Problem 3.13

Prove that if an ordered set A has the least upper bound property, then it has the greatest lower bound property

Suppose A_0 is a subset of A that is bounded below. Let B be the set of all lower bounds of A_0, so $B = \{b \in A : b \leq x \ \forall \, x \in A_0\}$.

Note that B is bounded above by any element of A_0, and B is a subset of A. Then, by the least upper bound property, B has a least upper bound $s \in A$. We show that s is the greatest lower bound of A_0.

Suppose $x \in A_0$. Then x is an upper bound of B, so that $s \leq x$. Hence, s is indeed a lower bound on A_0.

Suppose $s_0 > s$. Then s_0 is greater than any lower bound $b \in B$ of A_0, so it is not a lower bound of A_0. Hence, s is in fact the greatest lower bound of A_0.

Problem 9.8

We outline a proof. First note that the logistic function $x \mapsto \frac{1}{1+e^{-x}}$ is a bijection from $\mathbb{R} \to (0,1)$, so it suffices to show that $P(\mathbb{Z}^+)$ has the same cardinality as $[0,1]$, which has the same cardinality as $(0,1)$.

Now, for each set $A \subseteq \mathbb{Z}^+$ of positive integers, we view it as a sequence of binary digits, with the n^th element equal to 1 if $n \in A$, and 0 otherwise. Then this sequence is mapped to the real number in $[0,1]$ whose base 2 representation has as the n^th decimal place, the n^th element of the sequence. This is a bijection between $P(\mathbb{Z}^+)$ and $[0,1]$, with $\mathbb{Z}^+$ being represented by $111\cdots$ and thus mapped to $.111\cdots = 1$, and $\emptyset$ being represented by $000\cdots$ and hence mapped to $.000\cdots = 0$.

Problem 13.7

Determine which of the other topologies on $\mathbb{R}$ does each topology contain

$\mathcal{T}_1$ standard topology, $\mathcal{T}_2$ topology of $\mathbb{R}_K$, $\mathcal{T}_3$ finite complement topology, $\mathcal{T}_4$ the upper limit topology, $\mathcal{T}_5$ the topology with all sets $(-\infty, a)$ as basis.

$\underline{\mathcal{T}_1}$ contains $\mathcal{T}_3$. If $A \in \mathcal{T}_3$, then if $A^C = \mathbb{R}$, then $A = \emptyset \in \mathcal{T}_1$. If $A^C = \{x_1, \ldots x_n\}$, then reorder the indices so that $x_1 < \ldots < x_n$, and then $A = (-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_n, \infty)$, which is in the standard topology $\mathcal{T}_1$. Also, $\mathcal{T}_1$ contains $\mathcal{T}_5$, since every basis element $(-\infty, a)$ of $\mathcal{T}_5$ is an open ray so is contained in the standard topology. By lemma 13.4, the other two topologies are strictly finer (the proof for the lower limit topology generalizes for the upper limit topology) so are not contained in $\mathcal{T}_1$.

$\underline{\mathcal{T}_2}$, by lemma 13.4, contains $\mathcal{T}_1$ and hence $\mathcal{T}_3$ and $\mathcal{T}_5$, but is not comparable to $\mathcal{T}_4$

Problem 17.16

(**a**) *Determine the closure of the set $K = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$ in each of the 5 topologies above*

$\underline{\mathcal{T}_1}$ $\bar{K} = \{0\} \cup K$ by Munkres's word

$\underline{\mathcal{T}_2}$ $\bar{K} = \{0\} \cup K$ since $\mathcal{T}_2$ is finer than the standard topology, and any other x has an interval not intersecting K.

$\underline{\mathcal{T}_3}$ $\bar{K} = K$ because if $x \notin K$, then $\mathbb{R} \setminus K$ is a neighborhood of x disjoint from K.

$\underline{\mathcal{T}_4}$ $\bar{K} = \{0\} \cup K$ since $\mathcal{T}_4$ is finer than the standard topology and again any other x has an interval not intersecting K.

$\underline{\mathcal{T}_5}$ $\bar{K} = [0, \infty)$ since if $x \geq 0$, the any $(-\infty, a)$ containing x has $a > x \geq 0$, so that there is some n such that $a > \frac{1}{n} > 0$.

(**b**) *Which of the topologies satisfy the Hausdorff axiom or the* T_1 *axiom*

$\underline{\mathcal{T}_1}$ Is Hausdorff since for any $x \neq y \in \mathbb{R}$, say $x < y$ without loss of generality, then with $h = \frac{y-x}{2}$, we have $(x-h, x+h)$ and $(y-h, y+h)$ are disjoint, as $x+h = y-h$.

$\underline{\mathcal{T}_2}$ Is Hausdorff since it is finer than the standard topology.

$\underline{\mathcal{T}_3}$ Is not Hausdorff since any neighborhood of 1 has finite complement C, so that since every neighborhood of 0 is infinite, there can be no neighborhood of 0 contained in C. In other words, there is no neighborhood of 0 disjoint from any neighborhood of 1.

However, $\mathcal{T}_3$ is T_1, since for any $x \in \mathbb{R}$, we have $\mathbb{R} \setminus \{x\}$ is open, so that it's complement, $\{x\}$, is closed.

$\underline{\mathcal{T}_4}$ Is Hausdorff since it is finer than the standard topology.

$\underline{\mathcal{T}_5}$ Is not Hausdorff since every basis element containg 1 also contains 0. It is also not T_1 since $\mathbb{R} \setminus \{0\}$ is not an open set, since any open set containing 1 also contains 0, so this would lead to a contradiction of $\mathbb{R} \setminus \{0\}$ were open.

Problem 17.18

Determine the closures of the following subsets of the ordered square:

$A = \{(1/n) \times 0 : n \in \mathbb{Z}_+\}$. $\bar{A} = A \cup \{0 \times 1\}$.

$B = \{(1 - 1/n) \times \frac{1}{2} : n \in \mathbb{Z}_+\}$. $\bar{B} = B \cup \{1 \times 0\}$

$C = \{x \times 0 : 0 < x < 1\}$. $\bar{C} = C \cup [0,1) \times \{1\} \cup \{1 \times 0\}$

$D = \{x \times \frac{1}{2} : 0 < x < 1\}$. $\bar{D} = D \cup (0,1] \times \{0\} \cup [0,1) \times \{1\}$

$E = \{\frac{1}{2} \times y : 0 < y < 1\}$. $\bar{E} = E \cup \{\frac{1}{2} \times 0\} \cup \{\frac{1}{2} \times 1\}$