Problem 26.9

Prove that if A and B are subspaces of X and Y respectively, then for an open set N in $X \times Y$ containing $A \times B$, there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq N$.

Using the tube lemma, for every $a \in A$, choose W_a open in X and V_a open in Y such that

$$a \times B \subseteq W_a \times V_a \subseteq N$$

Since the (W_a) cover A, we can choose by compactness of A a finite subcover $W_{a_1}, \ldots, W_{a_n}$. Then the set $W_{a_1} \times V_{a_1}, \ldots, W_{a_n} \times V_{a_n}$ is an open cover of $A \times B$.

Let $W = \cup_{i=1}^n W_{a_i}$ and $V = \cap_{i=1}^n V_{a_i}$. Then for every $a \times b \in A \times B$, we have that a is in some W_{a_i} since the W_{a_i} cover A, and $b \in V$ since $B \subseteq V$. Hence, we have $A \times B \subseteq W \times V$.

Now, we see that since each $W_{a_i} \times V_{a_i}$ is a subset of N, we have

$$\begin{aligned} W_{a_i} \times V &\subseteq N \\ \cup_{i=1}^n (W_{a_i} \times V) &\subseteq N \\ = W \times V &\subseteq N \end{aligned}$$

so that we are done.

Problem 26.12

We use the following lemma here and in problem 31.7. *Nice Lemma: Let $p : X \to Y$ be a perfect map. If $U \subset X$ is a neighborhood of $p^{-1}(\{y\})$, then there exists a $W \subseteq Y$ such that $p^{-1}(W) \subseteq U$*

Let (U_α) be an open covering of X. For all $y_\beta \in Y$, we have that $p^{-1}(\{y_\beta\})$ is compact, so we can choose $U_{\beta,1}, \ldots, U_{\beta,n_\beta}$ covering it. Letting U_β be the union of these sets, we choose $W_\beta \subseteq Y$ a neighborhood of y_β such that $p^{-1}(W_\beta) \subseteq U_\beta$ by the nice lemma.

We have that (W_β) is an open covering of Y, so by compactness of Y we choose $W_{\beta_1}, \ldots, W_{\beta_k}$ that covering Y. Then we have that $(U_{\beta_j,i})$ is a finite subcover of X, since

$$\begin{aligned} X &= p^{-1}(Y) \quad \text{surjectivity} \\ &= \cup_{j=1}^k p^{-1}(W_{\beta_j}) \\ &\subseteq \cup_{j=1}^k U_\beta \quad \text{choice of } W_\beta \\ &= \cup_{j=1}^k \cup_{i=1}^{n_{\beta_j}} U_{\beta,i} \end{aligned}$$

Problem 28.4

We use the lemma that in a T_1 space, an element p is a limit point of a subspace A if and only if every neighborhood of p contains infinitely many points of A.

($\Longleftarrow$) Suppose that X is limit point compact, and let $(U_n)_{n \in \mathbb{N}}$ be a countable open cover of X. Suppose for sake of contradiction that there is no finite subcover.

For each $n \in \mathbb{N}$, choose an element $x_n \notin U_1 \cup \ldots \cup U_n$, which is possible since $U_1, \ldots, U_n$ does not cover X. Then, let $A = \{x_n : n \in \mathbb{N}\}$. By limit point compactness, there exists some limit point x

of A (noting that A is infinite since otherwise we could construct a finite subcover for X). Now, we have that $x \in U_k$ for some $k \in \mathbb{N}$. However, for $n \geq k$, we have that $x_n \notin U_1 \cup \ldots \cup U_n$, so that $x_n \notin U_k$. Thus, the neighborhood U_k of x has only finitely many points of A, contradicting the fact that it is a limit point.

($\implies$) Suppose that X is countably compact. For sake of contradiction, suppose also that we have an infinite set $B \subseteq X$ with no limit point. Choose a countably infinite subset $A \subseteq B$, that must also have no limit point.

For each $i \in \mathbb{N}$, let U_i be a neighborhood of a_i containing only finite many a_j where $j \neq i$, which is possible since if it were not then a_i would be a limit point of A. Then the U_i cover A, and is a countable collection, so we can choose a finite subcover $U_{i_1}, \ldots, U_{i_m}$ that covers A. However, each only covers a finite subset of A, so the union covers a finite subset of A which cannot be all of the infinite set A, so we have a contradiction.

Problem 30.5

(a)

Prove that every metrizable space with a countable dense subset has a countable basis.

Let d be a metric on X, and $Q \subseteq X$ a countable dense subset, and enumerate $Q = \{q_1, q_2, \ldots\}$. For every $m, n \in \mathbb{N}$, let $B_{m,n} = B_d(q_m, \frac{1}{n})$. Then $\mathcal{B} = (B_{m,n})$ is a countable collection of basis elements.

Hence, we have that the topology generated by $\mathcal{B}$ is a subset of the metric topology. Now, we show the other inclusion. Let U be open in X; we show that it is open in the topology generated by $\mathcal{B}$. Fix $x \in U$. Since $\overline{Q} = X$, we have that x is in the closure of Q. If $x \in Q$, then we know that some basis element $B(x, \epsilon)$ is contained in U, so that for $\frac{1}{n} < \epsilon$, we have $B(x, \frac{1}{n})$ is an element of $\mathcal{B}$ that is contained in U.

If $x \notin Q$, then it is a limit point of Q. Again, choose $\epsilon > 0$ so that $B(x, \epsilon)$ is contained in U. Now, choose some $k > 2$, and choose n so that $\frac{\epsilon}{k} < \frac{1}{n} < \frac{\epsilon}{2}$, increasing k if needed so that such an n exists. Note that we can always increase k large enough since $\frac{\epsilon}{k}$ converges to zero in k. Then choose a $q \in Q$ such that $q \in B(x, \frac{\epsilon}{k})$. Such a q exists since x is a limit point of Q. Then we have that $x \in B(q, \frac{1}{n})$ and $B(q, \frac{1}{n})$ is contained in $B(x, \epsilon)$ and hence in U. Thus, U is open in the topology generated by $\mathcal{B}$, and we are done.

(b)

Show that every metrizable Lindelöf space has a countable basis.

For each $n \in \mathbb{N}$, consider the open covering $(B(x, \frac{1}{n}))_{x \in X}$. By the Lindelöf property, let $\mathcal{C}_n$ be a countable subcovering of $(B(x, \frac{1}{n}))_{x \in X}$. then $\mathcal{C} = \cup_{n \in \mathbb{N}} \mathcal{C}_n$ is a countable collection of basis elements of the metric topology on X.

Now, to show that $\mathcal{C}$ is a basis for the metric topology, let $B(x, \epsilon)$ be a basis element of the metric topology, and choose $\frac{1}{n} < \frac{\epsilon}{2}$. Since $\mathcal{C}_n$ covers x, we have that $x \in B(y, \frac{1}{n}) \in \mathcal{C}$ for some $y \in X$. Then for every $z \in B(y, \frac{1}{n})$, we have that

$$d(x,z) \leq d(x,y) + d(y,z) < \frac{1}{n} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so that $z \in B(x, \epsilon)$ and hence $B(y, \frac{1}{n})$ is an element of $\mathcal{C}$ containing x that is contained in $B(x, \epsilon)$. From this, we have that $\mathcal{C}$ is indeed a countable basis of X.