Problem 18.13

Prove that if $A \subseteq X$, and $f : A \to Y$ is continuous for Y a Hausdorff space, that if f may be extended to a continuous function $g : \bar{A} \to Y$, then this g is unique.

Suppose g is an extension. We show that g is unique. Suppose h is another continuous extension of f with $g \neq h$, so let $x \in \bar{A}$ such that $g(x) \neq h(x)$.

Since every $a \in A$ has $g(a) = f(a) = h(a)$, we know that x is not in A so x is a limit point of A.

Let U and V be disjoint neighborhoods of $g(x)$ and $h(x)$ respectively. Then we have that $x \in g^{-1}(U)$ and $x \in h^{-1}(V)$.

By continuity of g we can choose a neighborhood $B_1 \subseteq \bar{A}$ such that $g(B_1) \subseteq U$. Also, there is a neighborhood $B_2 \subseteq \bar{A}$ such that $h(B_2) \subseteq V$. Then we have that x is in the open set $B_1 \cap B_2$.

Since x is a limit point of A, there is some limit point $a \in A$ such that $a \in B_1 \cap B_2$. Then for this a we have $g(a) = f(a) = h(a)$ with $f(a) \in U$ and $f(a) \in V$. This gives a contradiction since U and V were constructed to be disjoint. Thus, we conclude $g = h$.

Problem 20.6

(a)

Let $y = \max(x_1 + \epsilon - 1, x_1) \times \max(x_2 + \epsilon - 1/2, x_2) \times \max(x_3 + \epsilon - 1/3, x_3) \times \ldots$.

Then $y \in U(\mathbf{x}, \epsilon)$. However, we have that $\sup(x_\alpha, y_\alpha) = \epsilon$, so that y is not in the open ϵ ball about $\mathbf{x}$ in the uniform metric.

(b)

We show that y as defined above has no open ball in the uniform topology that is contained in $U(\mathbf{x}, \epsilon)$, so that $U(\mathbf{x}, \epsilon)$ is not open in the uniform topology.

Let $\delta > 0$, then there is some $n \in \mathbb{N}$ such that $1/n < \delta/2$. We have $z = y + \delta/2 \in B_\delta(y)$, where $y + \delta/2$ is y with $\delta/2$ added to each coordinate. However,

$$z_n = y_n + \delta/2 > y_n + 1/n \geq x_n + \epsilon - 1/n + 1/n = x_n + \epsilon$$

so that $z_n \notin (x_n + \epsilon)$. And thus $B_\delta(y)$ is not contained in $U(\mathbf{x}, \epsilon)$

(c)

Let $L = B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ and $R = \cup_{\delta < \epsilon} U(\mathbf{x}, \delta)$.

If $y \in L$, then we have that $\sup_n d(x_n, y_n) = d < \epsilon$, so that $d(x_n, y_n) \leq d < \epsilon$ for every $n \in \mathbb{N}$. Hence, $y_n \in (x_n - \frac{d+\epsilon}{2}, x_n + \frac{d+\epsilon}{2})$ for every $n \in \mathbb{N}$ and hence $y_n \in U(\mathbf{x}, \frac{d+\epsilon}{2})$, which is part of the union of R since $\frac{d+\epsilon}{2} < \epsilon$.

If $y \in R$, then we have that $y \in U(\mathbf{x}, \delta)$ for some $\delta < \epsilon$, so that $d(x_n, y_n) < \delta < \epsilon$ for every $n \in \mathbb{N}$. Hence, taking the supremum, we have that $\bar{\rho}(\mathbf{x}, y) \leq \delta < \epsilon$ so that $y \in L$.

Problem 20.8

Let $X \subseteq \mathbb{R}_\omega$ be the set of all sequences $\mathbf{x}$ such that $\sum_i x_i^2$ converges. Assume (a) as suggested.

(b)

We let the underlying space by the set $\mathbb{R}^\infty$ of all sequences that are eventually zero.

We have that product topology $\subseteq$ uniform topology $\subseteq$ l^2-topology $\subseteq$ box topology. Thus, if we show that each of these subset relations are proper subset relations, then we have that all 4 topologies are distinct.

First, we show that the product topology is a proper subset of the uniform topology. To see this, let $\epsilon > 0$. Then we have a subset of $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Now, for any element $C = C_1 \times C_2 \times \ldots$ of the product topology, we have that there is some $N \in \mathbb{N}$ such that $C_n = \mathbb{R}$ for $n \geq N$. Thus, we have that the point $0 \times 0 \times \ldots \times \epsilon + 1 \times 0 \times 0 \times \ldots$, where the $\epsilon + 1$ is in the Nth coordinate, is in C but not in $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Thus, there is no element of the product topology equal to this ball in the uniform topology.

Now, we show that the uniform topology is a proper subset of the l^2 topology. Choose the basis element $B_d(0,1)$ in l^2.

Let U be some open set in the uniform topology containing 0, so that U contains $B_{\bar{\rho}}(0, \delta)$ for some $\delta > 0$. Choose $N \in \mathbb{N}$ such that $\sqrt{N} \cdot \frac{\delta}{2} > 1$. Now, consider the sequence

$$x = \frac{\delta}{2} \times \frac{\delta}{2} \times \ldots \times \frac{\delta}{2} \times 0 \times 0 \times \ldots$$

where the every entry past the $(N+1)th$ is 0. Then we have that the distance from 0 in the uniform metric is $\frac{\delta}{2}$, but the distance from 0 in the l^2 metric is given by

$$\begin{aligned} d(0,x) &= \sqrt{\sum_{i=1}^{N} \left(\frac{\delta}{2}\right)^2} \\ &= \sqrt{N\left(\frac{\delta}{2}\right)^2} \\ &= \sqrt{N}\frac{\delta}{2} \\ &> 1 \end{aligned}$$

so that x is not contained in $B_d(0,1)$.

Finally, we show that the l^2 topology is a proper subset of the box topology. Consider the basis element

$$U = \prod_{n \in \mathbb{N}} (0, \frac{1}{n}) = (0,1) \times (0, \frac{1}{2}) \times (0, \frac{1}{3}) \times \ldots$$

of the box topology. Consider any basis element $B(0, \epsilon)$ for some $\epsilon > 0$ of the l^2 topology. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. Then we have the sequence

$$x = 0 \times 0 \times \ldots \times 0 \times \frac{\epsilon}{2} \times 0 \times \ldots$$

where the $\frac{\epsilon}{2}$ is in the nth position, has $d(x,0) = \frac{\epsilon}{2}$ so it is in the ball $B(0,\epsilon)$, but it is not in U since its nth coordinate is greater than $\frac{1}{n}$.

(c)

Compare the four topologies that the Hilbert cube $H = \prod_{n \in \mathbb{Z}_+} [0, \frac{1}{n}]$ inhertis as a subspace of X.

Product topology = Uniform topology. Since the notation is difficult we give a proof sketch. Let $B_{\bar{\rho}}(x, \epsilon)$ be a basis element of the uniform topology that H inherits, where $x \in H$ and $\epsilon > 0$. Then choose $n \in \mathbb{N}$ so that $\frac{1}{n} < \epsilon$.

Then the structure of $B_{\bar{\rho}}(x, \epsilon)$ is the cartesian product of first $n-1$ sets of either $[0, x_i - \epsilon), (x_i - \epsilon, x_i + \epsilon), [0, 1]$, or $(x - \epsilon, 1]$, and then the mth set in the product is of the form $[0, \frac{1}{m}]$ for all $m \geq n$. The product of the first $n-1$ sets and then the rest of the sets is in the product topology, since the first $n-1$ sets are all open in $\mathbb{R} \cap [0, \frac{1}{i}]$, and each set $[0, \frac{1}{m}]$ is exactly equal to $\mathbb{R} \cap [0, \frac{1}{m}]$. Thus, this basis element is part of the product topology, so that the two topologies are equal.

l^2 topology $\subsetneq$ box topology. To see this, we use the same U as above: $U = \prod_{n \in \mathbb{N}} (0, \frac{1}{n})$ in the box topology. The the point $x = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{6} \times \ldots$ is in U.

Suppose U belonged to the l^2 topology. Then there is some $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Then consider

$$y = x_1 \times x_2 \times \ldots \times x_{n-1} \times \frac{1}{n} \times x_{n+1} \times \ldots$$

where y differs from x only in the nth coordinate. Then the distance between x and y is the l^2 metric is $d(x, y) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} < \epsilon$, so $y \in B_d(x, \epsilon)$. However, $y \notin U$ because its nth coordinate is not less than $\frac{1}{n}$. Thus, we conclude that U is not in the l^2 topology and hence the l^2 topology is a proper subset.

Problem 24.4

Let X be an ordered set in the order topology. Prove that if X is connected, then X is a linear continuum

We first show that if $x < y \in X$, then there exists z such that $x < z < y$. Note that if this were not the case, then $(-\infty, y)$ and (x, ∞) are neighborhoods of x and y respectively that are disjoint since there are no points in the intersection (x, y) (no points greater than x and less than y) of these intervals. Moreover, these two intervals are nonempty, containing x and y respectively, and also cover all of X. This contradicts the connectedness of X, so we conclude that there is some z between x and y.

Now, we show that X has the least upper bound property. Suppose that A has no least upper bound. Let A be a nonempty subset of X that is bounded above, and let B be the set of all upper bounds. We show that B is both open and closed, so that B and B^C is a separation of X. Note that B is not empty since we are assuming that A has an upper bound, and we can assume B^C is not empty for if it were empty then A consists of only one element, so that singular element of A is the least upper bound.

First, we have that $B = \overline{B}$, since if $x \notin B$, then $x < a$ for some $a \in A$, so that x is in the open set $(-\infty, a)$ which is disjoint from B.

Now, we show that B is open. For $b \in B$, we have $b \geq a$ for every $a \in A$, so that there exists some b' such that $b > b' \geq a$ for all $a \in A$, for otherwise b would be the least upper bound. Then we have that $b \in (b', \infty) = I_b$, which is an open set contained in B. Moreover, we have that $B = \cup_{b \in B} I_b$, which is a union of open sets and hence open. Thus, B is both open and closed, and we have reached our contradiction.