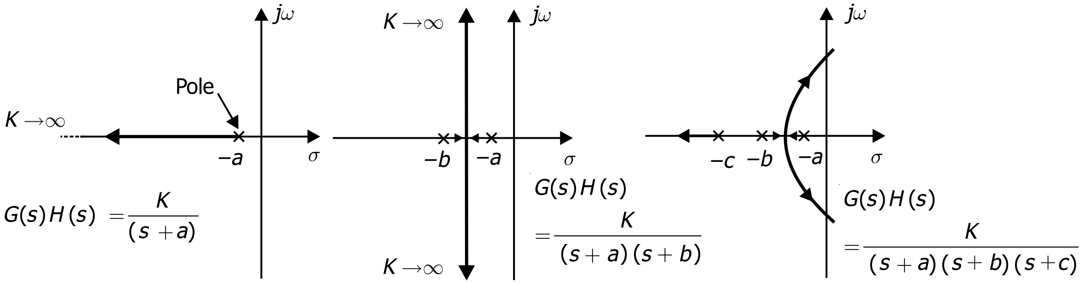
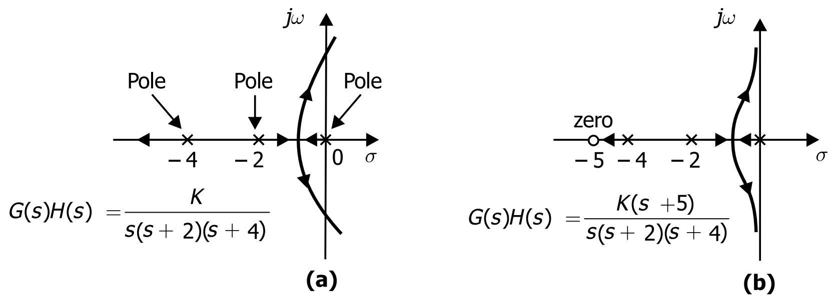


Fig. 9.37 *Root locus plots of single, double and triple pole systems*

The addition of a zero to the open-loop transfer function has the effect of shifting the root locus to the left, thereby increasing stability and decreasing settling time (fast). This effect may be compared with that of *derivative control* which makes the system fast and more anticipatory by adding a zero in the feed forward transfer function. Fig. 9.38(a) shows the root locus of a system which is unstable for large gain. Fig. 9.38(b), (c) and (d) show root locus plots for different locations of a zero added to the open-loop transfer function. Here it is evident that the inclusion of a zero makes the system of Fig. 9.38(a) stable for all values of gain.

The effect of adding poles and zeros to $G(s)$ $H(s)$ can be understood by solving few problems.



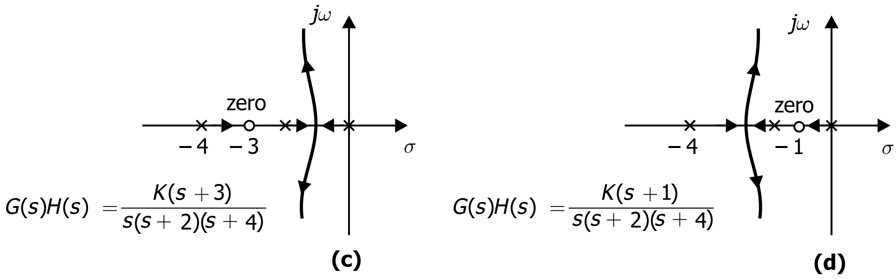


Fig. 9.38 *Root locus plots for different locations of a zero added to the three pole system in (a)*

9.7 ROOT-LOCUS PLOT WITH MATLAB

For plotting root loci with MATLAB, the characteristic equation must be written in the form given by

$$1 + K\frac{\text{num}}{\text{den}} = 0$$

where num and den are the numerator and denominator polynomials respectively.

A commonly used MATLAB command of plotting root locus is "rlocus (num, den)" with the gain vector K automatically determined. For plotting root loci with marks '0' or 'x', the commands to be followed are

r = rlocus (num, den);

plot (r, '0') or plot (r, 'x');

MATLAB uses the automatic axis scaling feature of plot command.

For manual axis scaling, the curve plotting limits are to be specified by

v = [x-min x-max y-min y-max]

'axis (v)' is the command to be entered next. Typing 'axis' resumes auto scaling. Entering 'axis square')' command sets the plotting region to square. Entering axis ('normal') will get back into normal mode. The common axis ('square') will set a line with slope 1 at a true 45°, not skewed by irregular shape of the screen.

Example 9.21 For a unity feedback control system with $G(s) = \dfrac{K(s+1)}{s(s-1)(s^2+4s+16)}$ plot the root loci with a square aspect ratio so that a line with slope 1 is a true 45° line.

Solution

The following program will generate the plot as shown in Fig. 9.39.

MATLAB PROGRAM 9.1

```
% – – – — – – Root-locus plot – – – – –
num = [0   0   0   1   1];
den = [1   3   12   –16   0];
rlocus (num, den)
v = [–6   6   –6   6]; axis (v); ('square')
grid
title ('Root-locus plot of G(s) = K(s + 1)/[s(s – 1)(s2 + 4s + 16)])
```

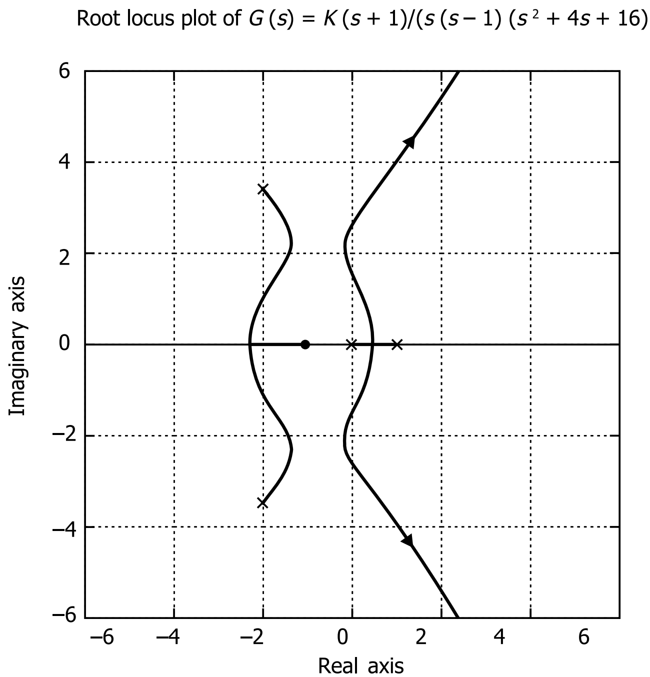


Fig. 9.39 *Root locus plot*

Note that the hand-copy plot may not be a square one as it depends on the printer.

Example 9.22 Using MATLAB, plot the root loci and their asymptotes for the system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}.$$

Solution

The open-loop transfer function may be written as

$$G(s)H(s) = \frac{K}{s^3 + 3s^2 + 2s}.$$

It may be noted that

$$= \lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 2s}$$

$$= \lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 3s + 1}$$

$$= \lim_{s\to\infty} \frac{K}{(s+1)^3}$$

So the equation for the asymptotes may be given by

$$G_a(s)H_a(s) = \frac{K}{(s+1)^3}.$$

The required MATLAB program is as follows.

MATLAB PROGRAM 9.2

```
% – – – — – – Root-locus plot – – – – –
% * * for the given system**
num = [0   0   0   1];
den = [1   3   2   0];
% ** for the asymptotes **
num a = [0   0   0   1];
den a = [1   3   3   1];
% *** gain constant K is included in the command to ensure that
% number of rows of r and that of a are the same***
K1 = 0: 0.1: 0.3;
K2 = 0.3: 0.005: 0.5;
K3 = 0.5: 0.5: 10;
K4 = 10: 5: 100;
K = [K1 K2 K3 K4];
r = rlocus (num, den, K);
a = rlocus (num a, den a, K);
y = [r a];
plot (y, '-')
v = (–4 4 –4 4]; axis (v)
grid
title ('Root-locus plot of G(s) = K/[s(s + 1)(s + 2] and asymptotes')
x label ('Real Axis')
y label ('Imng Axis')
```

The above program will generate the root locus plot of Fig. 9.40.

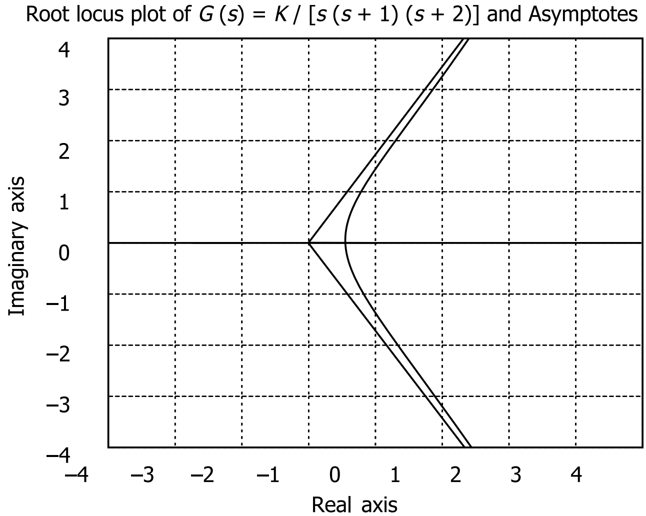


Fig. 9.40 *Root locus plot*

REVIEW QUESTIONS

9.1 Use the Routh-Hurwitz criterion to determine the stability of the systems with characteristic equations as follows. Also find the number of roots of each equation that lie on the right half of the s-plane.

i) $s^3 + 20s^2 + 9s + 200 = 0$

ii) $3s^4 + 10s^3 + 5s^2 + s + 2 = 0$

iii) $s^5 + 2s^4 + 2s^3 + 4s^2 + s + 1 = 0$

iv) $s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$

9.2 Determine the values of K so that the systems with the following characteristic equations will be stable.

i) $s^3 + as^2 + Ks + b = 0$, where a, b are constants.

ii) $s^4 + 20Ks^3 + 5s^2 + (10 + K)s + 15 = 0$

iii) $s(2s + 1)(3s + 1) + K(s + 1) = 0$

iv) $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

9.3 Derive the stability conditions which must be satisfied by the coefficients of the following characteristic equation.

$a_3s^3 + a_2s^2 + a_1s + a_0 = 0$, where $a_3 > 0$

9.4 A unity feedback control system has the forward transfer function

$$G(s) = \frac{K(2s+1)}{s(4s+1)(s+1)^2}.$$

Determine the stability condition of the system for the minimum value of K obtained from $e_{ss} \leq 0.1$ and $r(t) = 1 + t$.

9.5 Determine the stability condition for each of the systems with loop transfer functions as stated below:

i) $G(s)H(s) = \dfrac{1}{(s+2)(s+4)}$.

ii) $G(s)H(s) = \dfrac{Ks}{(s+1)^2}$.

iii) $G(s)H(s) = \dfrac{K(s-1)}{s(s^2+4s+4)}$.

iv) $G(s)H(s) = \dfrac{K(s+2)}{s(s+3)(s^2+2s+3)}$.

9.6 The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K(s+5)(s+40)}{s^3(s+200)(s+1000)}.$$

Discuss the stability of the closed-loop system in terms of K. What value of K will cause sustained oscillation? Find the frequencies of oscillations.

9.7 A certain system has the characteristic equation

$$s(Ts + 1) + K = 0$$

It is desired that all the roots lie to the left of the line $s = -a$ to guarantee a time constant of not larger than $1/a$. Use the Routh-Hurwitz criterion to find the values of K and T so that both roots meet this requirement.

9.8 Given a system with a characteristic equation

$$s^3 + 9s^2 + 26s + K = 0$$

Determine the value of K that will give a dominant time constant not larger than 0.5. The design of the system must be slightly underdamped. Is it possible? If yes, find the value of K needed and the resulting damping ratio.

9.9 $G(s) = K/(s + 1)\ (s + 2)$ is given for a two-input two-output control system as shown in the signal flow graph below.
Use the Routh-Hurwitz criterion to find the range of K (positive constant) so that the system is asymptotically stable.

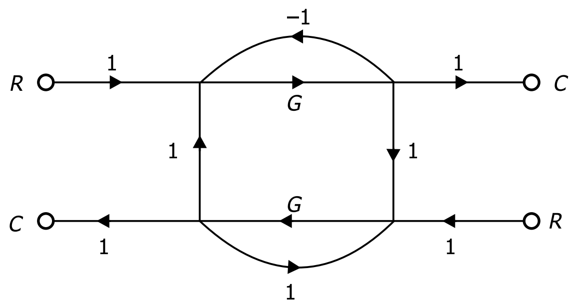


Fig. 9.41

9.10 A control system as shown below has a transfer function,

$$G(s) = \frac{K}{s(1+0.1s)(1+0.2s)}$$

and its amplifier exhibits a non-linear saturation characteristic. Determine the maximum value of K that keeps the whole system absolutely stable.

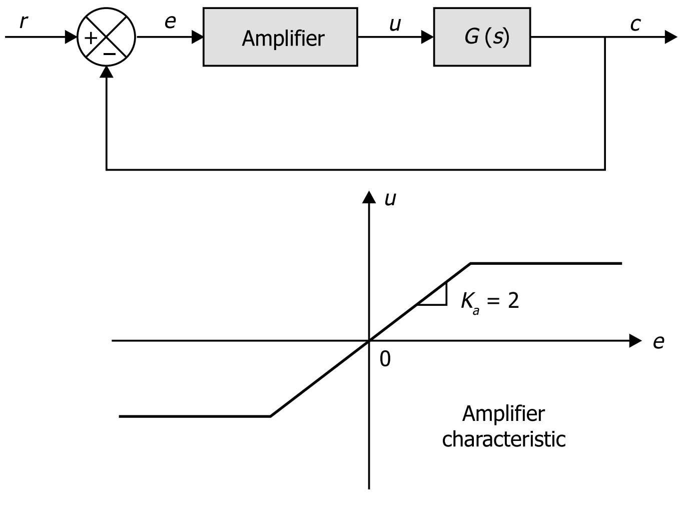


Fig. 9.42

9.11 Plot a root locus diagram with MATLAB for a system the open-loop transfer function of which is given by

$$G(s)H(s) = \frac{K}{(s^2+2s+2)(s^2+2s+5)}$$

9.12 With the help of a suitable example explain the effect of adding poles and zeros to the open loop transfer function of a control system.

9.13 The forward path transfer function of a unity feedback system is given below. Construct the root locus diagram for all values of gain K. Find the value of K at all the breakaway points.

9.14 Obtain $G(s)\,H(s)$ from the equation given below and draw the root locus for all values of K

$$s(s+4)\,(s^2+4s+20)+K=0$$

9.15 Draw the root locus for a system having

$$G(s)H(s)=\frac{K}{s(s+2)(s^2+2s+2)}$$

Determine the value of K for which the system is critically damped.

9.16 Obtain the root locus of a unity feedback system with $G(s)=\dfrac{K(s+4)}{s^2+2s+2}$.

9.17 Sketch the root locus of the system whose characteristic equation is given as $s^4+6s^3+8s^2+Ks+K=0$.

[Hint: Express the characteristic equation in the form, $1+G(s)\,H(s)=0$

or, $\quad 1+\dfrac{K(s+1)}{s^4+6s^3+8s^2}$

Thus, $\quad G(s)H(s)=\dfrac{K(s+1)}{s^2(s+2)(s+4)}$.

10

FREQUENCY RESPONSE ANALYSIS

10.1 INTRODUCTION

In the previous chapters we have studied the response of a system to step input, ramp input, etc. Since most of the input signals in practice are sinusoidal inputs, in this chapter; we will consider the steady state response of a system to a sinusoidal input test signal. The response will also be sinusoidal for a linear constant coefficient system once the transient die out. The output sinusoidal signal will also have the same frequency as the input signal. However, the magnitude and phase of the output will differ from the input signal once the frequency of the input signal is changed. *The frequency response analysis deals with the study of steady state response of a system to sinusoidal input of variable frequency*.

Thus we can define the frequency response of a system as the steady-state response of the system to a variable frequency sinusoidal input signal.

Analysis involves examining the transfer function $G(s)$ when $s = j\omega$ and graphically displaying $G(j\omega)$ as ω varies.

As mentioned, for analysis of many systems frequency response is of importance since most of the input signals are either sinusoidal or composed of sinusoidal components (harmonics). The starting point for frequency response analysis is the determination of system transfer function. Then in the frequency response the transfer function is expressed in terms of magnitude and phase angle as $M(s) = M\underline{|\phi}$.

Let us, consider the transfer function of a first order system as an example.

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{1+s\tau} \quad \text{or} \quad C(s) = \frac{1}{1+s\tau}R(s) \qquad \text{....(10.1)}$$

For frequency response analysis we replace s by $j\omega$.

Therefore, $$G(j\omega) = \frac{1}{1 + j\omega\tau}$$

The system is to be subjected to sinusoidal input and therefore, we take $R(s) = A_i \sin \boldsymbol{\omega} t$

From equation (10.1), the output $C(j\omega) = \dfrac{A_i \sin \omega t}{1 + j\omega\tau}$

$$\text{Magnitude of output, } A_0 = |C(j\omega)| = \frac{A_i}{\sqrt{1 + \omega^2\tau^2}}$$

$$\text{Magnitude of input} = A_i$$

The dimensionless ratio of output to input is given as

$$M = \left|\frac{A_0}{A_i}\right| = \frac{1}{\sqrt{1 + \omega^2\tau^2}} \qquad \text{....(10.2)}$$

and the phase angle

$$\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}\omega\tau \qquad \text{....(10.3)}$$

$$\text{Time lag} = \frac{1}{\omega}\tan^{-1}\omega\tau$$

Now let us plot M versus $\boldsymbol{\omega}$ and ϕ versus $\boldsymbol{\omega}$.

When $$\omega = 0,\ M = \frac{A_0}{A_i} = 1 \text{ and } \phi = 0$$

When $$\omega = \infty,\ M = \frac{A_0}{A_i} = 0 \text{ and } \phi = -90°$$

As $\boldsymbol{\omega}$ increases from 0 to ∞, the magnitude gradually decreases from 1 to 0 and the angle of lag, ϕ increases from 0 to $-90°$. Thus higher the frequency, higher is the attenuation (decay) of the output and greater is the angle of lag between output and input. The frequency response characteristic of a first order system has been shown in Fig. 10.1.

Let us again consider the first order system described earlier. The transfer function is $G(s)$

$$G(j\omega) = R(\omega) + jX(\omega)$$

where $$R(\omega) = Re[G(j\omega)] \text{ and } X(\omega) = I_m[G(j\omega)]$$

we had $$G(s) = \frac{C(s)}{R(s)} = \frac{1}{1 + s\tau}$$

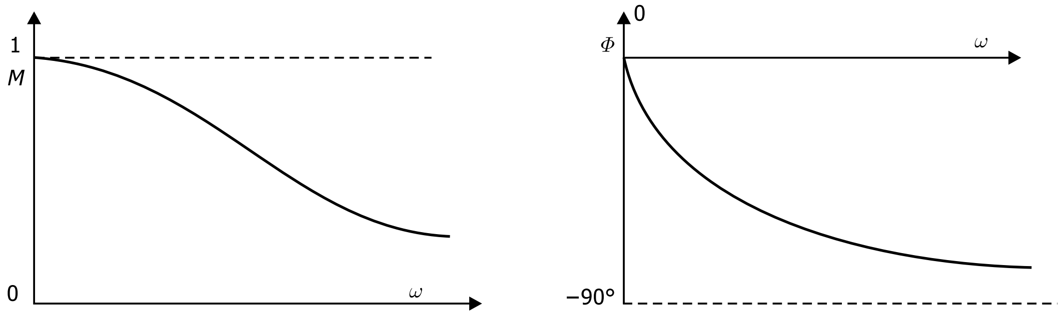


Fig. 10.1 *Frequency response of a first order system*

The polar plot is obtained from the relation

$$G(j\omega) = \frac{1}{1 + j\omega\tau}$$

After multiplying both numerator and denominator by $(1 - j\omega\tau)$ we write

$G(j\omega)$ as $\qquad G(j\omega) = R(\omega) + j\,X(\omega)$

$$= \frac{1 - j\left(\dfrac{\omega}{\omega_1}\right)}{1 + \left(\dfrac{\omega}{\omega_1}\right)^2} \qquad \text{where } \omega_1 = \frac{1}{\tau}$$

$$= \frac{1}{1 + \left(\dfrac{\omega}{\omega_1}\right)^2} - j\frac{\left(\dfrac{\omega}{\omega_1}\right)}{1 + \left(\dfrac{\omega}{\omega_1}\right)^2} = R(\omega) + jX(\omega)$$

To draw the polar plot we will determine $R(\omega)$ and $X(\omega)$ at $\omega = 0$ and $\omega = \infty$.
Again at $\omega = \omega_1$, the real part $R(\omega)$ is equal to the imaginary part $X(\omega)$, i.e. $\phi(\omega) = 45°$ as shown in Fig. 10.2. Thus, plot in Fig. 10.2 is a polar plot for a first order system.
Let us consider polar plot of another system, the transfer function of which is given as

$$G(s) = \frac{K}{s(s\tau + 1)}$$

Here a pole has been added to the first order system represented by

$$G(s) = \frac{1}{1 + s\tau}$$

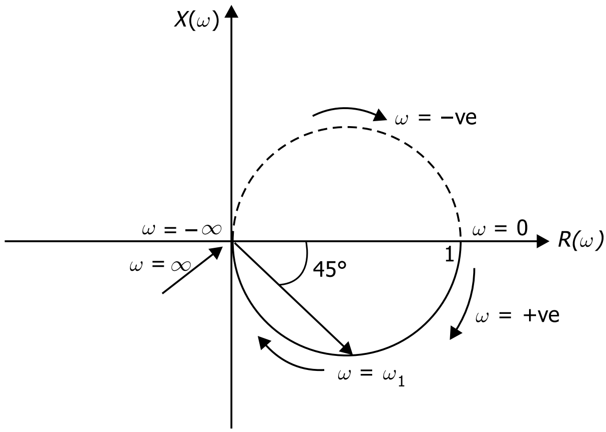


Fig. 10.2 *Polar plot for a first order system (e.g. a RC filter)*

$$\text{Given } T.F. = G(s) = \frac{K}{s(s\tau + 1)}$$

Putting $s = j\omega$,

$$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau} \quad (\because j^2 = -1)$$

The magnitude,

$$|G(j\omega)| = \frac{K}{\sqrt{\omega^2 + \omega^4\tau^2}}$$

and phase angle,

$$\phi(\omega) = -\tan^{-1}\left(\frac{1}{\omega\tau}\right)$$

The magnitude and phase angle at $\omega = 0$, $\omega = \dfrac{1}{2\tau}$, $\omega = \dfrac{1}{\tau}$, and $\omega = +\infty$ are calculated as:

For $\omega = 0$, $|G(j\omega)| = \infty$ and $\phi(\omega) = -90°$

For $\omega = \dfrac{1}{2\tau}$, $|G(j\omega)| = \dfrac{4K\tau}{\sqrt{5}}$ and $\phi(\omega) = -117°$

For $\omega = \dfrac{1}{\tau}$, $|G(j\omega)| = \dfrac{K\tau}{\sqrt{2}}$ and $\phi(\omega) = -135°$

For $\omega = \infty$, $|G(j\omega)| = 0$ and $\phi(\omega) = -180°$

The polar plot can now be drawn using the above values as shown in Fig. 10.3.

(The limitation of polar plot is that in case of adding poles or zeros to an existing system, we have to recalculate the values.)

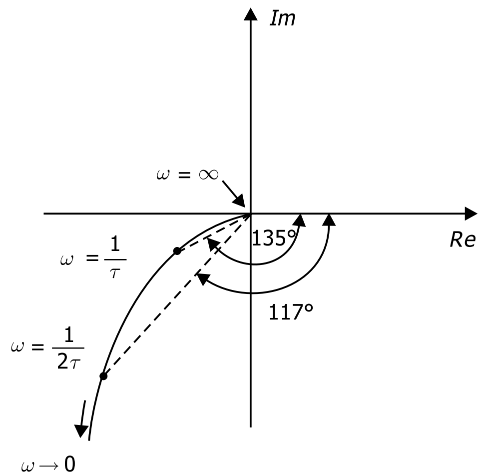


Fig. 10.3 Polar plot $G(j\omega) = \dfrac{K}{j\omega(j\omega\tau + 1)}$

By comparing Figs. 10.2 and 10.3 we observe that due to addition of a pole the polar plot gets a clockwise rotation by 90° as ω tends to infinity. **Nyquist** in 1932 related study of polar plots with system stability.

10.2 FREQUENCY RESPONSE SPECIFICATIONS

The frequency response specifications are the following:

a) Resonant peak M_r: This is the maximum value of M and is denoted as M_r. The magnitude of resonant peak M_r provides us information about the relative stability of the system. Large resonant peak corresponds to large overshoot in the transient response.

b) Resonant frequency ω_r: It is the frequency at which resonant peak occurs, that is, the maximum value of M_r occurs. High value of ω_r indicates that the time response of the output is faster as peak time is inversely proportional to ω_r.

c) Bandwidth BW: This is the range of frequencies in which the magnitude M expressed in dB does not drop more than -3 dB. As shown in Fig. 10.4, it is the band of frequencies at -3 dB points.

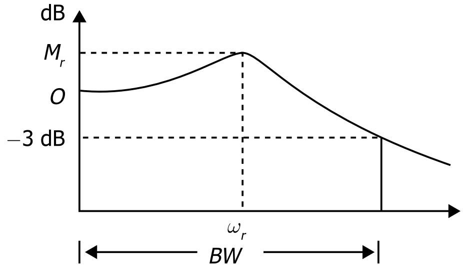


Fig. 10.4 *Frequency response specifications*

10.3 CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

For comparative study of time response and frequency response of a system, let us consider a second-order system. We had seen earlier that the transfer function of a second-order system can be expressed as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ is the damping ratio and ω_n is the undamped natural frequency of oscillations.
For the sinusoidal transfer function, we will put $s = j\omega$ in the above expression.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega}$$

Dividing by ω_n^2,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}}$$

Let $\dfrac{\omega}{\omega_n} = u$, then

$$M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1-u^2) + j2\zeta u} = M\angle\phi$$

$$|M(j\omega)| = M = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} \qquad \text{....(10.4)}$$

$$\underline{|M(j\omega)} = \tan^{-1}\frac{2\zeta u}{1-u^2} \qquad \text{....(10.5)}$$

From equations (10.4) and (10.5), it is seen that
if $u = 0$, $M = 1$ and $\phi = 0$

$$\text{if} \quad u = 1, M = \frac{1}{2\zeta} \text{ and } \phi = -\frac{\pi}{2}$$

if $u \to \infty$, $M \to 0$ and $\phi = -\pi$

As u changes from 0 to ∞, M changes from 1 to 0 and ϕ changes from 0 to $-\pi$.

The frequency at which M has maximum value is called the resonant frequency ω_r.

Let, $u_r = \omega_r/\omega_n$ where u_r is called the normalized resonant frequency.

We will differentiate M with respect to u and substitute $u = u_r$ and then equate to zero.

$$\frac{d}{du}\left[\left(1-u_r^2\right)^2 + (2\zeta u_r)^2\right]^{-1/2} = 0$$

or
$$-\frac{1}{2}\left[\left(1-u^2\right)^2 + (2\zeta u)^2\right]^{-3/2}\left[-4u + 4u^3 + 8\zeta^2 u\right] = 0$$

or
$$-\frac{4u^3 - 4u + 8\zeta^2 u}{\frac{1}{2}\left[(1-u^2) + (2\zeta u)^2\right]^{3/2}} = 0$$

Substituting $u = u_r$,

$$4u_r^3 - 4u_r + 8\zeta^2 u_r = 0$$

or
$$4u_r^3 = 4u_r - 8\zeta^2 u_r$$
$$4u_r^3 = 4u_r(1 - 2\zeta^2)$$
$$u_r^2 = 1 - 2\zeta^2$$
$$u_r = \sqrt{1-2\zeta^2}$$

or
$$\frac{\omega_r}{\omega_n} = u_r = \sqrt{1-2\zeta^2}$$

or

$$\omega_r = \omega_n \sqrt{1-2\zeta^2} \qquad \text{….(10.6)}$$

From equations (10.4) and (10.6), we have

$$M_r = \frac{1}{\sqrt{\left(1-u_r^2\right)^2 + (2\zeta u_r)^2}}$$

$$M_r = \frac{1}{\sqrt{[1-(1-2\zeta^2)]^2 + [4\zeta^2(1-2\zeta^2)]}}$$

$$M_r = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}}$$

$$M_r = \frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}} = \frac{1}{\sqrt{4\zeta^2(1-\zeta^2)}}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{….(10.6a)}$$

From equation (10.5),

$$\angle M(j\omega) = \phi = \tan^{-1}\frac{2\zeta u_r}{1-{}_r^2} = \tan^{-1}\frac{2\zeta\sqrt{1-2\zeta^2}}{1-1+2\zeta^2}$$

or

$$\phi = \tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}. \qquad \text{….(10.6b)}$$

The characteristics of magnitude M and phase angle ϕ for normalized frequency u for some value of ζ have been shown in Fig. 10.5.

10.3.1 Correlation Between Time Domain and Frequency Domain Parameters

We had calculated time domain specification parameters like maximum overshoot M_p, peak time t_p, rise time t_r, settling time t_s, etc. M_p is calculated as

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

and in the frequency domain resonant peak M_r has been calculated as

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

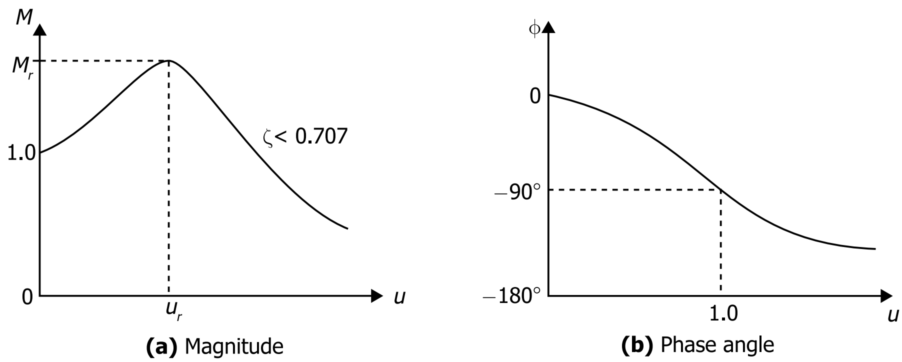


Fig. 10.5 *Frequency response magnitude and phase angle characteristics*

It is observed that both M_p and M_r are the functions of damping ratio ζ.

As ζ is increased, the value of maximum overshoot M_p goes on decreasing. When ζ is made equal to 1, the overshoot disappears, that is, no overshoot is produced by the system response. In the frequency domain, resonant peak M_r will disappear when $\zeta > 1/\sqrt{2}$, that is, $\zeta > 0.707$.

For lower value of ζ both M_p and M_r will be large which is undesirable. In practice, the value of ζ is kept such that both the performance indices, that is, M_p and M_r, are correlated. Therefore, ζ is generally designed as $0.4 < \zeta < 0.707$.

We had resonant frequency ω_r and damped frequency of oscillation ω_d as

$$\omega_r = \omega_n \sqrt{(1 - 2\zeta^2)}$$

$$\omega_d = \omega_n \sqrt{(1 - \zeta^2)}$$

By comparing these two values, it can be said that there exists correlation between resonant frequency and damped frequency of oscillations. The ratio of

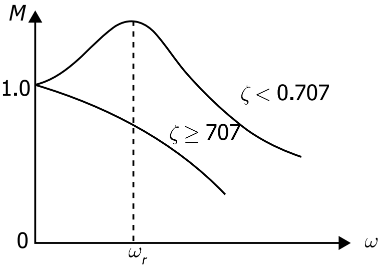
$$\frac{\omega_r}{\omega_d} = \sqrt{\frac{(1 - 2\zeta^2)}{(1 - \zeta^2)}}$$

is also a function of ζ. When ζ lies between 0.4 and 0.707, both ω_d and ω_r are comparable to each other.

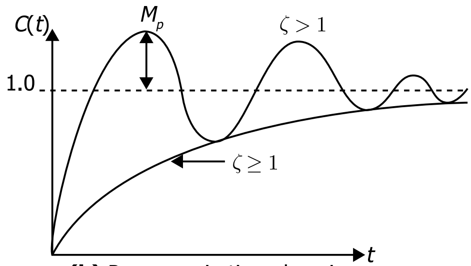
The magnitude characteristics in frequency domain and also in the time domain for two different values of ζ have been shown in Fig. 10.6.

10.3.2 Bandwidth

Bandwidth is the range of frequencies over which the value of M is equal to or greater than $1/\sqrt{2}$, that is, 0.707. We have from equation (10.4),



(a) Response in frequency domain



(b) Response in time domain

Fig. 10.6 *Magnitude characteristic in both frequency domain and time domain*

$$M = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

Let u_b = normalized bandwidth $= \frac{\omega_b}{\omega_n}$

Then,

$$M = \frac{1}{\sqrt{\left(1-u_b^2\right)^2 + (2\zeta u_b)^2}} = \frac{1}{\sqrt{2}}$$

or

$$(1-u_b^2)^2 + (2\zeta u_b)^2 = 2$$

or

$$1 + u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 = 2$$

or

$$u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 + 1 = 2$$

Let $u_b^2 = x$, then

$$x^2 - 2x + 4\zeta^2 x + 1 = 2$$

or

$$x^2 - 2x\,(1 - 2\zeta^2) - 1 = 0$$

$$x = \frac{2(1-2\zeta^2) \pm \sqrt{4(1-2\zeta^2)^2 + 4}}{2}$$

or

$$x = 1 - 2\zeta^2 \pm \sqrt{1 - 4\zeta^2 + 4\zeta^4 + 1}$$

or

$$x = 1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

or

$$u_b^2 = 1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

or

$$u_b = \sqrt{1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

As $u_b = \dfrac{\omega_b}{\omega_n}$

$$\omega_b = u_b \omega_n = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} \qquad \text{....(10.6c)}$$

$\boldsymbol{\omega_b}$ is the denormalized bandwidth.

10.3.3 Relative and Absolute Stability

Absolute stability indicates whether the system is stable or not. Relative stability indicates how stable the system is. Absolute stability can be checked by applying Routh-Hurwitz criterion. For relative stability Routh-Hurwitz criterion can also be applied by shifting the imaginary axis to the left in the s-plane successively and each time applying Routh-Hurwitz criterion. There are other tools and techniques like the root locus technique and Nyquist stability criterion to examine relative stability of a control system. These will be discussed in this chapter.

Example 10.1 The forward path transfer function of a unity feedback control system is given as

$$G(s) = \frac{64}{s(s+5)}$$

Calculate the resonant peak, resonant frequency and bandwidth of the closed-loop system

Solution

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + G(s)} \qquad [\because H(s) = 1]$$

$$= \frac{\dfrac{64}{s(s+5)}}{1 + \dfrac{64}{s(s+5)}} = \frac{64}{s(s+5) + 64}$$

$$= \frac{64}{s^2 + 5s + 64}$$

Comparing with the standard form of transfer function of a second-order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 64 \text{ or, } \omega_n = 8 \text{ rad/sec}$$

and

$$2\zeta\omega_n = 5$$

or

$$\zeta = \frac{5}{2\omega_n} = \frac{5}{2\times 8}$$
$$= 0.3125$$

Resonant frequency is (See eqn. 10.6)

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$
$$= 8\sqrt{1-2(0.3125)^2} = 7.1 \text{ rad/sec}$$

Resonant peak is (See eqn. 10.6a)

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2\times 0.3125\sqrt{1-(0.3125)^2}}$$
$$= 1.78$$

Bandwidth $\omega_b = \omega_n\sqrt{1-2\zeta^2 + (2-4\zeta^2+4\zeta^4)^{1/2}}$ (See eqn. 10.6c)

$$= 8\sqrt{1-2(0.3125)^2 + (2-4\times 0.3125^2 + 4\times 0.3125^4)^{1/2}}$$
$$= 11.25 \text{ rad/sec}$$

10.4 PRESENTATION OF FREQUENCY RESPONSE IN GRAPHICAL FORM

Sinusoidal transfer functions can be represented graphically from which the stability of the system can be studied. Three types of graphical representations are discussed in the following sections.

By now, we have studied that sinusoidal transfer function is a complex function of frequency ω. It is characterized by its magnitude and phase angle with frequency as the variable parameter. The following graphical representations for the study of system performance will be discussed:

a) Bode plot or logarithmic plot
b) Polar plot or Nyquist plot
c) Use of *M*-circles and *N*-circles.

10.5 BODE PLOT

Bode plot is one of the powerful graphical methods of analyzing and designing control systems. Introduction of logarithmic plots simplifies the determination of graphical representation of the frequency response of the system. Such logarithmic plots are popularly called Bode plots in honour of H.W. Bode. In Bode plot, we plot logarithm of magnitude versus frequency; and phase angle versus frequency. Both these plots are drawn on a semi-log graph paper. In this form of representation of a sinusoidal transfer function, the magnitude $G(j\omega)$ in dB which is 20 log $|G(j\omega)|$ is plotted against log ω.

The transfer function of a system in the frequency domain is expressed as

$$G(j\omega) = |G(j\omega)| \angle\phi(\omega)$$

By expressing the magnitude in terms of logarithm to the base 10, we can write Logarithmic gain = 20 $\log_{10}|G(j\omega)|$ where the units are in decibels.

For a Bode diagram logarithmic gain in dB versus ω and phase angle $\phi(\omega)$ versus ω are plotted separately on the semi-log paper.

Bode plots cover a wide range of frequencies due to log scale on x-axis. Due to log scale on y-axis Bode plots cover a wide range of gains. This is the advantage of using log scale.

We will now consider a generalized transfer function and explain the Bode plot in details. Let,

$$G(s)H(s) = \frac{K[(1+s\tau_1)(1+s\tau_2)(1+s\tau_3)............]\omega_n^2}{s^N\left[(1+s\tau_a)(1+s\tau_b).......(s^2+2\xi\omega_n s+\omega_n^2)\right]} \qquad ....(10.7)$$

The sinusoidal form of the transfer function is obtained by substituting s for $j\omega$.

$$\begin{aligned} G(j\omega)H(j\omega) &= \frac{K\left[(1+j\omega\tau_1)(1+j\omega\tau_2)(1+j\omega\tau_3).........\right]\omega_n^2}{(j\omega)^N\left[(1+j\omega\tau_a)(1+j\omega\tau_b)..........\left\{(\omega_n^2-\omega^2)+j2\xi\omega_n\omega)\right\}\right]} \\ &= \frac{K\left[(1+j\omega\tau_1)(1+j\omega\tau_2)(1+j\omega\tau_3).........\right]\omega_n^2}{(j\omega)^N\left[(1+j\omega\tau_a)(1+j\omega\tau_b)..........\left\{(\omega_n^2-\omega^2)+j2\xi\omega_n\omega)\right\}\right]} \end{aligned} \qquad ....(10.8)$$

Procedure for drawing Bode plot

Now we will explain the procedure for drawing the Bode plot as follows.

Magnitude of $G(j\omega)$ $H(j\omega)$ in decibel is given as:

$$\begin{aligned} 20\log_{10}|G(j\omega)H(j\omega)| = {} & 20\log_{10}K + 20\log_{10}|(1+j\omega\tau_1)| \\ & + 20\log_{10}|(1+j\omega\tau_2)| + 20\log_{10}|(1+j\omega\tau_3)| + ... \\ & - 20N\log_{10}|(j\omega)| - 20\log_{10}|1+j\omega\tau_a| \\ & - 20\log_{10}|1+j\omega\tau_b|............ \\ & - 20\log_{10}\left\|\left[1-\left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\frac{\omega}{\omega_n}\right]\right\| \end{aligned} \qquad ....(10.9)$$

For the phase angle we write,

$$\underline{|G(j\omega)H(j\omega)} = \tan^{-1}(0) + \tan^{-1}(\omega\tau_1) + \tan^{-1}(\omega\tau_2)\ldots\ldots\ldots$$

$$-N\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\omega\tau_a - \tan^{-1}(\omega\tau_b)\ldots\ldots\ldots - \tan^{-1}\left[\frac{2\xi\omega_n\omega}{\left(\omega_n^2 - \omega^2\right)}\right]$$

or,

$$\underline{|G(j\omega)H(j\omega)} = \tan^{-1}(\omega\tau_1) + \tan^{-1}(\omega\tau_2)\ldots\ldots\ldots$$

$$-N\times 90 - \tan^{-1}\omega\tau_a - \tan^{-1}(\omega\tau_b) - \tan^{-1}\left[\frac{2\xi\omega_n\omega}{\left(\omega_n^2 - \omega^2\right)}\right] \qquad \ldots.(10.10)$$

Bode plot is a graph obtained from equation (10.9) and (10.10) consisting of two parts as follows.

i) Plot of magnitude of $G(j\omega)$ $H(j\omega)$ in decibel versus $\log_{10}\omega$ and
ii) Plot of phase angle of $G(j\omega)H(j\omega)$ versus $\log_{10}\omega$.

10.5.1 Methods of Drawing Bode Plot

For plotting magnitude $|G(j\omega))H(j\omega)|$ in decibels versus $\log_{10}\omega$, we have to add the plots of all the individual factors as included in equation (10.9) which are listed below.

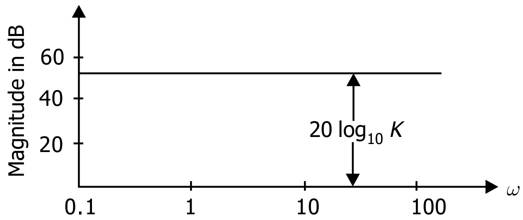
a) Plot of gain K which is a constant;
b) Plot of poles at the origin, $\left(\frac{1}{j\omega}\right)^N$;
c) Plot of poles on real axis, $\frac{1}{1+j\omega\tau}$;
d) Plot of zeros on real axis, $(1 + j\omega\tau)$;
e) Plot of complex conjugate poles, $\frac{1}{1+j2\xi\left(\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2}$
f) Plot of complex conjugate zeros, if present.

Now we will explain the Bode plots of the above individual factors.

a) *Plot for the constant gain K.*
The magnitude in decibel of the term K is given as

$$K(dB) = 20\log_{10}(K) \qquad \ldots.(10.11)$$

Equation (10.11) indicates that the magnitude is independent of $\log_{10}\omega$. Assuming K as positive and real, the Bode plot has been drawn as shown in Fig. 10.7(a).



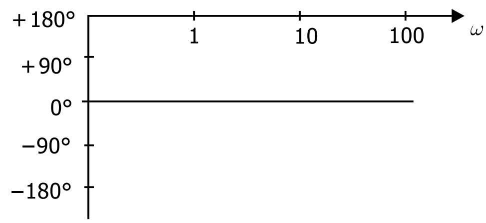


Fig. 10.7 *(a) Plot of constant gain K versus ω; **(b)** Plot of phase angle versus ω*

The phase angle is always zero for any value of ω as shown in Fig. 10.7(b).

b) *Plot of the term* $\dfrac{1}{(j\omega)^N}$ *representing a pole at the origin*

The magnitude of the term $\dfrac{1}{(j\omega)^N}$ in decibel is written as

$$20\log_{10}\left|\frac{1}{(j\omega)^N}\right| = -20N\log_{10}(\omega)$$

The magnitude curve will have a slope of -20 dB/decade for a pole ($N = 1$).
For $N = 2$ the slope will be -40 dB/decade.
The phase angle is given by

$$\underline{\left|\frac{1}{(j\omega)^N}\right.} = -N\tan^{-1}\left(\frac{\omega}{0}\right) = -N\tan^{-1}\infty = -N\times 90^\circ$$

Therefore, for a zero at the origin we have a logarithmic magnitude $+20\log_{10}\omega$ where the slope is $+20$ dB/decade; and the phase angle is, $\phi(\omega) = 90°$. The graphs have been shown in Fig. 10.8.

c) *Plot for the term* $\dfrac{1}{1+j\omega\tau}$, *i.e. for poles on the real axis*

The magnitude is written as

$$20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = 20\log_{10}\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)$$

$$= -20\log_{10}\left(1+\omega^2\tau^2\right)^{1/2}$$

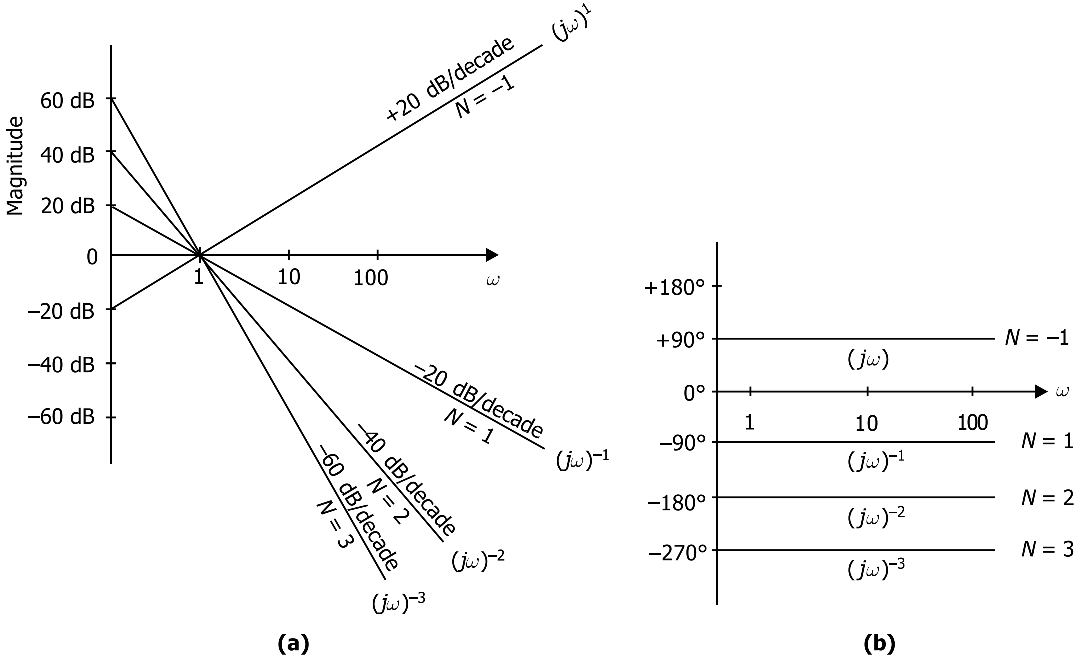


Fig. 10.8 *Bode diagram for* ***(jω)***$^{\pm N}$

Let us calculate the magnitude at very low and very high frequencies as when $\omega \ll \dfrac{1}{\tau}$, magnitude is

$$20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = -20\log_{10} 1 = 0 \text{ dB}. \qquad \text{....(10.12)}$$

when $\omega\tau \gg 1$;

$$\begin{aligned} \text{magnitude } &= 20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = 20\log_{10}(\omega\tau) \\ &= -20\log_{10}(\omega) - 20\log_{10}(\tau) \end{aligned} \qquad \text{....(10.13)}$$

Equation (10.13) is similar to the equation of a straight line $y = mx + c$.

Here, $\qquad m(\text{slope}) = -20$ dB/decade

$$c = -20\log_{10}(\tau) = 20\log_{10}\left(\frac{1}{\tau}\right).$$

Equations (10.12) and (10.13) are two curves. To determine where the two curves are intersecting on the 0 dB axis we have to equate equation (10.13) to zero.

Thus, we write

$$0 = -20 \log_{10}(\omega) - 20 \log_{10}(\tau)$$

or,
$$20\log_{10}(\omega) = -20\log_{10}(\tau) = 20\log_{10}\left(\frac{1}{\tau}\right)$$

or,
$$\omega = \frac{1}{\tau}$$

Hence the two curves intersect on 0 dB axis at $\omega = \omega_c = \dfrac{1}{\tau}$ where $\boldsymbol{\omega}_c$ is called the break frequency or *corner frequency.*

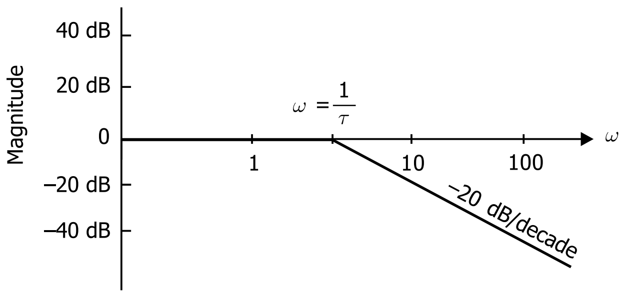
The phase angle, $\phi(\omega) = -\tan^{-1}(\omega\tau)$

At very low frequency, $\phi(\omega) = -\tan^{-1}(0) = 0°$

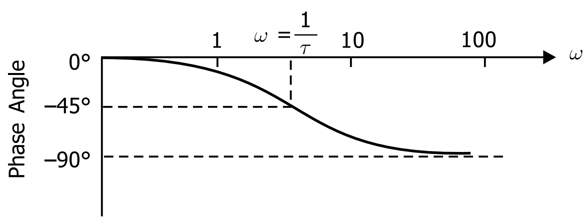
At very high frequency, $\phi(\omega) = -\tan^{-1}(\infty) = -90°$

At corner frequency, $\omega = \omega_c = \dfrac{1}{\tau};\ \phi(\omega) = -\tan^{-1}\left(\dfrac{1}{\tau} \times \tau\right) = -\tan^{-1}(1) = -45°$

The Bode diagram for the *pole factor* $\dfrac{1}{1 + j\omega\tau}$ has been shown in Fig. 10.9.



(a)



(b)

Fig. 10.9 *Bode diagram for* $(1 + j\omega\tau)^{-1}$

The Bode diagram for a zero factor (factor involving zero in the transfer function) $(1 + j\omega\tau)$ is determined in a similar way as

magnitude, $$20\log_{10}|(1+j\omega\tau)| = 20\log_{10}\left(1+\omega^2\tau^2\right)^{\frac{1}{2}} \quad \text{....(10.14)}$$

when $\omega\tau << 1$, $\omega\tau$ is negligible as compared to 1

therefore, $20 \log_{10} |(1 + j\omega\tau)| = 20 \log_{10}(1) = 0$ dB ....(10.15)

And when $\omega\tau >> 1$, $\omega\tau$ is much higher than 1

Therefore, $20 \log_{10} |(1 + j\omega\tau)| = 20 \log_{10} \omega\tau = 20 \log_{10}(\omega) + 20 \log_{10}(\tau)$ ....(10.16)

Equation (10.16) is similar to a general straight line equation

$$y = mx + c$$

Here, $$m(\text{slope}) = 20 \text{ dB/decade}$$

$$C = 20 \log_{10} (\tau)$$

Equations (10.15) and (10.16) represent two graphs. The graph of equation (10.15) lies on 0 dB axis, whereas graph of equation (10.16) has a slope of 20 dB/decade. The intersection of the two graphs is found by equating equation (10.16) to zero.

Thus,

$$0 = 20 \log_{10} (\omega) + 20 \log_{10} (\tau)$$

or, $$\omega = \frac{1}{\tau} = \omega_c$$

where ω_c is called the corner frequency. The Bode diagram for a *zero factor* of $(1 + j\omega\tau)$ has been shown in Fig. 10.10. The phase angle has been calculated as

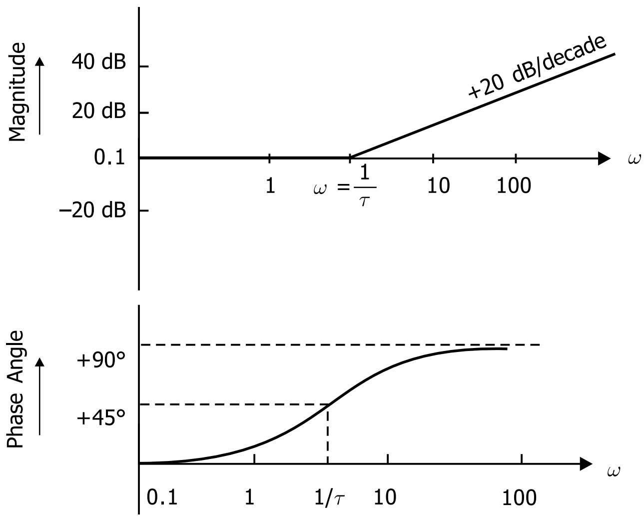


Fig. 10.10** Bode plot for zero factor **(1 + jωτ)

$$\phi = \tan^{-1}(\omega\tau)$$

at $\omega\tau << 1$, $\qquad \phi = \tan^{-1}(0) = 0$

at $\omega\tau >> 1$, $\qquad \phi = \tan^{-1}(\infty) = 90°$

at $\omega = \omega_c = \dfrac{1}{\tau}$, $\qquad \phi = \tan^{-1}\left(\dfrac{1}{\tau} \times \tau\right) = 45°$

The Bode plot for the above has been shown in Fig. 10.10. The magnitude versus $\log_{10}\omega$ graphs for both $(1 + j\omega\tau)^{-1}$ and $(1 + j\omega\tau)$ have been shown together in Fig. 10.11.

The exact or actual curves and the asymptotic curves differ from each other to some extent. They are matched at the corner frequency $\omega_c = \dfrac{1}{\tau}$ as shown in Fig. 10.11. The maximum error between the exact plot and the asymptotic plot occurs at the corner frequency. From exact plot the magnitude at $\omega = \omega_c = \dfrac{1}{\tau}$ is calculate as

$$20\log_{10}|(1 + j\omega\tau)| = 20\log_{10}\left|\left(1 + j\frac{1}{\tau} \times \tau\right)\right|$$

$$= 20\log_{10}\sqrt{2} = 3\,\text{dB}.$$

For the curve for $(1 + j\omega\tau)^{-1}$ the magnitude of exact plot at corner frequency will be -3 dB.

If the poles or zeros on the real axis are of the form $(1 + j\omega\tau)^{\pm N}$, the error in magnitude can be calculated to be equal to $\pm 3N$ dB and the phase angle error will be $\pm N\,45°$.

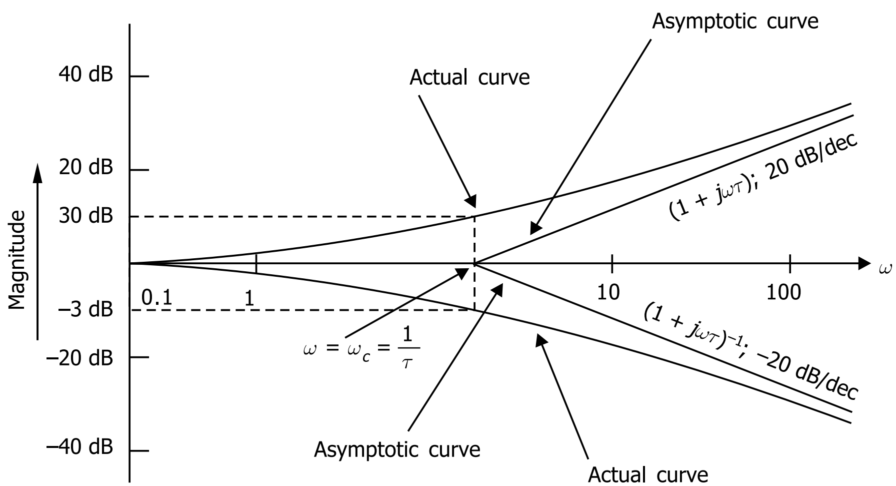


Fig. 10.11 *Actual Bode diagram for $(1 + j\omega\tau)$ and $(1 + j\omega\tau)^{-1}$ showing the magnitudes*

10.5.2 Initial Slope of Bode Plot

The initial slope of the Bode plot depends on the type of the system. Let us consider, for example type 0, type 1 and type 2 systems.

General sinusoidal transfer function for initial part of the Bode plot (for frequencies lower than the lowest corner frequency) is

$$G(j\omega)H(j\omega) = K\,(j\omega)^{\pm N}$$

If N is positive, then N represents the number of zero at the origin. If N is negative, then N represents the number of poles at the origin.

For a type 0 system,

$$\begin{aligned} 20\log_{10}|G(j\omega)H(j\omega)| &= 20\log_{10}\left|\frac{K}{(j\omega)^0}\right| \\ &= 20\log_{10} K \end{aligned} \qquad \text{....(10.17)}$$

The Bode plot has been shown in Fig. 10.12(a).

For a type 1 system, $N = 1$

$$\begin{aligned} 20\log_{10}|G(j\omega)H(j\omega)| &= 20\log_{10}\left|\frac{K}{(j\omega)^1}\right| \\ &= 20\log_{10} K - 20\log_{10}(\omega) \end{aligned} \qquad \text{....(10.18)}$$

Equation (10.18) is similar to the equation of a straight line,

$$y = mx + c$$

Here slope $\qquad m = -\,20$ dB/decade

and $\qquad c = 20\log_{10}(K)$

Intersection of the graph with 0 dB axis is calculated by equating equation (10.18) to 0. Therefore, $20\log_{10}(K) - 20\log_{10}(\omega) = 0$

or, $\qquad K = \omega$

Hence the graph will intercept 0 dB axis at $\omega = K$ as shown in Fig. 10.9(b).

For a type 2 system we take $N = 2$

We can write

$$\begin{aligned} 20\log|G(j\omega)H(j\omega)| &= 20\log_{10}\left|\frac{K}{(j\omega)^2}\right| \\ &= 20\log_{10} K - 40\log_{10}(\omega) \qquad \text{....(10.19)} \\ &= -40\log_{10}(\omega) + 20\log_{10}(K) \end{aligned}$$

Equation (10.18) is similar to the straight line equation $y = mx + c$

Here, $$m = -\ 40 \text{ dB/decade}$$

$$c = 20\log_{10}K$$

The intercept on the 0 dB axis is calculated by equating equation (10.18) to zero.
Thus

$$0 = 40 \log_{10}(\omega) + 20 \log_{10}(K)$$

or, $$20 \log_{10} K = 20 \log_{10} \omega^2$$

or, $$\omega = \sqrt{K}$$

The Bode plot for a type 2 system (initial part) has been shown in Fig. 10.12(c).

Table 10.1 shows the initial slope and intersection with 0 dB axis for different types of systems.

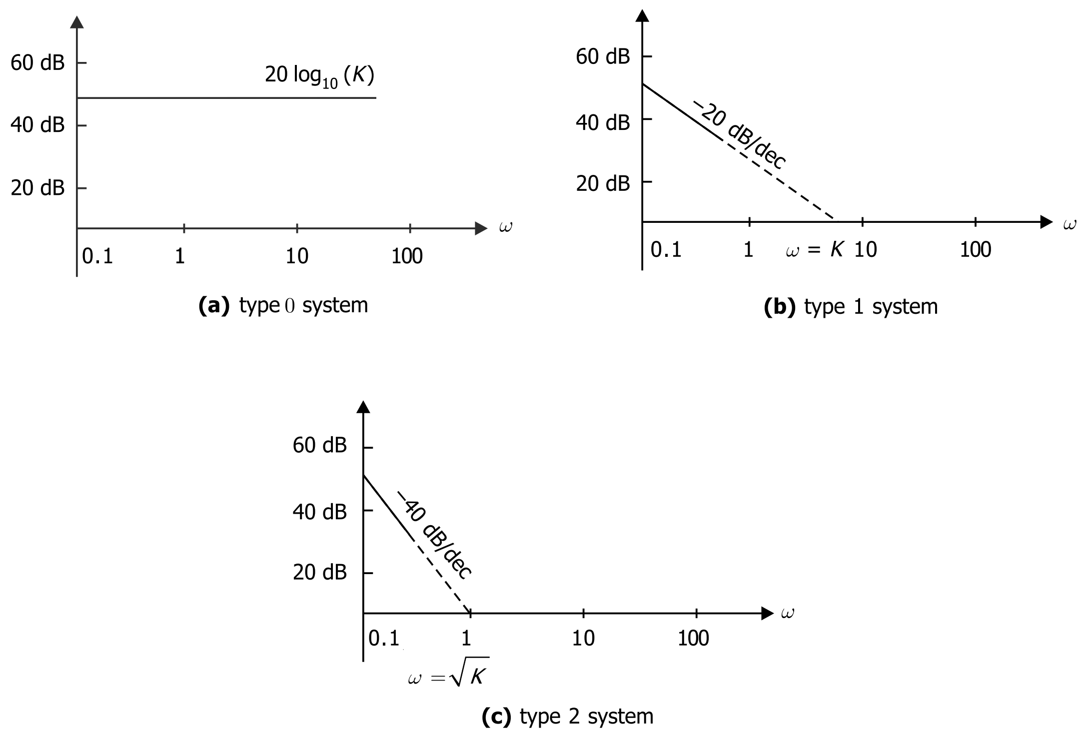


Fig. 10.12 *Initial part of Bode plots for type 0, type 1, and type 2 system*

Table 10.1 *Initial Slope of Bode Plot of Systems*

Type of System (*N*)	**Initial Slope (−20*N*) dB/dec**	**Intersection with 0 dB Axis at**
Type 0	0 dB/decade	Parallel to 0 dB axis
Type 1	–20 dB/decade	$\omega = \mathrm{K}$
Type 2	–40 dB/decade	$\omega = \sqrt{K}$
Type 3	–60 dB/decade	$\omega = \mathrm{K}1/3$
Type N	–20 N dB/decade	$\omega = \mathrm{K}1/\mathrm{N}$

10.5.3 Bode Plot for Quadratic Form of Transfer Function

Let us take transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The sinusoidal transfer function is

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{\left(\omega_n^2 - \omega^2\right) + j2\xi\omega_n\omega} \qquad \text{....(10.20a)}$$

$$= \left| \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j2\xi\frac{\omega}{\omega_n}} \right|$$

The magnitude of the above in decibel form is

$$20\log_{10}\left| \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j2\xi\frac{\omega}{\omega_n}} \right|$$

$$= 20\log_{10}\left|\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}\right| \qquad \text{....(10.20b)}$$

$$= -20\log_{10}\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}$$

When $\omega << \omega_n$, i.e. at very low frequency, the term $\left(\frac{\omega}{\omega_n}\right), \left(\frac{\omega}{\omega_n}\right)^2$ can be neglected. Equation (10.20) becomes equal to $-20 \log_{10}(1) = 0$ dB.

When $\omega >> \omega_n$, i.e. at very high frequency, the term 1 and $\left[2\xi\left(\frac{2}{2_n}\right)\right]^2$ can be neglected when compared to $\left[\left(\frac{\omega}{\omega_n}\right)^2\right]^2$. Equation (10.20) then becomes equal to

$$-20\log_{10}\left(\frac{\omega}{\omega_n}\right)^2$$

$$= -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$$

$$\text{Magnitude} = -40\log_{10}(\omega) + 40\log_{10}(\omega_n) \qquad \text{....(10.21)}$$

Equation (10.21a) is similar to the straight line. Equation, $y = mx + c$

Here, $m = -$ 40 dB/decade.

The magnitude in decibel versus $\log_{10}(\omega)$

Case I graph lies on 0 dB axis.

Case II graph has slope of $-$ 40 dB/decade.

Now, we have to find where the two graphs are intersected on the 0 dB axis. For finding the point on the 0 dB axis, we have to equate equation (10.21) to zero.

Therefore, $\qquad -40 \log_{10}(\omega) + 40 \log_{10} \omega_n = 0$

or $\qquad 40\log_{10}(\omega) = 40\log_{10}(\omega_n)$

or $\qquad \omega = \omega_n$, where ω_n is the corner frequency.

The two graphs will intersect at $\omega = \omega_n$ on the 0 dB axis. ω_n is called the natural frequency of oscillation.

10.5.4 Maximum Magnitude of the Second Order Transfer Function

The transfer function is written as

$$M = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}} \qquad \text{....(10.22)}$$

The magnitude,

$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \text{....(10.23)}$$

Let the frequency at which magnitude will be maximum be denoted by $\boldsymbol{\omega_r}$. $\boldsymbol{\omega_r}$ will be called the resonant frequency. The maximum value of M, also called the resonant peak M_r is determined by maximizing the expression for M with respect to $\boldsymbol{\omega}$. Therefore we will differentiate M with respect to $\boldsymbol{\omega}$ and equate it to zero.

$$\frac{d}{d\omega}\left[\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}\right] = 0$$

or,
$$-\frac{1}{2}\left[1 + \frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 4\xi^2\frac{\omega^2}{\omega_n^2}\right]^{-\frac{3}{2}}\left[0 + \frac{4\omega^3}{\omega_n^4} - 4\frac{\omega}{\omega_n^2} + 8\frac{\xi^2\omega}{\omega_n^2}\right] = 0$$

or,
$$-\frac{1}{2}\frac{\left(4\frac{\omega^3}{\omega_n^4} - 4\frac{\omega}{\omega_n^2} + 8\frac{\xi^2\omega}{\omega_n^2}\right)}{1 + \frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 4\xi^2\frac{\omega^2}{\omega_n^2}} = 0$$

or
$$\frac{4\omega}{\omega_n^2}\left[\frac{\omega^2}{\omega_n^2} - 1 + 2\xi^2\right] = 0$$

or
$$\frac{\omega^2}{\omega_n^2} = 1 - 2\xi^2$$

∴
$$\omega = \omega_r = \omega_n\sqrt{1 - \xi^2} \qquad \text{....(10.24)}$$

Now let the magnitude of M at $\boldsymbol{\omega} = \boldsymbol{\omega_r}$ be denoted as M_r, called the resonant peak amplitude. We can express M_r as

$$M_r = \frac{1}{\sqrt{\left(1-\frac{\omega_r^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega_r}{\omega_n}\right)^2}}$$

Let us put $\omega_r = \omega_n\sqrt{1-2\xi^2}$ in the above expression,

$$M_r = \frac{1}{\sqrt{\left[1-\frac{\omega_n^2(1-2\xi^2}{\omega_n^2}\right]^2\left[\frac{2\xi\omega_n\sqrt{1-2\xi^2}}{\omega_n}\right]^2}}$$

$$= \frac{1}{\sqrt{4\xi^2 = 4\xi^4}}$$

Therefore, $$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

Expressed in decibel,

$$M_r = 20\log_{10}\left[\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right]$$

From equation (10.22) the Phase angle of M_r is

$$= -\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)}$$

At corner frequency, $\boldsymbol{\omega} = \boldsymbol{\omega}_n$

Phase angle of $$M_r = -\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{\left(1-\frac{\omega_n^2}{\omega_n^2}\right)}$$

$$= -\tan\infty = -90°.$$

Thus for a second order system under consideration we find that

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\omega_r = \omega_n\sqrt{1-\xi^2}$$

And as discussed in the previous chapter, damped natural frequency, $\omega_d = \omega_n \sqrt{1-\xi^2}$ maximum overshoot, $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$

Example 10.2 Sketch the polar plot of a system whose transfer function is given as

$$G(s) = \frac{10}{s(s+1)(s+2)}$$

Solution

By putting $s = j\omega$ in the transfer function,

$$G(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+2)}$$

$$= \frac{10}{-3\omega^2 + j(2\omega - \omega^3)}$$

$$= \frac{10\left[-3\omega^2 - j(2\omega-\omega^3)\right]}{9\omega^4 + (2\omega-\omega^3)^2}$$

$$= \frac{-30\omega^2}{9\omega^4 + (2\omega-\omega^3)^2} + j\frac{10(\omega^3 - 2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$

$$= \frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} + j\frac{10(\omega^3 - 2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$

Equating respectively the real and imaginary parts,

$$-\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = 0;\ \text{from which, } \boldsymbol{\omega} = \infty$$

and $\dfrac{10(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)} = 0$; from which $\omega^3 - 2\omega = 0$ or $\omega = \pm\sqrt{2}$

Magnitude of $G(j\omega) = |G(j\omega)| = \dfrac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$

Phase angle of $G(j\omega) = |G(j\omega)| = \left(-90° - \tan^{-1}\omega - \tan^{-1}\dfrac{\omega}{2}\right)$

$G(j\boldsymbol{\omega})$ acts the real axis at $\omega = \pm\sqrt{2}$

The value of $G(j\boldsymbol{\omega})$ at $\omega = \pm\sqrt{2}$ is calculated as

$$-\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = -\frac{30\times 2}{(4+2)(4+2)} = -\frac{60}{36} = -1.67$$

The values of $G(j\omega)$, i.e. its magnitude and phase at $\omega \to 0$ and $\omega \to \infty$ are calculated as

$$\lim_{\omega\to 0}|G(j\omega)| = \lim_{\omega\to 0}\frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} = \infty$$

$$\lim_{\omega\to \infty}|G(j\omega)| = \lim_{\omega\to \infty}\frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} = 0$$

$$\lim_{\omega\to 0}\angle G(j\omega) = \lim_{x\to 0}\left(-90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}\right) = -90^\circ - 0^\circ - 0^\circ = -90^\circ$$

$$\lim_{\omega\to \sqrt{2}}\angle G(j\omega) = \lim_{\omega\to \sqrt{2}}\left(-90^\circ - \tan^{-1}\sqrt{2} - \tan^{-1}\frac{1}{\sqrt{2}}\right) = -180^\circ$$

$$\lim_{\omega\to \infty}\angle G(j\omega) = \lim_{\omega\to x}\left(-90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}\right) = -90^\circ - 90^\circ - 90^\circ = -270^\circ$$

Thus the informations for drawing the polar plot are shown below. The polar plot is shown in Fig. 10.13.

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
$\sqrt{2}$	-1.67	-180°
∞	0	-270°

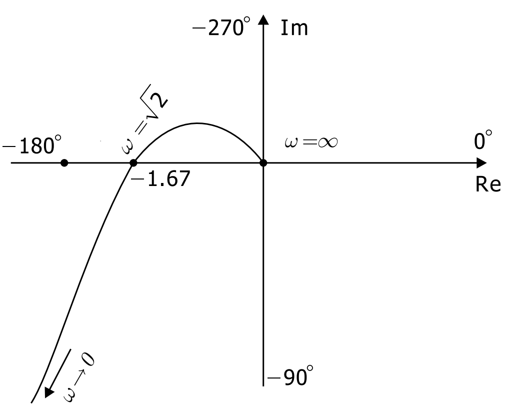


Fig. 10.13 *Polar plot of* $G(s) = \dfrac{10}{s(s+1)(s+2)}$

10.5.5 Determination of Gain Margin and Phase Margin for Stability Analysis

From the polar plot or from the Bode plot two quantities namely the gain margin and the phase margin are known. This information can be used to study the stability of the control system. The stability condition of the system can be improved by use of compensating networks (phase-lag or phase-lead networks). Knowledge of gain margin and phase margin becomes useful in the design and use of such compensating networks.

Gain Margin and Phase Crossover Frequency

Consider the polar plot of $G(j\omega)\ H(j\omega)$ as shown in Fig. 10.14.

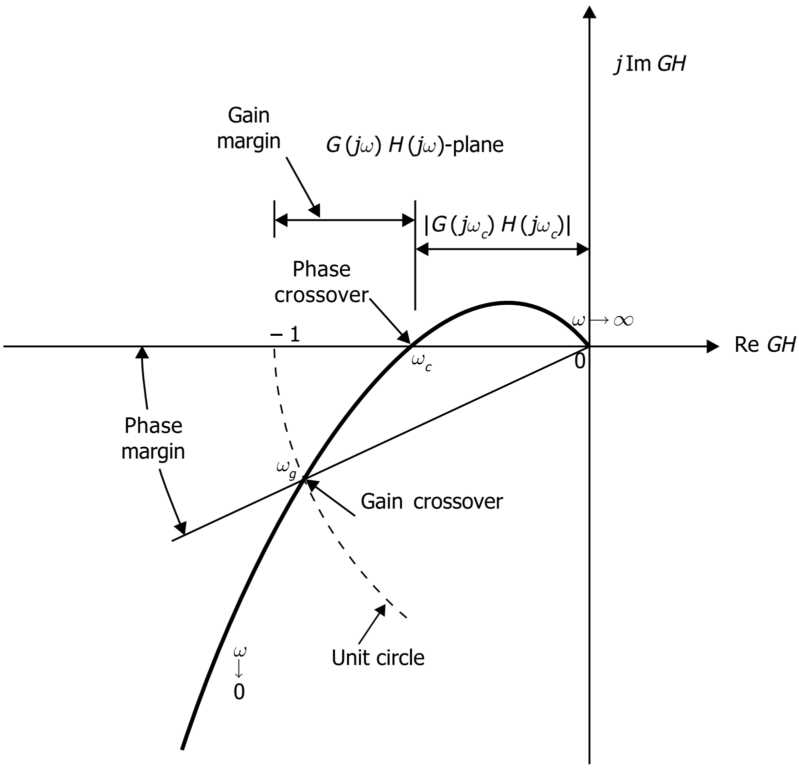


Fig. 10.14 *Polar plot of G(jω) H(jω)*

The frequency (ω_c) *at which the polar plot crosses the negative real axis is known as the phase crossover frequency. The gain margin is a measure of the closeness of the phase crossover point to the point* $(-1, j0)$ *in the* $G(s)\ H(s)$*-plane*. For a closed-loop system having $G(s)\ H(s)$ as its loop-transfer function, the gain margin is given by

$$G.M. = 20 \log \frac{1}{|G(j\omega_c)H(j\omega_c)|} \text{ dB.}$$

The following points may be observed from the above expression of *G.M.*

i) For stable systems as
$|G(j\omega_c)H(j\omega_c)| < 1$, the *G.M.* in dB is positive.

ii) For marginally stable systems as
$|G(j\omega_c)H(j\omega_c)| = 1$, the *G.M.* in dB is zero.

iii) For unstable systems as
$|G(j\omega_c)H(j\omega_c)| > 1$, the *G.M.* in dB is negative and the gain has to be reduced to make the system stable.

So, *gain margin may be defined as the amount of gain in decibels that can be allowed to increase in the loop before the closed loop-system reaches instability*.

Phase Margin and Gain Crossover Frequency

Referring to Fig. 10.14, the gain crossover frequency (ω_g) is the frequency at which $|G(j\omega_c) H(j\omega_c)|$, the magnitude of the open-loop transfer function, is unity. In other words, *the frequency at which the polar plot crosses the unit circle* (*whose centre is at origin and radius is unity*) *is known as gain crossover frequency* (ω_g).

The phase margin of a system is given by

$$P.M. = \angle G(j\omega_g)H(j\omega_g) + 180°.$$

Phase margin is measured positively in a counter-clockwise direction from the negative real axis. The phase angle at the gain crossover frequency is $(-180° + P.M.)$. If an additional phase-lag equal to *P.M.* is introduced at the gain crossover frequency, the polar plot will then pass through the point $(-1 + j0)$ driving the closed-loop system to the verge of instability. So, *the phase margin may be defined as the amount of additional phase-lag at the gain crossover frequency required to bring the system to the verge of instability.*

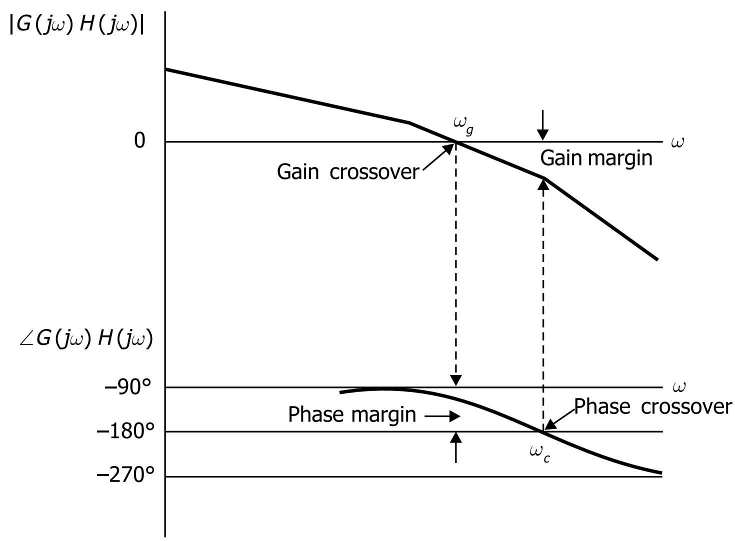


Fig. 10.15 *Bode plot of G(jω) H(jω)*

It may be noted that the phase margin is always positive for a stable closed-loop system. Usually a *G.M.* of about 6 dB or a *P.M.* of 30° to 35° results in a reasonably good degree of relative stability.

Graphical Methods for Obtaining Gain Margin and Phase Margin

As illustrated in Fig. 10.15, the gain margin measured at phase crossover frequency (ω_c) is given by

$$G.M. = -|G(j\omega_c)H(j\omega_c)| \text{ dB}.$$

The phase margin measured at gain crossover frequency (ω_g) is given by

$$P.M. = 180° + \angle G(j\omega_g)H(j\omega_g).$$

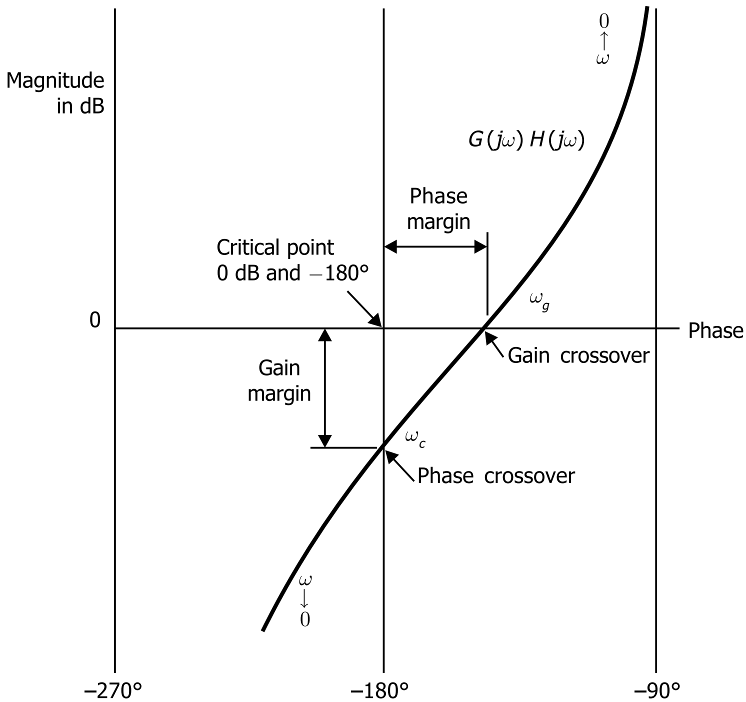


Fig. 10.16 *Magnitude versus phase plot of G(jω) H(jω)*

The gain and phase margins are even better illustrated on the magnitude versus phase plot as shown in Fig. 10.16. On this plot, the locus intersects the −180° axis at the phase crossover frequency (ω_c) and the locus intersects the 0 dB axis at the gain crossover frequency (ω_g). So the gain margin is simply the distance in decibels measured from the phase crossover to the critical point at 0 dB and −180°, and phase margin is the horizontal distance in degrees measured from the gain crossover to the critical point.

Example 10.3 Sketch the Bode plot and hence determine the gain cross over frequency and phase cross over frequency for the transfer function of a system represented by

$$G(s) = \frac{10}{s(s + 0.5s)(1 + 0.1s)}$$

Solution

By putting $s = j\omega$ in the transfer function we have

$$G(j\omega) = \frac{10}{j\omega(1 + j0.5\omega)(1 + j0.1\omega)}$$

a) The corner frequencies are:

$$\omega_1 = \frac{1}{0.5}$$
$$= 2 \text{ rad/sec}$$
$$\omega_2 = \frac{1}{0.1}$$
$$= 10 \text{ rad/sec}$$

b) As per procedure, the starting frequency of Bode plot is taken as lower than the lowest corner frequency. Here the lowest corner frequency is 2 rad/sec. Therefore, we can choose starting frequency as 1 rad/sec.

c) The system is type 1 system (type of the system is indicated by the power of s in the denominator of the transfer function). So the initial slope of the Bode plot is -20 dB/decade. This initial slope will continue till the lowest corner frequency of $\omega = 2$ rad/sec.

d) The corner frequency of $\omega = 2$ rad/sec is due to the term $\frac{1}{1 + j0.5\omega}$ of the transfer function. Therefore, the slope of the Bode plot will change by another -20 dB/decade after the corner frequency of $\omega = 2$ rad/sec. The total slope, therefore, will become -40 dB/dec. This slope will continue till the next corner frequency of $\omega = 10$ rad/sec. The corner frequency of $\omega = 10$ rad/sec is due to the term $\frac{1}{1 + j0.1\omega}$ of the transfer function. The slope of the Bode plot after the corner frequency of 10 rad/sec will change by another -20 dB/dec. Therefore, after corner frequency of 10 rad/sec, the total slope of the Bode plot becomes -60 dB/dec. and this slope will continue for higher frequencies.

e) The magnitudes of the Bode plot at different frequencies are calculated as shown below.

Frequency in rad/sec	Magnitude in dB
$\omega = 1$	$\left\lvert\frac{K}{j\omega}\right\rvert = 20\log K - 20\log\omega$ $= 20\log 10 - 20\log 1$ $= 20$ dB Here $K = 10$, given
$\omega = 2$	$\left\lvert\frac{K}{j\omega(1+j\omega 0.5)}\right\rvert = 20\log 10 - 20\log\omega$ $= 20\log\sqrt{1^2 + 2.5\omega^2}$ $= 20 - 20\log 2 - 10\log 2$ $= 11$ dB
$\omega = 10$	$\left\lvert\frac{K}{j\omega(1+j0.5\omega)(1+j0.1\omega)}\right\rvert = 20\log K - 20\log\omega$ $= 20\log\sqrt{1^2 + .25\omega^2}$ $-20\log\sqrt{1^2 + 0.1\omega^2}$ $= -16.16$ dB

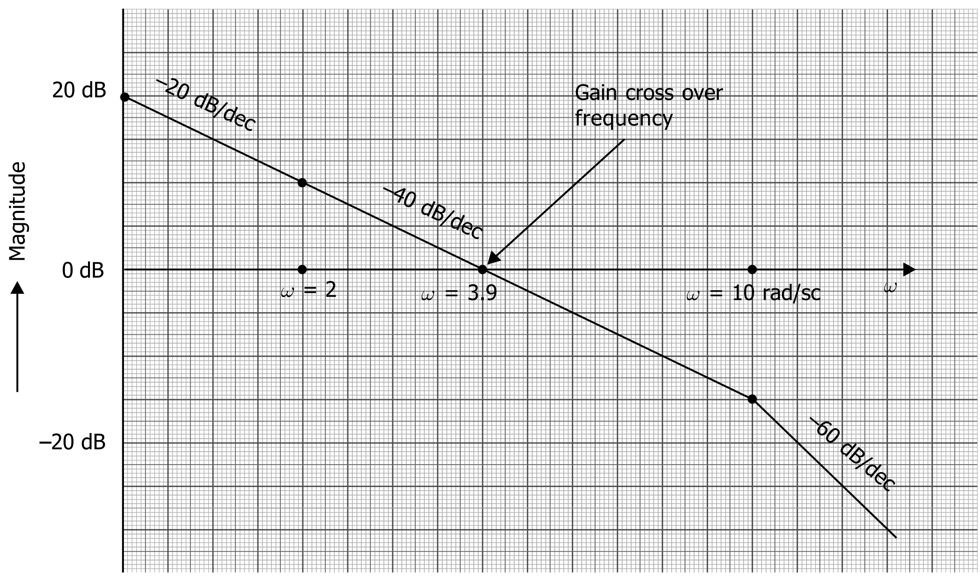


Fig. 10.17 *Bode magnitude plot for* $G(s) = \dfrac{10}{s(s + j0.5s)(1 + 0.1s)}$

f) Now, we will calculate the angle $\angle G(j\omega)$ for frequencies of $\omega = 1$ rad/sec to $\omega = 15$ rad/sec (say). The calculations are done using the relation, $\angle G(j\omega) = -90° - \tan^{-1} 0.5\,\omega - \tan^{-1} 0.1\,\omega$ (from the system transfer function)

ω	0	0.1	1.0	2	5	10	15
ϕ	−90°	−93.43°	−122.3°	−146.31°	−184.76°	−213.7°	−228.7°

g) We now sketch the Bode plot for magnitude and phase angle as shown in Figs. 10.17 and 10.18. From the figure we find,
Gain cross over frequency = 3.9 rad/sec
Phase cross over frequency = 4.5 rad/sec

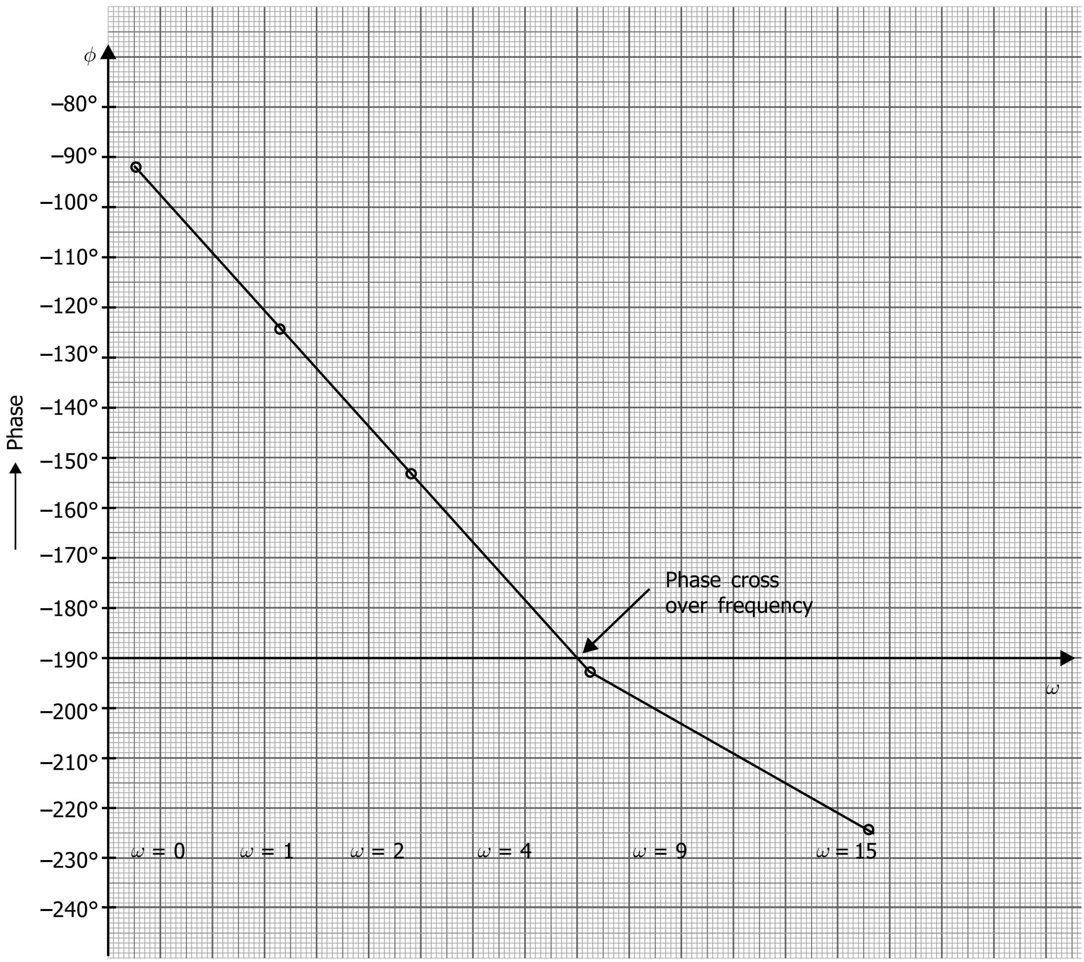


Fig. 10.18 *Bode phase plot for* $G(s) = \dfrac{10}{s(s + j0.5s)(1 + 0.1s)}$

Example 10.4 Find the open-loop transfer function of a system whose approximate Bode plot is shown below.

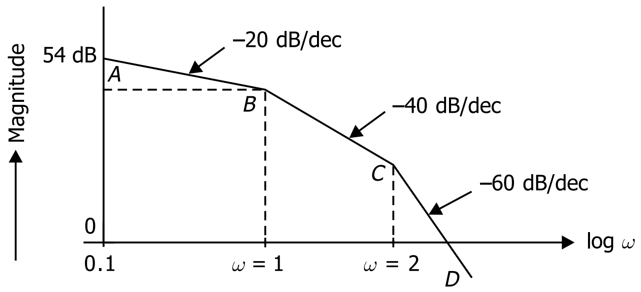


Fig. 10.19

Solution

a) The initial slope of the Bode plot is -20 dB/dec. We can write the equation of the line AB in the form $y = mx + c$*. Here, we write

$$y = -20 \log \omega + c$$

We put $\omega = 0.1$ and $y = 54$ dB and calculate the value of c as

$$54 \text{ dB} = -20 \log (0.1) + c$$

or, $$c = (54 - 20) \text{ dB} = 34 \text{ dB}.$$

For a slope of -20 dB/decade, the system is type 1 and its transfer function is $\frac{K}{j\omega}$. We have to find the value of K

$$c = 20 \log K$$

or, $$34 = 20 \log K$$

or, $$\log K = 1.7$$

$$K = 50.12$$

Therefore for the initial Bode plot we get the transfer function,

$$\frac{K}{s} = \frac{50.12}{s}$$

[*We know that starting part of Bode plot gives the transfer function of $\frac{K}{j\omega}$. This can be expressed in dB form, $T(s) = \frac{K}{s}$ or $T(j\omega) = \frac{K}{j\omega}$. The magnitude representation in dB form is

$$|T(j\omega)| = y = 20 \log K - 20 \log \omega.$$

or, $y = -20 \log \omega + 20 \log K.$

Viewing in $y = -mx + c$ form, $c = 20 \log K]$

At corner frequency, $\omega = 1$ the slope has changed by another -20 dB/dec. The slope is negative. The corresponding factor of the T.F is $\dfrac{1}{(1+s)}$.

At the corner frequency $\omega = 2$, the slope is increased by another -20 dB/dec. The slope is negative. Hence the corresponding factor of the T.F. is $\dfrac{1}{(1+0.5s)}$.

Thus the transfer function of the control system is

$$G(s) = \frac{K}{s(1+s)(1+0.5s)} = \frac{50.2}{s(1+s)(1+0.5s)}$$

Example 10.5 The Bode plot for open-loop transfer function of a system is shown in Fig. 10.20. Find the transfer function.

Solution

Initial slope of the Bode plot is -20 dB/decade. The system is a type 1 system. The corner frequencies are $\omega = 2.5$ rad/sec and $\omega = 40$ rad/sec.

Putting in $y = mx + c$ form as in the previous example,

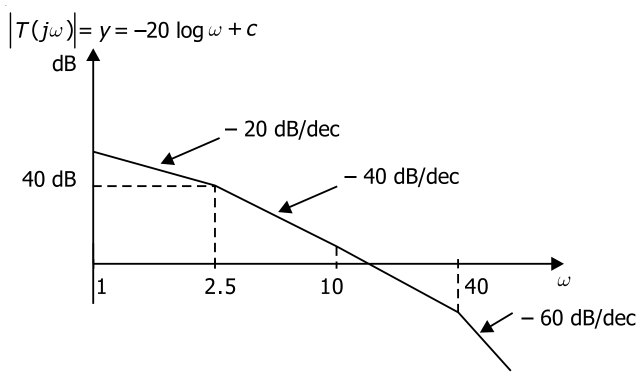


Fig. 10.20 *Bode plot for a given system; to find its transfer function*

at $\omega = 2.5$ rad/sec, $y = 40$ dB.

$$40 = -20 \log \omega + 20 \log K$$

or, $$40 = -20 \log(2.5) + 20 \log K$$

or, $$20 \log K = 40 + 7.95$$

$$= 47.95 \text{ dB}$$

or, $$K = 250.$$

The first factor of T.F is $\dfrac{K}{s} = \dfrac{250}{s}$.

At corner frequency of $\omega = 2.5$ rad/sec, slope changes by -20 dB/decade and the slope is negative. The transfer function component is $\dfrac{1}{\dfrac{s}{2.5}+1} = \dfrac{1}{(1+0.4s)}$

At corner frequency $\omega = 40$ rad/sec, slope again changes by -20 dB/decade. Slope is negative. Hence the transfer function component is $\dfrac{1}{\dfrac{s}{40}+1} = \dfrac{1}{(1+0.025s)}$

The open-loop transfer function is

$$G(s) = \frac{250}{s(1+0.4s)(1+0.025s)}$$

Example 10.6 Draw the Bode plot for a control system having transfer function,

$$G(s)H(s) = \frac{100}{s(s+1)(s+2)}$$

Determine from the Bode plot the following:
(a) Gain margin; (b) Phase margin; (c) Gain cross-over frequency; and (d) Phase cross-over frequency.

Solution

Let us put $s = j\omega$ in the transfer function as,

$$G(j\omega)H(j\omega) = \frac{100}{j\omega(1+j\omega)(2+j\omega)}$$
$$= \frac{50}{j\omega(1+j\omega)(1+j0.5\omega)}$$

a) Corner frequencies are, $\omega_1 = 1$ rad/sec

$$\omega_2 = \frac{1}{0.5} = 2 \text{ rad / sec}.$$

b) The starting of the Bode plot is taken as lower than the lowest frequency. Since lowest corner frequency here is 1 rad/sec, we can take starting frequency as 0.1 rad/sec.

c) By examining the transfer function we see that it represents a type 1 system (power of 5 in the denominator is 1). So the initial slope is -20 dB/decade and continues to corner frequency, $\omega = 1$ rad/sec.

d) Corner frequencies, $\omega = 1$ rad/sec is due to term $\dfrac{1}{1+j\omega}$ of the T.F. Therefore the Bode plot after this frequency will have a further slope of -20 dB/decade. Thus, the total slope will become -40 dB/dec. This slope will continue till the next corner frequency, $\omega_2 = 2$ rad/sec.

This corner frequency of ω_2 is due to the term $\dfrac{1}{1+j0.5\omega}$ of the T.F,; due to which there will be another increase of -20 dB in the slope of the Bode plot at $\omega = 2$ rad/sec. Thus the total slope at frequencies higher than $\omega = 2$ rad/sec will be -60 dB/decade.

e) The phase angle $\phi = \angle G(j\omega)H(j\omega)$ for a range of frequencies is calculated as follows.

$$G(j\omega)H(j\omega) = \frac{50}{(0+j\omega)(1+j\omega)(1+0.5j\omega)}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}\frac{\omega}{0} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{0.5\omega}{1}$$

or,
$$\phi = -90^\circ - \tan^{-1}\omega - \tan^{-1}0.5\omega.$$

Phase angle, ϕ at different values of ω have been calculated as

ω	0	0.1	0.2	0.5	1.0	1.3	1.5	2	4.5
ϕ	-90°	-98.6°	-107°	-130°	-161.6°	-175.5°	-183.2°	-198.4°	-233°

The magnitudes in dB at different frequencies, i.e. at initial and corner frequencies are calculated as follows:

$\omega = 0.1$	Magnitude $\left\lvert\dfrac{50}{j\omega}\right\rvert = 20\log 50 - 20\log\omega$ $= 20\log 50 - 20\log(0.1)$ $= 54$ dB
$\omega = 1.0$	Magnitude $\left\lvert\dfrac{50}{j\omega(1+j\omega)}\right\rvert = 20\log 50 - 20\log\omega - 20\log\sqrt{1+\omega}$ $= 20\log 50 - 20\log 1 - 20\log\sqrt{2}$ $= 30$ dB
$\omega = 2.0$	Magnitude $\left\lvert\dfrac{50}{j\omega(1+j\omega)(1+j0.5\omega)}\right\rvert = 20\log 50 - 20\log 2 = 20\log\sqrt{1^2+2^2}$ $-20\log\sqrt{1^2+(.5\times 2)}$ $= 5$ dB

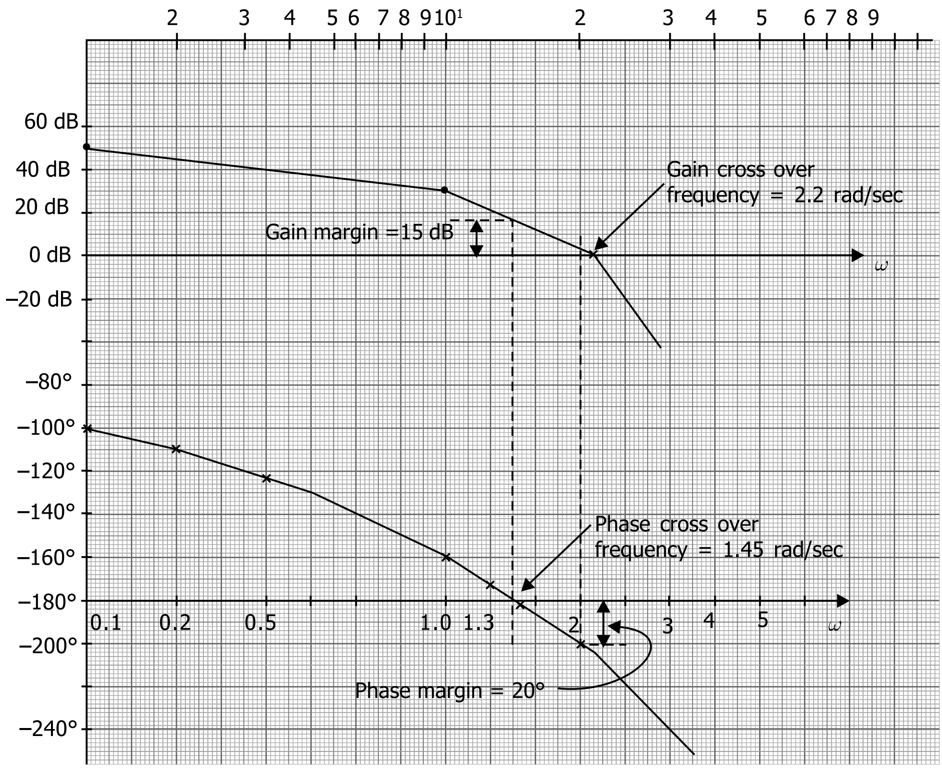


Fig. 10.21 *Bode plot for* $G(s) = \dfrac{100}{s(s+1)(s+2)}$

Fig. 10.21 shows the Bode plot for magnitude and phase angle drawn on log scale. The gain margin is calculated at the phase cross-over frequency and phase margin is calculated at gain cross over frequency. The results as found are
Gain margin = 15 dB; Phase margin = 20°
Gain cross-over frequency = 2.2 rad/sec; Phase cross-over frequency = 1.45 rad/sec.

Example 10.7 Draw the Bode plot for the unity feedback control system whose transfer function is given as

$$G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$

From the plot determine the values of gain margin and phase margin. State whether the system is stable or not.

Solution

$$G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$

$$= \frac{10(0.1s+1)}{s(0.5s+1)(0.2s+1)} \quad \text{[dividing numerator and denominator by 100]}$$

a) The corner frequencies are:

$$\omega_1 = \frac{1}{0.5} = 2 \text{ rad / sec}$$

$$\omega_2 = \frac{1}{0.2} = 5 \text{ rad / sec}$$

$$\omega_3 = \frac{1}{0.1} = 10 \text{ rad / sec}$$

b) Starting frequency of the Bode plot is taken as lower than the lowest corner frequency. Here the lowest corner frequency is 2 rad/sec. We can take starting frequency of say, 1 rad/sec.

c) The system is a type 1 system since the power of s in the denominator of the transfer function is 1. So, the initial slope will be -20 dB/decade. This slope will continue till the comer frequency of $\omega = 2$ rad/sec is reached.

d) The corner frequency of $\omega = 2$ rad/sec is due to the term $\dfrac{1}{0.5s+1}$ for which. $T(j\omega) = \dfrac{1}{1 + j\omega(0.5)}$. The slope of the Bode plot after this frequency will change by -20 dB/dec. Thus, the Bode plot after $\omega = 2$ rad/sec will have a slope of -40 dB/decade and continue till the next corner frequency of $\omega = 5$ rad/sec is reached. Corner frequency, $\omega = 5$ rad/sec is due to the term $\dfrac{1}{(0.2s+1)}$ of the transfer function for which we can write $T(j\omega) = \dfrac{1}{1 + j\omega(0.2)}$. The slope of the Bode plot after $\omega = 5$ rad/sec will change by another -20 dB/decade, making the total slope equal to -60 dB/decade. This slope will continue till the next corner frequency of $\omega = 10$ rad/sec is reached.

The corner frequency of $\omega = 10$ rad/sec is due to the term $(1 + 0.1s)$ appearing at the numerator of the transfer function which can be written as $T(j\omega) = 1/[1 + j\omega(0.1)]$. The Bode plot after this comer frequency will change by $+ 20$ dB/decade. The slope of the Bode plot after $\omega = 10$ rad/sec will therefore be -40 dB/dec and will continue for higher frequencies.

e) Now we will calculate the phase angle $\phi(\omega)$ for the transfer function.

$$G(j\omega) = \frac{10(1+j0.1\omega)}{j\omega(1+j0.5\omega)(1+j0.1\omega)}$$

$$\phi(\omega) = \underline{|G(j\omega)} = -90^\circ - \tan^{-1}\frac{0.5\omega}{1} - \tan^{-1}\frac{0.2\omega}{1} + \tan^{-1}\frac{0.1\omega}{1}$$

The values of ϕ at different values of ω are calculated as follows.

ω rad/sec	0	0.1	0.5	1	2	5	8	10	20
ϕ	-90°	-93°	-107°	-122°	-145°	-176°	-185.3°	-187°	-205°

The magnitudes are calculated as follows:

Frequency (rad/sec)	Magnitude
$\omega = 1$	$\left\lvert\frac{K}{j\omega}\right\rvert = 20\log K - 20\log\omega$ $= 20\log 10 - 20\log(1)$ $= 20$ dB
$\omega = 2$	$\left\lvert\frac{K}{j\omega(1+j\omega 0.5)}\right\rvert$ $= 20\log 10 - 20\log\omega - 20\log\sqrt{1^2 + 2.5\omega^2}$ $= 20\log 10 - 20\log 2 - 20\log\sqrt{1^2 + 2.5\times 2^2}$ $= 11$ dB
$\omega = 5$ rad/sec	$\left\lvert\frac{K}{j\omega(1+j\omega 0.5)(1+j\omega 0.2)}\right\rvert$ $= 20\log K - 20\log\omega - 20\log\sqrt{1^2 + .25\omega^2} - 20\log\sqrt{1^2 + .04\omega^2}$ $= 20\log 10 - 20\log 5 - 20\log\sqrt{1^2 + .25\times 25} - 20\log\sqrt{1^2 + .04\times 25}$ $= -20$ dB
$\omega = 10$	$\left\lvert\frac{10(1+j\omega 0.1)}{j\omega(1+j\omega 0.5)(1+j\omega 0.2)}\right\rvert$ $= 20\log 10 - 20\log 10 - 20\log\sqrt{1+25} - 20\log\sqrt{1+4} + 20\log\sqrt{1+1}$ $= -31$ dB

The Bode plot has been drawn using the above data (Fig. 10.22). The phase margin at gain cross over frequency = 18°. The gain margin calculated at phase cross over frequency = 24 dB.

$$\text{GM} = \text{Initial value} - \text{Final value} = (0)\text{dB} - (-24\text{ dB}) = 24\text{ dB}.$$

Since both phase margin and gain margin are positive, the system is stable.

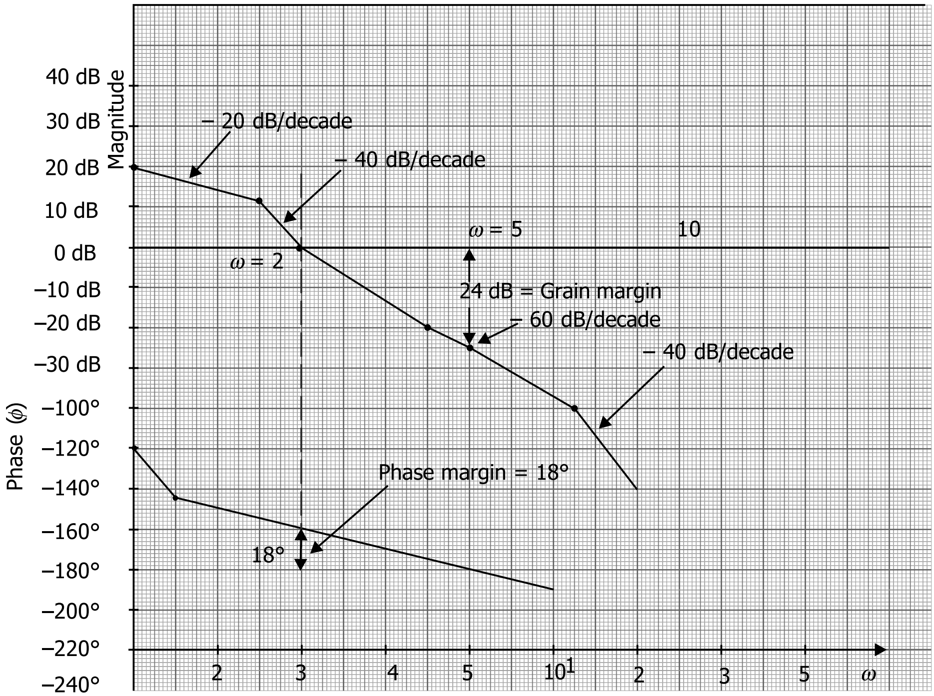


Fig. 10.22 Bode plot for $G(s) = \dfrac{10(s+10)}{s(s+2)(s+5)}$

Example 10.8 Find the open-loop transfer function of a control system whose approximate Bode plot is shown below. The Bode plot starts from a point -4.04 dB at $\omega = 1$ rad/sec.

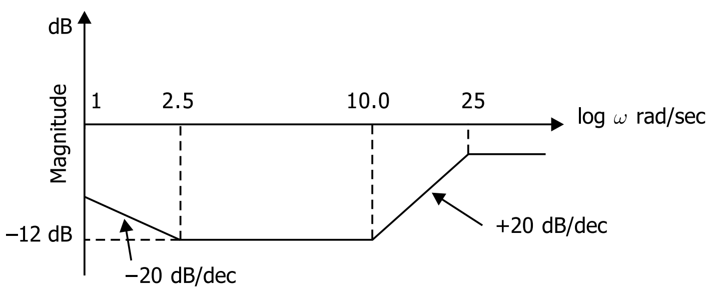


Fig. 10.23

Solution

The corner frequencies are, $\omega_1 = 2.5$ rad/sec, $\omega_2 = 10$ rad/sec, and $\omega_3 = 25$ rad/sec.
Initial slope of the Bode plot is -20 dB/decade. Thus, the system is a type one system whose transfer function will be $\dfrac{K}{s}$.

We calculate the value of K in the following way. Equation of the straight line is written as

$$\begin{aligned} y &= mx + c \\ &= -20 \log \omega + c \text{ (since the initial slope is } -20 \text{ dB/dec)} \\ &= -20 \log \omega + 20 \log K \end{aligned}$$

From Fig. 10.23 we find at $\omega = 2.5$ rad/sec, $y = -12$ dB
Putting these values,

$$i - 12 = -20 \log(2.5) + 20 \log K$$

or, $$20 \log K = 4.04 \text{ dB}$$

or, $$K = 0.63$$

a) Since the first line of the Bode plot has a slope of -20 dB/decade and starts from a point -4.04 dB at frequency $\omega = 1$ rad/sec, the factor contributing to the total transfer function of the system is

$$\frac{K}{j\omega} = \frac{0.6}{s}$$

b) At corner frequency $\omega = 2.5$ rad/sec, slope changes by $+20$ dB/dec. So the factor contributing this change is

$$\left(\frac{s}{2.5} + 1\right) = (1 + 0.4s)$$

and be placed at the numerator of the transfer function.

c) At corner frequency $\omega = 10$ rad/sec, slope changes by $+20$ dB/decade. Since the slope is again changing in the positive direction the factor contributing this has to be placed in the numerator of the transfer function and the factor is

$$\left(\frac{s}{10} + 1\right) = (1 + 0.1s)$$

d) At corner frequency, $\omega = 25$ rad/sec, the slope now changes in the negative direction by -20 dB/decade. The factor contributing this change has to be placed in the denominator of the T.F and is equal to

$$\left(\frac{s}{2.5}+1\right)=(1+0.04s)$$

e) Thus the open-loop transfer function is

$$G(s)=\frac{0.63(1+0.4s)(1+0.1s)}{s(1+0.04s)}$$

Example 10.9 Determine the open-loop transfer function of a system whose approximate Bode plot is shown in Fig. 10.24.

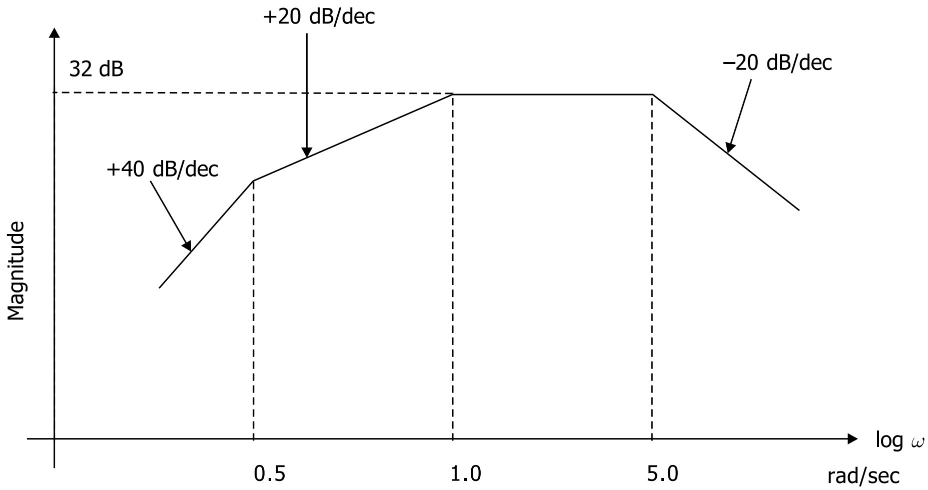


Fig. 10.24 Bode plot of a given control system

Solution

a) The first line of the Bode plot as above has a positive slope of +40 dB/dec. Therefore, this corresponds to s^2 term in the numerator of the transfer function this being representing a type 2 system. Thus, we have a transfer function component, $T_1(s) = Ks^2$

b) At corner frequency of $\omega = 0.5$ rad/sec, there is -20 dB/sec change of slope. This will contribute a term,

$$T_2(s)=\frac{1}{1+\left(\frac{s}{0.5}\right)}=\frac{1}{1+2s}$$

c) At corner frequency $\omega = 1$ rad/dec, there is further reduction of slope by -20 dB/dec. This negative change of slope at $\omega = 1$ rad/sec corresponds to a transfer function component of

$$T_3(s) = \frac{1}{1+\left(\frac{s}{1}\right)} = \frac{1}{1+s}$$

d) At corner frequency $\omega = 5$ rad/sec, there is -20 dB/dec reduction of slope which corresponds to

$$T_4(s) = \frac{1}{1+\left(\frac{s}{5}\right)} = \frac{1}{(1+0.2s)}$$

Thus the transfer function of the system is equal to $G(s) = T_1(s)\, T_2(s)\, T_3(s)\, T_4(s)$

i.e.,
$$G(s) = \frac{Ks^2}{(1+2s)(1+s)(1+0.2s)}$$

We have now to calculate the value of K.
We write the equation of the second line whose slope is $+20$ dB/decade. At $\omega = 1$ rad/sec, the magnitude is 32 dB. We write,

$$y = 20 \log \omega + c$$

$$32 \text{ dB} = 20 \log(l) + c$$

or,
$$c = 32 \text{ dB}$$

Then we have to find the magnitude at $\omega = 0.5$ rad

$$y = 20 \log \omega + c$$

$$= 20 \log (0.5) + 32$$

$$= 25.97 \text{ dB (see Fig. 10.25)}$$

Now we write the equation for + 40 dB line

$$y = 40 \log \omega + c$$

$$25.97 = 40 \log(0.5) + c$$

or,
$$c = 38.02 \text{ dB}$$

$$c = 201\text{og } K$$

or,
$$\log K = \frac{38.02}{20} = 1.9$$

or,
$$K = 79.62$$

Thus,

$$G(s) = \frac{79.62s^2}{(1+2s)(1+s)(1+.2s)}$$

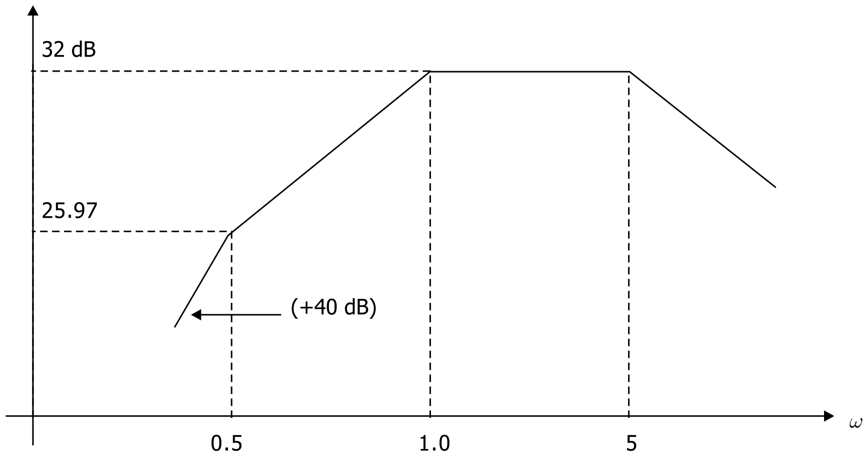


Fig. 10.25 *Determination of transfer function from the Bode plot*

Example 10.10 (a) State the advantages of Bode plot. Draw the Bode diagram for

$$G(s) = \frac{100(0.02s+1)}{(s+1)(1+0.1s)(1+0.01s)^2}$$

(b) Mark the following on the Bode diagram, recording the numerical values

a) Gain cross over frequency
b) Phase margin
c) Phase cross over frequency
d) Gain margin

Solution

(a) In Bode plot, $G(j\omega)$ in dB, i.e. $20\log|G(j\omega)|$ is plotted against log ω. Similarly, phase angle of $G(j\omega)$ is plotted against log ω. The following are its advantages:

i) Since $G(j\omega)$ consists of many multiplicative factors in both numerator and denominator, it is convenient to take logarithm of $|G(j\omega)|$ so that the factors can be converted into additions and subtractions, which can be carried out easily. This helps in simplified design modification procedure.

ii) The relative stability of the system can be studied by calculating gain margin and phase margin from the Bode plot.

iii) Transfer function can be obtained from Bode plot.

iv) The value of system gain K can be determined from Bode plot for desired gain margin and phase margin specifications.

Solution

(b) The students are to try this problem following the steps as:
First the corner frequencies are calculated and the initial slope is calculated as

$$20\log_{10} K = 20\log_{10} 100 = 40 \text{ dB.}$$

Step 1: Arrange $G(s)\ H(s)$ in time constant form.
The given system is already in time constant form, i.e.

$$G(s) = \frac{100(0.02s+1)}{(s+1)(1+0.1s)(1+0.01s)^2} \qquad \text{....(1)}$$

Step 2: Factors
As the system is type zero system $20\log_{10}|G(j\omega)H(j\omega)|$

$$20\log_{10} K = 20\log_{10}(\omega)$$
$$= 20\log_{10} = 40 \text{ dB}$$

The following are the corner frequencies:

i) Simple zero $(0.02s + 1)$, $\omega_{C_1} = \dfrac{1}{0.02} = 50$ rad/sec

ii) Simple pole $(s + 1)$, $\omega_{C_2} = 1$ rad/sec

iii) Simple pole $(1 + 0.1s)$, $\omega_{C_3} = 10$ rad/sec

iv) Simple pole $(1 + 0.01s)^2$, $\omega_{C_4} = 100$ rad/sec

Step 3: Magnitude plot

i) Contribution of K is $20 \log K = 40$ dB
ii) 1 pole at origin $(s + 1)$ starting slope becomes = – 20 dB/decade
iii) At $\omega_{C_2} = 10$

Slope will be = – 40 dB/decade

iv) At $\omega_{C_3} = 50$

Slope will be – 40 + 20 = – 20 dB/decade

v) At $\omega_{C_4} = 100$

Slope will be = – 20 – 40 = – 60 dB/decade.

Step 4: Phase angle plot
Substitute $s = j\omega$ in equation (1)

$$G(j\omega)H(j\omega) = \frac{100(0.02j\omega+1)}{(j\omega+1)(1+0.1j\omega)(1+0.01j\omega)}$$

$$\angle G(j\omega)H(j\omega) = \frac{\angle 1+0.02j\omega}{\angle + j\omega \angle 1+0.1j\omega < 1+0.01j\omega}$$

The calculation of phase angle is shown in a tabular form. The Bode plot for the given *TF.* is shown in Fig. 10.26.

Phase angle table

ω	$\tan^{-1} 0.02\omega$	$-\tan^{-1} \omega$	$-\tan^{-1} 0.1\omega$	$-2 \tan^{-1} 0.01\omega$	ϕ_R
0.1	0.114	−5.71	−0.57	−0.114	−6.28
1	1.145	−45	−57	−1.14	−50.695
5	5.71	−78.69	−26.56	−5.72	−105.26
10	11.30	−84.28	−45	−11.4	−129.38
20	21.80	−87.13	−63.43	−45.22	−173.98
50	45	−88.85	−78.69	−53.12	−175.16
100	63.43	−89.42	−84.28	−90	−200.02
150	71.56	−89.42	−86.42	−112.61	−216.84
∞	90°	−90°	−90°	−180°	−360°

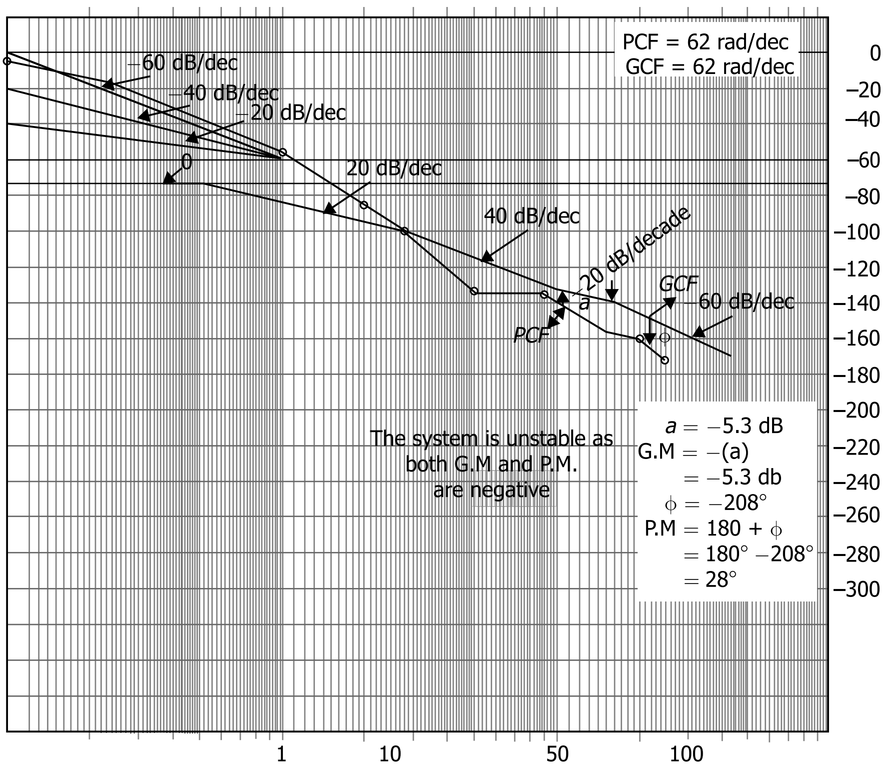


Fig. 10.26 *Bode plot for* $\dfrac{100(0.02s+1)}{(s+1)(1+0.1s)(1+0.01)^2}$

Example 10.11 Define gain margin and phase margin. Determine the gain margin and phase margin of the given transfer function $G(s)H(s) = \dfrac{2.6}{s(s+1)(s+4)}$.

Solution

Gain margin, *GM*, is the margin in gain that can be allowed till the system reaches a state of instability.

Phase margin, *PM*, is the amount of additional phase lag that can be introduced in the system before it becomes unstable.

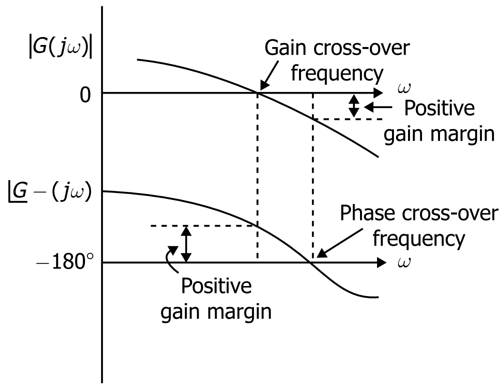
Mathematically, *GM* is defined as the reciprocal $G(j\omega)\ H(j\omega)$ when $\omega = \omega_{pc}$, i.e. at phase cross-over frequency.

i.e.
$$GM = \frac{1}{G(j\omega)H(j\omega)} \quad \text{at } \omega = \omega_{pc}$$

and
$$PM = 180^\circ + \angle G(j\omega)H(j\omega) \text{ at } \omega = \omega_{gc}$$

ω_{gc} is the gain cross-over frequency.

For a stable system, the phase margin and gain margin are indicated in the given diagram below. For a stable system, both gain margin and phase margin are positive.



At $\omega = \omega_{gc}$ the magnitude of $|G(j\omega)| = 0$ dB, i.e. equal to 1 absolute.

Therefore,
$$\frac{2.6}{|j\omega||1+j\omega||4+j\omega|} = 1$$

or,
$$\frac{2.6}{\omega\sqrt{1+\omega^2}\sqrt{16+\omega^2}} = 1$$

or,
$$6.76 = \omega^2(1+\omega^2)(16+\omega^2)$$

or, $$\omega^2(16+17\omega^2+\omega^4)=6.76$$

This equation can be solved by trial-and-error method as

$\omega = 0.4,$ L.H.S. $= 2.999$
$\omega = 0.5,$ L.H.S. $= 5.077$
$\omega = 0.55$ L.H.S. $= 5.707$
$\omega = 0.56,$ L.H.S. $= 6.720$
$\omega = 0.561,$ L.H.S. $= 6.750$
$\omega = 0.5615,$ L.H.S. $= 6.760$

Thus, $\omega_{gc} = 0.5615$ rad/sec

$$G(j\omega)H(j\omega)=\frac{2.6}{j\omega(1+j\omega)(4+j\omega)}$$

$$\angle G(j\omega)H(j\omega)=-90-\tan^{-1}\omega-\tan^{-1}\frac{\omega}{4}$$

Angle $\angle G(j\omega)H(j\omega)$ at $\omega=\omega_{gc}=0.5615$ is calculated as

$$\angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc=0.5615}}=-90-\tan^{-1}(0.5615)-\tan^{-1}\left(\frac{0.5615}{4}\right)$$
$$=-127.3^\circ$$

$$\text{Phase margin } =180+\angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$$
$$= 180 - 127.3^\circ$$
$$= 52.7^\circ$$

At phase cross-over frequency, the phase margin is -180°. Therefore, in the present case, we can write

$$-180^\circ=-90^\circ-\tan^{-1}\omega-\tan^{-1}\frac{\omega}{4}$$

or, $$\tan^{-1}\omega-\tan^{-1}0.25\omega=90^\circ$$

Taking tan of both sides,

$$\tan[\tan^{-1}\omega-\tan^{-1}0.25\omega]=\tan 90^\circ=\infty$$

or, $$\frac{\omega+0.25\omega}{1-\omega\times 0.25\omega}=\infty$$

Equating denominator of the *LHS* to 0,

$$1 - 0.25\omega^2 = 0$$

or, $$\omega^2 = 4$$

or, $$\omega = 2 = \omega_{pc}$$

The gain margin is calculated at $\omega = \omega_{pc} = 2$ by first rationalizing $G(j\omega)\ H(j\omega)$ as

$$G(j\omega)H(j\omega) = \frac{2.6}{j\omega(1+j\omega)(4+j\omega)}$$

$$= \frac{2.6(-j\omega)(1-j\omega)(4-j\omega)}{j\omega(1+j\omega)(4+j\omega)(-j\omega)(1-j\omega)(4-j\omega)}$$

$$= \frac{-13\omega^2}{\omega^2(1+\omega^2)(16+\omega^2)} - \frac{2.6j\omega(4-\omega^2}{\omega^2(1+\omega^2)(16+\omega^2)}$$

$$\left|G(j\omega)H(j\omega)\right| = \frac{-13\omega^2}{\omega^2(1+\omega^2)(16+\omega^2)}$$

at $\omega = \omega_{pc} = 2,\ \left|G(j\omega)H(j\omega)\right| = \dfrac{-13}{(1+4)(16+4)} = -0.13$

Gain margin, $GM = 20\log\dfrac{1}{0.13} = 17.72\ \text{dB}$

Since both phase margin and gain margin are positive, the system is stable.

Example 10.12 Sketch the Bode plot for the transfer function given by

$$G(s)H(s) = 2(s+0.25)/s^2(s+1)(s+0.5)$$

and from the plot find (a) phase and gain cross over frequencies; (b) gain margin and phase margin. Is the system stable?

Solution

$$G(s)H(s) = \frac{2(s+0.25)}{s^2(s+1)(s+0.5)}$$

Step 1: Arrange $G(s)\ H(s)$ in time constant form

$$G(s)H(s) = \frac{\dfrac{2\times 0.25}{0.5}\left[\dfrac{s}{0.25}+1\right]}{s^2(s+1)\left[\dfrac{s}{0.5}+1\right]}$$

$$= \frac{1(4s+1)}{s^2(s+1)(2s+1)}$$

Converting to sinusoidal form

$$G(j\omega)H(j\omega) = \frac{(j4\omega + 1)}{(j\omega)^2(j\omega + 1)(2j\omega + 1)}$$

The Bode plot for the transfer function has been drawn and shown in Fig. 10.27.

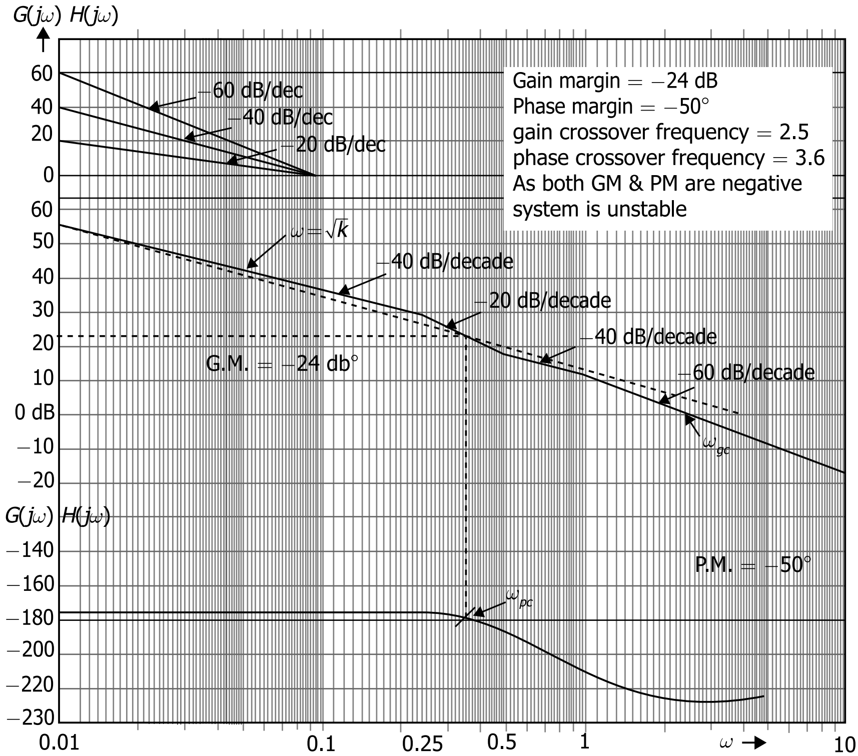


Fig. 10.27 *Bode plot for* $G(s)H(s) = \dfrac{2(s+0.25)}{s^2(s+1)(s+0.5)}$

Step 2: Factors are:

This is type 2 system, hence initial slope of Bode plot = −40 dB/decade and the plot intersects 0 dB axis at $\omega = \sqrt{k} = \sqrt{1} = 1\,\text{rad/sec}$.

The corner frequencies are:

a) $\omega = \dfrac{1}{4} = 0.25$ rad/sec due to numerator

b) $\omega = \dfrac{1}{2} = 0.5$ rad/sec due to denominator

c) $\omega = 1$ rad/sec due to denominator

Frequency range is considered from $\omega = 0.1$ rad/sec to $\omega = 10$ rad/sec.
The initial slope of plot is -40 dB/decade and corner frequency is 0.25 rad/sec. The plot after $\omega = 0.25$ has slope $-40 + 20 = -20$ dB/decade. The slopes after other two corner frequencies are
After $\omega = 0.5$, $(-20-20) = -40$ dB/decade
After $\omega = 1$, $(-40-20) = -60$ dB/decade
Step 3: Phase angle

$$\angle G(j\omega)H(j\omega) = \tan^{-1}(4\omega) - 180° - \tan^{-1} 2\omega$$

The phase angles for frequency range considered are calculated as

ω	0.01	0.1	– 179.52	0.25	0.5	1	5
$\angle G(j\omega)H(j\omega)$	-175.2	-175.6	-188	-212.4	-225		

Since both gain margin and phase margin are negative, the system is unstable.

10.5.6 Plotting Bode Diagrams with MATLAB

When the system is defined in the form given by

$$G(s) = \frac{\text{num}(s)}{\text{den}(s)}$$

where num(s) and den(s) are respectively the numerator and denominator polynomial in descending powers of *s,* the command bode (num, den) will draw the Bode diagram.

The command [mag, phase, ω] = bode (num, den, ω) will return the frequency response of the system in matrices mag, phase and ω. The phase angle is in degree. The mag and phase matrices is evaluated at user-specified frequency points.

The command mag db = 20 * log 10 (mag) will convert magnitude to decibels.

For specifying the frequency range, use the command ω = log space (dl, d2) to generate a vector of 50 points logarithmically equally spaced between decades 10^{dl} and 10^{d2}. To generate *n* number of frequency points use the command

ω = log space (dl, d2, *n*).

For example, the command
ω = log space (0, 3, 100) will generate 100 points between 1 rad/sec and 1000 rad/sec. Use the command bode (num, den, ω) to incorporate user-specified frequency points.

Example 10.13 The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{9(s^2 + 0.2s + 1)}{s(s^2 + 1.2s + 9)}$$

Plot the Bode diagram with MATLAB.

Solution

If we use the command bode (num, den), then the frequency range is automatically determined to b from 0.1 to 10 rad/sec. Here we will use the frequency range to be 0.1 to 1000 rad/sec and hence the command ω = log space (−2, 3,100) will be used.

The following MATLAB Program 10.1 will generate the Bode diagram as shown in Fig. 10.28.

MATLAB PROGRAM 10.1

```
num = [0   9      1.8   9];
den = [1    1.2    9     0];
w = log space (–2, 3, 100);
bode (num, den, w)
subplot (2, 1, 1);
title ('Bode Diagram of G(s) = 9(s ^ 2 + 0.2s + 1)/[s(s ^ 2 + 1.2s + 9]')
```

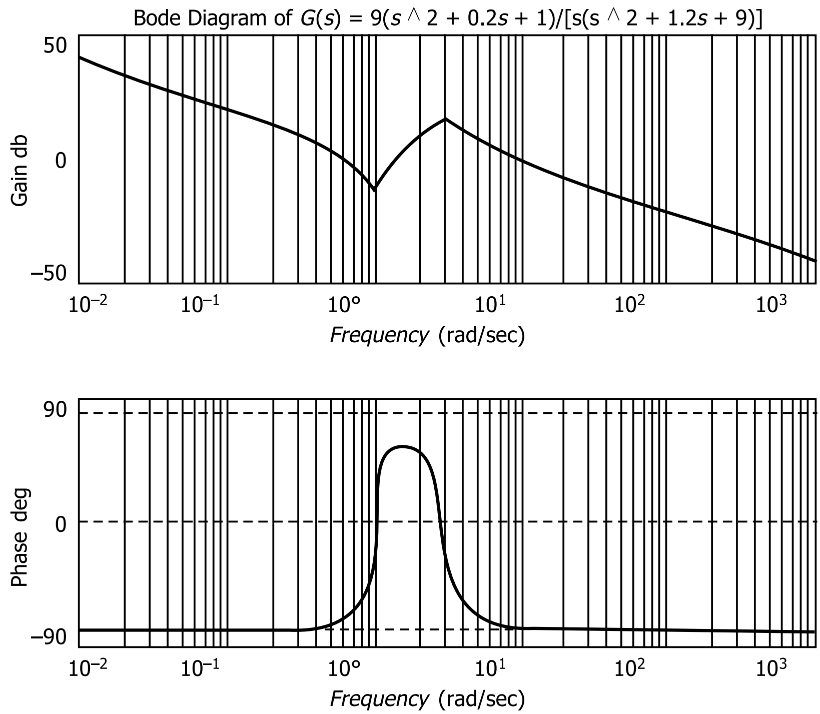


Fig. 10.28 *Bode plot for* $G(s) = \dfrac{9(s^2 + 0.2s + 1)}{s(s^2 + 1.2s + 9)}$

10.6 POLAR PLOT AND NYQUIST CRITERION

Nyquist used polar plots to study to stability of control system way back in 1932 and as a mark of respect to him polar plots are also called Nyquist plots. It is interesting to observe that the shapes of polar plots change due to addition of poles to the transfer function. As an illustration let us consider the polar plot of transfer function $\frac{1}{1+j\omega T_1}$ as shown in Fig. 10.29.

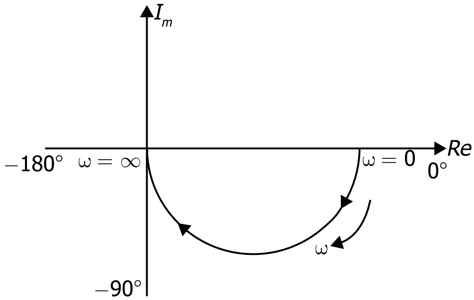


Fig. 10.29 *Bode plot of* $G(s) = \frac{1}{1+sT_1}$

It is observed that addition of poles to the transfer function changes the shapes of the polar plots as illustrated in Fig. 10.30(a), (b) and (c).

Let us add poles in this transfer function one more in succession making it $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$ and $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$.

Every time we add a pole, the polar plot gets rotated by $-90°$ as shown in Fig. 10.30(b) and (c) respectively.

We can observe that addition of a non-zero pole causes rotation of the polar plot at the high frequency portion by 90°.

If we add a pole at the negative real axis the transfer function $\frac{1}{1+j\omega T_1}$ will change to a transfer function of $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$.

The polar plots of these have been shown in Fig. 10.30(a) and (b) respectively. We can notice that addition of a pole has rotated the polar plot by $-90°$ at both zero and infinite frequencies.

Fig. 10.30(c) shows the polar plot for adding another pole with total transfer function as $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$. Fig. 10.30(d) is the polar plot for adding a pole at the origin with $T.F. = \frac{1}{j\omega(1+j\omega T_1)}$.

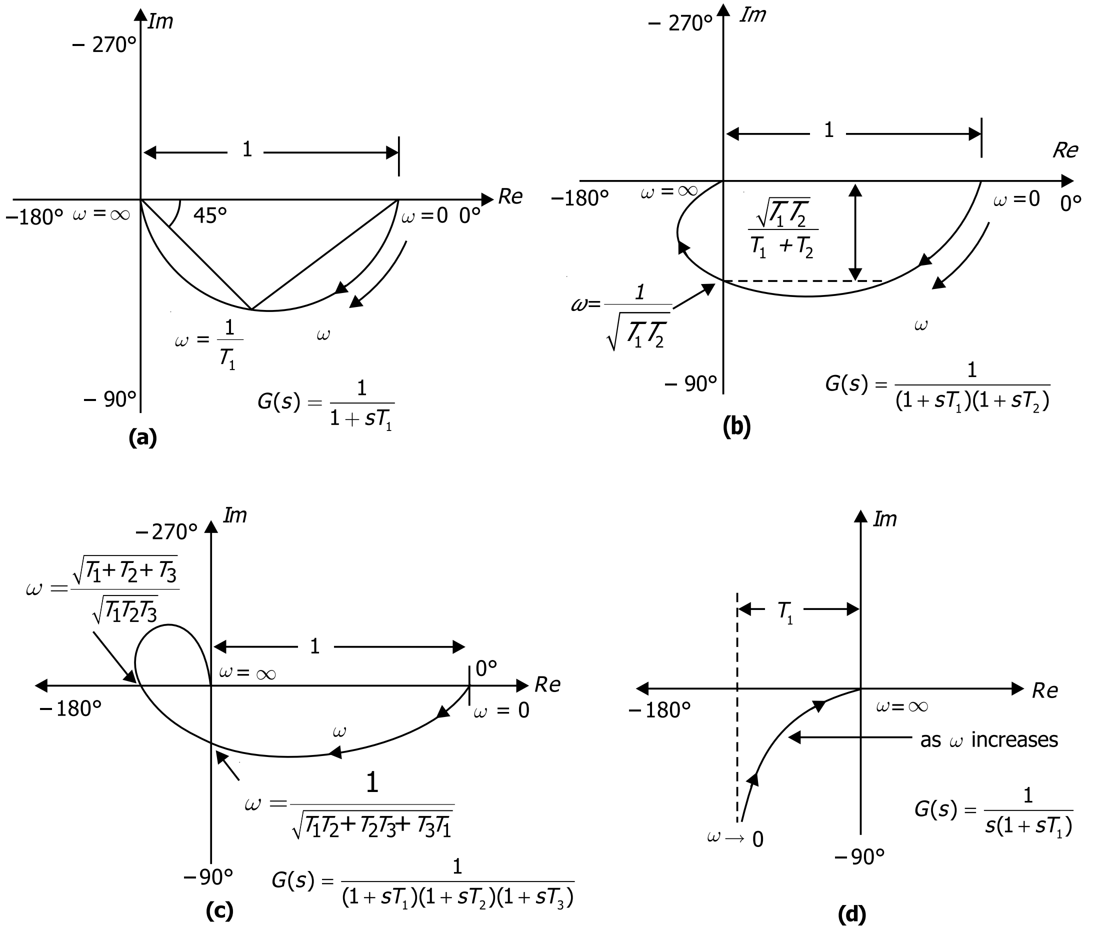


Fig. 10.30 *Effect of addition of poles on polar plot illustrated*

Nyquist Criterion

Nyquist criterion is based on the principle of argument which states that if there are P poles and Z zeros of the transfer function enclosed in s-plane contour, then the corresponding GH plane contour will encircle the origin Z-times in clockwise direction and P-times in anticlockwise direction, i.e. $(P-Z)$ times in anticlockwise direction.

To illustrate this let us consider a contour in the s-plane which encloses one zero of $G(s)$ $H(s)$, say $s = \alpha_1$. The remaining poles and zeros are placed in the s-plane outside the contour as shown in Fig. 10.31.

For any non-singular point in the s-plane contour, there is a corresponding point $G(s)$ $H(s)$ in the GH-plane contour. Let s follows a prescribed path as shown in the clockwise direction making one circle and returning to its original position. The phasor $(s - \alpha_1)$ will cover an angle of -2π while the net angle covered by the other two phasors, namely $(s - \beta_1)$

and $(s - \alpha_2)$ is zero. The phasor representing the magnitude of $G(s)\ H(s)$ in the GH-plane will undergo a net phase change of -2π.

That is, this phasor will encircle the origin once in the clockwise direction as shown in Fig. 10.31(b). If the contour in the s-plane encloses two zeros, α_1 and α_2 in the s-plane, the contour in the GH-plane will encircle the origin two-times. Generalizing this, it can be stated that for each zero of $G(s)\ H(s)$ in the s-plane enclosed, there is corresponding encirclement of origin by the contour in the GH-plane once in the clockwise direction.

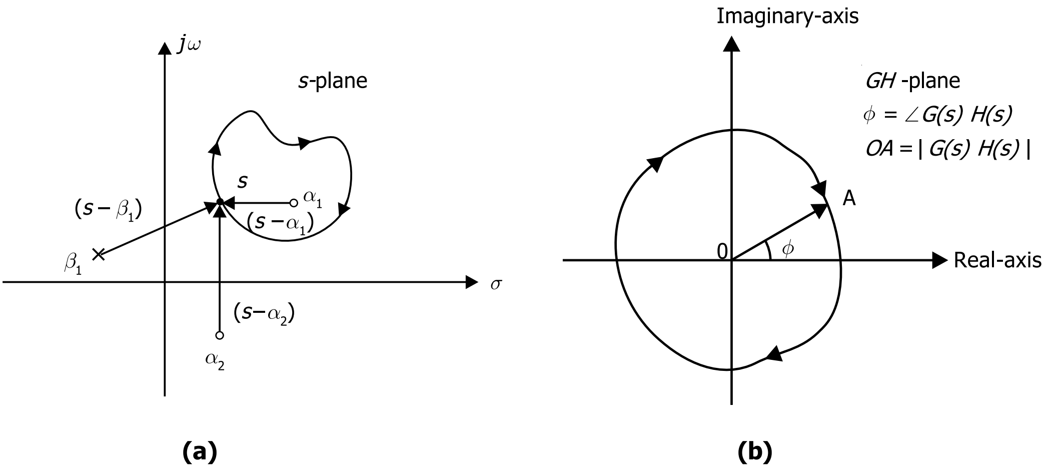


Fig. 10.31 *Nyquist stability-criterion illustrated*

When the contour in the s-plane encloses a pole and a zero say at α and β respectively, both the phasors $(s - \alpha)$ and $(s - \beta)$ will generate an angle of 2π as s traverses the prescribed path. However since $(s - \beta)$ will be in the denominator of G(s) $H(s)$, the contour in the GH-plane will experience one clockwise and one counter clockwise encirclement of the origin.

If we have p number of poles and z number of zeros enclosed by the contour in the s-plane, the: contour in the GH-plane will encircle the origin by $(p - z)$ times in counterclockwise direction [since $(s - p)$ is in the denominator of the open loop transfer function, $G(s)\ H(s)$].

For a system to be stable, there should not be any zeros of the closed loop transfer function in the right half of s-plane. Therefore, *a closed loop system is considered stable, if the number of encirclements of the contour in GH-plane around the origin is equal to the open-loop pole of the transfer function, $G(s)\ H(s)$.*

We can express

$$G(s)\ H(s) = [1 + G(s)\ H(s)] - 1$$

From this, it can be said that the contour $G(s)\ H(s)$ in GH-plane corresponding to Nyquist contour in the s-plane is the same as contour of $[1 + G(s)\ H(s)]$ drawn from the point $(-1 + j0)$.

Thus it can be stated that if the open loop transfer function $G(s)\ H(s)$ corresponding to Nyquist contour in the s-plane encircles the point at $(-1 + j0)$ in the counterclockwise direction the number of times equal to the right half s-plane poles of $G(s)\ H(s)$, it can then be said that the closed-loop control system is stable.

10.6.1 Nyquist Path or Nyquist Contour

The transfer function of a feedback control system is expressed as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The denominator is equated to 0 to represent the characteristic equation as

$$1 + G(s)H(s) = 0$$

The study of stability of the closed-loop system is done by determining whether the characteristic equation has any root in the right half of the s-plane. Or, in other words, we have to determine whether $C(s)/R(s)$ has any pole located in the right half of the s-plane. We use a contour in the s-plane which encloses the whole of the right hand half of the s-plane. The contour will encircle in the clockwise direction having a radius of infinity. If there is any pole on the $j\omega$-axis, these are bypasses with small semicircles taking around them. If the system does not have any pole or zero at the origin or in the $j\omega$-axis, the contour is drawn as shown in Fig. 10.32(b). A few examples on application of Nyquist criterion for stability study will be taken up now.

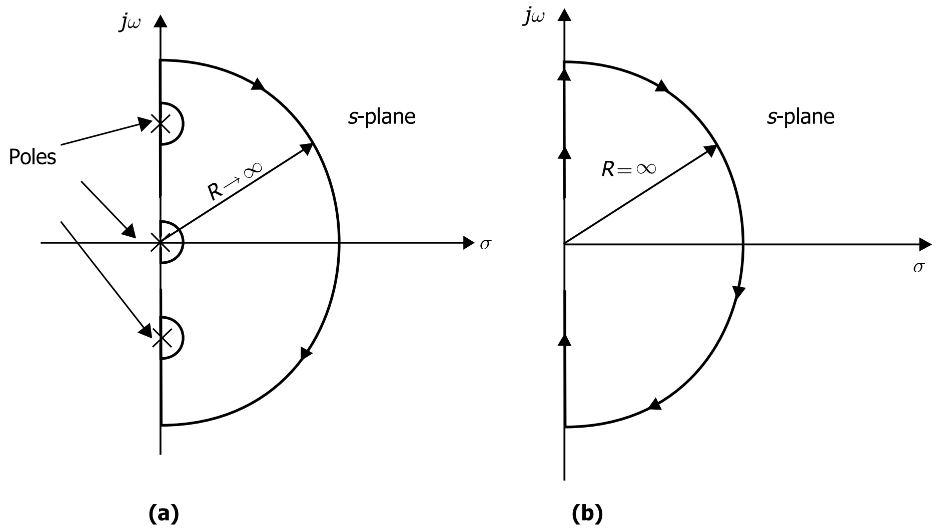


Fig. 10.32 *Nyquist contour*

Example 10.14 The open-loop transfer function of a system is given as

$$G(s)H(s) = \frac{K}{(1+\tau_1 s)(1+\tau_2 s)}$$

Examine the stability of the system by applying Routh Hurwitz criterion, and Nyquist stability criterion.

Solution

The closed-loop transfer function

$$\frac{C(S)}{R(S)} = \frac{\dfrac{K}{(1+\tau_1 s)(1+\tau_2 s)}}{1+\dfrac{K}{(1+\tau_1 s)(1+\tau_2 s)}}$$

$$= \frac{K}{s^2\tau_1\tau_2 + s(\tau_1+\tau_2)+(K+1)}$$

The characteristic equation is $s^2\tau_1\tau_2 + s(\tau_1+\tau_2)+(K+1)$

or,
$$s^2 + \frac{(\tau_1+\tau_2)}{\tau_1\tau_2}s + \frac{K+1}{\tau_1\tau_2} = 0$$

Routh Array is

$$\begin{array}{ccc} s^2 & 1 & \dfrac{K+1}{\tau_1\tau_2} \\ s^1 & \dfrac{\tau_1+\tau_2}{\tau_1\tau_2} & 0 \\ s^0 & \dfrac{K+1}{\tau_1\tau_2} & 0 \end{array}$$

Since $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are time constants which are positive, the system is stable for all values of K. Further, as the poles are lying on the left hand side at $-\dfrac{1}{T_1}$ and $-\dfrac{1}{T_2}$, in the s-plane, the system is stable. For drawing the Nyquist plot, we put $s = j\omega$ in the transfer function as

We have,
$$G(s)H(s) = \frac{K}{(1+s\tau_1)(1+s\tau_2)} = \frac{\dfrac{K}{\tau_1\tau_2}}{\left(s+\dfrac{1}{\tau_1}\right)\left(s+\dfrac{1}{\tau_2}\right)}$$

Put $s = j\omega$,

$$G(j\omega)H(j\omega)=\frac{\dfrac{K}{\tau_1\tau_2}}{\left(j\omega+\dfrac{1}{\tau_1}\right)\left(j\omega+\dfrac{1}{\tau_2}\right)}$$

at $\omega = 0$, $$|G(j\omega)H(j\omega)|=\frac{K}{\tau_1\tau_2}\times\tau_1\tau_2=K$$

at $\omega = \infty$, $$|G(j\omega)\,H(j\omega)| = 0$$

at $\omega=\dfrac{1}{\sqrt{\tau_1\tau_2}}$, $$|G(j\omega)H(j\omega)|=\frac{K\sqrt{\tau_1\tau_2}}{\tau_1+\tau_2}$$

The plot of $\dfrac{K}{(1+j\omega\tau_1)(1+j\omega\tau_2)}$ as ω increases from 0 to ∞ has been shown in Fig. 10.33.

The Nyquist plot is symmetrical about the real axis since $G(j\omega)\,H(j\omega) = G(-j\omega)\,H(-j\omega)$. Since the plot of $G(j\omega)\,H(j\omega)$ does not encircle $(-1 + j0)$ point, the system is stable for any positive value of τ_1, τ_2 and K.

If there is any open-loop poles in the $j\omega$ axis or an open-loop pole at the origin, as for example, in the case of $G(s)H(s)=\dfrac{K}{s(1+\tau s)}$, the Nyquist contour must be intended to bypass the origin or the point on the $j\omega$ axis where such a pole exists. This is because the s-plane contour should not pass through a singularity of $1 + G(s)\,H(s)$.

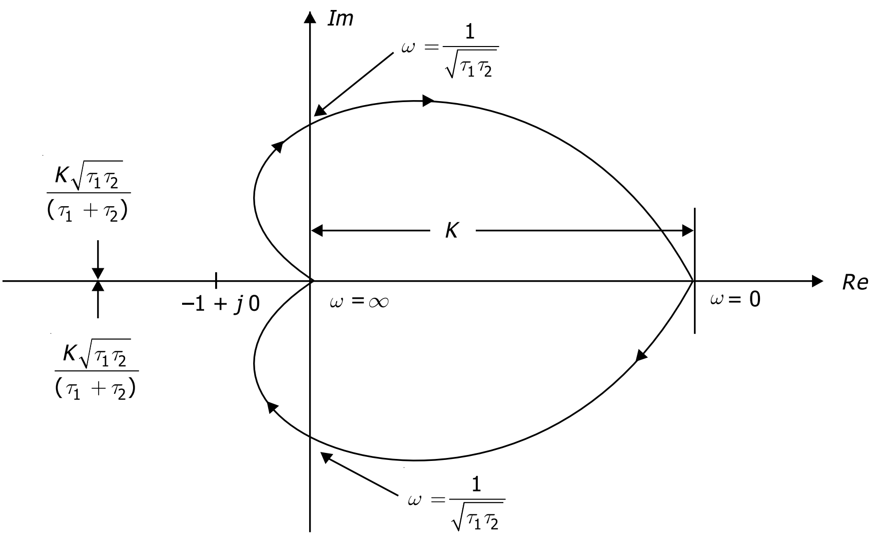


Fig. 10.33 *Nyquist plot of* $G(j\omega)H(j\omega)=\dfrac{K}{(1+j\omega\tau_1)(1+j\omega\tau_2)}$

Example 10.15 The open-loop transfer function of a unity feedback control system is given as

$$G(s)H(s)=\frac{K}{s(s^2+s+4)}$$

Determine the value of K for which the system is stable by applying (i) Routh Hurwitz criterion; (ii) Nyquist criterion; (iii) Root-locus technique.

Solution

i) The characteristic equation is written as

$$s(s^2+s+4)+K=0$$

or,
$$s^3+s^2+4s+K=0$$

The Routh Array is,

$$\begin{array}{ccc} s^3 & 1 & 4 \\ s^2 & 1 & K \\ s^1 & \dfrac{4-K}{1} & 0 \\ s^0 & K & 0 \end{array}$$

The condition for stability is, $K>0$ and $4-K>0$ that is, the system is stable if $0<K<4$

ii) We have, $G(s)H(s)=\dfrac{K}{s(s^2+s+4)}$

Putting $s=j\omega$,

$$G(j\omega)H(j\omega)=\frac{K}{j\omega\left[(j\omega)^2+j\omega+4\right]}$$

$$=\frac{K}{j\omega\left[4-\omega^2+j\omega\right]}$$

$$=\frac{K[(4-\omega^2)-j\omega]}{j\omega[(4-\omega)^2+j\omega][(4-\omega)^2-j\omega]}$$

$$=\frac{-(j^2)K[(4-\omega)^2-j\omega]}{j\omega[(4-\omega^2)^2+\omega^2]}$$

$$=\frac{-K[\omega+j(4-\omega^2)]}{j\omega[(4-\omega^2)^2+\omega^2]}$$

The crossing of Nyquist plot at the real axis can be found out by equating the imaginary part of the above to 0.

Thus, $$4 - \omega^2 = 0$$

or $$\omega = \pm 2$$

The magnitude $|G(j\omega)\ H(j\omega)|$ at $\omega = \pm 2$, i.e. at $\omega^2 = 4$ is found as

$$|G(j\omega)H(j\omega)| = \frac{-K[\omega]}{\omega[(4-4)+4]} = -\frac{K}{4}$$

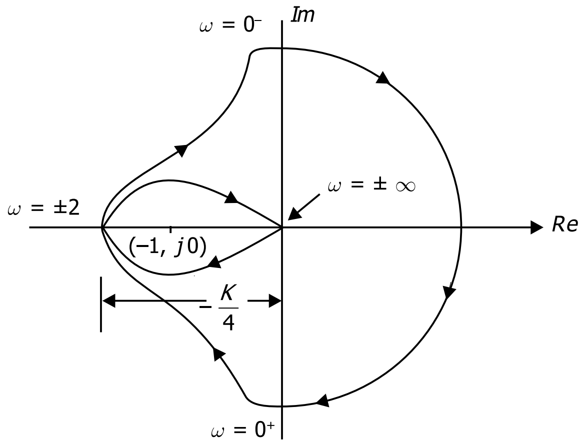


Fig. 10.34 *Nyquist plot of* $G(j\omega)H(j\omega) = \dfrac{K}{j\omega(4-\omega^2+j\omega)}$

The Nyquist plot for $G(j\omega)\ H(j\omega)$ has been drawn as in Fig. 10.34. From this figure we notice that the intercept on the real axis at $\omega = \pm 2$ is $-\dfrac{K}{4}$ and this $-\dfrac{K}{4}$ is greater than -1. The value of K is found as

$$-\frac{K}{4} > -1$$

or $$K > 4$$

According to Nyquist stability criterion, if there are P poles and Z zeros in the transfer function enclosed by the s-plane contour, the corresponding contour in the GH-plane must encircle origin Z times in the clockwise direction and P times in the anticlockwise direction resulting encircling $(P - Z)$ times in the anticlockwise direction. Again $G(s)\ H(s)$ contour around the origin is the same as contour of $1 + G(s)\ H(s)$ drawn from $(-1 + j0)$ point on the real axis.

Here the encirclement of $-1 + j0$ has been in the clockwise direction two times, hence $N = -2$. There is no pole on the right hand side of the s-plane and hence $P = 0$

Putting $$P - Z = N$$

we have, $$0 - Z = -2 \text{ or } Z = 2$$

Thus, for $K > 4$, the system is unstable. The system will be stable if $K < 4$.

The reader is advised to try the root-locus technique of solving this problem to arrive at the same result as above.

Example 10.16 The open-loop transfer function of a control system is given. Determine the closed-loop stability using Nyquist criterion.

$$G(s)H(s) = \frac{K}{s(1+\tau s)}.$$

Solution

The number of poles of $G(s)H(s)$ in the right half of s-plane, that is, with positive real part is zero, that is, $P = 0$.

To decide on the stability criterion as $N = -P$, that is, how many times the Nyquist plot should encircle $-1 + j0$ point for absolute stability, we write

$N = -P = 0$, that is, the Nyquist plot should not encircle $-1 + j0$ point for absolute stability in this case.

As there is one pole at the origin, it should be bypassed by small semicircle as shown in Fig. 10.35(a).

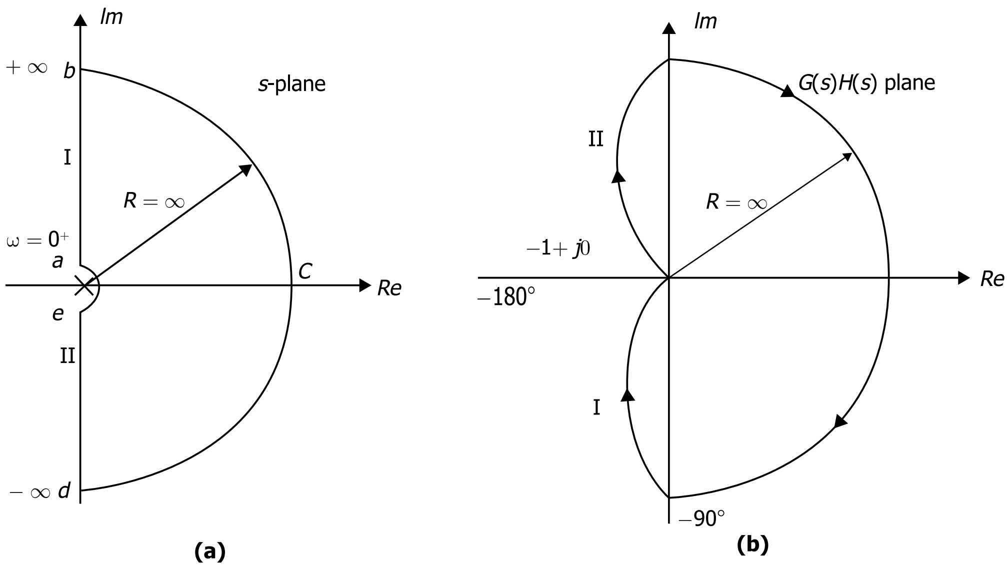


Fig. 10.35 *Nyquist plot (a) s-plane contour; (b) contour in G(s)H(s) plane*

Substituting $s = j\omega$,

$$
\begin{aligned}
G(j\omega)H(j\omega) &= \frac{K}{j\omega(1+j\omega\tau)} \\
&= \frac{K(-\omega^2\tau - j\omega)}{(-\omega^2\tau + j\omega)(-\omega^2\tau - j\omega)} \\
&= \frac{K(-\omega^2\tau - j\omega^2)}{\omega^4\tau^2 + \omega^2} \\
&= \frac{-K\tau}{\omega^2\tau^2 + 1} - j\frac{K}{\omega(1+\omega^2\tau^2)}
\end{aligned}
$$

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = -90°$$

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180°$$

From the above calculation, the section I of the Nyquist plot in the s-plane is represented by section I of the Nyquist plot in the $G(s)H(s)$ place as shown. This part relates to $0 < \omega < +\infty$. For $-\infty < \omega < 0$, part II is drawn which is the mirror reflection of part I of the portion of the Nyquist plot. The semicircle around the origin in s-place is mapped into semicircular path of infinite radius. This is represented a change of phase from $+\pi/2$ to $-\pi/2$. The point $-1 + j0$ is not circled by the Nyquist contour and hence $N = 0$.
The system is stable.

Example 10.17 A unity feedback control system is represented by its open loop transfer function

$$G(s)H(s) = \frac{8}{s(s+1)(s+2)}$$

Sketch the polar plot and examine whether the system is stable or not. Also calculate its gain margin from the polar plot.

Solution

First we write the transfer function in frequency domain by substituting $s = j\omega$. For a unity feedback system $H(s) = 1$

$$G(j\omega)H(j\omega) = \frac{8}{(j\omega)(1+j\omega)(2+j\omega)}$$

The magnitude and phase angle respectively are

$$\left|G(j\omega)H(j\omega)\right| = \frac{8}{\omega \times \sqrt{1+\omega^2} \times \sqrt{4+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \frac{8}{\angle 90 \tan^{-1}\omega \tan^{-1}\dfrac{\omega}{2}}$$

$$= -90° - \tan^{-1}\omega \tan^{-1}\frac{\omega}{2}$$

$$\lim_{\omega \to 0}\left|G(j\omega)H(j\omega)\right| = \infty \quad \text{and} \quad \angle G(j\omega)H(j\omega) = -90°$$

$$\lim_{\omega \to \infty}\left|G(j\omega)H(j\omega)\right| = 0 \quad \text{and} \quad \angle G(j\omega)H(j\omega) = -90° - 90° - 90° = -270°$$

From the above we notice that the locus starts at $-90°$ with magnitude at ∞ and terminates at the origin with 0 magnitude at $-270°$.
We also need to calculate the intersection of the locus on the negative real axis by equating the imaginary part $G(j\omega)H(j\omega)$ to 0.
For this, we have to rationalize $G(j\omega)H(j\omega)$

$$G(j\omega)H(j\omega) = \frac{8}{j\omega(1+j\omega)(2+j\omega)}$$

$$= \frac{8(-j\omega)(1-j\omega)(2-j\omega)}{(j\omega)(1+j\omega)(2+j\omega)(-j\omega)(1-j\omega)(2-j\omega)}$$

$$= \frac{-8j\omega(2-3j\omega-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2}$$

$$= \frac{24\omega^2}{\omega^2 \times (1+\omega^2)(4+\omega^2} - \frac{j8\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

Equating the imaginary part to 0,

$$8\omega(2 - \omega^2) = 0$$

$\therefore$

$$\omega = 0, \sqrt{2}$$

Substituting $\omega = \sqrt{2}$ in the real part of $G(j\omega)H(j\omega)$, value of $|G(j\omega)H(j\omega)|$ at the crossing of real axis is equal to

$$-\frac{24\omega^2}{\omega^2 \times (1+\omega^2)(4+\omega^2)} = -\frac{24 \times 2}{2 \times (1+2)(4+2)}$$

$$= -\frac{48}{36} = -1.33$$

The intersection of the locus with the real axis is at $-1.33 + j0$. The polar plot is shown in Fig. 10.36. As the locus is encircling the critical point $-1 + j0$, the system is unstable.

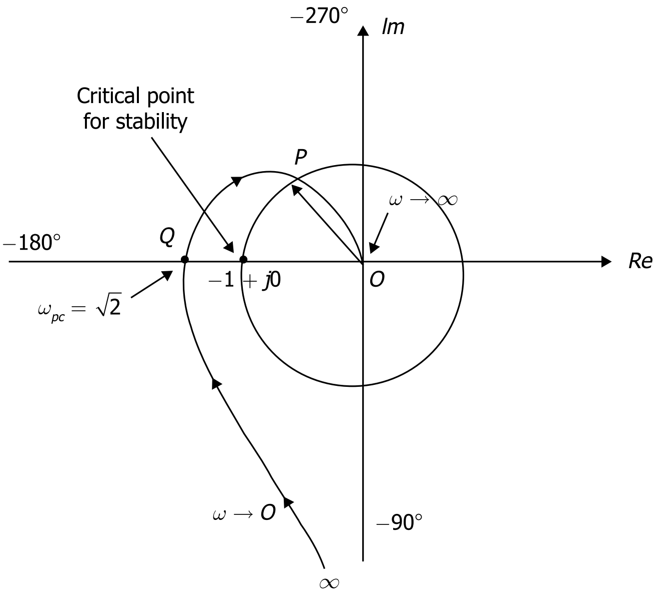


Fig. 10.36 *Polar plot of* $G(s)H(s) = \dfrac{8}{s(s+1)(s+2)}$

Determination of Gain Margin and Phase Margin from the Polar Plot

Let at $\omega = \omega_{pc}$ the polar plot intersects the negative real axis as shown in Fig. 10.36. It is also the frequency at which phase angle of $G(j\omega)H(j\omega)$ is $-180°$. Stability determination is done considering the critical factor, $|G(j\omega)H(j\omega)| = 1$ and $\angle G(j\omega)H(j\omega) = -180°$, that is the point $-1 + j0$ on the negative real axis. Gain margin GM is the gain that can be introduced in the system till it reaches instability.

$$\text{GM} = \left|\frac{1}{G(j\omega)H(j\omega)}\right|_{\text{at } \omega=\omega_{pc}}$$

In this example, the gain margin,

$$\text{GM} = \frac{1}{|OQ|} = \frac{1}{1.33} = 0.75$$

The gain in decibels is expressed as

$$\text{GM} = 20\log_{10}\frac{1}{|OQ|}\,\text{dB}$$

In this problem,

$$\mathrm{GM} = 20\log_{10}\frac{1}{1.33} = 20\log_{10} 0.75$$

Phase margin is obtained by drawing a circle of unit radius and locating the point of intersection *P* as shown in Fig. 10.36 on the locus. Let the frequency at this point be called ω_{gc}. Phase margin is $\angle G(j\omega)H(j\omega)$ at $\omega = \omega_{gc}$. By definition,

$$\mathrm{PM} = 180° + \angle G(j\omega)H(j\omega)\big|_{\text{at } \omega=\omega_{gc}}$$

Example 10.18 The open-loop transfer function of a control system is given as

$$G(s)H(s) = \frac{8}{(s+2)(s+4)}$$

Draw the complete Nyquist plot and comment on the stability of the system.

Solution

First we draw the Nyquist contour which consists of the entire $j\omega$-axis and a semicircle of infinite radius. On the $j\omega$ axis ω varies from $-\infty$ to $+\infty$. On the infinite semicircle $s = \lim_{R\to\infty} Re^{j\theta}$ where θ varies from $+\pi/2$ to $-\pi/2$. The contour, therefore, will enclose the whole of the right half of the *s*-plane encircled in the clockwise direction as shown in Fig. 10.37.

The transfer function in the frequency domain for the $G(s)H(s)$ is obtained by substituting $s = j\omega$ as

$$G(j\omega)H(j\omega) = \frac{8}{(j\omega+2)(j\omega+4)}$$

$$\lim_{\omega\to 0}\left|G(j\omega)H(j\omega)\right| = 1$$

$$\lim_{\omega\to 0}\angle G(j\omega)H(j\omega)\big| = 0$$

$$\lim_{\omega\to \infty}\left|G(j\omega)H(j\omega)\right| = 0$$

$$\lim_{\omega\to \infty}\angle G(j\omega)H(j\omega) = \frac{\angle 0}{\tan^{-1}\dfrac{\omega}{2}\tan\dfrac{\omega}{4}} = \frac{\angle 0}{90°\theta\ °} = -180°$$

The Nyquist locus in the *GH*-plane for $s = 0$ to $s = +\infty$ has been drawn in Fig. 10.37(b). For the infinite semicircle with radius $\mathrm{R}e^{j\theta}$ described by $s = +j\infty$ to $s = 0$ to $s = -j\infty$ for $\theta = +\pi/2$ to $\theta = -\pi/2$ is calculated as

$$\lim_{R\to 0}\frac{10}{(2+Re^{j\theta})(4+Re^{j\theta})} = 0\angle 180° \text{ to } 0\angle -180°$$

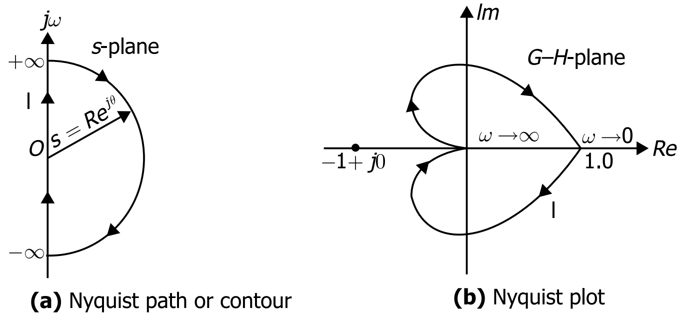


Fig. 10.37 *Nyquist plot for transfer function,* $G(s)H(s) = \dfrac{8}{(s+2)(s+4)}$

Therefore, the infinite semicircular path of Nyquist contour maps into the origin in the Nyquist plot.

For the portion $-\infty$ to 0, the locus will be the mirror image of portion I and is represented as shown in Fig. 10.37(b). The point $-1 + j0$ has also been shown. The locus does not encircle the point $-1 + j0$ and therefore $N = 0$. There is no pole of $G(s)H(s)$ in the right-hand side of the s-plane. Therefore, $P = 0$

$$N = P - Z$$

$$0 = 0 - Z$$

that is, $$Z = 0$$

There are no zeros of $1+ G(s)H(s)$. That is, the roots of the characteristic equation with positive real part is absent and hence the closed-loop system is stable.

Example 10.19 Determine the stability of the system using Nyquist criteria whose open-loop transfer function is

$$G(S)H(s) = \frac{1}{s(1+s)(1+2s)}$$

Solution

Since there is a pole at the origin, the Nyquist contour will be as shown its Fig. 10.38. The Nyquist contour will form an infinitely small semicircle of the centre *bypassing* the pole at the origin. The transfer function in the frequency domain is obtained by substituting $s = j\omega$ in $G(s)H(s)$ as

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

For drawing the Nyquist plot for the portion I, that is, for $s = 0+$ to $s = +j\infty$, we determine magnitude and phase angle of $G(j\omega)H(j\omega)$ as

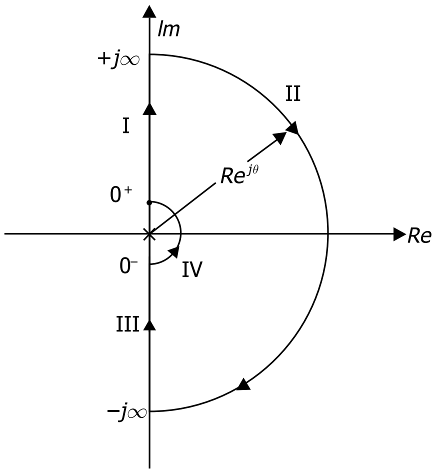


Fig. 10.38 *Nyquist contour*

$$\lim_{\omega\to 0}\left|G(j\omega)H(j\omega)\right| = \frac{1}{j0(1+j0)(1+2j0)} = \infty$$

$$\lim_{\omega\to 0}\angle G(j\omega)H(j\omega) = \frac{1}{(0+j\omega)(1+j\omega)(1+2j\omega)}$$

$$= \frac{1}{\tan^{-1}\frac{\omega}{0}\tan^{-1}\frac{\omega}{1}\tan^{-1}\frac{2\omega}{1}}$$

$$= \frac{1}{\tan^{-1}\infty\tan^{-1}0\tan^{-1}0}$$

$$= \frac{1}{\angle 90^\circ \angle 0^\circ \angle 0^\circ}$$

$$= -90^\circ$$

$$\lim_{\omega\to\infty}\left|G(j\omega)H(j\omega)\right| = 0$$

$$\lim_{\omega\to\infty}\angle G(j\omega)H(j\omega) = -90^\circ - 90^\circ - 90^\circ$$

$$= -270^\circ$$

For the portion I of the Nyquist contour, the Nyquist plot as shown in Fig. 10.39, will start from -90° with infinite magnitude and terminate at the origin at -270° with 0 magnitude. To determine whether the locus crosses the negative real axis, we have to rationalize $G(j\omega)$ $H(j\omega)$ and put its imaginary part to 0 as done below.

$$G(j\omega)H(j\omega) = \frac{(1-j\omega)(1-2j\omega)}{[j\omega(1+j\omega)(1+2j\omega)][(1-j\omega)(1-2j\omega)]}$$

$$= \frac{1-3j\omega-2\omega^2}{j\omega(1+\omega^2)(1+4\omega^2)}$$

$$= -\frac{3}{(1+\omega^2)(1+4\omega^2)} - \frac{j(1-2\omega^2)}{\omega(1+\omega^2)(1+4\omega^2)}$$

Equating imaginary part to 0,

$$1 - 2\omega^2 = 0$$

or

$$\omega = \frac{1}{\sqrt{2}} = 0.707$$

For $\omega = 0.707$,

$$\left|G(j\omega)H(j\omega)\right| = -\frac{3}{\left(1+\frac{1}{2}\right)(1+2)} - j0$$

$$= -\frac{1}{1.5} = -0.66$$

At $\omega = 0.707$, the Nyquist plot crosses the negative real axis at -0.66.
For the infinite semicircle, that is, portion II of the Nyquist contour, we put

$$\lim_{R\to\infty} \frac{1}{\mathrm{Re}^{j\theta}(1+\mathrm{Re}^{j\theta})(1+2\,\mathrm{Re}^{j\theta})} = 0\angle -270^\circ$$

Thus, the infinite semicircle will map at the origin itself. For the portion III of the Nyquist contour, that is, for $s \to -j\infty$ to $s = j0$, the locus will be the mirror image of the locus for portion I as has been shown in Fig. 10.39.
Now we will do the mapping of the small semicircle at the origin.
Considering the radius of the infinitely small semicircle as ε, and as θ varies from $-\pi/2$ to $+\pi/2$

$$\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon e^{j\theta}(1+\varepsilon e^{j\theta})(1+2\varepsilon e^{j\theta})}\right] = \infty\angle 90^\circ \text{ to } \infty\angle 0^\circ \text{ to } \infty\angle -90^\circ$$

This part of the Nyquist locus will be on infinite semicircle spread over the entire right hand side of the *GH*-plane.
The complete Nyquist plot for parts I, II, III and IV of the Nyquist contour has to be drawn as shown in Fig.10.39. Note that $OB = -0.66$. Since the critical point $-1 + j0$ is not encircled by the Nyquist locus, the system is stable.

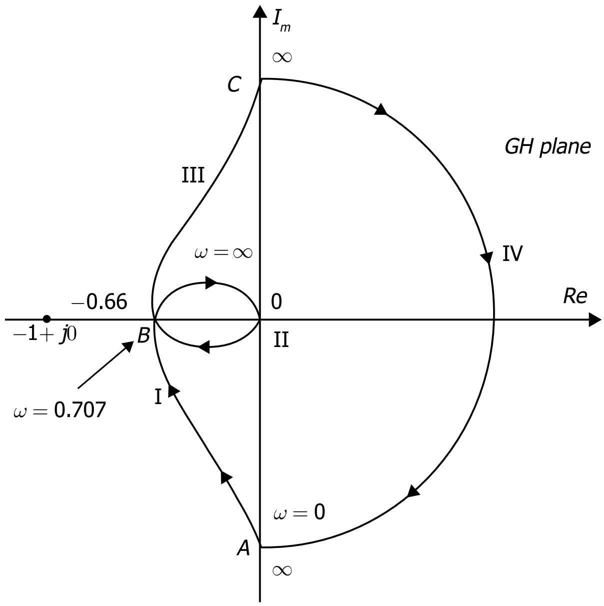


Fig. 10.39 Nyquist plot for $G(s)H(s) = \dfrac{1}{s(1+s)(1+2s)}$

Example 10.20 The open-loop transfer function of a system is

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$

Determine the value of K for which the system will remain stable with (a) phase margin of 60° and (b) gain margin of 6 dB.

Solution

a) The sinusoidal transfer function is

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(2+j\omega)(3+j\omega)}$$

Phase margin = $\angle G(j\omega)H(j\omega) + 180°$

$$\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3}$$

Given phase margin = 60°
Therefore,

$$60° = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3} + 180°$$

or

$$30° = \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{3}$$

Taking tangent of both the sides,

$$\tan 30° = \frac{\left(\frac{\omega}{2} + \frac{\omega}{3}\right)}{\left(1 - \frac{\omega}{2} \times \frac{\omega}{3}\right)}$$

or

$$0.57 = \frac{5\omega}{6 - \omega^2}$$

or $\qquad \boldsymbol{\omega} = 0.5$

Magnitude $|G(j\boldsymbol{\omega})|$ at $\boldsymbol{\omega} = 0.5$ has to be equated to 1 to determine the maximum value of gain K.

We rationalize $G(j\boldsymbol{\omega})H(j\boldsymbol{\omega})$ as

$$\begin{aligned}
G(j\omega)H(j\omega) &= \frac{K}{j\omega(2 + j\omega)(3 + j\omega)} = \frac{K(2 - j\omega)(3 - j\omega)}{j\omega(4 + \omega^2)(9 + \omega^2)} \\
&= \frac{K(6 - 5j\omega - \omega^2)}{j\omega(4 + \omega^2)(9 + \omega^2)} \\
&= -\frac{jk5\omega}{j\omega(4 + \omega^2)(9 + \omega^2 +)} + \frac{K(6 - \omega^2)}{j\omega(4 + \omega^2)(9 + \omega^2)} \\
&= -\frac{5K}{(4 + \omega^2)(9 + \omega^2)} - j\frac{K(6 - \omega^2)}{\omega(4 + \omega^2)(9 + \omega^2)}
\end{aligned} \qquad \text{....(10.25)}$$

$$\left|G(j\omega)H(j\omega)\right|_{\omega=0.5} = \frac{5K}{(4 + .25)(9 + .25)} = 1$$

or $\qquad 5K = 39.3$

or $\qquad K = 7.86$

b) Given gain margin = 16 dB
   Let the Nyquist plot intersect the negative real axis at a distance a from the origin.
   Then, gain margin = $20 \log_{10} 1/a$

or $\qquad 6 = 20 \log_{10} 1/a$

or $\qquad a = 0.5$

at the point of intersection the imaginary part of the $G(j\boldsymbol{\omega})H(j\boldsymbol{\omega})$ is equal to 0.

Therefore, from equation (10.25), we can write

$$\frac{K(6-\omega^2)}{\omega(4+\omega^2)(9+\omega^2)}=0$$

or $$\omega^2 = 6$$

or $$\omega = \sqrt{6} = 2.4$$

$|(j\omega)H(j\omega)|$ at $\omega = 2.45$ has to be calculated and equated to $a = 0.5$ to get the critical value of gain K for the gain margin of 6 dB. Thus,

$$\left|\frac{5K}{(4+\omega^2)(9+\omega^2)}\right| \text{at } \omega^2 = 6 \text{ is } \frac{5K}{(4+6)(9+6)}$$

Equating this value to $a = 0.5$

$$\frac{5K}{150} = 0.5$$

or $$5K = 75$$

or $$K = 15$$

The system will remain stable with a gain margin of 6 dB for $K = 15$.

Example 10.21 The open-loop transfer function of a unity feedback control system is

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

Draw the Nyquist plot and examine the stability of the closed-loop system. Check also the stability condition by Routh–Hurwitz criterion.

Solution

For unity feedback system, $H(s) = 1$

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

There is one pole on the right-hand side of the s-plane at $s = 1$. The open-loop system is therefore, unstable. Let us examine stability of the closed-loop system, first by Nyquist criterion and then by Routh–Hurwitz criterion.

Substituting $s = j\omega$,

$$G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$$

$$= -\frac{(2+j\omega)}{(1+j\omega)(1-j\omega)}$$

$$= -\frac{(2+j\omega)(1-j\omega)(1+j\omega)}{(1+\omega^2)(1+\omega^2)}$$

$$= -\left[\frac{2(1+\omega^2)+j\omega(1+\omega^2)}{(1+\omega^2)(1+\omega^2)}\right]$$

$$= -\frac{2}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

We will draw the Nyquist contour and then the Nyquist plot.

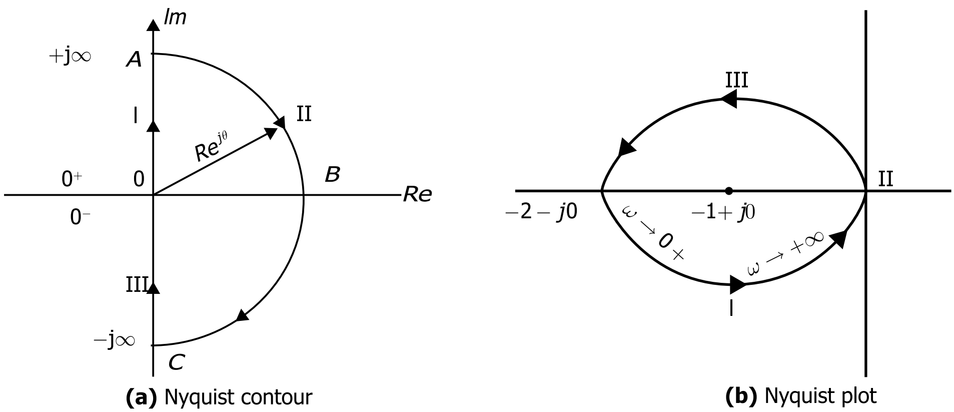


Fig. 10.40 *Nyquist plot for* $G(s)H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$

From the transfer function and the Nyquist contour,
as $\omega \to 0+$, $G(j\omega)H(j\omega) = -2 - j0$
$\omega \to +\infty$, $G(j\omega)H(j\omega) = -0 - j0$
$\omega \to 0-$, $G(j\omega)H(j\omega) = -2 + j0$
$\omega \to -\infty$, $G(j\omega)H(j\omega) = 0 + j0$
For the infinite semicircle

$$\lim_{R\to\infty} \frac{\mathrm{Re}^{j\theta}+2}{(\mathrm{Re}^{j\theta}+1)(\mathrm{Re}^{j\theta}-1)} \text{ as } \theta \text{ changes from } +\pi/2 \text{ to } -\pi/2$$

$$= 0$$

The infinite semicircle will map into the origin in the Nyquist plot as θ will change from $+\pi/2$ to $-\pi/2$. The complete Nyquist plot for the Nyquist contour considering part I, II, and III has been drawn as in Fig. 10.40(b). It is observed that the point $-1 + j0$ has been encircled by the locus once in the anticlockwise direction. Therefore,

$$N = -1$$

$$N = Z - P, P = 1$$

Hence,

$$-1 = Z - 1$$

or

$$Z = 0$$

$Z = 0$ means that there are no zeros of $1 + G(s)H(s)$ in the right-hand side of s-plane. Hence, the system in closed-loop configuration is stable.

For applying Routh-Hurwitz criterion, we will write the characteristic equation of the closed-loop system as

$$1 + G(s)H(s) = 0$$

or
$$1+\frac{(s+2)}{(s+1)(s-1)}=0$$

or
$$(s + 1)(s - 1) + (s + 2) = 0$$

or
$$s^2 + s + 1 = 0$$

The Routh array is

s^2	1	1
s^1	1	0
s^0	1	0

As there is no change of sign in the first column of Routh array, the system is stable.

10.7 SUMMARY OF NYQUIST STABILITY CRITERION AND MORE EXAMPLES

Nyquist stability criterion employs a graphical method of analyzing a system for its stability. We know that the closed-loop transfer function of a system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Putting $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

The characteristic equation is $1 + G(j\omega)H(j\omega) = 0$. For the system to be stable, the roots of the characteristic equation must lie on the left-hand side of the s-plane.

The Nyquist stability criterion relates the number of poles and zeros of $1 + G(s)H(s)$ that lie on the right-hand side of the s-plane to the open-loop frequency response of $G(j\omega)H(j\omega)$. This is explained below.

Suppose the open-loop transfer function $G(s)H(s)$ of a closed-loop system is given. We draw the polar plot of $G(j\omega)H(j\omega)$ as ω varies from $+\infty$ to 0 and its mirror reflection for 0 to $-\infty$.

The number of encirclements by the polar plots of $(-1 + j0)$ is observed. We take clockwise encirclement as negative and anti-clockwise encirclement as positive.

Nyquist criterion for stability is stated as:

For a closed-loop control system to be stable, the Nyquist plot (polar plot) of $G(j\omega)H(j\omega)$ must encircle the point $(-1 + j0)$ as many times as the number of poles of $G(s)H(s)$ that are in the right-hand side of the s-plane.

If N is the number of encirclements of point $(-1 + j0)$ and P is the number of poles of $G(s)H(s)$ that are on the RHS of s-plane, the for stability of the closed loop,

$$N = P - Z$$

For example, suppose the open-loop transfer function of a closed-loop system

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$$

Here, open-loop pole on the RHS of s-plane, $P = 1$

$$\begin{aligned} \text{Let} \quad N &= +1 \\ N &= P - Z \\ 1 &= 1 - Z \\ Z &= 0 \end{aligned}$$

$Z = 0$ means that there are no zeros of $1 + G(s)H(s)$ in the right hand side of s-plane. Hence the system is stable.

Example 10.22 The open-loop transfer of a closed-loop system (feedback system) is given as

$$G(s)H(s) = \frac{5}{s(1-s)}$$

Using, Nyquist stability criterion, check for the stability of the closed-loop system. Also verify your result using Routh's criterion.

Solution

We put $s = j\omega$ in the given transfer function.

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(1 - j\omega)}$$

$$\text{Magnitude, } M = \frac{5}{\omega\sqrt{1+\omega^2}}$$

$$\text{Phase angle, } \phi = -90° - \tan^{-1}\frac{-\omega}{1}$$

And

We calculate the values of ϕ and M for varying values of ω as

$\omega = 0$	$\phi = -90°$	$M = \infty$
$\omega = 1$	$\phi = -90° - (-45°) = -45°$	$M = 3.54$
$\omega = 2$	$\phi = -90° - (-63.5°) = -26.5°$	$M = 1.12$
$\omega = \infty$	$\phi = -90° - (-90°) = 0$	$M = 0$

From the $G(s)H(s)$, we find that one pole is on the right-hand side of the s plane and hence $P = 1$

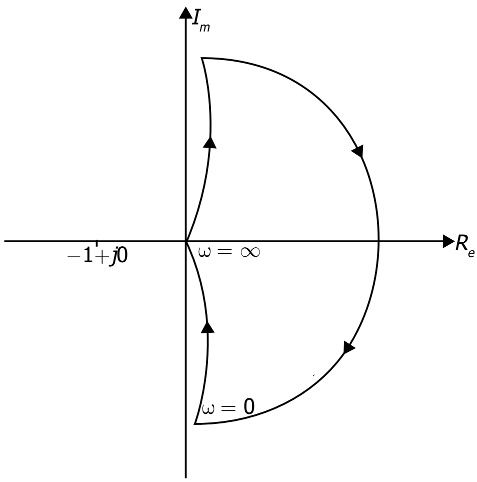


Fig. 10.41 *Nyquist plot of G(s)H(s) = 5/s(1 − s)*

We draw the Nyquist plot by considering the values of M and ϕ for different values of ω from $\omega = 0$ to $\omega = \infty$ and $\omega = -\infty$ by taking the mirror reflection. The number of encirclement by the Nyquist plot of $(-1 + j0)$ point is nil. Therefore, $N = 0$. The stability criterion,

$$N = P - Z$$

$$0 = 1 - Z$$

or, $$Z = 1$$

This shows that the closed-loop system is unstable.
To check for stability of the closed-loop system, by Routh's criterion, we write the characteristic as

$$1 + G(s)H(s) = 0$$

or,
$$1+\frac{5}{s(1-s)}=0$$

or,
$$s - s^2 + 5 = 0$$

or,
$$-s^2 + s + 5 = 0$$

or,
$$s^2 - s - 5 = 0$$

Routh's array:

$$\begin{array}{c|cc|} s^2 & 1 & -5 \\ s^1 & -1 & 0 \\ s^0 & -5 & 0 \end{array}$$

Since there is one sign change, i.e. from plus value to minus value in the first column of the array, the closed-loop system is unstable.

Example 10.23 The open-loop transfer function of a feedback control system is given as

$$G(s)H(s)=\frac{K}{s(s^2+2s+2)}$$

Using the Nyquist stability criterion, determine the maximum values of K for which the closed-loop system will be stable. Check your answer by Routh's stability criterion.

Solution

Putting $s = j\omega$,

$$G(j\omega)H(j\omega)=\frac{K}{j\omega[(j\omega)^2+2j\omega+2]}$$

$$=\frac{K}{j\omega[2-\omega^2+2j\omega]}$$

$$M=\frac{K}{\omega\sqrt{(2-\omega^2)^2+4\omega^2}}$$

$$\phi=-90^\circ-\tan^{-1}\frac{2\omega}{2-\omega^2}$$

For various values of ω, we calculate M and ϕ and tabulate them as shown below.

$\omega = 0$	$\phi = 90°$	$M = \infty$
$\omega = 1$	$\phi = \angle{-153°}$	$M = \dfrac{K}{\sqrt{5}}$
$\omega = 2$	$\phi = \angle{-206°}$	$M = \dfrac{k}{2\sqrt{18}}$
$\omega = \infty$	$\phi = \angle{-270°}$	$M = 0$

The intersection of the locus with negative real axis is to be calculated. We equate imaginary part of $G(j\omega)H(j\omega)$ to 0.
Alternately, we equate ϕ with $-180°$ to determine the phase crossover frequency as,

$$-180° = -\tan^{-1}\frac{2\omega}{2-\omega^2} = 90°$$

or, $$\tan^{-1}\frac{2\omega}{2-\omega^2} = 90°$$

or, $$\frac{2\omega}{2-\omega^2} = \tan 90° = \infty$$

Therefore $$2 - \omega^2 = 0$$

or, $$\omega = \sqrt{2}$$

$$\omega_{pc} = \sqrt{2}$$

The magnitude of M at ω_{pc} is calculated by putting $\omega = \sqrt{2}$ in the expression for M

$$M = \frac{K}{\sqrt{2}\left[\sqrt{(2-2)^2 + 4\times 2}\right]} = \frac{K}{\sqrt{2}\sqrt{8}} = \frac{K}{\sqrt{16}} = \frac{K}{4}$$

The Nyquist plot is drawn as shown in Fig. 10.42.
For the system to be stable M should be less than 1. That is

$$\frac{K}{4} < 1$$

or $$K < 4$$

The value of gain K should be less than 4 for the closed-loop system to remain stable. To check using Routh's criterion,

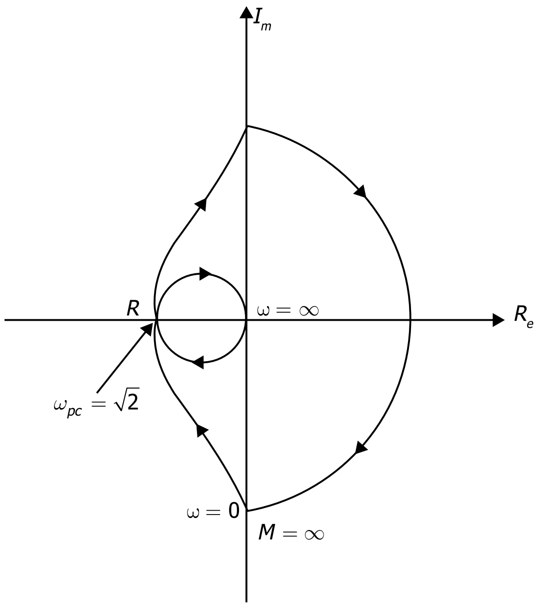


Fig. 10.42 *Nyquist plot of $K/s(s^3 + 2s + 2)$*

The characteristic equation of the closed-loop system is

$$1 + G(s)H(s) = 0$$

or,
$$1 + \frac{K}{s(s^2 + 2s + 2)} = 0$$

or,
$$s(s^2 + 2s + 2) + K = 0$$

or,
$$s^3 + 2s^2 + 2s + K = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

Routh's array

$$\begin{array}{c|cc|} s^3 & 1 & 2 \\ s^2 & 2 & K \\ s^1 & 4 - K/2 & 0 \\ s^0 & K & 0 \end{array}$$

$$K > 0$$

$$\frac{4 - K}{2} > 0$$

$$4 - K > 0$$

or $$K < 4$$

Thus the result is checked to be true.

Example 10.24 Using Nyquist criterion, examine the closed-loop stability of a system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{50}{(s+1)(s+2)}$$

Solution

Given $$G(s)H(s) = \frac{50}{(s+1)(s+2)}$$

Put $$s = j\omega$$

$$G(j\omega)H(j\omega) = \frac{50}{(1+j\omega)(2+j\omega)}$$

$$= \frac{50(1-j\omega)(2-j\omega)}{(1+j\omega)(2+j\omega)\,(1-j\omega)(2-j\omega)}$$

[rationalizing]

$$= \frac{50(2-j^3\omega-\omega^2)}{(1+j\omega^2)(4+j\omega^2)}$$

$$= \frac{50(2-\omega^2)-j150\omega}{(1+\omega^2)(4+\omega^2)}$$

separating real and imaginary parts

$$= \frac{50(2-\omega^2)}{(1+\omega^2)(4+\omega^2)} + j\frac{-150\omega}{(1+\omega^2)(4+\omega^2)}$$

so

$$\text{Re}[G(j\omega)H(j\omega)] = \frac{50(2-\omega^2)}{(1+\omega^2)(4+\omega^2)}$$

and

Now, $$\text{Im}[G(j\omega)H(j\omega)] = \frac{-150\omega}{(1+\omega^2)(4+\omega^2)}$$

The intersection of Nyquist plot with real axis can be found by equating $$\text{Im}[G(j\omega)H(j\omega)] = 0$$

$$\text{i.e., } \frac{-150\omega}{(1+\omega^2)(4+\omega^2)} = 0$$

i.e., $$\text{At } \omega = 0 \text{ or } \omega = \pm\infty$$

and the intersection with imaginary axis can be found by,

equating $$\text{Re}[G(j\omega)H(j\omega)] = 0$$

i.e., $$\frac{50(2-\omega^2)}{(1+\omega^2)(4+\omega^2)} = 0$$

or $$2 - \omega^2 = 0$$

or $$\omega^2 = 2$$

or $$\omega = \pm\sqrt{2} = \pm 1.414$$

Let us find some values Re[$G(j\omega)H(j\omega)$] and Im$G(j\omega)H(j\omega)$ for different value of ω.

ω	**0**	**1**	$\sqrt{2}$	∞
Re[$G(j\omega)H(j\omega)$]	25	5	0	0
Im$G(j\omega)H(j\omega)$	0	−j15	−j11.7	0

Let us draw Nyquist plot using these values

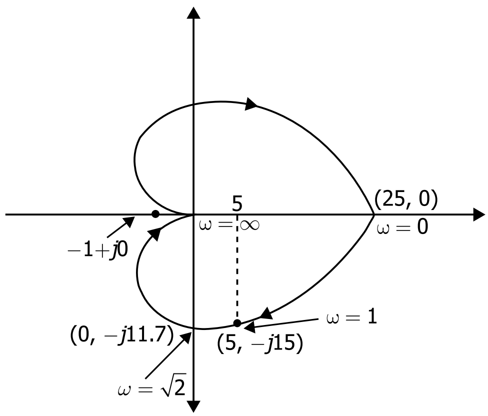


Fig. 10.43 *Nyqiust plot for* $G(s)H(s) = \dfrac{50}{(s+1)(s+2)}$

Since the point $(-1 + j0)$ is not encircled by the Nyqiust plot, there are no roots of closed-loop characteristic equation having positive real parts; so the system is stable.

Example 10.25 Determine the close-loop stability of the system whose open-loop transfer function is given as

$$G(s)H(s)=\frac{(s+0.25)}{s^2(s+1)(s+0.5)}$$

by applying the Nyqiust criterion.

Solution

Given,

$$G(s)H(s)=\frac{(s+0.25)}{s^2(s+1)(s+0.5)}$$

Put $$s=j\omega$$

$$G(j\omega)H(j\omega)=\frac{(j\omega+0.25)}{(j\omega)^2(j\omega+1)(j\omega+0.5)}$$

$$=\frac{(j\omega+0.25)(j\omega-1)(j\omega-0.5)}{-\omega^2(j\omega+1)(j\omega+0.5)(j\omega-1)(j\omega-0.5)}$$

[Rationalizing]

$$=\frac{(j\omega+.25)(-\omega^2-j1.5\omega+0.5)}{-\omega^2(1+\omega^2)(\omega^2-.5^2)}$$

$$=\frac{1.25\omega^2+.125-j\omega^3+j.125\omega}{-\omega^2(\omega^2+1)(\omega^2+.25)}$$

$$=\frac{1.25(\omega^2+0.1)-j\omega(\omega^2-0.125)}{-\omega^2(\omega^2+1)(\omega^2+0.25)}$$

$$=\frac{1.25(\omega^2+0.1)}{-\omega^2(\omega^2+1)(\omega^2+.25)}+j\frac{-\omega(\omega^2-0.125)}{-\omega^2(\omega^2+1)(\omega^2+0.25)}$$

$$=\frac{-1.25(\omega^2+0.1)}{\omega^2(\omega^2+1)(\omega^2+0.25)}+j\frac{(\omega^2-0.125)}{\omega(\omega^2+1)(\omega^2+0.25)}$$

Real part, $$\mathrm{Re}[G(j\omega)H(j\omega)]=\frac{-1.25(\omega^2+0.1)}{\omega^2(\omega^2+1)(\omega^2+0.25)}$$

Imaginary part, $$\mathrm{Im}[G(j\omega)H(j\omega)]=\frac{\omega^2-0.125}{\omega(\omega^2+1)(\omega^2+0.25)}$$

For finding the point of intersection of Nyquist plot:

i) with real axis

Put $$\text{Im}[G(j\omega)H(j\omega)] = 0$$

or $$\frac{\omega^2 - 0.125}{\omega(\omega^2+1)(\omega^2+0.25)} = 0$$

or $$\omega^2 - 0.125 = 0$$

or $$\omega^2 = 0.125$$

or $$\omega = \pm\sqrt{0.125}$$

ii) with imaginary axis

Put $$\text{Re}[G(j\omega)H(j\omega)] = 0$$

$$\frac{-1.25(\omega^2+0.1)}{\omega^2(\omega^2+1)(\omega^2+0.25)} = 0$$

or $$\omega^2 + 0.1 = 0$$

or $$\omega^2 = -0.1$$

or $$\omega = \pm j\sqrt{0.1}$$

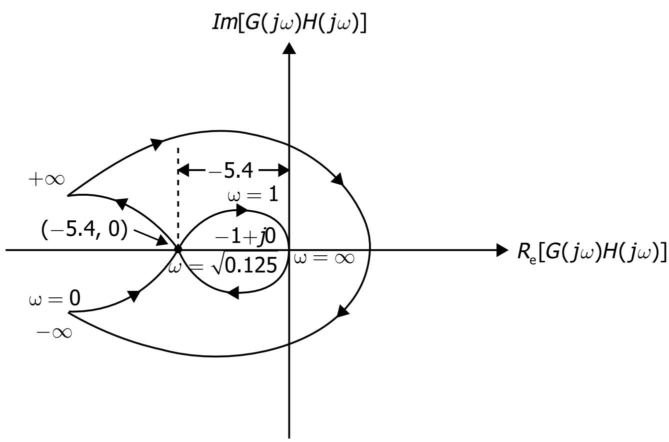


Fig. 10.44 *Nyquist plot for* $G(s)H(s) = \dfrac{(s+0.25)}{s^2(s+1)(s+0.5)}$

or $$\omega^2(\omega^2 + 1)(\omega^2 + 25) = \infty$$

or $$\omega = \pm\infty$$

Let us calculate some more values for different values of ω

ω	0	$\sqrt{0.125}$	1	∞
Re[$G(j\omega)H(j\omega)$]	$-\infty$	-5.4	-0.55	0
Im[$G(j\omega)H(j\omega)$]	$+\infty$	$-j0$	$j0.35$	0

Using the above values, the Nyquist plot is drawn as shown in Fig. 10.44.
Since $(-1 + j0)$ is encircled by Nyquist plot two roots of the system have positive real part; hence the system is unstable.

Example 10.26 Using Nyquist criterion, determine the stability of the closed-loop system, whose open-loop transfer function is given by

$$G(s)H(s) = \frac{2(1-s)}{s+1}$$

Solution

Given,

$$G(s)H(s) = \frac{2(1-s)}{s+1}$$

put $$s = j\omega$$

$$G(j\omega)H(j\omega) = \frac{2(1-j\omega)}{j\omega+1}$$

$$= \frac{2(1-j\omega)(1-j\omega)}{(j\omega+1)(1-j\omega)}$$

$$= \frac{2(1-j\omega)^2}{1+\omega^2}$$

$$= \frac{2[1+j^2\omega^2 - j^2\omega]}{1+\omega^2}$$

$$= \frac{2[1-\omega^2 - j^2\omega]}{1+\omega^2}$$

$$= \frac{2(1-\omega)^2}{1+\omega^2} + j\frac{-4\omega}{1+\omega^2}$$

So

$$\mathrm{Re}[G(j\omega)H(j\omega)] = \frac{2(1-\omega)^2}{1+\omega^2}$$

and

$$\mathrm{Im}[G(j\omega)H(j\omega)] = \frac{-4\omega}{1+\omega^2}$$

Nyquist plot will cut the real axis at a point at which

$$\mathrm{Im}[G(j\omega)H(j\omega)] = 0$$

i.e.,
$$\frac{-4\omega}{1+\omega^2} = 0$$

or
$$\omega = 0$$

And point of intersection of Nyquist plot with imaginary axis can be found by

Putting
$$\mathrm{Re}[G(j\omega)H(j\omega)] = 0$$

i.e.,
$$\frac{2(1-\omega)^2}{1+\omega^2} = 0$$

or
$$\omega^2 = 0$$

or
$$\omega = \pm 1$$

So

At
$$\omega = 0$$

$$\mathrm{Re}[G(j\omega)H(j\omega)] = \frac{2(1-0)}{1+0^2} = 2$$

$$\mathrm{Im}[G(j\omega)H(j\omega)] = \frac{-2(0)}{1+0^2} = 0$$

At
$$\omega = 1$$

$$\mathrm{Re}[G(j\omega)H(j\omega)] = \frac{2(1-1^2)}{1+1^2} = 0$$

$$\mathrm{Im}[G(j\omega)H(j\omega)] = \frac{-4(1)}{1+1^2} = \frac{-4}{2} = -2$$

Calculating for more values of real and imaginary parts for increasing value of ω, the Nyquist plot is drawn as shown in Fig. 10.45.

Since $(-1 + j0)$ is encircled by Nyquist plot, the given system is unstable.

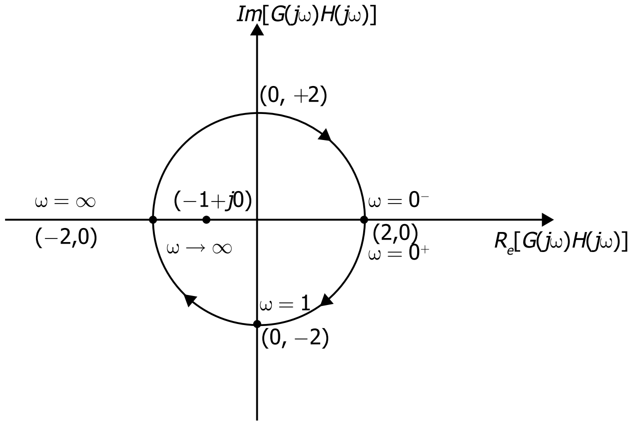


Fig. 10.45 *Nyquist plot of* $G(s)H(s) = \dfrac{2(1-s)}{s+1}$

Example 10.27 The transfer function of a system is given as

$$G(s)H(s) = \frac{K}{s(1+s)(2+s)}$$

The system is stable for positive values of *K*. Draw the Nyquist plot for the given transfer function and find the range of gain *K* for which the system is stable. Calculate the value of *K* for a gain margin of 3 dB. For this gain margin find the phase cross over frequency and the phase margin.

Solution

For mapping the Nyquist contour we may follow the procedure as below.

a) Semicircular path around the pole at the origin is represented by

$$s = \lim_{r \to 0} re^{j\theta}$$

We have,

$$G(s)H(s) = \frac{K}{s(1+s)(2+s)}$$

Substituting,

$$G(s)H(s) = \lim_{r \to 0} \frac{K}{re^{j\theta}(1+re^{j\theta})(2+re^{j\theta})}$$

$$G(s)H(s) = \infty e^{-j\theta}$$

The above shows that the semicircular arc of radius *r* in the s-plane corresponds to semicircular arc of infinite radius on the *G*(*H*) plane.

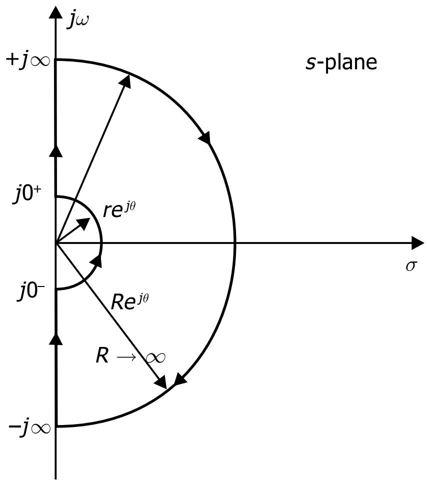


Fig. 10.46 *The Nyquist contour*

The phase of the semicircular indent varies from $-90°$ to $0°$ to $+\ 90°$. If we substitute the value of $s = \lim_{r \to 0} re^{j\theta}$, the magnitude of $G(s)\ H(s)$ will become infinity and the phase of the corresponding arc will vary from $+\ 90°$ to $0°$ to $-90°$.

b) Now let us consider the positive part of the imaginary axis of the Nyquist contour from $\omega = j0+$ to $\omega = +j\infty$. This is obtained by drawing polar plot of the transfer function,

$$G(s)H(s) = \frac{K}{j\omega(1 + j\omega)(2 + j\omega)}$$

We will provide some easy method for drawing the polar plot. Students may find this method very useful. We have known that the type of the system is indicated by the power of term s in the denominator of the T.F. The order of the system depends upon the power of s of the denominator of the transfer function. The starting point of the polar plot at $\omega = 0$ rad/sec depends on the type of the system and the terminating point of the polar plot depends upon the order of the system. For example, the T.F of this problem represents a type one system and its order is three. Type one because there is one s in the denominator of the T.F. Order is three because highest power of s is 3. For easy sketching of the polar plot we may use the diagram shown in Fig. 10.47.

Since the T.F. of the system in this example is type 1 and order 3, the starting of the polar plot at $\omega = 0+$ will be at $-90°$ and terminating point at $\omega = +\ \infty$ rad/sec will be in the second quadrant as shown in Fig. 10.48.

As mentioned earlier, if there is no zero present in the transfer function,

Order of the system = Number of poles.

If there are zeros in the T.F.

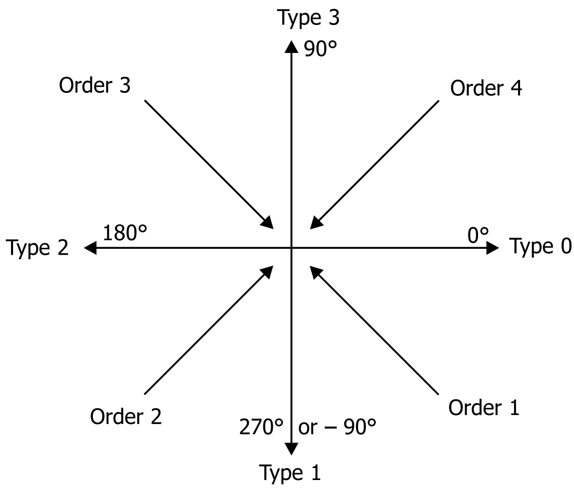


Fig. 10.47 *Identifying location of Nyquist plot for any type and any order system—a suggested guideline*

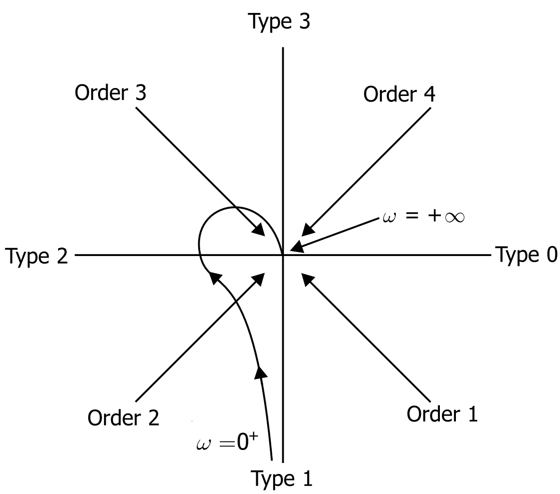


Fig. 10.48 *Polar plot of* $G(s)H(s) = \dfrac{K}{s(1+s)(2+s)}$ *as* ω *varies from* 0^+ *to* $+\infty$

Order of the system $= P - Z$

In general if

$$G(s)H(s) = \frac{K(s-z_1)(s-z_2)\ldots}{s^N(s-P_1)(s-P_2)(s-P_3)\ldots}$$

Then $N = 0, 1, 2, 3$ represent Type 0, 1, 2, 3 system and $(P - Z)$ represent order of the system.

c) Now, we will consider the infinite semicircular path from $\omega = j\infty+$ to $j\infty^-$ of the Nyquist contour. The contour can be represented by

$$s = \lim_{R\to 0} Re^{j\theta}$$

The infinite semicircular arc of the Nyquist contour represented by $s = \lim_{R\to\infty} Re^{j\theta}$ is mapped into

$$G(s)H(s) = \lim_{R\to\infty} \frac{K}{Re^{j\theta}(1+Re^{j\theta})(2+Re^{j\theta})}$$
$$= 0e^{-j3\theta}$$

The above indicates that the radius representing the semicircular arc is zero. Phase of the Nyquist contour when varied from $+90°$ to $0°$ to $-90°$, the phase of the corresponding mapping varies from $270°$ to $0°$ to $-270°$. The locus will, therefore, circle around the origin.

d) Now we have, lastly to do mapping for the negative imaginary axis, i.e. from $\omega = j\infty$ to $\omega = j0^-$ point. This is obtained simply by taking the mirror image of the polar plot when frequency is varied from $j0+$ to $\omega = \infty$. The complete Nyquist plot for the whole range of frequencies is shown in Fig. 10.49.

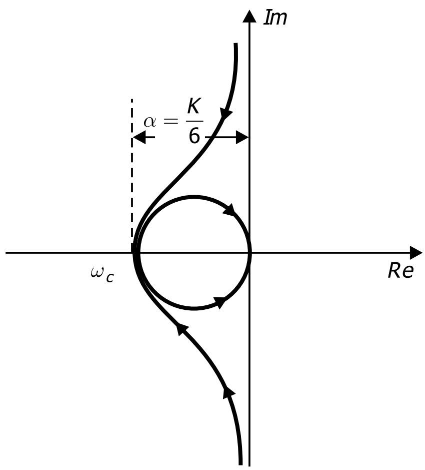


Fig. 10.49 *Nyquist plot for* $G(s)H(s) = \dfrac{K}{s(1+s)(2+s)}$

We know that if $G(s)H(s) = \dfrac{K}{s(1+sT_1)(1+sT_2)}$

$$\omega_c = \frac{1}{\sqrt{T_1T_2}}$$

Here, in this problem, $T_1 = 1,\ T_2 = \frac{1}{2} = 0.5$

At $\omega = \omega_c$ we can find the gain margin, say α

$$\alpha = \left|\frac{K}{j\omega(1+j\omega)(2+j\omega)}\right|_{\omega=\omega_c}$$

$$= \frac{K}{\sqrt{2}\sqrt{1+2}\sqrt{4+2}} = \frac{K}{\sqrt{2}\sqrt{3}\sqrt{6}} = \frac{K}{6}$$

For stability gain margin should be less than 1.

Therefore, $\frac{K}{6} < 1$

or, $K < 6$

In the problem it has been stated that K is positive. Therefore the condition for stability is established as $0 < K < 6$.

Now let us calculate the value of K for a gain margin of 3 dB.

$$\text{Gain margin} = 20\log\left(\frac{1}{\alpha}\right)$$

or, $$3\text{ dB} = 20\log\left(\frac{6}{K}\right)$$

or, $$\log\frac{6}{K} = \frac{3}{20} = 0.15;\ K = 4.25$$

Therefore, the value of gain K for 3 dB gain margin is 4.25

The transfer function can be written as,

$$G(j\omega)H(j\omega) = \frac{4.25}{j\omega(1+j\omega)(1+2j\omega)}$$

Gain cross over frequency is the frequency at which magnitude is unity. Therefore,

$$\frac{4.25}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} = 1$$

Squaring both sides,

$$\omega^2(1+\omega^2)(1+4\omega^2) = (4.25)^2$$

Assuming $\omega^2 = x$, we get

$$x(1 + x)(1 + 4x) = 18.06$$

or, $$x = 1.4.$$

Therefore, $$\omega = \sqrt{x} = \sqrt{1.4} = 1.18 \text{ rad/sec}.$$

$$\text{Phase margin} = -90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} + 180^\circ$$

$$= -90^\circ - \tan^{-1}(1.18) - \tan^{-1}\left(\frac{\omega}{2}\right) + 180^\circ = 9.75^\circ$$

Nyquist stability criterion has been dealt with in more details in Section 10.4.

Example 10.28 Explain how we can have an understanding about the relative stability by using Nyquist criterion: Construct Nyquist plot for the system whose open loop transfer function is $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$. Find the range of K for stability condition.

Solution

From the Nyquist plot (polar plot), we can have an idea about the gain margin and phase margin of the open-loop transfer function and can predict about the degree of stability of the control system.

Gain margin is the amount of gain that could be increased to produce instability of the system. Phase margin is measured positively in counterclockwise direction from the negative real axis.

i) For a stable system, $|G(j\omega_c)H(j\omega_c)| < 1,$ the GM in dB is positive.
ii) For a marginally stable system, $|G(j\omega_c)H(j\omega_c)| = 1,$ the GM in dB is zero.

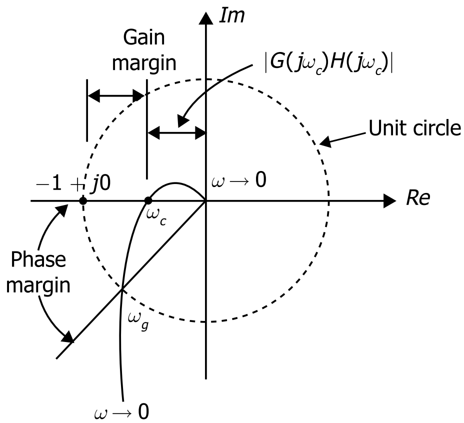


Fig. 10.50 Gain margin and phase margin illustrated

i) For $|G(j\omega_c)H(j\omega_c)| > 1,$ the system is unstable, the *GM* in dB is negative and the gain has to be reduced to make the system stable.

[For further details, you may refer to the topic of gain margin and phase margin explained earlier.]

Solution of the Numerical Problem:

We have, $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$

The Nyquist contour has been drawn as shown in Fig. 10.51.

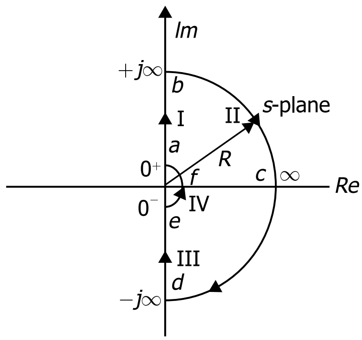


Fig. 10.51 *Nyquist contour*

The open-loop system has three poles at the origin. We will draw the Nyquist plot for the sections I, II, III, and IV of the Nyquist contour in the *GH* plane.
For section I

Substituting $s = j\omega, G(j\omega)H(j\omega) = \dfrac{K(1+j\omega)^2}{(j\omega)^3}$

for $\omega = 0^+, G(j\omega)H(j\omega) = \dfrac{K(1+j0^+)^2}{(j0^+)^3} = \infty\angle -270°$, i.e. the magnitude is ∞ at −270°.
(map of point *a*)

for $\omega = +\infty, G(j\omega)H(j\omega) = \dfrac{K(1+j\infty)^2}{(j\infty)^3} = 0\angle -90°$, i.e. the magnitude is *o* at −90°.
(map of point *b*)

From the above, we can see that the Nyquist plot will start from −270° at ∞ and come to −90° at 0 magnitude. The point of intersection with the negative real axis is calculated by putting the imaginary part of $G(j\omega)H(j\omega)$ to zero.
We have,

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3}$$

$$= -\frac{K(1+2j\omega-\omega^2)}{j\omega^3}$$

$$= -\frac{K(1-\omega^2)}{j\omega^3} - \frac{2K}{\omega^2}$$

$$= \frac{jK(1-\omega^2)}{\omega^3} - \frac{2K}{\omega^2} \quad [\because j^2 = -1]$$

Putting the imaginary part to zero,

$$K(1-\omega^2) = 0$$

or $$\omega = 1, K > 0.$$

At $\omega = 1$ the Nyquist plot will intersect the negative real axis. See Fig. 10.52. The point of intersection is

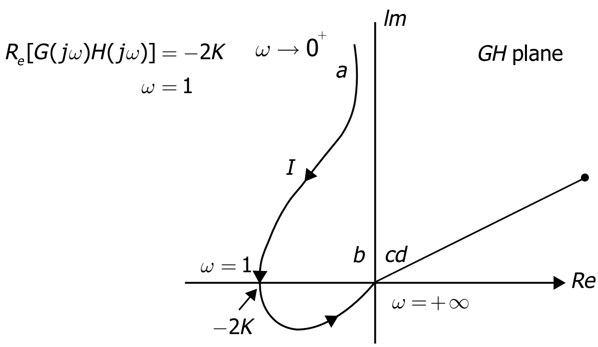


Fig. 10.52 *Nyquist plot for part I of the Nyquist contour of Fig. 10.51*

For part II of Fig. 10.51, i.e. for path *bcd* we put $s = \lim_{R\to\infty} Re^{j\theta}$. The Nyquist plot will map on to the origin and hence points *b*, *c*, *d* will lie at the origin in GH plane. For part III, i.e. path *de*, the plot will be an image reflection of part I as shown in Fig. 10.53.

For part or section IV, where the Nyquist contour forms a infinitesimal small semi-circle at the origin, we can consider,

$$s = \lim_{\in\to\infty} \in e^{j\theta}$$

where θ changes from $-\pi/2$ to 0 to $+\pi/2$

$$G(s)H(s)\big|_{s=\in e^{j\theta}} = \frac{K(\in e^{j\theta}+1)^2}{(\in e^{j\theta})^3}$$

As $\in \to 0$,

$$G(s)H(s) = \infty e^{-j3\theta}$$

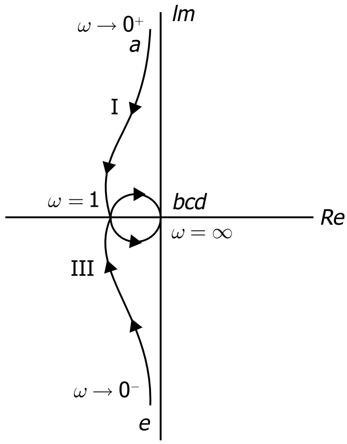


Fig. 10.53 *Partial Nyquist plot of* $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$

Considering the values $\boldsymbol{\theta}$ for $-90°$ to $+\,90°, \angle G(s)H(s)$ at $s = \in e^{j\theta}$ is calculated as

θ	$-$**90°**	$-$**60°**	$-$**30°**	$-$**0°**	$+$**30°**	$+$**60°**	$+$**90°**
$\angle G(s)H(s)$	$+270°$	$+180°$	$+90°$	$+0°$	$-90°$	$-180°$	$-270°$

The $G(s)$ $H(s)$ for the part IV of the Nyquist contour will complete three semicircles moving clockwise through an angle $+270°$ to $-270°$ as angle θ varies from -90 to $+90°$ in s-plane. This has been shown in Fig. 10.54.

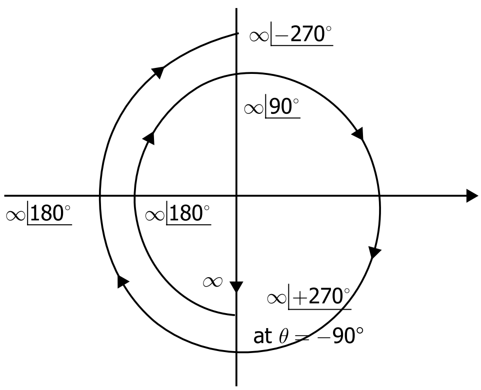


Fig. 10.54

The complete Nyquist plot considering parts I, II, III, and IV of the Nyquist contour is shown in Fig. 10.55.

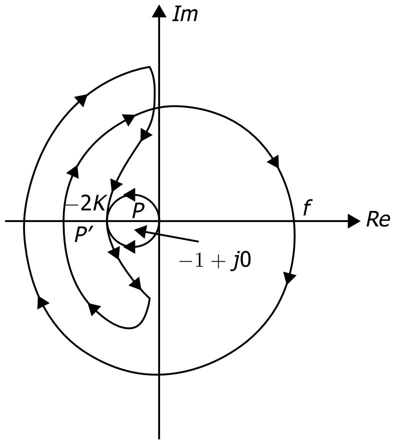


Fig. 10.55

Now, let us assume $2K > 1$. The point $-1 + j0$ will lie at point P as shown. The Nyquist locus will encircle $-1 + j0$ point, once in anticlockwise direction and once in clockwise direction. This can be observed from the Nyquist plot as above. The net encirclement of point P by the locus is therefore zero. There are no poles in the right-hand half of the s-plane, the system is stable for values of K greater than 0.5.

Here $P = 0, N = 0$ if $K > 0.5$

$N = Z - P$, so, $Z = 0$.

Note that Z must be zero for the closed loop system to be stable.

For $2K < 1$, i.e. for $K < 0.5$, the system will be unstable because point P will lie at P'.

10.8 DRAWING NYQUIST PLOTS WITH MATLAB

Nyquist plots are polar plots, while Bode diagrams are rectangular plots. The command Nyquist (num, den) will draw the Nyquist plot of $G(s)$ = num (s)/den (s) whereas the command Nyquist (num, den, ω) uses the user-specified frequency vector ω in radians per second.

MATLAB returns the frequency response of the system in matrices re, im and ω and no plot is drawn when the following command is invoked.

[re, im, ω] = nyquist (num, den)

or [re, im, ω] = nyquist (num, den, ω)

Example 10.29 Draw a Nyquist plot with MATLAB for the open-loop transfer function given by

$$G(s) = \frac{1}{s(s+1)}.$$

Solution

The MATLAB command
num = [0 0 1];
den = [1 1 0];
Nyquist (num, den)
will produce an erroneous Nyquist plot. A corrected Nyquist plot is obtained by entering the axis command
Nyquist (num, den)
v = [−2 2 −5 5]; axis (v)
MATLAB Program 10.2 will produce a correct Nyquist plot as shown in Fig. 10.56 even though a warning message "Divide by Zero" may appear on the screen.

MATLAB PROGRAM 10.2

%............. Nyquist plot...........
 num = [0 0 1];
 den = [1 1 0];
 Nyquist (num, den)
 v = (–2 2 –5 5); axis (v)
 grid
 title ('Nyquist plot of G(s) = l/([s(s+l)]')

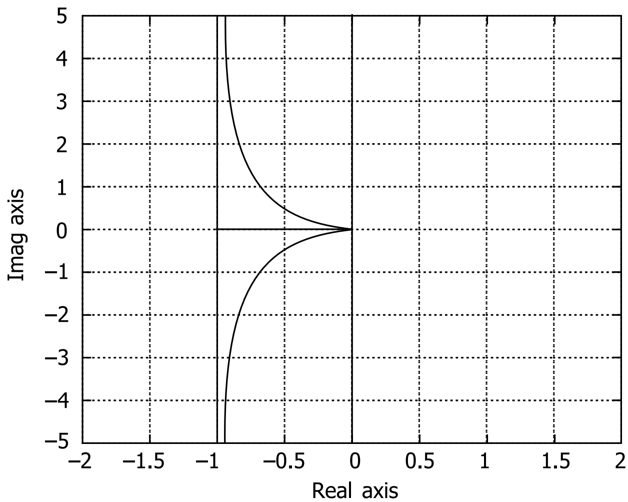


Fig. 10.56 *Nyquist plot of* $G(s)\dfrac{1}{s(s+1)}$

The above Nyquist plot includes the loci for both $\omega > 0$ and $\omega < 0$. For drawing a Nyquist plot in the positive frequency region ($\omega > 0$) we have to use the command [re, im, ω] = Nyquist (num, den, ω).

MATLAB Program 10.3 will draw the Nyquist plot (Fig. 10.57) in the positive frequency region ($\omega > 0$) only.

MATLAB PROGRAM 10.3

```
%...........Nyquist plot...........
num = [0    0   1];
den = [1   1   0];
w = 0.1:0.1:100;
[re, im, w] = Nyquist (num, den, w);
plot (re, im)
v = [–2   2   –5   5]; axis (v)
grid
title ('Nyquist plot of G(s) = l/[s(s+l)]')
x label ('Real Axis')
y label ('Imag Axis')
```

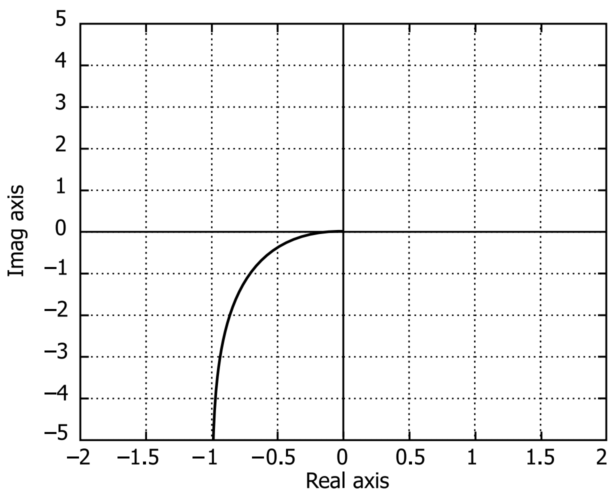


Fig. 10.57 *Nyquist plot of* $G(s) = \dfrac{1}{s(s+1)}$ *for* $\omega > 0$

10.9 RELATIVE STABILITY

In addition to knowing whether a system is stable or not, it is also desirable to know how stable the system is. It is possible to get this information from the open-loop frequency response of the system. This topic has been discussed earlier. Here we summarize the concept once again.

The nearness of the open-loop frequency response (the polar plot) to the point $-1 + j0$ in the *GH*-plane provides information regarding the degree of stability or the relative stability of the system.

Two quantities, namely the gain margin and the phase margin, are used to measure the relative stability. *Gain margin is defined as the additional gain that can be allowed which makes the system just unstable. Phase margin is defined as the additional phase lag that can be allowed to make the system just unstable*. From Fig. 10.58, it is observed that the gain margin is 2. Location of point *A* on the negative real axis is at -0.5. This is to be multiplied by 2 to make it -1, that is, the point of intersection being shifted from *A* to *B*. Here the gain margin in dB is expressed as 20 log 2 = 6 dB.

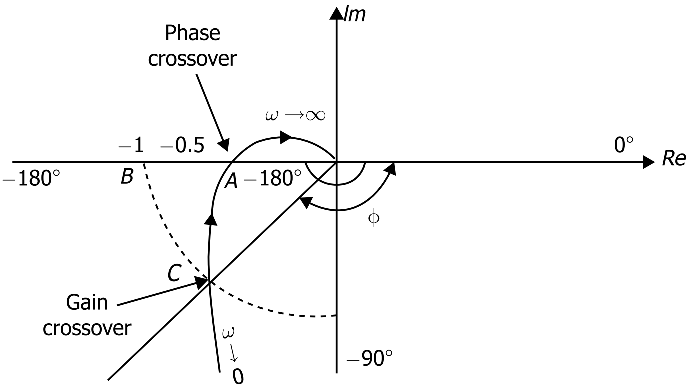


Fig. 10.58 *Determination of relative stability*

Phase margin is $180 - \phi$ as determined from point C. An additional phase lag of $180 - \phi$ is required to make the system just unstable.

The gain margin is determined from the gain at phase crossover frequency, that is, the frequency at which phase lag is 180°. Phase margin is determined from the phase lag at the gain crossover frequency. This is the frequency at which the gain is 0 dB.

10.10 FREQUENCY RESPONSE OF A CLOSED-LOOP SYSTEM USING *M*-CIRCLE AND *N*-CIRCLE

We have seen how stability condition of a control system is studied by using Bode plot and by using polar plot or Nyquist plot. Now, we will study another graphical method of determining stability condition by using *M*-circle and *N*-circle.

Transfer function of a closed-loop system is written as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

If it is a unity feedback system, $H(s) = 1$.

And for frequency response, we substitute $s = j\omega$. Therefore,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

If we draw the polar plot, this will have magnitude $M(\omega)$ and phase angle $\phi(\omega)$. *Frequency response will consist of magnitude and phase angle at different values of* ω.

Constant magnitude loci (M-circles) and constant phase angle loci (N-circles) can be drawn in complex $G(j\omega)$ plane and using them, the frequency response of a closed-loop system can be studied. These are described as follows.

10.10.1 Constant Magnitude Loci or Constant M-circle

The constant magnitude loci is a circle and hence it is also called M-circle. For different values of M, we can draw constant M-circles.

We write $G(j\omega) = x + jy$

$$\text{Magnitude, } M = |M(\omega)| = \left|\frac{G(j\omega)}{1+G(j\omega)}\right| = \frac{|x+jy|}{|1+x+jy|} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \quad \text{....(10.20)}$$

or

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

or

$$M^2[1+2x+x^2+y^2] = x^2+y^2$$

Rearranging,

$$x^2(1-M^2)+y^2(1-M^2)-2xM^2 = M^2$$

Dividing both sides by $(1 - M^2)$,

$$x^2+y^2-2x\frac{M^2}{1-M^2} = \frac{M^2}{1-M^2}$$

or

$$x^2-2x\frac{M^2}{1-M^2}+\left(\frac{M^2}{1-M^2}\right)^2+y^2 = \frac{M^2}{1-M^2}+\left(\frac{M^2}{1-M^2}\right)^2 \quad \text{adding} \left(\frac{M^2}{1-M^2}\right)^2 \text{ on both sides}$$

or

$$\left(x-\frac{M^2}{1-M^2}\right)^2+(y-0)^2 = \frac{M^2}{(1-M^2)^2} = \left[\frac{M}{1-M^2}\right]^2$$

This equation is of the form

$$(x-a)^2+(y-b)^2=c^2$$

This is the equation of a circle with centre at (a, b) and radius as c.

Here, the circle has the centre at $M^2/(1-M^2)$, 0 and radius as $M/(1-M^2)$.

For construction of M-circles for different values of M, the calculations of centre of the circles and their radius have been made as shown.

Value of *M*	**Centre of the *M*-circle** $\left(\frac{M^2}{1-M^2}, 0\right)$	**Radius of the *M*-circle** $\left(\frac{M}{1-M^2}\right)$
0	(0, 0)	0
$0 < M < 1$, say $M = 0.5$	(0.33, 0)	0.66
$M = 0.7$	(0.96, 0)	1.37
$M = 0.8$	(1.77, 0)	2.22

When $M = 1$, from equation (10.20), the value of x can be calculated as

$$M=1=\frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$$

or

$$(1+x)^2+y^2=x^2+y^2$$

or

$$1+2x+x^2+y^2=x^2+y^2$$

or

$$2x=-1$$

or

$$x=-\frac{1}{2}$$

For $M = 1$, we, therefore, find that $x = -1/2$ which is a straight line parallel to the y-axis at a distance of $-1/2$, i.e. -0.5 from the origin. Such a straight line can be considered to have an infinite radius

	Centre of Circle	**Radius**
For $M > 1$ say $M = 1.5$	(−1.8, 0)	−1.2
$M = 2$	(−1.33, 0)	−0.66
$M = 3$	(−1.125, 0)	−0.375
$M = 4$	(−1.066, 0)	−0.266

The constant M-circles for different values of M as calculated are shown in Fig. 10.59. It is observed that the loci are symmetrical about the locus of $M = 1$. For $M > 1$, the M-circles lie to the left of $M = 1$ and for $M < 1$, the M- =circles lie to the right of line for $M = 1$. For $M = 0$, the M-circle is the origin itself.

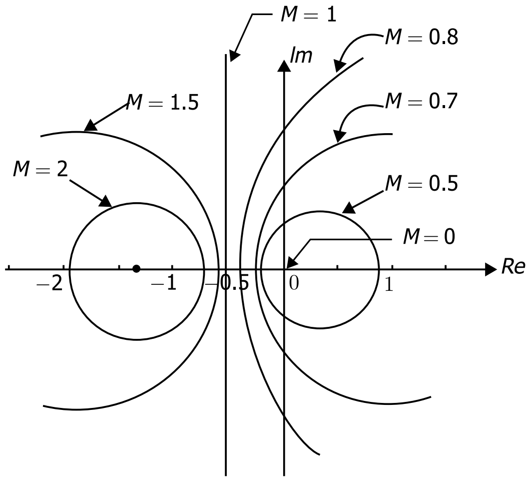


Fig. 10.59 *Constant M-circles*

10.10.2 Constant *N*-circles

$$\text{We again write, } G(j\omega) = x + jy$$

$$\text{Phase angle, } \phi = \angle \frac{G(j\omega)}{1+G(j\omega)} = \frac{\angle x + jy}{\angle 1 + x + jy} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1+x}$$

Taking tangent of the angles on both the sides,

$$\tan\phi = \frac{\frac{y}{x} - \frac{y}{1+x}}{1+\frac{y}{x}\left(\frac{y}{1+x}\right)} = \frac{\frac{y(1+x)-yx}{x(1+x)}}{\frac{x(1+x)+y^2}{x(1+x)}} \qquad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right]$$

$$= \frac{y}{x^2+y^2+x}$$

Let us assume $\tan\phi = N$
Then,

$$N = \frac{y}{x^2+y^2+x}$$

or

$$x^2 + y^2 + x - yN = 0$$

Adding $1/4 + 1/(2N)^2$ on both the sides,

$$x^2 + x + y^2 - \frac{y}{N} + \frac{1}{4} + \frac{1}{(2N)^2} = \frac{1}{4} + \frac{1}{(2N)^2}$$

or

$$x^2 + x + \frac{1}{4} + y^2 - \frac{y}{N} + \frac{1}{(2N)^2} = \frac{1}{4} + \frac{1}{(2N)^2}$$

or

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \left[\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}\right]^2$$

This is the equation of a circle of the form

$$(x - a)^2 + (y - b)^2 = c^2$$

Thus, the equation represents a family of circles whose centre is at $(-1/2, 1/2N)$ and radius of $\sqrt{1/4 + 1/(2N)^2}$.

For various values of ϕ, the centre points and radius can be calculated and the N-circles drawn as shown in Fig. 10.60.

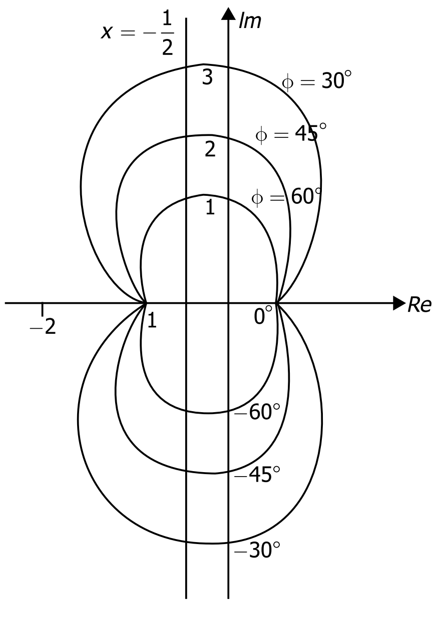


Fig. 10.60 *Constant N-circles*

10.10.3 Uses of *M*-circles and *N*-circles

The following are the uses of *M*-circles and *N*-circles:

a) Gain adjustment using *M*-circle
Consider polar plot in the diagram in Fig. 10.61.

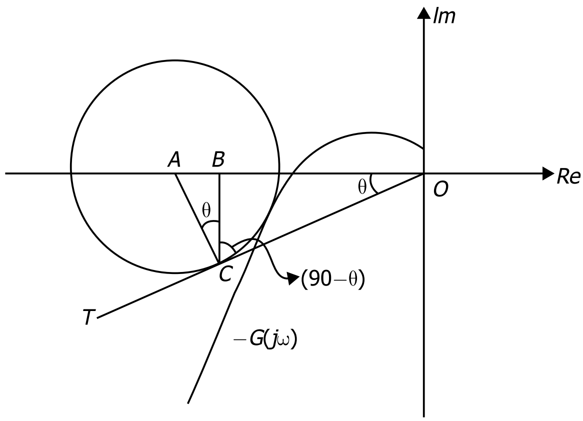


Fig. 10.61 *Gain adjustment by using M-circle drawn on the polar plot*

The circle has been drawn with centre on the negative real axis at ($M^2/1 - M^2$, 0). The circle is tangent to the line *OT*. The circle is also touching the polar plot. The radius of the circle is $M/(1 - M^2)$. The line *OT* is drawn at an angle $\boldsymbol{\theta}$ with the negative real axis where $\boldsymbol{\theta} = \sin^{-1} 1/M$.

$$\sin\theta = \frac{AC}{OA}$$

where AC = radius of the circle = $M/(1 - M^2)$ ....(10.26)
and OA = distance of the centre of the circle

$$OA = \frac{M^2}{1 - M^2} \quad \text{....(10.27)}$$

$$\sin\theta = \frac{\dfrac{M}{1-M^2}}{\dfrac{M^2}{1-M^2}} = \frac{1}{M}$$

$$\therefore OB = OA - AB \quad \text{....(10.28)}$$

$$= OA - AC\sin\theta$$

Substituting the values from equations (10.26), (10.27) and (10.28),

$$OB = \frac{M^2}{1-M^2} - \frac{M}{1-M^2}\frac{1}{M} = -1$$

The point B thus gets located on the negative real axis as $(-1, 0)$. The factor by which gain can be adjusted to obtain the desired value of M can be ascertained.

The steps of determining the gain are mentioned below.

- Draw the polar plot for $K = 1$ (or any assumed value)
- Calculate θ from $\sin\theta = \frac{1}{M}$
- Draw a line from origin at angle θ with negative real axis
- Draw a circle with centre on negative real axis, which is tangent to the line OT and to the polar plot
- From point C, draw a line CB perpendicular to the negative real axis
- Calculate the required value of gain, K as,

$$K = \frac{\text{Assumed vlaue of gian for which the polar plot was drawn}}{\text{Length } OB}$$

The above procedure is illustrated through the following example.

Example 10.30 The open-loop transfer function with unity feedback is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

Determine the value of K for which M will be equal to 2.

Solution

We will first draw the polar plot of $G(s)H(s)$ assuming $K = 1$

$$\left|G(j\omega)H(j\omega)\right| = \frac{1}{(j\omega)(1+j\omega)(5+j\omega)}$$

$$\omega = 0, \left|G(j\omega)H(j\omega)\right| = \infty, \angle G(j\omega)H(j\omega) = -90^\circ$$

$$\omega = \infty, \left|G(j\omega)H(j\omega)\right| = 0, \angle G(j\omega)H(j\omega) = -270^\circ$$

To determine the value phase cross-over frequency, we will equate the phase angles of $G(j\omega)$ $H(j\omega)$ to $-180°$. Thus

$$\phi = -180^\circ = -90^\circ - \tan^{-1}\omega - \tan^{-1}0.2\omega$$

or, $$\tan^{-1}\omega + \tan^{-1}0.2\omega = 90^\circ$$

Taking tan of both sides,

$$\tan(\tan^{-1}\omega + \tan^{-1} 0.2\omega) = \tan 90^\circ = \infty \qquad \text{....(10.24)}$$

We know, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Therefore, from equation (10.24), we can write

$$\frac{\tan\tan^{-1}\omega + \tan\tan^{-1} 0.2\omega}{1 - \tan\tan^{-1}\omega \tan\tan^{-1} 0.2\omega} = \infty$$

or, $$\frac{\omega + 0.2\omega}{1 - \omega(0.2\omega)} = \infty$$

or, $$1 - 0.2\omega^2 = \frac{1.2\omega}{\infty} = 0$$

or, $$\omega = \sqrt{\frac{1}{0.2}} = \sqrt{5} = 2.24$$

We draw the polar plot as in Fig. 10.62. We then draw line OT at an angle of 30° with the negative real axis. A circle is drawn with the centre on the negative real axis and tangent to both the polar plot and line OT. Note that the M-circle for $M > 1$ are on the left side of $M = 1$.

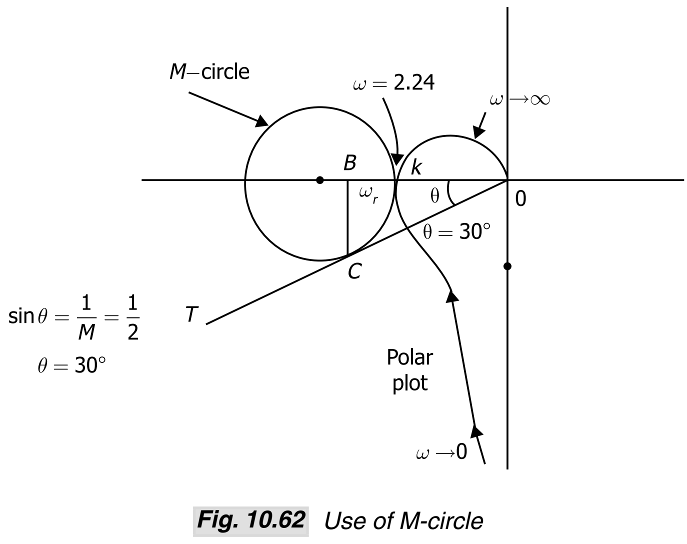


Fig. 10.62 *Use of M-circle*

The circle makes tangent to line OT at point C. From C draw a perpendicular line CB on the negative real axis. With $\omega = 2.24$, calculate length OK. Then measure length OB. In this case the length OB is approximately calculated as 0.033. The value of K is determined as

$$K = \frac{1}{OB} = \frac{1}{0.033} = 30.3$$

The intersection of the polar plot and the constant M-circle provides the value of M for a particular value of ω. When the M-circle is tangent to the polar plot, the value of M is the value of resonant peak M_r the value of $\boldsymbol{\omega}$ is the resonance frequency $\boldsymbol{\omega}_r$ as has been shown.

Use of *M*-circles and *N*-circles to determine the closed loop frequency response
Assume the open-loop transfer function of a control system as

$$G(s) = \frac{25}{s(1+2s)(1+5s)}$$

We write

$$G(j\omega) = \frac{25}{j\omega(1+2j\omega)(1+5j\omega)}$$

We can draw the polar plot of the system as shown in Fig. 10.63. We now draw the M-circles for different values of M. The circles with $M > 1$ will lie on the left side of $M = 1$ line and circles with $M < 1$ will lie on the right side of $M = 1$ line, as has been shown. The point of intersection of the polar plot with the M-circles are noted. The intersections give us the values of M and the corresponding value of frequency $\boldsymbol{\omega}$. The M-circle which becomes tangent to the polar plot gives the value of resonance peak M_r.

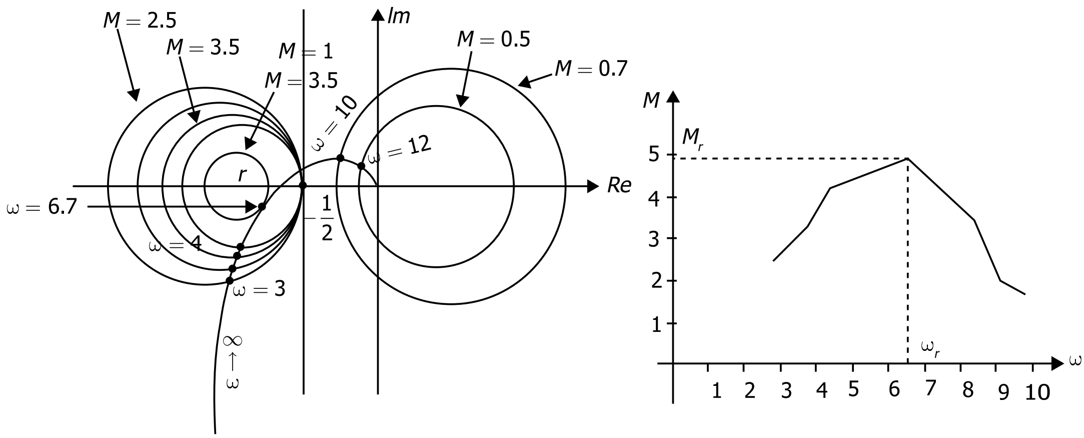


Fig. 10.63 *(a) Polar plot and M-circle; (b) Frequency response from polar plot and M-circles*

We now draw the N-circles and the polar plot as shown in Fig. 10.64(a). The point of intersection of the N-circles and the polar plot provides the values of phase angle at different values of $\boldsymbol{\omega}$. The frequency response of phase angle versus ω has been shown in Fig. 10.64(b).

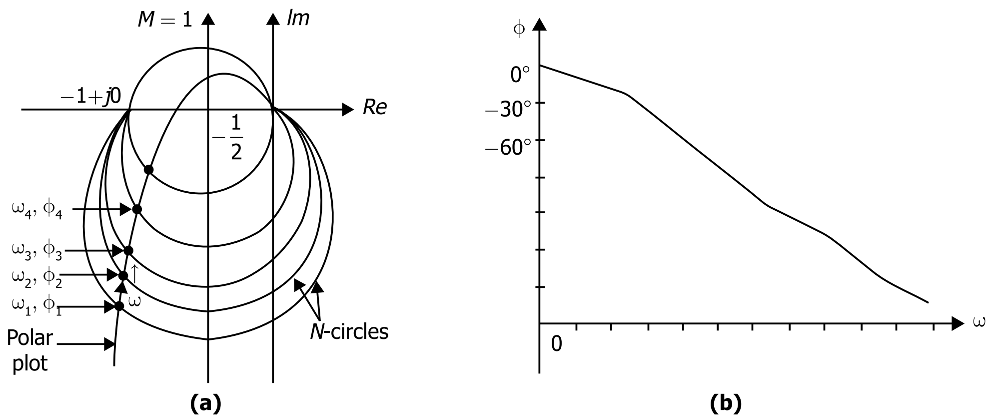


Fig. 10.64 *(a) Polar plot and the N-circles; (b) Frequency response of the system*

If we draw the polar plot superimposed on both constant M and constant N-circles, the magnitude of M and phase angle ϕ can be noted for different values of frequency, ω. This way using constant M and constant N-circles, the closed loop frequency response of the system can be drawn.

The Nichols Chart

For a given system, we have seen how by using constant M and N-circles and the polar plot, the frequency response of the system can be drawn.

However, since drawing of polar plot is a bit lengthy process than Bode plot, *N.B. Nichols* used constant M and N-circles on logarithmic co-ordinates by transformation of constant M and N-circles into log-magnitude and log-phase angle co-ordinates. The chart prepared is named after Nichols contribution and is called Nichol's chart. The Nichols chart can be made use of for drawing the closed-loop frequency response of any system.

REVIEW QUESTIONS

10.1 Plot the Bode diagram for each of the following transfer functions and find gain crossover frequency.

i) $G(s) = \dfrac{75(1+0.2s)}{s(s^2+16s+100)}$

ii) $G(s) = \dfrac{75(s^2+0.4s+1)}{s(s^2+0.8s+9)}$

10.2 Given $G(s) = \dfrac{Ks^2}{(1+0.2s)(1+00.2s)}$.

Sketch the Bode diagram and obtain the system gain K for the gain crossover frequency to be 5 rad/sec.

10.3 The open-loop transfer function of a system is given by

$$G(s)H(s) = \frac{100}{s(s+1)s+2}$$

Determine:

i) Gain margin and the corresponding gain crossover frequency.
ii) Phase margin and the corresponding phase crossover frequency.
iii) The stability of the system.

10.4 The test data of a unity feedback control system is given below.

ω	$\lvert G(j\omega)\rvert$, dB	$\angle G(j\omega)$, deg.
0.10	46.02	−179.1
0.50	18.14	−175.8
1.0	6.34	−171.8
2.0	−4.90	−166.1
4.0	−15.0	−162.9
8.0	−24.60	−173.1
10.0	−27.96	−180.0
14.0	−33.56	−192.8
20.0	−40.34	−207.9
40.0	−55.74	−234.2
80.0	−72.92	−251.2
140.0	−87.29	−259.1
200.0	−96.53	−262.4
240.0	−101.3	−236.6

i) Plot the data on a semi-log paper.
ii) Determine the transfer function represented by the above data.
iii) What type of system does it represent?

10.5 The asymptotic Bode plot of an open-loop minimum-phase transfer function for a unity feedback control system is shown in Fig.10.65.

i) Evaluate the open-loop transfer function.

ii) What is the frequency at which $|G(j\omega)|\text{dB} = 1$? What is the phase angle at this frequency?

iii) Draw the polar diagram of the open-loop control system.

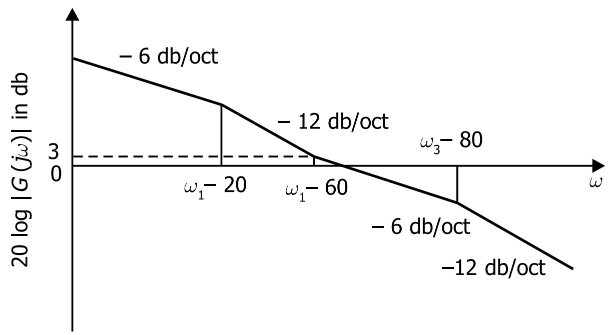


Fig. 10.65

10.6 The open-loop transfer function of a unity feedback control system is

$$G(s) = \frac{K}{s(s^2 + s + .5)}$$

Determine the value of K such that the resonant peak magnitude in the frequency response is 2 dB.

10.7 The block diagram of a control system is shown in the figure below. Draw a Bode plot of the open-loop transfer function and find the value of gain K that produces a phase margin of 50°. What is the gain margin for the same value of K?

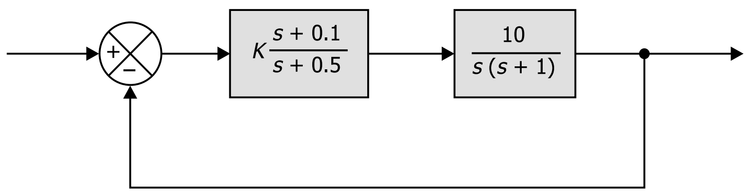


Fig. 10.66

10.8 A unity feedback control system has the transfer function

$$G(s) = \frac{K}{s(s+a)}.$$

i) Find the value of K and a to satisfy the frequency domain specification of $M_r = 1.04$ and $\omega_r = 11.55$ rad/sec.

ii) Evaluate the settling time and bandwidth of the system for the values of K and a determined in part (i).

10.9 Examine the stability of a system the open-loop transfer function of which is given by

$$G(s)H(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)},$$

for the two cases, that is,

i) the gain K is small and

ii) the gain K is large.

10.10 Using the Nyquist stability criterion determine the range of gain K (positive) for the stability of the system shown below.

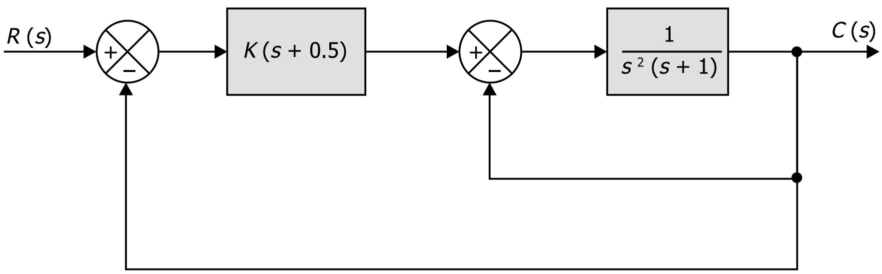


Fig. 10.67

10.11 Using MATLAB draw a Bode diagram for the system with open-loop transfer function

$$G(s) = \frac{25}{s^2 + 4s + 25}.$$

10.12 Given the open-loop transfer function

$$G(s) = \frac{1}{s(s^2 + 0.8s + 1)}.$$

draw a Nyquist plot with MATLAB.

10.13 Construct the Bode plot for the system whose open loop transfer function with unity feedback is given as

$$G(s) = \frac{1}{s(1 + 0.2s)(1 + 0.02s)}$$

Find phase margin and gain margin of the system.

10.14 The open loop transfer function of a system is given as

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+0.1j\omega)(1+j\omega)}.$$

Draw Bode plot and determine the value of K so that the phase margin is 60° and gain margin is 15 dB.

10.15 A unity feedback system has loop transfer function

$$G(s) = \frac{(s+2)}{(s+1)(s-1)}.$$

Use Nyquist criterion to determine the system stability in the closed loop configuration. Is the open loop system stable?

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11

DESIGN AND COMPENSATION

11.1 NECESSITY OF COMPENSATION

It has often been observed that the performance of a control system does not satisfy the given specifications in terms of accuracy, stability, and so on. After design and testing if the system does not perform satisfactorily some changes may need to be introduced to achieve the desired results. The changes could be in the form of adjustment of forward path gain or insertion of a compensating device in the control system.

To reduce steady-state error, gain can be increased. However, it results in an oscillatory transient response or even instability. Under such circumstances, it may be necessary to introduce some kind of corrective sub-systems to force the chosen plant (system) to meet the given specifications. These sub-systems are known as compensators and their job is to compensate for the deficiency in the performance of the plant. The desired behaviour of a system is specified in terms of transient response and steady-state error.

Compensators are required in the following two cases, viz (i) system is unstable, compensation is required to stabilize it and also to achieve the desired performance specifications; (ii) system is stable, compensation is required to achieve improved performance specifications.

System performance is specified in terms of steady state response as well as transient response. In time domain, mention of stability criterion is made in terms of damping factor, peak overshoot and speed of system response. In frequency domain, mention is made of resonant peak, phase margin, resonant frequency, etc.

The design of a *feed-back* control system involves a compromise between the magnitude of allowable steady-state error and the degree of stability that is desired in the system performance.

11.2 EFFECT OF ADJUSTMENT OF GAIN

The performance of a control system is made more satisfactory either by changing the loop gain or by adding poles and zeros, or by changing time constants so as to reduce the steady-state error and make the system stable within the limit of specifications.

Let us take the example of a position servo system. Assume that the limit of steady-state error is met by the system but the damping is too low. This means that the system will be oscillatory. The immediate suggestion would be to reduce the gain of the amplifier which is connected to the servo motor as shown in Fig. 11.1. The reduction of gain of the amplifier would increase the damping ratio which can be seen by drawing the root locus plot.

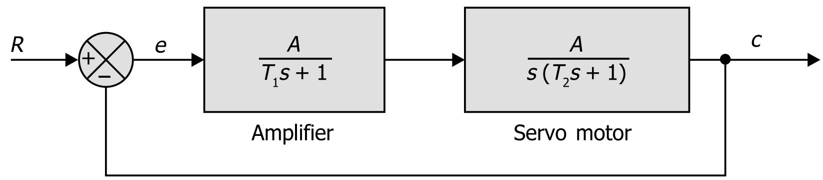
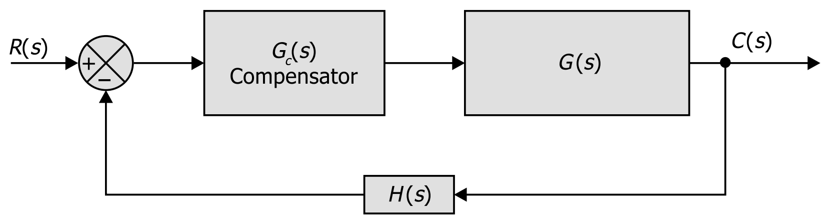


Fig. 11.1 *Block diagram of a position servo system*

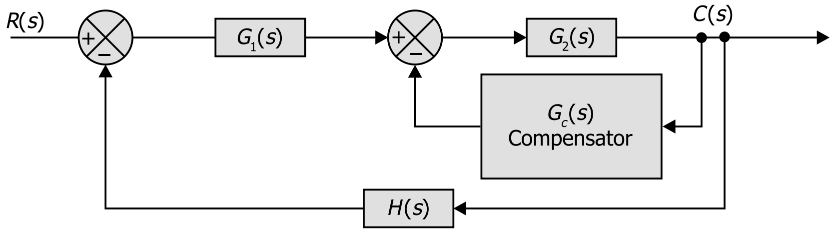
It may be discovered that the reduction of gain of the amplifier would cause improved stability but the velocity error of the system will increase. Thus we see that gain variation alone may not be sufficient to achieve the desired result. When a system requires improvement in the steady-state error or a degree of stability greater than what can be obtained by the adjustment of gain, inclusion of a network in the loop may be required. Such a network is called a compensating network.

11.3 COMPENSATION BY INSERTING A NETWORK

Compensation for improved performance may be provided to the control system by inserting a network either in series as a cascade or series compensation or in parallel as feedback compensation as shown in Fig. 11.2(a) and (b) respectively.



(a) *Cascade or series compensation*



(b) *Feedback compensation*

Fig. 11.2 *Use of compensating networks*

Let us consider some basic compensating networks which are also called compensators. Compensators may be made by using electrical, mechanical, or pneumatic components. Most often, electrical components such as resistors and capacitors are used to construct compensators. In this section, we will discuss three types of electrical compensators, namely, lead compensator, lag compensator, and lag-lead compensator.

11.4 LEAD COMPENSATOR

The lead compensator is a R-C network, as shown in Fig. 11.3. The transfer function of the lead compensator is calculated as

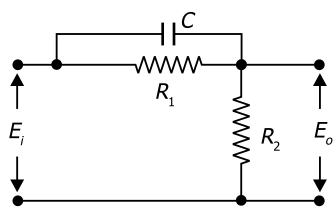


Fig. 11.3 *A lead network*

$$G_c(s) = \frac{E_o(s)}{E_i(s)} = \frac{R_2}{R_2 + \dfrac{R_1/Cs}{(R_1Cs+1)/Cs}} = \frac{R_2}{R_2 + \dfrac{R_1}{R_1Cs+1}}$$

$$= \frac{R_2(R_1Cs+1)}{R_1R_2Cs + R_1 + R_2} = \frac{R_2}{(R_1+R_2)} \frac{(R_1Cs+1)}{\left[1 + \dfrac{R_2}{R_1+R_2} R_1Cs\right]}$$

Now, considering $R_1C = \tau$, and $\dfrac{R_2}{R_1 + R_2} = \alpha$

$$G_c(s) = \frac{E_o(s)}{E_i(s)} = \frac{\alpha(1+\tau s)}{(1+\alpha\tau s)} = \frac{\alpha\tau\left(s+\frac{1}{\tau}\right)}{\alpha\tau\left(s+\frac{1}{\alpha\tau}\right)}$$

$$= \frac{\left(s+\frac{1}{\tau}\right)}{\left(s+\frac{1}{\alpha\tau}\right)} = \frac{s+Z_c}{s+P_c}, \tau > 0 \alpha = \frac{Z_c}{P_c} < 1$$

where Z_c and P_c are, respectively, the zero and pole of the compensator.

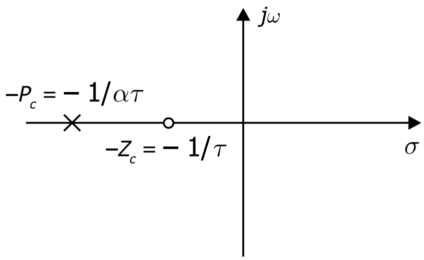


Fig. 11.4 *Positions of pole and zero of a lead compensator in s-plane*

The s-plane representation of the lead compensation with the positions of pole and zero is shown in Fig. 11.4. Note that a lead compensator is essentially a high pass filter.

The lead compensator produces an effect of adding a zero in the system. Since it is difficult to realise in practice a compensator with only one zero alone, a compensating network with a zero and a pole, as shown in Fig. 11.3 is generally used.

Such a compensator is used usually in association with an amplifier as shown in Fig. 11.5.

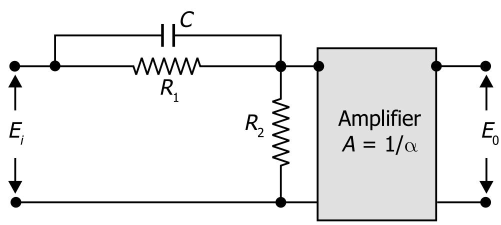
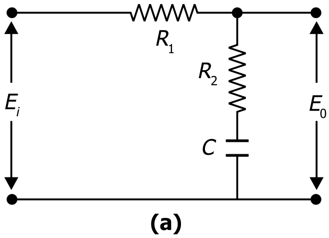


Fig. 11.5 *Modified lead compensator with amplifier*

At zero frequency, the network has a gain of $\alpha < 1$ or an attenuation of $1/\alpha$. It is convenient to cancel the DC attenuation of the lead network with an amplification, $A = \frac{1}{\alpha}$. This arrangement will place the compensating pole far from the origin in the left half of the s-plane so that this pole will have relatively less effect on the root loci as compared to the dominant complex poles. The effect of adding of lead compensator to a control system will be studied with examples a little later.

11.5 LAG COMPENSATOR

A phase-lag compensator has been shown in Fig. 11.6(a). The compensator has a simple pole and a simple zero in the left half of the s-plane, with the pole to the right of the zero. Note that a lag compensator is essentially a low pass filter.



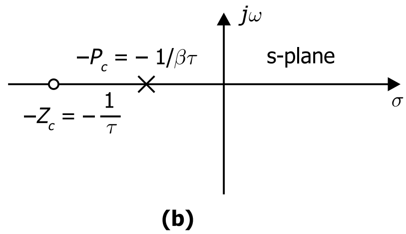


Fig. 11.6 *(a) Phase-lag network; (b) Positions of pole and zero of a phase-lag compensator is s-plane*

The transfer function of the compensator is calculated as follows.

$$G_c(s) = \frac{E_o(s)}{E_i(s)} = \frac{R_2 + \dfrac{1}{Cs}}{R_1 + R_2 + \dfrac{1}{Cs}}$$

$$= \frac{R_2Cs + 1}{Cs\,R_1 + Cs\,R_2 + 1}$$

$$= \frac{R_2Cs + 1}{R_2Cs\dfrac{(R_1 + R_2)}{R_2} + 1}$$

Defining, $R_2C = \tau$ and $\dfrac{R_1 + R_2}{R_2} = \beta,$

$$G_c(s) = \frac{\tau s + 1}{\tau s\beta + 1} = \frac{s + \dfrac{1}{\tau}}{\beta\left(s + \dfrac{1}{\beta\tau}\right)}$$

The s-plane representation of a lag compensator has been shown in Fig. 11.6(b). The lag compensator along with the amplifier of gain K_c has been shown in Fig. 11.7.

$$G_c(s) = \frac{E_o(s)}{E_i(s)} = K_c \frac{\tau C + 1}{(\beta\tau C + 1)} = \frac{K_c}{\beta} \frac{\left(s + \frac{1}{\tau}\right)}{\left(s + \frac{1}{\beta\tau}\right)} = K \frac{\left(s + \frac{1}{\tau}\right)}{\left(s + \frac{1}{\beta\tau}\right)}$$

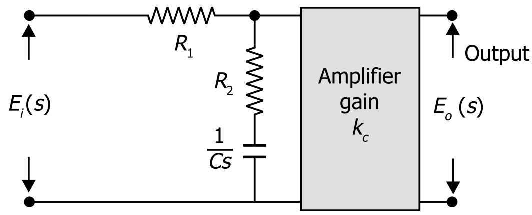


Fig. 11.7 *Modified lag compensator with an amplifier*

where $\tau = R_2C,\ \beta = \frac{R_1 + R_2}{R_2},\ K = \frac{K_c}{\beta}$.

If the pole zero locations are known, τ and β can be determined directly. The location of zero is at $-(1)/\tau$ and that of pole is at $-(1)/\beta\tau$. Thus, we can calculate β.

We have four parameters, R_1, R_2, C, K to be determined from three parameters, namely, K_c, τ, and β as determined from the design considerations.

Note that in a lead compensator the zero is located to the right of the pole whereas in a lag compensator the zero is located to the left of the pole.

11.6 LAG-LEAD COMPENSATOR

The lag-lead compensator comprises a lead compensator with a pole zero pair and a lag compensator with another pole zero pair connected in series.

These compensators are used when error in both transient and steady-state performance of the system have to be neutralised or reduced. A typical lag-lead compensator is shown in Fig. 11.8.

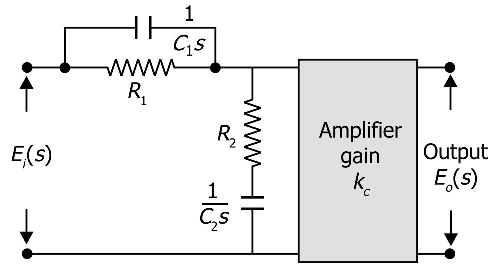


Fig. 11.8 *A phase lag-lead compensating network*

The overall transfer function of the network is

$$G_c(s) = \frac{E_o(s)}{E_i(s)}$$

$$= k_c \frac{R_2 + \dfrac{1}{C_2 s}}{R_2 + \dfrac{1}{C_2 s} + \dfrac{R_1 \dfrac{1}{C_1 s}}{R_1 + \dfrac{1}{C_1 s}}}$$

$$= k_c \left[\frac{\left(s + \dfrac{1}{R_1 C_1}\right)\left(s + \dfrac{1}{R_2 C_2}\right)}{s^2 + \left(\dfrac{1}{R_1 C_1} + \dfrac{1}{R_2 C_1} + \dfrac{1}{R_2 C_2}\right) s + \dfrac{1}{R_1 R_2 C_1 C_2}} \right]$$

$$= k \left[\underbrace{\frac{\left(s + \dfrac{1}{\tau_1}\right)}{\left(s + \dfrac{1}{\beta \tau_1}\right)}}_{\text{Lag section}} \underbrace{\frac{\left(s + \dfrac{1}{\tau_2}\right)}{\left(s + \dfrac{1}{\alpha \tau_2}\right)}}_{\text{Lead section}} \right], \beta > 1, \alpha < 1$$

where, $\tau_1 = R_1 C_1$, $\tau_2 = R_2 C_2$

The s-plane representation of the lag-head compensator is shown in Fig. 11.9.

$$G_c = \frac{(s + Z_{c1})}{(s + P_{c1})} \frac{(s + Z_{c2})}{(s + P_{c2})}$$

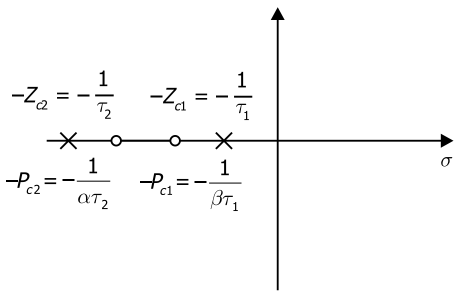


Fig. 11.9 *s-plane representation of a phase lag-lead compensator*

Fig. 11.10 shows the different types of networks with their transfer functions, pole zero locations and Bode plot.

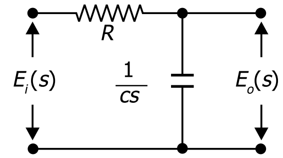
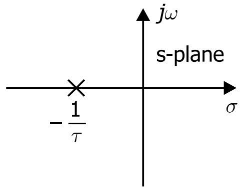
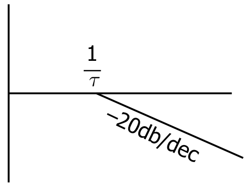
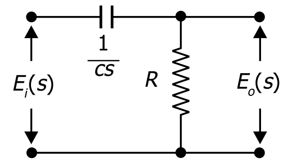
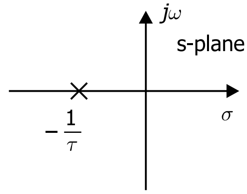
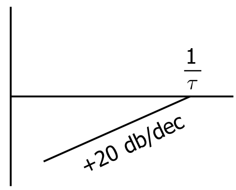
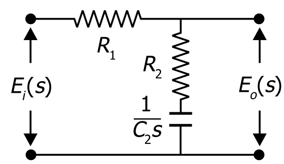
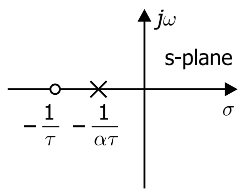
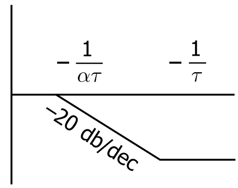
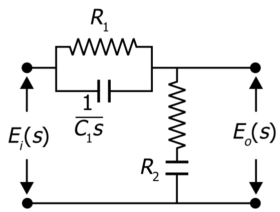
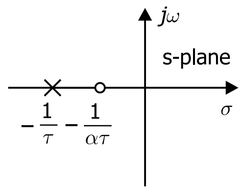
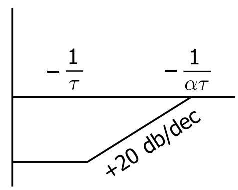
Compensating network	Transfer function	Pole zero location	Bode plot
	$G_c(s) = \dfrac{1}{\tau s + 1}$ $\tau = Rc$		
	$G_c(s) = \dfrac{\tau s}{\tau s + 1}$ $\tau = Rc$		
	$G_c(s) = \dfrac{\tau s + 1}{\alpha \tau s + 1}$ $\tau = c_2 R_2,\ \alpha = 1 + \dfrac{R_1}{R_2}$		
	$G_c(s) = \dfrac{(\alpha \tau s + 1)}{(\tau s + 1)}$		

Fig. 11.10 *Characteristics of frequently used compensating networks*

Example 11.1 Draw a phase lead compensating network. How is the effect of zero dominated in it?

Solution

A phase lead compensation network is as shown in Fig. 11.11.

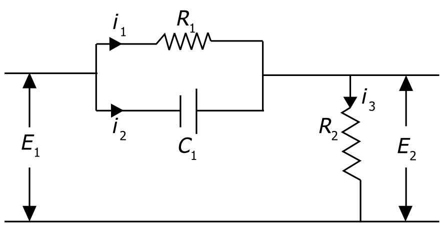


Fig. 11.11 *Phase lead compensation network*

From the circuit, using KVL, we get

$$\frac{1}{R^2}E_2(s) = \frac{1}{R_1}[E_1(s)] + sC[E_1(s) - E_2(s)]$$

Simplifying
$$\frac{E_2(s)}{E_1(s)} = \frac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs}$$

$$= \frac{R_2}{R_1 + R_2}\left[\frac{1 + R_1Cs}{1 + \frac{R_1R_2Cs}{R_1 + R_2}}\right]$$

Putting
$$= \frac{R_1 + R_2}{R_2}$$
$$= a \text{ where } a > 1$$

and
$$T = \frac{R_1 + R_2}{R_1 + R_2}C$$

we get
$$\frac{E_2(s)}{E_1(s)} = \frac{1}{a}\left[\frac{1 + aTs}{1 + Ts}\right]\frac{1}{a}\left[\frac{1 + aTs}{1 + Ts}\right]$$

Now if we plot pole and zero for the above transfer function, we get

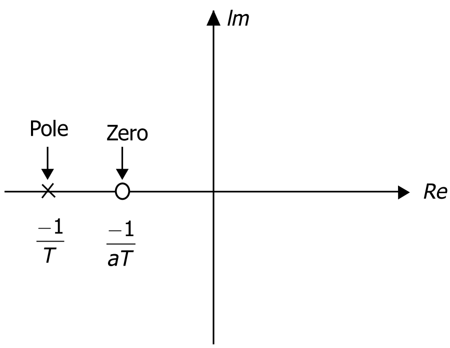


Fig. 11.12 *Pole zero plot of the transfer function of phase lead network*

Now, zero is more close to imaginary axis as compared to the pole. Zero becomes more dominant and gives a positive phase shift. Phase lead network allows the high frequencies to pass and attenuate low frequencies.

11.7 DESIGN PROCEDURE

A control system has been shown in Fig. 11.13. Examine the effect of adding a series phase lag network of transfer function $G_c = \dfrac{1+2s}{1+20s}$

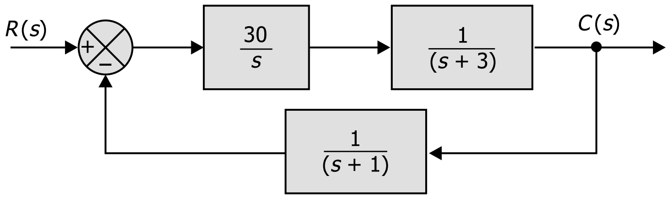


Fig. 11.13 *A feedback control system*

A portion of the Nyquist plot for the system has been drawn in Fig. 11.14. The polar plot intersects the negative real axis at −2.5. The system is therefore unstable (encircles −1). The system can perhaps be made stable by reducing the gain from 30 to a lower value. However, keeping the system parameters unchanged, let us use the series phase lag network. The transfer function of the compensated system becomes equal to

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G(s)G_c(s)H(s)} = \frac{(1+2s)}{(1+20s)}\frac{30(s+1)}{s(s+1)(s+3)+30(1+2s)}$$

The polar plot of the compensated system has been shown in Fig. 11.14. It can be seen that the compensated system has made the system perform better in terms of the following parameters.

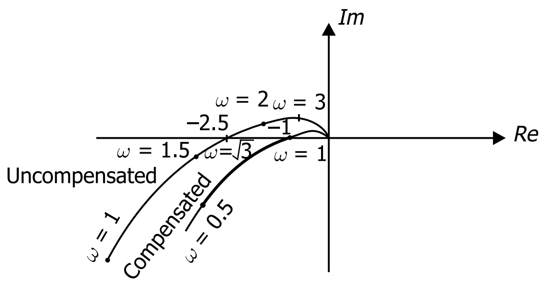


Fig. 11.14 *Polar plot of uncompensated and compensated systems*

The compensated system is now stable as the polar plot intersects the negative real axis to the right of −1. Though the stability of the system could be improved by lowering the value of gain, the same has been achieved by using the series compensating network. Keeping the value of gain high helps reduce the steady-state error to certain inputs which is also advantageous from the point of view of reduced errors, caused due to friction, backlash, and so on.

Let us now consider through an example the effect of adding open-loop poles and zeros which will be helpful in understanding design of feedback control system.

i) Effect of addition of poles: The root loci of configuration with two real poles at $s = 0$ and $s = -1$, has been drawn in Fig. 11.15(a). The addition of a pole at $s = -2$, has been shown in Fig. 11.15(b).

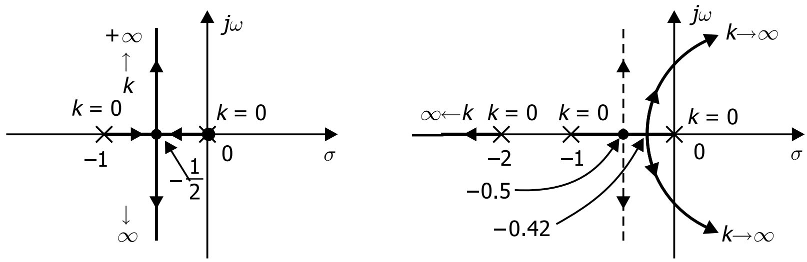


Fig. 11.15 *(a) Uncompensated root locus diagram; (b) The effect of addition of poles*

This will move the root loci to the right (break away point being moved to the right) and also causes the root loci to bend towards the right half of the s-plane as shown in Fig. 11.15(b). The break away point has been moved from -0.5 to -0.42. The addition of another pole at a point further to the left of the real axis will move the root loci more to the right. The root loci diagram shown in Fig. 11.15(a) shows that the system is stable, but the addition of poles makes the system unstable even for smaller values of k. The addition of more poles will make the system unstable even for further reduction of values of gain k.

ii) Effect of addition of zeros: Addition zeros to the system transfer function produces a phase lead which tends to stabilise the closed loop control system. The root loci of a two-pole configuration has been shown in Fig. 11.16(a). If a zero is added at $s = -b$ as shown in Fig. 11.16(b), the resultant root loci will bend towards the left. The stability margin is therefore increased.

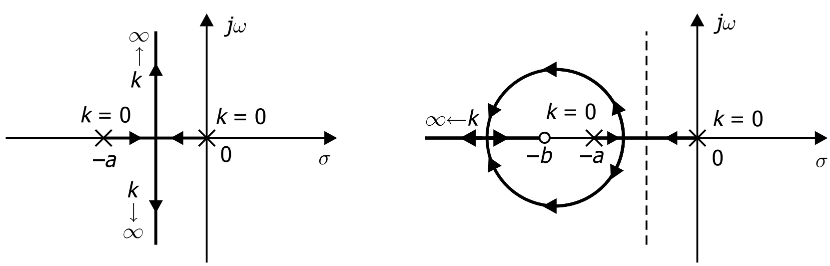


Fig. 11.16 *(a) Uncompensated root locus diagram; (b) The effect of addition of poles*

The performance of a control system is generally defined either in terms of time domain performance measures or frequency domain performance measures. The performance of a system can be specified in terms of maximum allowable steady-state error for several test signal inputs, maximum overshoot and settling-time for a step input, and so on. The performance specifications can be defined in terms of the desirable locations of the poles and zeros of the closed-loop transfer function.

The location of the s-plane poles and zeros of the system transfer function is first specified. The locus of the roots of the closed-loop system then obtained for the variation of one of the system parameters. On examination when it is found that the locus of roots does not satisfy the required configuration, compensating networks are used to alter the locus of the roots of the uncompensated system. We can, therefore, use the root locus method and use a portable compensator so that the resultant root locus satisfies the desired closed-loop root configuration. We can also use frequency performance measures to describe the performance of a feedback control system.

In such a case we can describe a system in terms of the peak of the closed-loop frequency response, resonant frequency, bandwidth and phase margin of the system. We can use a suitable compensating network to improve and satisfy system performance. As we have already seen, the compensating network is a *R*-*C* circuit.

We shall now take up some design examples using root locus.

The phase lead network has a transfer function

$$G_c(s) = \frac{s + Z_c}{s + P_c} = \frac{\left(s + \dfrac{1}{\tau}\right)}{\left(s + \dfrac{1}{\alpha\tau}\right)}$$

where α and τ have been defined for the RC network. The locations of pole and zero are selected to obtain satisfactory root locus for the compensated control system.

The procedure for compensation design is as follows:

i) Note the system specifications and translate them into a desired root location for the dominant roots.

ii) Draw the uncompensated root locus.

iii) If a compensator is necessary, place the zero of the phase lead network directly below the desired root location or to the left of the first two real poles.

vi) Determine the pole location so that the total angle at the desired root location is 180° and therefore is on the compensated root locus.

v) Calculate the total system gain at the desired root location and calculate the error constant. If the error constant is not satisfactory, all the steps from (i) to (iv) above need to be repeated.

These steps are illustrated as in Fig. 11.17.

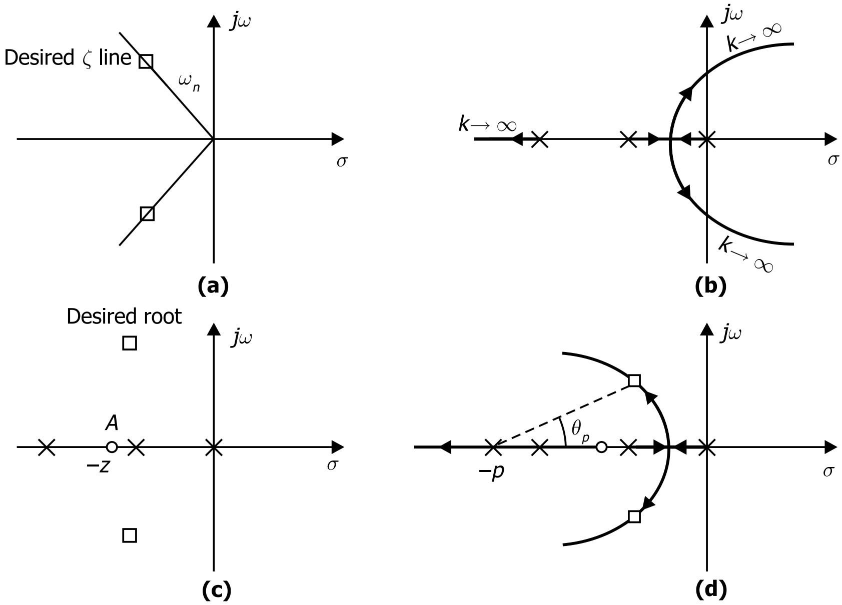


Fig. 11.17 *Compensation on s-plane using a phase lead network: (a) Desired root location; (b) uncompensated root locus; (c) Addition of zero; (d) Location of new pole*

Fig. 11.17(a) shows the dominant root locations satisfying the specifications in terms of ξ and ω_n. The root locus of the uncompensated system has been shown in Fig. 11.17(b). A zero is added at *A* on the real axis to provide a phase lead as shown in Fig. 11.17(c). The zero has been placed to the left of the first two real poles. Fig. 11.17(c) shows that the desired root is directly above the second pole, the zero, *z*, is placed somewhat to the left of the second real pole. The designer must continually see that the compensated system will respond to the influence of real roots and zeros of the system and that the dominant roots will not by themselves dictate the response. When the final compensation is achieved, the desired root will obviously be on the root locus and it is expected that the algebraic sum of the vector angles will be equal to 180° at that point. The angle θ_p is calculated such that the total angle is 180°. By locating a line at an angle θ_p on the real axis and intersecting the desired root, it is possible to locate the compensator pole *p*.

The advantage of this root-locus method is that the designer is able to specify the location of dominant roots and hence the dominant transient response.

The disadvantage is that it is not possible to directly specify an error constant. After the design is complete, the designer has to calculate the gain of the system at the root location which depends on *p* and z and also calculate the error constant for the compensated system. He has to repeat the process if the result is not satisfactory.

Now let us consider the phase-lead compensation design using Bode diagram. Here the frequency response of the series compensation network is added to the frequency response of the uncompensated system. The procedure is as follows. First, the Bode diagram for the

$G(j\omega)\ H(j\omega)$ is drawn. After examining the plot for $G(j\omega)\ H(j\omega)$, we determine a suitable location for p and z of $G_c\,(j\omega)$ in order to reshape the frequency response. The uncompensated $G\ (j\omega)\ H(j\omega)$ is plotted with the desired gain to allow an acceptable steady-state error, the phase margin, and then the expected M_{pw} are examined to see whether the system satisfies the specifications. If the phase margin is not sufficient, phase lead can be provided by adding $G_c\,(j\omega)$ at a suitable location.

The compensating network is determined through the following steps.

i) Determine the phase margin of the uncompensated system when the error constants requirements are satisfied.

ii) Determine the requirement of additional phase lead, θ_m (allowing for small amount of safety).

iii) Evaluate α from the relation,

$$\sin\theta_m = \frac{\alpha - 1}{\alpha + 1}.$$

iv) Evaluate 10 log α and determine the frequency where the uncompensated magnitude curve is equal to $-$ 10 log α dB. Since the compensation network provides a gain of 10 log α at ω_m, this frequency is now the zero dB cross over frequency and also ω_m.

v) Calculate the pole $p = \omega_m\ \sqrt{\alpha}$ and $Z\ P/\alpha$.

vi) Draw the compensated frequency response and check the phase margin and repeat this process if necessary.

We shall now take up design of control systems using compensating networks of different types discussed earlier.

Example 11.2 The characteristic equation of an uncompensated system is given as

$$1 + G(s)H(s) = 1 + \frac{K}{s^2} = 0$$

whose root locus is on they $j\omega$ axis. Design a compensator so that the system meets the specification of settling time at 2 percent criterion of $T_s \leq 4$ seconds and the damping ratio $\xi \geq 0.4$.

Solution

We know that settling time is the time required for the system to settle within a certain percentage of the input. Here, settling time at 2% criterion, $T_s \leq 4$ seconds. This occurs when

$$e^{-\xi\omega_n T_s} < 0.0$$

or,

$$\xi\omega_n T_s \simeq 4$$

or

$$T_s = \frac{4}{\xi\omega_n}$$

Here, T_s is given as 4.

$$\therefore \quad T_s = \frac{4}{\xi\omega_n} = 4$$

or $$\xi\omega_n = 1$$

Let us choose a dominant root location as $-1 \pm j2$ as shown in Fig. 11.18 so that value of ξ satisfies the specification.
Referring to Fig. 11.18,
$\boldsymbol{\omega_n \cos\theta = \xi\omega_n} = 1$

$$\text{Damping Ratio, } \xi = \cos\theta = \frac{1}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}} = 0.4\ 5 \ \ (\xi \geq 0.4)$$

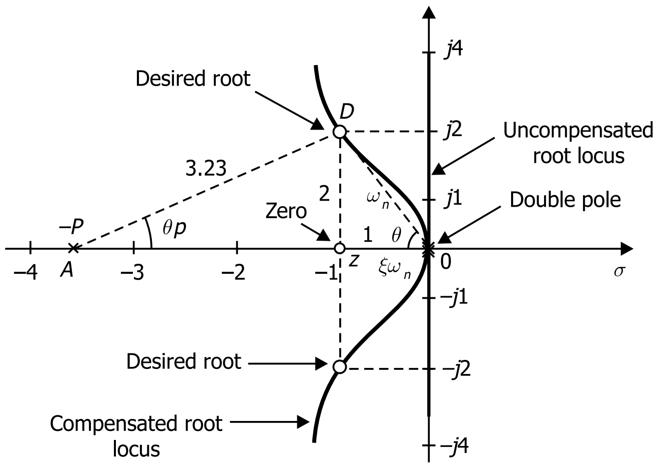


Fig. 11.18 *Phase lead compensation of Example 11.1*

We have to follow the procedure for compensation design step by step. After we have found the desired root location and seen that they satisfy the specifications, we place the zero of the phase lead compensator just below the desired root on the real axis which is at -1. Therefore, $s = -z = -1$.
Now we will calculate $\boldsymbol{\theta_p}$. For this we will consider the angle contribution of all the roots at the desired root and equate it to $-180°$.

$$-180° = -\boldsymbol{\theta p} - 115° - 115° + 90°(115° \text{ each for the double pole at the origin})$$

or, $$\boldsymbol{\theta_p} = 270° - 230° = 40°.$$

Angles contributed by the poles are taken as negative.
We now draw a line (shown as dotted line) from the desired root making an angle of 40° with the negative real axis and intersecting it at point A. The point of intersection is noted as -3.65. Therefore, $s = -p = -3.65$.

$$\text{The lead compensator is } G_c(s) = \frac{s+z}{s+p} = \frac{s+1}{s+3.65}$$

$$\text{The transfer function } G(s)H(s) \text{ was } = \frac{K}{s_2}$$

Therefore for series compensation, the transfer function of the compensated system is $\dfrac{K(s+1)}{s^2(s+3.65)}$. The gain K can be calculated by measuring the distance of the poles and zeros from the desired root location. The value of K is calculated using the relation,

$$K = \frac{\text{vector length of poles}}{\text{vector length of zeros}} = \frac{\text{Lengths DA} \times \text{ DO } \times \text{ DO}}{\text{Length DZ}}$$

$$= \frac{3.23 \times \sqrt{5} \times \sqrt{5}}{2} = 8.05.$$

Example 11.3 A control system is represented by open-loop transfer function

$$G(s) = \frac{K}{s(s+3)}$$

It is desired that damping ratio of the dominant roots of the system be $\boldsymbol{\xi} = 0.5$ and that the velocity error constant be equal to 10.

Solution

We know, error constant $K_v = \lim_{s\to 0} sG(s)$. Therefore, $\lim_{s\to 0} \dfrac{sK}{s(s+3)} = \dfrac{K}{3}$

Substituting, $\dfrac{K}{3} = 10,$

the gain K for satisfying error constant requirement must be 30. With $K = 30$, we can calculate the roots of the uncompensated system as

$$s^2 + 3s + 30 = 0$$

or,
$$s = \frac{-3 \pm \sqrt{3^2 - 4 \times 30}}{2} = -1.5 \pm j5.3$$

For the uncompensated system,

$$\xi\omega_n = 1.5$$

$$\xi = \cos\theta = \frac{\text{OT}}{\text{OD}}$$

$$= \frac{1.5}{\sqrt{5.3^2 + 1.5^2}} = 0.28.$$

Since the damping ratio of the uncompensated network is 0.28 and the desired value is 0.5, a compensating network has to be used. Let us assume a reasonable settling time of 2 seconds.

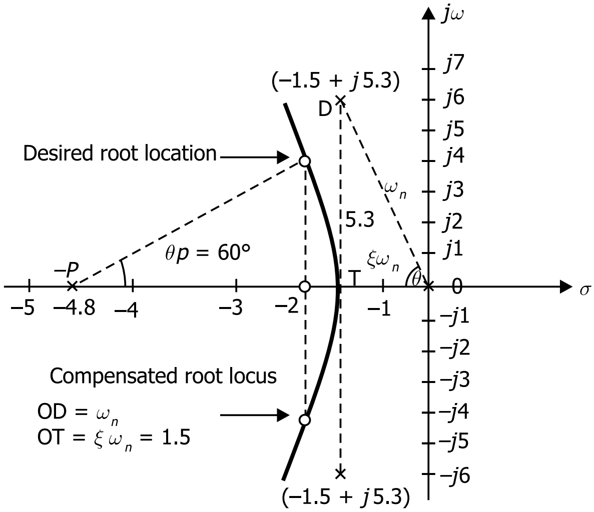


Fig. 11.19 *Phase lead compensation for Example 11.3*

We know, setting time $T_s = \dfrac{4}{\xi\omega_n}$. Here $T_s = 2$ seconds (given)

$$T_s = 2 = \frac{4}{\xi\omega_n}$$

$\therefore \quad \boldsymbol{\xi\omega_n} = 2$

$$\omega_n = \frac{2}{0.5} = 4.$$

The location of the desired roots with $\boldsymbol{\xi} = 0.5$, $\boldsymbol{\omega}_n = 4$, $\boldsymbol{\omega}_n = 2$ has been shown in Fig. 11.19. The angle at the desired root location has been calculated approximately and equated as

$$-180° = -\boldsymbol{\theta}_p - 138° - 72° + 90°$$

or, $$\boldsymbol{\theta}p = 60°$$

The angle of intercept $\boldsymbol{\theta}p = 60°$. The intersection point on the negative real axis is at -4.8.
The compensating network is
The value of K is calculated as

$$K = \frac{\text{Distance of poles}}{\text{Distance of zeros}} = \frac{4.2 \times 4.1 \times 5.1}{3.8} = 22.6.$$

The compensated system is

$$G_c(s)G(s)H(s) = \frac{22.6(s+2)}{s(s+3)(s+4.8)}$$

Example 11.4 A control system is represented by an open loop transfer function.

$$G(s)H(s) = \frac{K}{s(s+2)}$$

The damping ratio has to be made equal to 0.5 and the velocity error constant restricted to 18. Design the lead compensating network for the above.

Solution

Given, error constant $K_v = 18$

We know,

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

$$18 = \lim_{s\to 0} \frac{sK}{s(s+2)} = \frac{K}{2}$$

or $$K = 36$$

Therefore, to satisfy the requirement of error constant, gain K should be equal to 36.
With this value of K, let us calculate the roots of the uncompensated system.

$$s\,(s + 2) + 36 = 0$$

$$s^2 + 2s + 36 = 0$$

$$s = \frac{-2 \pm \sqrt{2^2 - 4\times 36}}{2} = -1 \pm j5.9$$

$$s_1, s_2 = (s+1+j5.9)(s+1-j5.9)$$

Damping ratio of the uncompensated system is calculated as, $\omega_n \cos\theta = \xi\omega_n$

$$\xi = \cos\theta = \frac{1}{\sqrt{5.9^2 + 1^2}} = 0.165$$

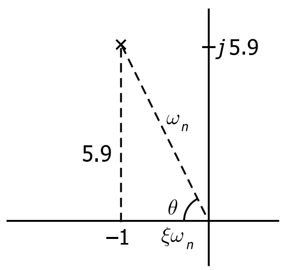


Fig. 11.20 Calculation of damping ratio for the uncompensated system with $G(s)H(s) = \dfrac{36}{s(s+2)}$

This shows that the value of ξ of the uncompensated system is less than the desired value, i.e. 0.5.

We know, settling time $T_s = \dfrac{4}{\xi\omega_n}$

If we consider settling time as 1 second, then

$$T_s = 1 = \frac{4}{\xi\omega_n}$$

Therefore, $\xi\omega_n = 4$ which will be on the real axis.
Considering the new value of $\xi = 0.5$

$$\omega_n = \frac{4}{\xi} = \frac{4}{0.5} = 8.$$

The desired root location has been shown in Fig. 11.21 at R and R'.
Now we place the zero of the compensator just below the desired pole location on the negative real axis at $s = -z = -4$ so that numerator of the compensator transfer function is $(s + 4)$.
We have now to calculate the angle, ϕ at the desired root location due to the other poles and zeros and equate to 180° by assuming the pole location of the compensator on the real axis at $-p$. The line joining R and $-p$ makes an angle θ_p with the negative real axis.
The angle θ_p of the compensator pole location is determined as

$$-180° = \theta - p$$

Angle, $$\phi = -114° - 105° + 90 = -129°$$

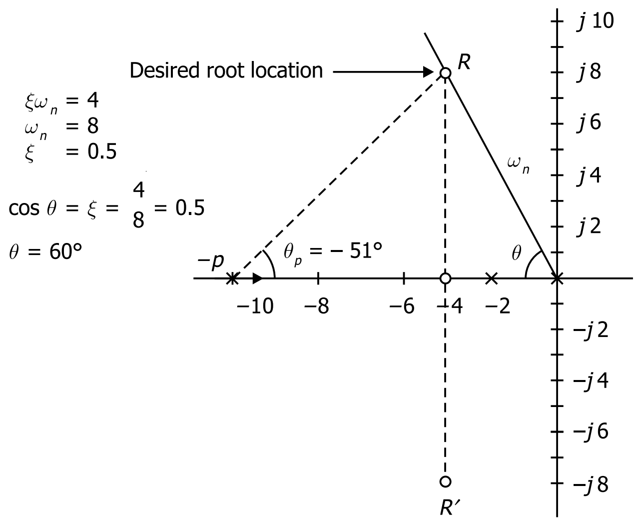


Fig. 11.21 *Determination of the desired root location of the compensated system*

$$\therefore \quad \theta_p = -180° + 129° = -51°.$$

The-point of intersection of the dotted line from R to $-p$ on the negative real axis is at -10.8. Thus the denominator of the compensator with

$$s = -p = -10.8 \text{ is } (s + 10.8)$$

The transfer function $G_c(s)$ of the compensator is

$$G_c(s) = \frac{(s+4)}{(s+10.8)}$$

The value of K is calculated as

$$K = \frac{8 \times 7.9 \times 10.2}{7.8} = 83$$

The compensated system,

$$G_c(s)G(s)H(s) = \frac{83(s+4)}{s(s+2)(s+10.8)}$$

The velocity error constant of the compensated system has to be calculated and checked with the desired value.

$$K_v = \lim_{s \to 0} \frac{s \times 83(s+4)}{s(s+2)(s+10.8)} = 15.5$$

This is somewhat less than desired value of 18. We will have to repeat the design process by making another choice for the desired root. We may choose $\xi\omega_n = 4.5$ so that ω_n become 9 and continue with the set procedure.

We can calculate the values of the components of the phase lead network considering the design already made (the students will take values of the repeat design and calculate in the same way).

$$G_c(s) = \frac{s+4}{s+10.8} = \frac{s + \frac{1}{\tau}}{s + \frac{1}{\alpha\tau}}$$

where $$\tau = R_1 C \text{ and } \alpha = \frac{R_2}{R_1 + R_2}$$

Comparing,

$$\tau = \frac{1}{4} = R_1 C$$

$$\frac{R_2}{R_1 + R_2} = \frac{4}{10.8} = 0.37$$

We have to choose a suitable value of C and calculate values of R_1 and R_2 from the above relations.

Example 11.5 The open-loop transfer function of a feedback control system is given as

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$

Design a compensating network so that the steady state error is not more than 0.3 for unit ramp input and the damping ratio is 0.6.

Solution

$R(s) = \frac{1}{s^2}$ (for unit ramp input)

The steady state error $$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

Substituting,

$$e_{ss} = \frac{s\frac{1}{s^2}}{1+\frac{K}{s(s+2)(s+3)}}$$

$$= \frac{1}{s}\frac{s(s+2)(s+3)}{s(s+2)(s+3)+K}$$

$$= \frac{6}{K}$$

The value of K for the uncompensated system for a steady state error comes to

$$0.3 = \frac{6}{K}$$

$$\therefore K = 20$$

Therefore for the steady state error to be less than 0.3 the value of K for the uncompensated system is calculated as 20.

Let us draw the root locus diagram for the system (Fig. 11.22)

To fix the $\boldsymbol{\xi}$-line,

$$\tan\beta = \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\xi}{\sqrt{1-\xi^2}}$$

$$= \frac{0.6}{\sqrt{1-0.6^2}} \simeq 0.95$$

$$\beta = 43°$$

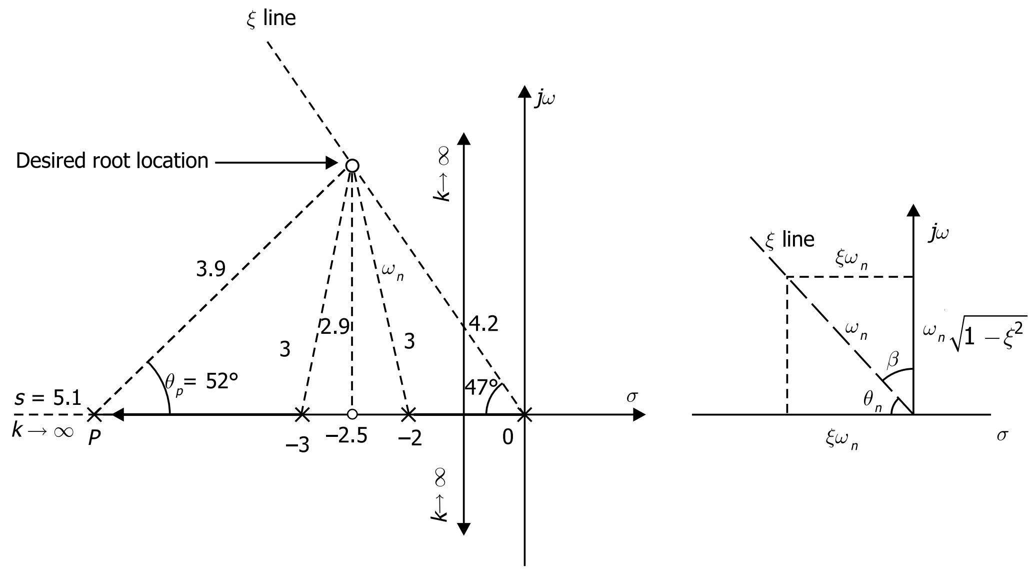


Fig. 11.22 *The design procedure for a compensator for a control system*

$$\therefore \quad \theta = 47°$$

Therefore, we draw the ξ-line at an angle of 47° with the negative real axis. Let us place the zero on the negative real axis to the left of two poles at $s = -2.5$. The desired root location on the ξ-line is shown. Thus the requirement of $\xi = 0.6$ is satisfied and the settling time is calculated as

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{2.5} = 1.6 \text{ seconds}$$

We have
$$\xi\omega_n = 2.5, \omega_n = \frac{2.5}{0.6} = 4.2$$

Settling time of $T_s = 1.6$ seconds is quite good. The numerator of the compensating network is $(s + 2.5)$. We are to fix the pole position by drawing a dotted line from the desired pole position to the negative real axis at an angle θ_p which is calculated as

$$-180° = -\theta p - 85° - 133° + 90°$$

or,
$$\theta p = 52°.$$

We draw a line (as shown dotted) with $\theta p = 52°$ and the point of intersection is at -5.1. The denominator of the compensating network is $(s + 5.1)$

Thus,

$$G_c = \frac{(s+2.5)}{(s+5.1)}$$

The transfer function of the compensated system is

$$G(s)H(s)G_c(s) = \frac{K(s+2.5)}{s(s+2)(s+3)(s+5.1)}$$

The value of K is calculated as (refer to Fig. 11.19)

$$K = \frac{\text{Distance of poles}}{\text{Distance of zero}} = \frac{3.9 \times 3 \times 4.2 \times 3}{2.9} = 54$$

$$G(s)H(s)G_c(s) = \frac{54(s+2.5)}{s(s+2)(s+3)(s+5.1)}$$

Let us now check for the steady state error.

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\dfrac{1}{s^2}}{1+\dfrac{54(s+2.5)}{s(s+2)(s+3)(s+5.1)}}$$

$$= \frac{1}{s}\frac{s(s+2)(s+3)(s+5.1)}{s(s+2)(s+3)(s+5.1)+54(s+2.5)}$$

$$= \frac{2\times 3\times 5.1}{54\times 2.5} = 0.13$$

Thus the steady state error is well below 0.3 and is acceptable.
The components of the compensating network can now be calculated using the relation.

$$G_c(s) = \frac{(s+2.5)}{(s+5.1)} = \frac{\left(s+\dfrac{1}{\tau}\right)}{\left(s+\dfrac{1}{\alpha\tau}\right)}$$

where $\tau = R_1 C$ and $\alpha = \dfrac{R_2}{R_1+R_2}$ of the phase lead network shown in Fig. 11.23 below.

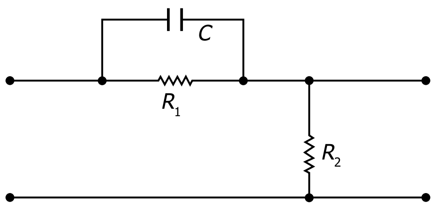


Fig. 11.23 *Phase lead network designed for a system*

We have to choose a value of C_1 say 1 micro-Farad to calculate values of R_1 and R_2.

Example 11.6 The open-loop transfer function of a unity feedback system is given as

$$G(s) = \frac{1}{s(1+s)(1+0.1s)}$$

Find the gain margin and phase margin. If a phase lag element with transfer function $\frac{(1+s)}{(1+0.4s)}$ is added in the forward path, determine by how much the gain must be changed to maintain the same gain margin.

Solution

$$G(s)H(s) = \frac{1}{s(1+s)(1+0.1s)}$$

The corner frequencies are 1 and 10 rad/sec.

1. The lowest corner frequency is 1 rad/sec. So, we assume the starting point of the Bode plot at $\omega = 0.1$ rad/sec. The starting slope of the Bode plot is -20 dB/decade due to $1/j\omega$. This slope continues till the corner frequency of $\omega = 1$ rad/sec.
2. At $\omega = 1$ rad/sec, slope changes by another -20 dB/dec. due to $\frac{1}{1+j\omega}$. Thus the slope of Bode plot after $\omega = 1$ rad/sec becomes -40 dB/dec. This slope continues till the next corner frequency of $\omega = 10$ rad/sec.
3. At corner frequency $\omega = 10$ rad/sec slope changes by another -20 dB/dec due to $\frac{1}{1+j0.1\omega}$, making the total slope of -60 dB/sec. This slope continues for higher frequencies.

The magnitudes of the transfer function components represented in the Bode plot at different frequencies are calculated as

ω	**0.1**	**1.0**	**5**	**10**
ϕ	$-96.28°$	$-140.71°$	$-195.25°$	$-219.28°$

With the above values we can now draw the Bode plot as shown in Fig. 11.25 and determine the gain. With a compensating network introduced in the forward path we will have

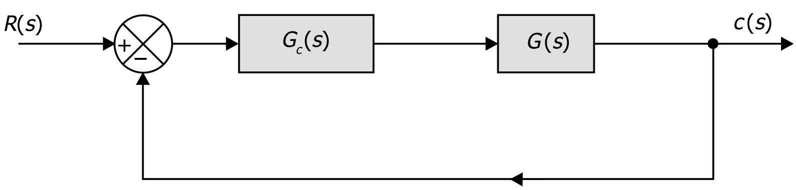


Fig. 11.24 *A compensating network introduced in the forward path of the control system*

$$T(s) = G_c(s)G(s) = \frac{(1+s)}{(1+0.4s)} \times \frac{1}{s(1+s)(1+0.1s)}$$
$$= \frac{1}{s(1+0.1s)(1+0.4s)}$$

$\omega = 0.1$ rad/sec $\left|\frac{k}{j\omega}\right| = \frac{1}{\omega} = \frac{1}{0.1} = 10$

In terms of dB,

20 log 10 = 20 dB

$\omega = 1$ rad/sec $\left|\frac{1}{j\omega(1+j\omega)}\right| = \frac{1}{\omega\sqrt{1+\omega^2}}$

In terms of dB,

$-20 \log \omega - 10 \log (1 + \omega^2)$
$= -20 \log (1) - 10 \log (2)$
$= 0 - 3.0 = -3$ dB

$\omega = 10$ rad/sec $\left|\frac{1}{j\omega(1+j\omega)(1+0.1j\omega)}\right|$

$= -20 \log \omega - 10 \log (1 + \omega^2) - 10 \log (1 + 0.01\, \omega^2)$
$= -20 \log 10 - 10 \log (101) - 10 \log (2)$
$= - 20 - 20.04 - 3.0$
$= -43.04$ dB

The phase angle, i.e. $\underline{|G(j\omega)1+(j\omega)}$ is calculated using

$0 = -90° - \tan^{-1} \omega - \tan^{-1} 0.1\omega$.

We will draw the Bode plot for the above $T(s)$. The corner frequencies are,
$\omega_1 = 2.5$ rad/sec and $\omega_2 = 10$ rad/sec
Let us take starting frequency as the same as 0.1 rad/sec. The slope of the Bode point is -20 db/dec till corner frequency of $\omega = 2.5$ rad/sec is reached. The slope of the Bode plot will change by -20 dB/dec at corner frequencies $\omega_1 = 2.5$ rad/sec and $\omega_2 = 10$ rad/sec. The total slope beyond $\omega = 10$ rad/sec will be -60 dB/dec.
Phase angles, ϕ of $G(j\omega)\, H(j\omega)$ is calculated as

$$\phi = -90° - \tan^{-1} 0.1\, \omega - \tan^{-1} 0.4\, \omega$$
$$\omega = 0.1,\ \phi = -92.80°;\ \omega = 1,\ \phi = 117.51°;$$
$$\omega = 2.5,\ \phi = -149.03°;\ \omega = 10 \text{ rad/sec},\ \phi = -210°.$$

Magnitudes are:

i) At $\omega = 0.1$ rad/sec, $\left|\frac{1}{j\omega}\right| = \frac{1}{0.1} = 10$

I terms of dB, 20 log 10 = 20 dB

ii) At $\omega = 2.5$ rad/sec,

$$\left|\frac{1}{j\omega(1 + j0.4\omega)}\right|$$

I terms of dB,

$$-20\log\omega - 20\log\sqrt{1+(.4\omega)^2}$$

Putting $\omega = 2.5$
Magnitude = −10 dB

iii) At $\omega = 10$ rad/sec

$$\left|\frac{1}{j\omega(1 + j0.4\omega)(1 + j0.1\omega)}\right|$$

$$= -20\log\omega - 20\log\sqrt{1+(.4\omega)^2} - \log\sqrt{1+(.1\omega)^2}.$$

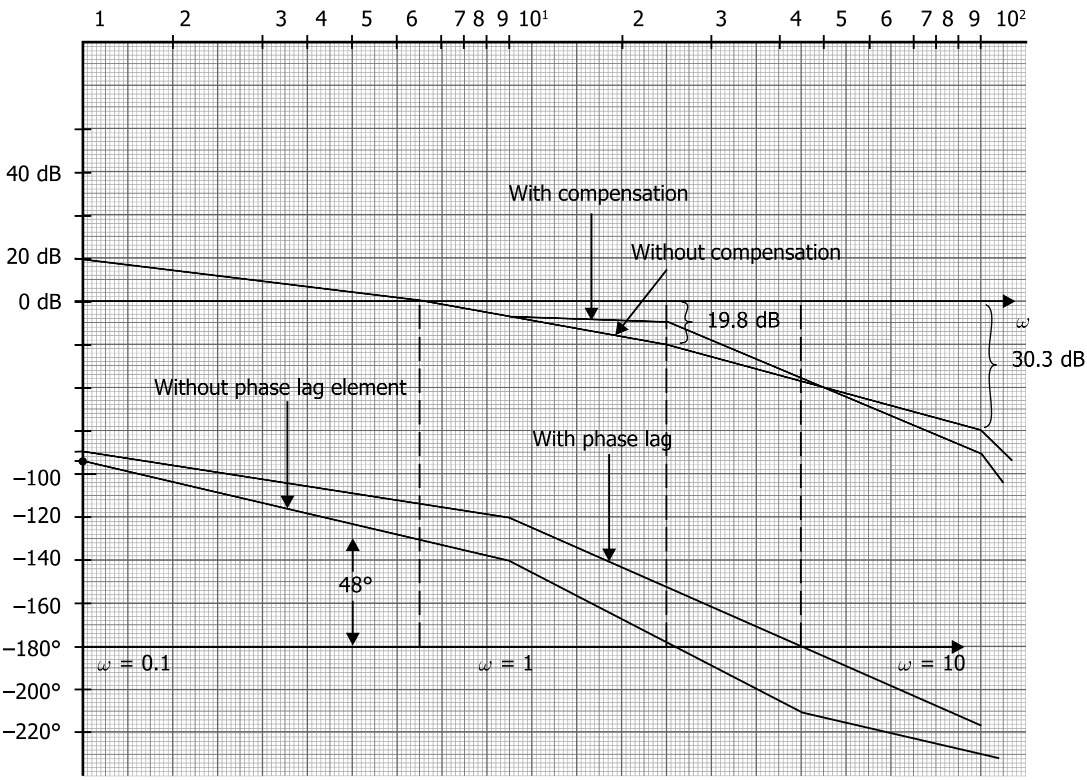


Fig. 11.25 *Bode plot for uncompensated and compensated systems as in Example 11.6*

Substituting $\omega = 10$ rad/sec
Magnitude $= -35.05$ dB.
We now draw the Bode plot for the compensated network with the above values. The findings are:
Without compensation, gain margin $= 0 - (-19.8) = 19.8$ dB and Phase margin $= 48°$.
With the phase lag network introduced, the gain margin is 30.3 dB. Since the gain margin is required be kept at 19.5 dB only, we change the value of K

$$20 \log K = (30.3 - 19.8) \text{ dB} = 19.5 \text{ dB}$$

or, $$\log K = 0.525$$

or, $$K = 3.349.$$

Example 11.7 The open loop transfer function of a control system is

$$G(s) = \frac{K}{s(s+1)(s+1)}$$

The system is to be compensated so that the following specifications are met Damping ratio, $\xi = 0.5$.
Undamped natural frequency, $\omega_n = 2$
Calculate the steady state error which should be within some reasonable limit.
Also calculate the components of the compensating network.

Solution

We have $\omega_n = 2$, $\xi = 0.5$, $\xi\omega_n = 0.5 \times 2 = 1.0$

$$\omega_n\sqrt{1-\xi^2} = 2\sqrt{1-(0.5)^2} = 1.73.$$

The positions of the poles and the desired dominant closed loop poles are shown as in Fig. 11.26.

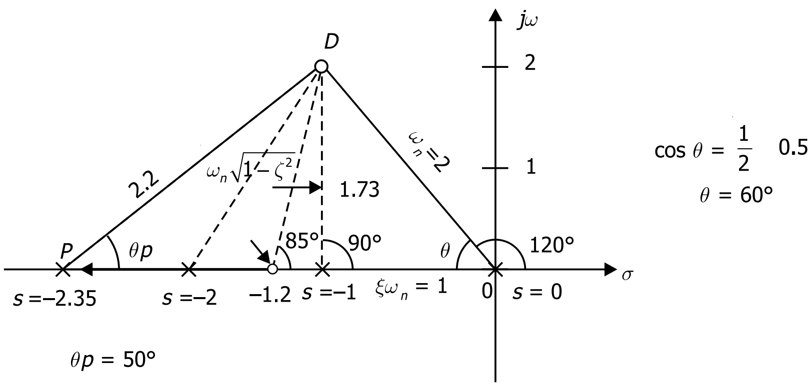


Fig. 11.26 Location of poles and zeros of the compensated control system

We have drawn a line at $\theta = 60°$. The vertical line from point D to the real axis must be equal to 1.73. The location of the dominant pole is thus made. We have now to locate the zero on the negative part of the real axis just below the dominant root. However, since one pole is lying just below the dominant root, we locate the zero somewhat to the left of the first two poles. One pole is at $s = 0$ and the second one is at $s = -1$. The position of zero may be located at a point say -1.2. The numerator of the compensating network is thus decided as $(s + 1.2)$. Now to locate the position of the new pole we have to draw a line from the point D to the negative real axis at an angle θp with that axis. Angle θp is calculated by taking into consideration the angle contribution of all the roots as

$$-180° = -\theta p - 120° - 90° + 85°$$

or,
$$\theta p = 55°$$

Distance of p from the origin is calculated as

$$OP = 1 + \frac{1.73}{\tan 55°} = 2.35$$

The value of K is calculated as

$$K = \frac{2.2 \times 2 \times 1.73 \times 2}{1.85} = 8.4$$

The denominator of the compensating network is $(s + 2.35)$. Thus

$$G_c = \frac{(s+1.2)}{(s+2.35)}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = \frac{(s+1.2)8.2}{(s+2.35)s(s+1)(s+2)}$$

Velocity error constant

$$K_v - \lim_{s\to 0} sG(s)$$

The steady state error $e_{ss} = \lim_{s\to 0} \dfrac{sR(s)}{1+G(s)H(s)}$

Assuming unit ramp input, $R(s) = \dfrac{1}{s^2}$

Substituting,

$$e_{ss} = \lim_{s\to 0} \frac{s\dfrac{1}{s^2}}{1 + \dfrac{s(s+1.2)8.2}{(s+2.35)s(s+1)(s+2)}}$$

$$= 0.47$$

Steady state error of 0.47 is somewhat on the higher side. The position of zero could be changed somewhat. This would change the value of K and e_{ss}.
The components of the compensating network can be calculated from

$$G_c(s) = \frac{(s+1.2)}{(s+2.35)} = \frac{\left(s+\frac{1}{\tau}\right)}{\left(s+\frac{1}{\alpha\tau}\right)} \text{ where } \tau = R_1C \text{ and } \alpha = \frac{R_2}{R_1+R_2}.$$

Example 11.8 The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{4}{s(s+2)}.$$

Design a compensator to meet the following specifications and compare the responses through MATLAB.

a) State velocity error constant $K_v = 20$
b) Phase margin of at least 50°
c) Gain margin of at least 10 db

Solution

Here we shall use a lead compensator given by

$$G_c(s) = K_c\alpha\frac{\tau s+1}{\alpha\tau s+1} = K_c\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$$

where, $0 < \alpha < 1$.
Let us define

$$G_1(s) = KG(s) = \frac{4K}{s(s+2)}$$

where, $K = K_c\alpha$
To obtain $K_v = 20$, we must have

$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(c) \\ &= \lim_{s\to 0} sK_c\alpha\left(\frac{\tau s+1}{\alpha\tau s+1}\right)\frac{4}{s(s+2)} \\ &= 2K = 20 \end{aligned}$$

or, $$K = K_c\alpha = 10$$

so, $$G_1(j\omega) = \frac{40}{j\omega(j\omega+2)}.$$

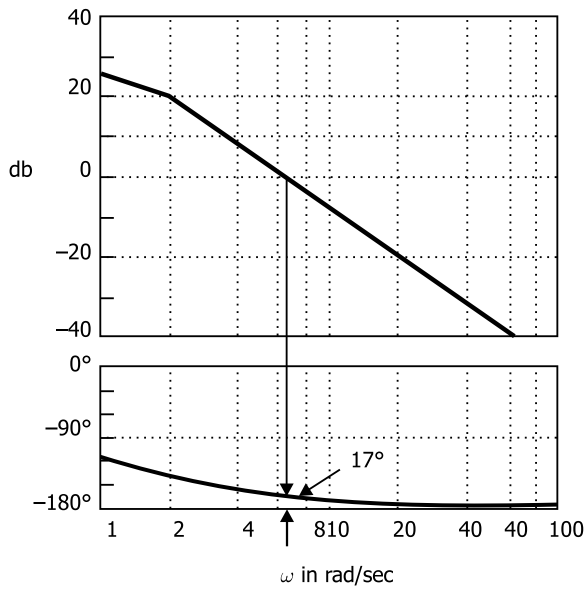


Fig. 11.27 Bode plot of $G_1(j\omega) = \dfrac{40}{j\omega(j\omega+2)}$

From Fig. 11.27 we observe that the phase and gain margins of the system are 17° and $+\infty$ dB respectively. The phase margin of 17° indicates that the system is oscillatory.

As the addition of a lead compensator shifts the gain crossover frequency towards the right, we must offset the increased phase lag of $G_1(j\omega)$ by an amount of 5°. So the maximum phase lead (ϕ_m) required to achieve a phase margin of 50° is given by

$$\phi_m = 50° - 17° + 5° = 38°$$

$$= \frac{1-\alpha}{1+\alpha} \sin\phi_m = \sin 38°$$

gives $\alpha = 0.24.$

As the maximum phase lead angle ϕ_m occurs at the geometric mean of two corner frequencies of the lead network, we have

$$\omega_m = \sqrt{\left(\frac{1}{\tau}\right)\left(\frac{1}{\alpha\tau}\right)} = \frac{1}{\tau\sqrt{\alpha}}$$

since, $$\left|\frac{1+j\omega\tau}{1+j\omega\alpha\tau}\right|_{\omega=\omega_m} = \frac{1}{\sqrt{\alpha}} = \frac{1}{0.49} = 6°2 \text{ db}$$

and $|G, (j\omega)| = -6.2$ db corresponds to the new gain crossover frequency of $\omega_m = 9$ rad/sec., we get the corner frequencies as follows:

$$\frac{1}{\tau} = \omega_m \sqrt{\alpha} = 4.41$$

and
$$\frac{1}{\alpha\tau} = \frac{\omega_m}{\sqrt{\alpha}} = 18.4.$$

Also,
$$K_c = \frac{K}{\alpha} = \frac{10}{0.24} = 41.7.$$

So the transfer function of the lead compensator becomes

$$G_c(s) = 10\frac{0.227s+1}{0.054s+1} = 41.7\frac{s+4.41}{s+18.4}.$$

Since
$$\frac{G_c(s)}{K}G_1(s) = \frac{G_c(s)}{10}10G(s)$$

$$= G_c(s)G(s) = 41.7\left(\frac{s+4.41}{s+18.4}\right)\frac{4}{s(s+2)}$$

is the open-loop transfer function of the Compensated system, the Bode plots of $G_c/10$, G_1 and G_cG are shown in Fig. 11.28.

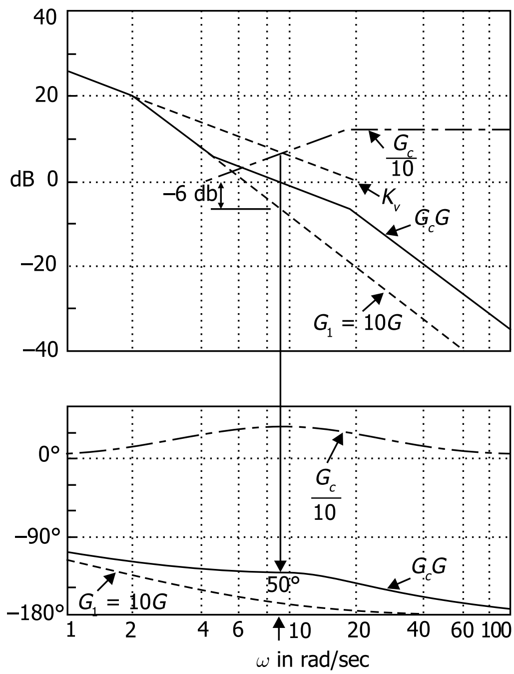


Fig. 11.28 *Bode plot of compensated system*

The lead compensator causes the gain crossover frequency to increase from 6.3 to 9 rad/sec. So the bandwidth increases and thus the speed of response increases. The phase and gain margin of the compensated system are now 50° and + ∞ db respectively. Thus the compensated system shown in Fig. 11.29 meets both the steady-state and relative stability requirements.

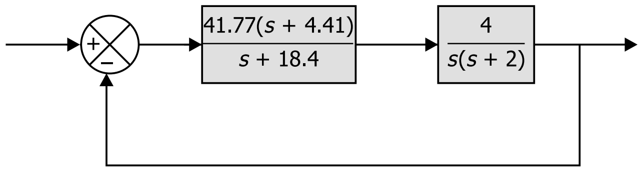


Fig. 11.29 *Compensated system*

Now we shall compare through MATLAB the unit step and unit ramp responses of both the compensated and uncompensated system the closed-loop transfer functions of which are as follows:

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4} \text{ for uncompensated system.}$$

$$= \frac{166.8s + 735.588}{s^2 + 20.4s^2 + 203.6s + 735.588} \text{ for compensated system}$$

and, MATLAB Program 11.1 gives the unit-step and unit-ramp response curves of both the compensated and uncompensated system as shown in Figs. 11.30 and 11.31.

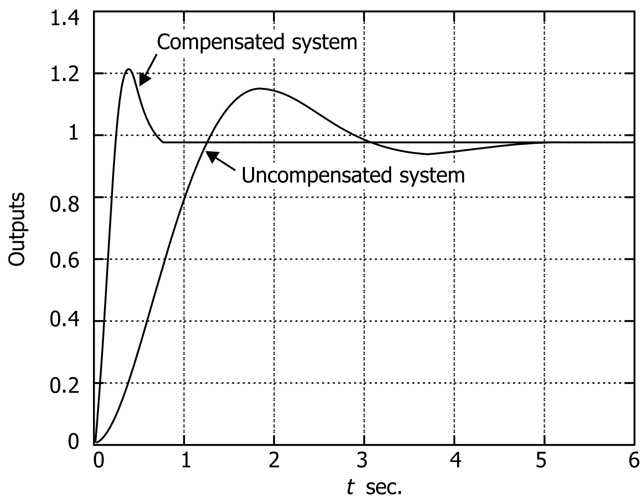


Fig. 11.30 *Unit-step response curves of the compensated and uncompensated systems*

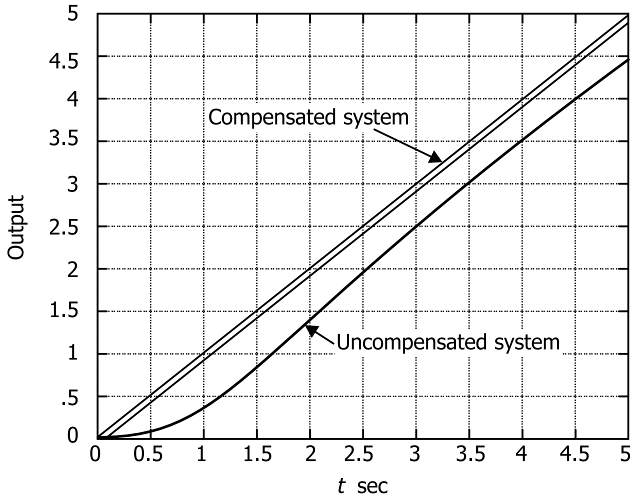


Fig. 11.31 *Unit-ramp response curves of the compensated and uncompensated systems*

MATLAB PROGRAM 11.1

```
% ***** Unit-step responses*****
num = [0   0   4];
den = [1   2   4];
numc = [0   0   166.8   735.588];
denc = [1   20.4   203.6   735.588];
t = 0:0.02:6;
t = 0:0.02:6;
[cl, xl,t] = step (num,den,t);
[c2, x2,t] = step (numc, denc,t);
plot (t,c1,'.',t,c2,'-')
grid
title ('Unit-Step Responses of Compensated and Uncompensated Systems')
xlabel ('t Sec')
ylabel [('Outputs')
text(0.35,1.3, 'Compensated system')
text(1.55, 0.88,'Uncompensated system')
%*****Unit-ramp responses*****
numl =[0   0   0   4];
denl = [1   2   4   0];
numlc =[0   0   0   166.8   735.588];
denlc = [1   20.4   203.6   735.588   0];
t = 0:0.02:5;
[yl;zl,t] = step(numl,denl,t);
[y2,z2,t] = step(numlc,denlc,t);
plot(t,yl,'.',t,y2,'-',t,t, '—')
grid
```

```
title('Unit-Ramp Responses of Compensated and Uncompensated Systems')
xlabel('t, Sec')
ylabel('Outputs')
text(0.77,3.7,'Compensated system')
text(2.25,1.1,'Uncompensated system')
```

Example 11.9 The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}.$$

Design a compensator to meet the following specifications and compare the responses through MATLAB.

a) Static velocity error constant $K_v = 5$/sec.
b) Phase margin of at least 40°
c) Gain margin of at least 10 db.

Solution

Here we shall use a lag compensator given by

$$G_c(s) = K_c\beta \frac{\tau s+1}{\beta\rho s+4} = K_c \frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$$

where $\beta > 1$.
Let us define

$$G_1(s) = KG(s) = \frac{K}{s(s+1)(0.5s+1)}$$

where, $K = K_c\beta$.
To obtain $K_v = 5$/sec., we must have

$$= \lim_{s\to 0} sK_c\beta\left(\frac{\tau s+1}{\beta\tau s+1}\right)\frac{1}{s(s+1)(0.5s+1)} = K_c\beta = K = 5$$

or, $$K = K_c\beta = 5.$$

So, $$G_1(jw) = \frac{5}{j\omega(j\omega+1)(0.5j\omega+1)}.$$

From the Bode plot (Fig. 11.32) of $G_1(j\omega)$ it is observed that phase margin is $-20°$ which implies that the system is unstable.

$$K_v = \lim_{s\to 0} s\, G_c(s)\, G(s)$$

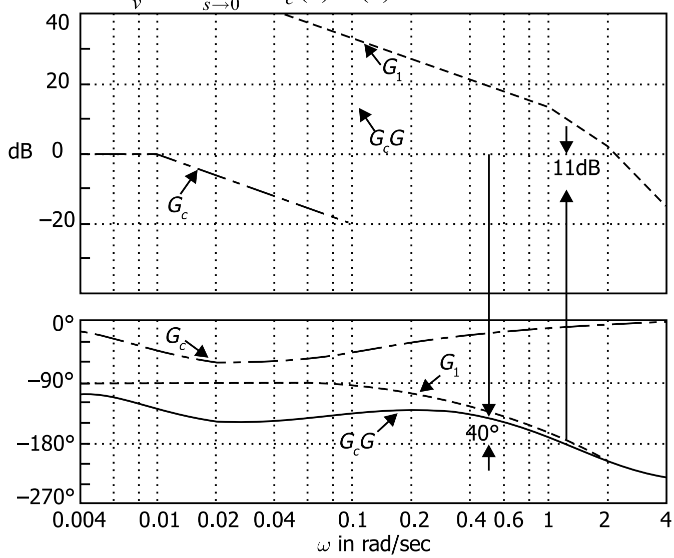


Fig. 11.32 *Bode plots of $G(j\omega)$, $G_c(j\omega)$ and $G_c(j\omega)\, G(j\omega)$*

Since the frequency of 0.7 rad/sec corresponds to the phase margin of 40°, the corner frequency $(1/\tau) = 0.1$ rad/sec is chosen to prevent an overly large time constant of the lag compensator. As the corner frequency is not far below the frequency of 0.7 rad/sec, we may add 12° to the specified phase margin of 40° to compensate for the phase lag due to lag compensator. In order to meet the specified phase margin of 40°, the required phase angle of the uncompensated system $G_1(j\omega)$ should be $(-180° + 40° + 12°)$ or $-128°$ which corresponds to the new gain crossover frequency of 0.5 rad sec. To bring the magnitude curve of $G_1(j\omega)$ down to 0 db at this frequency the attenuation needed by the lag compensator is -20 dB.

$$\therefore \qquad 20 \log 1/\beta = -20$$

$$\text{or,} \qquad \beta = 10.$$

So the other corner frequency of the compensator is $(1/\beta\tau) = 0.01$ rad/sec.

$$\therefore \quad K_C = \frac{K}{\beta} = \frac{5}{10} = 0.5$$

So the open-loop transfer function of the compensated system is

$$G_c(s)G(s) = \frac{5(10s+1)}{s(100s+1)(s+1)(0.5s+1)}.$$

From the Bode plot of Fig. 11.32 we observe that the compensated system now satisfies the requirements of both the steady-state and relative stability.

Now we shall compare through MATLAB the unit step and unit ramp responses of both the compensated and uncompensated system whose closed-loop transfer functions are as follows.

$$\frac{C(s)}{R(s)} = \frac{50s+5}{50s^4+150.5s^3+101.5s^2+51s+5} \text{ for compensated system}$$

and $$= \frac{1}{0.5s^3+1.5s^2+s+1} \text{ for uncompensated system.}$$

MATLAB Program 11.2 gives the unit-step and unit-ramp response curves of both the compensated and uncompensated system as shown in Figs. 11.33 and 11.34.

MATLAB PROGRAM 11.2

```
% ***** Unit-step response*****
num = [0   0   0   1];
den = [0.5   1.5   1   1];
numc = [0   0   0   50   5];
denc = [50   150.5   101.5   51 5];
t = 0:0.1:40;
[cl, xl,t] = step (num,den,t);
[c2, x2,t] = step (numc, denc,t);
plot (t,c1,'.',t,c2,'-')
grid
title ('Unit-Step Responses of Compensated and Uncompensated Systems')
xlabel ('t Sec')
ylabel[('Outputs')
text(12.2, 1.27,'Compensated system')
text(12.2, 0.7,'Uncompensated system')
%*****Unit-ramp response*****
numl = [0   0   0   0   1];
denl = [0.5   1.5   1   1   0];
numlc =[0   0   0   0   50   5];
denlc = [50   150.5   101.551   5   0];
t = 0:0.1:20;
[yl,zl,t] = step(numl,denl,t);
[y2,z2,t] = step(numlc,denlc,t);
plot(t,yl,'.',t,y2,'-',t,t,'-')
grid
title('Unit-Ramp Responses of Compensated and Uncompensated Systems')
xlabel('t Sec')
ylabel('Outputs')
text(8.4.3,'Compensated system')
text(8.4, 5,'Uncompensated system')
```

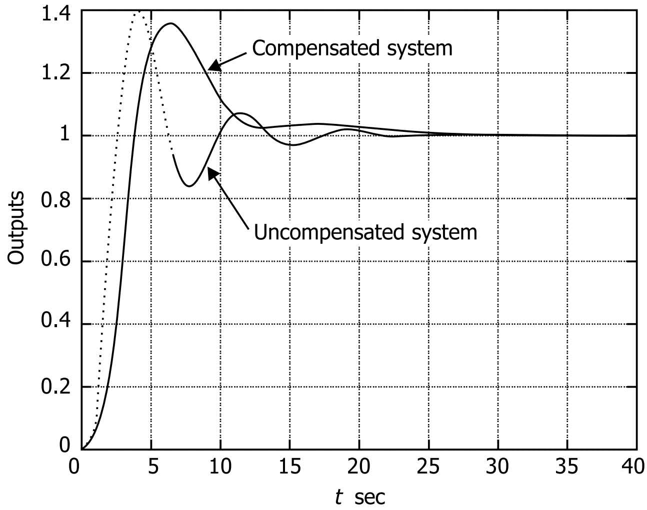


Fig. 11.33 Unit-step response curves for the compensated and uncompensated systems

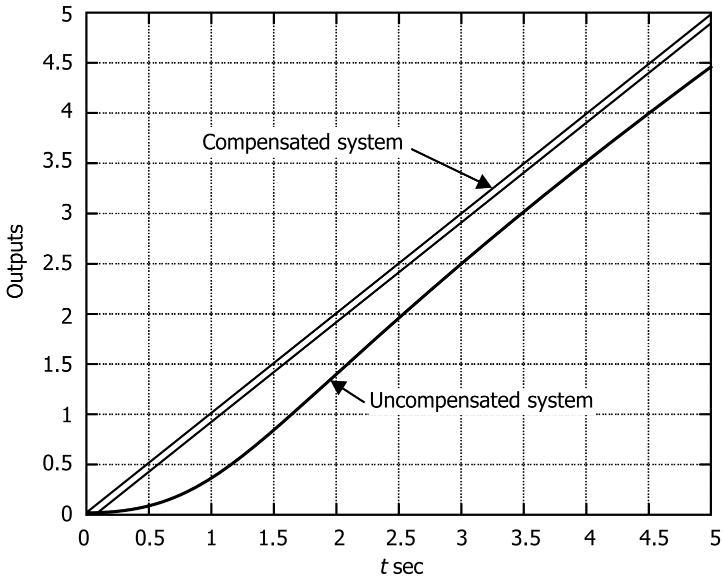


Fig. 11.34 Unit-ramp response curves for the compensated and uncompensated systems

Example 11.10 The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s+1)(s+2)}.$$

Design a compensator to meet the following specification and draw the response curves through MATLAB.

a) Static velocity error constant $K_v = 10$/sec
b) Phase margin of at least 50°
c) Gain margin of at least 10 db Solution

Solution

Here we shall use a lag lead compensator given by

$$G_c(s) = K_c \frac{(s+\frac{1}{\tau_1})(s+\frac{1}{\tau_2})}{(s+\frac{\beta}{\tau_1})(s+\frac{1}{\beta\tau_2})}$$

where, $\beta > 1$.
As the gain K of the system is adjustable we may assume $K_c = 1$.
Thus, to satisfy $K_v = 10$/sec., we must have

$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(s) \\ &= \lim_{s\to 0} G_c \lim_{s\to 0} sG(s) \\ &= \lim_{s\to 0} s\frac{K}{s(s+1)(s+2)} \\ &= K/2 = 10 \\ K &= 20 \end{aligned}$$

From the Bode diagram (Fig. 11.35) of $G(j\omega)$ with $K = 20$, the phase margin is found to be $-32°$ which indicates an unstable system. Also we see that $\angle G(j\omega) = -180°$ at $\omega =$ 1.5 rad/sec, which may be considered as the new gain crossover frequency to provide a phase lead angle of 50°.
So the corner frequency of the phase lag portion of lag-lead compensator is chosen as $(1/\tau_2)$ = 0.15 rad/sec. which is 1 decade below the new gain crossover frequency of 1.5 rad/sec.
As $\alpha = 1/\beta$ in the phase lead portion of lag-lead compensator, the maximum phase lead angle ϕ_m is given by

$$\sin\phi_m = \frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}} = \frac{\beta-1}{\beta+1}$$

Since we need a 50° phase margin and $\beta = 10$ gives $\phi_m = 54.9°$ we may take $\beta = 10$. So the other corner frequency of the phase lag portion becomes $(1/\beta\tau_2) = 0.015$ rad/sec.

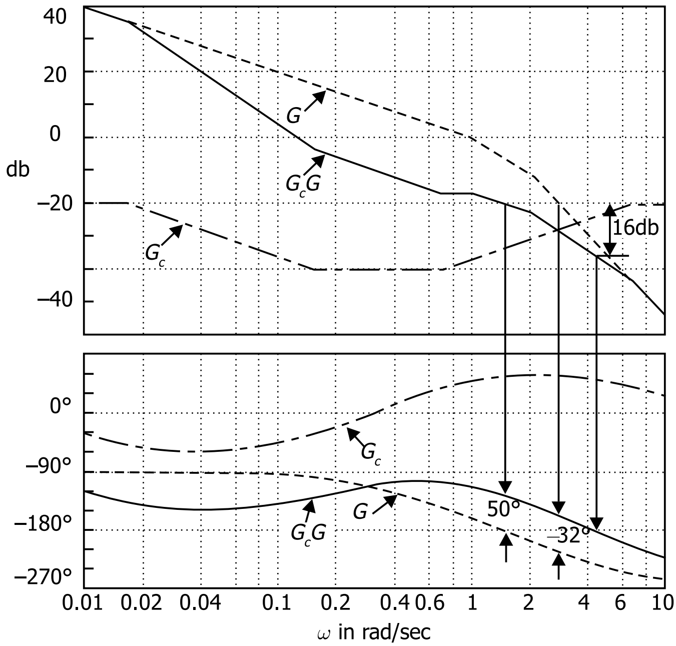


Fig. 11.35 *Bode diagrams of G, G_c and G_cG*

From Bode diagram (Fig. 11.35) of $G(j\omega)$ the gain margin is found to be 13 db at $\omega = 1.5$ rad/sec. So the lag lead compensator must contribute -13 db at $\omega = 1.5$ rad/sec. to make the new gain crossover frequency at $\omega = 1.5$ rad/sec.

The corner frequencies of the phase lead portion are thus obtained by intersecting the straight line of slope 20 db/decade through the point (-13 db, 1.5 rad/sec.) with -20 db line at $(1/\tau_1) = 0.7$ rad/ sec. and with 0 db line at $(\beta/\tau_1,) = 7$ rad/sec.

With $K_c = 1$, the open-loop transfer function of the compensated system now becomes

$$G_c(s)G(s) = \left(\frac{s+0.7}{s+7}\right)\left(\frac{s+0.15}{s+0.015}\right)\frac{20}{s(s+1)(s+2)}.$$

The Bode diagrams of $G_c(j\omega)$ and $G_c(j\omega)\,G(j\omega)$ are also drawn in Fig. 11.35. The phase margin (50°), gain margin (16 db) and static velocity error constant (10/sec.) of the compensated system now satisfy the specified value.

The closed-loop transfer function of the compensated system is now given by

$$\frac{C(s)}{R(s)} = \frac{95.381s^2 + 81s + 10}{4.7691s^5 + 47.7287s^4 + 110.3026s^3 + 163.724s^2 + 82s + 10}.$$

MATLAB Program 11.3 will produce the unit-step and unit-ramp response curves of the compensated system as shown in Figs. 11.36 and 11.37.

MATLAB PROGRAM 11.3

```
% ***** Unit-step response*****
num = [0   0   0   95.381   81   10];
den = [4.7691   47.7287   110.3026   163.724   82   10];
t = 0:0.05:20;
c = step (num,den,t);
plot (t,c,'-')
grid
title ('Unit-Step Responses of Compensated System')
xlabel ('t Sec')
ylabel[('Outputs')
%*****Unit-ramp response*****
numl = [0   0   0   0 95.381   81   10];
denl = [4.7691   47.7287   110.3026   163.724   82   10   0];
t = 0:0.05:20;
c = step(numl,denl,t);
plot(t,c,'-'t,t,'.')
grid
title ('Unit-Ramp Response of Compensated System')
xlabel('t Sec')
ylabel('Output')
```

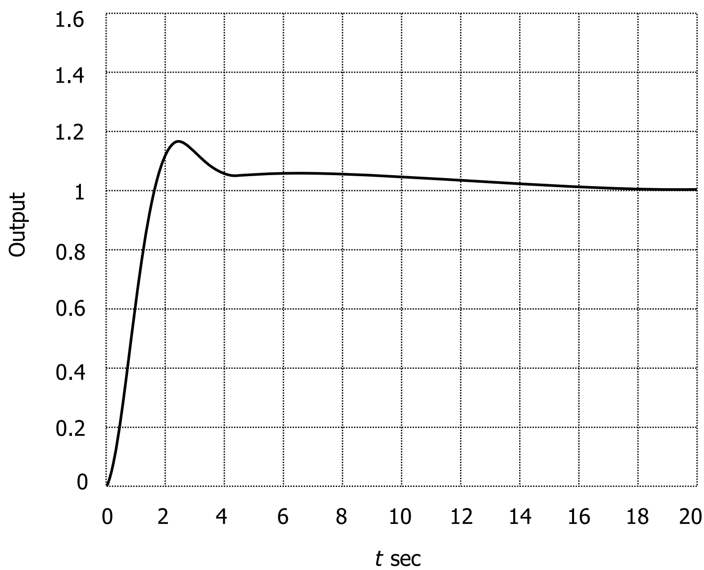


Fig. 11.36 *Unit step response of compensated system*

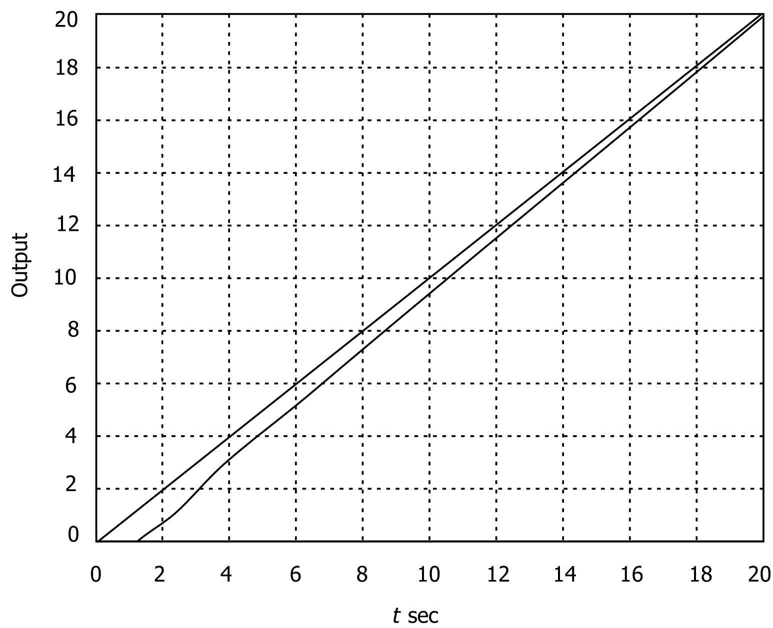


Fig. 11.37 *Unit ramp response of compensated system*

11.8 PID CONTROLLERS

The position of a controller in a control system has been shown in Fig. 11.38. The controller modifies the input to the system on the basis of feedback.

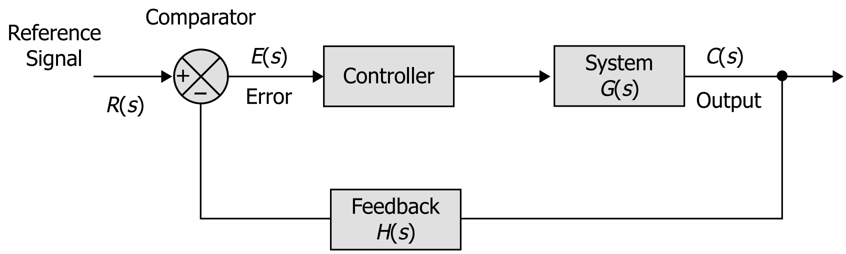


Fig. 11.38 *Controller in a control system*

Controllers are of different types:

i) ON/OFF controllers
ii) Proportional controllers (P controllers)
iii) Proportional derivative controllers (PD controllers)
iv) Proportional integral controllers (PI controllers)
v) Proportional integral derivative controllers (PID controllers)

These are conventional controllers used mainly in linear control applications. These are not preferred in nonlinear control system applications.

The other state-of-the-art controllers are:

i) fuzzy logic controllers

ii) artificial neural network (ANN)-based controllers

iii) neuro-fuzzy system, expert systems and genetic algorithms

In this section, we discuss PID controllers in detail. We start with proportional controllers.

11.8.1 Proportional Controllers

A proportional controller in a closed-loop control system has been shown in Fig. 11.39. A electronic P controller made with the help of an op-amp has been also shown in Fig. 11.39.

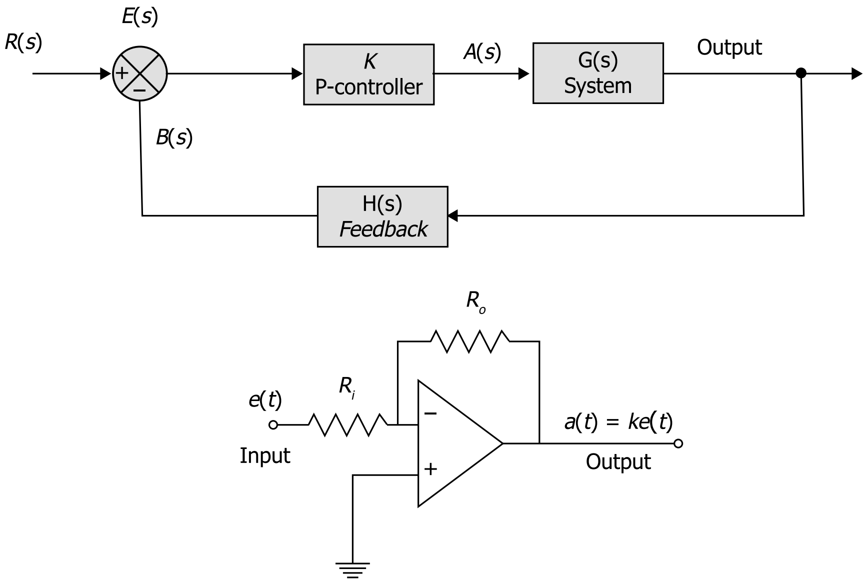


Fig. 11.39 *A p-controller introduced in a closed-loop control system*

The output of the P controller, $a(t) = ke(t)$, i.e., the output is proportional to the error signal $e(t)$.

The output by input of the controller is expressed as

$$\frac{A(s)}{E(s)} = K = \frac{K}{s^0}$$

Thus a P controller is a type zero system.

11.8.2 Proportional Derivative Controllers

The actuating signal $a(t)$ in terms of the error signal $e(t)$ for a PD controller is expressed as

$$a(t) = Ke(t) + T_d \frac{de(t)}{dt}$$
$$= \text{proportional force} + \text{derivative force}$$

Taking Laplace transform,

$$A(s) = KE(s) + T_d SE(s)$$

Transfer function, $$\frac{A(s)}{E(s)} = (K + ST_d)$$

This transfer function of the PD controller has no poles. Such a controller is difficult to be made in practice.

A block diagram of a closed-loop control system with a PD controller has been shown in Fig. 11.40.

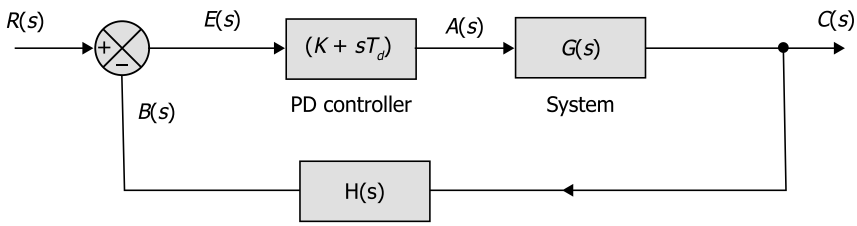


Fig. 11.40 *Block diagram of a PD Controller*

The overall transfer function,

$$\frac{C(s)}{R(s)} = \frac{(K + sT_d)G(s)}{1 + (K + sT_d)G(s)H(s)}$$

$$\text{Let } G(s) = \frac{\omega_n^2}{s^2 + 2s_1\omega_n s}, H(s) = 1$$

(assuming a second-order unity feedback system)

$$\frac{C(s)}{R(s)} = \frac{(1 + sT_d)\left[\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}\right]}{1 + (s + ST_d)\left[\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}\right] \times 1}$$

$$= \frac{(1 + ST_d)\omega_n^2}{s^2 + (2\zeta\omega_n + T_d\omega_n^2)s + \omega_n^2}$$

The characteristic equation is

$$s^2 + (2\zeta\omega_n + T_d\omega_n^2)s + \omega_n^2 = 0$$

$$s^2 + 2\zeta\omega_n\left(\zeta + \frac{T_d\omega_n}{2}\right)s + \omega_n^2 = 0$$

By comparing this characteristic equation with the standard equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Effective damping ratio without PD controller is ζ.

Effective damping ratio with PD controller $= \zeta + \dfrac{T_d\omega_n}{2}$

This shows that with PD controller, the damping ratio increases. As a consequence, system becomes less oscillatory.

11.8.3 Proportional Integral Controllers

The actuating signal, $a(t)$, for a PI controller is expressed as

$$a(t) = Ke(t) + Ki\int e(t)dt$$

$$= \text{proportional force} + \text{integral force}$$

Taking Laplace transform,

$$A(s) = KE(s) + \frac{K_i}{s}E(s)$$

$$\text{Transfer function, } \frac{A(s)}{E(s)} = \left[K + \frac{K_i}{s}\right]$$

$$= \frac{(K_i + sK)}{s}$$

A PI controller is easily realizable as it contains a pole and a zero. Since the pole is at the origin, it increases the type of the system by one.

11.8.4 Basic Elements of a PID Controller

It is possible to improve the performance of a control system by introducing a PID controller. The controller output provided by a PID controller is described by the following mathematical relation:

$$e_a(t) = K_p e(t) + K_i\int e(t)dt + K_d\frac{d}{dt}e(t)$$

In Laplace transform form,

$$E_a(s) = E(s)\left[K_p + \frac{K_i}{s} + sK_d\right]$$

The PID controller connected to the system is shown in block diagram form in Fig. 11.41.

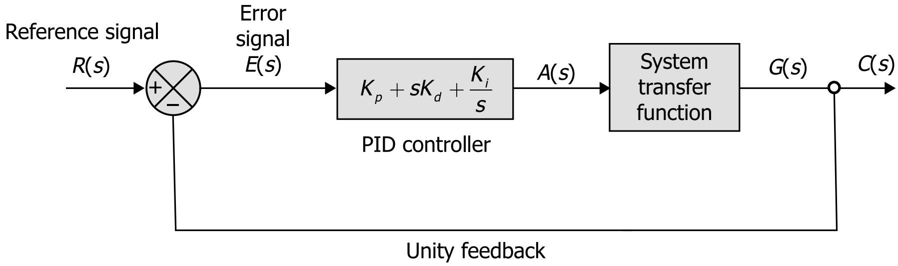


Fig. 11.41 *PID Controller is a unity feedback control system*

PID controllers are mostly used in a variety of industrial control systems like in precision position control system, temperature control in a thermal system, motor driving a gear train, in pressure control, level control, etc. The PID controller consists of three components, namely:

(i) $K_pE(s)$ which is proportional to the error signal, (ii) $sK_dE(s)$ which is proportional to the derivative of the error and (iii) $\frac{K_i}{s}E(s)$ which is proportional to the integral of the error.

A PID controller has all the advantages of proportional, integral and derivative control. Their individual disadvantages also get neutralized as they are used together.

Proportional integral increased the loop gain of the system and hence reduces the sensitivity to the variation of system parameters.

The integral component increased the order of the system and reduces the steady-state error of the system since a pole gets added at the origin in the s plane.

The main advantage of PID controller is that the gain constants K_p, K_d, K_i can be adjusted to improve the system performance when there is large variation in the system parameters, This is called PID tuning.

The transfer function of PID controller is expressed as

$$\frac{A(s)}{E(s)} = \left[K_p + sK_d + \frac{K_i}{s}\right]$$

$$= \frac{\left[s^2K_d + sK_p + K_i\right]}{s}$$

The transfer function has two zeros and one pole. Since it is difficult to realize such a system, a filter element having a transfer function of $1/(Xs+1)$ is often added as a dummy element to the PID controller.

Example 11.11 The open-loop transfer function of a system is given as

$$G(s) = \frac{5}{s(s+2)}$$

It is desired to locate the poles of this transfer function at –8 and $-3 \pm j4$ by using a PID controller. Calculate the values of K_p, K_d and K_i of the controller so as to achieve this modification.

Solution

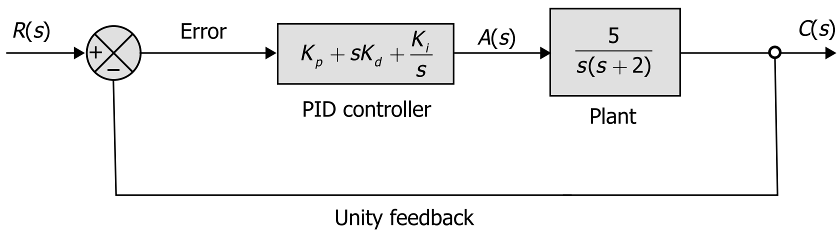


Fig. 11.42 Use of PID Controller

For the unity feedback system with a PID controller, the system transfer function is

$$\frac{C(s)}{R(s)} = \frac{\left(K_p + sK_d + \frac{K_i}{s}\right)\frac{5}{s(s+2)}}{1+\left(K_p + sK_d + \frac{K_i}{s}\right)\frac{5}{s(s+2)}}$$

$$\text{or,} \quad \frac{C(s)}{R(s)} = \frac{5(K_d s^2 + K_p s + K_i)}{s^3 + 2s^2 + 5K_d s^2 + 5K_p s + 5K_i}$$

In order to achieve poles at $s = -8$ and $s = -3 \pm j4$, the denominator of $C(s)/R(s)$ should be equated as

$$s^3 + s^2(2+5K_d) + s5K_p + 5K_i = (s+8)(s^2+6s+25)$$

$$\text{or } s^3 + (2+5K_d)s^2 + 5K_p s + 5K_i = s^3 + 14s^2 + 73s + 200$$

Equating the coefficient on both sides,

$$2 + 5K_d = 14$$

$$\text{or,} \quad K_d = \frac{14-2}{5} = 2.4$$

$$\text{And,} \quad 5K_p = 73$$

$$\text{or,} \qquad K_p = \frac{73}{5} = 14.6$$

$$\text{And,} \qquad 5K_i = 200$$

$$\text{or,} \qquad K_i = 40$$

Thus the modified transfer function is

$$\frac{C(s)}{R(s)} = \frac{5(K_d s^2 + K_p s + \frac{K_i}{s})}{(s+8)(s^2+6s+25)}$$

$$\text{or,} \frac{C(s)}{R(s)} = \frac{5(2.4s^2 + 14.6s + 40)}{(s+8)(s^2+6s+25)}$$

For a unit step function, $R(s) = 1/s$ and hence the output $C(s)$ is

$$C(s) = \frac{5(2.4s^2 + 14.6s + 40)}{s(s+8)(s^2+6s+25)}$$

11.8.5 An Electronic PID Controller

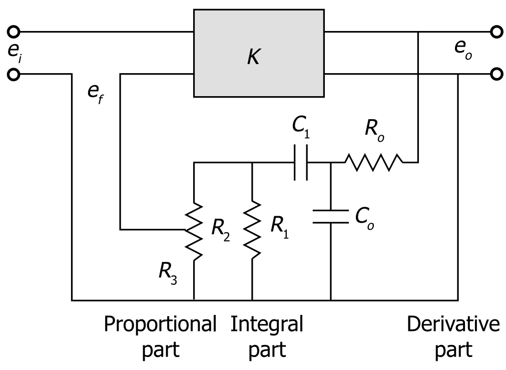


Fig. 11.43 *An electronic PID Controller*

An electronic PID controller has been shown in Fig. 10.43. The transfer function of this PID controller can be derived as

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2}{R_3}\left[sT_0 + \left(1 + \frac{R_0}{R_1} + \frac{T_0}{T_1}\right) + \frac{1}{sT_1}\right]$$

where $T_0 = R_0 C_0, T_1 = R_1 C_1$

The proportional controller is a simple system which is an amplifier with adjustable gain control and is placed in the forward path in series with G(s). The output of the controller is proportional to the error. Increase in gain reduces steady-state error but increases oscillations. With PD controller damping ratio is increased, steady-state error is not affected, and transient response is improved. PI control meets the high accuracy requirement and improves steady-state performance. Using a PID controller, we can achieve the following advantages: (i) oscillations are reduced to zero, (ii) transient response and (iii) steady-state response are improved.

REVIEW QUESTIONS

11.1 For the system shown below design a lead compensator such that $\omega_n = 4$ rad/sec and $\zeta = 0.5$ for the compensated system

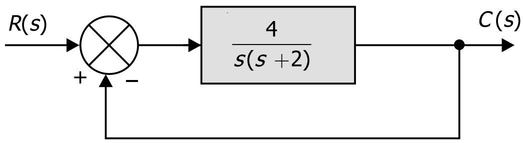


Fig. 11.44

11.2 Consider a plant whose open-loop transfer function is

$$G(s)H(s) = \frac{1}{s(s^2 + 4s + 13)}.$$

The complex poles near the origin give slightly damped oscillations which are undesirable. Insert a gain K_c and a compensator $G_c(s)$ in series to speed up the closed-loop response. Using the root locus plot for the compensated system, determine which of the following compensators will give the best performance.

i) $G_c(s) = \dfrac{s + \dfrac{1}{aT}}{s + \dfrac{1}{T}}, a > 1$

ii) $G_c(s) = a\dfrac{s + \dfrac{1}{aT}}{s + \dfrac{1}{T}}, a < 1$

iii) $G_c(s) = \dfrac{1 - T_1 s}{1 + T_2 s}$

11.3 A unity feedback control system has the open-loop transfer function of

$$G(s)=\frac{K}{s(s+2)}$$

with $K = 4$, damping ratio $\zeta = 0.5$, natural frequency $\omega_n = 2$ rad/sec and $K_v = 2$/sec.

i) Design a compensator to obtain $\omega_n = 4$ rad/sec. while keeping $\zeta = 0.5$. Find the compensator's resistances if $C = 1\ \mu F$.
ii) Design a compensator that increases K_v to $K_v = 20$/sec. while the original system gives a satisfactory transient response with $K = 4$.

11.4 A system with velocity feedback is shown in Fig. 11.45.

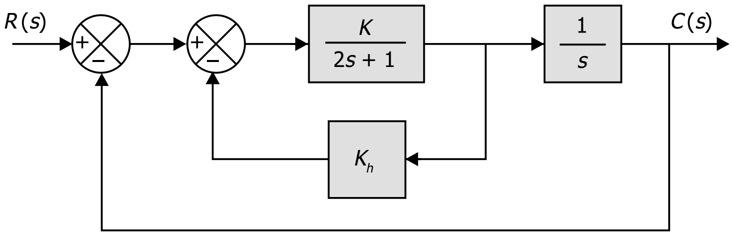


Fig. 11.45

Determine the value of K and K_h *so* that the following specifications are satisfied.

i) Damping ratio of closed-loop poles is 0.5
ii) Settling time < 2 sec
iii) Static velocity error constant ≥ 50/sec
iv) Static velocity error constant $0 < K_h < 1$

11.5 A unity feedback system has an unstable plant transfer function of

$$G(s)=\frac{1}{1000(s^2-1.1772)}$$

Design a series compensator with $G_c(s) = K_p\ (1+T_d s)$ such that the damping ratio of the closed-loop system is 0.7 the and undamped natural frequency is 0.5 rad/sec.

11.6 Given the open-loop transfer function of a plant as

$$G(s)=\frac{5}{s(0.5s+1)}$$

design a lead compensator in series so that the dominant closed-loop poles are located at

$$s=-2\pm j2\sqrt{3}.$$

11.7 The dominant closed-loop poles of an angular position control system (Fig. 11.46) are at $s = -3.6 \pm j4.8$. The damping ratio ξ of dominant closed-loop poles is 0.6. The static velocity error constant K_v is 4.1/sec., which means that a ramp input of 360°/sec will produce a steady-state error of 87.8°.

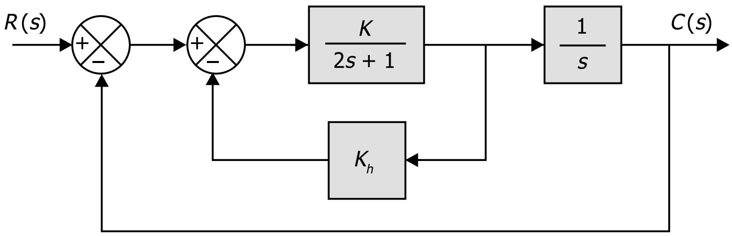


Fig. 11.46

A small change in the undamped natural frequency of the dominant closed-loop pole is allowed. Design a lag compensator to decrease the steady-state error to one-tenth of the present value or to increase the static velocity error constant to 41/sec while the damping ratio is kept at 0.6.

11.8 The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{2s+1}{s(s+1)(s+2)}.$$

Design a compensator that gives dominant closed-loop poles at $s = -2 \pm j2\sqrt{3}$ and static velocity error constant K_v = 80/sec.

11.9 The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{2s+1}{s(s+1)(s+2)}.$$

Design a series compensator so that the unit-step response will exhibit a maximum overshoot of 30 per cent or less and a settling time of 3 sec or less.

11.10 The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{1}{s(s+1)(s+5)}.$$

Design a lag lead compensator so that K_v = 20/sec., phase margin is 60° and gain margin is not less than 8 db. Draw the unit-step and unit-ramp response curve of the compensated system with MATLAB.

11.11 Design a lag compensating network for a system represented by

$$G(s) = \frac{K}{s(s+2)(s+20)}$$

So that the system performance meets the following specifications.
Phase Margin ≥ 3.5° and velocity error constant, k_u = 20/sec.

12

CONCEPT OF STATE VARIABLE MODELLING

12.1 INTRODUCTION

The concept of modelling, analysis and design of control systems discussed, so far was based on their transfer functions which suffer from some drawbacks as stated below.

i) The transfer function is defined only under zero initial conditions.
ii) It is only applicable to linear time-invariant systems and generally restricted to single-input single-output (SISO) systems.
iii) It gives output for a certain input and provides no information about the internal state of the system.

To overcome these drawbacks in the transfer function, a more generalised and powerful technique of state variable approach in the time domain was developed. The state variable method of modelling, analysis and design is applicable to linear and non-linear, time-invariant or time varying multi-input multi-output (MIMO) systems. The placement of closed-loop poles for improvement of system performance can be done with state feedback. We shall now discuss the state variable techniques for linear continuous time systems.

12.2 CONCEPTS OF STATE, STATE VARIABLES AND STATE MODEL

We will define first the concepts of state, state variables, state vector and state space before proceeding for modelling in state space.

State: The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t \geq t_0$.

State Variables: The state variables of a dynamic system are the smallest set of variables that completely determines the state of the dynamic system.

State Vector: This is a vector consisting of n number of state variables that completely determine the behaviour of a dynamic system.

State Space: The state space is an n-dimensional space whose coordinate axes are the n number of state variables that completely determine the behaviour of a dynamic system. Any state can be represented by a point in the state space.

If $x_1, x_2, \ldots\ldots\ldots x_n$ are the n state variables of a dynamic system, then its state vector (X) is written by an $n\ x_1$, matrix as follows.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ X_n \end{bmatrix}$$

where $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ all are functions of time t.

State Space Equations: The input variables, output variables and state variables are the three types of variables used in state space modelling of dynamic systems.

Let us now assume that there are r inputs $u_1(t), u_2(t), \ldots\ldots u_r(t)$, m outputs $y_1(t), y_2(t), \ldots\ldots y_m(t)$ and n state variables $x_1(t), x_2(t), \ldots\ldots, x_n(t)$ that completely describe a multiple-input multiple-output (**MIMO**) system. Then the state equations of a time-invariant system are given by the following n differential equations of first order.

$$\frac{dx_i(t)}{dt} = f_i(x_1, x_2, \ldots.x_n, u_1, u_2, \ldots..u_r)$$

or,
$$\dot{x}_i(t) = f_i(x_1, x_2, \ldots..x_n, u_1, u_2, \ldots\ldots., u_r)$$

for $\quad i = 1, 2, \ldots\ldots, n.$

The above state equations may be written in terms of vector equation as $\dot{\boldsymbol{X}}(t) = \boldsymbol{f}(x(t)\boldsymbol{u}(t))$, where $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$ and $\boldsymbol{f}(x(t), \boldsymbol{u}(t))$ are state vector, input vector and function vector, respectively, and can be written as

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ : \\ x_n(t) \end{bmatrix}, \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ : \\ : \\ u_r(t) \end{bmatrix} \text{ and}$$

$$\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \begin{bmatrix} f_1(x_1, x_2, \ldots\ldots x_n, u_1, u_2, \ldots\ldots, u_r) \\ f_2(x_1, x_2, \ldots\ldots x_n, u_1, u_2, \ldots\ldots, u_r) \\ : \\ : \\ f_n(x_1, x_2, \ldots\ldots, x_n, u_1, u_2, \ldots\ldots, u_n) \end{bmatrix}$$

The output equations of the above time invariant system are given by the following m equations.

$$y_j(t) = g_j(x_1, x_2, \ldots\ldots, x_n, u_1, u_2, \ldots\ldots, u_r)$$

for $\quad j = 1, 2, \ldots\ldots, m.$

These output equations may also be written in terms of vector equations as

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

where $\boldsymbol{y}(t)$ and $\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ are output vector and function vector respectively and can be written as

$$\boldsymbol{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ : \\ : \\ y_m(t) \end{bmatrix} \text{and}$$

$$\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \begin{bmatrix} g_1(x_1, x_2, \ldots\ldots x_n, u_1, u_2, \ldots\ldots, u_r) \\ g_2(x_1, x_2, \ldots\ldots x_n, u_1, u_2, \ldots\ldots, u_r) \\ : \\ : \\ g_m(x_1, x_2, \ldots\ldots, x_n, u_1, u_2, \ldots\ldots, u_r) \end{bmatrix}$$

Similarly, the state equations and output equations of a time-varying system may be written by vector equations as

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \text{ and}$$
$$\boldsymbol{Y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), t).$$

The state equations and output equations together form the state model of a system.

12.2.1 State Model of Linear Systems

The state model of a linear time-invariant system can be written as

$$\dot{x}_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n +$$
$$b_{i1}u_1 + b_{i2}u_2 + \ldots + b_{ir}u_r$$

for, $\quad i = 1, 2, \ldots., n$

and,

$$y_j = c_{j1}x_1 + c_{j2}x_2 + \ldots + c_{jn}x_n + d_{j1}u_1 + d_{j2}u_2 + \ldots + d_{jr}u_r$$

for, $\quad j = 1, 2, \ldots., m$

The set of state equations and output equations of the above state model may be written in a vector-matrix form as given below.

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t)$$
$$\dot{\boldsymbol{Y}}(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t)$$

where $\boldsymbol{x}(t)$ is $n \times 1$ state vector, $\boldsymbol{u}(t)$ is $r \times 1$ input vector, $\boldsymbol{A}$ is the $n \times n$ state matrix of constant elements, $\boldsymbol{B}$ is the $n \times r$ input matrix of constant elements, $\boldsymbol{C}$ is the $m \times n$ output matrix of constant elements and $\boldsymbol{D}$ is the $m \times r$ transmission matrix of constant elements. The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are defined as follows

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ \vdots & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}. \boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & & & \\ \vdots & & & \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}$$

So, the block diagram of the above linear time-invariant MIMO (r input and m output) system can be drawn as follows.

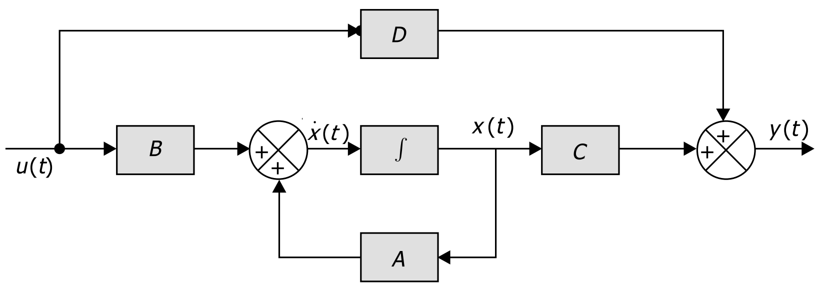


Fig. 12.1 *Block diagram of a linear time-invariant continuous time MIMO control system in state space*

In Fig. 12.1, the broad arrows represent vector quantities and state vector is the output of the integrator block.

12.2.2 State Model of Single-Input Single-Output Linear Systems

The state model for a linear time-invariant SISO system can be written as follows by taking $r = 1$ and $m = 1$.

$$\dot{x}(t) = Ax(t) + Bu$$

and $$y(t) = Cx(t) + du$$

where d is a constant and u is a scalar control variable. Also $\boldsymbol{B}$ and $\boldsymbol{C}$ are row $n \times 1$ and $1 \times n$ matrices respectively. So the block diagram of the linear time-invariant SISO system can be drawn as follows

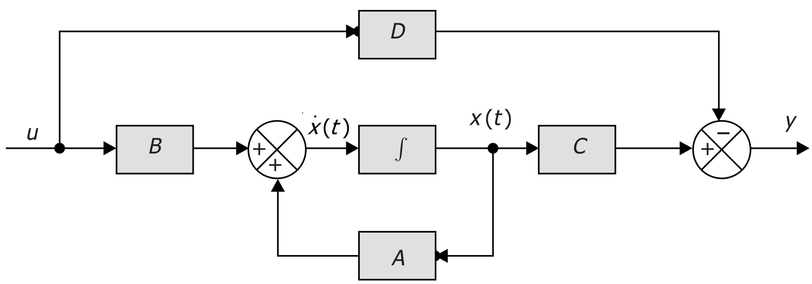


Fig. 12.2 *Block diagram of a linear time-invariant continuous time SISO control system in state space*

It may be noted that state vector as well as state model of a control system in state space are not unique, although the minimal number (n) of elements in the state vector of the given system is constant and the system is said to be of the order n.

12.3 STATE MODELS OF LINEAR CONTINUOUS TIME SYSTEMS

Linear time-invariant continuous time dynamic systems are described by linear differential equations with constant coefficients. The general form of an nth-order linear differential equation relating the input $u(t)$ and the output $y(t)$ of a linear time invariant continuous time system is given by

$$\begin{aligned} y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y \\ = b_0 u^{(m)} + b_1 u^{(m-1)} + \ldots + b_{m-1}\dot{u} + b_m u \end{aligned}$$

where a_i (for $i = 1, 2, \ldots, n$) and b_j (for $j = 1, 2, \ldots, m$) are constants; m and n are integers such that $m \le n$; $y^{(n)}$ and $u^{(m)}$ denote respectively the n^{th} and m^{th} order derivatives with respect to time t. The initial conditions are $y(0), \dot{y}(0), \ddot{y}(0), \ldots\ldots y^{(n-1)}(0)$.

The state model of a given differential equation is not unique and any two models can be uniquely related by a linear transformation of $\boldsymbol{x} = \boldsymbol{Pz}$ where '$\boldsymbol{x}$ and '$\boldsymbol{z}$' are two different sets of state variables belonging to two different state model of the same system and '$\boldsymbol{P}$' is a non-singular (that is, $|\boldsymbol{P}| \neq 0$) constant matrix. We shall now discuss some commonly used forms of state models in the following sub-sections.

12.3.1 State Space Representation Using Physical Variables

The state model obtained from physical variables as state variables can be realised through an example on real physical system. Let us consider a simple electrical system of RLC network shown in Fig. 12.3.

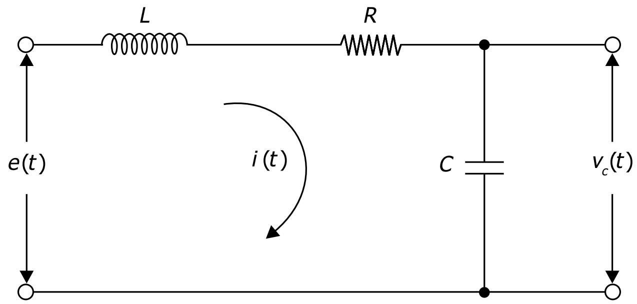


Fig. 12.3 *RLC network*

From the figure it is evident that if we have a knowledge of the initial conditions of $v(0)$, $i(0)$ and the input signal $e(t)$ for $t \geq 0$, then the behaviour of the network is completely specified for $t \geq 0$. We may now assign the current $i(t)$ through the inductor and the voltage $v(t)$ across the capacitor as state variables $x_1(t)$ and $x_2(t)$ respectively.

That is, $$x_1(t) = i(t) \qquad \text{....(12.1)}$$

and $$x_2(t) = v(t) \qquad \text{....(12.2)}$$

From Fig. 12.3, the state equation of the RLC network can be written as

$$L\frac{di(t)}{dt} = -Ri(t) - v(t) + e(t)$$

and $$C\frac{dv(t)}{dt} = i(t).$$

Using equations (12.1) and (12.2) the above state equations are as follows.

$$\frac{dx_1(t)}{dt} = -\frac{R}{L}x_1(t) = \frac{1}{L}x_2(t) + \frac{1}{L}e(t) \qquad \text{....(12.3)}$$

$$\frac{dx_2(t)}{dt} = \frac{1}{C} x_1(t) \qquad \text{....(12.4)}$$

If we assume the voltage across the capacitor as the output variable y, then the output equation can be written as

$$y = x_2(t) \qquad \text{....(12.5)}$$

So the state model using physical variables as state variables can be written in a vector-matrix form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $u = e(t)$ is the input.

12.3.2 State Space Representation Using Phase Variables

Let us consider a dynamic, linear, time-invariant and continuous time system the nth order linear differential equation of which is given by

$$\begin{aligned} y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1}\dot{y} + a_n y \\ = b_0 u^{(n)} + b_1 u^{n-1} + ... + b_{n-1}\dot{u} + b_n u \end{aligned} \qquad \text{....(12.6)}$$

where b_0, b_i and a_i are constants for $i = 1, 2, ..., n$ and $y(0), \dot{y}(0), ..., y^{(n-1)}(0)$ and $u(0), \dot{u}(0), ...,$ $u^{(n-1)}(0)$ are the initial conditions to completely specify the system.

In order to obtain the state model of the above linear differential equation, the following state variables are chosen in such a way as to eliminate the derivatives of the forcing function u in the state equations.

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u \\ &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ &\vdots \\ &\vdots \end{aligned} \qquad \text{....(12.7)}$$

$$x_{n-1} = \dot{x}_{n-2} - \beta_{n-2}u = y^{(n-2)} - \beta_0 u^{(n-2)} - \beta_1 u^{(n-3)} - \ldots.. - \beta_{n-3}\dot{u} - \beta_{n-2}u$$

$$x_n = x_{n-1} - \beta_{n-1}u$$

$$= y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \ldots.. - \beta_{n-2}\dot{u} - \beta_{n-1}u$$

Putting the values of $y, \dot{y}, \ddot{y}, \ldots.., y^{(n-1)}, y^{(n)}$ from equation (12.7) into equation (12.6) we will get the value of $\dot{x}_n$ which can be made free from derivatives of forcing function u by writing the values of $\beta_0, \beta_1, \beta_2, \ldots.., \beta_{n-1}, \beta_n$ as follows.

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2 b_1 - a_3\beta_0 \\
\beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \ldots. - a_{n-2}\beta_1 - a_{n-1}\beta_0 \\
\beta_n &= b_n - a_1\beta_{n-1} - \ldots.. - a_{n-1}\beta_1 - a_n\beta_0
\end{aligned} \qquad \ldots(12.8)$$

So from equations (12.7) and (12.8) the state equations and output equation can be written as

$$\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\vdots \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1}u \\
\dot{x}_n &= -a_n x_1 - a_{n-1}x_2 - \ldots.. - a_1 x_n + \beta_n u
\end{aligned}$$

and $$y = x_1 + \beta_0 u.$$

Hence the state space representation of equation (12.6) in a vector-matrix form can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

and
$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \beta_0 u$$

or
$$\left.\begin{aligned} \dot{X} &= Ax + Bu \\ Y &= Cx + Du \end{aligned}\right\} \qquad \text{....(12.9)}$$

wher/e,
$$x = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } D = \beta_0.$$

The form of matrix 'A' is known as the Bush or Companion form.

The initial conditions of state variables can be written in a vector-matrix form as follows.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ \vdots \\ x_n(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-2)}(0) \\ y^{(n-2)}(0) \end{bmatrix} +$$

$$\begin{bmatrix} -\beta_0 & 0 & 0 & \cdots & 0 & 0 \\ -\beta_1 & -\beta_0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ -\beta_{n-2} & -\beta_{n-3} & -\beta_{n-4} & \cdots & -\beta_0 & 0 \\ -\beta_{n-1} & -\beta_{n-2} & -\beta_{n-3} & \cdots & -\beta_1 & -\beta_0 \end{bmatrix} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \vdots \\ u^{(n-2)}(0) \\ u^{(n-1)}(0) \end{bmatrix}$$

The block diagram of equation (12.9) is shown in Fig. 12.4.

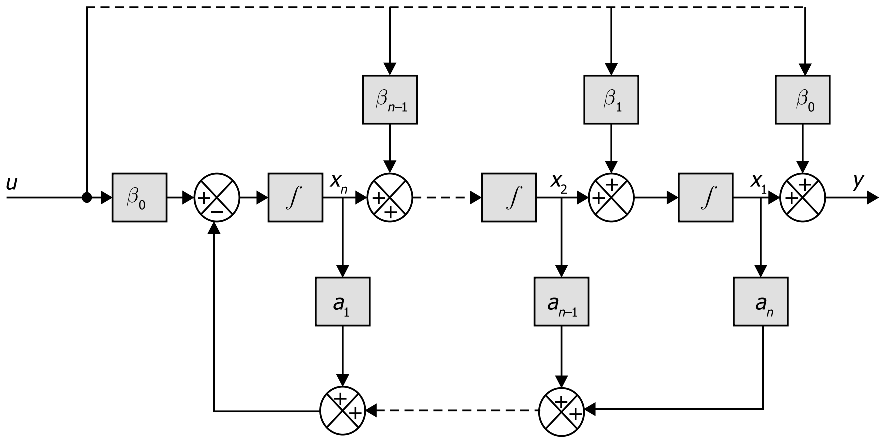


Fig. 12.4 *Block diagram of linear differential equation (12.9) in state space*

Phase variables are defined as those particular state variables which are obtained from one (often the system output) of the system variables and its derivatives. The state model in phase variables form for the linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1}\dot{y} + a_n y = b_n \qquad ....(12.10)$$

with zero initial conditions may be obtained from equation (12.9) by putting $\beta_0 = \beta_1 = \beta_2 = ... = \beta_{n-2} = \beta_{n-1} = 0$ and $\beta_n = b$. The corresponding phase variables will be obtained from equation (12.7).

Two other ways of phase variable formulation of differential equation (12.6) with zero initial conditions are discussed in sections (*a*) and (*b*).

(a) Observable phase variable form: The transfer function *Y*(*s*)/*U*(*s*) can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 + \sum_{i=1}^{n}(b_i / s^i)}{1 + \sum_{i=1}^{n}(a_i / s^i)}$$

or,
$$Y(s) = b_0 U(s) + \sum_{i=1}^{n} \frac{b_i U(s) - a_i Y(s)}{s^i} \qquad ....(12.11)$$

Now let us define the state variables as follows.

$$sX_i(s) - X_i + 1(s) = b_i U(s) - a_i Y(s) \qquad ....(12.12)$$

for $\quad i = 1, 2, 3, ..., n$

and $$X_{n+1}(s) = 0$$

From equations (12.11) and (12.12) we get

$$Y(s) = b_0 U(s) + \sum_{i=1}^{n} \frac{sX_i(s) - X_{i+1}(s)}{s^i}$$

$$= b_0 U(s) + \sum_{i=1}^{n} \left(\frac{X_i(s)}{s^{i-1}} - \frac{X_{i+1}(s)}{s^i} \right)$$

$$= b_0 U(s) + X_1(s) - \frac{X_{n+1}(s)}{s^n}$$

$\therefore$ $$Y(s) = b_0 U(s) + X_1(s)$$

Taking inverse Laplace transform we get,

$$y = b_0 u + x_1 \qquad \text{....(12.13)}$$

Taking inverse Laplace transform of (12.12) we get

$$\dot{x}_1 - x_{i+1} = b_i u - a_i y$$

or, $$\dot{x}_1 - x_{i+1} = b_i u - a_i (b_0 u + x_1)$$

$\therefore$ $$\dot{x}_1 = x_{i+1} - a_i x_1 + (b_i - a_i b_0) u \qquad \text{....(12.14)}$$

for $i = 1, 2, 3, \ldots\ldots, n$

and $$x_{n+1} = 0$$

From equations (12.13) and (12.14) we may write the observable phase variable form of representation of transfer function in state space in a vector-matrix form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

and $$y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

(b) Controllable phase variable form: The transfer function $Y(S)/U(S)$ can be written as

$$\frac{Y(s)}{U(s)} = b_0 + \frac{\sum_{i=1}^{n}(b_i - a_i b_0)s^{n-i}}{s^n + \sum_{i=1}^{n} a_i s^{n-i}}$$

$\therefore$
$$Y(s) = b_0 U(s) + \sum_{i=1}^{n}(b_i - a_i b_0)s^{n-1}Q(s) \qquad \text{....(12.15)}$$

where
$$Q(s) = \frac{U(s)}{s^n + \sum_{i=1}^{n} a_i s^{n-i}} \qquad \text{....(12.16)}$$

Let us now take the state variables as follows.

$$X_{n-i+1}(s) = s^{n-i}Q(s) \qquad \text{....(12.17)}$$

for $i = n, (n-1), \ldots, 3, 2, 1$

$$sX_{n-i+1}(s) = s^{n-i+1}Q(s)$$

or
$$sX_{n-i+1}(s) = X_{n-i+2}(s)$$

Taking inverse Laplace transform, we may write

$$= \begin{bmatrix} 1 & 0 & 0 \\ (p_5 - 17) & (p_6 - 56) & -160 \\ (p_8 + 196) & p_9 + 784 & 2240 \end{bmatrix} \qquad \text{....(12.18)}$$

for $i = n, (n-1), \ldots\ldots\ldots 3, 2.$

Equation (12.16) may be written as

$$s^n Q(s) + \sum_{i=1}^{n} a_i s^{n-i} Q(s) = U(s)$$

Using equation (12.17), the above equation may be written as

$$sX_n(s) + \sum_{i=1}^{n} a_i X_{n-i+1}(s) = U(s)$$

Taking inverse Laplace transform we get

$$\dot{x}_n = -\sum_{i=1}^{n} a_i x_{n-i+1} + u \qquad \text{....(12.19)}$$

From equations (12.15) and (12.17) we get

$$Y(s) = b_0 U(s) + \sum_{i=1}^{n} (b_i - a_i b_0) X_{n-i+1}(s)$$

Taking inverse Laplace transform we get

$$y = b_0 u + \sum_{i=1}^{n} (b_i - a_i b_0) x_{n-i+1} \qquad \text{....(12.20)}$$

From equations (12.18), (12.19) and (12.20) we may write the controllable phase variable form of representation of transfer function in state space in a vector-matrix form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \left[b_n - a_n b_0 ,\, b_{n-1} - a_{n-1} b_0 ,\, ...,\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Thus phase variables provide a powerful method of state variable formulation. But they are not a practical set of state variables from a measurement, control and analysis point of view.

12.3.3 State Space Representation Using Canonical Variables

The linear differential equation (12.6) with zero initial conditions has the transfer function given by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

The state space representations of transfer function discussed so far are not in a canonical or normal form. Since the canonical form of representation of transfer function is most suitable from an analysis point of view, we will discuss them in the following cases.

Case I: When the poles of a transfer function are distinct

Let us now assume that the transfer function $Y(s)/U(s)$ has n distinct poles of $\lambda_1, \lambda_2, \ldots, \lambda_n$ and is of the following form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n}{(s-\lambda_1)(s-\lambda_2)....(s-\lambda_n)}$$

or,
$$Y(s) = b_0 U(s) + \sum_{i=1}^{n} \frac{C_i U(S)}{(s-\lambda_i)} \qquad ....(12.21)$$

where $C_i's$ (for $i = 1, 2, ..., n$) are constants called residues of the poles at $s = \lambda_i$.

Now let us define the state variables as follows.

$$X_1(s) = \frac{U(s)}{s-\lambda_i} \qquad ....(12.22)$$

for $\quad i = 1, 2, ..., n$

or
$$sX_i(s) = \lambda_i X_i(s) + U(s)$$

Taking inverse Laplace transform, we get

$$\dot{x}_i = \lambda_i x_i + u \qquad ....(12.23)$$

for $\quad i = 1, 2, ..., n$

From equations (12.21) and (12.22) we get

$$Y(s) = b_0 U(s) + \sum_{i=1}^{n} C_i X_i(s)$$

Taking inverse Laplace transform, we get

$$y = b_0 u + \sum_{i=1}^{n} C_i x_i \qquad ....(12.24)$$

From equations (12.23) and (12.24) we may write the diagonal canonical form of representation of transfer function in state space in a vector-matrix form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u$$

and
$$y = \begin{bmatrix} C_1 & C_2 & \cdots & C_{n-1} & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Case II: When the poles of a transfer function are not distinct

Let us assume that the transfer function $Y(s)/U(s)$ has a pole at $s = \boldsymbol{\lambda}_r$ of order r and the other poles are assumed to be distinct and located at $s = \boldsymbol{\lambda} j$ for $j = (r + 1), (r + 2), \ldots\ldots, n$. So the transfer function can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots. b_{n-1} s + b_n}{(s + \lambda_r)^r (s + \lambda_{r+1}) \ldots. (s + \lambda_n)}$$

or,
$$Y(s) = b_0 U(s) + \sum_{i=1}^{r-1} \frac{C_i U(s)}{(s - \lambda_r)^{r-i+1}} + \sum_{j=r}^{n} \frac{C_j U(s)}{(s - \lambda_j)} \qquad \ldots.(12.25)$$

Let us now define the state variables as follows.

$$X_i(s) = \frac{U(s)}{(s - \lambda_r)^{r-i+1}} \qquad \ldots.(12.26)$$

for $\quad i = 1, 2, \ldots\ldots.., r$

and
$$X_j(s) = \frac{U(s)}{(s - \lambda_j)} \qquad \ldots.(12.27)$$

for $\quad j = r, (r + 1), \ldots\ldots, n.$

Replacing i by $(i-1)$ in equation (12.26), we get

$$X_{i-1}(s) = \frac{U(s)}{(s - \lambda_r)^{r-i+2}}$$

or, $$X_{i-1}(s) = \frac{1}{(s-\lambda_r)} \cdot \frac{U(s)}{(s-\lambda_r)^{r-i+1}}$$

or $$X_{i-1}(s) = \frac{X_i(s)}{(s-\lambda_r)}$$

or $$sX_{i-1}(s) = \lambda_r X_{i-1}(s) + X_i(s)$$

for $i = 2, 3, \ldots\ldots, r.$

Taking inverse Laplace transform, we get

$$\dot{x}_{i-1} = \lambda_r x_{i-1} + x_i \qquad \ldots(12.28)$$

for $i = 2, 3, \ldots\ldots\ldots, r.$

From equation (12.27), we get

$$sx_j(s) = \lambda_j x_j(s) + U(s)$$

for $j = r, (r+1), \ldots\ldots, n.$

Taking inverse Laplace transform, we get

$$\dot{x}_j = \lambda_j x_j + u \qquad \ldots(12.29)$$

for $j = r, (r+1), \ldots.., n.$

From equations (12.25), (12.26) and (12.27) we get

$$Y(s) = b_0 U(s) + \sum_{i=1}^{r-1} C_i x_i(s) + \sum_{j=r}^{n} C_j x_j(s)$$
$$= b_0 U(s) + \sum_{i=1}^{n} C_i x_i(s).$$

Taking inverse Laplace transform, we get

$$y = b_0 u + \sum_{i=1}^{n} C_i x_i \qquad \ldots(12.30)$$

From equations (12.28), (12.29) and (12.30) we may write the Jordan canonical form of representation of transfer function in state space in a vector-matrix form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{r-1} \\ \dot{x}_r \\ \dot{x}_{r+1} \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_r & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_r & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_r & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{r+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

and
$$y = \begin{bmatrix} C_1 & C_2 & \cdots & C_{r-1} & C_r & C_{r+1} & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

The $n \times n$ matrix, whose diagonal elements are poles at $s = \lambda_r, \lambda_{r+1}, \ldots, \lambda_n$ of the transfer function, is known as the Jordan canonical matrix. This matrix also has the property that all the elements below the principal diagonal are zero and certain elements to the immediate right of the principal diagonal are unity when the adjacent elements in the principal diagonal are equal. The square sub-matrix, the diagonal elements of which are all equal to λ_r, is known as the Jordan Block.

12.4 CORRELATION BETWEEN STATE MODEL AND TRANSFER FUNCTION

Let us consider a single-input single-output system the transfer function of which is given by

$$\frac{Y(s)}{U(s)} = G(s) \qquad \text{....(12.31)}$$

The state model of the above system may be given by the following equations.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}x + \boldsymbol{B}u \qquad \text{....(12.32)}$$

$$y = \boldsymbol{C}x + dU \qquad \text{....(12.33)}$$

where x is the state vector, u is the input and y is the output and all are functions of time t.

The Laplace transform of equations (12.32) and (12.33) are given by

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{AX}(s) + \boldsymbol{B}U(s) \quad \text{....(12.34)}$$

and

$$Y(s) = \boldsymbol{CX}(s) + dU(s) \quad \text{....(12.35)}$$

As transfer function is defined with zero initial condition, putting $\boldsymbol{x}(0) = 0$ in equation (12.34), we get,

$$s\boldsymbol{X}(s) = \boldsymbol{A}X(s) + \boldsymbol{B}U(s)$$

or,

$$s\boldsymbol{X}(s) - \boldsymbol{AX}(s) = \boldsymbol{B}U(s)$$

or,

$$(s\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{X}(s) = \boldsymbol{B}U(s)$$

Pre-multiplying both sides by $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$, we get,

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A}) - 1\boldsymbol{B}U(s) \quad \text{....(12.36)}$$

Putting the value $\boldsymbol{X}(s)$ of in equation (12.35), we get,

$$Y(s) = [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + d]U(s) \quad \text{....(12.37)}$$

Comparing equations (12.31) and (12.37), we get

$$\begin{aligned} G(s) &= \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + d \\ &= \frac{\boldsymbol{C}[adj(s\boldsymbol{I} - \boldsymbol{A})]\boldsymbol{B}}{|(s\boldsymbol{I} - \boldsymbol{A})|} + d \end{aligned} \quad \text{....(12.38)}$$

So the transfer function is expressed in terms of the parameters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and d of the state model. It is also evident that $|(s\boldsymbol{I} - \boldsymbol{A})|$ is the characteristic polynomial of $G(s)$, that is, in other words, the poles of $G(s)$ are the eigen values of matrix 'A'. It may also be noted that while the state model of a system is non-unique, its transfer function is unique.

12.5 DIAGONALISATION OF STATE MATRIX

Let us consider the state model of a system given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where 'A' is a non-diagonal state matrix.

If there exists a non-singular matrix 'P' such that the transformation $x = Pz$ transforms the given state model into a diagonal canonical form given by

$$\dot{z} = (P^{-1}AP)z + (P^{-1}B)u$$

$$y = (CP)z + Du,$$

then the matrix 'P' is called diagonalising or modal matrix and the technique used in transforming the state matrix 'A' into a diagonal matrix ($P^{-1}AP$) is called diagonalisation.

For determining modal matrix 'P', let λ_i (for $i = 1, 2, \ldots, n$) be the distinct roots (known as the characteristic roots or eigen values of matrix 'A' of the characteristic equation given by

$$|\lambda I - A = 0|$$

or, $$\lambda^n + a_1\lambda^{n-1} + \ldots\ldots. + a_{n-1}\lambda + a_n = 0$$

Referring to equation (12.38) it can be concluded that the eigenvalues of the state matrix 'A' are in fact the poles of the system transfer function.

Let $x = p_i$ (for $i = 1, 2, \ldots, n$) which are $(n \times 1)$ vectors be the solution of the matrix equation

$$(\lambda_i I - A)x = 0.$$

Then p_i are called the eigenvector of 'A' corresponding to the eigenvalue λ_i and

$$AP_i = \lambda_i p_i \qquad \ldots.(12.39)$$

Let the $(n \times n)$ matrix $P - [p_1 p_2 \ldots\ldots p_n)$ be formed by the eigenvectors of 'A'. Then

$$\begin{aligned} AP &= A\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \\ &= \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} \end{aligned}$$

Using equation (12.39),

$$= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

$$\therefore \qquad AP = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ : & : & \cdots & : \\ : & : & \cdots & : \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Pre-multiplying both sides by P^{-1} we get

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ : & : & \cdots & : \\ : & : & \cdots & : \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

So $P = [p_1 p_2 \ldots p_n]$ the required diagonalising or modal matrix. Also, it may be noted that the characteristic equation and eigenvalues are invariant under non-singular transformation.

If the state matrix 'A' is of the Bush or Companion form as shown below

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ : & : & : & \cdots & : \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \qquad \text{....(12.40)}$$

then the modal matrix 'P' is known as the Vander Monde matrix and is given by

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ : & : & \cdots & : \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

In general, when the state matrix 'A' has eigenvalues λ_1 of order m and all other eigenvalues of λ_{m+1}, λ_{m+2}, ..., λ_n are distinct, then diagonalisation is not possible, but the transformation $x = p_z$ will yield the matrix ($P^{-1}AP$) in Jordan canonical form.

The column vectors associated with eigenvectors λ_1 of order m are determined from the transformation

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_m \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ : & : & \cdots & : & : \\ 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix} = A\begin{bmatrix} p_1 & p_2 & \cdots & p_m \end{bmatrix}$$

or, $$p_{i-1} + \boldsymbol{\lambda}_1 p_i = Ap_i$$

or, $$(\boldsymbol{\lambda}_1 I - A)p = -p_{i-1} \qquad \text{….(12.41)}$$

The column vectors associated with distinct eigenvectors $\boldsymbol{\lambda}_{m+1}$ $\boldsymbol{\lambda}_{m+2}$, ……, $\boldsymbol{\lambda}_n$ are determined from

$$(\boldsymbol{\lambda}_i I - \mathrm{A})p_i = 0 \qquad \text{….(12.42)}$$

for $i = (m + 1), (m + 2), \ldots\ldots, n$

So the matrix P can be formed as follows.

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_m & p_{m+1} & \cdots & p_n \end{bmatrix}$$

In particular, if the state matrix 'A' is in Bush or Companion form (12.40), then the modified Vander Monde matrix can be written as

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \cdots & \lambda_{m+1} & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 1 & \cdots & \lambda_{m+1}^2 & \cdots & \lambda_n^2 \\ : & : & : & \cdots & : & \cdots & : \\ \lambda_1^{n-1} & \frac{d}{d\lambda_1}(\lambda_1^{n-1} & \frac{1}{2!}\frac{d^2(\lambda_1^{n-1})}{d\lambda_1^2} & \cdots & \lambda_{m+1}^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

12.6 SOLUTION OF STATE EQUATION

The state equation of a linear time-invariant system is of the form given below.

$$\dot{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \qquad \text{….(12.43)}$$

Taking Laplace transform on both sides of equation (12.43), we get

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s)$$

or, $$(s\boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{B}\boldsymbol{U}(s)$$

Pre-multiplying both sides by $(s\boldsymbol{I} - A)^{-1}$, we get

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{BU}(s)$$

or, $$\boldsymbol{X}(s) = \boldsymbol{F}_1(s)\,\boldsymbol{x}(0) + \boldsymbol{F}_1(s)\,\boldsymbol{F}_2(s) \qquad \text{....(12.44)}$$

where $F_1(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$

and $\boldsymbol{F}_2(s) = \boldsymbol{BU}(s)$

Now, $\boldsymbol{F}_1(s) = (s\boldsymbol{I} - A)^{-1}$ is defined as follows.

$$\boldsymbol{F}_1(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$$
$$= \frac{\boldsymbol{I}}{s} + \frac{\boldsymbol{A}}{s^2} + \frac{\boldsymbol{A}^2}{s^3} + \frac{\boldsymbol{A}^3}{s^4} + \ldots\ldots.$$

Taking inverse Laplace transform, we get,

$$\boldsymbol{f}_1(t) = L^{-1}[s\boldsymbol{I} - \boldsymbol{A})]^{-1}$$
$$= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots\ldots..$$
$$= e^{At} = \phi(t) \text{ (assumed)} \qquad \text{....(12.45)}$$
$$\phi(t) = e^{At} = L^{-1}[sI - A)^{-1}]$$
$$= \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i$$

where, $\phi(t)$ is known as the state-transition matrix and $A^0 = \mathrm{I}$.

Also we know that,

$$L\left[\int_0^t \boldsymbol{f}_1(t-\tau)\boldsymbol{f}_2(\tau)d\tau\right] = \boldsymbol{F}_1(s)\boldsymbol{f}_2(s)$$

$$\therefore \quad L^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-t)f_2(t)dt$$
$$= \int_0^t e^{A(t-t)}Bu(t)dt \qquad \text{....(12.46)}$$

Now taking inverse Laplace transform on both sides of equation (12.44), we get

$$\boldsymbol{x}(t) = \boldsymbol{f}_1(t)\,\boldsymbol{x}(0) + L^{-1}\,[\boldsymbol{F}_1(s)\,\boldsymbol{F}_2(s)]$$

Using equations (12.45) and (12.46) we may write

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}\boldsymbol{BU}(\tau)d\tau$$
$$= \phi(t)x(0) + \int_0^t \phi(t-\tau)Bu(\tau)d\tau \qquad \text{....(12.47)}$$

So equation (12.47) is the solution of state equation (12.43). The first term on the right-hand side of equation (12.47) is the zero input response and the second term is the zero state response of the systGeneralising the initial time as $t_0 \neq 0$, solution (12.47) may be written as

$$s(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$= \phi(t-t_0)x(t_0) + \int_{t_0}^{t} \phi(t-\tau)BU(\tau)d\tau \qquad \text{....(12.48)}$$

It may be noted from equation (12.47) that if $\boldsymbol{B} = 0$, then

$$\boldsymbol{x}(t) = e^{At}x(0) = \phi(t)\,x(0) \qquad \text{....(12.49)}$$

is the solution of the state equation $\dot{x} = Ax$ which is homogeneous. So, it is clear that $\phi(t) = e^{At}$ is called the state transition matrix because it relates the system state at time t to that at time zero.

12.6.1 Computation of State Transition Matrix

The state transition matrix $\phi(t)$ can be computed by several methods. Two of them are given in formula (12.45). The third one, based on the Cayley-Hamilton theorem, will be discussed here.

If A be the state matrix of order $(n \times n)$, then its characteristic equation is given by

$$q(\boldsymbol{\lambda}) = |\boldsymbol{\lambda}I - A| = \boldsymbol{\lambda}^n + a_1\boldsymbol{\lambda}^{n-1} + ... + a_{n-1}\,\boldsymbol{\lambda} + a_n = 0 \qquad \text{....(12.50)}$$

The power series of $e^{\lambda t}$ converges for all finite values of $\boldsymbol{\lambda}$ and t and let it be denoted by $f(\boldsymbol{\lambda})$.

$$\therefore \qquad f(\boldsymbol{\lambda}) = e^{\lambda t}$$

$$= 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + ... + \frac{\lambda^n t^n}{n!} + \frac{\lambda^{n+1} t^{n+1}}{(n+1)!} + ... \text{ to } \infty \qquad \text{....(12.51)}$$

If $Q(\boldsymbol{\lambda})$ be the quotient polynomial and $R(\boldsymbol{\lambda})$ be the remainder polynomial when $f(\boldsymbol{\lambda})$ is divided by the characteristic polynomial $q(\boldsymbol{\lambda})$, then we have

$$f(\boldsymbol{\lambda}) = Q(\boldsymbol{\lambda})\,q(\boldsymbol{\lambda}) + R(\boldsymbol{\lambda}) \qquad \text{....(12.52)}$$

where $R(\boldsymbol{\lambda})$ is of the form given by

$$R(\boldsymbol{\lambda}) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\boldsymbol{\lambda} + \boldsymbol{\alpha}_2\boldsymbol{\lambda}^2 + ... + \boldsymbol{\alpha}_{n-1}\boldsymbol{\lambda}^{n-1} \qquad \text{....(12.53)}$$

If λ_i (for $i = 1, 2, \ldots, n$) be the distinct eigenvalues of matrix A, then from equation (12.50) we may write so from equation (12.94), we have $q(\lambda_i) = 0$ for all λ_i.

$$f(\lambda_i) = Q(\lambda_i)q(\lambda_i) + R(\lambda_i)$$

or,
$$f(\lambda_i) = R(\lambda_i)$$

or,
$$e^{\lambda_i t} = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \ldots.. + \alpha_{n-1}\lambda_i^{n-1} \qquad \ldots.(12.54)$$

for $i = 1, 2, \ldots, n$. Solving the above equations we obtain the coefficients $\alpha_0, \alpha_1, \ldots\ldots, \alpha_{n-1}$ of the remainder polynomial $R(\lambda)$.

It should be noted that if eigenvalues of A are not distinct and if λ_1 be an eigenvalue of order m, then only one independent equation can be obtained by putting $i = 1$ into equation (12.54). The remaining $(m - 1)$ linear equations can be obtained by differentiating both sides of equation (12.54) as follows.

$$\left.\frac{d^j f(\lambda_i)}{d\lambda_n^j}\right|_{i-1} = \left.\frac{d^j R(\lambda_i)}{d\lambda_i^j}\right|_{i=1}$$

or,
$$\left.\frac{d^j}{d\lambda_i^j}\left(e^{\lambda_i}t\right)\right|_{i=1} = \left.\frac{d^j}{d\lambda_i^j}\left(\sum_{k=0}^{n-1}\alpha_k\lambda_i^k\right)\right|_{i=1} \qquad \ldots.(12.55)$$

for
$$j = 1, 2, \ldots\ldots., (m-1)$$

Replacing λ by A in equation (12.52), we get

$$f(A) = Q(A)q(A) + R(A) \qquad \ldots.(12.56)$$

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. So from equation (12.50) we may write

$$q(A) = A^n + a_1A^{n-1} + \ldots\ldots + a_{n-1}A + a_n I = 0 \qquad \ldots.(12.57)$$

From equations (12.56) and (12.57), we get

$$f(A) = R(A)$$

Thus, the state transition matrix $\phi(t)$ can be obtained by replacing λ by A in equation (12.53) as follows.

$$\phi(t) = e^{At} = f(A) = R(A)$$

$$= \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \ldots.. + \alpha_{n-1}A^{n-1} \qquad \ldots.(12.58)$$

12.6.2 Properties of State Transition Matrix

The important properties of the state-transition matrix $\phi(t)$ for the time-invariant system $\dot{x} = Ax$ are stated below and their proofs are left as an exercise to the students.

a) $\phi(0) = e^{A0} = I$

b) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi\,(-t)]^{-1}$

or, $\phi^{-1}(t) = \phi(-t)$

c) $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

d) $[\phi(t)]^n = \phi(nt)$

e) $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\,\phi(t_2 - t_1)$

12.7 CONCEPT OF CONTROLLABILITY AND OBSERVABILITY

The concepts of controllability and observability were originally introduced by R. Kalman and play an important role in the design of modern control system in state space. Precise definitions of controllability and observability are given below.

A system is said to be controllable at time t_0 if it is possible by an unconstrained control vector $u(t)$ to transfer the system from any initial state $x(t_0)$ to any other state $x(t_1)$ in a finite interval of time.

A system is said to be observable at time t_0 if it is possible to determine the state $x(t_0)$ from the observation of the output $y(t)$ over a finite time interval.

The conditions of controllability, and of observability and the principle of durability will be discussed in the following subsections.

12.7.1 Controllability

Let us consider the linear continuous time system be given by

$$\dot{x} = Ax + Bu \qquad \text{....(12.59)}$$

The dimension of the matrices and vectors are as stated in Section 12.2.1.

A system is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state $x(t_0)$ to any final state $x(t_1)$ in a finite time interval $t_0 \leq t_1$. If every state is controllable, then the system will be completely state controllable.

Now we shall derive the condition of complete state controllability. The solution of the given state equation may be written from equation (12.48) as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

At the final time ($t = t_1$) we have

$$x(t) = e^{A(t_1 - t_0)} x(t_0) + \int_{t_0}^{t_1} e^{A(t_1 - \tau)} Bu(\tau) d\tau$$

or,
$$e^{-At} x(t_1) - e^{-At_0} x(t_0) + \int_{t_0}^{t_1} e^{-A\tau} Bu(\tau) d\tau$$

Using Cayley-Hamilton theorem, we may write

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau) A^k$$

where $\alpha_k(\tau)$ for $k = 0, 1, \ldots\ldots, (n-1)$ are scalar time functions.

Thus, the previous equation can be written as

$$e^{-At} x(t_1) - e^{-At_0} x(t_0) = \sum_{k=0}^{n-1} A^k B \int_{t_0}^{t_1} \alpha_k(\tau) u(\tau) d\tau$$

or,
$$e^{-At} x(t_1) - e^{-At_0} x(t_0) = [B : AB : \ldots A^{n-1} B] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

where,
$$\beta_k = \int_{t_0}^{t_1} \alpha_k(\tau) u(\tau) d\tau.$$

The left-hand side of the above equation is a ($n \times 1$) known constant vector. For the system to be completely the state controllable, the above equation must be satisfied with the given constant vector on the left-hand side.

This requires that the ($n \times m$) matrix [B : AB : ... A^{n-1} B] must be of rank n. This matrix is known as the controllability matrix and we may denote it as Q_c.

So, the condition of complete state controllability of the system

$$\dot{x} = Ax + Bu$$

is that the controllability matrix Q_c given by

$$Q_c = [B : AB : \ldots /: A^{n-1} B] \text{ be of rank } n.$$

In other words, it may be said that the pair (AB) is controllable if the rank of Q_c is n.

Alternative condition of complete state controllability: Let λ_i (for $i = 1, 2, \ldots\ldots, n$) be the distinct eigenvalues of matrix 'A'. Also assume that the transformation $x = Pz$ makes $P^{-1} AP$ into a diagonal matrix given by

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Substituting $x = Pz$ into equation (12.59), we have

$$\dot{z} = (P^{-1}AP)z + 1(P^{-1}B)u.$$

If the elements of any one row of the $(n \times r)$ matrix $(P^{-1}B)$ are all zero, then the corresponding state variable cannot be controlled by any of the r elements of control vector u. So for distinct eigenvalues of matrix 'A', the condition of complete state controllability of the system $\dot{x} = Ax + Bu$ is that no row of $P^{-1}B$ has all zero elements.

If the eigenvalues of 'A' are of the form $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \dots, \lambda_n$ with $(n - 3)$ distinct eigenvalues and if there exists a transformation $x = SZ$ such that $AS = J$ changes to Jordan canonical form given by

$$J = \left[\begin{array}{ccc|cc|ccc} \lambda_1 & 1 & 0 & & & & & \\ 0 & \lambda_1 & 1 & & & & & \\ 0 & & \lambda_1 & & & & & \\ \hline & & & \lambda_4 & 1 & & & \\ & & & 0 & \lambda_4 & & & \\ \hline & & & & & \lambda_6 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{array}\right]$$

then substituting $X = SZ$ into equation (12.59) we have,

$$\begin{aligned} \dot{Z} &= (S^{-1}AS)Z + (S^{-1}B)u \\ &= JZ + (S^{-1}B)u. \end{aligned}$$

Thus the system $x = Ax + Bu$ is completely state controllable if and only if the following conditions hold.

a) No two Jordan blocks in J are associated with the same eigenvalues.

b) The elements of any row of $S^{-1}B$ corresponding to the last row of each Jordan block are not all zero.

c) The elements of each row of $S^{-1}B$ corresponding to distinct eigenvalues are not all zero.

Output controllability: Let us consider the linear continuous time system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The dimensions of vectors and matrices are as stated in Section 12.2.1.

The system is said to be completely output controllable if it is possible to construct an unconstrained control vector $u(t)$ that will transfer any given initial output $y(t_0)$ to any final output $y(t_1)$ in a finite time interval $t_0 \leq t \leq t_1$.

It may be stated without proof that the condition of complete output controllability is that the $m \times (n+1)\, r$ matrix

$$[CB : CAB : CA^2B : \ldots : CA^{n-1}B : D]$$

be of rank m.

12.7.2 Observability

Let us consider the linear continuous time system given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where the dimensions of matrices and vectors are as stated in Section 12.2.1. The system is said to be completely observable if every state $x(t_0)$ can be determined from the observation of $y(t)$ over a finite interval of time $t_0 \leq t \leq t_1$ Since control function u does not affect the complete observability, we may assume without loss of generality that $u = 0$ and $t_0 = 0$. So the system now reduces to

$$\dot{x} = Ax \qquad \text{....(12.60)}$$

$$y = Cx \qquad \text{....(12.61)}$$

Since the solution of the homogeneous equation (12.60) is $x(t) = e^{At}x(0)$, substituting this into equation (12.61) we get,

$$y(t) = Ce^{At}x(0) \qquad \text{....(12.62)}$$

Using the Cayley-Hamilton theorem, we may write

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)A^k$$

Substituting this into equation (12.62) we get,

$$y(t) = \sum_{k=0}^{n-1} \alpha_k(t)CA^k x(0)$$

or $$y(t) = [a_0 I : \alpha_1 I : \cdots : \alpha_{n-1} I] \begin{bmatrix} C \\ CA \\ CA^2 \\ : \\ CA^{n-1} \end{bmatrix} x(0)$$

Therefore, knowing the output $y(t)$ over the time interval $0 \le t \le t_1$, $x(0)$ is uniquely determined from the above equation if and only if the $mn \times n$ matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ : \\ CA^{n-1} \end{bmatrix}$$

or its conjugate transpose is of rank n.

Thus the condition of complete observability of the system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

is that the observability matrix Q_0 given by

$$Q_0 = [C^* : A^* C^* : (A^*)^2 C^* : \ldots : (A^*)^{n-1} C^*]$$

is of rank n, where A^* and C^* are conjugate transpose of matrices 'A' and 'C' respectively. In other words, it may be said that pair (AC) is observable if rank of Q_0 is n.

Alternative condition of complete observability: Let λ_i.(for $i = 1, 2, \ldots, n$) be the distinct eigenvalues of matrix 'A'. Also assume that the transformation $x = Pz$ makes $A = P^{-1}AP$ into diagonal matrix given by

$$A = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_1 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Substituting $x = Pz$ into equations (12.60) and (12.61), we get

$$\dot{z} = P^{-1}APz = Az \qquad \text{....(12.63)}$$

$$y = CPz \qquad \text{....(12.64)}$$

Putting the solutions $z = e^{At}z(0)$ equation (12.63) into equation (12.64), we get

$$y(t) = CPe^{At}z(0)$$

$$= CP\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ : & : & \dots & : \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} = CP\begin{bmatrix} e^{\lambda_1 t}z_2(0) \\ e^{\lambda_2 t}z_2(0) \\ : \\ e^{\lambda_n t}z_n(0) \end{bmatrix}$$

If all the elements of the ith (for $i = 1, 2, \dots, n$) column of $m \times n$ matrix CP are zero, then the state variable $z_i(0)$ cannot be determined from the observation of $y(t)$. Thus $x(0)$ cannot be determined by the linear transformation $x = Pz$.

Therefore, for distinct eigenvalues of matrix 'A', the condition of complete observability of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is that no column of $m \times n$ matrix CP has all zero elements.

In case of repeated eigenvalues of matrix 'A' we may substitute $x = Sz$ into equations (12.60) and (12.61) to obtain

$$\dot{Z} = S^{-1}ASZ = Jz \qquad \text{....(12.65)}$$

$$y = Csz \qquad \text{....(12.66)}$$

where the matrix $J = S^{-1}AS$ is in the Jordan canonical form as stated in the alternative condition of controllability. Substituting the solution $z = e^{Jt}z(0)$ of equation (12.65) into equation (12.66), we get

$$y(t) = CS\, e^{Jt}z(0)$$

Thus the system given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is completely observable if and only if the following conditions hold.

1) No two Jordan blocks in J are associated with the same eigenvalues.
2) The elements of any column of CS corresponding to the first row of each Jordan block are not all zero.
3) The elements of each column of CS corresponding to distinct eigenvalues are not all zero.

12.7.3 Principle of Duality

The principle of duality, due to Kalman, can be used to establish analogies between controllability and observability.

Let us consider the system S_1 and its dual system S_2 defined as follows:

System S_1 $$\dot{x} = Ax + Bu$$
$$y = Cx$$

Dual system S_2 $$\dot{z} = A^* z + C^* v$$
$$n = B^* z$$

where z is an n-dimensional state vector, v is an m-dimensional control vector, n is a r-dimensional output vector, A^* is the conjugate transpose of matrix 'A' The dimensions of vectors and matrices of system S_1, are as stated in Section 12.2.1.

The principle of duality states that the system S_1 is completely state controllable if and only if the dual system S_2 is completely observable and vice-versa. In other words, the principle of duality may be stated as follows.

a) The pair (AB) is completely state controllable implies that the pair (A^* B^*) is completely observable.

b) The pair (AC) is completely observable implies that the pair (A^* C*) is completely state controllable. This principle can easily be verified from the necessary and sufficient conditions of complete state controllability and complete observability for systems S_1 and S_2.

Example 12.1 The mechanical system shown in Fig. 12.5 has u_1 and u_2 as its inputs and y_1 and y_0 is its outputs. Obtain a state space representation of the system.

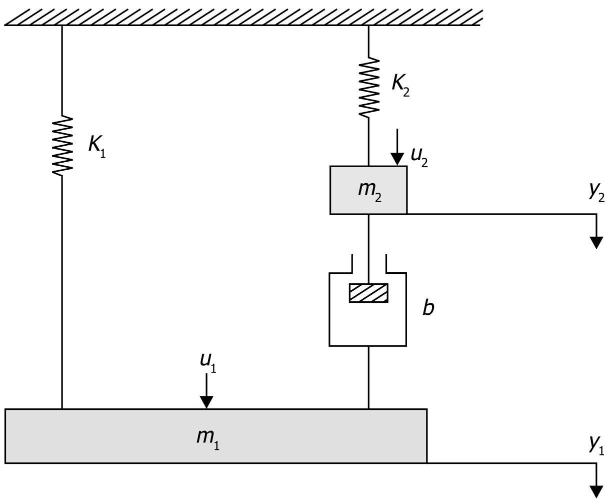


Fig. 12.5

Solution

Here u_1 and u_2 are the force inputs to the system and its outputs y_1 and y_2 are the displacements measured from the equilibrium position of rest. Applying Newton's second law of Motion, we may write the equations of motion for the two masses m_1 and m_2 as follows.

$$u_1 - k_1 y_1 - b(\dot{y}_1 - \dot{y}_2) = m_1 \ddot{y}_1 \qquad \text{....(12.67)}$$

and

$$u_2 - k_2 y_2 + b(\dot{y}_1 - \dot{y}_2) = m_2 \ddot{y}_2 \qquad \text{....(12.68)}$$

Now, we may define displacement and velocity to be state variables as follows.

$$x_1 = y_1 \text{ and } x_2 = y_2$$

$$x_3 = \dot{y}_1 \quad \text{and} \quad x_4 = \dot{y}_2.$$

Thus the state equations may be written as

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1}{m_1}x_1 - \frac{b}{m_1}x_3 + \frac{b}{m_1}x_4 + \frac{u_1}{m_1} \\
\dot{x}_4 &= -\frac{k_2}{m_2}x_2 - \frac{b}{m_2}x_3 + \frac{b}{m_2}x_4 + \frac{u_2}{m_2}
\end{aligned}$$

The output equations are

$$y_1 = x_1$$

$$y_2 = x_2$$

So the state space representation of the system in a vector-matrix form can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & 0 & -\dfrac{b}{m_1} & \dfrac{b}{m_1} \\ 0 & -\dfrac{K_2}{m_2} & \dfrac{b}{m_2} & -\dfrac{b}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{m_1} & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Example 12.2 For the liquid-level system shown in Fig. 12.6, q_1 q_2, q_3, q_4 are the deviations of flow rate from their respective steady-state value, h_1 and h_2 are the deviations of heads from their respective steady-state heads, C_1 and C_2 are the capacitances of two tanks and R_1 R_2 are the resistances for liquid flow at the two valves shown in the diagram.

Assuming all the deviations to be small, obtain a state space representation for the system the two inputs of which are q_1 and q_2 and two outputs are h_1 and h_2.

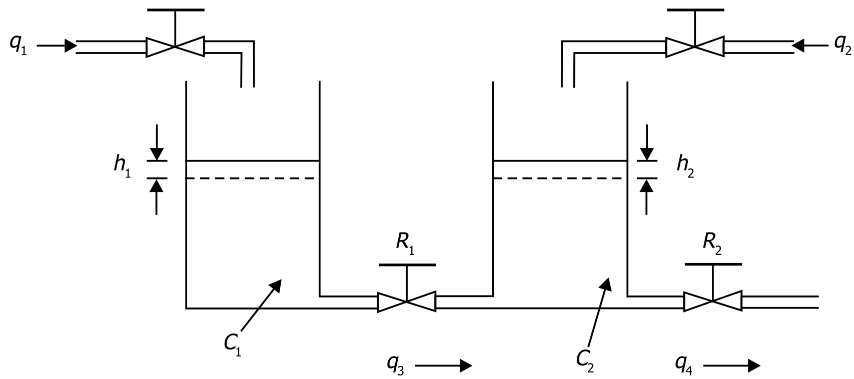


Fig. 12.6

Solution

Since all the deviations are small, the system can be considered to be linear and hence the equations for the system can be written as

$$C_1 \frac{dh_1}{dt} = q_1 - q_3 \qquad \text{....(12.69)}$$

$$C_2 \frac{dh_2}{dt} = q_3 - q_2 - q_4 \qquad \text{....(12.70)}$$

$$\frac{h_1 - h_2}{R_1} = q_3 \qquad \text{....(12.71)}$$

and

$$\frac{h_2}{R_2} = q_{4.} \qquad \text{....(12.72)}$$

Eliminating q_3 from equations (12.69) and (12.71), we get

$$C_1 \frac{dh_1}{dt} = q_1 - \frac{h_1 - h_2}{R_1}$$

or, $$\frac{dh_1}{dt} = -\frac{h_1}{R_1C_1} + \frac{h_2}{R_1C_1} + \frac{q_1}{C_1} \qquad \text{....(12.73)}$$

Eliminating q_4 from equations (12.70), (12.71) and (12.72), we get

$$C_2\frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} + q_2 - \frac{h_2}{R_2}$$

Or, $$\frac{dh_2}{dt} = -\frac{h_1}{R_1C_2} - \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)h_2 + \frac{q_2}{C_2} \qquad \text{....(12.74)}$$

Let us now define the state variables x_1, x_2, input variables u_1, u_2 and output variables y_1, y_2 as follows.

$x_1 = h_1$; $\qquad x_1 = h_2$;

$u_1 = q_1$; $\qquad u_1 = q_2$;

$y_1 = h_1 = x_1$;

$y_2 = h_2 = x_2$.

Using the above definitions, equations (12.73) and (12.74) can be written as

$$\dot{x}_1 = -\frac{1}{R_1C_1}x_1 + \frac{1}{R_1C_1}x_2 + \frac{1}{C_1}u_1.$$

$$\dot{x}_2 = \frac{1}{R_1C_2}x_1 - \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)x_2 + \frac{1}{C_2}u_2.$$

So, the state model in the vector-matrix form of representation of the system can be written as follows.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\dfrac{-1}{R_1C_1} & \dfrac{1}{R_1C_1}\\ \dfrac{1}{R_1C_2} & -\left(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2}\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\dfrac{1}{C_1} & 0\\ 0 & \dfrac{1}{C_2}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

and $$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Example 12.3 Consider the electrical circuit shown in Fig. 12.7. Obtain a state model of the system with zero initial condition where $e_i(t)$ and $e_0(t)$ are input and output voltages respectively.

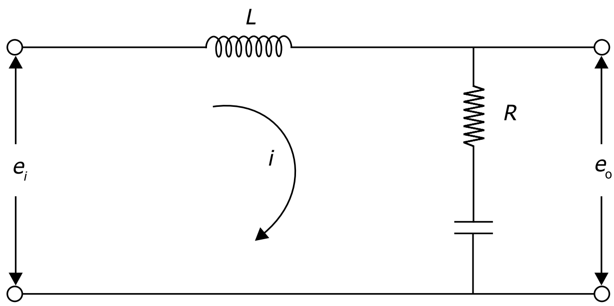


Fig. 12.7

Solution

From Kirchoff's voltage law, we get

$$L\frac{di}{dt} + iR + \frac{1}{C}\int idt = e_i \qquad \text{....(12.75)}$$

$$iR + \frac{1}{C}\int idt = e_o \qquad \text{....(12.76)}$$

Taking Laplace transform of above equations, we get

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E_i(s) \qquad \text{....(12.77)}$$

$$\left(R + \frac{1}{Cs}\right)I(s) = E_o(s) \qquad \text{....(12.78)}$$

Dividing equations (12.78) by (12.77), we get

$$\frac{E_0(s)}{E_i(s)} = \frac{R + \dfrac{1}{Cs}}{Ls + R + \dfrac{1}{Cs}}$$

or, $$\left(Ls + R + \frac{1}{Cs}\right)E_0(s) = \left(R + \frac{1}{Cs}\right)E_i(s)$$

or, $$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)E_0(s) = \left(\frac{R}{L}s + \frac{1}{CL}\right)E_i(s)$$

Now taking inverse Laplace transform, we get

$$\ddot{e}_0 + \frac{R}{L}\dot{e}_0 + \frac{1}{LC}e_0 = \frac{R}{L}\dot{e}_i + \frac{1}{CL}e_i$$

or,
$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u \qquad \text{....(12.79)}$$

where,

$$e_0 = y; \qquad e_i = u$$

$$a_1 = \frac{R}{L}; \qquad a_2 = \frac{1}{LC}$$

$$b_0 = 0; \qquad b_1 = \frac{R}{L}; \qquad b_2 = \frac{1}{LC}$$

Now from equation (12.8) we may write

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = b_1 = \frac{R}{L}$$

$$\begin{aligned}\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ &= b_2 - a_1\beta_1 \\ &= \frac{1}{LC} - \left(\frac{R}{L}\right)^2\end{aligned}$$

From Section 12.4 the state variables x_1 and x_2 are given by

$$x_1 = y - b_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{R}{L}u$$

From equation (12.9) the state model of the given circuit may be written as

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix}u$$

$$= \begin{bmatrix}0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}\frac{R}{L} \\ \frac{1}{LC} - \left(\frac{R}{L}\right)^2\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \beta_0 u = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

Example 12.4 Consider the following transfer function of a system.

$$\frac{Y(s)}{U(s)} = \frac{s+12}{s^2+7s+12}$$

Obtain the state space representation of this system in

a) controllable phase variable form and

b) observable phase variable form.

Solution

The given transfer function can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

where, $b_0 = 0$; $b_1 = 1$;

$b_2 = 12$; $a_1 = 7$;

$a_2 = 12$.

a) The state space representation of the given system in controllable phase variable form can be written as follows from Section 12.3.2(b).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

$$= \begin{bmatrix} 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The state variables here are defined by

$$X_1(s) = \frac{U(s)}{s^2+7s+12}$$

$$X_2(s) = \frac{sU(s)}{s^2+7s+12} = sX_1(s)$$

b) From Section 12.3.2(a), the state space representation of the system in observable phase variable form is given as follows.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-a_1 & 1\\ -a_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}b_1-a_2b_0\\ b_2-a_2b_0\end{bmatrix}u$$

$$=\begin{bmatrix}-7 & 1\\ -12 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 12\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+b_0u=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The state variables here are defined by

$$sX_1(s)-X_2(s)=b_1\,U(s)-a_1\,Y(s)$$
$$=[U(s)-7Y(s)]$$

and

$$sX_2(s)=b_2U(s)-a_2Y(s)$$
$$=12[U(s)-Y(s)]$$

or,

$$X_2(s)=\frac{12}{s}[U(s)-Y(s)]$$

$$X_1(s)=\frac{s+12}{s^2}U(s)-\frac{7s+12}{s^2}Y(s)$$

or,

$$X_2(s)=\frac{12}{s}[U(s)-X_1(s)]$$

$$X_1(s)=\frac{s+12}{s^2}U(s)-\frac{7s+12}{s^2}X_1(s)\qquad [\text{as } y=x_1]$$

or,

$$X_2(s)=\frac{12(s+6)}{s^2+7s+12}U(s)$$

$$X_1(s)=\frac{s+12}{s^2+7s+12}U(s)$$
$$=Y(s).$$

Example 12.5 Consider the following system with differential equation given by

$$\dddot{y}+6\ddot{y}+11\dot{y}+6y=6u.$$

Obtain a state model of the given system in a diagonal canonical form.

Solution

Taking Laplace transform of the given differential equation we get

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$$

$$= \frac{6}{(s+1)(s+2)(s+3)}$$

$$= \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{s+3}$$

$\therefore$ $$Y(s) = b_0 U(s) + \frac{C_1 U(s)}{s+\lambda_1} + \frac{C_2 U(s)}{s+\lambda_2} + \frac{C_3 U(s)}{s+\lambda_3}$$

where,

$$b_0 = 0;\ C_1 = 3;$$

$$C_2 = -60;\quad C_3 = 3;$$

$$\boldsymbol{\lambda}_1 = -1;\quad \boldsymbol{\lambda}_2 = -2;\quad \boldsymbol{\lambda}_3 = -3;$$

From Case I of Section 12.3.3, the diagonal canonical form of state model of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

or, $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

and, $$y = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

or, $$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the state variables are defined by

$$X_1(s) = \frac{U(s)}{s-\lambda_1} = \frac{U(s)}{s+1}$$

$$X_2(s) = \frac{U(s)}{s-\lambda_2} = \frac{U(s)}{s+2}$$

$$X_3(s) = \frac{U(s)}{s-\lambda_3} = \frac{U(s)}{s+3}$$

Example 12.6 Find the state space model of the following transfer function using the Jordan canonical form.

$$\frac{Y(s)}{U(s)} = \frac{sa^2 + 6s + 8}{(s+1)^2 + (s+3)}$$

Solution

From the given transfer function we get

$$Y(s) = b_0 U(s) + \frac{C_1 U(s)}{(s-\lambda_2)^2} + \frac{C_2 U(s)}{s-\lambda_2} + \frac{C_3 U(s)}{s-\lambda_3}$$

where,

$$b_0 = 0; C_1, = 1.5;$$

$$C_2 = 1.25; C_3 = -0.25;$$

$$\boldsymbol{\lambda}_2 = -1; \boldsymbol{\lambda}_3 = -\ 3$$

From Case II of Section 12.3.3 we may write the state model in Jordan canonical form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

$$= \begin{bmatrix} 1.5 & 1.25 & -0.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here the state variables are defined as

$$X_1(s) = \frac{U(s)}{(s-\lambda_1)^2} = \frac{U(s)}{(s+1)^2}$$

$$X_2(s) = \frac{U(s)}{s-\lambda_2} = \frac{U(s)}{s+1}$$

$$X_3(s) = \frac{U(s)}{s-\lambda_3} = \frac{U(s)}{s+3}.$$

Example 12.7 Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function of the system.

Solution

The given system may be written as

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$y = \boldsymbol{Cx} + du$$

where, $$A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and $$d = 0$$

$\therefore$ $$(s\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$$

Thus the adjoint of matrix **($s\boldsymbol{I}$–$\boldsymbol{A}$)** is given by

Adj $$(s\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

Also, $$|s\boldsymbol{I}-\boldsymbol{A}| = (s+4)(s+1) + 3$$

$$= s^2 + 5s + 7$$

We may write the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\boldsymbol{C}[adj(s\boldsymbol{I} - \boldsymbol{A})]\boldsymbol{B}}{|(s\boldsymbol{I} - \boldsymbol{A})|} + d$$

Now, C [adj $(s\boldsymbol{I} - \boldsymbol{A})]\boldsymbol{B}$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ s+7 \end{bmatrix} = s$$

$$\frac{Y(s)}{U(s)} = \frac{s}{s^2 + 5s + 7}$$

Example 12.8 The state space model of a system is given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Derive a suitable transformation by which another state space model of the same system can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Solution

Let $\boldsymbol{x} = \boldsymbol{Pz}$ be the required transformation, where

$$\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}$$

The given state model can be written as

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

where, $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

and, $$B = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix}$$

and, $$y = \boldsymbol{Cx}$$

where $$\boldsymbol{C} = [1 \quad 0 \quad 0].$$

Applying the above transformation, another state model can be written as

$$\dot{z} = (\boldsymbol{P}^{-1}\boldsymbol{AP})z + (\boldsymbol{P}^{-1}\boldsymbol{B})u$$
$$y = (\boldsymbol{CP})\boldsymbol{z}$$

$\therefore$ $$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{CP} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $$\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix}$$

or, $$\begin{bmatrix} p_1 \\ p_4 \\ p_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix}$$

Again, $$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $$p_1 = p_3 = 0; \qquad p_2 = p_4 = 1$$

and $$p_7 = -14$$

Again, $$\boldsymbol{AP} = \boldsymbol{P}\begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

or, $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}\begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

or, $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & p_5 & p_6 \\ -14 & p_8 & p_9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & p_5 & p_6 \\ -14 & p_8 & p_9 \end{bmatrix}\begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

or, $$\begin{bmatrix} 1 & p_5 & p_6 \\ -14 & p_8 & p_9 \\ 140 & (-160 - 56p_6 - 14p_8) & (-56p_6 - 14p_9) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ (p_5 - 17) & (p_6 - 56) & -160 \\ (p_8 + 196) & p_9 + 784 & 2240 \end{bmatrix}$$

Comparing elements of both the matrices, we get

$$p_5 = p_6 = 0$$
$$p_9 = -160$$
$$p_8 = p_6 - 56 = -56$$

Thus the required transformation is as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

or,
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -14 & -56 & -160 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Example 12.9 The state model of a system is given by

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$
$$y = \boldsymbol{Cx}$$

where,
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

and
$$\boldsymbol{C} = [1 \quad 0 \quad 0]$$

Obtain a diagonal canonical form of the state model by a suitable transformation matrix.

Solution

The characteristic equation of the matrix '$\boldsymbol{A}$' is given by

or,
$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = 0$$

or,
$$\boldsymbol{\lambda}^3 + 6\boldsymbol{\lambda}^2 + 11\boldsymbol{\lambda} + 6 = 0$$

or,
$$(\boldsymbol{\lambda} + 1)(\boldsymbol{\lambda} + 2)(\boldsymbol{\lambda} + 3) = 0$$

$\therefore$ $\boldsymbol{\lambda}_1 = -1, \therefore \boldsymbol{\lambda}_2 = -2$ and $\boldsymbol{\lambda}_3 = -3$ are the three distinct eigenvalues of matrix '$\boldsymbol{A}$'**.** So the Vander Monde transformation matrix can be written as

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\therefore \qquad P^{-1} = \frac{1}{|P|}(adj\ P) = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}$$

$$\therefore \qquad \boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{P}^{-1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 4 & 4 & 9 \end{vmatrix}$$

$$= \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -2 & -3 \\ 1 & 4 & 9 \\ -1 & -8 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$\boldsymbol{CP} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Applying the transformation $\boldsymbol{x} = \boldsymbol{Pz}$ we get the diagonal canonical form of the state model as

$$\dot{z} = (\boldsymbol{P}^{-1}\boldsymbol{AP})z + (\boldsymbol{P}^{-1}\boldsymbol{B})u$$
$$y = (\boldsymbol{CP})\boldsymbol{z}$$

or,
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Example 12.10 A system is given by the following vector-matrix equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

where the initial condition is given by $x(0) = [1 \quad 1]^T$

Determine a) State transition matrix, b) Zero input response, c) Zero state response for $u = 0$, d) Total response, and e) Inverse of state transition matrix.

Solution

The given state equation may be written as

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

where,
$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) State transition matrix $\boldsymbol{\Phi}(t)$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix}$$

$$|s\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} s & -1 \\ 4 & s+5 \end{vmatrix} = s^2 + 5s + 4$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{adj(s\boldsymbol{I} - \boldsymbol{A})}{|s\boldsymbol{I} - \boldsymbol{A}|}$$

$$= \frac{1}{s^2 + 5s + 4} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+5}{s^2+5s+4} & \dfrac{1}{s^2+5s+4} \\ \dfrac{-4}{s^2+5s+4} & \dfrac{s}{s^2+5s+4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4}{3(s+1)} - \dfrac{1}{3(s+4)} & \dfrac{1}{3(s+1)} - \dfrac{1}{3(s+4)} \\ \dfrac{-4}{3(s+1)} + \dfrac{4}{3(s+4)} & \dfrac{-1}{3(s+1)} + \dfrac{4}{3(s+4)} \end{bmatrix}$$

From equation (12.45) $\Phi(t)$ is given by $\Phi(t) = e^{At} = L^{-1}\,[(s\boldsymbol{I} - \boldsymbol{A})^{-1}]$

$$= \begin{bmatrix} \left(\dfrac{4}{3}e^{-t} - \dfrac{1}{3}e^{-4t}\right) & \left(\dfrac{1}{3}e^{-t} - \dfrac{1}{3}e^{-4t}\right) \\ \left(-\dfrac{4}{3}e^{-t} + \dfrac{4}{3}e^{-4t}\right) & \left(-\dfrac{1}{3}e^{-t} + \dfrac{4}{3}e^{-4t}\right) \end{bmatrix}$$

b) Zero input response is obtained from (12.47) by putting $u(\tau) = 0$ and letting it be x_1.

$\therefore \quad x_1 = e^{At}x(0) = \Phi(t)\, x(0)$

$$= \begin{bmatrix} \frac{1}{3}(4e^{-t} - e^{4t}) & \frac{1}{3}(e^{-t} - e^{4t}) \\ \frac{4}{3}(-e^{-t} + e^{-4t}) & \frac{1}{3}(-e^{-t} + 4e^{-4t}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(5e^{-t} - 2e^{-4t}) \\ \frac{1}{3}(-5e^{-t} + 8e^{-4t}) \end{bmatrix}$$

c) Zero state response for $u = 1$ is obtained from equation (12.47) by putting $x(0)$ and letting it be x_2.

$\therefore \quad x_2 = \int_0^t e^{A(t-\tau)} \boldsymbol{B} u(\tau) d\tau$

$$= \int_0^t \begin{bmatrix} \frac{1}{3}\left(4e^{-(t-\tau)} - e^{-4(t-\tau)}\right) & \frac{1}{3}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right) \\ \frac{4}{3}\left(-e^{-(t-\tau)} + e^{-4(t-\tau)}\right) & \frac{1}{3}\left(-e^{-(t-\tau)} + 4e^{-4(t-\tau)}\right) \end{bmatrix} \boldsymbol{B} d\tau$$

$$= \int_0^t \begin{bmatrix} \frac{1}{3}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right) \\ \frac{1}{3}\left(e^{-(t-\tau)} + 4e^{-4(t-\tau)}\right) \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \int_0^t \frac{1}{3}\left(e^{-t+\tau} - e^{-4t+4\tau}\right) d\tau \\ \int_0^t \frac{1}{3}\left(e^{-t+\tau} + 4e^{-4t+4\tau}\right) d\tau \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{1}{4}\right) - \left(e^{-t} - \frac{1}{4}e^{-4t}\right) \\ (-1+1) - (-e^{-t} + e^{-4t}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} - e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-t} - e^{-4t} \end{bmatrix}$$

d) Total response is given by

$$x(t) = x_1 + x_2$$

$$= \begin{bmatrix} \frac{1}{3}\left(5e^{-t} - 2e^{-4t}\right) \\ \frac{1}{3}\left(-5e^{-t} + 8e^{-4t}\right) \end{bmatrix} + \begin{bmatrix} \frac{3}{4} - e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-t} - e^{-4t} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{2}{3}e^{-t} - \frac{5}{12}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{5}{3}e^{-4t} \end{bmatrix}$$

$$\therefore \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{2}{3}e^{-t} - \frac{5}{12}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{5}{3}e^{-4t} \end{bmatrix}$$

e) Inverse of $\Phi(t)$ as shown below.

$$\Phi^{-1}(t) = e^{-At} = \Phi(-t)$$

$$= \begin{bmatrix} \frac{1}{3}(4e^{t} - e^{4t}) & \frac{1}{3}(e^{t} - e^{4t}) \\ \frac{4}{3}(e^{4t} - e^{t}) & \frac{1}{3}(4e^{4t} - e^{t}) \end{bmatrix}$$

Example 12.11 Consider the state matrix 'A' given by

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

If it is not possible to diagonalise matrix 'A' determine matrix P so that $P^{-1}AP$ is in the Jordan canonical form.

Solution

The characteristic equation of matrix 'A' is

$$|\lambda I - A| = 0$$

or,
$$\begin{vmatrix} \lambda - 4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda - 3 \end{vmatrix} = 0$$

or,
$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

or,
$$(\lambda - 1)(\lambda - 3)^2 = 0$$

So matrix has eigenvalues of $\lambda_1 = 3$ of order 2 and $\lambda_3 = 1$.
Using equation (12.41), the column vectors p_1 and p_2 associated with $\lambda_1 = 3$ of order 2 are obtained from

$$(\lambda_1 I - A)p_1 = 0 \qquad \text{....(12.80)}$$

and
$$(\lambda_1 I - A)p_2 = -p_1 \qquad \text{....(12.81)}$$

From equation (12.80) we get

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By elementary row transformation this equation can be reduced to

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$p_{13} = p_{12} = p_{11} = 1 \ (say)$$

From equation (12.81), we get (as any constant can be assumed)

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

By elementary row transformation this equation can be reduced to

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$p_{23} = p_{22} = 1 \text{ (say)}$$

and $$p_{21} = p_{23} + 1 = 2.$$

Using equation (12.42), the column vector p_3 associated with $\lambda_3 = 1$ is obtained from

$$(\lambda_3 I - A)p_3 = 0$$

or, $$\begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By elementary row transformation, this equation can be reduced to

$$\begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming $p_{33} = 1$, we get

$$p_{31} = 0 \text{ and } p_{32} = 2\,p_{33} = 2$$

So matrix 'P' can be written as follows

$$P = \begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,
$$P^{-1} = \frac{1}{|P|} Adj\ A = \begin{bmatrix} -1 & -2 & 4 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$\therefore$
$$P^{-1}AP = \begin{bmatrix} -1 & -2 & 4 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} P$$

$$= \begin{bmatrix} -2 & -5 & 10 \\ 3 & 3 & -6 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in the Jordan canonical form.

Example 12.12 Find the state transition matrix using Cayley-Hamilton theorem for the state matrix 'A' given below.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Solution

The characteristic equation of state matrix 'A' is

$$q(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{vmatrix}$$

$$= (\lambda + 1)^2 = 0$$

Since 'A' is of order 2, the remainder polynomial will be of the form given below.

$$R(\boldsymbol{\lambda}) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_0 \boldsymbol{\lambda} \qquad \text{....(12.82)}$$

Since $\boldsymbol{\lambda}_1, = -1$ is the eigenvalue of order 2, the only one independent equation obtained by putting $i = 1$ into equation (12.54) is

$$e^{\lambda_2 t} = \alpha_0 + \alpha_1 \lambda_1 \qquad \text{....(12.83)}$$

or,
$$e^{-t} = \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 \qquad \text{....(12.84)}$$

The other equation is obtained by differentiating both sides of equation (12.83) with respect to $\boldsymbol{\lambda}_1$ as follows.

$$te^{\lambda_1 t} = \alpha_1$$

or,
$$te^{-t} = \boldsymbol{\alpha}_1 \qquad \text{....(12.85)}$$

Solving equations (12.84) and (12.85), we get,

$$\boldsymbol{\alpha}_0 = (1 + t)e^{t} \text{ and } \boldsymbol{\alpha}_1 = te^{-t}$$

Using formula (12.58) we may write the state transition matrix ϕ(t) as given below.

$$\phi(t) = e^{At} = R(A)$$

$$\phi(t) = \alpha_0 I + \alpha_1 A$$

$$= \begin{bmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 2\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Example 12.13 Consider the system defined by

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$y = \boldsymbol{Cx}$$

where,
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 10 & 5 & 1 \end{bmatrix}$$

Check the system for (a) complete state controllability and (b) complete observability.

Solution

a) Test for complete state controllability

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

So the controllability matrix Q_c is given by

$$Q_c = \begin{bmatrix} B: & AB: & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

Now, $$|Q_C| = -84 \neq 0$$

So the rank of matrix Q_c is equal to its order, that is, 3. This indicates that according to Kalman's test the system is completely state controllable.

b) Test for complete observability

$$C^* = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$A^*C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$(A^*)^2C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

So the observability matrix Q_0 is given by

$$Q_0 = [C^* : A^*C^* : (A^*)^2C^*]$$

$$= \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

Now, $$|Q_0| = 96 \neq 0$$

So, the rank of matrix Q_0 is equal to its order, that is, 3. This indicates that due to Kalman the system is completely observable.

REVIEW QUESTIONS

12.1 Develop a state variable model for the mechanical system shown below. The displacements y_1 and y_2 represent deviations from equilibrium positions when $u_1 = u_2 = 0$.

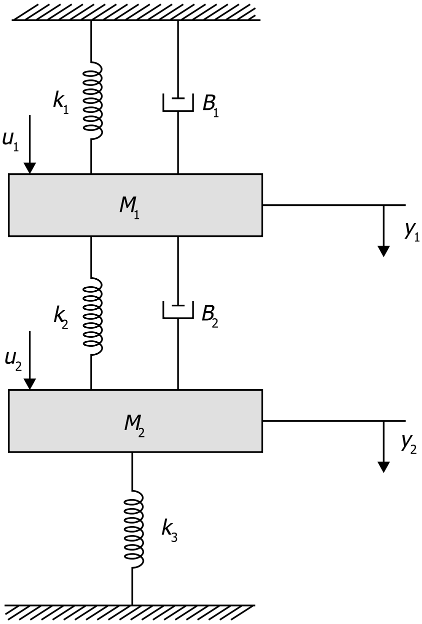


Fig. 12.8

12.2 Draw the state diagram for the following model in which the state equations are

$$\dot{x}_1 = -4x_1 + 2x_2 + 4u_1$$
$$\dot{x}_2 = -3x_2 + 5u_2$$

and the output equations are

$$y_1 = x_1 + x_2 + 3u_1$$
$$y_2 = x_2$$

Assume zero initial conditions.

12.3 Derive a state space representation of the following systems given by

i) $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3 + 6s^2 + 4s + 6}$.

ii) $\dddot{y} + 3\ddot{y} + 2\dot{y} = u$.

12.4 Determine the transfer function of the system which is represented by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boldsymbol{u} \ \text{ and } \ y = x_1.$$

12.5 For the system with transfer function

$$\frac{Y(s)}{U(s)} = \frac{s+6}{s^2+5s+6}$$

obtain the state space representation in (a) controllable canonical form and (b) observable canonical form.

12.6 Consider the following state model of a system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Determine a suitable transformation matrix that will transform the state equation in the form given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Then obtain y in terms of z_1, z_2 and z_3

12.7 Consider the following state space model of a linear system.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -6 & 4 \\ -2 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{u} \text{ and } y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x$$

Find the complete solution for $y(t)$ when $x_1(0) = 1$, $x_2(0) = 0$ and u is a unit step function.

12.8 Using the Cayley-Hamilton theorem, find the state transition matrix for the following state equations.

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

12.9 Consider the system defined by

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$y = \boldsymbol{Cx}$$

where, $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is the system completely state controllable, completely observable and completely output controllable?

12.10 Transfer function of a system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{s+3}{(s+1)(s+2)}$$

Represent the system by a state space model.

13

CONTROL COMPONENTS

13.1 INTRODUCTION

In any control system, a number of components are used to perform specific control functions. These control components are potentiometers and synchros, servo motors, gear trains, tachogenerators, magnetic amplifiers, stepper motors, various types of transducers and so on. These are described in brief in this chapter.

13.2 ERROR DETECTORS—POTENTIOMETERS AND SYNCHROS

In control systems, as you know, error detectors are used to sense and measure any error, which may occur due to the difference between the actual output and the desired output. The input level is held constant as reference input corresponding to the magnitude of the desired output. If there is any deviation in the actual output from the desired output, an error signal is created. Potentiometers and synchros are used as error detectors and are explained in the following sections.

13.2.1 Potentiometer Error Detector

As mentioned earlier, error detectors detect the difference between the desired output and actual output and produce a voltage signal proportional to the difference, that is, the error. A potentiometer error detector consists of two similar potentiometers connected in parallel and supplied from a voltage source as shown in Fig. 13.1.

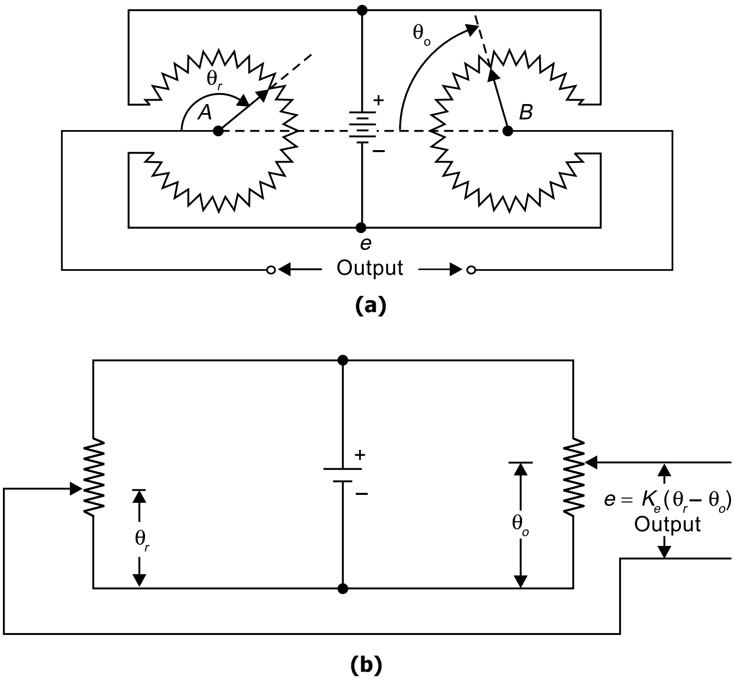


Fig. 13.1 *Potentiometer error detector*

Fig. 13.1(a) shows the input shaft of the control system connected to the potentiometer marked A and its setting fixed at a desired angular position of θ_r. The output shaft is shown making an angle of θ_o. Fig. 13.1(b), shows the arrangement of Fig. 13.1(a) in a circuit diagram. Here, if angle θ_o equals θ_r, then their difference, that is, the error, is zero. However, if there is some error in the system response, then the difference of θ_o and θ_r, which is proportional to an electrical potential, e, can be expressed as

$$e \, \alpha(\theta_r - \theta_o)$$

or

$$e = K_e(\theta_r - \theta_o) = K_e\theta_e$$

where K_e is the gain of the error detector and expressed as V/rad and θ_e is the angular error $(\theta_r - \theta_o)$.

The Laplace transform of the equation is

$$\begin{aligned} E(s) &= K_e[\theta_r(s) - \theta_o(s)] \\ &= K_e\theta_e(s) \end{aligned}$$

The block diagram representation is shown in Fig. 13.2.

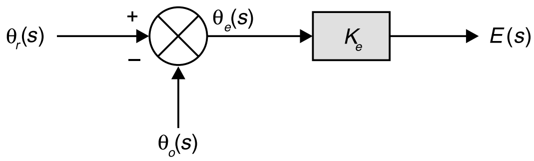


Fig. 13.2 *Block diagram representation of a potentiometer-type error detector*

13.2.2 Synchro Transmitter and Synchro Control Transformer

We have seen earlier how in a potentiometer-type error detector, angular difference of the alignment of two shafts is converted into a voltage signal which is used for correcting the the system output to achieve minimal error.

A synchro transmitter is similar to a three-phase alternator. The stator carries three-phase windings connected in star. The rotor is provided with a single-phase supply through brush and slip-ring arrangement. The rotor is of salient type. A synchro control transformer is also similar to a synchro transmitter except that the rotor is cylindrical type.

A synchro transmitter and a synchro control transformer are used together as an error detector.

In a synchro error detector, the angular difference of shaft positions is converted into a proportional a.c. voltage using a synchro transmitter at the input and a synchro control transformer at the output. Fig. 13.3 shows the arrangement of a synchro error detector.

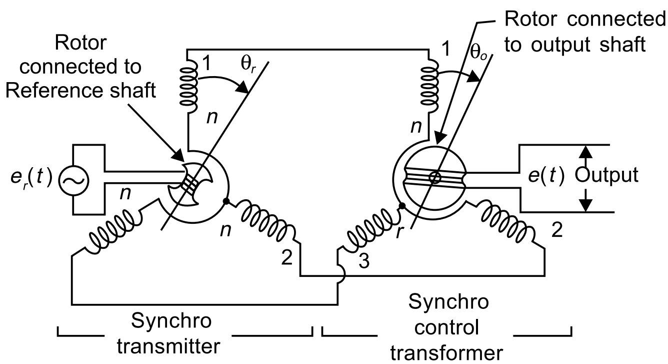


Fig. 13.3 *Connection diagram of a synchro (selsyn error detector)*

The rotor winding of the synchro transmitter is connected across an a.c. voltage source. The rotor shaft is held fixed at a desired angular position θ_r (called input or reference). The three-stator windings of the synchro transmitter are 120° apart. Similarly, the three-stator windings of the synchro control transformer are wound 120° apart. These three sets of windings are connected in parallel as shown in Fig. 13.3. It may be noted that the rotor of the synchro transmitter is of the projected (salient) type whereas the rotor of the synchro control transformer is of the cylindrical type. It may be mentioned here that both in the case of potentiometer error detector and in synchro error detector, the physical location of the input and output shafts and the corresponding potentiometers or the rotors may be at a distance (as in the case of remote position control). The angular position of the input and output shafts have been shown by θ_r and θ_o, respectively. The error $\theta_e = \theta_r - \theta_o$.

This angular displacement is converted into a proportional voltage signal through electromagnetic action. When a single-phase voltage, say $e_r(t) = e_m \sin \omega t$, is applied across the winding terminals of the rotor of the synchro transmitter, three emfs are induced in the three-stator windings and the magnitude of the induced emfs depends on the rotor position. In Fig. 13.3, the rotor is shown making an angle θ_r with the axis of winding, "1 – *n*". Since the three-stator windings of the synchro transmitter are connected respectively in parallel with the three-stator windings of the control transformer, the stator winding induced voltage of the synchro transmitter would cause the current to flow and produce a resultant flux in the air-gap of the synchro control transformer, which in turn will induce an emf in the rotor winding. The magnitude of the induced emf would be proportional to (θ_r) and (θ_o). If the difference $\theta_r - \theta_o$ is zero, the emf induced will be zero. At 90°, the induced emf will be maximum. In fact, the magnitude of the induced emf at the output, that is, across the rotor terminals, will have a relation,

$$e = k_s \sin(\theta_r - \theta_o)$$

where k_s is a constant of proportionality.

Normally, error $\theta_r - \theta_o$ is small and hence $\sin(\theta_r - \theta_o)$ can be equated as $(\theta_r - \theta_o)$. Hence,

$$e = K_s(\theta_r - \theta_o)$$
$$= K_s \theta_e$$

Taking Laplace transform,

$$E(s) = K_s[\theta_r(s) - \theta_o(s)]$$

or

$$E(s) = K_s \theta_e(s) \qquad [K_s \text{ is expressed in V/rad}]$$

Fig. 13.4 shows the block diagram representation of a synchro error detector.

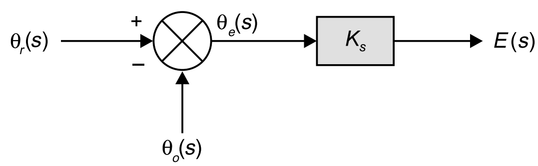


Fig. 13.4 *Block diagram representation of a synchro error detector*

We can express K_s as

$$K_s = \frac{E(s)}{\theta_e(s)} = \frac{E(s)}{\theta_r(s) - \theta_o(s)}$$

K_s is known as the sensitivity or the gain of the error detector.

13.3 TACHOGENERATORS

Tachogenerators are small generators mounted on the shaft of a rotating system. They convert rotational speed into proportional voltage. That voltage can be compared with the reference input (set) voltage and system error (in this case, in terms of the difference between actual speed and desired speed) can be detected and corrective action taken. There are two types of tachogenerators—d.c. tachogenerators and a.c. tachogenerators.

13.3.1 D.C. Tachogenerator

A d.c. tachogenerator converts rotational speed into proportional d.c. voltage. It is a permanent magnet type small d.c. generator. Such a tachogenerator is mounted on the same shaft of the rotating system. As the shaft rotates, emf is induced in the rotor which can be expressed as

$$e = K\phi\omega$$

where k is constant, ϕ is the field flux and ω is the angular speed. Since, permanent magnets are used for the field system, induced emf, e, becomes proportional to the speed of rotation only.

That is, $e \propto \omega$ or $e = K_{tg}\omega$ where K_{tg} is a tachogenerator constant expressed in V-sec/rad.

Taking Laplace transform, $E(s) = K_{tg}\omega(s)$.

The block diagram is shown in Fig. 13.5.

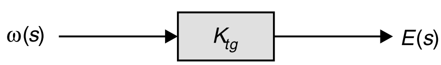


Fig. 13.5 Block diagram of a d.c. tachogenerator

A d.c. tachogenerator can also be used for an a.c. system by using a modulator to convert d.c. voltage signal into an a.c. voltage signal.

13.3.2 A.C. Tachogenerator

An a.c. tachogenerator converts rotational speed into proportional a.c. voltage. Here, the stator has two separate windings wound at right angles as shown in Fig. 13.6. The rotor is a thin aluminium disc which is free to rotate in an air-gap between a fixed magnetic field structure. Since the rotor is made of conducting material, it works like a short-circuited secondary winding. A reference alternating voltage is applied to the stator reference winding. The emf induced in the quadrature coil is proportional to the rotor speed and is in phase with the voltage applied to the reference coil.

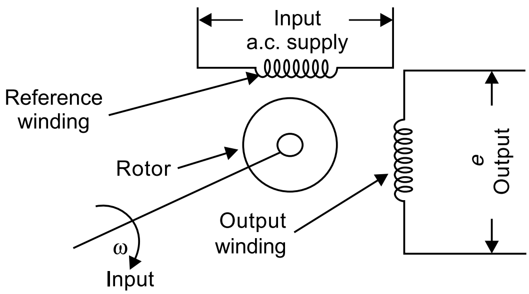


Fig. 13.6 *An a.c. tachogenerator*

The emf available at the terminals of the quadrature coil can be expressed as

$$e \propto \omega \quad \text{or, } e = k_{tg}\omega$$

where k_{tg} is the tachometer constant and ω is the angular speed of rotor.

Taking Laplace transform,

$$E(s) = K_{tg}\omega(s)$$

or

$$K_{tg} = \frac{E(s)}{\omega(s)}$$

The block diagram has been shown in Fig. 13.7.

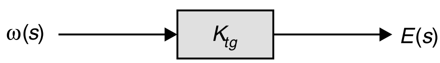


Fig. 13.7 *Block diagram of an a.c. tachogenerator*

13.4 SERVO MOTORS AND GEAR TRAINS

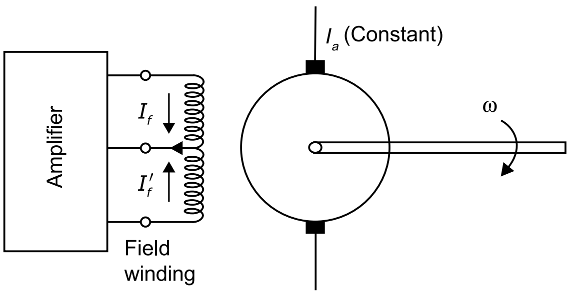
In control systems, a servo motor is used to convert the final control element into mechanical displacement, velocity, torque, etc. as the desired output. Servo motors can be either a.c. or a.c. type. Gear trains are used to reduce speed as per load requirements.

13.4.1 D.C. Servo Motors

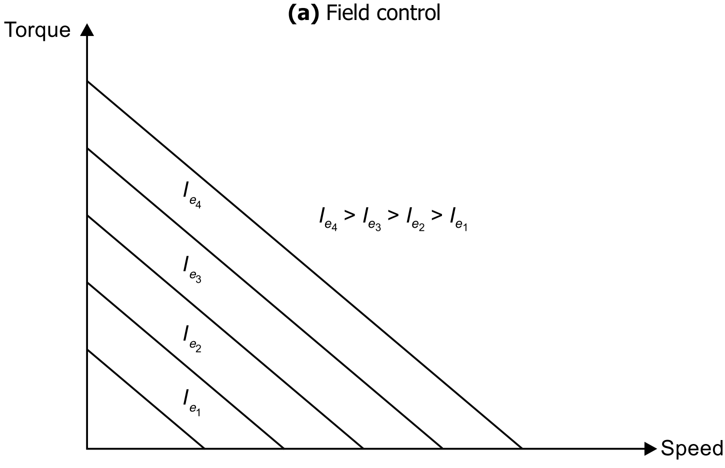
A d.c. servo motor is a specially designed d.c. motor with high starting torque and low inertia. They are used in systems where a high starting torque and wide speed range may be required to operate in either direction of rotation. The d.c. machines are used in control systems because of the ease with which their speed can be controlled.

In d.c. servo motors, the field winding is supplied from a constant voltage source, while the armature circuit is used for controlling the speed. The torque developed is given by $T = kI_f I_a$, where k is the motor constant, I_f is the field current, and I_a is the armature current. A split field motor is a separately excited d.c. motor in which the field winding is centrally tapped, with the two halves wound in opposite directions. The flux set up in the field winding is proportional to the effective current I_e which is the difference between the currents in each half of the field winding. When currents are equal, effective flux is zero and no torque will be developed. An unbalanced current is used to control the speed and the direction of the motor shaft rotation.

In Fig. 13.8, the armature circuit is fed from a constant current source and field current is supplied by an amplifier, a push–pull type. As the field current is small, it is necessary to have a large number of turns in the field windings.



(a) Field control



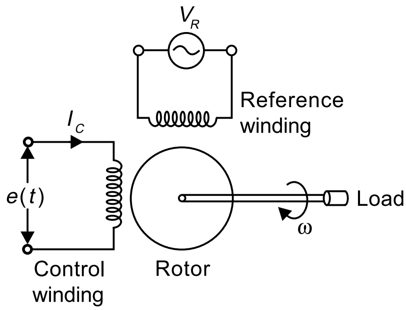
(b) Torque speed characteristics

Fig. 13.8 *A d.c. servo motor and its typical torque/speed characteristics for various values of effective current*

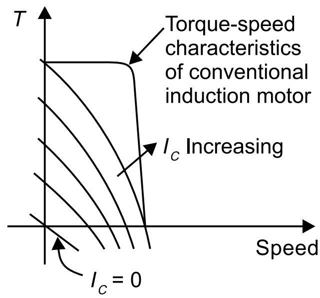
The d.c. servo motors find extensive applications in low power servo systems. The derivation of the transfer function of such a motor has already been discussed in Chapter 3.

13.4.2 A.C. Servo Motors

The a.c. servo motors are two-phase induction motors suitable for simple and low power applications. Fig. 13.9 shows the stator having two field windings placed at right angles to each other in order to produce a rotating field on which the motor action depends. One phase is supplied from a constant a.c. reference voltage V_R, the other phase acts as the controlling field and is supplied from the output of the servo amplifier. The speed of rotation is proportional to the control current I_c, the phase of which determines the direction of rotation. The torque/speed characteristics are shown in Fig. 13.9(b).



(a) Two-phase motor



(b) Torque speed characteristics of servo motor

Fig. 13.9 *Torque/speed characteristics of an a.c. servo motor*

It can be seen that the curve for zero control current goes through the origin and the slope is negative, that is, when the control current becomes zero, the motor develops a decelerating torque, causing it to stop. The curve also shows a large torque at zero speed at increased I_c.

For normal induction motor, the ratio of X_2/R_2 is high. But for servomotors X_2/R_2 ratio is kept low to obtain linear torque-speed characteristics. High value of R_2 will ensure high starting torque.

Let E be the rms value of sinusoidal control voltage $e(t)$. The torque (T_M) generated by the motor is a function of angular speed $(\dot{\theta})$ and rms control voltage (E).

That is,

$$T_M = f(\theta, E)$$

Applying Taylor's series expansion about the normal operating point (T_{M0}, E_o, $\dot{\theta}_o$) and neglecting terms involving second and higher order derivative, we get

$$T_M = T_{M0} + K(E - E_0) - f(\dot{\theta} - \theta_o)$$

where

$$K = \left.\frac{\partial T_M}{\partial E}\right|_{\substack{E=E_o \\ \dot{\theta}=\dot{\theta}_o}}$$

and

$$f = -\left.\frac{\partial T_M}{\partial \dot{\theta}}\right|_{\substack{E=E_o \\ \dot{\theta}=\dot{\theta}_o}} \text{ are constants}$$

or

$$T_M - T_{Mo} = K(E - E_0) - f(\dot{\theta} - \dot{\theta}_o)$$

or

$$\Delta T_M = K\Delta E - f\Delta\dot{\theta}$$

where

$$\Delta T_M = (T_M - T_{Mo}) \text{ and so on.}$$

If J and f_0 be the inertia and coefficient of viscous friction, respectively, of the load, the torque equation becomes

$$\Delta T_M = J\Delta\dot{\theta} + f_0\Delta\dot{\theta} = K\Delta E - f\Delta\dot{\theta}$$

Taking Laplace transform, we get

$$(Js^2 + f_0 s)\Delta\theta(s) = K\Sigma E(s) - fs\Delta\theta(s)$$

When an a.c. servomotor is used in a position control system, the operating point becomes $E_0 = 0$ and $\dot{\theta}_o = 0$ and hence $\Delta\theta = \theta$ and $\Delta E = E$.

So the motor transfer function now becomes

$$G(s) = \frac{\theta(s)}{E(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

where

$$K_m = \frac{K}{f_0 + f} \text{ motor gain constant}$$

and

$$\tau_m = \frac{J}{f_0 + f} \text{ motor time constant}$$

13.4.3 Gear Trains

Control systems with rotating elements have a system of gearing because it is economical to design a servo motor which would run at a higher speed than is required by the load. Further, it permits a higher load acceleration for a given motor (Fig. 13.10).

A gearbox is a mechanical system which reduces or steps down the speed of the motor to a lower value as required by the load, similar to a step-down transformer where voltage level is reduced as per requirement. Thus, for ease of calculation, quantities such as inertia can be 'referred' to the load side.

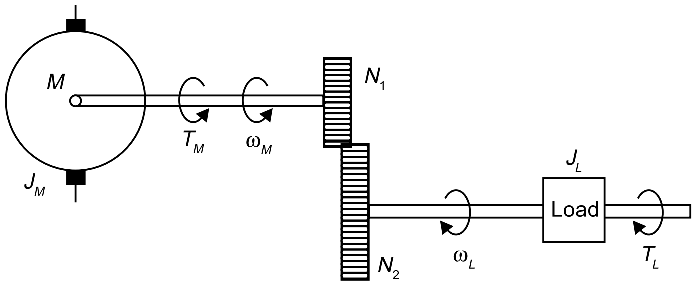


Fig. 13.10 *Gear train engaged between the driving motor and the driven load*

J_m, J_L are the inertia of motor and load; T_m, T_L are, respectively, the torque developed by motor and torque required by load; N_1 and N_2 are the number of tooths of the gears; ω_m and ω_L are the angular speed of motor and load as has been shown in Fig. 13.10.

For an ideal system, we can equate mechanical power developed by the motor and the mechanical power output of the load as

$$\omega_m T_m = \omega_L T_L$$

or

$$T_L = \frac{\omega_m}{\omega_L} \times T_m$$

Gear ratio, angular speed and angular acceleration $(\alpha_L$ and $\alpha_m)$ are related as

$$\frac{N_1}{N_2} = \frac{\omega_L}{\omega_m} = \frac{\alpha_L}{\alpha_m}$$

$$T_L = \frac{\omega_m}{\omega_L} T_m = \frac{N_2}{N_1} T_m$$

Again,

$$\alpha_L = \frac{T_L}{J_L} \text{ [since, torque} = J\alpha]$$

$$= \frac{\omega_m T_m}{\omega_L J_L} = \frac{N_2 T_m}{N_1 J_L}$$

$$\alpha_m = \frac{N_2}{N_1}\alpha_L = \frac{N_2}{N_1}\left(\frac{N_2}{N_1}\right)\frac{T_m}{J_L} = \left(\frac{N_2}{N_1}\right)^2 \frac{T_m}{J_L}$$

$$\alpha_m = \left(\frac{N_2}{N_1}\right)^2 \frac{T_m}{J_L}$$

$$\alpha_L = \left(\frac{N_2}{N_1}\right)\frac{T_m}{J_L}$$

$$T_n = J_n \alpha_m$$

$$T_L = J_L \alpha_L$$

$$J_L = \frac{T_L}{\alpha_L} = \frac{\omega_m}{\omega_L}\frac{T_m}{\alpha_L} = \frac{N_2}{N_1} \times \frac{T_m}{\alpha_L} = \frac{N_2}{N_1} \times \frac{\alpha_m J_m}{\alpha_L}$$

$$= \frac{N_2}{N_1}\frac{N_2}{N_1} J_m$$

$$J_L = \left(\frac{N_2}{N_1}\right)^2 J_m$$

Thus, J_m, the motor inertia, referred to load side is equal to $(N_2/N_1)^2 J_m$
Since Torque = Inertia $J \times$ Acceleration α,

$$\text{Load acceleration} = \frac{\left(\frac{N_2}{N_1}\right) T_m}{J_L + \left(\frac{N_2}{N_1}\right)^2 J_m}$$

For maximum load acceleration, the gear ratio will be

$$\frac{N_2}{N_1} = \sqrt{\frac{J_L}{J_m}}$$

13.5 TRANSDUCERS

The output of a control system represented in the block diagram form is measured in the form of position, speed, acceleration, force, pressure, temperature, volume, flow and so on.

The performance of the system is electrically controlled. As shown in Fig. 13.11, the measurement and feedback portions of the system take the feedback, whatever is its form,

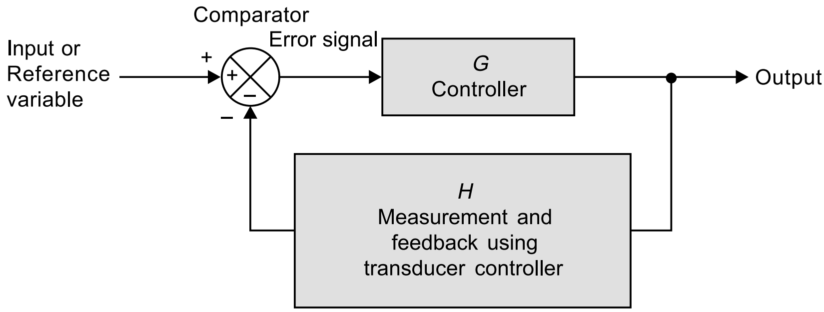


Fig. 13.11 *Block diagram of a feedback control system*

and convert it to an electrical signal so that it is easy to compare it with the reference variable. Transducers are devices which convert non-electrical physical quantities into electrical energy.

A transducer is a sensing device that converts physical phenomena and chemical composition into electric or hydraulic output signals. There are many types of transducers available for this purpose. For example, the speed of rotation of a shaft can be converted into an electrical signal by means of an electric tachometer (a tachogenerator); force, torque and stress can be converted into an electrical signal by means of a resistance strain-gauge. The choice of transducers depends on the application and circumstances. For example, a transducer in a space ship should be small, light and reliable. However, cost consideration in case of space ship application will be of less importance.

The following factors should be considered in the selection of a transducer. The relative importance of each function depends on the application.

(a) Accuracy
(b) Frequency response
(c) Range
(d) Sensitivity
(e) Resolution
(f) Reliability
(g) Linearity
(h) Size and weight
(i) Environmental effects
(j) Noise level
(k) Cost

Some of the transducers commonly in use are described below.

(1) Resistance Potentiometer: Linear resistance potentiometer (pot) is a resistance transducer in which the voltage obtained from the slider is directly proportional to the position of the slider. The pots are designed for a long working life, good resolution and low noise.

In wire-wound type potentiometers, the voltage at the slider changes in discrete steps as the slider moves from turn to turn.

The resolution of a pot is defined as the ratio of the voltage between adjacent turns of the wire to the input voltage. This is a measure of the degree to which small changes of the input can be discriminated by the potentiometer.

$$\text{Resolution} = \frac{\Delta V}{V} = \frac{\Delta V}{n \times \Delta V} = \frac{1}{n} \times 100\%$$

where ΔV is the voltage between adjacent turns, V is the total voltage across the pot and n is the number of turns of the potentiometer.

A pot can be calibrated and used for the measurement of angular displacement, for which a rotary pot can be used. In the wire-wound type, an insulated resistance wire is wound helically around an insulated bar and then bent into a circular shape. The pot is supplied from a constant voltage V_i. The wiper is connected to the rotating shaft and provides a voltage proportional to the angular position θ of the shaft.

(2) Thermocouples: The principle of operation of a thermocouple is based on the fact that when two dissimilar metals are joined at both ends, an electric current will flow if the junctions are at different temperatures. The current flow is due to an induced emf which is proportional to the difference in temperatures.

The cold junction is used as the reference junction and is kept at a constant temperature. Tungsten, platinum, rhodium, copper and iron are some of the metals used in thermocouples. For high accuracy measurements, they need constant reference temperature and accurate measuring apparatus. These are used over a wide range of temperatures. Thermocouples are also used for measuring temperatures in nuclear reactors.

(3) Strain gauges: The principle of operation of a resistance strain gauge is based on the variation of resistance of an element when its dimensions are altered due to a stress or a force.

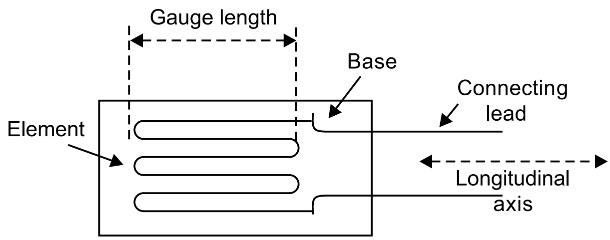
Bonded resistance strain gauges may be of the wire type, metal foil type, semi-conductor crystal type. However, all these operate on the same principle.

A bonded strain gauge consists of a thin piece of material (paper) on which a resistive element is fixed. The gauge is treated with adhesive on the back of it could be of self-adhesive type. When the gauge is bonded on the body under test and the body is subjected to stress, the gauge base will be elongated and the bonded element will be lengthened. There will be contraction along the axes at right angles to the direction of strain. The dimensional changes will alter the resistance of the element in proportion to the strain. The sensitivity factor or gauge factor k is the ratio of the change in resistance to the original resistance, to the change in length to the original length. It can be shown that

$$G = \frac{\Delta R/R}{\Delta \ell/\ell} = 1 + 2\mu + \frac{d\rho/\rho}{dl/l}$$

where ρ is the resistivity and μ is the Poisson's ratio of the resistive material. As $\mu = 0.3$ for most metals and $(d\rho/\rho)/(dl/l)$ is almost zero, the average value of gauge factor is 2 for most metals and metallic alloys. A gauge material with coefficient of expansion close to that of the body under test is chosen.

Resistance strain gauges are used in the measurements of static deformation and study of dynamic behaviour of systems. They are also used for measurement of strain in bone structures such as the human skull. Fig. 13.12 shows a resistance strain gauge.



(a) Wire type

Fig. 13.12 *Resistance strain gauge*

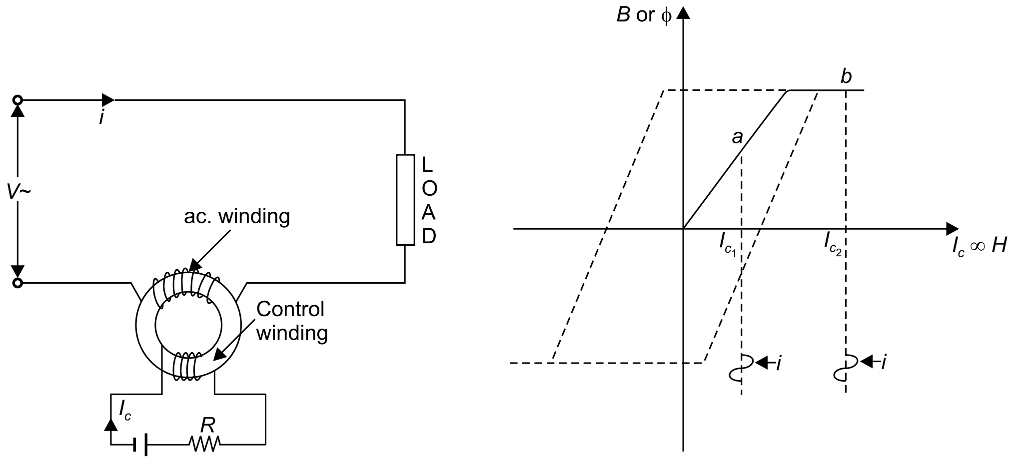
13.5.1 Magnetic Amplifier

Magnetic amplifier, called "mag amp" in short, is a static device which uses a saturable core having one d.c. winding and one a.c. winding. The B-H characteristic of the core material is approximately rectangular in shape.

Mag amps were invented in early twentieth century and were extensively used in control applications. However, they have now mostly been replaced with solid state amplifiers which are more compact and efficient than mag amps.

In Fig. 13.13(a), a simple saturable core and its B-H characteristic have been shown. On the core, two windings have been wound. A small amount of d.c. current flowing through the d.c. winding sets the point of operation of the core.

Figure 13.13(b) shows a d.c. current I_c flowing through the control winding. The a.c. winding has large number of turns. The load and the a.c. winding are connected in series. The a.c. supply voltage is V volts. By varying the control winding current, I_c, the voltage appearing across the load can be varied. The amount of control current through the control winding sets the level of saturation of the core. For example, when control current is I_{c_1}, the saturation level of the core will be at point 'a' and when the current is increased to I_{c_2}, the saturation level will be at point 'b' as shown in Fig. 13.13(b). The rate of change of flux ϕ with respect to current I at point 'a' is large as compared to such change at point 'b' due different level of saturation of the core by the control current. The inductance of the a.c. winding is $N\ d\phi/di$. The inductive reactance of the a.c. winding at point 'a' will be very high and at point 'b' will be very low. Thus, the a.c. winding connected in series with the load will go from a high impedance state to low impedance state as the control current is increased from a low value to a high value. The sum of voltage drop across the a.c. winding and the load is equal to the supply voltage V. The control current will be changing the voltage drop across the a.c. winding. Hence, the voltage drop across the load will change if the control current is changed. If the control current is low, inductive reactance of the a.c. winding will be high, the voltage drop across the winding will be high, and hence voltage appearing across the load will be low. As the control current is increased, the voltage across the load will increase. A relatively small change in the d.c. control current will change a large a.c. current flowing through the load resulting in amplifier action. The ability of a mag amp to control large load current with small control current made the device useful in control of lighting circuits such as stage lighting, lighting control of cinema halls, advertising signs, etc. Mag amps were also used in control of power in industrial furnaces, control of small motors, control of fan speed, etc. These applications have, however, been superseded by semiconductor-based solid-state devices.



(a) (b)

Fig. 13.13 *(a) Magnetic Amplifier circuit; (b) B–H characteristics of a mag amp*

However, magnetic amplifiers are still used in some welders, switch mode power supplies, measurement of high d.c. voltage in high voltage d.c. transmission system, etc.

13.5.2 Electronic Amplifiers

Operational amplifiers are basically high-gain direct coupled amplifiers with very high input impedance and very low output impedance. These amplifiers employ an odd number of stages so that their high gain, differential output and other properties are suitable for analog computers.

An inverting amplifier circuit or inverter consists of an op-amp and two resistors, an input resistor R_1 and a feedback resistor R_f as shown in Fig. 13.14(a). The node voltages are the input voltage V_i, the output voltage V_o and voltage V_1, all referred to ground. Therefore, $V_1 = 0$.

Output voltage is

$$V_o = \frac{-R_f}{R_1} V_i$$

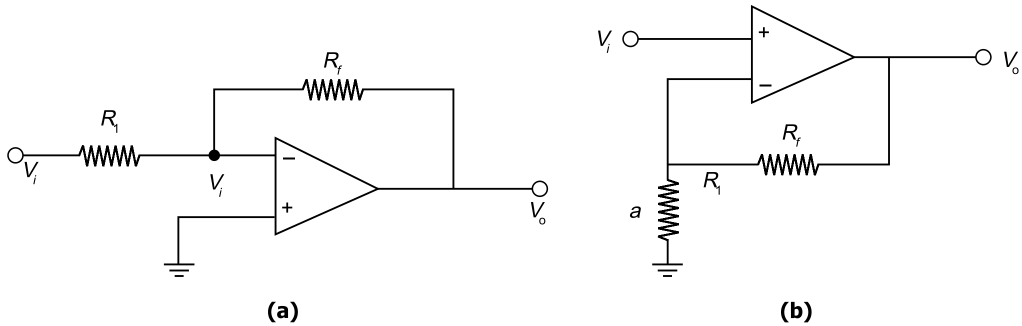


Fig. 13.14 *(a) Inverting amplifier; (b) Non-inverting amplifier*

Closed-loop gain is

$$A_f = \frac{-R_f}{R_1}$$

The negative sign indicates a sign reversal for a phase shift of 180°. The gain depends on the resistance ratio rather than on the value of each resistance.

A non-inverting amplifier circuit consists of an op-amp and two resistors, an input resistance R_i and feedback resistor R_f as shown in Fig. 13.14(b).

If the signal is applied to the non-inverting terminal and a feedback connection is made, the circuit will amplify without inverting the input signal. It is called a non-inverting amplifier. The output voltage equation is

$$V_o = \left(1 + \frac{R_f}{R_i}\right) V_i$$

The gain A_f $$= 1 + \frac{R_f}{R_i}$$

$$V_o = A_f V_i$$

Since the resistances are never negative, the gain is always 1 or more. The circuit is that of an amplifier where the output voltage has the same sign as the input voltage.

13.5.3 Rotary Amplifiers

High gain rotary amplifiers are cross field generators which can be used as either a constant-current or a constant-voltage source. A cross field generator is a separately excited d.c. generator driven by an induction motor. There are four brushes placed at right angles, two of which are short circuited. The output is taken from the other set of brushes.

The field current I_f sets off flux ϕ_f. Since the two brushes are short circuited, a large current I_a flows and sets up a large flux ϕ_a at right angles to the controlling flux. When the load current I_a flows, a reaction flux ϕ_L will be set up opposing flux ϕ_f (as per Lenz's law) without any load connected. The output voltage V_0 will be proportional to I_a and hence to I_f. This variation is similar to the magnetization curve of a d.c. machine but rotated through 90°.

When the load current flows, the effective ampere-turns will be $I_f N_f - I_L N_a$ where N_a is the effective armature turns. For a given field current, the load characteristics will be similar to the open circuit characteristics but shifted vertically, the values of current being dependent on the ampere turns. If the "effective" ampere-turns are small in the unsaturated range, the load current remains constant. This constant-current generator is known as the metadyne and is shown in Fig. 13.15.

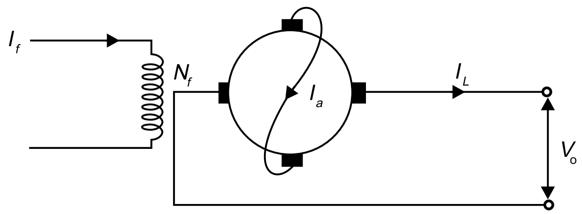


Fig. 13.15 A rotating amplifier

The effect of reaction flux can be reduced by including a set of compensating winding in the path of the load current, placed so as to aid the field flux ϕ_f.

Depending on the percentage cancellation of the fluxes, different characteristics will be obtained. A machine in which 100% compensation is used is called an amplidyne. In the unsaturated range, for a large variation in load current, the output voltage varies very little, thus providing constant voltage generation. An additional winding at right angles to the main field winding will provide a flux assisting the armature field.

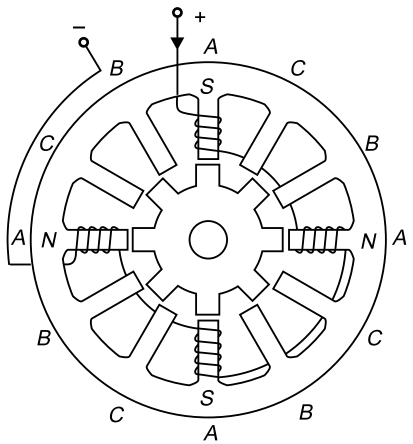
The ratio of output power to input power is a measure of the amplification. High power amplification can be achieved in rotating amplifiers.

13.6 STEPPER MOTORS

A stepper motor is a brushless d.c. motor, the rotor of which rotates in discrete angular increments (i.e., in steps of certain fraction of a revolution) when its stator windings are energized in a programmed manner. Torque is produced due to the magnetic interaction between the stator poles, which are sequentially energized by d.c. supply and the poles of the rotor. The rotor has no electrical windings but has projected poles.

Stepper motors may be divided into two major groups: (a) either stator or rotor not being a permanent magnet and (b) with permanent magnet on the rotor of the motor.

(1) Stepper motor without permanent magnet as variable reluctance stepper motor: The variable reluctance stepper motor is characterized by the fact that there is no permanent magnet either on rotor or stator. The rotor is made of soft iron stampings of variable reluctance and carries no windings as shown in Fig. 13.16. The stator is also made of soft iron stampings and is of salient (projected) poles type and carries stator windings.



Schematic diagram of a three-phase single stack variable reluctance stepper motor. Only the 'A' phase windings are shown for clarity

Fig. 13.16 *Schematic diagram of a variable reluctance-type stepper motor*

As shown in Fig. 13.16, when phase *A* is energized through supply, the rotor moves to the position in which the rotor teeth align themselves with the teeth of phase *A*. In this position, the reluctance of the magnetic circuit is the minimum. After this, if phase *A* is de-energized and phase *B* is energized by giving supply to its winding (not shown in figure), the rotor will rotate through an angle of 15° in a clockwise direction so as to align its teeth with those of phase *B*. After this, de-energizing phase *B* and energizing phase *C* will make the rotor rotate by another 15° in clockwise direction. Thus, by sequencing power supply to the phases, the rotor could be made to rotate by a step of 15° each time. The direction of rotation could be reversed by changing the sequence of supply to the phase, that is, for anti-clockwise rotation, supply should be given in the sequence of *ACB*.

(2) Permanent magnet type stepper motors: The most popular type of stepper motor of this type is known as permanent magnet hybrid (PMH) stepper motor. The simplest type of PMH motor is shown in Fig. 13.17. The stator has four poles, each pole carrying its own winding. Windings of two alternate poles are connected in series to form phase *A* winding. Similarly, windings of the other two alternate poles are connected in series to form phase *B* winding.

The axis of the two windings is perpendicular to each other in space. The rotor is a bar magnet having north and south poles.

When supply is given to phase AA, The rotor magnet will align with the stator magnetic field as shown. Keeping supply to phase AA unchanged, if supply is given to phase BB, a resultant magnetic field will be produced whose axis will rotate by 45°. The rotor, in its tendency to align with the stator field, will rotate by 45°. Next, the supply to phase AA is disconnected and supply to phase BB is kept unchanged.

The direction of the stator field will shift by another 45° and the rotor will rotate by another 45° to keep itself aligned with the stator field. Thus, it can be seen that by changing the supply to the two stator phases as also changing the direction of supply to the phases, the rotor can be made to rotate in steps.

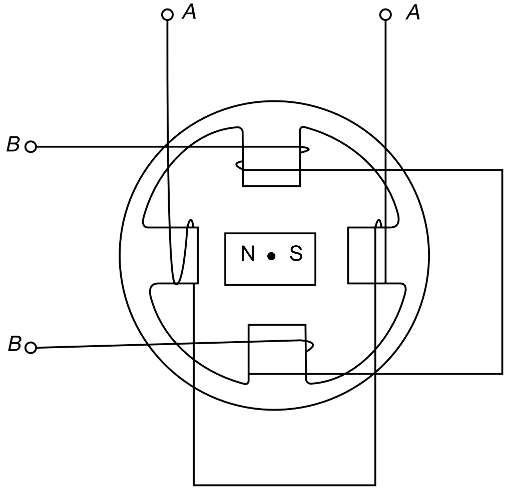


Fig. 13.17 *Principle of operation of hybrid stepper motor*

Electronic drive circuits for hybrid stepper motors such as logic sequence generators, pulse generators, current suppression circuits are used to feed the stator windings with certain pattern of power supply.

The direction of rotation of the rotor could be changed by changing the sequence of supply to the stator-phase windings.

Various types of stepper motors find applications as paper feed motor of a printer or head drive motor of a floppy disk drive, robotics, CNC machines, etc. Stepper motors can be used in control systems either in open-loop mode or in closed-loop mode. We have seen that the output of a stepper motor is in the form of angular displacement of the shaft which is determined by the input pulses. For this reason, we may not need a feedback mechanism to determine the position of the shaft and of the load and then provide feedback for correction. Thus, we can say that an open-loop stepper motor driven servo system can be designed to provide the same accuracy as the closed-loop system. It is obvious, therefore, that the system design will be simpler. In the closed-loop position control system, the stepper motor, however, is used as a conventional servo motor where a signal from the output is fedback, compared and control pulses generated from a pulse generator for correction of output.

13.7 MISCELLANEOUS CONTROL COMPONENTS

The miscellaneous control components are described in the following

(1) Modulators and demodulators: In feedback control system, it is often required to convert d.c. signals into a.c. signals and vice versa. Modulators and demodulators are used for such conversion. The outputs of transducers are often d.c. They are first converted into a.c. and then amplified using high gain a.c. amplifiers.

(2) Servo amplifiers: The d.c. and a.c. amplifiers are used in control systems for signal amplification. The desirable characteristics of such amplifiers are low output impedance, low noise and almost constant gain over working range of frequencies.

(3) Optical encoders: Optical encoders are often used in control systems for the conversion of displacement into digital code or pulse signals.

(4) Pneumatic and hydraulic systems: Pneumatic systems like pneumatic bellows, pneumatic flapper valve, pneumatic relay, pneumatic actuator, etc. find considerable applications in process control field. Considerable applications are in process control field. Similarly, hydraulically operated systems like hydraulic pumps, hydraulic valves and actuators are used in hydraulic feedback systems. In these elements, power is transmitted through the action of airflow or fluidflow under pressure.

REVIEW QUESTIONS

13.1 List a few control system components and state their possible uses.

13.2 Explain the working principle of a synchro used as an error detector in a control circuit. Draw its block diagram representation.

13.3 What is an a.c. tachogenerator? Where do you use it? How do you represent it in block diagram form?

13.4 Explain the principle of working of the following control circuit components:
(a) d.c. Tachogenerator; (b) d.c. Servo motor; (c) Transducer; and (d) Gear train.

13.5 Describe a two-phase servo motor and draw its torque–speed characteristics.

13.6 Name and explain the principle of working of three commonly used transducers.

13.7 Distinguish between inverting and non-inverting amplifiers.

13.8 Explain the working of a rotating amplifier.

13.9 With the help of diagrams, explain the working of a stepper motor. Mention its applications.

13.10 Represent in block diagram form a potentiometer-type error detector.

13.11 Draw and explain the working of a synchro or selsyn error detector and represent it in block diagram form.

13.12 Distinguish between a d.c. servomotor and an a.c. servomotor.

13.13 Deduce the transfer function of a servomotor.

13.14 State and explain the factors which influence the selection of a transducer in instrumentation and control applications.

13.15 Explain the principle of working of a hybrid stepper motor.

13.16 Write short notes on the following:

(a) Servo amplifier; (b) Modulators and demodulators; (c) Rotating amplifier; (d) magnetic amplifier; and (e) Resistance strain gauge.

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14

MATLAB BASED PROBLEMS AND THEIR SOLUTIONS

14.1 MATLAB FUNCTIONS FOR CONTROL SYSTEM

This chapter is completely devoted to solving control system problems through MATLAB programs.

MATLAB FUNCTIONS FOR CONTROL SYSTEM

	Function name	Description
1.	acker	Compute the K matrix to place the poles of A-BK, see also place
2.	bode	Draw the Bode plot, see also logspace, margin, nyquist 1
3.	c2dm	Continuous system to discrete system
4.	ctrb	The controllability matrix, see also obsv
5.	dimpulse	Impulse response of discrete-time linear systems, see also dstep
6.	dlqr	Linear-quadratic regulator design for discrete-time systems, see also lqr
7.	dlsim	Simulation of discrete-time linear systems, see also lsim
8.	dstep	Step response of discrete-time linear systems, see also stairs
9.	feedback	Feedback connection of two systems
10.	impulse	Impulse response of continuous-time linear systems, see also step, lism, dlsim
11.	jgrid	Generate grid lines of constant damping ratio (zeta) and settling time (sigma), see also sgrid, sigrid, zgrid
12.	Inyquist1	Product a Nyquist plot on a logarithmic scale, see also nyquist1
13.	lqr	Linear quadratic regulator design for continuous systems, see also dlqr
14.	lsim	Simulate a linear system, see also step, impulse, dlsim
15.	margin	Returns the gain margin, phase margin, and crossover frequencies, see also bode
16.	obsv	The observability matrix, see also ctrb

	Function name	Description
17.	place	Compute the K matrix to place the poles of A-BK, see also acker
18.	plot	Draw a plot, see also figure, axis, subplot
19.	polyadd	Add two different polynomials
20.	pzmap	Pole-zero map of linear systems
21.	rlocfind	Find the value of k and the poles at the selected point
22.	rlocus	Draw the root locus
23.	rscale	Find the scale factor for a full-state feedback system
24.	series	Series interconnection of Linear time-independent systems
25.	sgrid	Generate grid lines of constant damping ratio (zeta) and natural frequency (Wn)
26.	Sigrid	Generate grid lines of constant settling time (sigma), see also jgrid, sgrid, zgrid
27.	ss	Create state-space models or convert LTI model to state space, see also tf
28.	ss2tf	State-space to transfer function representation, see also tf2ss
29.	ss2zp	State-space to pole-zero representation, see also zp2ss
30.	staris	Stairstep plot for discrete response, see also dstep
31.	tf	Creation of transfer functions or conversion to transfer function, see also ss
32.	tf2ss	Transfer function to state-space representation, see also ss2tf
33.	tf2zp	Transfer function to Pole-zero representation, see also zp2tf
34.	wbw	Returns the bandwidth frequency given the damping ratio and the rise or settling time
35.	zgrid	Generate grid lines of constant damping ratio (zeta) and natural frequency (Wn), see also sgrid, jgrid, Sigrid
36.	zp2ss	Pole-zero to state-space representation, see also ss2zp
37.	zp2tf	Pole-zero to transfer function representation, see also tf2zp
38.	Conv	Multiply polynomials
39.	Deconv	Divide polynomials
40.	Roots	Find polynomial roots
41.	Residue	Partial fraction residue
42.	Polyval	Polynomial evaluation
43.	Polyder	Polynomial derivative
43.	Polyfit	Polynomial curve fitting
44.	nichols	Calculate Nichols plot

	Function name	Description
45.	nyquist	Calculate Nyquist plot
46.	feedback	Calculate the feedback connection of model
47.	obsv	To calculate the observability
48.	rank	To compute the rank of matrix
49.	evalfr	Evaluate the frequency response any given frequency
50.	initial	Plots the response of a state-space model to an initial condition
51.	margin	Computes gain and phase margins and associated crossover frequencies
52.	dcgain	Computes the low-frequency (DC) gain of an LTI system
53.	fminsearch	Find minimum of unconstrained multivariable function
54.	lqr	Calculates a linear-quadratic (LQ) state-feedback regulator for state-space System
55.	step	Plots the step response of LTI systems
56.	sigma	Plot singular values of LTI models
57.	svd	Computes the matrix singular value decomposition
58.	eig	Finds eigenvalues and eigenvectors
59.	frd	Creates or converts to frequency-response data model
60.	drss	Generate random discrete state-space model
61.	dss	Create descriptor state-space model
62.	filt	Create discrete filter with DSP convention
63.	rss	Generate random continuous state-space model
64.	frd	Create and FRD model
65.	frdata	Retrieve FRD model data
66.	tfdata	Retrieve transfer function data
67.	size	Get output/input/array dimensions or model order
68.	ssdata, dssdata	Retrieve state-space data (respectively, descriptor state-space data) or convert it to cell array format

Command summary for symbolic math command		
1.	ilaplace (X)	Find inverse Laplace transform of X (s)
2.	int(s, v, a, b)	Integrate s.w.r. to v from lower a limit to upper limit b
3.	Laplace(X)	Find L[X(t)]
4.	Pretty(x)	Pretty print x
5.	Sym(v)	Convert v to symbolic object
6.	Syms x y z	Declare x, y and z to be symbolic objects
7.	Ztrans(f)	Find Z transform of f(nT)

14.2 ASSORTED MATLAB-BASED PROBLEMS

Example 14.1 Determine the transfer function from the following data.

A ={0 1 −6 −5}, B= {0 1}, C=[2 1]

Solution

The MATLAB program is as follows:

```
>>A=[01;−6 −5]
A=
 0  1
−6 −5
>>B= [0;1]
B=
0
>>C=[2 1]
C=
2   1
>>D=[0]
D=
 0
>>[num,den] = ss2tf(A,B,C,D)
Num =
     0 1.0000 2.0000
Den=
     1.0000  5.0000  6.0000
>>printsys(num,den)
Num/den=
```

$$\frac{s+2}{s\,\hat{}\,2+5s+6}$$

Example 14.2 The State Equations of a system are given below:

X1=X1+X2+u

X2=−X2

Check for the controllability

[x1 x2] = [1 0 0 −1] [x1 x2]+[0 1]u

Hence the system matrix A = [1 0 0 −1]

B=[0 1]

Solution

The MATLAB script is as follows:

```
>>A=[1 1;0 −1]
A=
     1.   1
     2.   0 −1
>>B=[1;0]
```

```
B=
    1
    0
>>p=ctrb(A, B)
P =
1 1
0 0
>> RANK (P)
ans =
1
```

The controllability matrix P is a square matrix (2 * 2) and it is singular matrix, hence the rank of P matrix is 1, not equal to n = 2. The system is therefore uncontrollable.

Example 14.3 The block diagram of a unity feedback control system is shown in Fig. 14.1. Determine from the characteristic equation of the system w_n, ζ, t_p, $\%M_p$, the time at which the first overshoot occurs, the time period of oscillations, and the number of cycles completed before reaching the steady state condition.

Solution

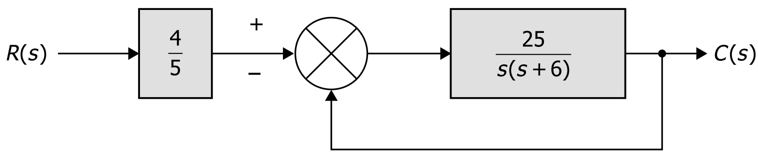


Fig. 14.1

```
>>num = 20; den = [165];
>>[n1, d1] = cloop (num, den −1)
n1 =
0   0   20
d1 =
1   6   25
>>sys = tf (n1, d1)
Transfer function:
```

$$\frac{20}{s\hat{}2 + 6s + 25}$$

```
>>wn = sqrt (25)
Wn =
5
>>zeta = 6/(2*wn)
zeta =
0.6000
>>wd = wn*sqrt(1-zeta^2)
wd =
4
>>tp = pi/wd
tp =
0.7854
>>mp = exp (−zeta*wn*tp)*100
Mp =
```

```
9.4780
>>tfirst = 2*pi/wd
Tfirst =
1.5708
>>period = 2*pi/wd
period =
1.5708
>>oscillations = (wd/(2*pi))*(4/(zeta*wn))
oscillations =
0.8488
```

The results obtained are as follows, wn = 5rad/sec, $\zeta = 0.6$, wd = 4 rad/sec $t_p = 0.7854$ $\%M_p = 9.478\%$, time to reach first over shoot = 1.5708 sec., the time period of oscillations = 1.5708 sec., the number of oscillations before reaching the steady state = 0.8488 cycle.

Example 14.4 The open–loop transfer function of a unity feedback control system is given by

G(s) = 10/s^2+6s+10

Determine the following:
a) Kp b) Kv c) Ka

Solution

The MATLAB coding to obtain the values of Kp, Kv and Ka is written below

```
>> % calculation of error coefficients
>> numg = 10
numg =
10
>> deng = [1    6    10]
deng =
1    6    10
>> % step input
>> % calculation of error coefficient kp
>> G = tf (numg, deng)
Transfer function:
```

$$\frac{10}{s^2+6s+10}$$

```
>> kp = dcgain(G)
kp =
1
>> Ess = 1/(1 + kp)
Ess =
0.5000
>> % ramp input
```

```
>> numsg = conv ([1 0]. Numg)
numsg =
10   0
>> densg = [1   6   10]
densg =
1   6   10
>> sg = tf (numsg, densg)
Transfer function
```

$$\frac{10s}{s^\wedge 2+6s+10}$$

```
>> kv = dcgain (sg)
kv =
0
>> ess = 1/kv
ess =
Inf
>> % parabolic input
>> nums2g = conv ([10   0], numg)
numg2g =
10   0   0
>> dens2g = [1   6   10]
dens2g =
1   6   10
>> s2g = tf (nums2g, dens2g)
Transfer function
```

$$\frac{10s^2}{s^2+6s+10}$$

```
s^2 + 6s + 10
>> ka = dcgain (s2g)
ka =
0
>> ess = 1/ka
ess =
Inf
```

Example 14.5 Obtain the State Model for the transfer function given below:

C(s)/R(s) = s+2/(s+1)(s+3)

Solution

```
>> % State model for the transfer function
>> % (s+2)/(s+1)(s+3)
```

```
>> num = [1    4    3];
>> [A, B, C, D] = tf2ss(num, den);
>> p = [0    1; 1    0]
p =
0   1
1   0
>> % phase variable form
>>Ap = inv(p)*A*p\
Ap =
0   1
−3   −4
>>Bp = inv(p)*B
Bp =
0
1
>> C = C*p
C =
2   1
>>Dp = D
Dp =
0
```

The row vectors corresponding to numerator and denominator polynomials of a given transfer function are assigned the variables num and den respectively. The state model corresponding to controllable canonical form is obtained using tf2ss command. This model is converted to phase variable form using the transformation. P is the transformation matrix. The matrices Ap, Bp, Cp and Dp in phase variables are obtained as above.

Example 14.6 The transfer function of a system is given below:

$G(s) = 8(s + 3)(s + 4)/s(s + 2)(s^2 + 2s + 5)$

Determine the poles and zeros.

Solution

The numerator and denominator polynomials in s for the given transfer function are as:

$P1 = 8s^2 + 56s + 96$

$Q1 = s^4 + 4s^3 + 9s^2 + 10s$

The numerator is assigned a variable name num and the coefficients of a are arranged in descending order, same procedure is followed for denominator polynomial.

MATLAB script for this problem is as follow

```
>> p1 =  [8    56    96]
>> Q1 = [1    4      9    10    0]
Q1 =
```

1 4 9 10 0
>> sys4 = tf (p1, Q1)
Transfer function:

$$\frac{8s^2 + 56s + 96}{s^4 + 4s^3 + 9s^2 + 10s}$$

>> pxmap (sys4)

Here pxmap is a MATLAB command. It describes the pole-zero plot of a linear system.

The pole-zero map is obtained as shown in fig.1. From the pole-zero map of the transfer function, poles and zeros are indicated as:

Poles: s = 0; s=−2; s=−1+j1; s=−1−j1
Zeros: s=−3; s=−4.

Example 14.7 Write MATLAB script to determine the State Transition Matrix for

$$A = \begin{pmatrix} 1 & 4 \\ -2 & -5 \end{pmatrix}$$

Solution

The MATLAB script for determination of the State Transition Matrix for >> % calculation of state transition matrix using inverse technique

Sys t
a = [1 4; −2 −5]
phi = exmp(a*t)
a =
1 4
−2 −5
Phi =
[2*exp(−t) − exp(−3*t), −2*exp(−3*t)+2*exp(−t)]
[exp (−3*t) − exp(−t), −exp(−t)+2*exp(−3*t)]

Example 14.8 A unity feedback control system has its open loop transfer function given by G(s) = (4s + 1)/4s^2. Determine an expression for the time response when the system is subjected to

a) unit impulse input function.
b) unit step input function.

Solution

The MATLAB script is as follows:
>> % time response to unit impulse input
Syms st
n1 = [4 1]
d1 = [4 0 0]

```
sys1 = tf(n1, d1)
[n2, d2] = cloop (n1, d1, −1)
F = (4*s+1)/4*s^2+4*s+1)
f = i laplace (F)
n1 =
4   1
d1 =
4   0   0
Transfer function:
```

$$\frac{4s+1}{4s\text{^}2}$$

```
n2 =
0   4   1
d2 =
4   4   1
F =
(4*s+1)/4*s^2+4*s+1)
F =
−1/4*t*exp(−1/2t)+exp(−1/2*t)
F =
1+1/2*t*exp(−1/2t) −exp(−1/2*t)
```

Example 14.9 Given the transfer functions of individual blocks shown in Fig. 14.2 below, generate the system transfer function of the block combinations.

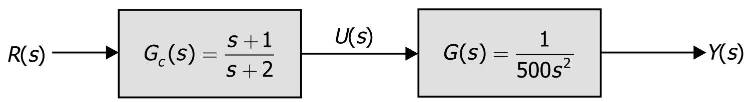


Fig. 14.2

Solution

```
>>numg = [1];
>>deng = [500   0   0];
>>sysg = tf (numg, deng);
>>numh = [1   1];
>>denh = [1   2];
>>sysh = tf (numh, denh);
>>sys = series (sysg, sysh);
>>sys
Transfer function:
```

$$\frac{s+1}{500s\text{ ^ }3+1000s\text{ ^ }2}$$

Example 14.10 For the multi-loop feedback system shown in Fig. 14.3, find the closed loop transfer function.

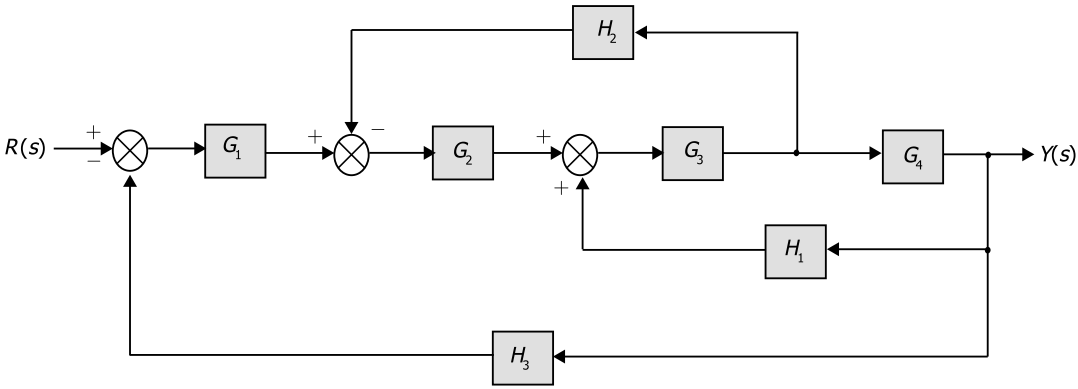


Fig. 14.3

Given

$$G_1 = \frac{1}{(s+10)}; G_2 = \frac{1}{(s+1)}; G_3 = \frac{s^2+1}{(s^2+4s+4)}; G_4 = \frac{s+1}{(s^2+6)}; H_1 \frac{s+1}{(s+2)}; H_2 - 2; H_3 = 1$$

$II_3 = 1$

Solution

MATLAB script for this problem is as follow;

```
>>ng1 = [1];
>>dg1 = [1   1   0];
>>sysg1 = tf(ng1, dg1);
>>ng2 = [1];
>>dg2 = [1   1];
>>sysg2 = tf(ng2, dg2);
>>ng3 = [1   0   1];
>>dg3 = [1   4   1];
>>sysg3 = tf(ng3, dg3);
>>ng4 = [1   1];
>>dg4 = [1   6];
>>sysg4 = tf(ng4, dg4);
>>nh1 = [1   1];
>>dh1 = [1   2];
>>sysh1 = tf(nh1, dh1);
>>nh2 = [2];
>>dh2 = [1];
>>sysh2 = tf(nh2, dh2);
>>nh3 = [1];
```

```
>>dh3 = [1];
>>sysh3 = tf(nh3, dh3);
>>sys1 = sys2/sys4;
>>sys2 = series(sysg3, sysg4);
>>sys3 = feedback (sys2, sysh1, + 1);
>>sys4 = series(sysg2, sysg3);
>>sys5 = feedback (sys4, sys1);
>>sys6 = series (sysg1, sys5);
>>sys = feedback (sys6, [1]);
```

Transfer function:

$$\frac{s\,\hat{}\,5+4s\,\hat{}\,4+6s\,\hat{}\,3+6s\,\hat{}\,2+5s+2}{12s\,\hat{}\,6+20s\,\hat{}\,5+1066s\,\hat{}\,4+2517s\,\hat{}\,3+3128s\,\hat{}\,2+2196s+712}$$

Example 14.11 Suppose you have following continuous transfer function

$$X(s)/F(s)=\frac{1}{Ms^2+bs+k}$$

M = 1kg
b = 10N.s/m
k = 20N/m
F(S) = 1

Find the discrete transfer function.

Solution

Assuming the closed-loop bandwidth frequency is greater than 1 rad/sec, we will choose the sampling time (Ts) equal to 1/100 sec.

MATLAB script for obtaining the discrete transfer function from the continuous transfer function is as follow

```
>> M = 1;
>> b = 10;
>> k = 20;
>>num = [1];
>>den = [M b k];
>> Ts = 1/100;
>> [numDz, denDz] = c2dm(num, den, Ts, 'zoh')
numDz =
1.0e−004*
0    0.4837    0.4678
```

From these matrices, the discrete transfer function can be written as

X(z)/F(z) = 0.0001(0.4873z + 0.4678)/z^2 − 1.9029 + 0.9048

Example 14.12 Suppose you have the following continuous state-space model.

$$\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/M & -k/M \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M \end{pmatrix} [F]$$

$$Y = [1 \quad 0] \begin{pmatrix} x \\ v \end{pmatrix} + [0][F]$$

All constant are same as in example 14.11. Determine the discrete time state-space model.

Solution

The following M-file converts the continuous state-space to discrete state space

```
>> M = 1;
>> b = 10;
>> k = 20;
>> A = [0 1;
>>−k/M –b/M];
>> B = [0];
>> 1/M];
>> C = [1 0];
>> D = [0];
>> Ts = 1/100;
>> [F, G, H, J] = c2dm (A, B, C, D, Ts, 'zoh')
F =
0.9990    0.0095
−0.1903    0.9039
G =
0.0000
0.0095
H =
1    0
J =
0
```

From these matrices, the discrete state-space can be written as

$$\begin{pmatrix} X(k) \\ V(k) \end{pmatrix} = \begin{pmatrix} 0.999 & 0.995 \\ -0.1903 & 0.9039 \end{pmatrix} \begin{pmatrix} X(k-1) \\ V(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0.0095 \end{pmatrix} [F(k-1)]$$

Now you have the discrete time state-space model as shown above.

Example 14.13 Sketch the root locus plot for the system when the open loop transfer function is given by:

$G(s) = K/s(s + 4)(s^2 + 4s + 13)$

Solution

p =
1
>> q = [1 8 29 52 0]
q =
1 8 29 52 0
>> sys 7 = tf (p, q)
Transfer function:

$$\frac{1}{S\,{}^\wedge 4 + 8s\,{}^\wedge 3 + 29s\,{}^\wedge 2 + 52s}$$

>>rlocus(p, q);

The root locus plot is shown in Fig. 14.4.

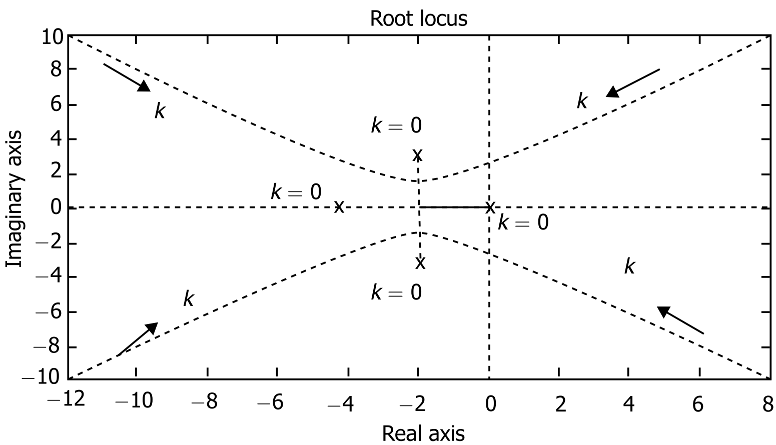


Fig. 14.4

Example 14.14 Sketch the root locus for

G(s) = k(S + 1)/S^2(S + 3.6)

Solution

>>clear
>>num = [1, 1];
>>den = conv([100],[13.6]);
>>G = tf(num, den);
>>rlocus(G);
Output
The root locus plot is shown in Fig. 14.5.

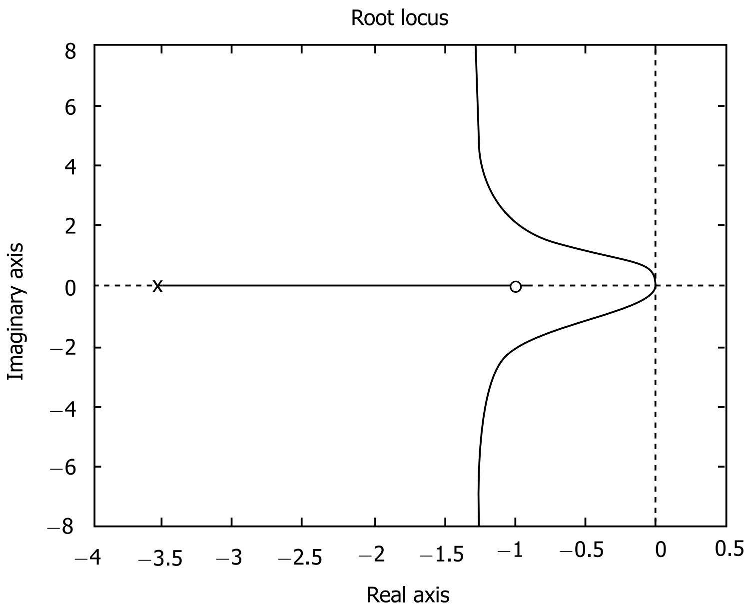


Fig. 14.5

Example 14.15 The open loop transfer function of a unity feedback system is given below

G(s) = K / S (S + 2)(S^2 + 2S + 2)

Plot the root locus.

Solution

\>> p = 1
P =
1
\>> q = [1 4 6 4 0]
q =
1 4 6 4 0
\>>sys5 = tf(p, q)
Transfer function:

$$\frac{1}{s\,^\wedge 4 + 4s\,^\wedge 3 + 6s\,^\wedge 2 + 4s}$$

\>>rlocus (sys5)
Output
The root locus plot is shown in Fig. 14.6

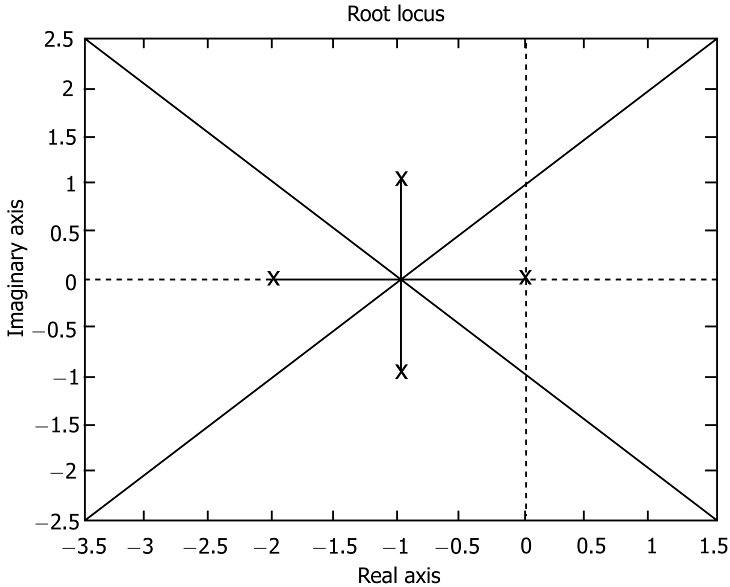


Fig. 14.6

Example 14.16 Sketch the root locus plot for the open loop transfer function

G(s) H(s) = K/s(s + 1)(s + 3)

(a) Determine the value of K at s = − 4
(b) The corner frequency of sustained oscillations

Solution

(a) >> p = 1
P =
1
>> q = [1 4 3 0]
q =
1 4 3 0
>> sys3 = tf(p, q)
Transfer function:

$$\frac{1}{s\,{}^\wedge 3 + 4s\,{}^\wedge 2 + 3s}$$

>>rlocus(p, q)
Output
The root locus plot is displayed in Fig. 14.7.

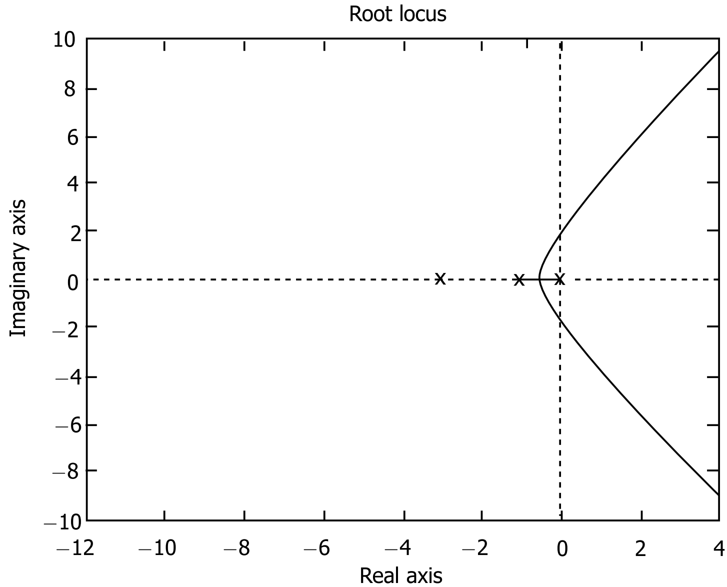


Fig. 14.7

The value of the gain K at s = – 4 obtained below.
Modulus of K/ (– 4)(– 4 + 1)(– 4 + 3) = 1
Modulus of K/ (– 4)(– 3)(– 1) = 1
K = 12 at s = – 4

(b) the root locus plot crosses the imaginary axis at s = j1.72, hence the frequency of sustained oscillations is w = 1.72 rad./sec.

Example 14.17 Sketch the root locus plot for the system when the pen loop transfer function is given by
$G(s) = K/s(s + 4)(s^2 + 4s + 13)$

Solution

>> p = 1
P =
1
>> q = [1 8 29 52 0]
q =
1 8 29 52 0
>> sys4 = tf(p, q)
Transfer function:

$$\frac{1}{s\,\hat{}\,4 + 8s\,\hat{}\,3 + 29s\,\hat{}\,2 + 52s}$$

```
>>rlocus(p, q)
```
Output

The root locus plot is shown in Fig. 14.8.

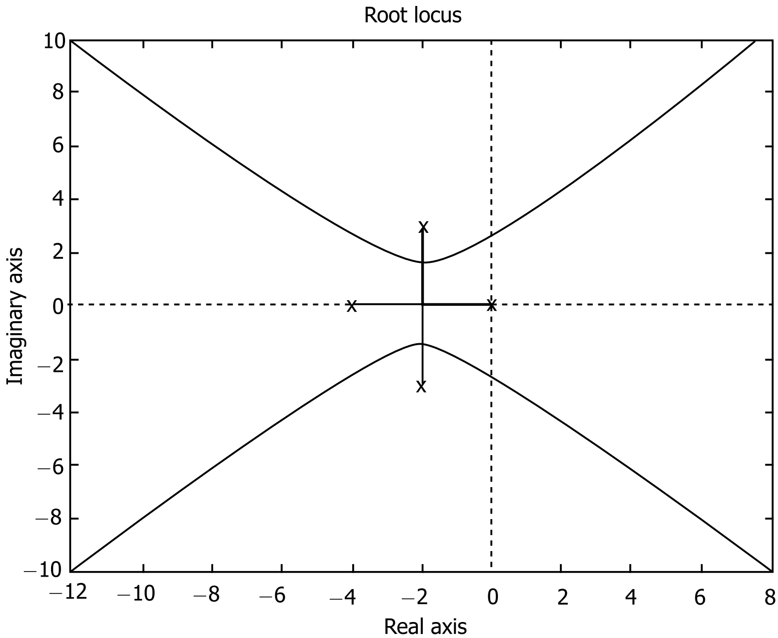


Fig. 14.8

Example 14.18 Sketch the root locus plot for the open-loop transfer function given below

G(S) H(S) = K (s^2 + 4)/s(s + 2)

Calculate the value of K at a) breakaway point and b) s = − 0.69 + j0.88

Solution

```
>> p = [1 0 4];
>> q = [1 2 0];
>> sys9 = tf(p, q)
```
Transfer function:

$$\frac{s\,^\wedge 2+4}{s\,^\wedge 2+2s}$$

```
>>rlocus(p, q)
>>rlocus(p, q)
```
Select a point in the graphics window

selected_point =

−0.6908 + 0.8773i
ans =
0.4512
Output
The root locus plot is shown in Fig. 14.9.

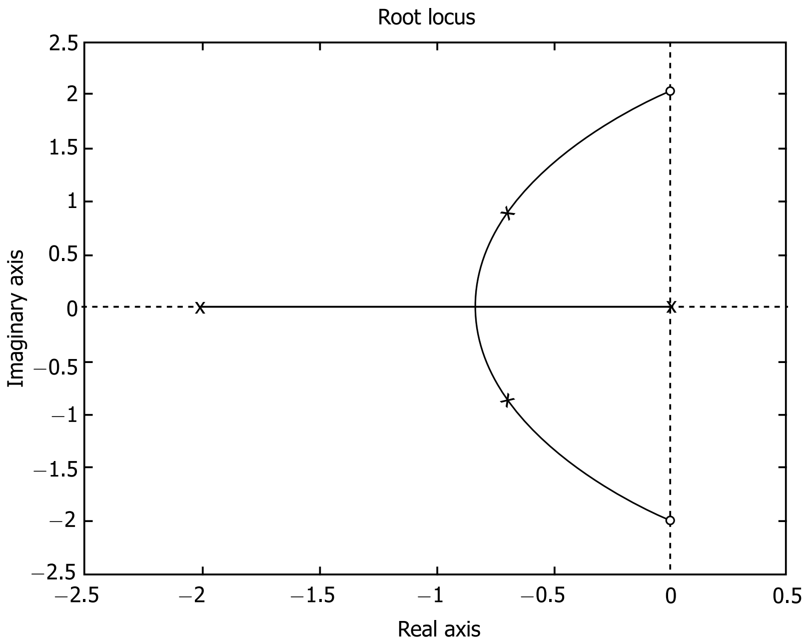


Fig. 14.9

Example 14.19 The open loop transfer function of a unity feedback control system is given by

G(s) = K/s(1 + 0.2s)

Design a lead compensator such that the system will have Kv = 10 and P.M = 57 degree.

Solution

$$Kv = \lim_{s\to 0} \text{S.G(s)}$$

$$\lim_{s\to 0} \text{S.K} / \text{s}(1+0.2\text{s})$$

K = 10
and in order to satisfy steady state error requirement
G(s) = 10/s(1 + 0.2s)
>> n = 10;

>> d = [0.2 1 0];
>> sys6 = tf(n, d)
Transfer function:

$$\frac{10}{0.2s \wedge 2 + s}$$

>>sisotool (sys6)

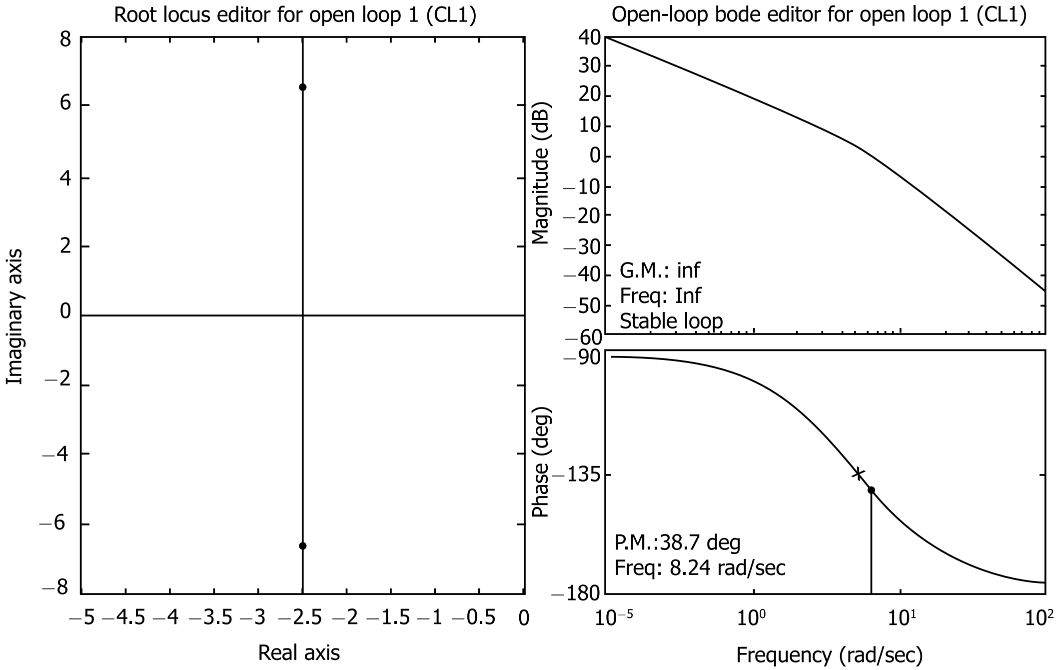


Fig. 14.10

Bode plot based numerical

Example 14.20 A unity feedback control system has its forward path transfer function as

G(S) = 210/ s(s + 2)(s^2 + 12s + 6)

Give a MATLAB programme to determine

(a) The roots of the characteristic equation
(b) The Bode plot for the open loop transfer function.

Solution

>> n1 = 210;
>> d1 = [1 44 390 192 0]
d1 =
1 44 390 192 0
>>sys = tf(n1, d1)

Transfer function:

$$\frac{210}{s\wedge 4 + 44s\wedge 3 + 390s\wedge 2 + 192s}$$

>>[num, den] = cloop(n1, d1)
num =
0 0 0 0 210
den =
1 44 390 192 210
>> sys1 = tf(num, den)
Transfer function:

$$\frac{210}{s\wedge 4 + 44s\wedge 3 + 390s\wedge 2 + 192s + 210}$$

>>roots (den)
ans =
− 31.9898

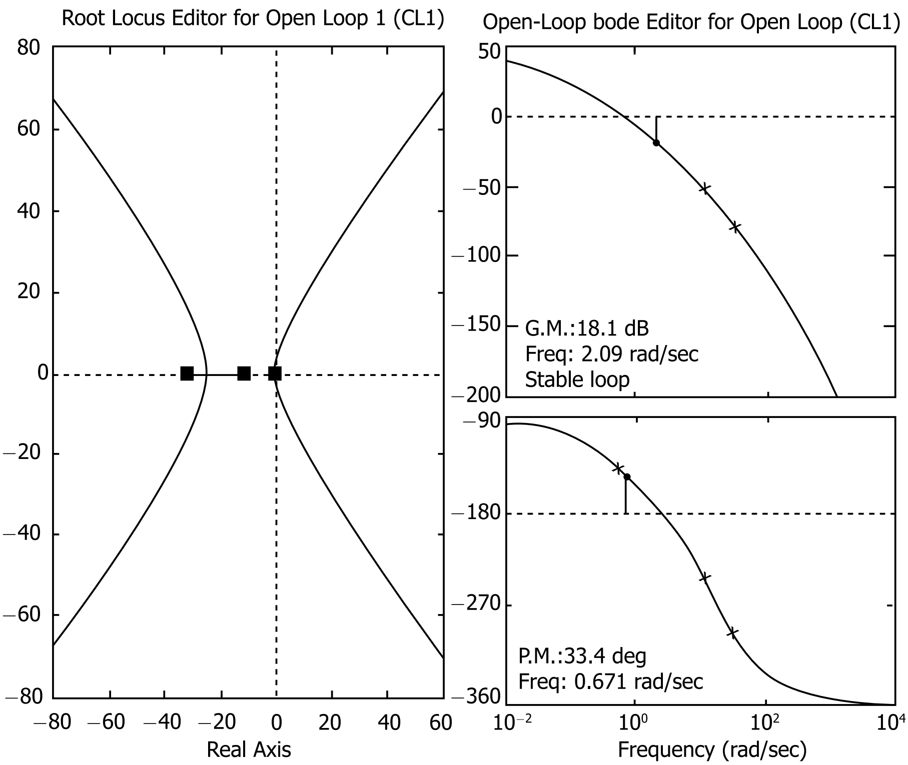


Fig. 14.11

– 11.5578
– 0.2262 + 0.7189i
– 0.2262 – 0.7189i
>>sisotool (sys)

For the given open-loop transfer function G(s), the closed-loop transfer function is calculated first. The characteristic equation is

s^4 + 44s^3 + 390s^2 + 192s + + 210 = 0

The roots of the characteristic equation are obtained using MATLAB function roots and the roots are

– 31.9898, – 11.5578, – 0.2262 + 0.7189i, – 0.2262 – 0.7189i

The Bode plot is obtained using functions isotool as has been shown.
Output

Example 14.21 Sketch the Bode plot for the transfer function given below:

$G(s)\,H(s) = 2(s + 0.25)/(s^2(s + 1)(s + 0.5))$

From the bode plot determine

a) The phase cross over frequency
b) The gain cross over frequency
c) The gain margin
d) The phase margin

Solution

>>num = [2 0.5]
num =
2.0000 0.5000
>>den = [1 1.5 0.5 0 0]
den =
1.0000 1.5000 0.5000 0 0
>> sys1 = tf(num, den)
Transfer function:

$$\frac{2s+0.5}{s\,^\wedge 4+1.5s\,^\wedge 3+0.5s\,^\wedge 2}$$

>>margin (sys1)
The function margin sys1 computes and plots the gain margin and phase margin directly as displayed in fig below
The phase cross over frequency = 0.35355 rad/sec
The gain cross over frequency = 1.117 rad/sec
The gain margin = – 20.561 dB
The phase margin = −36.665°

As both margins are negative, hence the system is unstable.
Output

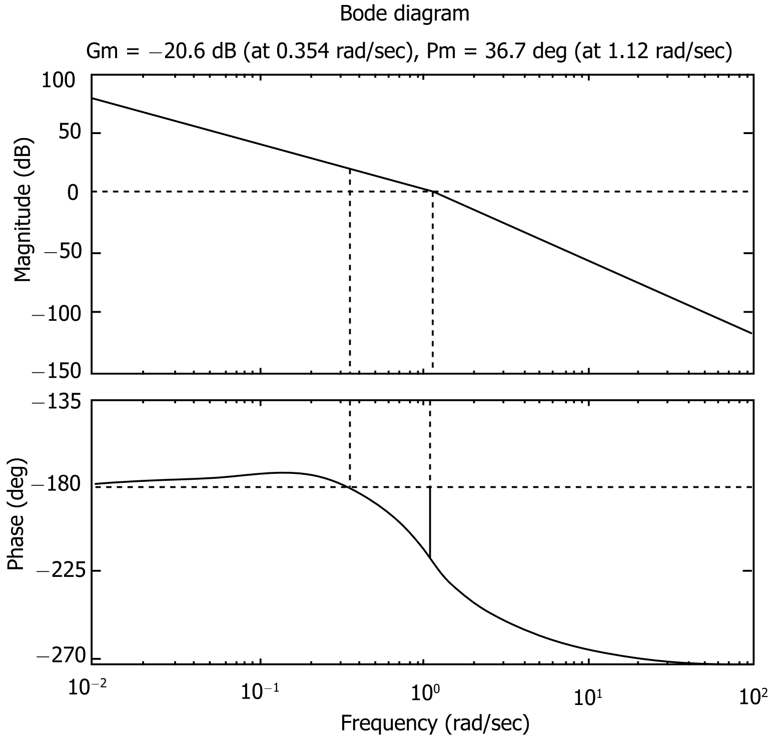


Fig. 14.12 Bode plot for example 14.21

Example 14.22 The open loop transfer function of a system is given by

G(S) H(S) = 4/S(1 + 0.5S)(1 + 0.08S)

Determine a) gain margin b) phase margin and c) closed loop stability

Solution

```
>>num = 4
num =
4
>>den = [0.0400 0.5800 1.0000 0]
den =
0.0400 0.5800 1.0000 0
>>sys = tf(num, den)
Transfer function:
```

$$\frac{4}{0.04s\,^\wedge 3 + 0.58s\,^\wedge 2 + s}$$

>> load ltiexamples
>>ltiview
>>margin (sys)
The gain margin and phase margin are positive, as shown in Fig. 14.13 below the system is stable.
Output

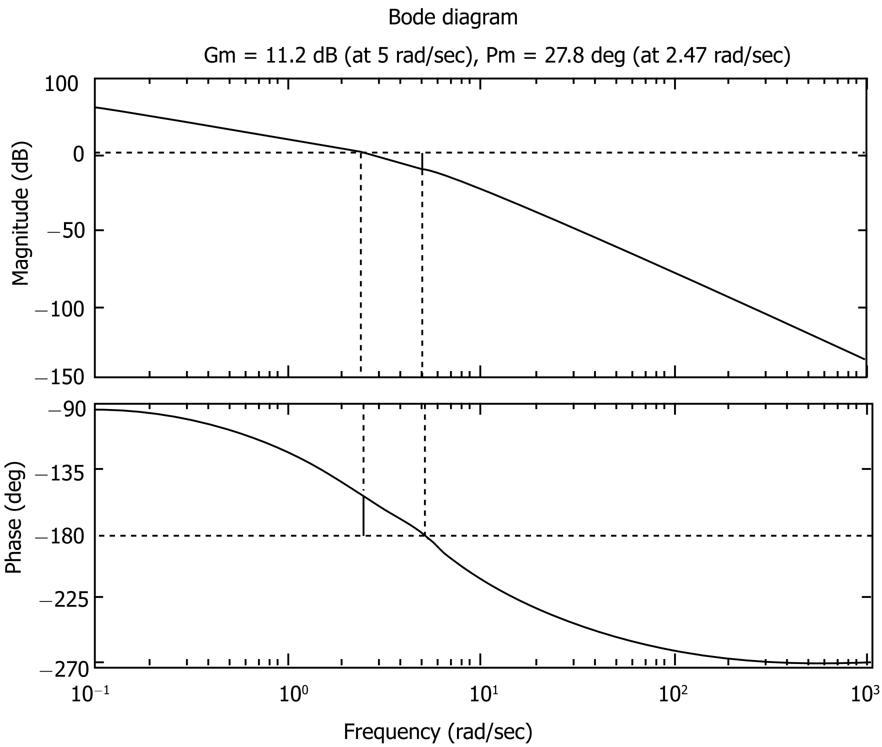


Fig. 14.13

Example 14.23 Examine the closed loop stability of a system whose open loop transfer function is given by:

G(s) H(s) = 50/(s + 1)(s + 2)

Solution
The commands are to be used sequentially as mentioned below
>>num = 50
num =
50
>>den = [1 3 2]
den =
1 3 2

>> sys3 = tf(num, den)
Transfer function:

$$\frac{50}{s\,^\wedge 2+3s+2}$$

>> load ltiexamples
>> ltiview
The transfer function model sys3 is obtained. The LTI examples are loaded. The LTI view browser is opened next. Under file menu, sys3 is imported. Nyquist plot is chosen. The closedloop stability for the given openloop transfer function as indicated is stable.
Output-

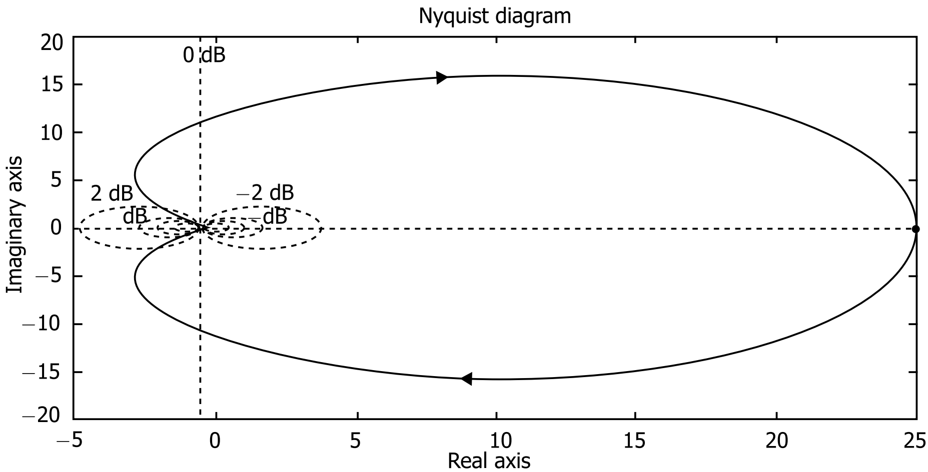


Fig. 14.14 *Nyquist plot for* $G(s)H(s)=\dfrac{50}{(s+1)(s+2)}$

Nyquist Plot

Example 14.24 Determine the damping ratio, damped frequency of oscillatory roots and % overshoot for a unit step response given that:

C(s)/E(s) = 1/s(1 + 0.5s)(1 + 0.2s)

The system is unity feedback type.

Solution

The closed loop transfer for the given condition is determined as below:

C(s)/R(s) = 10/(s + 5.5)(s^2 + 1.5s + 1.8)

C(s)/R(s) = 10/s^3 + 7s^2 + 10s + 10s + 10

Program:
num =
10
>> den = [1 7 10 10]
den =
1 7 10 10
>> sys12 = tf(num, den)
Transfer function:

$$\frac{10}{s\,^\wedge 3+7s\,^\wedge 2+10s+10}$$

>> load ltiexamples
>> ltiview
The time response is shown in Fig. 14.15, and following results are noted:

Output

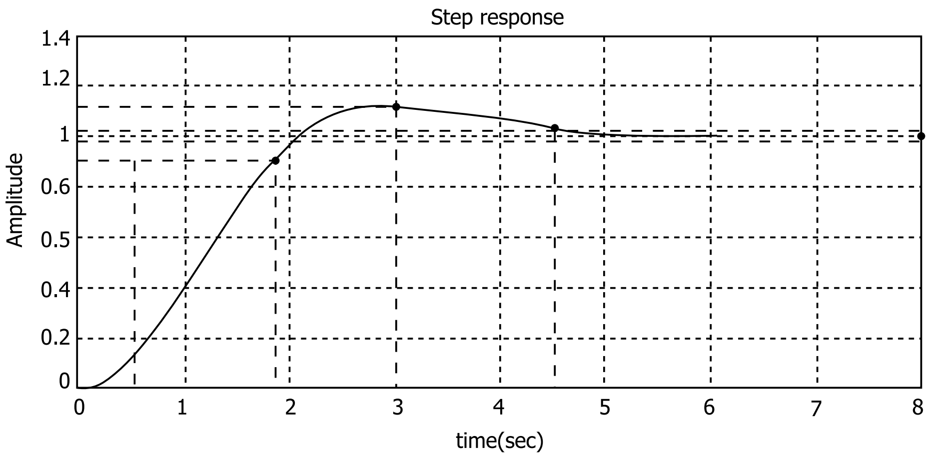


Fig. 14.15

Time response of the system
Calculation:
$\%M_p = 16\%$, $\zeta = 0.5$ (as per calculation)
$W_d = 1/2/2.72*2\text{pi} = 1.15$ rad/sec
$W_n = 1.32$rad/sec. $t_s = 5.6$ sec
$T = 1/\zeta\, W_n = 1.4$ sec

15

INTRODUCTION TO DIGITAL CONTROL SYSTEMS

15.1 INTRODUCTION

In the control systems, we have discussed so far, the signal at every point in the system is a continuous function of time. In such control systems analog controllers are used to produce continuous time varying control signals. The controller is the portion of the system that does the controlling operation. The portion of the system that is to be controlled is called the plant or the controlled system.

In analog control system the controller is made up of resistors, capacitors, operational amplifiers, etc. A digital controller is in the form of programmed digital computer. Digital controllers will have analog devices for interfacing with the plant or the control system. As compared to digital controllers, analog controllers have the following drawbacks:

1. Analog controllers are more complex.
2. Analog controllers are not flexible, adaptable and optional.
3. Analog controllers are not versatile, i.e., their control function cannot be easily changed.

The following are the advantages offered by digital control.

Advantages of digital control:

1. The controller software can be redesigned so as to adapt to the changing operating conditions of the plant, i.e., the control system. Thus a digital controller offers the flexibility of modifying the controller characteristics so as to respond to the changing needs.

2. The computer based control system has overcome the limitations of analog controllers. Further, artificial-intelligence are influencing the design and application of control systems.
3. Digital controllers can be used for controlling complex control systems which have high demand of flexibility, adaptability and optimality.
4. The control function of a digital controller can be easily changed by changing a few program instructions of the computer.

The study of digital control system, which is also called sampled data control system, is related to the effects of sampling and quantization. As we know, most of the systems that we are to control operate in continuous time. That is to say that most of the systems are analog in nature. To use a digital computer we need to interface the digital computer through which we want to control the system or the process. This interfacing is achieved through analog to digital (A/D) and digital to analog (D/A) converters. The A to D converters derive samples of analog signal at discrete instants of time at regular intervals called the sampling period. The D to A converters does the reverse process, i.e., it involves reconstructing the analog signal from the samples available from the output of the digital computer.

In this chapter we will discuss sampling, sampling theorem and method of finding pulse transfer function of the sampled data control systems.

15.2 CONFIGURATION OF SAMPLED DATA CONTROL SYSTEM

With the development of computer technology at a reasonable cost, the digital computer is now very conveniently and widely used to design the controller. A digital control scheme is shown in Fig. 15.1 in block diagram form. This digital control scheme is also called sampled data control system.

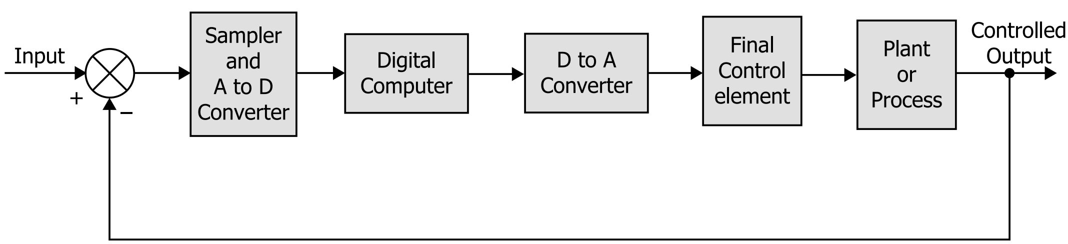


Fig. 15.1 *Basic digital control scheme or sampled data control system*

The function of the various blocks of the sampled data control system are mentioned below.

Sampler: It converts the continuous time signal into a sequence of pulses.

Analog to digital converter: It transforms analog signal to digital signal. The analog signal gets tied to finite number of quantization levels in the process of conversion from analog to digital form.

Digital computer: It performs designed manipulations of the input signal.

Digital to analog converter: It consists of the numerically coded output of the digital converter to piecewise continuous time signal. The output of the D to A converter is fed to the plant through the actuator, i.e., the final control element to control the plant.

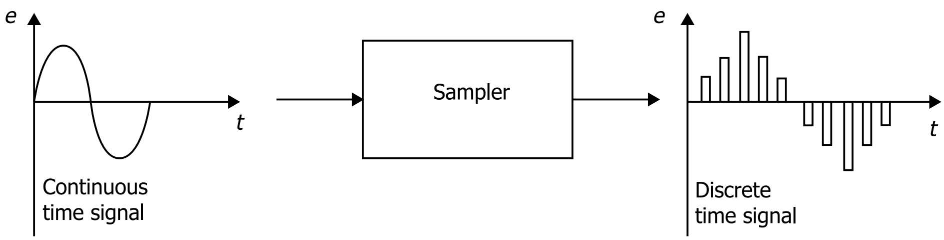
Plant: It is a process which is controlled by the continuous time signal that is finally developed by the sampled data control system. The control scheme shows a single feedback loop.

15.3 SAMPLING PROCESS

This is the conversion of a continuous time signal into a discrete time signal obtained by taking samples of the continuous time signal at discrete time instants as shown in Fig. 15.2. The continuous time signal shown in Fig. 15.2 cannot be stored in digital computer. The signal must be converted to a form that is acceptable by the digital computer. The sampling process involves recording of sample values of the signal at equally spaced instants. The rate at which sample values are measured and recorded is called sampling rate. The choice of sampling rate is an important consideration as it determines the accuracy of signal conversion.

Sampling Theorem: An analog signal is converted to digital signals through an A to D converter. The signal is reconstructed from its samples at the receiving end by a D to A converter. Sampling theorem states that a signal can be reconstructed from its samples at the receiving end if the sampling frequency is twice the highest frequency present in the signal. Sampling signals are quantified at different quantization levels. After sampling and quantization, coding is done using an encoder. The encoder maps each quantized sample value into digital word. Thus a A to D converter performs sampling, quantization, and coding.

The D to A converter generates the output samples from the binary form of digital signals of the computer and then converts these samples to analog form. The decoder maps each digital word into a sample value of the signal in discrete time form. The operations performed by a A to D converter and a D to A converter are shown in Fig. 15.3(a) and (b) respectively.



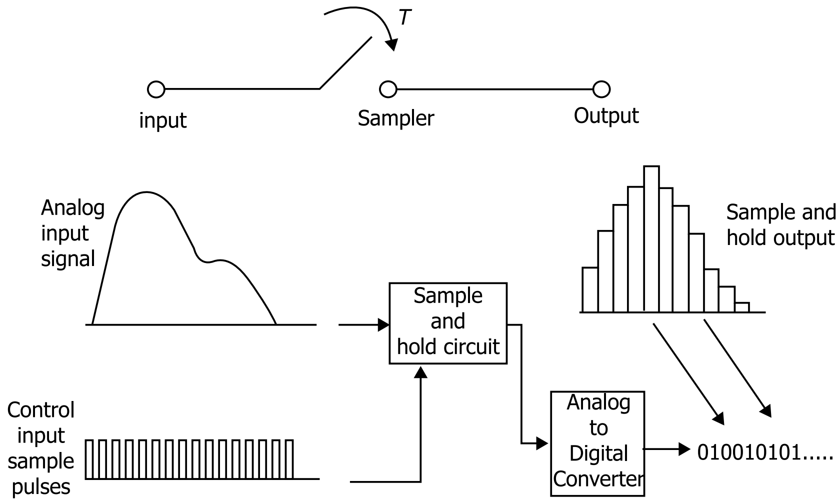


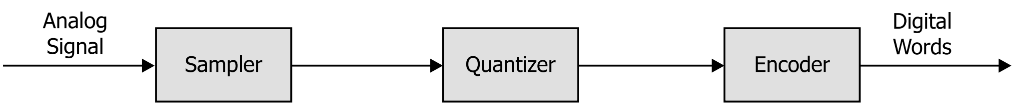
Fig. 15.2 *Basic idea of analog to digital conversion*

Advantages of Sampled Data Control System

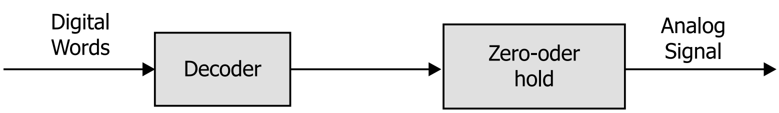
1. These are most appropriate for control systems requiring long distance data transmission.
2. Digital computer or microprocessor can be used as a part of the control loop. Microprocessor can handle large amount of data in any control system.

Advantages of Digital Controllers over Analog Controllers

1. Digital controllers are less complex than analog controllers.



(a) Operations of A to D Converter.



(b) Operations of D to A Converter.

Fig. 15.3 *Operations performed by converters*

2. A digital controller is more flexible, adaptable and optimal than analog controller.

15.4 *Z*-TRANSFORM

The Laplace transform and its discrete-time counterpart, the z-transform are essential mathematical tools for system design and analysis, and for monitoring the stability of a system. A working knowledge of the z-transform is essential for the study of discrete systems.

It is through the use of these transforms that we formulate closed-form mathematical description of a system in the frequency domain, design the system, and then analyse for the stability, the transient response and the steady state characteristics of the system.

$$z[f(t)] = z[f(KT)] = \sum_{k=-\infty}^{\infty} f(t)z^{-4}$$

$$= \sum_{k=-\infty}^{\infty} f(KT)z^{-4}$$

where $t = KT$

T is the sampling line

K is the number of samples

Region of Convergence (ROC): Since the z-transform is an infinite power series, it exists only for those values of the variable z for which the series converges to a finite sum. The region of convergence (ROC) of $x(z)$ is the set of all the values of z for which $x(z)$ attains a finite computable value.

Example 15.1 Find the z-transform of unit step function.

Solution

The step function is designed as

$$x(t) = 1 \quad (t \leq 0)$$

It is shown graphically in Fig. 15.4.

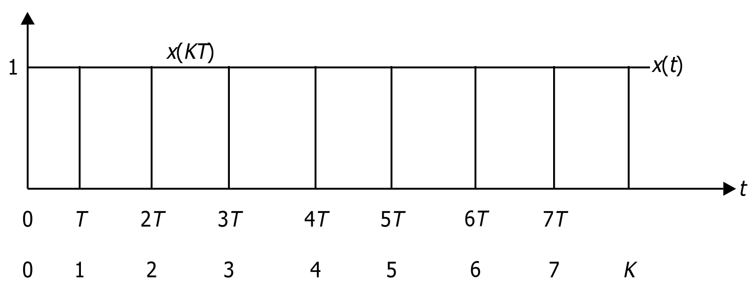


Fig. 15.4 *Representation of a step function*

$$z[x(t)] = \sum_{k=0}^{\infty} x(t) z^{-4}$$
$$= \sum_{k=0}^{\infty} x(KT) z^{-4}$$
$$= \sum_{k=0}^{\infty} 1 \cdot z^{-4}$$
$$= 1 \cdot z^{0} + z^{-1} + 1z^{-2} + z^{-3} + \ldots\infty$$
$$= 1 + z^{-1} + 1z^{-2} + z^{-3} + \ldots\infty$$

Here, s is a G.P. with first term and common ratio is z^{-1}

$$= \frac{1}{1 - z^{-1}}$$
$$= \frac{1}{1 - \frac{1}{z}}$$
$$= \frac{z}{z - 1}$$

We will now calculate the z-transform of some other functions.

a) z-transform of Unit Function
We know unit ramp function is defined as

$$x(t) = t \; (t \geq 0)$$

By definition

$$z[x(t)] = \sum_{k=0}^{\infty} x(t) z^{-4}$$
$$= \sum_{k=0}^{\infty} t z^{-4}$$

Now put $t = KT$

$$= \sum_{k=0}^{\infty} (KT) z^{-4}$$
$$= 0 + 1.Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \ldots\infty$$
$$z[x(t)] = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \ldots\infty$$

Multiplying both sides by $(z - 1)$, we get

$$\begin{aligned}(z-1)z[x(t)] &= (z-1)\left(Tz^{-1}+2Tz^{-2}+3Tz^{-3}+\ldots\infty\right)\\ &= T+Tz^{-1}+Tz^{-2}+Tz^{-3}+\ldots\infty\\ &= T\left[1+z^{-1}+z^{-2}+z^{-3}+\ldots\infty\right]\\ &= T.\frac{1}{1-z^{-1}}\end{aligned}$$

$$(z-1)z[x(t)]=\frac{Tz}{z-1}$$

Or

$$z[x(t)]=\frac{T}{(z-1)^2}\Rightarrow \text{Required } z\text{-transform}$$

b) z-transform of a^k
Here $x(t) = a^k$
By definition

$$\begin{aligned}z[x(t)] &= \sum_{k=0}^{\infty}x(t)z^{-k}\\ &= \sum_{k=0}^{\infty}a^{k}z^{-k}\\ &= 1+a^1z^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots\infty\\ &= 1+\left(az^{-1}\right)+\left(az^{-1}\right)^2+\left(az^{-1}\right)^3+\ldots\infty\\ &= \frac{1}{1-az^{-1}}\end{aligned}$$

$$z[x(t)]=\frac{z}{z-a}\Rightarrow \text{Required } z\text{-transform}$$

c) z-transform of e^{at}
Here, $x(t) = e^{at}$
By definition,

$$\begin{aligned}z[x(t)] &= \sum_{k=0}^{\infty}x(t)z^{-k}\\ &= \sum_{k=0}^{\infty}e^{at}z^{-k}\end{aligned}$$

Put,

$$t = KT$$

$$= \sum_{k=0}^{\infty} e^{aKT} z^{-\mathrm{k}}$$

$$= 1 + e^{aT} z^{-1} + e^{2aT} z^{-2} + e^{3aT} z^{-3} + \ldots\infty$$

$$= 1 + \left(e^{aT} z^{-1}\right) + \left(e^{aT} z^{-1}\right)^2 + \left(e^{aT} z^{-1}\right)^3 + \ldots\infty$$

$$= \frac{1}{1 - e^{aT} z^{-1}}$$

$$= \frac{z}{z - e^{aT}} \Rightarrow \text{Required } z\text{-transform}$$

d) z-transform of $e^{-\mathrm{at}}$
By definition

$$z\left[e^{-\mathrm{at}}\right] = \sum_{k=0}^{\infty} e^{-\mathrm{at}} z^{-\mathrm{k}}$$

Put

$$t = KT = \sum_{k=0}^{\infty} e^{-\mathrm{aKT}} z^{-\mathrm{k}}$$

$$= \sum_{k=0}^{\infty} \left(e^{-\mathrm{aT}} z^{-1}\right)^k$$

$$= 1 + \left(e^{-\mathrm{aT}} z^{-1}\right) + \left(e^{-\mathrm{aT}} z^{-2}\right) + \left(e^{-\mathrm{aT}} z^{-3}\right) + \ldots\infty$$

$$= \frac{1}{1 - e^{-\mathrm{aT}} z^{-1}}$$

$$= \frac{z}{z - e^{-\mathrm{aT}}}$$

e) z-transform of sin $\boldsymbol{\omega t}$ or cos $\boldsymbol{\omega t}$
First, let us find z-transform of $e^{-\delta\omega t}$
Now,

$$z\left[e^{\delta\omega t}\right] = \sum_{k=0}^{\infty} e^{\delta\omega t} z^{-\mathrm{k}}$$

Put $t = KT$

$$= \sum_{k=0}^{\infty} e^{\delta\omega KT} z^{-k} = \sum_{k=0}^{\infty} \left(e^{\delta\omega KT} z^{-k}\right)^k$$

$$= 1 + \left(e^{\delta\omega T} z^{-1}\right) + \left(e^{\delta\omega T} z^{-1}\right)^2 + \ldots\infty$$

$$= \frac{1}{1 - e^{\delta\omega T} z^{-1}}$$

$$= \frac{z}{z - e^{\delta\omega T}}$$

$$= \frac{z\left(z - e^{-\delta\omega T}\right)}{\left(z - e^{\delta\omega T}\right)\left(z - e^{-\delta\omega T}\right)}$$

Multiplying and dividing by $z - e^{-\delta\omega t}$

$$= \frac{z^2 - ze^{-\delta\omega T}}{z^2 - ze^{-\delta\omega T} - z^2 - ze^{\delta\omega T} + 1}$$

$$= \frac{z^2 - ze^{-\delta\omega T}}{z^2 - z\left(e^{\delta\omega T} + ze^{-\delta\omega T}\right) + 1}$$

$$= \frac{z^2 - ze^{-\delta\omega T}}{z^2 - 2z\left(\dfrac{e^{\delta\omega T} + e^{-\delta\omega T}}{2}\right) + 1}$$

$$= \frac{z^2 - z\left(\cos\omega T - j\sin\omega T\right)}{z^2 - 2z\cos\omega T + 1} \quad \because \frac{e^{\delta\theta} + e^{-\delta\theta}}{2} = \cos\theta$$

$$= \frac{z^2 - z\cos\omega T + jz\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

$$z\left[e^{\delta\omega t}\right] = \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1} + j\frac{z\sin\omega}{z^2 - 2z\cos\omega T + 1}$$

$$z\left[\cos\omega t + j\sin\omega t\right] = \frac{z\left(z - \cos\omega T\right)}{z^2 - 2z\cos\omega T + 1} + j\frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

Comparing real and imaginary parts on both sides, we get

$$z\left[\cos\omega t\right] = \frac{z\left(z - \cos\omega T\right)}{z^2 - 2z\cos\omega T + 1}$$

$$z\left[\sin\omega t\right] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

f) z-transform of sinh ωt

By definition

$$z\left[\sinh\omega t\right] = \sum_{k=0}^{\infty} \sinh\omega t\, z^{-k} = \sum_{k=0}^{\infty}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right)z^{-k}$$

$$\left[\because \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(e^{\omega t} z^{-k} - e^{-\omega t} z^{-k}\right)$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{\omega KT} z^{-k} - \sum_{k=0}^{\infty} e^{-\omega KT} z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty}\left(e^{\omega t} z^{-1}\right)^{k} - \sum_{k=0}^{\infty}\left(e^{-\omega t} z^{-1}\right)^{k}\right]$$

$$= \frac{1}{2}\left[\left\{1+\left(e^{\omega T} z^{-1}\right)+\left(e^{\omega T} z^{-1}\right)^{2}+\ldots\infty\right\}\left\{1+\left(e^{-\omega T} z^{-1}\right)+\left(e^{-\omega T} z^{-1}\right)^{2}+\ldots\infty\right\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{1-e^{\omega T} z^{-1}} - \frac{1}{1-e^{\omega T} z^{-1}}\right)$$

g) z-transform of cosh $\boldsymbol{\omega t}$

$$z[\cosh \omega t] = \sum_{k=0}^{\infty} \cosh \omega t \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty}\left(\frac{e^{\omega t} + e^{-\omega t}}{2}\right) z^{-k}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(e^{\omega t} z^{-k} + e^{-\omega t} z^{-k}\right)$$

Put, $t = KT$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} e^{\omega KT} z^{-k} + \sum_{k=0}^{\infty} e^{-\omega KT} z^{-k}\right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty}\left(e^{\omega T} z^{-1}\right)^{k} + \sum_{k=0}^{\infty}\left(e^{-\omega T} z^{-1}\right)^{k}\right)$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{\omega t}} + \frac{z}{z-e^{-\omega t}}\right)$$

$$= \frac{1}{2}\left(\frac{z^{2} - ze^{-\omega T} + z^{2} - ze^{\omega T}}{\left(z-e^{\omega T}\right)\left(z-e^{\omega T}\right)}\right)$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\omega T}} - \frac{z}{z-e^{-\omega T}}\right]$$

$$= \frac{1}{2}\left[\frac{z^{2} - ze^{-\omega T} - z^{2} + ze^{\omega T}}{\left(z-e^{e\omega T}\right)\left(z-e^{-\omega T}\right)}\right]$$

$$= \frac{1}{2}\left(\frac{z\left(e^{\omega T} - e^{-\omega T}\right)}{z^{2} - ze^{-\omega T} - ze^{\omega T} + 1}\right)$$

$$=\frac{1}{2}\left[\frac{2z\left(\frac{e^{\omega T}-e^{-\omega T}}{2}\right)}{z^2-2z\left(\frac{e^{-\omega T}+e^{\omega T}}{2}\right)+1}\right]$$

$$=\frac{1}{2}\left[\frac{2z\sinh\omega T}{z^2-2z\cosh\omega T+1}\right]$$

$$=\frac{z\sinh\omega T}{z^2-2z\cosh\omega T+1},\ \text{(final result)}$$

$$=\frac{1}{2}\left[\frac{2\left(z^2-z\cosh\omega T\right)}{z^2-2z\cosh\omega T+1}\right]$$

$$=\frac{z^2-z\cosh\omega T}{z^2-2z\cosh\omega T+1}$$

$$=\frac{z\left(z-\cosh\omega T\right)}{z^2-2z\cosh\omega T+1}$$

$$=\frac{1}{2}\left[\frac{2z^2-z\left(e^{-\omega T}+e^{\omega T}\right)}{z^2-ze^{-\omega T}-ze^{\omega T}+1}\right]$$

$$=\frac{1}{2}\left[\frac{2z^2-2z\left(\frac{e^{-\omega T}+e^{\omega T}}{2}\right)}{z^2-2z\left(\frac{e^{-\omega T}+e^{\omega T}}{2}\right)+1}\right]$$

This is the required expression.

Table 15.1 *Laplace Transform and z-Transform of Certain Functions*

$x(t)$	$L[x(t)]$	$z[x(t)]$
1	$1/s$	$\frac{z}{z-1}$
t	$1/s^2$	$\frac{Tz}{(z-1)^2}$, $T=$ Sample time
e^{at}	$\frac{1}{s-a}$	$\frac{z}{z-e^{at}}$

15.5 CONVERSION OF LAPLACE TRANSFORM TO Z-TRANSFORM

We will show how to find z-transform of a function whose Laplace transform is given.

Let us assume that we have been given $G(s)$ and we want to find z-transform of $G(s)$, i.e., $G(z)$ or $z[G(s)]$.

Steps to be followed are as follows:

Step 1: Take inverse Laplace transform of $G(s)$, i.e. convert $G(s)$ into time domain
Let $L^{-1}[G(s)] = g(t)$

Step 2: Take z-transform of $g(t)$
i.e. $z[g(t)]$
so

$$\begin{aligned} z[g(s)] &= z\left[L^{-1}\{G(s)\}\right] \\ &= z[g(t)] \end{aligned}$$

Example 15.2 Find the z-transform of $G(s) = \dfrac{1}{s(s+1)}$.

Solution

$$\begin{aligned} z[G(s)] &= zL^{-1}\left[\left\{\frac{1}{s(s+1)}\right\}\right] \\ &= z\left[L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}\right] \\ &= z\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}\right] \\ &= z[1 - e^{-t}] \\ &= z[1] - z[e^{-t}] \\ &= \frac{z}{z-1} - \frac{z}{z-e^{-T}} \end{aligned}$$

Example 15.3 Find the z-transform of $G(s) = \dfrac{1}{s^2(s+2)}$.

Solution

$$z[G(s)] = z\left[L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}\right]$$

$$= z\left[L^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}\right\}\right]$$

$$= z\left[L^{-1}\left\{\frac{-1/4}{s}+\frac{1/2}{s^2}+\frac{1/4}{s+2}\right\}\right]$$

$$= z\left[-\frac{1}{4}\cdot 1+\frac{1}{2}\cdot t+\frac{1}{4}e^{-2t}\right]$$

$$= -\frac{1}{4}z[1]+\frac{1}{2}z[t]+\frac{1}{4}z[e^{-2t}]$$

$$= -\frac{1}{4}\cdot\frac{z}{z-1}+\frac{1}{2}\frac{Tz}{(z-1)^2}+\frac{1}{4}\frac{z}{(z-e^{-2T})}$$

Example 15.4 Find the z-transform of

$$x(KT)=\{2,3,\underset{\uparrow}{4},1,7\}$$

Solution

Given

$$\left.\begin{aligned} x(0)&=4\\ x(T)&=1\\ x(2T)&=7\\ x(-T)&=3\\ x(-2T)&=2 \end{aligned}\right\}\quad \begin{aligned} &x(KT)=\{2,3,\underset{\uparrow}{4},1,7\}\\ &x(-2)x(-1)\text{ arrow means } x(0)=4x(T)x(2T) \end{aligned}$$

The vertical arrow below 4 means $x(0) = 4$ to the right of the arrow, we have $x(T) = 1$, $x(2T) = 7$ and to the left of the arrow we have $x(-T) = 3$, $x(-2T) = 2$.
Now by definition

$$z[x(KT)]=\sum_{k=m}^{\infty}x(KT)z^{-k}$$

$$=\sum_{k-2}^{2}x(KT)z^{-k}$$

$$=x(-2T)z^2+x(-T)z^1+x(0)z^0+x(T)z^{-1}+x(2T)z^{-2}$$

$$=2z^2+3z+4z^0+1z^{-1}+7z^{-2}$$

$$=2z^2+3z+4+z^{-1}+7z^{-2}$$

15.6 INVERSE Z-TRANSFORM

There are two methods of taking inverse z-transform.
These methods are namely partial fraction method and long division method.

a) Inversion of the z-transform Using Partial Fraction

This method of finding z-transform has been illustrated through examples as follows:

Example 15.5 Find the inverse z-transform of

$$H(z)=\frac{1-\frac{21}{5}z^{-1}}{1+\frac{1}{10}z^{-1}-\frac{18}{25}z^{-2}}, \quad |z|>\frac{9}{10}.$$

Solution

Given

$$H(z)=\frac{1-\frac{21}{5}z^{-1}}{1+\frac{1}{10}z^{-1}-\frac{18}{25}z^{-2}},$$

It can be written as

$$=\frac{1-\frac{21}{5}z}{z^2+\frac{1}{10}z-\frac{18}{25}}=\frac{z\left(z-\frac{21}{5}\right)}{z^2+\frac{1}{10}z+\frac{18}{25}}$$

Example 15.6 Find the z-transform of $x(KT) = \{2, 4, 1, 3\}$.

Solution

Here arrow is not given, and by default it is on first position,

$$\begin{array}{cccc} x(KT)=\{2, & 4, & 1, & 3\} \\ \uparrow & \uparrow & \uparrow & \uparrow \\ k=0 & k=1 & k=2 & k=3 \end{array}$$

By definition

$$\begin{aligned} z[x(KT)] &= \sum_{k=0}^{3} x(KT)z^{-k} \\ &= 2z^0+4z^{-1}+z^{-2}+3z^{-3} \\ &= 2+4z^{-1}+z^{-2}+3z^{-3} \end{aligned}$$

Now making partial fractions of $\frac{H(z)}{z}$,

$$\frac{H(z)}{z} = \frac{z - \frac{21}{5}}{z^2 + \frac{1}{10}z - \frac{18}{25}} = \frac{z - \frac{21}{5}}{z^2 + \frac{9}{10}z - \frac{4}{5}z - \frac{18}{25}}$$

$$= \frac{z - \frac{21}{5}}{\left(z + \frac{9}{10}\right)\left(z - \frac{4}{5}\right)}$$

$$= \frac{A}{z + \frac{9}{10}} + \frac{B}{z - \frac{4}{5}}$$

Now,

$$A = \left.\frac{z - \frac{21}{5}}{z - \frac{4}{5}}\right|_{z=-\frac{9}{10}}$$

$$= \frac{-\frac{9}{10} - \frac{21}{5}}{-\frac{9}{10} - \frac{4}{5}}$$

$$= \frac{-\frac{51}{10}}{-\frac{17}{10}}$$

$$A = 3$$

Also,

$$B = \left.\frac{z - \frac{21}{5}}{z + \frac{9}{10}}\right|_{z=\frac{4}{5}}$$

$$= \frac{\frac{4}{5} - \frac{21}{5}}{-\frac{4}{5} + \frac{9}{10}}$$

$$= \frac{-\frac{17}{5}}{\frac{17}{10}}$$

$$B = -2$$

$$\frac{H(z)}{z} = \frac{3}{z + \frac{9}{10}} - \frac{2}{z - \frac{4}{5}}$$

$$\frac{H(z)}{z} = \frac{3}{z + \frac{9}{10}} - \frac{2}{z - \frac{4}{5}}$$

$$H(z) = 3 \cdot \frac{z}{z - \left(-\frac{9}{10}\right)} - 2 \cdot \frac{z}{z - \left(\frac{4}{5}\right)}$$

Taking inverse z-transform on both the sides,

$$h(K) = 3z^{-1}\left[\frac{z}{z - \left(\frac{-9}{10}\right)}\right] - 2z^{-1}\left[\frac{z}{z - \left(\frac{4}{5}\right)}\right]$$

$$= 3\left(-\frac{9}{10}\right)^k u(k) - 2\left(\frac{4}{5}\right)^k u(k),$$

$$\left[\because z^{-1}\left[\frac{z}{z - a^k}\right] = a^k u(k)\right]$$

where $u(k)$ is unit step.

Example 15.7 Find the inverse z-transform of $H(z) = \dfrac{1z^{-1}}{\left(1 - \frac{2}{3}z^{-1}\right)^2}, \quad |2| > \dfrac{2}{3}$.

Solution

Given

$$H(z) = \frac{1z^{-1}}{\left(1 - \frac{2}{3}z^{-1}\right)^2}$$

It can also be written as

$$= \frac{z^2 + z}{\left(z - \frac{2}{3}\right)^2}$$

Now,

$$\frac{H(z)}{z}=\frac{z+1}{\left(z-\frac{2}{3}\right)^2}$$

Taking partial fractions

$$=\frac{A}{z-\frac{2}{3}}+\frac{B}{\left(z-\frac{2}{3}\right)^2} \qquad \text{....(15.1)}$$

Now,

$$A=\frac{d}{dz}\left[\left(z-\frac{2}{3}\right)^2\cdot\frac{z+1}{\left(z-\frac{2}{3}\right)^2}\right]_{z=\frac{2}{3}}$$

$$=\frac{d}{dz}[z+1]_{z=\frac{2}{3}}=1$$

And

$$B=\left(z-\frac{2}{3}\right)^2\cdot\frac{z+1}{\left(z-\frac{2}{3}\right)^2}\Bigg|_{z=\frac{2}{3}}$$

$$=[z+1]_{z=\frac{2}{3}}$$

$$=\frac{2}{3}+1$$

$$=\frac{5}{3}$$

$$\left.\begin{array}{l}\text{So put } A=1\\ B=\frac{5}{3}\end{array}\right\}\text{in equation (13.1)}$$

$$\frac{H(z)}{z}=\frac{1}{z-\frac{2}{3}}+\frac{\frac{5}{3}}{\left(z-\frac{2}{3}\right)^2}$$

or

$$H(z) = \frac{z}{z-\frac{2}{3}} + \frac{5}{3}\frac{z}{\left(z-\frac{2}{3}\right)^2}$$

Take up inverse z-transform

$$h(k) = z^{-1}\left[\frac{z}{z-\frac{2}{3}}\right] + \frac{5}{3}z^{-1}\left[\frac{z}{\left(z-\frac{2}{3}\right)^2}\right]\omega$$

$$= \left(\frac{2}{3}\right)^k u(k) + \frac{5}{3}k\left(\frac{2}{3}\right)^k u(k)$$

$$\because \left[z^{-1}\left[\frac{z}{(z-a)^2}\right] = ka^k u(k)\right]$$

Example 15.8 Find inverse z-transform of $X(z) = \dfrac{8z-19}{(z-2)(z-3)}$.

Solution

Divide both sides by z, we get

$$\frac{X(z)}{z} = \frac{8z-19}{z(z-2)(z-3)}$$

Converting to partial fraction, we get

$$= \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

Multiply both sides by z

$$X(z) = \frac{-19}{6} + \frac{3}{2} + \frac{z}{z-2} + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

Obtain inverse z-transform of each term

$$x(n) = \frac{-19}{6}s(n) + \frac{3}{2}(2)^n u(n) + \frac{5}{3}(3)^n u(n)$$

b) Inverse *z*-transform by Long Division Method

Example 15.9 Find inverse *z*-transform of Long Division Method.

$$X(z) = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)}$$

Solution

Given

$$X(z) = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)}$$

$$= \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0.8z^{-0.1}}$$

Perform long division,

$$
\begin{array}{r|l}
 & 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} \\
\hline
z^3 - 1.7z^2 + 0.8z - 0.1 & 7z^3 - 2z^2 \\
 & \underline{7z^3 - 11.9z^2 + 5.6\,0z - 0.7} \\
 & 9.9z^2 - 5.6\,0z + 0.7 \\
 & \underline{9.9z^2 - 16.83z + 7.92 - 0.99z^{-1}} \\
 & 11.23z - 7.22 + 0.99z^{-1} \\
 & \underline{11.23z - 19.09 + 8.98z^{-1}} \\
 & 11.87 - 7.99z^{-1} \\
 & \vdots
\end{array}
$$

Thus,

$$X(z) = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)}$$

$$= 7 + 9.9z^{-1} + 11.23z^{-2} + 11.8z^3 + \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots$$

So,

$$x(0) = 7$$

$$x(1) = 9.9$$

$$x(2) = 11.23$$

$$x(3) = 11.87$$

$$\vdots$$

$$\text{so } x(n) = \{7, 9.9, 11.23, 11.87\}$$

15.7 PROPERTIES OF *Z*-TRANSFORM

a) Linearity Property

Statement: Let $x_1(n)$, $x_2(n)$ be two discrete sequences and

$$z[x_1(n)] = X_1(z)$$
$$z[x_2(n)] = X_2(z)$$

Then according to linearity property

$$z[ax_1(n) + bx_1(n)] = \sum_{n=\infty}^{\infty} [ax_1(n) + bx_2(n)]z^{-n}$$

$$= \sum_{n=\infty}^{\infty} ax_1(n)z^{-n} + bx_2(n)z^{-n}$$

$$= \sum_{n=\infty}^{\infty} ax_1(n)z^{-n} + \sum_{n=\infty}^{\infty} bx_2(n)z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$= aX_1(z) + bX_2(z)$$

Hence proved.

b) Time Shifting Property

Statement: Let $x(n)$ be a discrete time sequences and $z[x(n)] = X(z)$, then according to time shifting property of z-transform

$$z[x(n-k)] = z^{-k}X(z)$$

By definition

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Replace $x(n)$ by $x(n - k)$

$$z[x(n-k)] = \sum_{n=\infty}^{\infty} x(n-k) z^{-n}$$

Let $n - k = m$

$$\Rightarrow n = k + m$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-k-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-k} \cdot z^{-m}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-m}$$

$$\downarrow$$

$$= z^{-k} X(z) \ [\text{By definition}]$$

Hence proved.

c) Time Reversal Property

Statement: Let $x(n)$ be a discrete time sequences and $z[x(n)] = X(z)$, then according to time reversal property,

$$z[x(-n)] = X\left(\frac{1}{z}\right)$$

Proof: By definition

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Replace $x(n)$ by $x(-n)$

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let $-n = -m$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) \left(z^{-1}\right)^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \frac{1}{z}^{-m} = X\left(\frac{1}{z}\right)$$

Hence proved.

d) Time Convolution Theorem

Statement: Let $x_1(n)$, $x_2(n)$ be two discrete time sequences.

and $z[x_1(n)] = X_1(z)$

$$z[x_2(n)] = X_2(z),$$

Then according to the convolution theorem,

$$z[x_1(n) * x_2(n)] = X_1(z) \times X_2(z)$$

Proof:

By definition

$$z\left[x_1(n) * x_2(n)\right] = \sum_{n=-\infty}^{\infty}\left[x_1(n) * x_2(n)\right] z^{-n}$$

$$\Downarrow \text{(definition of conv)}$$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m)\right) z^{-n}$$

Change the order of summation

$$= \sum_{m=-\infty}^{\infty} x_1 m \left(\sum_{\substack{n=-\infty \\ \Downarrow}}^{\infty} x_2(n-m) z^{-n}\right)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) z\left[x_2(n-m)\right]$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} z\left[x_2(n)\right] \quad \text{time shifting property}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} X_2(z)$$

$$= X_1(z) \cdot X_2(z)$$

e) Derivative Property

Statement: Let $x(n)$ be a time discrete sequence and $z[x(n)] = X(z)$, then according to derivative property of z-transform $z\left[nx(n)\right] = -\dfrac{d}{dz}\left[X(z)\right]$

Proof:

By definition

$$z\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\text{or } X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiate w.r.t. z,

$$\frac{d}{dz}[X(z)] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n)(z^{-n-1})$$

$$= -\sum_{n=-\infty}^{\infty} n\,x(n)z^{-n} \cdot z^{-1}$$

$$= z^{-1}\sum_{\substack{n=-\infty \\ \downarrow}}^{\infty} [n\,x(n)]z^{-n}$$

$$= -\frac{1}{z}(nx(n))$$

or
$$-z\frac{d}{dz}[X(z)] = z[nx(n)]$$

f) Initial Value Theorem

Statement: Let $x(n)$ be discrete time casual sequence and $z[nx(n)] = X(z)$, then according to initial value theorem

$$x(0) = \underset{n\to 0}{Lt}\, x(n) = \underset{z\to\infty}{Lt}\, X(n)$$

Proof:

By definition

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Or
$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \ldots$$

Apply as $z \to \infty$

$$\underset{z\to\infty}{Lt}\, X(z) = \frac{x(1)}{\infty} + \frac{x(2)}{(\infty)^2} + \ldots$$

$$= x(0) + 0 + 0 + \ldots$$

$$= x(0)$$

$$= \underset{n\to 0}{Lt}\, x(n)$$

Hence proved.

$$x(0)=\underset{n\to 0}{Lt}\,x(n)=\underset{z\to\infty}{Lt}\,X(z)$$

g) Final Value Theorem

Statement: Let $x(n)$ be a discrete time casual sequence and $z[x(n)] = X(z)$, then according to final theorem

$$x(\infty)=\underset{n\to\infty}{Lt}\,x(n)=\underset{z\to 1}{Lt}\left(1-z^{-1}\right)X(z)$$

Proof:
By definition

$$z\left[x(n)\right]=\sum_{n=0}^{\infty}x(n)z^{-n}$$

Replace $x(n)$ by $[x(n) - x(n - 1)]$

$$z\left[x(n)-x(n-1)\right]=\sum_{n=0}^{\infty}\left[x(n)-x(n-1)\right]z^{-n}$$

$$X(z)z^{-1}X(z)=\sum_{n=0}^{\infty}\left\{x(n)-x(n-1)\right\}z^{-n}$$

$$1-z^{-1}X(z)=\sum_{n=0}^{\infty}\left\{x(n)-x(n-1)\right\}z^{-n}$$

Apply $z\to 1$

$$\underset{z\to 1}{Lt}\left(1-z^{-1}\right)X(z)=\underset{z\to 1}{Lt}\sum_{n=0}^{\infty}\left\{x(n)-x(n-1)\right\}z^{-n}$$

$$=\sum_{n=0}^{\infty}\left\{x(n)-x(n-1)\right\}\underset{z\to 1}{Lt}\,z^{-n}$$

$$=\sum_{n=0}^{\infty}\left\{x(n)-x(n-1)\right\}$$

$$=\left[x(0)-x(-1)\right]+\left[x(1)-x(0)\right]+\left(x(2)-x(1)\right)+\ldots$$

$$-x(-1)+x(\infty)=X(\infty)$$

15.8 HOLD CIRCUITS

Signal reconstruction from sample at the receiver end is done by different types of hold circuits. These hold circuits are also called Extrapolators.

The simplest hold circuit is called zero-order hold (ZOH).

15.8.1 Zero-order Hold (ZOH)

Zero-order hold means that the reconstructed signal has the same value as the last received sample for the entire sampling period.

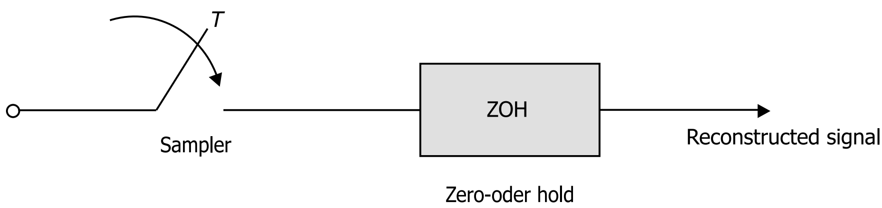


Fig. 15.5 Zero-order hold

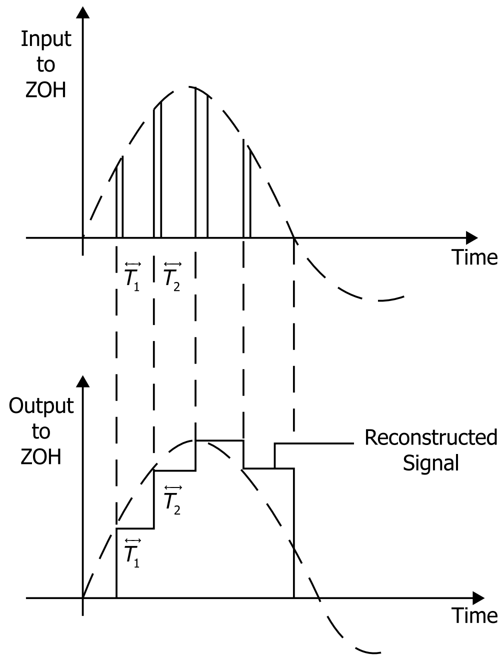


Fig. 15.6 Signal reconstruction by using ZOH

15.8.2 First-order Hold (FOH)

In a first-order hold, the last two signal samples are used to reconstruct the signal for the current sampling period. First order hold offers no particular advantage over the ZOH.

The simple ZOH when used in conjunction with a high sampling rate provides a satisfactory performance.

15.9 OPEN LOOP SAMPLED DATA CONTROL SYSTEM

It is the control system having no feedback loop. General block diagram of the system is shown in Fig. 15.7.

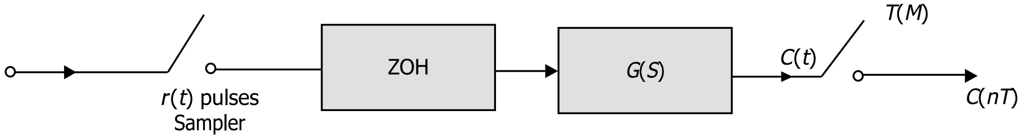


Fig. 15.7 *Open Loop control system*

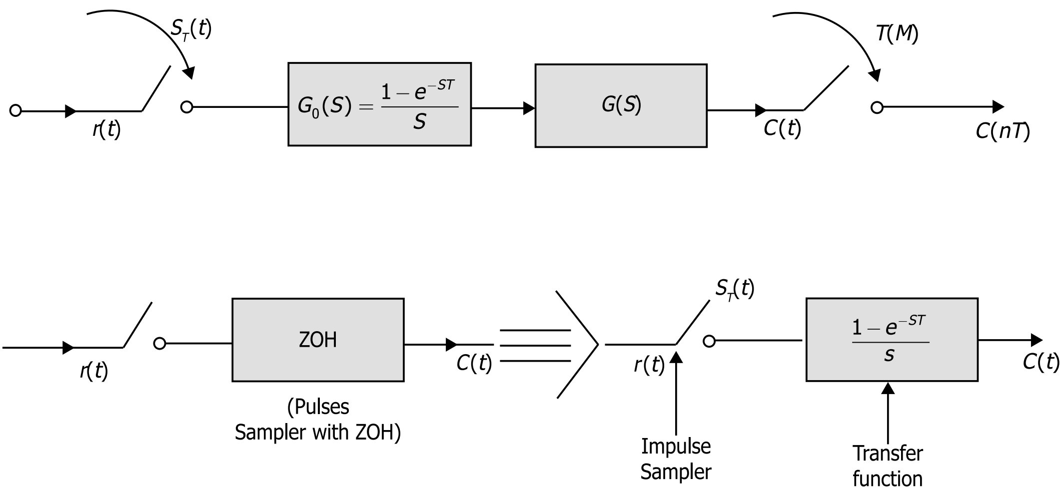


Fig. 15.8 *Pulse transfer function of a system*

Now z-transform of $G_0(s) \times G(s)$

$$= \left[z \left\{ \frac{1-e^{-ST}}{s} \right\} G(s) \right]$$

$$= \left[\frac{G(s)}{s} - \frac{e^{-ST} G(s)}{s} \right] \qquad \text{....(15.2)}$$

Suppose

$$L^{-1} \left[\frac{G(s)}{s} \right] = f_1(t)$$

$$\text{and } L^{-1} \left[\frac{e^{-ST} G(s)}{s} \right] = f_1(t-T)$$

Then

$$Z\left[\frac{e^{-ST}G(s)}{s}\right] = z\left[f_1(KT-T)\right]$$

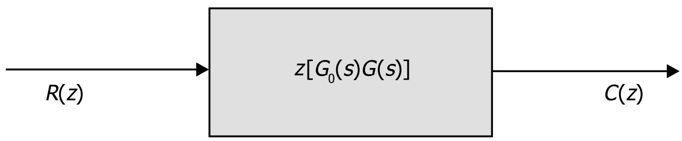
$$= z^{-1}z\left[f_1(KT)\right]$$

$$= z^{-1}z\left[\frac{G(s)}{s}\right]$$

From equation (15.2),

$$z\left[G_0(s)\cdot G(s)\right] = z\left[\frac{G(s)}{s}\right] - z\left[\frac{e^{-ST}G(s)}{s}\right]$$

$$= z\left[\frac{G(s)}{s}\right] - z^{-1}z\left[\frac{G(s)}{s}\right]$$



This system is equivalent to the block diagram representation as shown in Fig. 15.9.

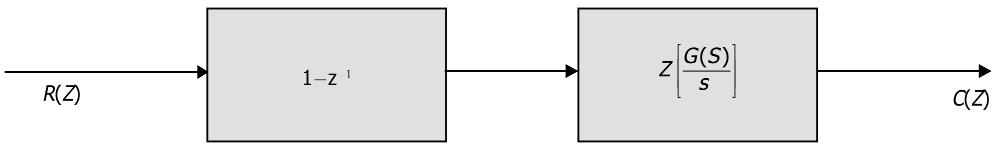


Fig. 15.9 Pulse transfer function

Hence, the pulse transfer function of open loop sampled data control system is written as

$$H(z) = \frac{C(z)}{R(z)}$$

$$= \left(1-z^{-1}\right)z\left[\frac{G(s)}{s}\right]$$

15.10 CLOSED LOOP SAMPLED DATA CONTROL SYSTEM

It is the sampled data control system with feedback. General block diagram of closed loop sampled data control system is shown in Fig. 15.10.

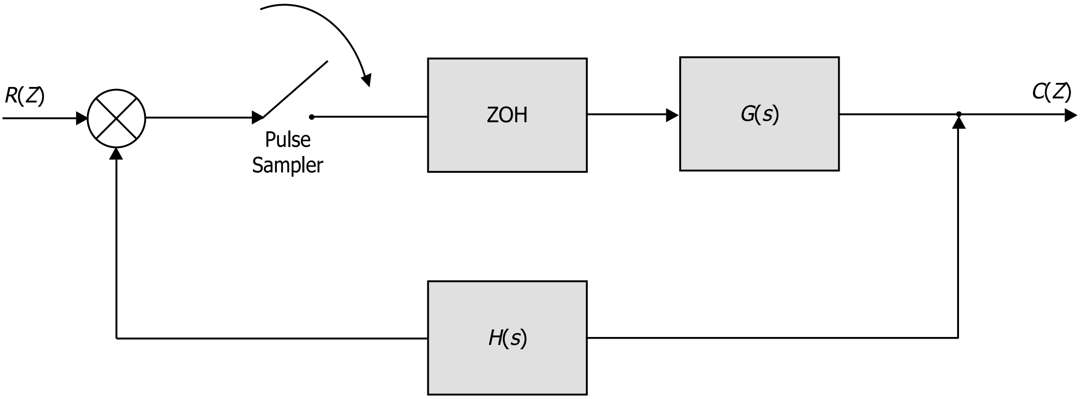


Fig. 15.10 *Closed loop controlled system*

Let us find its pulse transfer function, i.e., $H(z) = \dfrac{C(z)}{R(z)}$.

Converting pulse sampler into impulse sampler, we get the representation as shown in Fig. 15.11.

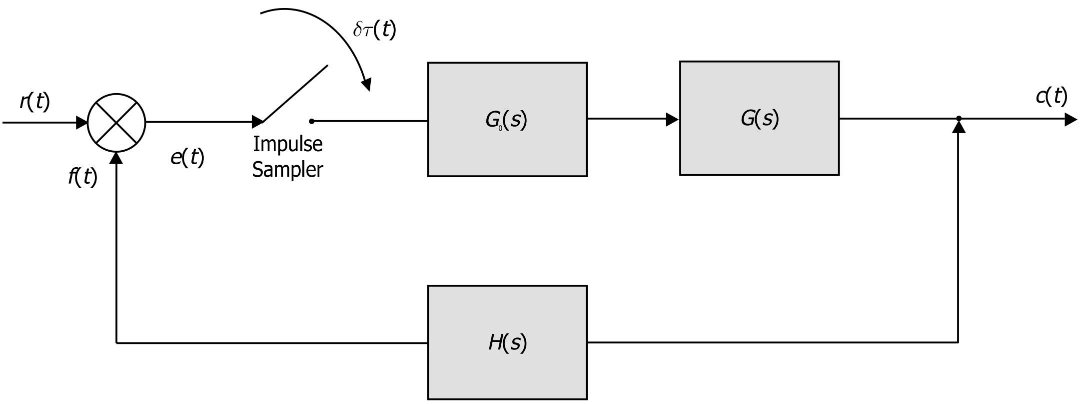


Fig. 15.11 *Conversion of pulse sampler into impulse sampler in the closed loop control system*

Now

$$C(z) = \{z[G_0(s)G(s)]\}E(z) \qquad \text{....(15.3)}$$

and

$$F(z) = \{z[G_0(s)G(s)H(s)]\}E(z) \qquad \text{....(15.4)}$$

and

$$e(t) = r(t) - f(t) \qquad \text{....(15.5)}$$

Take up z-transform, we get

$$E(z) = R(z) - F(z)$$

$$E(z) = R(z) - \{z[G_0(s)G(s)H(s)]\}E(z) \quad [\text{using}(13.4)]$$

$$E(z)\{1 + z[G_0(s)G(s)H(s)]\} = R(z) \qquad \text{....(15.6)}$$

Again $C(z) = z[G_0(s)G(s)H(s)]E(z)$ {from equation (15.3)}. Substituting the value of $E(z)$ from equation (15.6), we get

$$C(z) = z[G_0(s)G(s)] \times \frac{R(z)}{1 + z[G_0(s)G(s)H(s)]}$$

$$\frac{C(z)}{R(z)} = \frac{z[G_0(s)G(s)]}{1 + z[G_0(s)G(s)H(s)]}$$

$$H(z) = \frac{G_0G(z)}{1 + G_0GH(z)}$$

This is the required pulse transfer function for the given closed-loop system.

15.11 STATE SPACE REPRESENTATION OF DISCRETE TIME SYSTEMS

There are basically two methods for state space representation for discrete time systems:

a) Signal flow graph method
b) Partial fraction method

a) Signal flow graph method

We will first draw the signal flow graph for a general transfer function of the form,

$$T(z) = \frac{Y(z)}{U(z)} = \frac{b_0z^3 + b_1z^2 + b_2z + b_3}{z^3 + a_1z^2 + a_2z + a_3}$$

Dividing denominator and numerator by z^3,

$$T(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}}$$

The signal flow graph for the above transfer function is shown in Fig. 15.12.

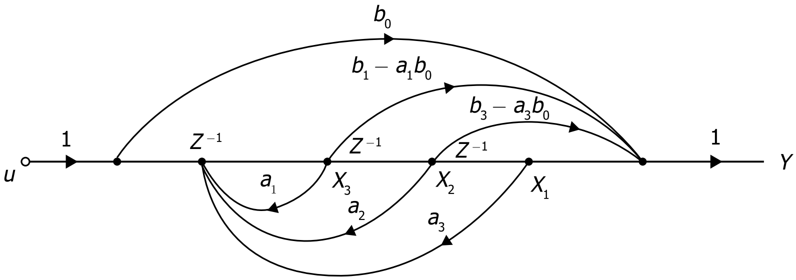


Fig. 15.12 *Signal flow graph for a transfer function of general form*

The method of state space representation by signal flow graph method is illustrated through an example.

Example 15.9 Find the state model for a system represented by the transfer function.

$$T(z) = \frac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6}$$

Solution

Dividing both denominator and numerator by Z^3,

$$T(z) = \frac{Y(z)}{U(z)} = \frac{1 + 8z^{-1}17z^{-2} + 8z^{-3}}{1 + 6z^{-1}11z^{-2} + 6z^{-3}}$$

The signal flow graph for the transfer function is drawn with reference to the signal flow graph drawn earlier for a general transfer function and is shown in Fig. 15.13.

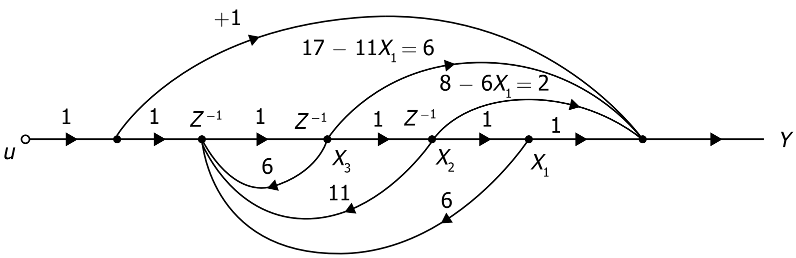


Fig. 15.13 *Signal flow graph for the given transfer function*

From the signal flow graph, the state equation are written as

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = x_3(k)$$

$$x_3(k+1) = u(k) - 6x_3(k) - 11x_2(k) - 6x_1(k)$$

And the output equation is written as

$$y(k) = u(k) - 2x_3(k) - 6x_2(k) - 2x_1(k)$$

The state model is represented in matrix form as

$$\begin{bmatrix} x_1 & (k+1) \\ x_2 & (k+1) \\ x_3 & (k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 k \\ x_2 k \\ x_3 k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

b) Partial Fraction Method

This method is best illustrated by an example as follows.

Example 15.10 Find the state model for using partial fraction method.

$$T(z) = \frac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6}$$

Solution

Given,

$$T(z) = \frac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6}$$

$$= \frac{z^3 + 8z^2 + 17z + 8}{(z+1)(z+2)(z+3)}$$

$$\frac{Y(z)}{U(z)} = 1 - \frac{1}{z+1} + \frac{1}{z+2} + \frac{1}{z+3}$$

Hence, so required state model is,

$$\begin{bmatrix} x_1 & (k+1) \\ x_2 & (k+1) \\ x_3 & (k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 k \\ x_2 k \\ x_3 k \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y(k)$$

and

$$y(k) = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 k \\ x_2 k \\ x_3 k \end{bmatrix}$$

15.12 STABILITY ANALYSIS

Let us have the block diagram of closed-loop digital control system as shown in Fig. 15.14.

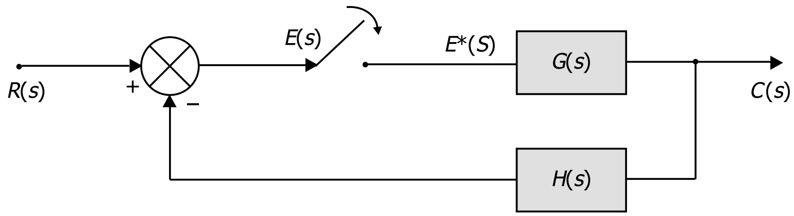


Fig. 15.14 *Block diagram of digital control system*

The stability can be determined by roots of the equation

$$1 + z[G(s)H(s)] = 0$$

This is also known as characteristic equation of the system.

The most commonly used method to find the stability of digital control systems is Jury's stability test.

Jury's stability test

It is an algebraic criterion for determining whether or not the roots of the characteristic equation lie within a unit circle thereby determining the stability of the system.

Let us have the characteristic polynomial of digital control system as follows:

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 = 0$$

Then according to Jury's test, necessary conditions for stability are:

1. $F(1) > 0$
2. $(-1)^n F(-1) > 0$

If the above conditions are satisfied, then the sufficient conditions are tested as follows. We prepare a table of coefficients of the characteristic polynomial as below.

Table 15.2 *Coefficients of the Characteristics Polynomial*

Row	z^0	z^1	z^2	...	z^{n-2}	z^{n-1}	z^n
1	a_0	a_1	a_2	...	a_{n-2}	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	...	a_2	a_1	a_0
3	b_0	b_1	b_2	...	b_{n-2}	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	...	b_1	b_0	
5	c_0	c_1	c_2	...	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	...	c_0		

Where

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, c_k$$

$$= \begin{vmatrix} b_0 & b_{n-1} \\ b_{n-1} & b_k \end{vmatrix}$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \ldots$$

The sufficient conditions for stability are as follows:

$$\left.\begin{array}{l} |a_0| < |a_n| \\ |b_0| > |b_{n-1}| \\ |c_0| > |c_{n-2}| \end{array}\right\} (n-1) \text{constraints.}$$

Example 15.11 Apply Jury's stability test to find stability of a digital control system whose characteristic equation is

$$F(z) = 4z^4 + 6z^3 + 12z^2 + 5z + 1$$

Here, $\boldsymbol{n} = 4$ (order of equation)

Solution

Necessary conditions are:

$$F(1) = 4 + 6 + 12 + 5 + 1 = 28 > 0$$

$$(-1)^n F(-1) = (-1)^4 \{4 - 6 + 12 - 5 + 1\}$$

$$= 1.\{6\}$$

$$= 6 < 0$$

Since

$$\left.\begin{array}{r} F(1) > 0 \\ (-1)^n F(-1) > 0 \end{array}\right\} \text{hence so both the necesssary conditions are satisfied.}$$

Let us check the sufficient conditions

Row	z^0	z^1	z^2	z^3	z^4
1	$a_0 = 1$	$a_1 = 5$	$a_2 = 12$	$a_3 = 6$	$a_4 = 4$
2	$a_4 = 4$	$a_3 = 6$	$a_2 = 12$	$a_1 = 5$	$a_0 = 1$
3	$b_0 = -15$	$b_1 = -19$	$b_2 = -36$	$b_3 = -14$	
4	$b_3 = -14$	$b_2 = -36$	$b_1 = -19$	$b_0 = -15$	
5	$c_0 = 29$	$c_1 = 219$	$c_2 = 274$		

Calculations are as follows:

$$b_0 = \begin{vmatrix} a_0 & a_4 \\ a_4 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} = 1 - 16 = -15$$

$$b_1 = \begin{vmatrix} a_0 & a_3 \\ a_4 & a_1 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ 4 & 5 \end{vmatrix} = -19$$

$$b_2 = \begin{vmatrix} a_0 & a_2 \\ a_4 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 12 \\ 4 & 12 \end{vmatrix} = -36$$

$$b_3 = \begin{vmatrix} 1 & 5 \\ 4 & 6 \end{vmatrix} = -14$$

and

$$c_0 = \begin{vmatrix} b_0 & b_3 \\ b_3 & b_0 \end{vmatrix} = \begin{vmatrix} -15 & -14 \\ -14 & -15 \end{vmatrix} = 29$$

$$c_1 = \begin{vmatrix} -15 & -36 \\ -14 & -14 \end{vmatrix} = -219$$

$$c_2 = \begin{vmatrix} -15 & -19 \\ -14 & -36 \end{vmatrix} = 274$$

Sufficient conditions for stability will be as follows:

$$|a_0| < |a_n|$$

$$|b_0| > |b_{n-1}|$$

$$|c_0| > |c_{n-2}|$$

Here

$$|a_0| < |a_4| \ (satisfied)$$

$$|b_0| > |a_3| \ (satisfied)$$

$$|c_0| > |c_2| \ (not\ satisfied)$$

Since all the sufficient conditions are not satisfied so system is unstable.

Example 15.12 Check the stability of the sampled data control system represented by the characteristic equation,

$$z^3 - 0.2z^2 - 0.25z + 0.05 = 0$$

Solution

Here, $n = 3$,

Degree of characteristic equation and

$$a_0 = 0.05, \quad a_1 = -0.25$$
$$a_2 = -0.2, \quad a_3 = 1$$

Necessary conditions are,

i) $F(1) = 1 - 0.2 - 0.25 + 0.05$
$= 0.6 > 0$

ii) $(-1)^n F(-1) = (-1)^3 [-1 - 0.02 + 0.25 + 0.05]$
$= 0.9 > 0$

Hence, necessary conditions are satisfied.
Let us check sufficient conditions as follows:

Row	z^0	z^1	z^2	z^3
1	$a_0 = 0.05 +$	$a_1 = -0.25$	$a_2 = -0.2$	$a_3 = 1$
2	$a_3 = 1$	$a_3 = -0.2$	$a_1 = -0.25$	$a_1 = 0.05$
3	$b_0 = -09975$	$b_1 = .1875$	$b_2 = .24$	
4	$b_2 = .24$	$b_1 = .1875$	$b_0 = -.9975$	

Now

$$|a_0| = 0.05 \text{ and } |a_3| = 1$$
$$|b_0| = 0.9975 \text{ and } |b_2| = 0.241$$

$$\text{Here } |a_0| < |a_3| \text{ (satisfied)}$$
$$|b_0| > |b_2| \text{ (satisfied)}$$

Since all the necessary and sufficient conditions are satisfied, hence according to Jury's test the given system is stable.

REVIEW QUESTIONS

15.1 Explain the advantages of digital control system over linear control system.

15.2 Using block diagram explain a sampled data control system.

15.3 Explain the sampling process. State sampling theorem.

15.4 What is *z*-transform and what is the use of *z*-transform in the analysis of control system?

15.5 Find the *z*-transform of a unit step function.

15.6 Find the *z*-transform of a ramp function.

15.7 Calculate the *z*-transform of (a) $x(t) = a^k$; (b) e^{-at}.

15.8 Find *z*-transform of sin ωt and cos ωt.

15.9 Find the *z*-transform of sinh ωt and cosh ωt.

15.10 Show how *z*-transform can be obtained from the Laplace transform of a system.

15.11 Find the *z*-transform of $G(s) = \dfrac{1}{s(s+1)}$ and $G(s) = \dfrac{1}{s^2(s+1)}$.

15.12 Explain the two methods of finding inverse *z*-transform.

15.13 State and prove final value theorem.

15.14 Find the pulse transfer function of a closed-loop control system.

15.15 State and explain Jury's stability test.

15.16 The characteristic polynomial of a system is $F_1(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$. Employing stability criteria, comment on the system stability.

APPENDIX 1
LAPLACE TRANSFORM

A1.1 INTRODUCTION

For a proper understanding of the behaviour of a system, one has to represent the system in the form of mathematical relations. The voltage equation in algebraic form or a differential equation of an electrical circuit are a few such mathematical relations.

The Laplace transformation offers a very simple and elegant method of solving linear or linearised differential equations which result from the mathematical modelling of a process or a system. The Laplace transform also allows us to develop input-process-output models which are very useful for the study of control systems. It is because of the above reasons that the Laplace transform is used extensively in control engineering.

A1.2 DEFINITION OF LAPLACE TRANSFORM

Let there be a function $f(t)$. The Laplace transform of $f(t)$, denoted by $\mathscr{L}[f(t)]$ is defined as

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt \qquad \text{....(A1.1)}$$

It can also be represented as

$$\mathscr{L}[f(t)] = F(s) \qquad \text{....(A1.2)}$$

Laplace transformation is a transformation of a function from the time domain (where time is the independent variable) to the 5 domain (where s is the independent variable). s is a variable defined in the complex plane as $s = a + ib$.

Laplace transformation of a function $f(t)$ exists if the integral $\int_0^\infty f(t)e^{-st}dt$ remains bound, that is, takes, a finite value. For example, consider a function

$$f(t) = e^{at}.$$

The Laplace transform of the above is given as

$$\mathcal{L}[f(t)] = \int_0^\infty e^{at} e^{-st} dt \int_0^\infty e^{(a-s)t} dt$$

Now, if $a - s > 0$ or $a > s$, then the integral remains unbound.

However, if $s > a$ then there will be a finite solution. Thus the Laplace transform of the function exists for $s > a$ only.

The Laplace transformation is a linear operation. For example,

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = [a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)]$$

where a_1 and a_2 are constant parameters.

Proof:

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)] e^{-st} dt$$

$$= a_1 \int_0^\infty f_1(t) e^{-st} dt + a_2 \int_0^\infty f_2(t)] e^{-st} dt$$

$$= a_1 + [f_1(t)] + a_2 + [f_2(t)] \qquad \text{....(A1.3)}$$

A1.3 LAPLACE TRANSFORM OF SOME BASIC FUNCTIONS

Let us consider the following three types of functions:

1. Exponential functions;
2. Trigonometric functions; and
3. Standard Test Signals.

1. Exponential functions: An exponential function as shown in Fig. A1.1 is defined as

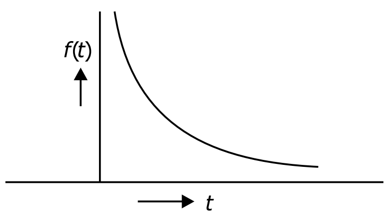


Fig. A1.1 *Exponential function representation*

$$f(t) = e^{-at} \text{ for } t \geq 0.$$

Then, $\mathscr{L}[e^{-at}] = \dfrac{1}{s+a}$.

Proof:

$$\mathscr{L}[e^{-at}] = \int_0^\infty e^{-at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = \frac{1}{s+a} \qquad \text{....(A1.4)}$$

Similarly, $$\mathscr{L}[e^{at}] = \frac{1}{s-a} \qquad \text{....(A1.5)}$$

2. Trigonometric functions: Consider the sinusoidal function $f(t) = \sin \omega t$

Then, $$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} \qquad \text{....(A1.6)}$$

Proof:

$$\mathscr{L}[\sin \omega t] = \int_0^\infty \sin \omega t\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t}) e^{-st}\, dt \qquad \left[\because \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

$$= \int_0^\infty \frac{1}{2j}\left[e^{-(s-j\omega)t} dt - e^{-(s+j\omega)t}\right] dt = \frac{1}{2j}\left[-\frac{e^{-(s-j\omega)t}}{s-j\omega} + \frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{s+j\omega-s+j\omega}{s^2-(j\omega)^2}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

Similarly, $$\mathscr{L}[\cos \omega t] = \frac{s}{s^2+\omega^2} \qquad \text{....(A1.7)}$$

$$\left(\text{Note} : \cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]\right)$$

A1.4 STANDARD TEST SIGNALS

In time-domain analysis, the dynamic response of the system to different types of inputs, which are a function of time, is analysed at different intervals of time after the application of the input signal. In practice, the input signals applied to a measurement system are not known ahead of time. In most cases, the actual input signals vary in random fashion with respect to time and, therefore, cannot be mathematically defined. However, for the purpose of analysis and design, it is necessary to assume some basic types of input signals which can be easily defined mathematically, so that the performance of a system can be analysed with these standard signals.

In the time-domain analysis, the following standard test signals are used:

i) Step signal;
ii) Ramp signal;
iii) Parabolic signal; and
iv) Impulse signal.

i) Step signal: A step signal is one the value of which changes from one level (usually zero) to another level, A, in zero time.

Mathematically, it is defined as

$r(t) = A;\ t \geq 0.$

$= 0;\ t < 0.$

The step signal is plotted in Fig. A1.2.

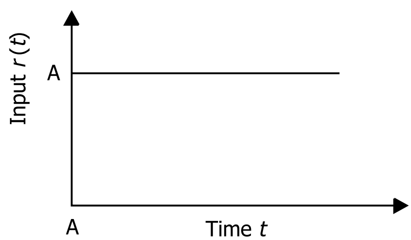


Fig. A1.2 *A step signal representation*

In the Laplace form,

$$\mathcal{L}[r(t)] = R(s) = \int_0^\infty Ae^{-st}\,dt$$

$$= A\int_0^\infty e^{-st}\,dt$$

$$= A\left[\frac{e^{-st}}{-s}\right]_0^\infty = A\left[e - \frac{e^0}{-s}\right]$$

$$R(s) = \frac{A}{s} \qquad \text{....(A1.8)}$$

If step signal is a unit step, then $A = 1$.

So, $$R(s) = \frac{1}{s}$$

ii) Ramp signal: A ramp signal is one which starts at a value of zero and increases linearly with time like velocity and therefore is also known as velocity signal as shown in Fig. A1.3.

Mathematically, $r(t) = At;\ t > 0.$
$= 0;\ t < 0.$

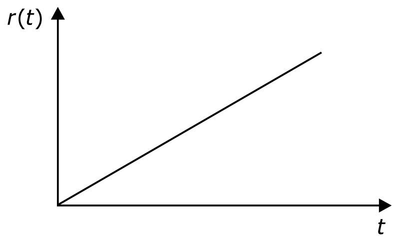


Fig. A1.3 *A ramp signal*

In the Laplace form,

$$\mathcal{L}[r(t)] = R(s) = \int_0^\infty At\, e^{-st} dt = A\int_0^\infty t\, e^{-st} dt$$

$$= A\left[\frac{te^{-st}}{-s}\right]_0^\infty - A\int_0^\infty \frac{e^{-st}}{-s} dt = A\left[\frac{te^{-st}}{-s}\right]_0^\infty + A\left[\frac{e^{-st}}{-s.s}\right]_0^\infty$$

$$\left[\because \int UVdt = U\int Vdt - \int\left(\frac{du}{dt}\int Vdt\right)dt\right]$$

From equations (A1.8) and (A1.9) we see that the ramp signal is a integral of the step signal.

$$= 0 + A\left[\frac{e^{-\infty}}{-s^2} - \frac{e^{-0}}{-s^2}\right] = 0 + \frac{A}{s^2} \quad [\because e^{-\infty} = 0]$$

So, $$\mathcal{L}[r(t)] = R(s) = \frac{A}{s^2} \qquad \text{....(A1.9)}$$

iii) Parabolic signal: A parabolic signal is one which is proportional to the square of time and therefore, represents a constant acceleration. Hence the signal is also known as acceleration signal.

Mathematically, $r(t) = t^2; t > 0.$
$= 0; t < 0.$

Fig. A1.4(a) and (b) show the parabolic signals.

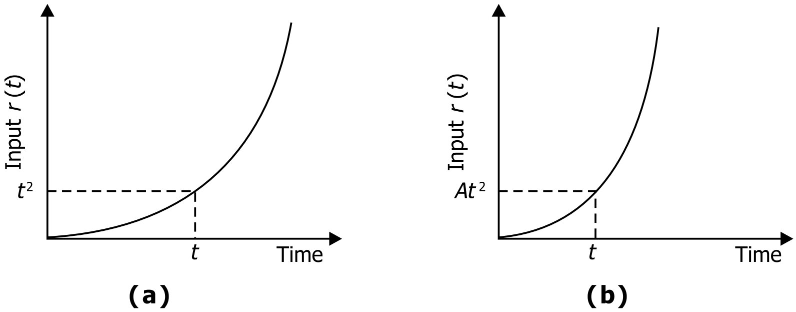


Fig. A1.4 *Parabolic signal representation*

The Laplace transform of a unit parabolic signal is

$$R(s) = \int_0^\infty t^2 e^{-st} dt = \frac{2}{s^3} \qquad \text{....(A1.10)}$$

if the parabolic signal has the magnitude A as shown in Fig. A1.4(b).

Mathematically, $r(t) = At^2 - 1 > 0$
$= 0 \; t < 0$

The Laplace transform

$$R(s) = \int_0^\infty At^2 e^{-st} dt = A\int_0^\infty t^2 e^{-st} dt$$

$$\frac{2!}{S^3} \qquad \text{....(A1.11)}$$

From expressions (A1.9) and (A1.11) it is clear that the parabolic signal is the integral of the ramp signal.

iv) Impulse signal: A unit impulse is defined as a signal which has zero value everywhere except at $t = 0$ where the magnitude is finite. This function is generally called as (delta) function. The δ function has the following properties:

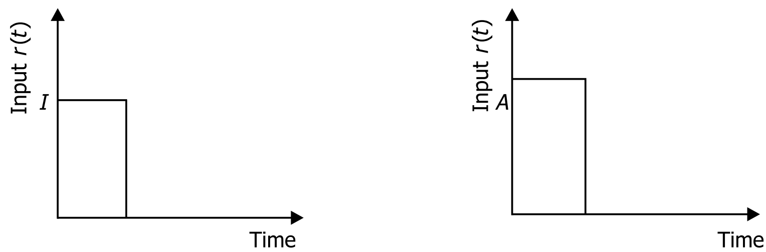
$$\delta t = = 0; t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1$$

where $\varepsilon \to 0$.

Since a perfect impulse function is not practically realisable, it is approximated by a pulse of small width having a unit area as shown in Fig. A1.5(a). The Laplace transform of unit impulse is

$$\mathscr{L}[d(t)] = 1 \qquad \text{....(A1.12)}$$



(a) *Unit impulse function* **(b)** *Impulse function of strength A*

Fig. A1.5

A1.5 TRANSLATED FUNCTIONS

Consider the function $f(t)$ shown in Fig. A1.6(a). If this function is delayed by t_0 seconds, it will be as shown in Fig. A1.6(b) and if it is advanced by t_0 seconds, it will be as shown in Fig. A1.6(c).

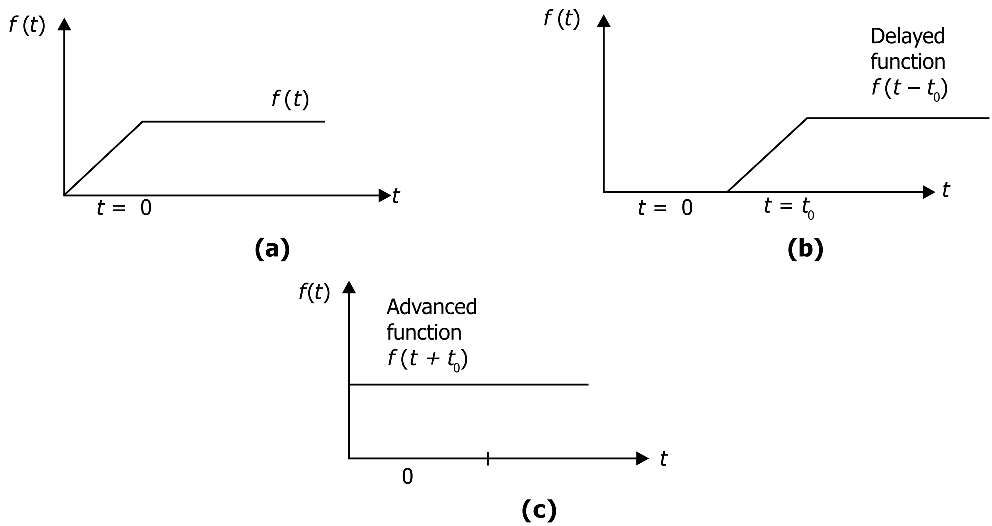


Fig. A1.6 *Translated functions*

The relationship among the three curves is given as

$$f(t + t_0) = f(t) = f(t -- t_0)$$

A 1.6 SOME LAPLACE TRANSFORMS

a) Let $\mathscr{L}[f(t)] = F(s)$ be the Laplace transform of $f(t)$. Then,

$$\mathscr{L}[f(t-t_0)] = e^{-st_0}F(s) \qquad \text{....(A1.13)}$$

and

$$\mathscr{L}[f(t+t_0)] = e^{-st_0}F(s) \qquad \text{....(A1.14)}$$

Proof:

$$\mathscr{L}[f(t-t_0)] = \int_0^\infty f(t-t_0)e^{-st}dt$$

$$= \int_0^\infty f(t-t_0)e^{-st}e^{st_0}e^{-st_0}dt$$

$$= e^{-st_0}\int_0^\infty [f(t-t_0)]e^{-st}e^{st_0}dt$$

$$= e^{-st_0}\int_0^\infty [f(t-t_0)]e^{-s(t-t_0)}d(t-t_0)$$

Since $dt = d(t - t_0)$
let $t - t_0 = \tau$, then

$$e^{-st_0}\int_0^\infty f(t-t_0)e^{-s(t-t_0)}d(t-t_0) = e^{-st_0}\int_{-t_0}^\infty f(\tau)e^{-s\tau}d\tau$$

$$= e^{-st_0}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-st_0}F(s)$$

[The lower limit $- t_0$ is replaced by 0. This will not change the value of the integral since $f(\tau) = 0$ for $\tau < 0$].

Similarly we can find for, $\mathscr{L}[f(t + t_0)]$ as equal to $e^{st_0}F(s)$.

b) Unit pulse function Consider a pulse signal as shown in Fig. A1.7.
The pulse has a height of $1/A$ and width equal to A. Therefore, the area under the curve is given as, area $= A\frac{1}{A} = 1$.
This function is a unit pulse function of duration A and is defined by

$$\delta_A(t) = \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{1}{A} & \text{for } 0 < t < A \\ 0 & \text{for } t > A \end{cases}$$

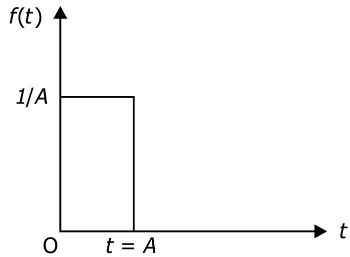


Fig. A1.7 *Unit pulse function*

It can also be defined as the difference of two step functions of equal size $1/A$. The first function occurs at time $t = 0$ while the second one is delayed by A units of time. Thus, if

first step function $$f_1(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{A} & t > 0 \end{cases}$$

and second step function $$f_2(t) = \begin{cases} 0 & t < A \\ \dfrac{1}{A} & t > A \end{cases}$$

then $\boldsymbol{\delta}_A(t)$ = unit pulse of duration A

$$= f_1(t) - f_2(t) = f_1(t) - f_1(t - A)$$

$$\mathscr{L}[\delta_A(t)] = +[f_1(t) - f_1(t - A)] = +[f_1(t)] - +[f_1(t - A)]$$

$$= \frac{1}{AS} - e^{-sA} + [f_1(t) = \frac{1}{AS} = -e^{-sA}\,\frac{1}{AS} = \frac{1}{A}\left[\frac{1 - e^{-sA}}{s}\right]$$

Thus the Laplace transform of a unit pulse function of duration A is given as,

$$\mathscr{L}[\delta_A(t)] = \frac{1}{A}\left[\frac{1 - e^{-sA}}{s}\right].$$

c) Laplace transform of derivatives

$$\mathscr{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

where, $F(s) = \mathcal{L}[f(t)]$

Proof:

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$= \left[e^{-st}f(t)\right]_0^\infty + \int_0^\infty se^{-st}f(t)d \quad \left[\text{using UV formula for integration}\right]$$

$$= [0 + f(0)] + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Similarly, it can be proved that

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0) - f'(0)$$

where, $f'(0) = \dfrac{df(t)}{dt}$ at $t = 0$.

In general,

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - sf^{(n-2)}(0) - f^{(n-1)}(0). \quad \text{....(A1.15)}$$

Thus, in order to find the Laplace transform of an *nth* order derivative we need *n* initial conditions.

$$f(0), f'(0), f''(0), \ldots, f^{(n-1)}(0).$$

d) Laplace transforms of integrals

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

where, $F(s) = \mathcal{L}[f(t)]$

Proof:

$$\left[\int_0^t f(t)dt\right] = \int_0^\infty \left[\int_0^t f(t)dt\right]e^{-st}dt$$

Integrating by parts,

put $U = e^{-st}$ and $V = \int_0^t f(t)\,dt$ then,

$$dU = -se^{-st}\,dt \text{ and } dV = f(t)\,dt$$

$$e^{-st}\,dt = -\frac{1}{s}\,dU$$

Now $$\int_0^\infty \left[\int_0^t f(t)dt\right] e^{-st}\,dt.$$

Putting the values of $e^{-st\,dt}$ and $\int_0^t f(t)dt$,

$$= \int_0^\infty \frac{VdU}{-s} = -\frac{1}{s}\int_0^\infty VdU = -\frac{1}{s}\,|(VU)|_0^\infty + \frac{1}{s}\int_0^\infty UdV$$

$$= -\frac{1}{s}\left[\int_0^t f(t)dte^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$

$$= -\frac{1}{s}(0-0) + \frac{1}{s}\overline{F}(s) = \frac{1}{s}\overline{F}(s)$$

So, $$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s).$$

A1.7 THEOREMS OF LAPLACE TRANSFORM

a) Final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$$

where, $$F(s) = \mathcal{L}[f(t)]$$

Proof:

Using Laplace transform of derivative we have,

$$\int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt = sF(s) - f(0)$$

Take the limit of both sides as $s \neq 0$

$$\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt = \lim_{s\to 0}[sF(s) - f(0)]$$

Since variable s is independent of time,

$$\int_0^\infty \lim_{s\to 0} \frac{df(t)}{dt} e^{-st} dt = \lim_{s\to 0} [sF(s) - f(0)]$$

or,
$$\int_0^\infty \frac{df(t)}{dt} dt = \lim_{s\to 0} [sF(s) - f(0)]$$

So,
$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{....(A1.16)}$$

b) Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

where,
$$G(s) = \int_0^\infty f(t) e^{-st} dt = +[f(t)].$$

Proof:
We know that

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

or,
$$\int_0^\infty \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0)$$

Taking $\lim\limits_{s\to\infty}$ to both sides,

$$\lim_{s\to\infty} \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \lim_{s\to\infty} [sF(s) - f(0)]$$

$$\int_0^\infty \lim_{s\to\infty} \frac{df(t)}{dt} e^{-st} dt = \lim_{s\to\infty} [sF(s) - f(0)]$$

$$0 = \lim_{s\to\infty} sF(s) - f(0)$$

$$f(0) = \lim_{s\to\infty} sF(s)$$

or,
$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{....(A1.17)}$$

Laplace transforms of various functions are shown in Table A1.1.

Table A1.1 ***Laplace Transforms of Various Function***

Time Function ($t \geq 0$)	Laplace Transform
Unit impulse, $\delta(t)$	1
Unit pulse, $\delta_A(t)$	$\frac{1}{A}\frac{1-e^{-sA}}{s}$
Unit step	$\frac{1}{s}$
Ramp, $f(t) = t$	$\frac{1}{s^2}$
t^2	$\frac{2!}{S^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
sin (ωt)	$\frac{\omega}{s^2+\omega^2}$
cos (ωt)	$\frac{s}{s^2+\omega^2}$
sin $h(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
cos $h(\omega t)$	$\frac{s}{s^2-\omega^2}$
e^{-at} sin (ωt)	$\frac{\omega}{(s+a)^2+\omega^2}$
e^{-at} cos (ωt)	$\frac{(s+a)}{(s+a)^2+\omega^2}$

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APPENDIX 2
MATLAB FUNDAMENTALS

A2.1 INTRODUCTION

MATLAB, an abbreviation for Matrix Laboratory, is an interactive matrix-based program for scientific and engineering calculations. The base program together with a toolbox (m-files) meant for a particular topic extends the capability to use MATLAB in various fields such as Control Systems, Statistics, Fuzzy Logic, Neural Network and so on.

A MATLAB session can be started on most systems by executing the command MATLAB. To end the MATLAB session simply execute the command exit or quit.

A MATLAB program line that begins with '%' are comments, or remarks which are not executed. In the event of comments or remarks being more than one line, each line must begin with %.

A semicolon at the end of a statement is used to suppress printing, that is, its result is not displayed without hampering its execution. While entering a matrix, the semicolon is also used to indicate the end of a row, except at the last row.

The colon operator is used in MATLAB to create vectors, to subscript matrices and to specify for iteration. As an example, $m : n$ means $[m\ m + 1\ \ldots\ldots..n]$, $B(:, j)$ means the jth column of B, and $B(i, :)$ means ith row of B.

In the following sections we will deal with four objects that allow us to interact with the programs.

A2.2 STATEMENTS AND VARIABLES

In MATLAB, statements are formed by assigning an expression to a variable as follows: variable = expression

For example, $A = [1\ 2;\ 3\ 4]$ is a statement whose execution results in the matrix

$$A = \begin{bmatrix} 1.0000 & 2.0000 \\ 3.0000 & 4.0000 \end{bmatrix}$$

But the statement $A = [1\ 2;\ 3\ 4]$ will produce no output after its execution.

The usual mathematical operators like + for addition, – for subtraction, * for multiplication, / for division and ^ for power can be used in expressions and to alter order of operation, parentheses are to be used.

A convenient feature of MATLAB is that the variables need not be dimensioned before use. A variable name begins with a letter and is followed by letters and numbers comprising the maximum length of 19 characters. If no variable is assigned to an expression, the result is assigned to a generic variable.

MATLAB is case sensitive. The variables *n* and *N* are recognised as different quantities. The command 'who' in lowercase gives a list of all variables in workspace. The command 'clear' will erase all non-permanent variables from the workspace. The command 'clear' *x* will erase a particular variable *x*.

MATLAB has many pre-defined variables like *pi* (for π), *Inf* (*for* $+\infty$), *NaN* (for Not-a-Number) (for $\sqrt{-1}$), *j* (for $\sqrt{-1}$). One may use *i* as an integer and reserve *j* for complex arithmetic. Predefined variables can be reset to their default value by using 'clear' name (for example, clear *pi*).

To continue a long statement in the next line an ellipsis consisting of three or more periods (…) followed by the carriage return can be used. Many statements can be placed in one line by using commas or semicolons.

In MATLAB all computations are performed in double precision. The default output format contains four decimal places for non-integers. However, there are four possible ways of displaying output by using 'format' command as follows:

a) pi or format short; pi
   will display a 5-digit scaled fixed point number.
   ans =
   3.1416.
b) format long; *pi*
   will display a 15-digit scaled fixed point number.
   ans =
   3.14159265358979.
c) format short *e; pi*
   will display a 5-digit scaled floating point number.
   ans =
   $3.1416e + 00$.
d) format long *e*; *pi*
   will display a 15-digit scaled floating point number.
   ans =
   $3.141592653589793e + 00$.

The variables used in the workspace can be stored in a file named matlab.mat by using 'save' command.

A2.3 MATRICES

The basic computational unit in MATLAB is the matrix and its functions. No dimension or type statements are needed in using matrices. The elements of a matrix can contain real and complex numbers, elementary math functions, as well as trigonometric and logarithmic functions. A typical matrix expression is written within square brackets, [.]. The column elements are separated by blanks or commas and the rows are separated by semicolon or carriage returns. Matrices can be input across many lines by using a carriage return following the semicolon or in place of the semicolon. For an example,

$$\begin{aligned}&A=-2*j \quad \sin(\text{pi}/s \quad 3;\\&\quad 2.5 \quad -6 \quad 0.5 \qquad\qquad\qquad \text{carriage return}\\&\quad \cos(\text{pi}/3) \quad \exp(0.8) \quad 0]\end{aligned}$$

will produce,

$$A=\begin{bmatrix}0-2.0000i & 1.0000 & 3.0000\\ 2.5000 & -6.0000 & 0.5000\\ 0.5000 & 2.2255 & 0.0000\end{bmatrix}$$

Basic matrix operations are addition, subtraction, multiplication, transpose, powers and array operations. Some examples follow.

Example 1 Addition of matrices

$$\begin{aligned}&A=[1 \quad 2; \quad 3 \quad 4];\\&B=[0 \quad 2; \quad -1 \quad 1];\\&A+B\end{aligned}$$

will produce
ans =

$$\begin{bmatrix}1.0000 & 4.0000\\ 2.0000 & 5.0000\end{bmatrix}$$

Example 2 Multiplication of matrices
$C = [1; 2]$
$A * C$
will generate
ans =

$$5.0000$$
$$11.0000$$

Example 3 Transpose of matrix

A
will produce
ans =

$$\begin{bmatrix} 1.0000 & 3.0000 \\ 2.0000 & 4.0000 \end{bmatrix}$$

If u and v are $m \times 1$ vectors, then $u' * v$ is called the inner product (or dot product) which is a scalar and $u * v'$ is called the outer product which is a $m \times m$ matrix.

The modified element-by-element operations of matrices are known as array operation. Array multiplication (.*), division (./) and power (.^) operators require the preceding dot. Example of array multiplication

$$A = [1; \quad 3; \quad 2];$$
$$B = [-1; \quad 2; \quad -5];$$
$$A.*\,B$$

will produce
ans =

$$\begin{bmatrix} -1.0000 \\ 6.0000 \\ -10.0000 \end{bmatrix}$$

The colon notation is used to generate a row vector $x = [x_i : dx : x_f]$, where x_i is the initial value, dx is the increment and x_f is the final value.

Example 4

$x = [0.2 : 0.2 : 1];$ $\quad$ $y = x .* \sin(x);$
$[x \quad y]$
will produce two vectors x and y
ans =

$$\begin{bmatrix} 0.2000 & 0.0397 \\ 0.4000 & 0.1558 \\ 0.6000 & 0.3388 \\ 0.8000 & 0.5793 \\ 1.0000 & 0.5415 \end{bmatrix}$$

If A is an $n \times n$ matrix, the n scalars λ that satisfy $Ax = \lambda x$ are the eigenvalues of A. They are found by using the command eig(A). For example,

$$A = [2 \quad 1; \quad 0 \quad 3]$$

eig (A)
will produce
ans =

$$\begin{bmatrix} 2.0000 \\ 3.0000 \end{bmatrix}$$

Eigenvectors are obtained from the double assignment statement

$$[X, D] = \text{eig}\ (A)$$

where the diagonal elements of D are eigenvalues and columns of X are the corresponding eigenvectors, so that $AX = XD$. For example, if

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Then the statement
$[X, D]$ = eig (A)
will produce

$$\sqrt{-1}$$

The coefficients of the characteristic polynomial of a matrix 'A' are obtained by using the statement p = poly (A). For example,

$$A = [1 \quad 2 \quad 3 \quad ; \quad 4 \quad 5 \quad 6 \quad ; \quad 7 \quad 8 \quad 0]$$

p = poly (A)
will produce the characteristic polynomial $s^3 - 6s^2 - 12s - 27$ by giving the output

$$P = 1.0000 \quad -6.0000 \quad -72.0000 \quad -27.0000$$

The roots of the characteristic equation $s^3 - 6s^2 - 12s - 27 = 0$ will be obtained as follows.
r = roots (p)
produces
r =

$$\begin{bmatrix} 12.1229 \\ -5.7345 \\ -0.3884 \end{bmatrix}$$

The following command will get back the original polynomial from the given roots.
q = poly (r)

will produce

$$q = 1.0000 \quad -6.0000 \quad -72.0000 \quad -27.0000$$

To evaluate the polynomial $p(s) = s^4 + 5s^2 + 7$ at $s = 1$, the following command

$$p = [1 \quad 0 \quad 5 \quad 0 \quad 7];$$

polyval (p, 1)
produces
ans = 13.0000

The command [r, p, k] = residue (num, den) will express the ratio $b(s)/a(s)$ of two polynomial into partial fraction given by

$$\frac{b(s)}{a(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \ldots + \frac{r(n)}{s - p(n)} + k(s)$$

where, $r(1)$, $r(2)$, … etc., are the residues, $p(1)$, $p(2)$, … etc., are the poles and $k(s)$ is the direct (or constant) term. For example,
num = [2 5 3 6];
den = [1 6 11 6];
[r, p, k] = residue (num, den)
will produce

$$r = \begin{bmatrix} -6.0000 \\ -4.0000 \\ 3.0000 \end{bmatrix}$$

$$p = \begin{bmatrix} -3.0000 \\ -2.0000 \\ -1.0000 \end{bmatrix}$$

$k = [2]$
This MATLAB result gives the following partial fraction

$$\frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s + 11s + 6} = \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

The command c = conv (a, b) will produce the multiplication of two polynomials $a(s)$ and $b(s)$. For example,
a = [1 2 3];
b = [4 5 6];
c = conv (a, b)

produces

$$c = 4.0000 \quad 13.0000 \quad 28.0000 \quad 27.0000 \quad 18.0000$$

The command deconv (c, a) will divide polynomial $c(s)$ by $a(s)$ to give quotient and remainder polynomials of $q(s)$ and $r(s)$ respectively.

A2.4 GRAPHICS

Graphics play an important role in the design and analysis of control systems. MATLAB uses a graph display to present plots. Plot formats and the customised plots are the two basic groups of graphic functions which are given in Tables A2.1 and A2.2. *A* basic *x-y* plot is generated with the combination of functions plot, *x* label, *y* label, title and grid.

Table A2.1 *Plot Format*

plot (*x*, *y*)	plots the vector *x* versus the vector *y*
semilog *x* (*x*, *y*)	plots the vector *x* versus vector *y* where *x*-axis is $\log_{10}$ and *y*-axis is linear
semilog *y* (*x*, *y*)	plots the vector *x* versus vector *y* where *x*-axis is liner and *y*-axis is $\log_{10}$
loglog (*x*, *y*)	plots the vector *x* versus vector y where both axes are $\log_{10}$

Table A2.2 *Functions for Customised Plots*

title ('text')	puts 'text' at the top of the plot
*x*label ('text')	labels *x*-axis with 'text'
*y*label (text')	labels *y*-axis with 'text'
text (*p*1, *p*2, 'text')	puts 'text' at (*p*1, *p*2) in screen co-ordinates
subplot	sub divides the graphics window
-grid	draws grid lines on the current plot

To plot multiple curves on a single graph the following plot command with multiple arguments is to be used.

$$\text{plot}\ (x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

The default line (or point) types can also be altered by the line type chosen from Table A2.3 as given by the following command.

$$\text{plot}\ (x_1, y_1, \text{':'}\ x_2, y_2, \text{'+'})$$

The hold command freezes the current plot and inhibits erasure and rescaling. This command also allows us to plot multiple curves on a single graph. The current plot will be released by entering the hold command again.

Table A2.3 *Line and Point Types*

Line Types		**Point Types**	
solid	-	point	.
dashed	- -	plus	+
dotted	:	star	*
dash-dot	- .	circle	O
		x-mark	X

Colour statements are used as plot (x, y, r) which indicates a red line. Similarly, the command plot (x, y, '$+g$') indicates that the line is formed by green + marks. Other available colours are b for blue, w for white and i for invisible.

In MATLAB, the plot is automatically scaled. For manual axis scaling, the following commands will freeze the current axis scaling for subsequent plots.

$v = [x\text{-min} \quad x\text{-max} \quad y\text{-min} \quad y\text{-max}]$

axis (v)

To resume auto scaling, type axis again, axis ('square') sets the plot region on the region in the square; axis ('normal') sets back the plot region to be normal.

A2.5 SCRIPTS

A long sequence of commands to be executed for control system design and analysis is stored in a file of the form filename.m. These files are called m-files. Scripts are a type of m-files containing the ASCII text of codes necessary to perform some function. A text editor is required to create and edit m-files. A script is invoked at the command prompt by simply typing its filename. A script file should have a header and descriptive comments that begin with '%'. For example, the program stored in a script file plotdata.m is given below.

```
% This is a script for plotting
% the function y = sin (alpha*t).
% The value of alpha must be
% given before invoking script.
t = [0 : 0.01 : 1];
y = sin (alpha *t);
plot (t, y)
xlabel ('Time [sec]')
ylabel ('y(t) = sin (alpha *t) ' )
grid
```

The command

Alpha = 50;
plotdata

will produce the following graph after execution of the above script.

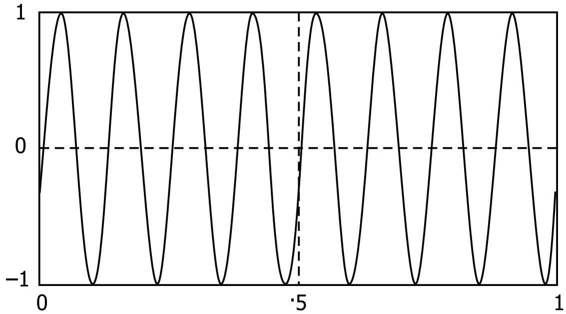


Fig. A2.1 *Graph of y(t) = sin (αt) for α = 50 using the script plotdata.m*

Some MATLAB functions used to write script are given in Table *A*2.4.

Table A2.4 *Some MATLAB Functions*

Function Name	Function Description
abs	Computes the absolute value
acos	Computes the arccosine
ans	Variable created for expressions
asin	Computes the arcsine
atan	Computes the arctangent (2 quadrant)
atan2	Computes the arctangent (4 quadrant)
axis	Specifies the manual axis scaling on plots
bode	Generates Bode frequency response plots
c2dm	Converts a continuous-time state variable system representation to a discrete-time system representation
clear	Clears the workspace
clg	Clears the graph window
cloop	Computes the closed-loop system with unity feedback
conj	Computes the complex conjugate
conv	Multiplies two polynomials (convolution)
cos	Computes the cosine

Function Name	Function Description
ctrb	Computes the controllability matrix
diary	Saves the session in a disk file
d2cm	Converts a discrete-time state variable system representation to a continuous time system representation
dstep	Computes the unit step response of a discrete-time system
eig	Computes the eigenvalues and eigenvectors
end	Terminates control structures
exp	Computes the exponential with base e
expm	Computes the matrix exponential with base e
eye	Generates an identity matrix
feedback	Computes the feedback interconnection of two systems
for	Generates a loop
format	Sets the output display format
grid	Adds a grid to the current graph
help	Prints a list of HELP topics
hold	Holds the current graph on the screen
i	$\sqrt{-1}$
imag	Computes the imaginary part of a complex number
impulse	Computes the unit impulse response of a system
inf	Represents infinity
j	$\sqrt{-1}$
linspace	Generates linearly spaced vectors
load	Loads variables saved in a file
log	Computes the natural logarithm
log10	Computes the logarithm base 10
loglog	Generates log-log plots
logspace	Generates logarithmically spaced vectors
abs	Computes the absolute value
lsim	Computes the time response of a system to an arbitrary input and initial conditions
margin	Computes the gain margin, phase margin, and associated crossover frequencies from frequency response data
max	Determines the maximum value

Function Name	Function Description
mesh	Creates three-dimensional mesh surfaces
meshdom	Generates arrays for use with the mesh function
min	Determines the minimum value
minreal	Transfer function pole-zero cancellation
NaN	Representation for Not-a-Number
ngrid	Draws grid lines on a Nichols chart
nichols	Computes a Nichols frequency response plot
num2str	Converts numbers to strings
nyquist	Calculates the Nyquist frequency response
obsv	Computes the observability matrix
ones	Generates a matrix of integers where all the integers are 1
pade	Computes an *nth*-order Pade approximation to a time delay
parallel	Computes a parallel system connection
plot	Generates a linear plot
poly	Computes a polynomial from roots
polyval	Evaluates a polynomial
printsys	Prints state variable and transfer function representations of linear systems in a readable form
pzmap	Plots the pole-zero map of a linear system
rank	Calculates the rank of a matrix
real	Computes the real part of a complex number
residue	Computes a partial fraction expansion
rlocfind	Finds the gain associated with a given set of roots on a root locus plot
rlocus	Computes the root locus
roots	Determines the roots of a polynomial
roots 1	Same as the roots function, but gives more accurate answers when there are repeated roots
semilogx	Generates an x-y plot using semilog scales with the x-axis log10 and the y-axis linear
semilogy	Generates an x-y plot using semi log scales with the y-axis log10 and the x-axis linear
series	Computes a series system connection

Function Name	Function Description
shg	Shows graph window
sin	Computes the sine
sqrt	Computes the square root
ss2tf	Converts state variable form to transfer function form
step	Calculates the unit step response of a system
abs	Computes the absolute value
subplot	Splits the graph window into subwindows
tan	Computes the tangent
text	Adds text to the current graph
title	Adds a title to the current graph
tf2ss	Converts a transfer function to state variable form
who	Lists the variables currently in memory
whos	Lists the current variables and sizes
xlabel	Adds a label to the x-axis of the current graph
ylabel	Adds a label to the y-axis of the current graph
zeros	Generates a matrix of zeros

APPENDIX 3
FUZZY LOGIC

A3.1 CONCEPT OF FUZZY LOGIC

A classical logic system requires a deep understanding of a system, exact mathematical equations and numerical values of constants.

Fuzzy logic provides an alternative way of thinking, which allows modelling of complex systems using knowledge and experience of operating a control system. It provides a simple way to draw definite conclusions from vague or imprecise information and resembles human decision-making in its ability to work with of approximate data and find precise control system solutions. Fuzzy logic allows expression of knowledge in a subjective way, for example, in a temperature control system, subjective concepts like very hot, very cold, moderately hot, moderately cold and so on are mapped into exact numeric ranges.

There are a large number of commercially available products using fuzzy logic, ranging from washing machines to high-speed trains. Nearly every application can potentially realise some of the benefits of fuzzy logic, such as performance, simplicity, lower cost and productivity. Manufacturers in the automotive industry are using fuzzy technology to improve quality and reduce development time. In aerospace, fuzzy enables very complex real-time problems being tackled using simple approach. In consumer electronics, fuzzy improves the time to market the product and helps reduce costs. In manufacturing, fuzzy is proven to be invaluable in increasing equipment efficiency and identifying malfunctioning.

The concept of fuzzy logic is not presented as a control methodology, but as a way of processing data by allowing partial set membership rather than crisp set membership or non-membership. If feedback controllers could be programmed to accept noisy, imprecise input, they would be much more effective and perhaps easier to implement. It lends itself to implementation in systems ranging from simple, small, embedded microcontrollers to large, networked, multi-channel PC or workstation-based data acquisition and control systems. It can be implemented in hardware, software, or a combination of both. Fuzzy logic provides a simple way to arrive at a definite conclusion based on vagueness, ambiguity, imprecision, noise, or missing input information. Approach to control problems, using fuzzy logic, mimics how a human being would take decisions, but at a faster rate. Fuzzy logic works well in applications such as industrial controls, transportation systems, video equipment, washing machines and so on.

A3.2 BASIC NOTIONS OF FUZZY LOGIC

Fuzzy set: Fuzzy set is a range of values, which form the basis of fuzzy logic. Each value has a grade of membership between zero and one. Logic expression defines values as either True or False. To express degrees of intensity of some control, variable fuzzy logic uses labels such as 'slow', 'medium', 'fast' or 'moderate', 'somewhat', 'a little', and so on. This is illustrated in Fig. A3.1(a) represents a fuzzy membership whereas Fig. A3.1(b) which represents a Boolean set.

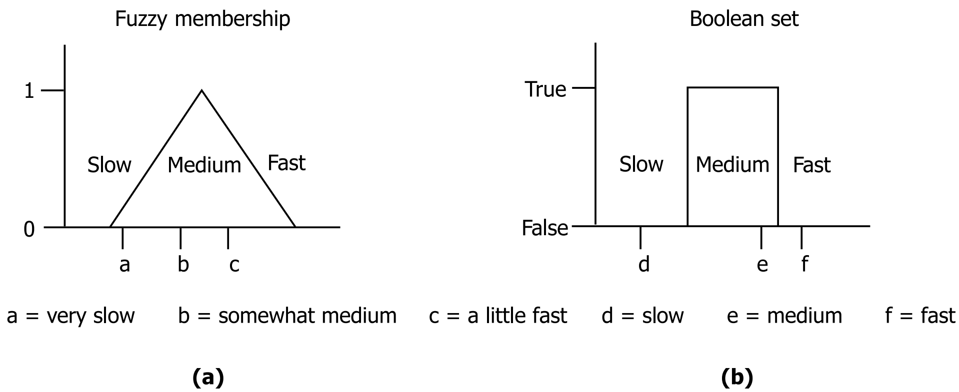


Fig. A3.1 Defining speed in terms of (a) Fuzzy logic and (b) Boolean

The fuzzy set in Fig. A3.1(a) is triangular in shape as is easy to draw. The set can be of any shape including a trapezoid or a bell curve. When fuzzy sets cover a whole range of possible values, they overlap so that a given value may be the member of more than one set. This makes a value unique.

A3.3 LINGUISTIC VARIABLES

Linguistic or fuzzy variables are adjectives like large positive error, small positive error, zero error, small negative error and large negative error, that modify the variable. If we want to have minimum of variables, we can simply have positive, zero and negative as variables for each of the control parameters. Additional range such as very large and very small can also be added to extend the responsiveness to exceptional or non-linear conditions. The sensor inputs like temperature, displacement, velocity, flow, pressure, and so on can therefore be expressed in terms of linguistic variables. Since the error is just a difference, it can also be considered in the same way.

a) **Membership functions:** The membership function is a graphical representation of the magnitude of participation of each input. It associates with each of the inputs that are processed, defines functional overlap between inputs and ultimately determines an output response.

b) **Fuzzy rules:** We take decisions based on rules. Even though, we may not be aware of it, all the decisions we take are based on If-Then statements. If the weather is fine, then we may decide to go the market. If the forecast predict that the weather will be bad today, but fine tomorrow, then we take decision not to go today and postpone it till tomorrow. Rules associate ideas and relate one event to another. Thus rules control a system.
Fuzzy machines, which always tend to mimic the behaviour of man, work the same way. Onh this time the decision and the means of choosing that decision are replaced by fuzzy sets and the rules are replaced by fuzzy rules. Fuzzy rules define fuzzy patches, which is the key idea in fuzz\logic.

c) **Understanding fuzzy logic:** Fuzzy logic uses a set of rules to define its behaviour. The rules define the conditions expected and outcomes desired with If/Then statements. These rules replace formulas. They must cover all situations that may occur but are not to be written for even possible combination.
Fuzzy logic can understand statements such as:
"If the temperature is close to the set point"
"If temperature change is very slow"
"Then add a little heat"

d) **Need for the use of fuzzy logic:** Fuzzy logic offers several unique features that makes it a good choice for many control problems.

1. It is inherently robust since it does not require precise, noise-free inputs and can be programmed to fail safely if a feedback sensor is destroyed. The input control is a smooth control function despite a wide range of input variations.
2. Since the fuzzy logic controller, processes user-defined rules governing the target control system, it can be modified easily to improve or drastically alter the system performance. New sensors can easily be incorporated into the system simply by generating appropriate governing rules.
3. Fuzzy logic is not limited to a few feedback inputs and one or two control inputs, nor is it necessary to measure or compute the rate of change of parameters in order, for it to be implemented. Any sensor data that provides some indication of a system's

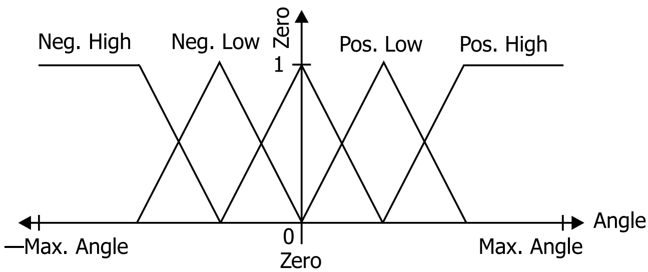
actions and reactions is sufficient. This allows the sensors to be inexpensive and imprecise thus keeping the overall system cost and complexity low.

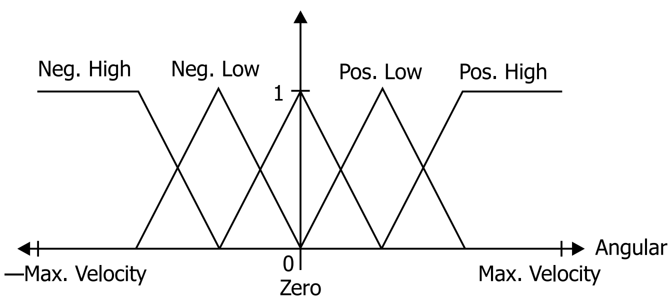
4. Because of the rule-based operation, any reasonable number of inputs can be processed and numerous outputs generated. However, defining the rule base quickly becomes complex if too many inputs and outputs are chosen for a single implementation as rules defining their interrelations must also be defined. It would be better to break the control system into smaller units and use several smaller fuzzy logic controllers distributed on the system, each with limited responsibilities.
5. Fuzzy logic can control non-linear systems that would be difficult or impossible to model mathematically.

A3.4 FUZZY CONTROL

Fuzzy control, which directly uses fuzzy rules, is the most important application in fuzzy theory. Using a procedure originated by Ebrahim Mamdani in the late 1970's three steps are taken to create a fuzzy controlled machine:

Fuzzification (Using membership functions to graphically describe a situation)
Rule evaluation (Application of fuzzy rules)
Defuzzification (Obtaining crisp results)





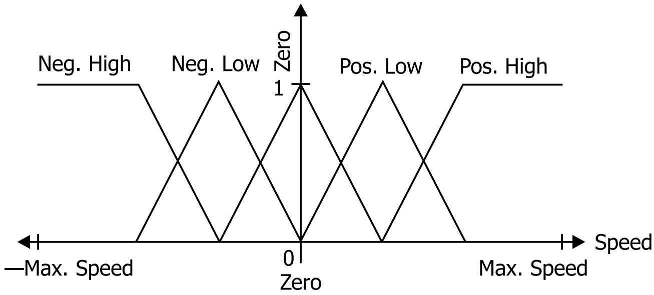


Fig. A3.2 *Membership functions of each fuzzy subset for an inverted pendulum*

As a simple example on how fuzzy controls are constructed, let us consider the following classic situation: In an inverted pendulum there is a need to balance a pole on a mobile platform that can move only in two directions, to the left or to the right. The angle between the platform and the pendulum and the angular velocity of this angle are chosen as the inputs of the system. The speed of the platform is chosen as the corresponding output. The procedure is as follows:

a) **First Step:** First of all, the input of the system and the angle between the platform and the pendulum, is defined by four linguistic variables partitioned on its universe of discourse. These fuzzy subsets assigned with linguistic labels are defined as negative high, negative low, zero, positive low and positive high. The membership functions for each fuzzy subset is shown in Fig. A3.2. In a similar way, the other input, angular velocity is also partitioned into five fuzzy subsets on its universe of discourse, as negative high, negative low, zero, positive low and positive high as shown in Fig. A3.2. The same procedure is applied to the output of the system, that is, the speed of the platform.

b) **Second step:** The next step is to define the fuzzy rules. The fuzzy rules are merely a series of If-Then statements as mentioned below. These statements are usually derived by an expert who has operated such a control system to achieve optimum results. The rules formulated are:

IF angle is zero AND angular velocity is zero then speed is also zero.

IF angle is zero AND angular velocity is negative low THEN the speed shall be negative low.

IF angle is zero AND angular velocity is positive low THEN the speed shall be positive low.

IF angle is negative low and angular velocity is zero THEN the speech shall be negative low.

IF angle is zero AND angular velocity is positive low THEN the speed shall be positive low.

IF angle is positive low and angular velocity is zero THEN the speed shall be positive low.

IF angle is negative low AND angular velocity is positive low THEN the speed shall be zero.

IF angle is zero and angular velocity is positive low THEN the speed shall be positive low.

IF angle is positive low AND angular velocity is negative low THEN the speed shall be zero.

IF angle is zero and angular velocity is positive low THEN the speed shall be positive low.

These rules are depicted in Table A3.1.

Table A3.1 *Fuzzy Rules for an Inverted Pendulum*

Speed		**Angle**				
ANGULAR VELOCITY		**Neg. High**	**Neg. Low**	**Zero**	**Positive Low**	**Positive High**
	Negative High	–	–	Negative High	–	
	Negative Low	–	–	Negative Low	Zero	–
	Zero.	Negative High	Negative Low	Zero	Positive Low	Positive
	Positive Low	–	Zero	Positive Low	–	–
	Positive High	–	–	Positive High	–	–

An application of these rules is shown using specific values for angle and angular velocities. The values are seen as in the graphs in Fig. A3.3.

As seen in the Fig. A3.3 the result patch yielded by the rule "if angle is zero and angular velocity is zero, the speed is zero", is a bit tricky since there are two variables relating to one point. The actual value belongs to the fuzzy set zero to a degree of 0.75 for "angle" and 0.4 for "angular velocity" respectively. Since this is an AND operation, minimum criterion is used and the fuzzy set zero of the variable "speed" is cut at 0.4 and the patches are shaded up to that area. This is illustrated in Fig. A3.4.

The following figures show the result patches yielded by the rule "if angle is zero and angular velocity is negative low, the speed is negative low", "if angle is positive low and angular velocity is zero, then speed is positive low" and "if angle is positive low and angular velocity is negative low, the speed is zero".

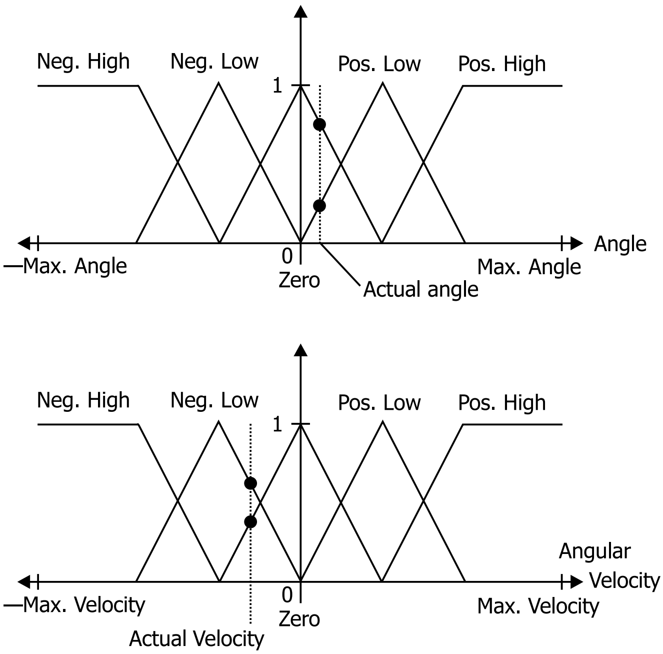


Fig. A3.3

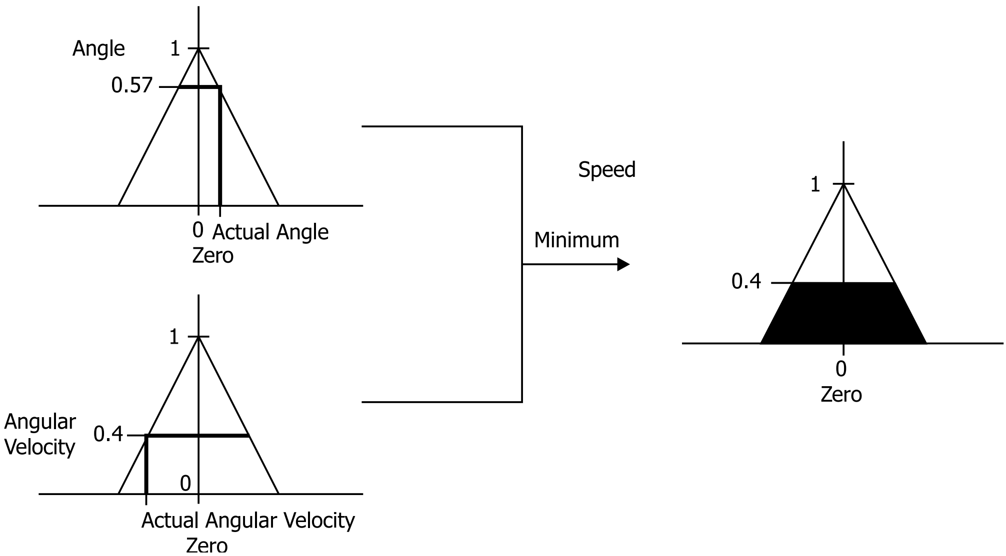


Fig. A3.4

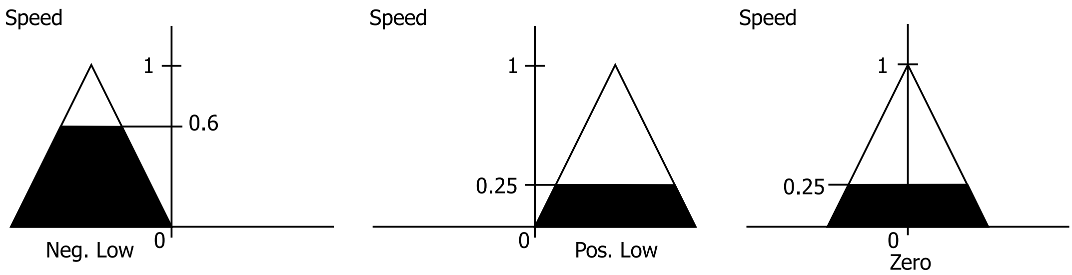


Fig. A3.5

The four results are overlapped and is reduced to what has been shown in Fig. A3.6.

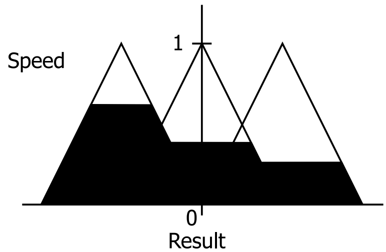


Fig. A3.6

c) **Third step:** The result of the fuzzy controller so far is a fuzzy set (of speed). To choose an appropriate representative value as the final output (crisp values), defuzzification must be done. This can be done in many ways, but the most common method used is the determination of the centre of gravity of the set as shown in Fig. A3.7

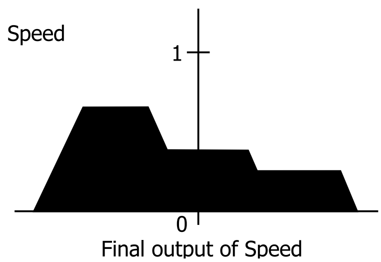


Fig. A3.7

A3.5 COMPARISON OF DESIGN METHODOLOGIES

Fuzzy logic is a paradigm for an alternative design methodology which can be applied in developing both linear and non-linear systems of embedded control. By using fuzzy logic, designers can realise lower development costs, superior features and better product performance. Furthermore, products can be brought to the market faster and more cost-effectively. In order to appreciate why a fuzzy based design methodology is very attractive in control applications, let us examine a typical design flow. Fig. A3.8 illustrates a sequence of design steps required to develop a controller using a conventional and a fuzzy approach.

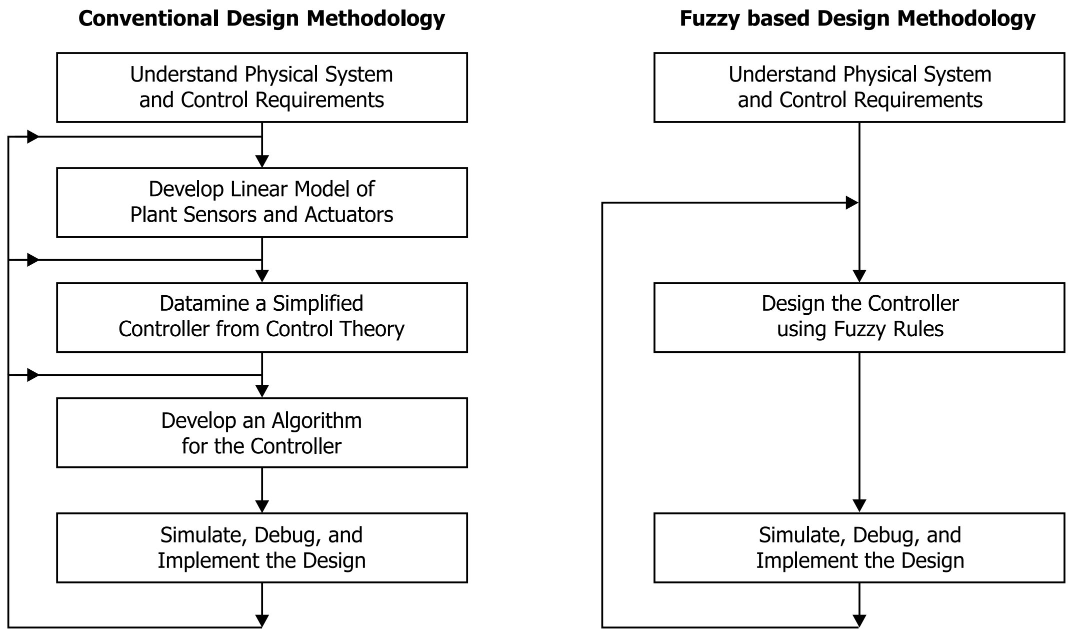


Fig. A3.8 *Conventional and fuzzy design*

Using the conventional approach, the first step is to understand the physical system and its control requirements. Based on this understanding, the second step is to develop a model which includes the plant, sensors and actuators. The third step is to use linear control theory in order to determine a simplified version of the controller, such as the parameters of a PID controller. The fourth step is to develop an algorithm for the simplified controller. The last step is to simulate the design including the effects of non-linearity, noise and parameter variations. If the performance is not satisfactory we need to modify our system modelling, re-design the controller, re-write the algorithm and try again.

With fuzzy logic, the first step is to understand and characterise the system behaviour by using our knowledge and experience. The second step is to directly design the control algorithm using fuzzy rules, which describe the principles of the controller's regulation in terms of the relationship between its inputs and outputs. The last step is to simulate and debug the design. If the performance is not satisfactory we only need to modify some fuzzy rules and try again.

Although the two design methodologies are similar, the fuzzy-based methodology substantially simplifies the design loop. This results in some significant benefits, such as reduced development time, simpler design and faster time to market.

With a fuzzy logic design methodology, some time consuming steps are eliminated. Moreover, during the debugging and tuning cycle we can change our system by simply modifying rules instead of redesigning the controller. In addition, fuzzy helps us to focus more on our application instead of programming. As a result, fuzzy logic substantially reduces the overall development cycle.

Fuzzy logic describes complex systems using our knowledge and experience in simple English like rules. It does not require any system modelling or complex mathematical equations governing the relationship between inputs and outputs. Fuzzy rules are very easy to learn and use, even by non experts. It typically takes only a few rules to describe systems. As a result, fuzzy logic significantly simplifies design complexity. Commercial applications in embedded control system require a significant development effort, majority of which is spent on the software aspects of the project. Development time is a function of design complexity and the number of iterations required in the debugging and tuning cycle.

Most real life physical systems are actually non-linear systems. Conventional design approaches use different approximation methods to handle non-linearity. Some typical choices are linear, piece- wise linear, and look-up table approximations to trade off factors of complexity, cost and system performance.

A linear approximation technique is relatively simple, however, it tends to the limit control performance and may be costly to implement in certain applications. A piece-wise linear technique works better, although it is tedious to implement because it often requires the design of several linear controllers. A look-up table technique may help improve control performance, but it is difficult to debug and tune. Furthermore in complex systems where multiple inputs exist, a look-up table may be impractical or very costly to implement due to its large memory requirements.

Fuzzy logic provides an alternative solution to non-linear control because it is close to the real world. Non-linearity is handled by rules, membership functions and the inference process which results in improved performance, simpler implementation and reduced design costs.

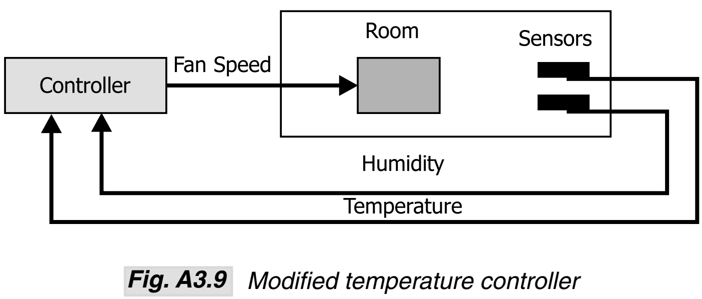


Fig. A3.9 *Modified temperature controller*

Most control applications have multiple inputs and require modelling and tuning of a large number of parameters which makes implementation very tedious and time consuming. Fuzzy rules can help you simplify implementation by combining multiple inputs into single If-Then statements while still handling non-linearity. Consider a temperature controller with two inputs, temperature and humidity and the same output, fan speed as shown in Fig. A3.9. This example can be described with a small set of rules as follows:

IF temperature is cold AND humidity is high THEN fan speed is high
IF temperature is cold AND humidity is high THEN fan speed is medium
IF temperature is cold AND humidity is high THEN fan speed is low
IF temperature is cold AND humidity is high THEN fan speed is zero
IF temperature is cold AND humidity is medium THEN fan speed is medium
IF temperature is cold AND humidity is medium THEN fan speed is low
IF temperature is warm AND humidity is medium THEN fan speed is zero
IF temperature is hot AND humidity is medium THEN fan speed is zero
IF temperature is cold AND humidity is low THEN fan speed is medium
IF temperature is cold AND humidity is low THEN fan speed is low
IF temperature is cold AND humidity is low THEN fan speed is zero
IF temperature is cold AND humidity is low THEN fan speed is zero

Using fuzzy logic we can describe the output as a function of two or more inputs linked with operators such as AND. This relationship can also be represented in a tabular form as shown in Table A3.2. The fuzzy approach requires significantly less entries than a look-up table depending upon the number of labels for each input variable. Rules are much easier to develop and simpler to debug and tune compared to a look-up table.

Table A3.2 *Fuzzy Logic Membership Functions*

Temperature ↓

Humidity →

Cool	Cool	Warm	Hot	
Low	Medium	Low	Zero	Zero
Medium	Medium	Low	Zero	Zero
High	High	Medium	Low	Zero

↓ Fan speed

A3.6 EXAMPLES OF FUZZY CONTROLLERS

a) Fuzzy traffic light controller: Traffic light signals are installed on road crossings to control the movement of vehicles in different directions. Normally the illumination period of the lamps (Red, Yellow, and Green) is fixed and a cyclic time period is provided by using time delay devices for automatic operation. It is often observed that irrespective of whether there are enough - vehicles to cross the junction in a particular direction, the vehicles on the other road have to wait. Thus, providing constant time period of cyclic operation, irrespective of the number of vehicles to pass through, the function in different directions is not an optimum solution. It should be possible to design a control system where more cars can pass at the green interval if there are fewer cars waiting behind the red signals on the other road. Mathematical decisions for such a control are difficult. With fuzzy logic, it is easier to develop such a control.

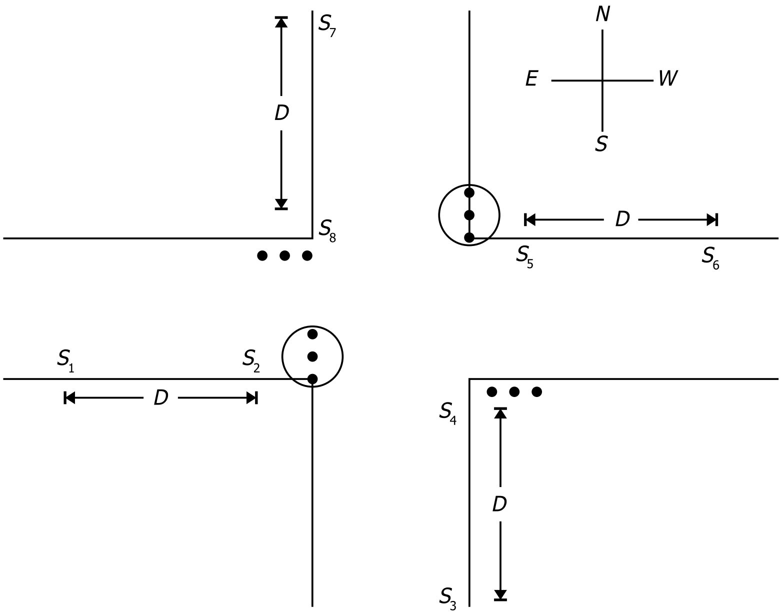


Fig. A3.10

Fig. A3.10 shows such an arrangement. Eight incremental sensors S_1 S_2, S_3...S_8 are put in specific positions as shown in Fig. A3.10. The first sensor behind each traffic light (S_1 S_3, S_5, S_7) counts the number of cars coming to the intersection and the other set of sensors (S_2, S_4, S_6 ... S_8) counts the cars passing the traffic lights. The amount of cars between the traffic lights is determined by the difference of the reading of the two sensors. For example, the number of cars behind traffic light north is $S_7 - S_8$.

The distance D, chosen to be 200 ft., is used to determine the maximum density of cars allowed to wait in a very crowded situation. This is done by adding the number of cars between two paths and dividing it by the total distance. For instance, the number of cars between the east and west street is $(S_1 - S_2) + (S_5 - S_6)/400$.

Next comes the fuzzy decision process which uses the three steps mentioned earlier (fuzzification, rule evaluation and defuzzification).

1. Fuzzification

As before, first the inputs and outputs of the design are to be determined. Assuming red light is shown to both north and south streets and distance D is constant, the inputs of the mode consist of:

i) Cycle time
ii) Cars behind red light
iii) Cars behind green light

The cars behind the light are the maximum number of cars in the two directions. The corresponding output parameter is the probability of change of the current cycle time. Once this is done, the input and output parameters are divided into overlapping member functions, each function corresponding to different levels. For inputs one and two, the levels and their corresponding ranges are zero (0, 1), low (0, 7), medium (4,11), high (7, 18), and chaos (14, 20). For input 3, the levels are very short (0, 14), short (0, 34), medium (14, 60), long (33, 88), very long (65, 100), limit (85,100). The levels of output are no (0), probably no (0. 25), may be (0.5), probably yes (0.75), and yes (1.0).

Note For the output, one value (singleton position) is associated to each level instead of a range of values. The corresponding graphs for each of these membership functions are drawn in the similar way as shown in Fig. A3.10.

2. Rule Evaluation

The rules, are formulated using a series of If-Then statements, combined with AND/OR operators. For example, if cycle time is medium AND cars behind red are low AND cars behind green are medium then the change is 'probably not'. With three inputs, each having 5, 5 and 6 membership functions, there are a combination of 150 rules. However using the minimum or maximum criterion, some rules are combined to a total of 86.

3. Defuzzification

This process converts the fuzzy set output to real crisp value. The method used for this system is called centre of gravity.

Crisp Output = [(Sum (Membership Degree * Singleton Position)/Sum (Membership degree].

For example, after the rule evaluation, if the membership degree for the output parameter probability of the change of cycle time is yes = 0, probably yes = 0.6, may be = 0.9, probably no = 0.3 and no 0.1. Then the crisp value will be: crisp output = (0.1 * 0.00) + (0.3*0.25) + (0.9*0.50) + (0.6*0.75) + (0* 1.00)/0.1 +0.3 + 0.9 + 0.6 + 0 = 0.51.

The fuzzy controller has been tested under seven different kinds of traffic conditions from very heavy traffic to very lean traffic. Thirty-five random chosen car densities were grouped according to different periods of the day representing those traffic conditions.

The performance of the controller was compared with that of a conventional controller and a human expert. The criteria used for comparison were number of cars allowed to pass at one time and average waiting time. The fuzzy controller passed 31 per cent more cars, with an average waiting time shorter by 5 per cent than the theoretical minimum of the conventional controller. However, in comparison, with a human excerpt, the fuzzy controller passed 14 per cent more cars with 14 per cent shorter waiting time.

b) Fuzzy server motor controller: An example of a fuzzy logic controller for controlling a sevomotor is shown in Fig. A3.11

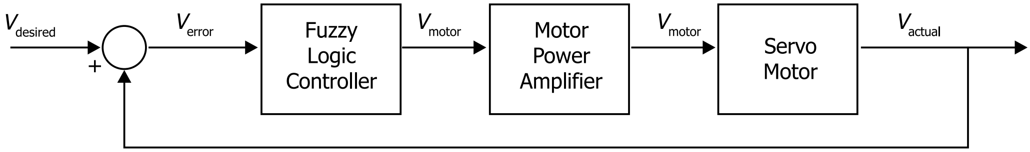


Fig. A3.11

The actual voltage of the servomotor is compared with the desired voltage and the difference (error) in voltage in supplied as input to the fuzzy logic controller. The other input is the rate of change in error (d/dt). The output of the fuzzy logic controller is the motor voltage V_{motor}. The fuzzy sets for V_{error} $\frac{d}{dt}V_{error}$, and V_{motor} are shown in Fig. A3.12. The linguistic variables which define the fuzzy sets are *LN* (large negative), *SN* (slow regular) *ZE* (Zero) *SP* (slow positive) *LP* (large positive). The rules for the fuzzy logic controller are:

1. If V_{error} is *LP* and $\frac{d}{dt}V_{error}$ is any then V_{motor} is *LP*.

2. If V_{error} is *SP* and $\frac{d}{dt}V_{error}$ is *SP* or *ZE* then V_{motor} is *SP*.

3. If V_{error} is *ZE* and $\frac{d}{dt}V_{error}$ is *SP* then V_{motor} is *ZE*.

4. If V_{error} is *ZE* and $\frac{d}{dt}V_{error}$ is *SN* then V_{motor} is *SN*.

5. If V_{error} is *SN* and $\frac{d}{dt}V_{error}$ is *SN* then V_{motor} is *SN*.

6. If V_{error} is *SB* and $\frac{d}{dt}V_{error}$ is any then V_{motor} is *LN*.

Counter the case where $V_{error} = 30$ rps and $\frac{d}{dt}V_{error} = 1$ rps/s. Rules 1 to 6 are calculated as depicted in Fig. A3.13.

The sets for V_{error}, $\frac{d}{dt}V_{error}$, and V_{motor} are:

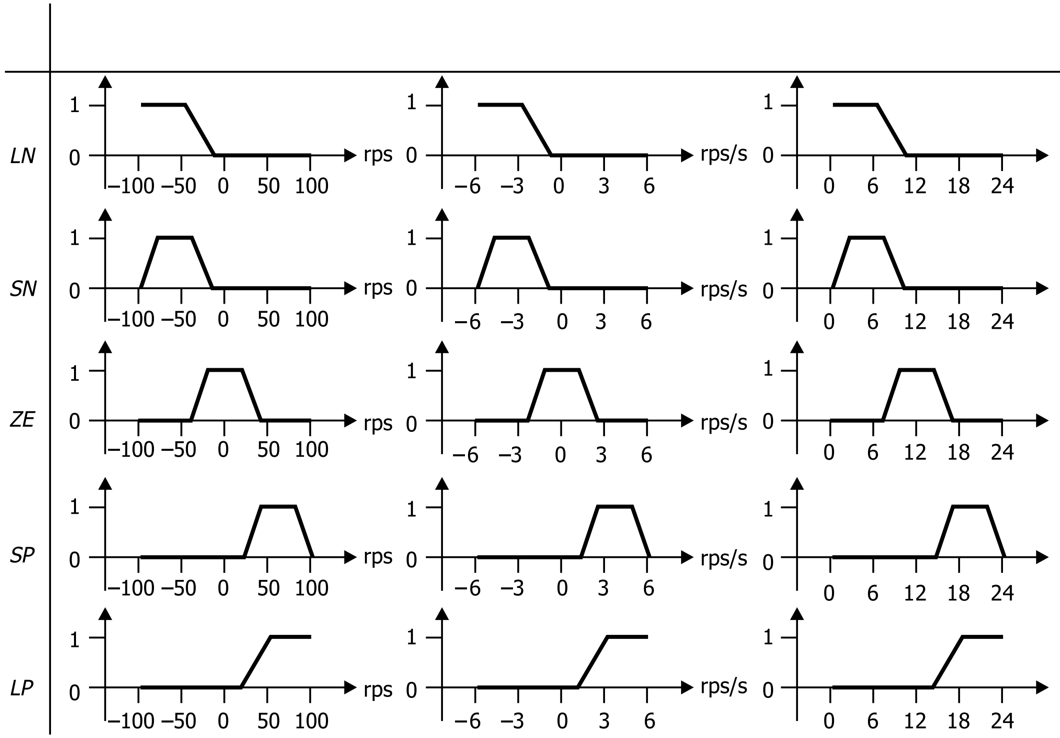
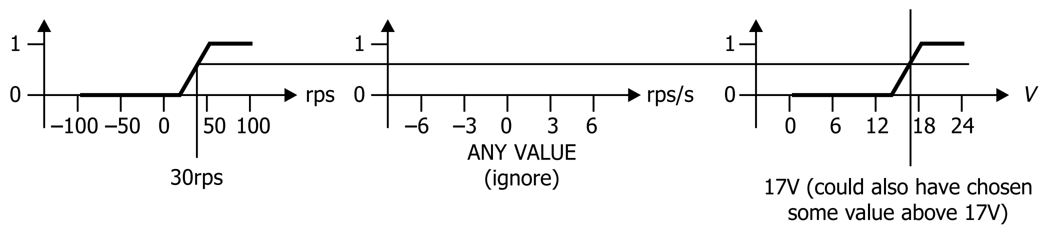


Fig. A3.12

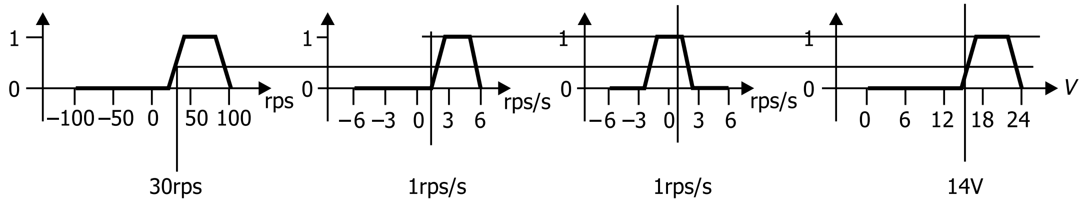
Consider the case where $V_{\text{error}} = 30$ rps and $\frac{d}{dt}V_{\text{error}} = 1$ rps/s. Rule 1 to 6 are calculated in Fig. A3.13(a)–(f)

1. If V_{error} is *LP* and $\frac{d}{dt}V_{\text{error}}$ is any then V_{motor} is *LP*.



(a)

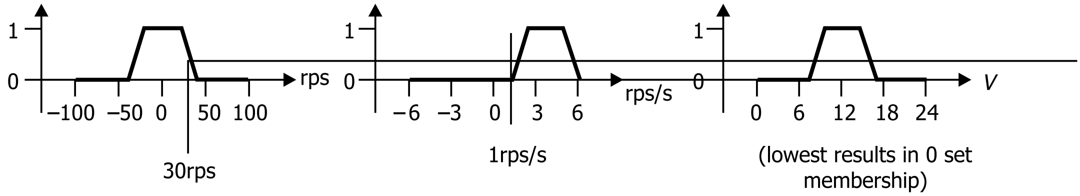
2. If V_{error} is *SP* and $\frac{d}{dt}V_{\text{error}}$ is *SP* or *ZE* then V_{motor} is *SP*.
   Or means take the highest of the two memberships
   AND means take the lowest of the two memberships



(b)

This has about 0.4 (out of 1) membership

3. If V_{error} is *ZE* and $\dfrac{d}{dt}V_{\text{error}}$ is *SP* then V_{motor} is *ZE*.

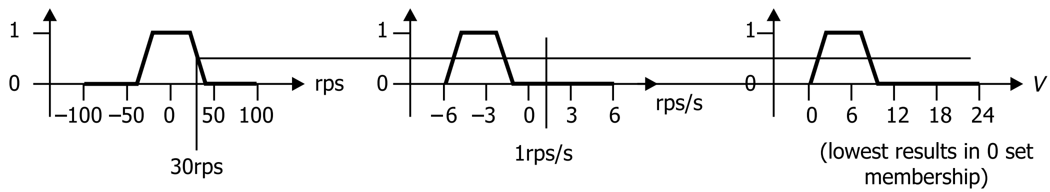


(c)

This has about 0.0 (out of 1) membership.

4. If V_{error} is *ZE* and $\dfrac{d}{dt}V_{\text{error}}$ is *SN* then V_{motor} is *SN*.

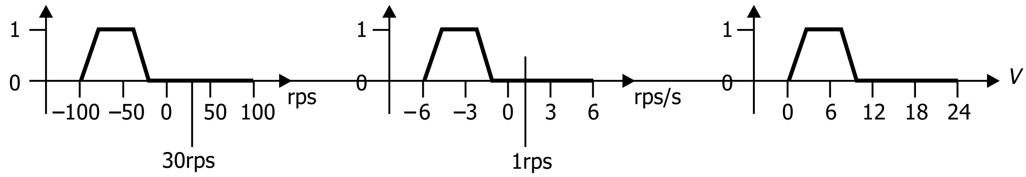
This has about 0.0 (out of 1) membership



(d)

5. If V_{error} is *SN* and $\dfrac{d}{dt}V_{\text{error}}$ is *SN* then V_{motor} is *SN*.

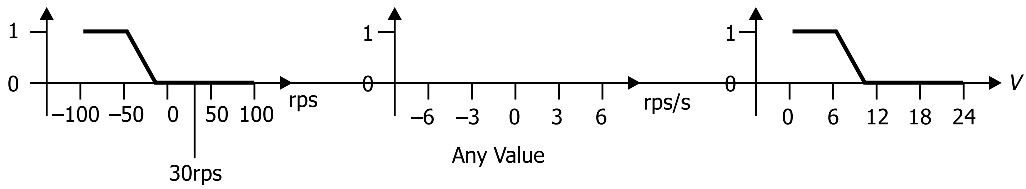
This has about 0.0 (out of 1) membership



(e)

6. If V_{error} is LN and $\dfrac{d}{dt}V_{\text{error}}$ is any then V_{motor} is LN.

   This has about 0 (out of 1) membership



(f)

Fig. A3.13 *Rule calculation*

The results from the individual rules can be combined using the following calculation. In this case only two rules matched, hence only two terms are used, to give a final motor control voltage.

$$V_{\text{motor}} = \frac{\sum_{i=1}^{n}(V_{\text{motor}_i})(\text{membership}_i)}{\sum_{i=1}^{n}(\text{membership}_i)}$$

$$V_{\text{motor}} = \frac{0.6(17\text{ V}) + 0.4(14\text{ V})}{0.6 + 0.4}$$

Thus, the final motor control voltage calculated as above is 15.8 volts.

APPENDIX 4
OBJECTIVE TYPE QUESTIONS

A4.1 INTRODUCTION

Q.1 State whether the following statements are True or False.

i) Feedback control systems are also referred to as closed-loop control systems

ii) A washing machine where the washing is done on a time basis through a timer is an example of closed-loop control system

iii) In an open-loop the control operation is independent of the output

iv) In an error detector the feedback signal is added with the reference signal to obtain error signal

v) A refrigerator is an example of closed-loop control system

vi) Performance of an open-loop system depends on the settings of the system components

vii) Fixed-time traffic light control system is an example of closed-loop control system

viii) In a linear system the response produced by simultaneous action of two different forcing functions is the difference of individual responses

ix) In stochastic control system the response is not predictable and repeatable

x) Servomechanism is an automatic control system

xi) An open-loop system is a system without feedback

xii) A control system is an interconnection of components forming a system configuration that provides a desired system response

xiii) A closed-loop control system uses a measurement of the output and compares it with the desired input (reference or command)

xiv) A negative feedback control system is the one where the output signal is fed back so that it is added to the input signal

xv) The student-teacher learning process is inherently a feedback process intended to reduce the system error to a minimum

Q.2 The response of a control system for step input is shown in Fig. A4.1 .The various performance measures of the system had been indicated as 1, 2, 3, 4, and 5. Identify these from the following alternatives.

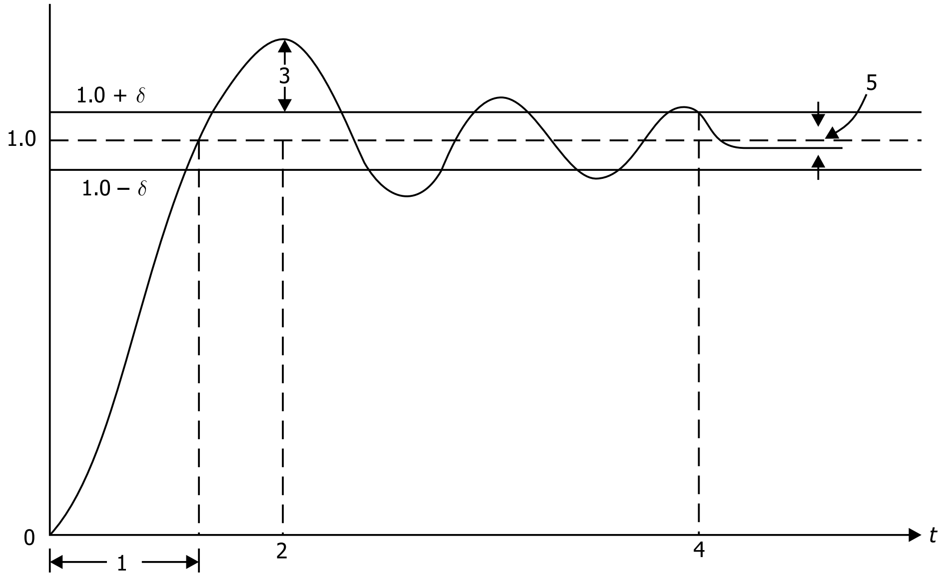


Fig. A4.1

(a) 1 = Rise time; 2 = Peak time; 3 = Overshoot; 4 = Steady-state error; 5 = Settling time

(b) 1 = Rise time; 2 = Peak time; 3 = Overshoot; 4 = Settling time; 5 = Steady-state error

(c) 1 = Steady-state error; 2 = Peak time; 3 = Overshoot; 4 = Rise time; 5 = Settling time

(d) 1 = Rise time; 2 = Peak time; 3 = Steady-state error; 4 = Overshoot; 5 = Settling time

Q.3 The equations representing various test signals are represented as

(a) Step: $\gamma(t) = A,\ t > 0$ $\quad R(s) = A/s$

$= 0,\ t < 0$

Ramp: $\gamma(t) = At,\ t > 0$ $\quad R(s) = A/s^2$

$= 0,\ t < 0$

Parabolic: $\gamma(t) = At^2, t > 0$ $R(s) = 2A/s^2$
$= 0, t < 0$

(b) Step: $\gamma(t) = A, t > 0$ $R(s) = A/s$
$= 0, t < 0$

Ramp: $\gamma(t) = At, t > 0$ $R(s) = A/s^2$
$= 0, t < 0$

Hyperbolic: $\gamma(t) = At^3, t > 0$ $R(s) = A/s^3$
$= 0, t < 0$

(c) Step: $\gamma(t) = At, t > 0$ $R(s) = A/s^2$
$= 0, t < 0$

Ramp: $\gamma(t) = At^2, t > 0$ $R(s) = A/s^3$
$= 0, t < 0$

Hyperbolic: $\gamma(t) = At^2, t > 0$ $R(s) = A/s^4$
$= 0, t < 0$

(d) Step: $\gamma(t) = A, t > 0$ $R(s) = A/s$
$= 0, \mathrm{t} < 0$

Ramp: $\gamma(t) = At, t > 0$ $R(s) = A/s^2$
$= 0, t < 0$

Hyperbolic: $\gamma(t) = At^2, t > 0$ $R(s) = A/s^3$
$= 0, t < 0$

Q.4 Increasing the amplifier gain of a control system
(a) usually reduces the steady state-error but results in oscillatory transient response
(b) usually increases the steady state-error and results in oscillatory transient response
(c) usually increases the steady state-error but reduces oscillations of the transient response
(d) usually reduces steady state-error and oscillations of the transient response

Q.5 In time domain, the performance measure of a system for relative stability and speed of response are
(a) damping factor or peak overshoot and rise time, settling time or natural frequency
(b) peak time, percent overshoot, damping ratio
(c) damping factor, rise time and settling time, damping ratio
(d) damping factor, per cent overshoot, and rise time

Q.6 In frequency domain the measure of relative stability and speed of response respectively are
(a) peak time, per cent overshoot and damping ratio
(b) phase margin, gain margin, and per cent overshoot
(c) phase margin, peak time and damping ratio
(d) resonant peak or phase margin, resonant frequency or bandwidth

Q.7 A closed-loop control system can be defined as
(a) system that directly generates the output in response to an input signal
(b) system with a measurement of the output signal and a comparison with the desired output to generate an error signal that is applied to the actuator
(c) an interconnection of components forming a system configuration that will provide a desired response, the output having no effect upon the signal to the process
(d) a system that utilises a device to control the process without using feedback

Q.8 In a multivariable control system there is
(a) more than one input variable but one unique output
(b) one input variable but variable outputs
(c) more than one input variable or more than one output variable
(d) more than one input variable and more than one output variable

Q.9 A negative feedback control system is the one where
(a) the output signal is fed back as it subtracts from the input signal to provide desired response
(b) the output signal is fed back so that it adds to the input signal to reduce the output
(c) the input signal is reduced so as to reduce the output when it is more than the desired value
(d) control of the process is achieved without using any feedback

Q.10 The characteristic equation of a control system is the relation formed by
(a) equating to zero the numerator of the transfer function
(b) equating to zero the denominator of the transfer function
(c) subtracting the denominator from the numerator of the transfer function and equating to zero
(d) adding the denominator and numerator of the transfer function and equating to zero

Q.11 In control system, damped oscillation of the output means
(a) oscillation where damping is on the boundary between underdamped and overdamped
(b) oscillation in which the amplitude remains steady
(c) oscillation in which the amplitude increases with time
(d) oscillation in which the amplitude decreases with time

Q.12 Dominant roots are
(a) the roots of the characteristic equation of the closed-loop system that cause dominant transient response of the system
(b) the complex conjugate roots away from the origin of the S-plane relative to the other roots of the closed-loop system
(c) the roots of the characteristic equation of the closed-loop system that cause minimum transient response of the system
(d) the roots of the characteristic equation of the closed-loop system that cause minimum steady-state error of the system

Q.13 Frequency response of a control system means
(a) the transient response of a system to a sinusoidal input signal
(b) the steady-state response of a system to a sinusoidal input signal
(c) the steady-state response of a system to a non-sinusoidal input signal
(d) the transient response of a system to a non-sinusoidal input signal

Q.14 A control system is stable if
(a) all the roots of the characteristic equation have negative real parts
(b) all the roots of the characteristic equation have positive time real parts
(c) any root of the characteristic equation has a positive real part
(d) any root of the characteristic equation has a negative real part

Q.15 A control system is unstable if
(a) any root of the characteristic equation has a negative real part
(b) any root of the characteristic equation has a positive real part or if there is repeated root on the $j\omega$-axis
(c) all the roots of the characteristic equation have negative real parts
(d) all the roots of the characteristic equation must have positive real parts

ANSWERS

Q.1 i) True ii) False iii) True iv) False v) True vi) True vii) False viii) False ix) True x) True xi) True xii) True xiii) True xiv) False xv) True Q.2 (b) Q.3 (a) Q.4 (a) Q.5 (a) Q.6 (b) Q.7 (b) Q.8 (c) Q.9 (a) Q.10 (b) Q.ll (d) Q.12 (a) Q.13 (b) Q.14 (a) Q.15 (b)

A4.2 MODELLING A CONTROL SYSTEM—TRANSFER FUNCTION APPROACH AND BLOCK DIAGRAM APPROACH

Q.l Complete the following sentences.

i) The transfer function of a linear system is defined as the ratio of the _____ of the output variable to the _____ of the _____
ii) Transfer function of an integrating circuit is $G(s) =$ _____
iii) Transfer function of a differentiating circuit is $G(s) =$ _____
iv) First step in determining the transfer function of a control system is to formulate the _____ for the system
v) If the transfer function of a system is known, we can determine the behaviour of the system for _____ to understand the nature of system
vi) A system is described by the differential equation $4y + 2y + y = 5x + 3\text{x} + 2x + x$, where x is the system output and y is the system input. The transfer function of the system is _____
vii) Transfer function of a mechanical system having mass, spring and damper is calculated as $G(s) =$ _____
viii) The S-plane poles and zeros of the transfer function represent the _____ response of the system

Q.2 Transfer function of a linear time-invariant system is defined as

(a) the ratio of the Laplace transform of the input variable to the Laplace transform of the output variable when all initial conditions are zero

(b) the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable when all initial conditions are zero

(c) the Laplace transform of the output variable when all initial conditions are zero

(d) the Laplace transform of the input variable when all initial conditions are zero

Q.3 The force equation for a mass-spring-dashpot system is written as

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; q = 0,1,2,\text{--},(n\text{-}m\text{-}1)$$

Assuming zero initial conditions, the transfer function can be expressed as

(a) $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{Ms^2 + fs + K}$

(b) $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{M}{s^2} + \dfrac{f}{s} + K$

(c) $G(s) = \dfrac{F(s)}{X(s)} = \dfrac{1}{Ms^2 + fs + K}$

(d) $G(s) = \dfrac{X(s)}{F(s)} = Ms^2 + fs + K$

Q.4 The block diagram of a closed-loop system is shown in Fig. A4.2 below. The transfer function is

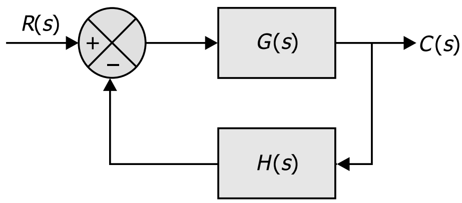


Fig. A4.2

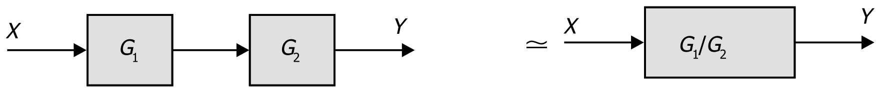
(a) $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 - G(s)H(s)}$

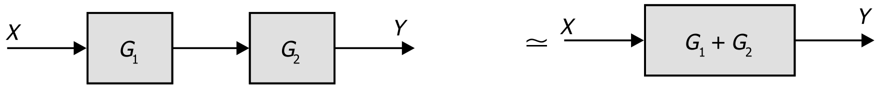
(b) $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)H(s)}$

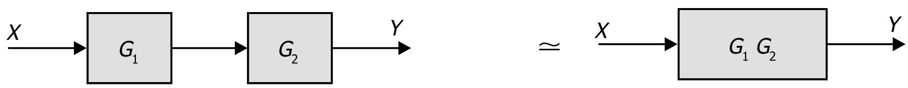
(c) $\dfrac{C(s)}{R(s)} = \dfrac{1+G(s)H(s)}{G(s)}$

(d) $\dfrac{C(s)}{R(s)} = \dfrac{1-G(s)H(s)}{G(s)}$

Q.5 In block diagram representation, when two blocks are in series, they can be represented by an equivalent block as

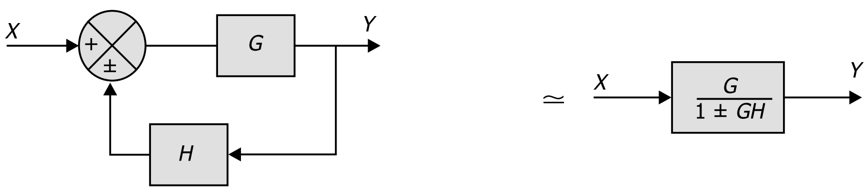
(a) 

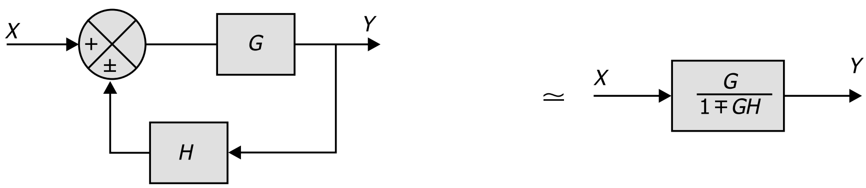
(b) 

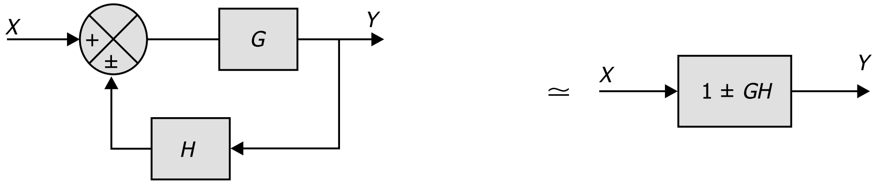
(c) 

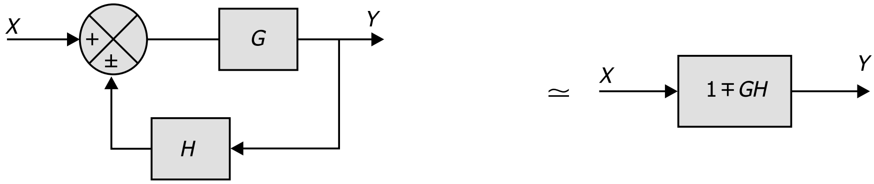
(d) 

Q.6 In a block diagram representation of a control system, when eliminating a feedback loop, one can show

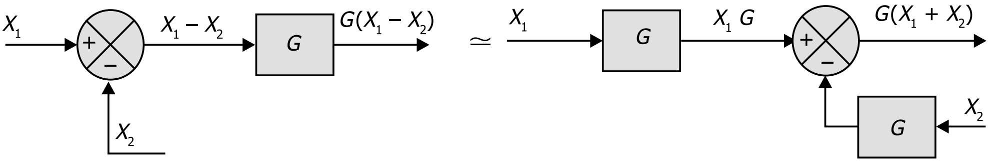
(a) 

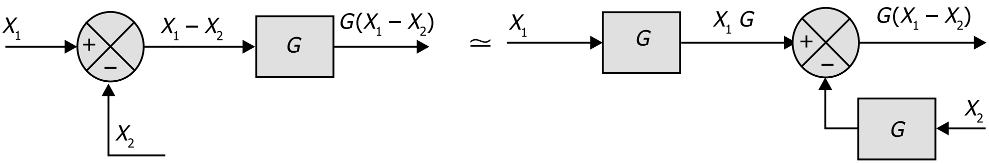
(b) 

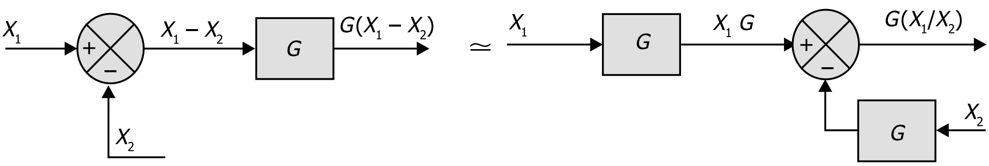
(c) 

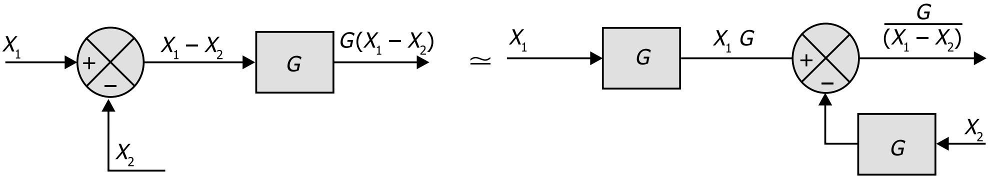
(d) 

Q.7 The rule for block diagram algebra for moving a summing point after a block is

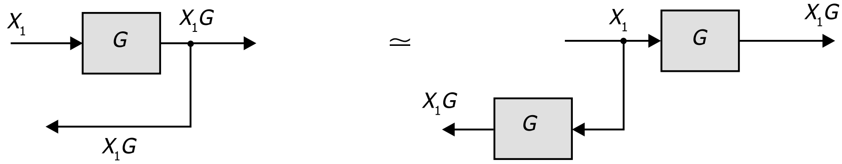
(a) 

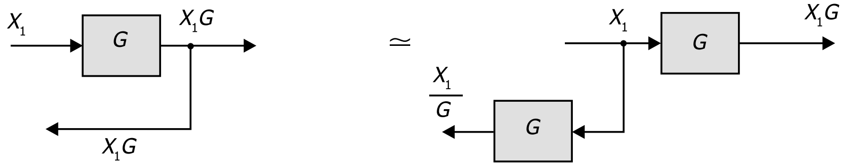
(b) 

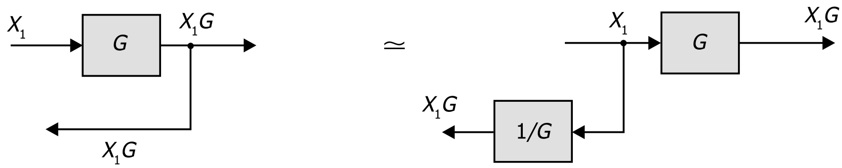
(c) 

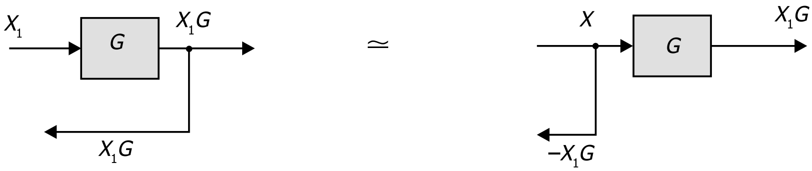
(d) 

Q.8 The rule for block diagram algebra for moving a take off point ahead of a block is

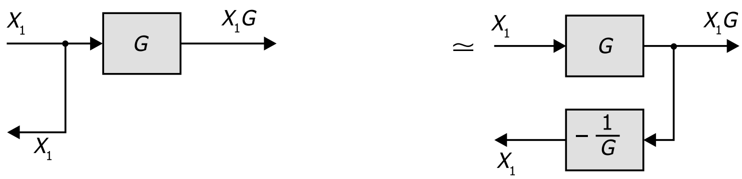
(a) 

(b) 

(c) 

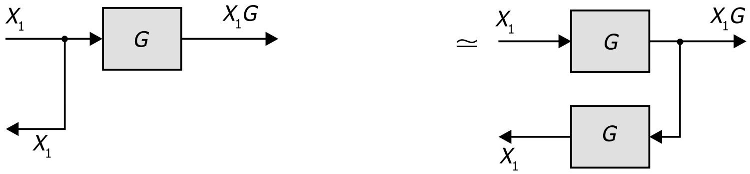
(d) 

Q.9 In block diagram algebra the rule for moving a take off point after a block is

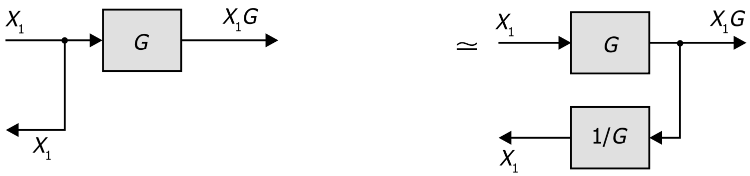
(a) 

(b) 

(c)

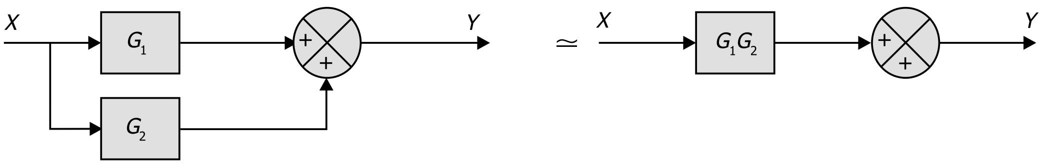


(d)

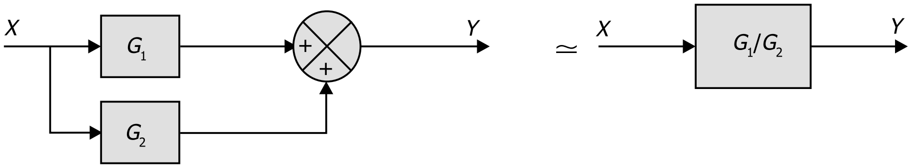


Q.10 The rule for block diagram algebra for eliminating two parallel blocks is

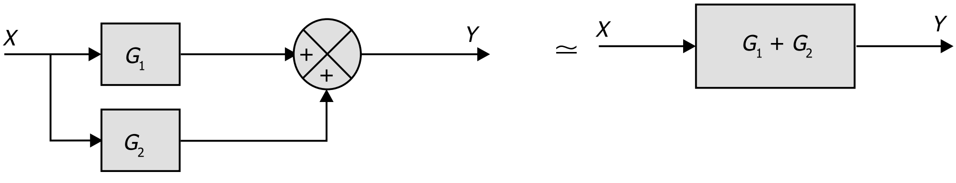
(a)



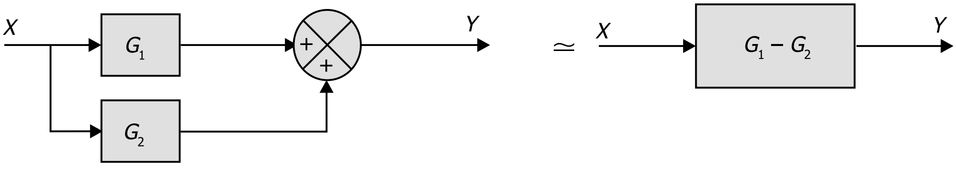
(b)



(c)



(d)



Q.11 The transfer function, *TF* of the network shown in the Fig. A4.3 below

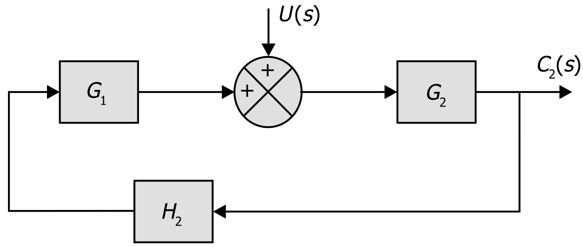


Fig. A4.3

is expressed as

(a) $TF = \dfrac{C_2(s)}{U(s)} = \dfrac{G_2}{1+G_1G_2H_2}$

(b) $TF = \dfrac{C_2(s)}{U(s)} = \dfrac{G_2}{1-G_1G_2H_2}$

(c) $TF = \dfrac{C_2(s)}{U(s)} = \dfrac{G_2}{1+\dfrac{G_1H_2}{G_1}}$

(d) $TF = \dfrac{C_2(s)}{U(s)} = \dfrac{G_2}{1-\dfrac{G_1H_2}{G_1}}$

Q.12 The transfer function of the feedback control system is shown in Fig. A4.4

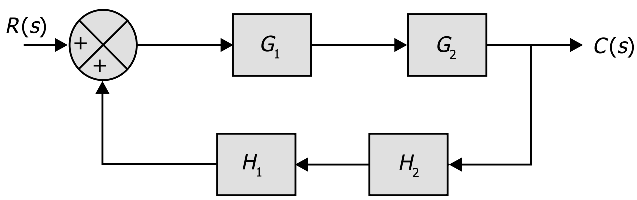


Fig. A4.4

is given as

(a) $TF = \dfrac{G_1G_2}{1+G_1G_2H_1H_2}$

(b) $TF = \dfrac{G_1/G_2}{1+G_1G_2H_1H_2}$

(c) $TF = \dfrac{G_1G_2}{1 - G_1G_2H_1H_2}$

(d) $TF = \dfrac{G_1G_2}{1 + \dfrac{G_1G_2}{H_1H_2}}$

Q.13. Block diagram of a control system is shown in Fig. A4.5

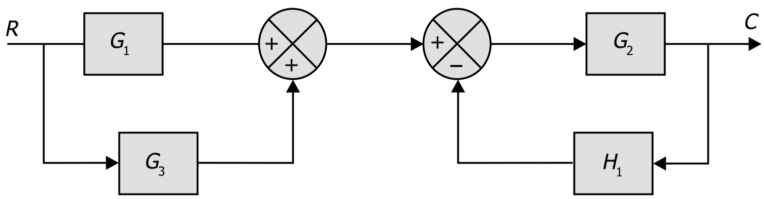
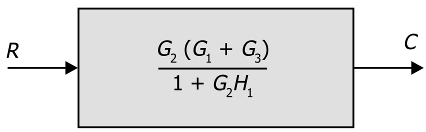


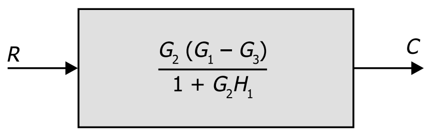
Fig. A4.5

Reduce the block diagram to

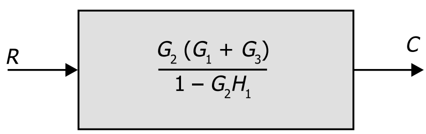
(a)



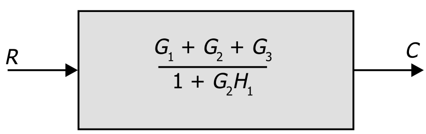
(b)



(c)



(d)



Q.14 The transfer function of the *RC* network is shown in Fig. A4.6

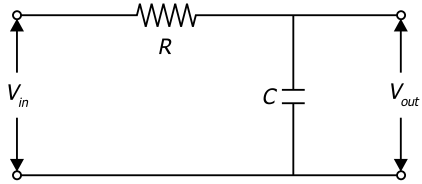


Fig. A4.6

(a) $G(s) = \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{1}{RCS+1}$

(b) $G(s) = \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{R}{CS+1}$

(c) $G(s) = \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{S}{RCS+1}$

(d) $G(s) = \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{1}{R(CS+1)}$

Q.15 The transfer function of the RC network is shown in Fig. A4.7

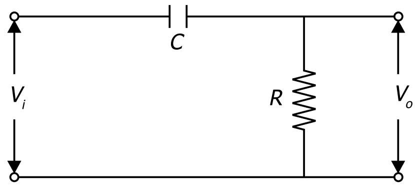


Fig. A4.7

(a) $G(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{1}{RCS+1}$

(b) $G(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{RCS+1}{RC}$

(c) $G(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{R}{RCS+1}$

(d) $G(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{RCS}{RCS+1}$

ANSWERS

Q.1 i) Laplace transform, Laplace transform input variable ii) $G(s)=\dfrac{1}{RCS+1}$ iii) $G(s)=\dfrac{RCS}{RCS+1}$ iv) mathematical equations v) various or different inputs vi) $G(s)=\dfrac{4s^2+2s+1}{5s^3+3s^2+2s+1}$ vii) $G(s)=\dfrac{1}{Ms^2+fs+K}$ viii) Transient Q.2 (b) Q.3 (a)

Q.4 (b) Q.5 (c) Q.6 (a) Q.7 (b) Q.8 (a) Q.9 (d) Q.10 (c) Q.11 (b) Q.12 (c) Q.13 (a) Q.14 (a)

Q.15 (d)

A4.3 MODELLING A CONTROL SYSTEM—SIGNAL FLOW GRAPHS

Q.1 A signal-flow graph is a
- (a) diagram that consists of nodes connected by several direct branches and is a graphical representation of a set of linear relations
- (b) diagram that consists of nodes connected by unidirectional branches and is a graphical representation of a set of linear relations
- (c) diagram that consists of nodes connected by unidirectional branches and is a graphical representation of a set of non-linear relations
- (d) diagram that consists of nodes connected by several direct branches and is a graphical representation of a set of non-linear relations

Q.2 In a signal-flow graph the term node represents
- (a) a system variable which is equal to the sum of incoming and outgoing signals at the node
- (b) a system variable which is equal to the sum of incoming signals minus the outgoing signals at the node
- (c) a system variable which is equal to the sum of all incoming signals at the node and outgoing signals from the node not affecting the value of the node variable
- (d) a system variable which is equal to the product of all the incoming and outgoing signals meeting at a point of the signal flow graph

Q.3 Signal flow graphs are primarily useful for
- (a) analysing open-loop control systems
- (b) feedback control systems because feedback theory is primarily concerned with the flow and processing of signals in systems
- (c) analysing non-linear control systems
- (d) analysing non-linear feedback control systems

Q.4 The basic element of a signal-flow graph is
- (a) unidirectional path segment called a Branch which relates the dependency of an input and an output variable in a manner equivalent to a block of a block diagram

(b) a set of nodes with incoming and outgoing branches
(c) a traversal of interconnected branches in the direction of the branch arrows
(d) a unidirectional path segment called a Branch which relates the dependency of an input and an output variable and has no equivalency with a block of a block diagram

Q.5 The block diagram representation of the branch relating to the output of a DC motor, $\boldsymbol{\theta}$ (s) and field voltage, $V_f(s)$ is shown in Fig. A4.8. The corresponding signal flow graph will be

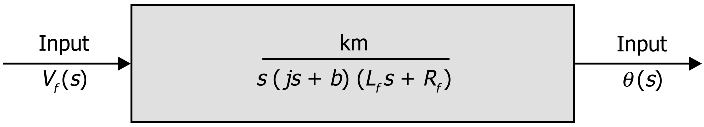
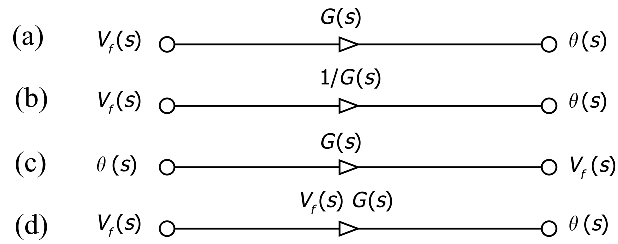


Fig. A4.8



Q.6 In a signal-flow graph, a loop is a
(a) closed path that originates and terminates on the same node and along the path not more than one node meets
(b) closed path that originates and terminates on the same node and along the path no node meets twice
(c) closed path that originates at the input node and terminates at the output node
(d) closed path forming the feedback loop

Q.7 A signal-flow graph of an interconnected system is shown in Fig. A4.9

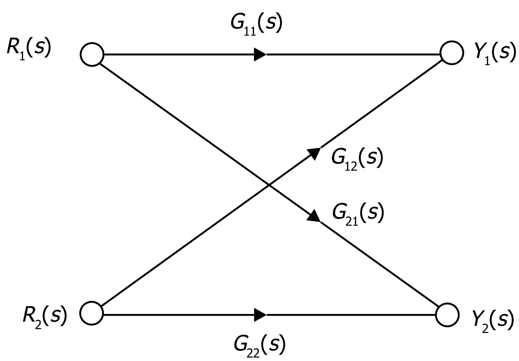


Fig. A4.8

The corresponding algebraic equations are written as

(a) $Y_1(s) = G_{11}(s)\, R_1(s) - G_{12}(s)\, R_2(s)$
$Y_2\,(s) = G_{12}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$

(b) $Y_1(s) = G_{11}(s)\, R_1(s) - G_{12}(s)\, R_2(s)$
$Y_2\,(s) = G_{12}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$

(c) $Y_1(s) = G_{12}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$
$Y_2\,(s) = G_{11}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$

(d) $Y_1(s) = G_{12}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$
$Y_2\,(s) = G_{11}(s)\, R_1(s) - G_{22}(s)\, R_2(s)$

Q.8 For two input variables y, and y_2 and the output variables x_x and x_2, a signal-flow graph is shown in Fig. A4.10. The corresponding simultaneous algebraic equation are

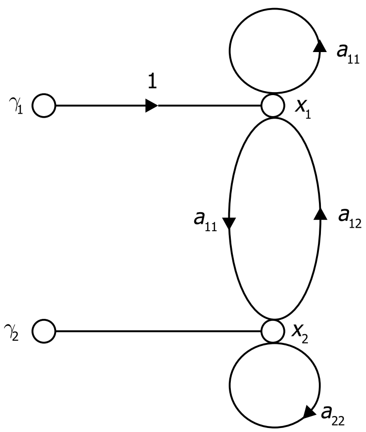


Fig. A4.10

(a) $x_1\,(1 + a_{11}) + x_2(-a_{12}) = \gamma_1$
$x_1\,(-a_{21}) + x_2(1 + a_{22}) = \gamma_2$

(b) $x_1\,(1 + a_{11}) - x_2(-a_{12}) = \gamma_1$
$x_1\,(-a_{21}) + x_2(1 + a_{22}) = \gamma_2$

(c) $x_1\,(1 - a_{11}) - x_2(-a_{12}) = \gamma_1$
$x_1\,(-a_{21}) + x_2(1 - a_{22}) = \gamma_2$

(d) $x_1\,(1 - a_{11}) - x_2(-a_{12}) = \gamma_1$
$x_1\,(-a_{21}) + x_2(1 + a_{22}) = \gamma_2$

Q.9 Manson's gain formula for overall gain T can be expressed as

$$T = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

The value of Δ is calculated as

(a) 1 − (sum of all different loop gains)
+ (sum of the gain products of all combinations of two non-touching loops)
− (sum of the gain products of all combinations of three non-touching loops)
+ …...

(b) 1 + (sum of all different loop gains)
− (sum of the gain products of all combination of two non-touching loops)
+ (sum of the gain products of all combinations of three non-touching loops) +…

(c) (sum of all different loop gains) + (sum of the gain products of all combinations of two non-touching loops) − (sum of the gain products of all combinations of three non-touching loops) +…

(d) (sum of all different loop gains) − (sum of the gain products of all combinations of two non-touching loops) + (sum of the gain products of all combination of three non-touching loops) −….

Q.10 In a signal-flow graph loops are said to be non-touching if
(a) they do not touch alternate nodes
(b) they do not touch any node
(c) they do not posses any common node
(d) they posses one common node

Q.11 In a signal-flow graph forward path is a
(a) path from the input node to the output node
(b) traversal of connected branches in the direction of the branch arrows
(c) trace of outgoing branches from the input node
(d) path from the input node to the output node including all the non-touching loops

Q.12 In the signal-flow graph shown in Fig. A4.11

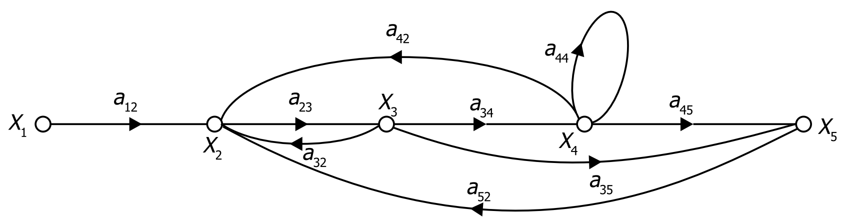


Fig. A4.11

the forward path gains are
(a) $P_1 = a_{12} + a_{23} + a_{34} + a_{45}$ and $P_2 = a_{12} + a_{23} + a_{35}$
(b) $P_1 = a_{12}\, a_{23} + a_{34} + a_{45}$ and $P_2 = a_{12} + a_{23} + a_{35}$
(c) $P_1 = a_{12}\, a_{23}\, a_{34} a_{45}$ and $P_2 = a_{12} + a_{23} + a_{35}$
(d) $P_1 = a_{12}\, a_{23} + a_{34}\, a_{45}$ and $P_2 = a_{12} + a_{23} + a_{35}$

Q.13 In the signal-flow graph shown in Fig. A4.11, the determinant of the graph, Δ is
(a) $\Delta = 1 - (a_{23}\, a_{32} + a_{23}\, a_{34}\, a_{42} + a_{44} + a_{23}\, a_{34}\, a_{45}\, a_{52} + a_{23}\, a_{35}\, a_{52}) + (a_{23}\, a_{32}\, a_{44} + a_{23}\, a_{35}\, a_{52}\, a_{44})$
(b) $\Delta = 1 - (a_{23}\, a_{32} + a_{23}\, a_{34}\, a_{42} - a_{44} - a_{23}\, a_{34}\, a_{45}\, a_{52} + a_{23}\, a_{35}\, a_{52}) + (a_{23}\, a_{32}\, a_{44} + a_{23}\, a_{35}\, a_{52}\, a_{44})$
(c) $\Delta = 1 + (a_{23}\, a_{32} + a_{23}\, a_{34}\, a_{42} + a_{44} + a_{23}\, a_{34}\, a_{45}\, a_{52} + a_{23}\, a_{35}\, a_{52}) - (a_{23}\, a_{32}\, a_{44} + a_{23}\, a_{35}\, a_{52}\, a_{44})$
(d) $\Delta = 1 - (a_{23}\, a_{32} - a_{23}\, a_{34}\, a_{42} + a_{44} + a_{23}\, a_{34}\, a_{45}\, a_{52} - a_{23}\, a_{35}\, a_{52}) + (a_{23}\, a_{32}\, a_{44} + a_{23}\, a_{35}\, a_{52}\, a_{44})$

ANSWERS

Q.1 (a) Q.2 (c) Q.3 (b) Q.4 (a) Q.5 (a) Q.6 (b) Q.7 (b) Q.8 (d) Q.9 (a) Q.10 (c) Q.11 (a) Q.12 (c) Q.l3 (a)

A4.4 FEEDBACK CONTROL SYSTEM—CHARACTERISTICS AND PERFORMANCE

Q.1 Frequency response of a system means
(a) a steady-state response of the system to sinusoidal input signal
(b) a transient response of the system to sinusoidal input signal
(c) a steady-state response of the system to impulse input signal
(d) a steady-state response of the system to unit step input signal

Q.2 The steady-state error is the error when
(a) the time period is not less than one microsecond
(b) the time period is large and the transient response has decayed
(c) the time period is large but the transient response is still persisting
(d) the time period is large but there is oscillating behaviour of the output

Q.3 A system sensitivity can be expressed as
(a) the ratio of the change in the system transfer function to the change of process transfer function (or parameter) for a small incremental change
(b) product of the change in the system transfer function to the change of process transfer function (or parameter) for a small incremental change
(c) the effect of change on performance of the system as a consequence of change of parameters
(d) a number which signifies the sensitivity with respect to the response of the system to any change

Q.4 The effect of introducing the feedback can be expressed as
(a) increasing sensitivity, improving transient response and removing all possibilities of the system becoming unstable
(b) reducing sensitivity, empowering transient response, but increasing the effects of transient response

(c) reducing sensitivity, improving transient response, minimising effects of disturbance signals, reducing the gain of the system and introducing the possibility of instability

(d) increasing sensitivity, improving transient response, minimising effects of disturbance signals, increasing the gain of the systems and introducing possibility of instability

Q.5 For a system with process transfer function $G(s) = \dfrac{K}{TS+1}$ the steady-state error for the open-loop system will be zero when

(a) $K = 1$

(b) $K = 0$

(c) $K < 1$

(d) $K > 1$

Q.6 For a system with process transfer function $G(s) = \dfrac{K}{TS+1}$, the steady-state error for the closed-loop system will be zero when

(a) $K = \mu$

(b) $K = 0$

(c) $K > 1$

(d) $K < 1$

Q.7 For a feedback control system transient response and steady-state response respectively means

(a) response that disappears with time and that exists after a long passage of time

(b) response that does not die out so easily and that exists after a long passage of time

(c) response that occurs as a result of disturbance signal and that exists after a long passage of time

(d) response that occurs as a result of sinusoidal input signal and the response as a result of unit step input

Q.8 The step, ramp, and parabolic test input signals can respectively be expressed as

(a) $R(s) = A/s$; $R(s) = A/s^2$; and $R(s) = \dfrac{2A}{s^2}$

(b) $R(s) = A/s$; $R(s) = \dfrac{A}{s^2}$; and $R(s) = \dfrac{2A}{s^3}$

(c) $R(s) = A$; $R(s) = \dfrac{2A}{s}$; and $R(s) = \dfrac{2A}{s^3}$

(d) $R(s) = A$; $R(s) = \dfrac{A}{s}$; and $R(s) = \dfrac{A}{s^2}$

Q.9 Due to the use of feedback, the steady-state error
(a) gets increased
(b) gets reduced
(c) remains constant
(d) remains independent of parameter variations

ANSWERS

Q.1 (a) Q.2 (b) Q.3 (a) Q.4 (c) Q.5 (a) Q.6 (a) Q.7 (a) Q.8 (B) Q.9 (b)

A4.5 ERROR ANALYSIS

Q.1 The various test input signals applied to the study of the performance of control systems are
(a) step signal representing instantaneous or sudden change ramp signal representing constant change; parabolic signal representing accelerating change and impulse signal representing sudden shock
(b) step signal representing constant change; ramp signal representing sudden change; parabolic signal representing accelerating change and impulse signal representing sudden shock
(c) step signal representing instantaneous or sudden change; ramp signal representing accelerating change; parabolic signal representing constant change and impulse signal representing sudden shock
(d) step signal representing sudden shock; ramp signal representing constant change; parabolic signal representing accelerating change and impulse signal representing instantaneous or sudden change

Q.2 The block diagram of a control system is shown in Fig. A4.12

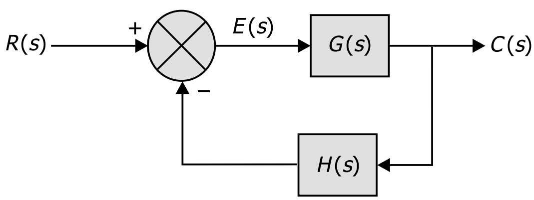


Fig. A4.12

The steady-state error is given as

(a) $\lim_{s \to 0} \dfrac{SR(s)}{1+G(s)H(s)}$

(b) $\lim_{s \to 0} \dfrac{R(s)}{1-G(s)H(s)}$

(c) $\lim\limits_{s\to 0}\dfrac{R(s)}{1+G(s)H(s)}$

(d) $\lim\limits_{s\to 0}\dfrac{R(s)}{s[1-G(s)H(s)]}$

Q.3 For a unit step input the steady-state error is

(a) $\lim\limits_{s\to 0}\dfrac{s+1}{1+G(s)H(s)}$

(b) $\lim\limits_{s\to 0}\dfrac{1}{1-G(s)H(s)}$

(c) $\lim\limits_{s\to 0}\dfrac{1}{1+G(s)H(s)}$

(d) $\lim\limits_{s\to 0}\dfrac{1}{s[1+G(s)H(s)]}$

Q.4 The error constants k_p, k_v, and k_a stands respectively for

(a) static position error, static velocity error and static acceleration error
(b) static proportional error, static velocity error and static acceleration error
(c) static position error, static velocity error and static acceleration error
(d) static proportional error, static velocity error and static acceleration error

Q.5 The steady-state errors of a Type-0 system

(a) For unit-step it is $1/(1 - k_p)$; for unit ramp it is μ ; and for unit parabolic it is μ where k_p = position error constant = $\lim\limits_{s\to 0} G(s)$
(b) For unit-step it is $1/(1 - k_p)$; for unit ramp it is μ; and for unit parabolic it is μ; where k_p = position error constant = $\lim\limits_{s\to 0} G(s)$
(c) For unit-step it is $1/(1 + k_p)$; for unit ramp it is 0 and for unit parabolic it is μ ; where k_p = position error constant = $\lim\limits_{s\to 0} G(s)$
(d) For unit-step it is $1/(1 + k_p)$; for unit ramp it is μ ; and for unit parabolic it is 0; where k_p = position error constant = $\lim\limits_{s\to 0} G(s)$

Q.6 Steady-state error for unit step for various types of system are

(a) Type-0 system = $\dfrac{1}{1+k_p}$; Type-1 system = μ; Type-2 system = μ

(b) Type-0 system = $\dfrac{1}{1+k_p}$; Type-1 system = 0; and Type-2 system = μ

(c) Type-0 system $= 0$; Type-1 system $= 0$; Type-2 system $= \mu$

(d) Type-0 system $= \dfrac{1}{1+k_p}$; Type-1 system $= 0$; and Type -2 system $= 0$

Q.7 Steady-state error for unit ramp input for type 0, type 1, and type 2 systems respectively are [where $k_s = \lim_{s\to 0} SG(s)$]:

(a) μ, k_v, μ
(b) μ,$1/k_v$, μ
(c) μ $1/k_v$, μ
(d) μ, $1/k_v$, μ

Q.8 Steady-state error for unit parabolic input for type 0, type 1, and type 2 systems respectively are [where k_a $\underset{s\to 0}{\text{Limit}}\, S^2G(s)$]:

(a) 0 k_a μ
(b) k_a μ μ
(c) μ μ k_a
(d) μ μ $1/k_a$

Q.9 Choose the correct statement from the following four alternative statements

(a) Higher is the type of system, less is the steady-state error and better is the system stability
(b) Higher is the type of system, less is the steady-state error but less stable is the system
(c) Higher is the type of system, higher is the steady-state error but less stable is the system
(d) Higher is the type of system, higher is the steady-state error but more stable is the system

Q.10 For a unity feedback system having an open-loop transfer function, $G(s) = \dfrac{K(S+2)}{s^2(s^2+7s+12)}$, the error constants k_p, k_v, and k_a respectively are

(a) μ, μ, $k/6$
(b) 0, 0, $k/6$
(c) μ, 0, $k/6$
(d) 0, μ, k

Q.11 The steady-state error constants suffers from the following drawbacks

(a) They provide no information on steady state-errors when inputs are other than step, ramp, or parabolic; and how the error varies with time.
(b) They provide information on steady-state error for all kinds of inputs and how the error varies with time.

(c) They provide information of steady-state error for all kinds of inputs but do not provide information as to how the error varies with time.
(d) They provide no information on steady-state error when input is other than step; and how the error varies with time.

Q.12 A system is considered an optimum control system when
(a) the system parameters are adjusted so as to remain at an extreme constant value
(b) the system parameters are adjusted so that the performance index reaches an value, commonly a minimum value
(c) the system parameters are kept constant
(d) the system parameters are adjusted so that the performance index reaches a maximum value

Q.13 Performance indices for a control system are
(a) integral of square of the error, ISE; integral of the absolute magnitude of the error, IAE; integral of time multiplied by squared error, ITSE
(b) integral of step input error, ISE; integral of the absolute magnitude of the error, IAE; integral of time multiplied by squared error, ITSE
(c) integral of square of the error, ISE; integral of the amplitude error IAE; integral of time multiplied by squared error, ITSE
(d) integral of square of the error, ISE; integral of the absolute magnitude of the error; IAE, integral of true stable error; ITSE

Q.14 Performance index is a quantitative measure of the performance of a system. Some of the indices are ISE, ITSE, IAE. These are expressed respectively as

(a) $\mathrm{ISE} = \int_0^T e^2(t)dt; \mathrm{ITSE} = \int_0^T t\, e^2(t)dt;\ \mathrm{IAE} = \int_0^T |e(t)|dt$

(b) $\mathrm{ISE} = \int_0^T e(t)dt; ITSE = \int_0^T t\, e(t)dt;\ \mathrm{IAE} = \int_0^T |e(t)|\, dt$

(c) $\mathrm{ISE} = \int_0^T e^2(t)dt; \mathrm{ITSE} = \int_0^T t^2\, e^2(t)dt;\ \mathrm{IAE} = \int_0^T |e(t)|dt$

(d) $ISE = \int_0^T e(t)dt; ITSE = \int_0^T t^2 e(t)dt;\ IAE = \int_0^T |e(t)|\, dt$

Q.15 *PD, PI,* and *PID* are respectively known as
(a) proportional plus derivative controller, proportional plus instability controller and proportional plus instability plus derivative controller
(b) proportional plus derivative controller, proportional plus integral controller and proportional plus integral plus derivative controller

(c) position plus derivative controller, position plus integral controller and position plus integral plus derivative controller
(d) position plus derivative controller, proportional plus integral controller and position plus integral plus derivative controller

ANSWERS

Q.1 (a) Q.2 (a) Q.3 (c) Q.4 (a) Q.5 (a) Q.6 (d) Q.7 (d) Q.8 (d) Q.9 (b) Q.10 (a) Q.11 (a) Q.12 (b) Q.13 (a) Q.14 (a) Q.15 (b)

A4.6 TIME RESPONSE ANALYSIS

Q.1 Four standard test signals are shown in Fig. A4.13

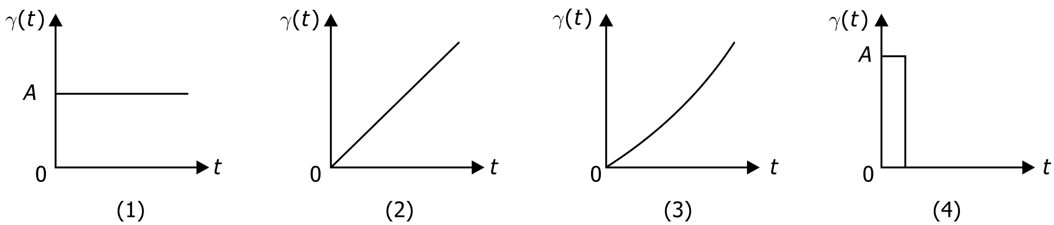


Fig. A4.13

These test signals are respectively called
(a) impulse signal; step signal; parabolic signal; and step signal
(b) ramp signal; step signal; parabolic signal; and impulse signal
(c) step signal; ramp signal; parabolic signal; and impulse signal
(d) impulse signal; ramp signal; parabolic signal; and step signal

Q.2 A *R*-*C* network is shown in Fig. A4.14

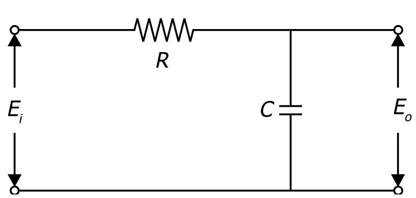


Fig. A4.14

The transfer function $E_0(s)/E_1(s)$ can be written as

(a) $\dfrac{1}{1+RCS}$ or $\dfrac{1}{1+TS}$ where $T = RC$

(b) $\dfrac{1}{1+\dfrac{S}{RC}}$ or $\dfrac{1}{1+\dfrac{S}{T}}$ where $T = RC$

(c) $\dfrac{1}{1+R^2C^2S}$ or $\dfrac{1}{1+T^2S}$ where $T = RC$

(d) $\dfrac{RCS}{1+RCS}$ or $\dfrac{TS}{1+TS}$ where $T = RC$

Q.3 Block diagram of a first order system is shown in Fig. A4.15

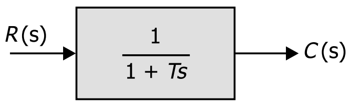


Fig. A4.15

The output response for a unit-step input is expressed as
(a) $C(t) = 1 + e^{t/T}$
(b) $C(t) = 1 - e^{-t/T}$
(c) $C(t) = 1 + e^{-ttT}$
(d) $C(t) = 1 + e^{t/T}$

Q.4 The time constant T is indicative of
(a) how fast the system tends to reach the final value
(b) how slow the system is
(c) how fast the system output decays to zero value
(d) the system's steady-state behaviour

Q.5 A large time constant and small time constant of a system corresponds respectively to
(a) oscillatory system and stable system
(b) sluggish system and fast responsive system
(c) fast responsive system and sluggish system
(d) large output and minimum output

Q.6 The steady state error for the first order system is shown in Fig. A4.16

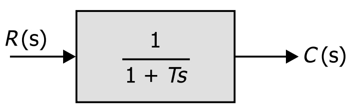


Fig. A4.16

Unit-step input and unit-ramp input respectively are
(a) T and T
(b) 0 and 0
(c) T and 0
(d) 0 and T

Q.7 The steady-state error for a first order system having transfer function $\frac{1}{1+Ts}$ for unit-step, ramp and impulse test signals respectively are
(a) 0, T, 0
(b) 0, 0, T
(c) T, 0, 0
(d) 0, T, T

Q.8 The general form of transfer function of a second order system can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2s\omega_n s + \omega_n^2}$$

where s = damping factor or damping ratio and $\boldsymbol{\omega}_n$ = undamped natural frequency The characteristic equation is
(a) $\omega_n^2(s^2 + 2s\omega_n s + \omega_n^2) = 0$
(b) $s^2 + 2s\omega_n s + \omega_n^2 = 0$
(c) $s^2 + 2s\omega_n s - \omega_n^2 = 0$
(d) $s^2 + 2s\omega_n s = 0$

Q.9 For second order system when value of damping factor is varied from less than 1 to 0 to 1 to greater than 1, the system behaviour will respectively be
(a) underdamped, critically damped, oscillatory, overdamped
(b) underdamped, oscillatory, critically damped overdamped
(c) overdamped, oscillatory, critically damped, underdamped
(d) underdamped, critically damped, overdamped, oscillatory

Q.10 Time response specifications of a system are
(a) Delay time, rise time, maximum error, peak overshoot, settling time, and steady state-error
(b) Delay time, rise time, peak time, peak overshoot, settling time and transient and steady-state error
(c) Delay time, rise time, peak time, peak overshoot, settling time, and maximum error
(d) Delay time, rise time, peak time, peak overshoot, settling time, and steady-state error

Q.11 Delay time in time response specification of a system indicates
(a) the time required for the response to reach 70.7 per cent of the final value in the first-attempt
(b) the time required for the response to reach 100 per cent of the final value in the first-attempt
(c) the time required for the response to reach 50 per cent of the final value in the first-attempt
(d) the time required for the response to reach 80 per cent of the final value in the first-attempt

Q.12 Rise time in time response specification of a system indicates the time required for the response to rise from
(a) 10 per cent to 90 per cent of the final value of overdamped systems; and 0 to 100 per cent of the final value for underdamped systems
(b) 20 per cent to 80 per cent of the final value for overdamped systems; and 0 to 100 per cent of the final value of under damped systems
(c) 10 per cent to 90 per cent of the final value of underdamped systems; and 0 to 100 per cent of the final value of overdamped systems
(d) 20 per cent to 80 per cent of the final value of underdamped systems; and 0 to 100 per cent of the final value for overdamped systems

Q.13 Settling time in time response specification of a system indicates the time required for the response to reach and stay within a specified tolerance band of usually
(a) 1 per cent or 2 per cent
(b) 5 per cent or 10 per cent
(c) 2 per cent or 5 per cent
(d) 0.1 per cent or 0.2 per cent

ANSWERS

Q.l (c) Q.2 (a) Q.3 (b) Q.4 (a) Q.5 (b) Q.6 (d) Q.7 (a) Q.8 (b) Q.9 (b) Q.10 (d) Q.11 (c) Q.12 (a) Q.l3 (c)

A4.7 CONCEPT OF STABILITY AND ROUTH–HURWITZ CRITERIA

Q.1 A system is stable if
(a) all the poles of the transfer function have positive real parts
(b) all the poles of the transfer function have negative real parts
(c) all the poles of the transfer function have positive real parts near zero
(d) one of the poles of the transfer function lies near zero

Q.2 A system is stable if
(a) all the poles of the characteristic equation are towards the left-hand side of the s-plane
(b) one of the poles of the characteristic equation is in the left-hand side of s-plane
(c) one of the poles of the characteristic equation is towards the right-hand side of s-plane
(d) one of the poles of the characteristic equation lie at the origin

Q.3 If the characteristic equation has simple roots on the imaginary axis ($j\omega$-axis) with all other roots in the left half of s-plane, for bounded input the steady-state output will be
(a) stable
(b) sustained oscillations, unless the input is a sinusoid whose frequency equals the magnitude of $j\omega$-axis roots
(c) unstable
(d) stable provided that the input is a sinusoid

Q.4 The location of roots in the s-plane are shown in Fig. A4.17. State whether the system is stable or not.

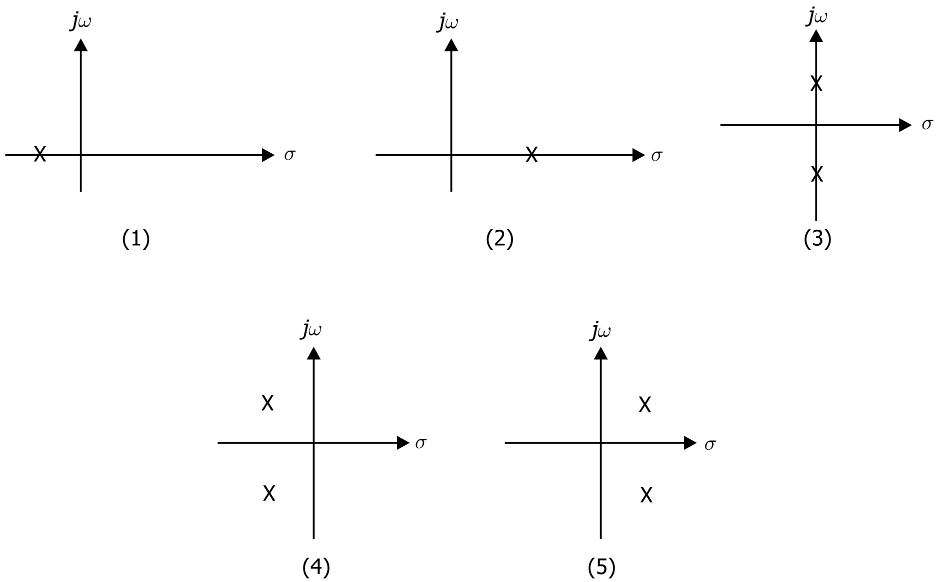


Fig. A4.17

(a) (1) - stable, (2) - unstable, (3) - stable, (4) - unstable, (5) - unstable
(b) (1) - unstable, (2) - unstable, (3) - stable, (4) - unstable, (5) - stable
(c) (1) - stable, (2) - stable, (3) - oscillatory, (4) - unstable, (5) - stable
(d) (1) - stable, (2) - unstable, (3) - oscillatory, (4) - stable; (5) - unstable

Q.5 If any root of the characteristic equation has a positive real part of the system is
(a) stable (b) unstable (c) oscillatory (d) conditionally stable

Q.6 If there are repeated roots of the characteristic equation on the $j\omega$-axis, the system is
(a) conditionally stable (b) oscillatory
(c) stable (d) unstable

Q.7 The characteristic equation of a first order system is $a_0s + a_1 = 0$. The condition for stability of the system is
(a) either a_0 or a_1 must be negative
(b) both a_0 or a_1 must be positive
(c) either a_0 or a_1 must be positive
(d) both a_0 or a_1 must be positive

Q.8 Routh array for the characteristic equation
$q(s) = a_0s^n + a_1s^{n-1} + a_2s^{n-2} + ... + a_{n-1}\, s + a_n = 0$
is shown as

$$\begin{array}{c|ccccc} s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & & \\ s^{n-2} & b_1 & b_2 & b_3 & & \\ s^{n-3} & c_1 & c_2 & & & \\ \vdots & \vdots & \vdots & & & \\ s^2 & & & & & \\ s^1 & & & & & \\ s^0 & & & & & \end{array}$$

The value of b_1 and b_2 are calculated as
(a) $b_1 = (a_1 a_2 - a_0a_3)/a_1$
$b_2 = (a_1 a_4 - a_0 a_3)/a_1$
(b) $b_1 = (a_1 a_2 + a_0 a_3)/a_1$
$b_2 = (a_1 a_4 - a_0 a_3)/a_1$
(c) $b_1 = (a_1 a_2 + a_0 a_3)/a_1$
$b_2 = (a_1 a_4 - a_0 a_5)/b_1$
(d) $b_1 = (a_1 a_2 + a_0 a_3)/a_1$
$b_2 = (a_1 a_4 - a_0 a_5)/b_1$

Q.9 Characteristic equation for a second order system is $a_2s^2 + a_1s + a_0 = 0$
The Routh array can be represented as

(a) $\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s & a_1 & 0 \\ s^0 & a_0 & 0 \end{array}$ (b) $\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s & a_1 & 0 \\ s^0 & 0 & 0 \end{array}$

(c) $\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s & a_1 & 0 \\ s^0 & 0 & a_0 \end{array}$

(d) $\begin{array}{c|cc} s^2 & a_0 & a_2 \\ s & 0 & a_1 \\ s^0 & 0 & a_0 \end{array}$

Q.10 The characteristic equation of a third order system is
$a_3s^2 + a_2s^2 + a_1s + a_0 = 0$
The necessary and sufficient condition for stability using Routh's criteria is
(a) All the coefficients be positive and $a_2a_1 < a_0a_3$
(b) All the coefficients be positive and $a_2a_1 > a_0a_3$
(c) All the coefficients be negative and $a_2a_1 < a_0a_3$
(d) All the coefficients be negative and $a_2a_1 < a_0a_3$

Q.11 The characteristic equation for a third order system is $s^3 + 2s^2 + 4s + k = 0$
For the system to be stable, what should be the value of gain k?
(a) $0 < k < 8$
(b) $0 < k < 4$
(c) $2 < k < 8$
(d) $0 < k < 2$

Q.12 The characteristic equation for a system is $s^2 + bs + 1 = 0$.
What is the condition for stability of the system according to Routh's stability criterion?
(a) $b = 1$
(b) $b > 0$
(c) $b < 0$
(d) $b > 1$

Q.13 The characteristic equation for a system is $s^3 + bs^2 + cs + 1 = 0$
(a) $bc - 1 > 0$
(b) $bc - 1 < 0$
(c) $b > c$
(d) $b < c$

Q.14 The characteristic equation for a system is expressed as $(K + 1)\, s^2 + (3K - 0.9)\, s + (2K + 0.1) = 0$
The condition for stability is
(a) $K < 1$
(b) $K > 0.9$
(c) $K > 0.3$
(d) $K < 0.3$

Q.15 The characteristic equation for a system is given as $2s^3 + s^2 - 3s + 10 = 0$.
State why the system is stable or unstable?

(a) stable because no root is in the right-half of s-plane
(b) unstable because there are two roots in the right-half of s-plane
(c) stable because all the roots are in the left-side of s-plane
(d) unstable because one root is in the right-half of s-plane

ANSWERS

Q.1 (b) Q.2 (a) Q.3 (b) Q.4 (d) Q.5 (b) Q.6 (d) Q.7 (d) Q.8 (c) Q.9 (a) Q.10 (b) Q.11 (a) Q.12 (b) Q.13 (a) Q.14 (c) Q.15 (b)

A4.8 ROOT LOCUS TECHNIQUE

Q.1 Root locus is defined as
(a) the path of the roots of the characteristic equation traced out in the S-plane as a system parameter is changed
(b) the path of the roots of the characteristic equation traced out in the S-plane as the system parameters are kept constant
(c) the path of the roots of the transfer function traced out in the S-plane as a system parameter is changed
(d) the path of the roots of the transfer function traced out in the S-plane as a system parameter is kept constant

Q.2 The open-loop transfer function of a system is

$$G(s) = \frac{K}{S(S+A)}$$

where 'K' and 'A' are constants.
How many poles and zeros are there in transfer function?
(a) one, zero
(b) two, zero
(c) three, one
(d) zero, two

Q.3 The closed-loop transfer function of a system is

$$\frac{C(s)}{R(s)} = \frac{K}{S^2 + AS + K}$$

The characteristic equation of the system is
(a) $S^2 + AS + 2K = 0$
(b) $S^2 + AS + K = 0$
(c) $S^2 + AS + K^2 = 0$
(d) $S^2 + AS = 0$

Q.4 Root locus technique is applicable to
(a) single-loop system
(b) multiple-loop system
(c) single as well as multiple loop system
(d) not more than two loop systems

Q.5 Root locus technique provides a graphical method of plotting the locus of the roots in the S-plane for
(a) constant system parameters
(b) a given parameter that varies within a limited range
(c) a given parameter that varies over a complete range of values
(d) variable system parameters over a limited range of values

Q.6 Root locus method of determining the stability of a system may be defined as
(a) a method of plotting the roots of the characteristic equation of the open-loop system in the S-plane with the gain parameter varying from zero to infinity
(b) a method of plotting the roots of the characteristic equation of the closed-loop system in the S-plane with the gain parameter varying from zero to infinity
(c) a method of plotting the roots of the characteristic equation of the closed-loop system in the S-plane with the gain parameter kept constant
(d) a method of plotting the roots of the characteristic equation of the open-loop system is the S-plane with the gain parameter kept constant

Q.7 As the gain K increases from zero to infinity
(a) each branch of the root locus will originate from an open-loop pole with $K = 0$ and terminate either on an open-loop zero or on infinity with $K = \mu$ and the number of branches terminating on infinity will be equal to the number of open-loop poles minus zeros
(b) each branch of the root locus will originate from an open-loop pole with $K = 0$ and terminate either on an open-loop pole or on infinity with $K = \mu$ and the number of branches terminating on infinity will be equal to the number of open-loop poles minus zeros
(c) each branch of the root locus will originate from an open-loop pole with $K = 0$ and terminate either on an open-loop zero or on infinity with $K = \mu$ and the number of branches terminating on infinity will be equal to the number of open-loop poles plus zeros
(d) each branch of the root locus will originate from an open-loop zero with $K = 0$ and terminate either on an open-loop pole or an infinity with $K = \mu$ and the number of branches terminating on infinity will equal to the number of open-loop poles plus zeros

Q.8 For an open-loop transfer function of a system expressed as

$$\frac{K(S+1)(S+2)}{S(S+3)(S+4)}$$

the information regarding the root locus plot that can be obtained is

(a) there are three branches of the root locus since there are three open-loop poles; the branches start with $K = 0$ at each of the poles, $S = 0$, $S = -3$, and $S = -4$; as K increases the branches will leave the open-loop poles and seek open-loop zeros; two of the branches will terminate on the open-loop zeros at $S = -1$ and $S = -2$ and one will terminate at infinity

(b) there are three branches of the root locus since there are open-loop poles; the branches start with $K = 0$ at each of the poles, $S = 0$, $S = -1$, and $S = -2$; as K increases the branches will leave the open-loop poles and seek open-loop zeros; two of the branches will terminate on the open-loop zeros at $S = -1$ and $S = -2$ and one will terminate at infinity

(c) there are three branches of the root locus since there are three open-loop poles; the branches start with $K = 0$ at each of the poles, $S = 0$, $S = -3$, and $S = -4$; as K increases the branches will leave the open-loop poles and seek open-loop zeros; all the branches will terminate at infinity

(d) there are three branches of the root locus since there are three open-loop poles; the branches start with $K = 0$ at each of the poles, $S = 0$, $S = -1$, and $S = -2$; as K increases the branches will leave the open-loop poles and seek open-loop zeros; all the branches will terminate at infinity

Q.9 For a feedback control system having a characteristic equation

$$1 + \frac{K}{S(S+1)(S+2)} = 0,$$

The information regarding the root locus that can be obtained is:

(a) there are three branches of the root locus; with $K = 0$, the branches of the root locus originate from the open-loop poles at $S = 0, -1, -2$; there are no open-loop poles and all the three branches terminate at infinity

(b) there are three branches of the root locus; with $K = 0$, the branches of the root locus originate from the open-loop poles at $S = 0, -1, -2$; there is one open-loop pole; and two of the branches terminate at infinity

(c) there are three branches of the root locus; with $K = 0$, all the branches of the root locus originate from $S = 0, -1, -2$; there are no open-loop poles; and all the three branches terminate at infinity

(d) there are two branches of the root locus; with $K = 0$, the branches originate from the open loop poles at $S = -1$, $S - 2$; there is one open-loop pole; and the two branches terminate at infinity

Q.10 The (number of roots n, number of zeros m) branches of a root locus which tend to infinity, do so along straight-line asymptotes whose angles are given by the relation

(a) $\phi_A = \dfrac{(2q+1)90^\circ}{n-m}; q = 0,1,2,\text{--},(n\text{-}m\text{-}1)$

(b) $\phi_A = \dfrac{(2q+1)180^\circ}{n-m}; q = 0,1,2,\text{--},(n\text{-}m\text{-}1)$

(c) $\phi_A = \dfrac{(2q+1)180^\circ}{n-m}; q = 0,1,2,\text{--},(n\text{-}m\text{-}1)$

(d) $\phi_A = \dfrac{(2q+1)180^\circ}{n-m}; q = 0,1,2,\text{--},(n\text{-}m\text{-}1)$

Q.11 For a feedback system having the characteristic equation

$$1 + \frac{K}{S(S+1)(S+2)} = 0,$$

the angles of the straight line asymptotes with the real axis, along with the root locus tend to infinity are given by

(a) 39°, 90°, 180°
(b) 30°, 180°, 300°
(c) 60°, 180°, 300°
(d) 30°, 60°, 120°

Q.12 The open-loop transfer function of a feedback system is expressed as

$$G(s)H(s) = \frac{K}{S(S+4)(S^2+4S+20)}$$

Applying the rules for drawing the root locus we get the following information as

(a) There are four branches of root locus; there is no open-loop-zero; all the four branches terminate on infinity along the asymptotes whose angles with the real axis are 40°, 135°, 225°, and 315°.
(b) There are three branches of root locus; there is no open-loop zero; all the three branches terminate on infinity along the asymptotes whose angles with the real axis are 40°, 135°, 225°, and 315°.
(c) There are four branches of root locus; there is no open-loop zero; two of the four branches terminate on infinity along the asymptotes whose angles with the real axis are 40°, 315°, 225°, and 315°.
(d) There are four branches of root locus; there is no open-loop zero; all the four branches terminate on infinity along the asymptotes whose angles with the real axis are 30°, 180°, and 270°.

Q.13 The transfer function of a unit feedback system is represented as

$$G(S) = \frac{K}{S(S+1)}$$

The following information can be written for drawing the root locus

(a) Number of branches of the root locus is two; number of asymptotes is two, making angles of 90° and 270° respectively with the real axis; the branches start at $S = 0$ and $S = -1$ and both terminate on infinity along the asymptotes; the break-away point is at $S = -1/2$.

(b) Number of branches of the root locus is two; number of asymptotes is two, making angles of 180° and 270° respectively with the real axis; the branches start at $S = 0$ and $S = -1$ and both terminate on infinity along the asymptotes; the break-away point is at $S = -1/4$.

(c) Number of branches of the root locus is two; number of asymptotes is two, making angles of 90° and 270° respectively with the real axis; the branches start at $S = 0$ and $S = -1$ and both terminate on infinity along the asymptotes; the break-away point is at $S = -1/2$.

(d) The number of branches of the root locus is two; number of asymptotes is two, making angles of 90° and 270° respectively with the real axis; both the branches start at $S = 0$ and both terminate on infinity along the asymptotes; the break-away point is at $S = -1/2$.

Q.14 Addition of a plea to the open-loop transfer function has the effect of

(a) shifting the root-locus to the left, thereby decreasing the relative stability and increasing the settling time

(b) shifting the root-locus to the right, thereby decreasing the relative stability and increasing the settling time

(c) shifting the root-locus to the left, thereby decreasing the relative stability and the settling time

(d) shifting the root-locus to the right, thereby increasing the relative stability and the settling time

Q.15 Addition of a zero to the open-loop transfer function has the effect of

(a) shifting the root-locus to the right thereby increasing stability and decreasing settling time

(b) shifting the root-locus to the right, thereby increasing stability and settling time

(c) shifting the root-locus to the left, thereby increasing both stability and settling time

(d) shifting the root-locus to the left, thereby increasing stability and decreasing settling time

Q.16 For the third-order feedback control system whose characteristic equation is

$$1 + G(S)H(S) = 1 + \frac{K(S+1)}{S(S+2)(S+3)} = 0,$$

There root-locus information such as (i) number of poles; (ii) number of zeros; (iii) number of branches; (iv) number of asymptotes and their angles with the real axis; (v) the centered are calculated as:

(a) (i) 3; (ii) 1; (iii) 3; (iv) 2 at ± 90°; (v) – 2;
(b) (i) 3; (ii) 1; (iii) 3; (iv) 2 at ± 180°; (v) – 2;
(c) (i) 3; (ii) 1; (iii) 3; (iv) 2 at ± 90°; (v) – 1;
(d) (i) 3; (ii) 1; (iii) 2; (iv) 2 at ± 90°; (v) – 2;

Q.17 For a feedback control system with a characteristic equation

$$1+\frac{K}{S(S+1)(S+2)}=0,$$

The branches originating at $S = 0$ and $S = -1$, will break-away on real axis as K increases on a point

(a) −1.577 (b) −0.605
(c) −0.423 (d) −0.005

ANSWERS

Q.1 (d) Q.2 (b) Q.3 (b) Q.4 (c) Q.5 (c) Q.6 (b) Q.7 (a) Q.8 (a) Q.9 (a) Q.10 (b) Q.11 (c) Q.12 (a) Q.13 (a) Q.14 (b) Q.15 (d) Q.16 (a) Q.17 (c)

A4.9 FREQUENCY RESPONSE ANALYSIS

Q.1 Frequency response of a system is defined as

(a) the transient response to a sinusoidal input signal
(b) the steady-state response to a sinusoidal input signal
(c) the transient response to a parabolic input signal
(d) the steady-state response to a unit step input signal

Q.2 The advantages and disadvantages of frequency response method of analysis can be expressed as

(a) ready availability of sinusoidal test signals; transfer function can be obtained by replacing s by $j\omega$ in the transfer function $G(s)$; no direct link between frequency and time domain
(b) ready availability of step input signal; transfer function can be obtained by replacing s by $j\omega$ in the transfer function $G(s)$; no direct link between frequency and time domain
(c) ready availability of all types of input signals; transfer function can be obtained by replacing s by $j\omega$ in the transfer function $G(s)$; no direct link between frequency and time domain
(d) ready availability of sinusoidal input signal; transfer function can be obtained by replacing s by $j\omega$ in the transfer function $G(s)$; having direct link between frequency and time domain

Q.3 A second-order system is shown in Fig. A4.18

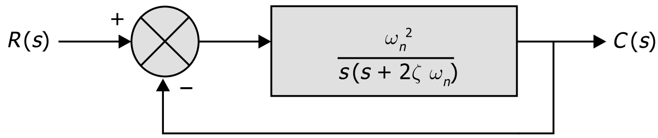


Fig. A4.18

The transfer function is

(a) $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$

(b) $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

(c) $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s-\omega_n^2}$

(d) $\dfrac{C(s)}{R(s)} = \dfrac{1}{s^2+2\zeta\omega_n s-\omega_n^2}$

Q.4 In the general transfer function of a second-order system expressed as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

the terms ζ and ω_n stands respectively for
(a) error coefficient and undamped natural frequency of oscillations
(b) damping factor and undamped natural frequency of oscillations
(c) phase margin and undamped natural frequency of oscillations
(d) damping factor and angular velocity

Q.5 For the system having transfer function as in 4 above, the resonant frequency, ω_r and resonant peak, M_r respectively are

(a) $\omega_r = \omega_n\sqrt{1-2\zeta^2}; M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

(b) $\omega_r = \omega_n\sqrt{1+2\zeta^2}; M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

(c) $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}; M_r = \dfrac{1}{2\zeta\sqrt{1+\zeta^2}}$

(d) $\omega_r = \omega_n (1 - 2\zeta^2); M_r = \dfrac{1}{2\zeta\sqrt{1+\zeta^2}}$

Q.6 The two important performance indices for a second-order system are
(a) Band width and break frequency
(b) Resonant peak, M_r and resonant frequency, $\boldsymbol{\omega}_r$
(c) Error coefficient and resonant frequency, $_r$
(d) Resonant peak, M_r and angular velocity, $\boldsymbol{\omega}$

Q.7 A typical frequency response curve of a feedback control system has been shown in Fig. A4.19

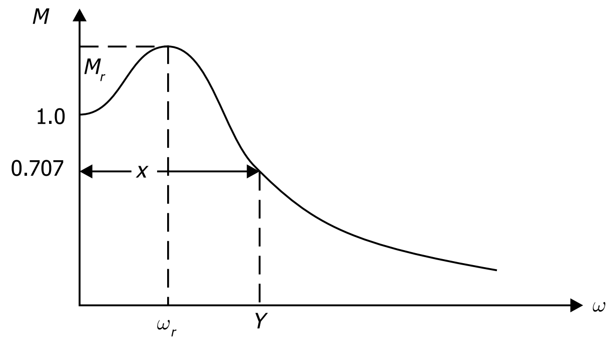


Fig. A4.19

The term x and y in the graph represents respectively
(a) bandwidth and cut-off frequency
(b) cut-off frequency and bandwidth
(c) gain margin and phase margin
(d) bandwidth and phase margin

Q.8 Phase margin may be defined as
(a) the amount of additional phase-lag at the gain cross-over frequency required to bring the system to the verge of instability
(b) the amount of additional phase-lead at the gain cross-over frequency required to bring the system to the verge of instability
(c) the amount of additional phase-lag at the gain cross-over frequency required to bring the system to the verge of stability
(d) the amount of additional phase-lead at the gain cross-over frequency required to bring the system to the verge of stability

Q.9 Gain margin is
(a) the factor by which the system gain can be increased to drive it to stability
(b) the factor by which the system gain can be increased to drive it to the verge of instability
(c) the factor by which the system gain must be increased to keep the system stable
(d) the factor by which the system gain must be increased to drive the system to oscillatory mode

Q.10 Gain margin in the magnitude versus phase plot of $G(j\omega)$ $H(j\omega)$ is the distance measured in decibels from
(a) the phase cross-over to the critical point at 0 dB and $-$ 180°
(b) the gain cross-over to the critical point at 0 dB and $-$ 180°
(c) the gain cross-over to the point at 0 dB and $-$ 90°
(d) the gain cross-over to the point at 0 db and $-$ 270°

Q.11 Phase margin in the magnitude versus phase plot of $G(j\omega)$ $H(j\omega)$ is the
(a) vertical distance in dB measured from the phase cross-over to the point at 0 dB and $-$ 180°
(b) horizontal distance in degrees measured from the gain cross-over to the critical point at 0 dB and $-$ 180°
(c) horizontal distance in degree's measured from the gain cross-over to the point at 0 dB and $-$90°
(d) horizontal distance in degrees measured from gain cross-over to the point at 0 dB and $-$270°

Q.12 A closed-loop system is unstable if
(a) both gain margin and phase margin are negative
(b) gain margin is positive and phase margin is negative
(c) gain margin is negative and phase margin is positive
(d) both gain margin and phase margin are positive

Q.13 A closed-loop system is stable if
(a) gain margin is negative and phase margin is positive
(b) gain margin is positive and phase margin is negative
(c) both gain margin and phase margin are negative
(d) both gain margin and phase margin are positive

ANSWERS

Q.1 (b) Q.2 (a) Q.3 (b) Q.4 (b) Q.5 (a) Q.6 (b) Q.7 (a) Q.8 (a) Q.9 (b) Q.10 (a) Q.11 (b) Q.12 (a) Q.13(d)

A4.10 DESIGN AND COMPENSATION

Q.1 A compensator used in control systems is
(a) an additional component or circuit that is inserted into the system to compensate for its deficient performance
(b) to reduce the steady-state error keeping other output variables unchanged
(c) to compensate the inefficient performance of any of the system components
(d) a device which controls the gain of the amplifier used in the feedback circuit

Q.2 Performance specification of a control system can be defined in terms of
(a) its transfer function
(b) desirable location of the poles and zeros of the closed-loop transfer function
(c) its characteristic equation
(d) desirable location of the poles and not the zeros of the closed-loop transfer function

Q.3 A compensating network is added to
(a) keep the locus of the roots Constant as a system parameter is varied
(b) alter the locus of the roots as a system parameter is varied
(c) alter the locus without changing the position of poles and zero
(d) keep the locus of the roots constant and not alter the position of the poles and zeros

Q.4 Performance of a control system can be described in terms of
(a) time-domain performance measures and not frequency-domain performance measures
(b) frequency-domain performance measures and not time-domain performance measures
(c) time-domain performance measures or frequency-domain performance measures
(d) only time-domain performance measures

Q.5 In time-domain, the performance of a system can be specified in terms of
(a) Peak time, maximum overshoot and settling time for a step input and steady-state error for several test signal inputs
(b) Peak time, resonant frequency and settling time
(c) Phase margin, maximum overshoot and settling time
(d) Resonant frequency, maximum overshoot and bandwidth

Q.6 The performance of a feedback control system in terms of frequency performance measures can be described by
(a) peak of the closed-loop frequency response, resonant frequency, bandwidth and phase margin of the system
(b) peak time, maximum overshoot and phase margin
(c) peak time phase margin, and bandwidth
(d) phase margin, peak time and settling time

Q.7 In frequency response approach, compensation network is used to alter and reshape the system's characteristics represented on an

(a) s-plane
(b) root locus diagram
(c) polar graph
(d) bode diagram and Nichols chart

Q.8 The design of the control system can be accomplished in the s-plane by root locus method
(a) altering and reshaping the root locus so that the roots of the system lie in the desired position
(b) fixing the roots of the system on the right-hand side of the imaginary axis
(c) so that the roots lie on the left-hand side on the imaginary axis
(d) altering and reshaping the root locus so that at least one pair of roots lie on the imaginary axis

Q.9 Addition of compensating networks are done when the designer
(a) is not able to alter the design of the process or the basic design of the system including the control components
(b) wants to change the basic design of the system with a compensating network added to the system
(c) does not want to compromise on the basic design of the system or change any component
(d) wants to study the performance of the system under variable input conditions

Q.10 The transfer function of a compensating network is given as

$$G_c(s) = \frac{s+z}{s+p}$$

when $|z| < |p|$, the network is called the
(a) phase-lag network
(b) phase-lead network
(c) phase lead-lag network
(d) phase shifting network

Q.11 Fig. A4.20 below shows a compensating network

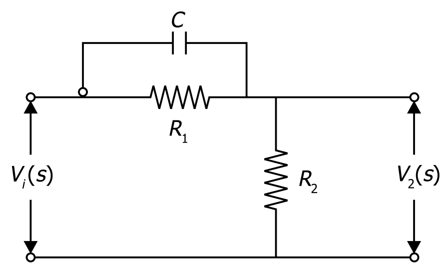


Fig. A4.20

The above network is called
(a) phase lag network
(b) phase-lead-lag network
(c) phase-lead network
(d) phase correcting network

Q.12 The network as shown in Fig. A4.21 is a

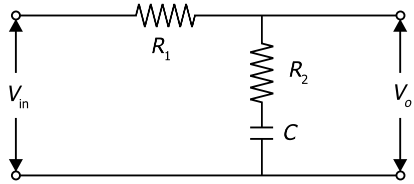


Fig. A4.21

(a) phase-lag network
(b) phase-lead network
(c) differentiator
(d) phase lead-lag network

Q.13 Addition of phase-lead compensating network
(a) increases system bandwidth, increases gain at higher frequencies and improves dynamic response but requires additional amplifier gain
(b) decreases system bandwidth, increases gain at higher frequencies and improves dynamic response, but requires additional amplifier gain
(c) increases system bandwidth, decreases gain at higher frequencies and improves dynamic response but requires additional amplifier gain
(d) increases system bandwidth, increases gain at higher frequencies but decreases dynamic response

Q.14 Addition of phase-lag compensating network produces
(a) decreased error constant, decreased system bandwidth and reduced steady-state error
(b) increased error constant, decreased system bandwidth and reduced steady-state error
(c) increased error constant, increased system bandwidth and reduced steady-state error
(d) increased error constant, decreased system bandwidth and increased steady-state error

Q.15 Introduction of phase-lead and phase-lag compensating network leads to the following features. Choose the correct alternative.
(a) phase-lead compensation improves speed; phase-lag compensation improves steady-state error but speed of response gets reduced
(b) phase-lead compensation improves speed of response, reduces steady-state error; phase lag compensation improves steady-state error but speed of response gets reduced

(c) phase-lead compensation improves speed of response but not the steady-state error; phase-lag compensation improves both the speed of response and steady-state error
(d) phase-lead compensation improves speed of response but not the steady-state error; phase-lag compensation improves speed of response but increases, steady-state error

ANSWERS

Q.1 (a) Q.2 (b) Q.3 (b) Q.4 (c) Q.5 (a) Q.6 (a) Q.7 (d) Q.8 (a) Q.9 (a) Q.10 (b) Q.11 (c) Q.12 (a) Q.13 (a) Q.14 (b) Q.15 (a)

A4.11 CONCEPT OF STATE-VARIABLE MODELLING

Q.1 State variable method of modelling, analysis and design is applicable to
(a) linear, time varying, multi-input multi-output system
(b) linear and non linear, time-invariant or time varying, multi-input multi-output systems
(c) linear, time-invariant, single-input single-output systems
(d) linear, time-invariant, multi-input multi-output systems

Q.2 Transfer function approach to modelling, analysis and design has limitations. Choose the correct statement.
(a) Transfer function is defined only under zero initial conditions, only applicable to linear time-invariant systems and generally applicable to SISO systems
(b) Transfer function is defined for zero initial conditions, only applicable to non-linear time-invariant system and generally applicable to SISO systems
(c) Transfer function is defined only under zero initial conditions, only applicable to linear times-varying systems and generally applicable to SISO systems
(d) Transfer function is defined only under zero initial conditions, only applicable to linear time-varying systems and generally applicable to MIMO systems

Q.3 State variable modelling utilises
(a) state feedback to introduce pole replacement technique in the design for improvement of system performance
(b) state variable techniques for time invariant systems for improvement of system performance
(c) state feedback to replace transfer functions approach in the design for improvement of system performance
(d) state feedback to introduce pole placement technique in the design for improvement of system performance

Q.4 The closed-loop system pole locations can be arbitrarily placed if and only if the system is

(a) time invariant
(b) linear
(c) controllable
(d) stable

Q.5 A system is said to be controllable if the state variable flow graph is drawn and it is determined whether the control signal
(a) has a path to each state variables
(b) has a path to almost all the state variables
(c) has a path to at least one state variable
(d) has a path which does not touch all the state variables

Q.6 The three types of variables used in state-space modelling of dynamic systems are
(a) the controllable phase variables, state-space variables and output variables
(b) the input variables, output variables and state variables
(c) the state-space variables output variables and state variables
(d) the controllable phase variables, output variables and state variables

Q.7 State variables of a dynamic system are
(a) the smallest set of variables that completely determine the state of the system
(b) canonical form of the state model expressed as a suitable transformation matrix
(c) a set of variables that partially describes a system
(d) a set of differential equations that describes the system

Q.8 Phase variables are defined as those state variables which are obtained from
(a) one of the system variables and its limitations
(b) one of the system variables and its derivatives
(c) two of the system variables and their derivatives

Q.9 Principle of duality, according to Kalman, can be used to establish analogies between
(a) controllability and sensitivity
(b) controllability and responsiveness
(c) controllability and responsiveness
(d) controllability and susceptibility

ANSWERS

Q.1 (b) Q.2 (a) Q.3 (d) Q.4 (c) Q.5 (a) Q.6 (b) Q.7 (a) Q.8 (b) Q.9 (c)

A4.12 CONTROL COMPONENTS

Q.1 An error detector produces an error signal which is
(a) the difference between the desired output and the actual output
(b) the difference between the desired output and the set input

(c) the sum of all the errors of the system converted in terms of error voltage signal
(d) the ratio of the output minus the input and the input

Q.2 As error detectors are used
(a) magnetic amplifiers and potentiometers
(b) servomotors and transducers
(c) tachogenerators and transducers
(d) potentiometers and synchros

Q.3 The angular difference of the alignment of two shafts is converted into voltage signals in an error detector which is
(a) a thermistor
(b) a potentiometer
(c) set of two stepper motors
(d) set of two synchros

Q.4 In a synchro error detector, the angular difference of shaft positions are converted into
(a) AC voltage using a synchro control transformer at the input and a synchro transmitter at the output
(b) DC voltage using a synchro control transformer at the input and a synchro transmitter at the output
(c) AC voltage using a synchro transmitter at the input and a synchro control transformer at the output
(d) DC voltage using a synchro transmitter at the input and a synchro control transformer at the output

Q.5 Tachogenerators are used as transducers where
(a) rotational speed is converted into proportional DC or AC voltage
(b) rotational speed is converted into proportional DC voltage only
(c) rotational speed is converted into proportional AC voltage only
(d) rotational speed is converted into proportional angular displacement

Q.6 DC Servomotors are specially designed
(a) DC motors with high starting torque and low inertia and operate in either directions
(b) DC motors with very low starting torque and inertia and can operate in either directions
(c) DC motors with high starting torque and high inertia and can operate in either directions
(d) DC motors with high starting torque and low inertia and operate in a particular direction only

Q.7 AC servo motors are two-phase induction motors where
(a) field windings are placed at right angles, one phase winding is supplied from a constant DC reference voltage
(b) there is no phase difference between the two stator phase windings, one phase winding is supplied from a constant AC reference voltage

(c) field windings are placed at right angles, one phase winding is supplied from a constant AC reference voltage
(d) there is no phase diffidence between the two stator phase windings, one phase winding is supplied from a constant DC reference voltage

Q.8 In an AC servomotor
(a) one phase winding is supplied from a constant AC reference voltage and the other from the output of a servoamplifier which acts as a controlling field
(b) one phase winding is supplied from a constant DC reference voltage and the other from the output of a servo amplifier which acts as a controlling filed
(c) both the windings are supplied from two-phase AC so as the create a rotating magnetic field effect
(d) one phase winding is supplied from a constant DC voltage as reference voltage and the other from a sinusoidal AC voltage

Q.9 Gear trains are used between the servomotors and the load to
(a) provide variable torque and variable speed as required by the load
(b) reduce the speed of the motor as required by the load
(c) create a flexible coupling between the motor and the load
(d) provide constant torque at reduced speed required by the load

Q.10 Transducers are devices which convert
(a) non-electrical physical quantities into electro-mechanical energy
(b) any electrical quantity into a proportional feedback signal
(c) any mechanical quantity into a proportional electrical signal
(d) non-electrical physical quantities into electrical signals

Q.11 Which of the conversion tables using three transducers are correct?

(a)

Transducer	Conversion
i) Piezo-electric ii) Strain-gauge iii) potentiometer	i) Pressure to voltage ii) Displacement to resistance iii) Displacement to resistance

(b)

i) Piezo-electric crystal ii) Strain-gauge iii) Potentiometer	i) Light to resistance ii) Displacement to voltage iii) Displacement to resistance

(c)

i) Piezo-electric crystal ii) Strain-gauge iii) Potentiometer	i) Pressure to resistance ii) Displacement to voltage iii) Displacement to voltage

(d)

i) Piezo-electric crystal ii) Strain gauge iii) Potentiometer	i) Pressure to voltage ii) Displacement to voltage iii) Displacement to current

Q.12 Which of the following sets represent transducers?
(a) Thermocouple, solar cell, microphone, thermistor and strain gauge
(b) Thyristor, solar cell, loudspeaker, microphone and strain gauge
(c) Transistor, resistor, solar cell, strain gauge and microphone
(d) Thermocouple, diode, loud speaker, thermistor and strain gauge

Q.13 Type of measurement and the corresponding type of transducers required are shown. Which of these is the correct?

(a)

i) Measurement of temperature ii) Measurement of light iii) Measurement of speed of rotation iv) Measurement of linear displacement	i) Solar cells ii) Thermocouples iii) Tachogenerator iv) LDVT

(b)

i) Measurement of temperature ii) Measurement of light iii) Measurement of speed of rotation iv) Measurement of linear displacement	i) Thermocouple ii) Solar cells iii) Tachogenerators iv) LVDT

(c)

i) Measurement of temperature ii) Measurement of light iii) Measurement of speed of rotation iv) Measurement of linear displacement	i) LVDT ii) Thermocouple iii) Solar cells iv) Tachogenerators

(d)

i) Measurement of temperature ii) Measurement of light iii) Measurement of speed of rotation iv) Measurement of linear displacement	i) Solar cells ii) Thermocouple iii) Servomotor iv) Tachogenerator

Q.14 Magnetic amplifiers are made of
(a) saturable core reactors which have narrow and steep hysteresis loop
(b) saturable core reactors which have broad but steep hysteresis loop
(c) grain oriented silicon steel laminations having wide hysteresis loop
(d) saturable core reactors which have narrow hysteresis loop and how situation flux density

Q.15 An operational amplifier is basically a
(a) low gain direct coupled amplifier with very high input impedance and very low output impedance
(b) high gain direct coupled amplifier with very low input impedance and very high output impedance
(c) high gain direct coupled amplifier with very low input and output impedances
(d) high gain direct coupled amplifier with very high input impedance and very low output impedance

Q.16 A rotary amplifier is a
- (a) cross field generator, separately excited and driven by an induction motor
- (b) cross field generator, self excited and driven by an induction motor
- (c) DC generator having multiple excitation windings and is run by a DC motor
- (d) brushless DC generator driven by an induction motor

Q.17 A stepper motor is a
- (a) brushless DC motor whose rotor windings are kept short circuited
- (b) brushless DC motor whose rotor has no electrical windings
- (c) brushless DC motor whose rotor winding is fed from a constant DC voltage source
- (d) small DC motor which is separately excited from a constant DC voltage source

ANSWERS

Q.1 (a) Q.2 (d) Q.3 (b) Q.4 (c) Q.5 (a) Q.6 (a) Q.7 (c) Q.8 (a) Q.9 (b) Q.10 (d) Q.11 (a) Q.12 (a) Q.13 (b) Q.14 (d) Q.15 (a) Q.16 (b)

APPENDIX 5
KEY TERMS

Actuating signal It is also called error signal (see error signal).

Asymptote The path the root locus follows as the parameter becomes very large and approaches infinity.

Bandwidth The frequency at which the frequency response declines to 3 dB from its low-frequency value.

Bode plot The logarithm of the magnitude of the transfer function is plotted versus the logarithm of ω, the frequency. The ϕ, of the transfer function is plotted versus the logarithm of the frequency.

Characteristic equation The relation formed by equating to zero the denominator of a transfer function.

Closed-loop feedback control system A system that uses a measurement of the output and compares it with the desired output.

Compensation The alteration or adjustment of a control system in order to provide a suitable performance.

Compensator An additional component or circuit that is inserted into the system to compensate for a performance deficiency.

Control system An interconnection of components that forms a system configuration providing a desired response.

Controlled variables Quantity or condition of the control system which is directly controlled or measured.

Dominant roots The roots of the characteristic equation that cause the dominant transient response of the system.

Damped oscillation An oscillation in which the amplitude decreases with time.

Damping ratio A measure of damping. A dimensionless number for the second-order characteristic equation.

Error signal The difference between the desired output, $R(s)$ and the actual output, $C(s)$. Therefore, $E(s) = R(s) - \mathrm{C(s)}$.

Frequency response The steady-state response of a system to a sinusoidal input signal.

Gain margin The increase in the system gain when phase $= -180°$ will result in a marginally stable system with the intersection of the $-1 + j0$ point on the nyquist diagram.

Laplace transform A transformation of a function $f(t)$ from the time domain into the complex frequency domain, yielding $F(s)$.

Lead-lag network A network with dual characteristics of a lead network and a lag network.

Linear system A system that satisfies the properties of superposition and homogeneity.

Natural response The frequency of natural oscillation that would occur for two complex poles if the damping were equal to zero.

Negative feedback The output signal is fed back so that it is subtracted from the input signal.

Nichols chart A chart displaying the curves for the relationship between the open-loop and closed-loop frequency response.

Nyquist stability criterion A feedback system is stable if and only if, the contour in the $G(s)$-plane does not encircle the $(-1, 0)$ point when the number of poles of $G(s)$ in the right-hand S-plane is zero. If $G(s)$ has P poles in the right-hand plane, then the number of counterclockwise encirclements of the $(-1, 0)$ point must be equal to P for a stable system.

Open-loop control system A system that utilises a device to control the process without using the feedback. The output has no effect on the signal process.

Optimisation The adjustment of the parameters to achieve the most favourable or advantageous design.

Optimum control system A system whose parameters are adjusted so that the performance index reaches an extreme value.

Overshoot The amount by which the system output response proceeds beyond the desired response.

PD controller A controller with three terms in which the output is the sum of a proportional term, an integrating term and a differentiating term, with an adjustable gain for each term.

Peak time The time for a system to respond to a step input and finally rise to a peak response.

Performance index A quantitative measure of the performance of a system.

Phase margin The amount of phase shift of the $GH(j\omega)$ at unity magnitude will result in a marginally stable system with intersections of the $-1 + j0$ point on the nyquist diagram.

Phase-lag network A network that provides a negative phase angle and a significant attenuation over the frequency range of interest.

Phase-lead network A network that provides a positive phase angle over the frequency range of interest. Thus phase lead can be used to cause a system to have an adequate phase margin.

PI controller Controller with a proportional term and an integral term (proportional-integral).

Polar plot A plot of the real part of $G(j\omega)$ versus the imaginary part of $G(j\omega)$.

Positive feedback The output signal is fed back so that it adds to the input signal.

Resonant frequency The frequency ω_r, at which the maximum value of the frequency response of a complex pair of poles is attained.

Rise time The time for a system to respond to a step input and attain a response equal to a percentage of the magnitude of the input. The 0–100 per cent rise time T_r, measures the time to 100 per cent of the magnitude of the input.

Root locus The locus or path of the roots traced out on the s-plane as a parameter is changed.

Root locus method The method for determining the locus of roots of the characteristic equation $1 + KP(s) = 0$ as K varies from 0 to ∞.

Routh-Hurwitz criterion A criterion for determining the stability of a system by examining the characteristic equation of the transfer function. The criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes of sign of the coefficients in the first column of the routh array.

Sensors These are low-powered transducers which produce output signal as a measure of the controlled variable.

Settling time The time required for the system output to settle within a certain percentage of the input amplitude.

Signal-flow graph A diagram that consists of nodes connected by several direct branches and is a graphical representation of a set of linear relations.

Simulation Model of a system that is used to investigate the behaviour of a system by utilising actual input signals.

Stability Performance measurement of a system. A system is stable if all the poles of the transfer function have negative real parts.

State variable The set of variables that describes a system.

Steady-state error Error when the time period is large and the transient response has decayed, leaving a continuous response.

System An interconnection of elements and devices for a desired purpose.

Test input signal An input signal used as a standard test of a system's ability to respond adequately.

Time domain The mathematical domain that incorporates the time response and the description of a system in terms of time t.

Transfer function The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable.

Transfer function in the frequency domain The ratio of the output to the input signal where the input is a sinusoid. It is expressed as $G(j\omega)$.

Transient response The response of a system as a function of time.

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Note: Locator followed by *"f"* and *"t"* represents figures and tables

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