概率论作业七

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习题二第12题

1. 由概率密度的性质得$\int_0^1 Ax^3dx = \frac{A}{4}X^4|_0^1 = \frac{A}{4} = 1$,
   解得$A = 4$.
2. $F(x) = \begin{cases} 0 & ,x \le 0 \\ x^4 & ,0 \le x \le 1 \\ 1 & ,x \ge 1 \end{cases}$
3. $P(X < B) = F(B), P(X > B) = 1 - F(B),$
   $\therefore P(X < B) = P(X > B) \Rightarrow F(B) = \frac{1}{2} \Rightarrow B = \sqrt[4]{\frac{1}{2}}$

习题二第17题

1. X~$N(220, 25^2)$,则$Z = \frac{X-220}{25}$~$N(0，1)$,
   设事件A_1事件为电压不超过200V，A_2事件为电压200~240V，
   A_3事件为电压超过240V，B事件为电子元件损坏，则有
   $P(B) = P(A_1) * 0.1 + P(A_2) * 0.001 + P(A_3) * 0.2$
   $= (1 - 0.7881) * 0.1 + (0.7881 * 2 - 1) * 0.001 + (1 - 0.7881) * 0.2$
   $= 6.41462\%$
2. $P(A_2|B) = \frac{P(A_2)P(B|A_2)}{P(B)} = 1.17887\%$

习题二第21题

$P_Y(y) = P_X(g^{-1}(y)) * |[g^{-1}(y)]'| = \frac{1}{8\sqrt{\pi}}e^{-\frac{1}{64}(x-8)^2}, -\infty < x < +\infty$

习题二第24题

$x = (1-y)^3$
$P_Y(y) = P_X(g^{-1}(y)) * |[g^{-1}(y)]'| = \frac{1}{\pi(1+(1-y)^6)}|-3(1-y)^2|$
$\quad = \frac{3(1-y)^2}{\pi(1+(1-y)^6)}, -\infty < x < +\infty$

习题三第4题

1. $P(X > 2Y) = \int_0^1 dx \int_0^{\frac{1}{2}x}(2-x-y)dy = \frac{1}{2}x^2 - \frac{5}{24}x^3|_0^1 = \frac{7}{24}$
2. 当x小于零或y小于零时，易得等于0；当x,y大于零时，易得z=1为分界条件，
   当z小于1的时候，$F_Z(z) = \int_0^z dx \int_0^{z-x}(2-x-y)dy = z^2 - \frac{z^3}{3}$,
   当z大于1小于2的时候，$F_Z(z) = 1 - \int_{z-1}^1 dx \int_{z-x}^1 (2-x-y)dy$
   $\because P_Z(z) = F_Z(z)', \therefore P_Z(z) = \begin{cases} 0 & ,others \\ 2z - z^2 & ,0 \le z \le 1 \\ (z-2)^2 & ,z \le 2 \end{cases}$

习题三第5题

1. $p_x(x)=\int_{-\infty}^{+\infty}p(x,y)dy=\int_0^{2x}dy=2x,0<x<1$
   $p_y(y)=\int_{-\infty}^{+\infty}p(x,y)dx=\int_{\frac{y}{2}}^{1}dx=1-\frac{y}{2},0<y<2x$
2. $F_Z(z)=1-\int_{\frac{z}{2}}^{1}dx\int_{)}^{2x-z}dy=z-\frac{z^2}{4}$
   $P_Z(z)=F_Z(z)'=1-\frac{z}{2},0<z<2$
3. $P(Y\le\frac{1}{2}|X\le\frac{1}{2})=\frac{P(Y\le\frac{1}{2},X\le\frac{1}{2})}{P(Y\le\frac{1}{2},X\le 1)}=1-\frac{\int_{\frac{1}{4}}^{\frac{1}{2}}dx\int_{\frac{1}{2}}^{2x}dy}{\int_0^{\frac{1}{2}}dx\int_0^{2x}dy}=\frac{3}{4}$

习题三第12题

1. $1=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p(x,y)dxdy=\frac{4}{3}c\int_0^{+\infty}x^3e^{-x}=8c\Rightarrow c=\frac{1}{8}$
2. $p_x(x)=\int_{-\infty}^{+\infty}p(x,y)dy=\int_{-x}^{x}\frac{1}{8}(x^2-y^2)e^{-x}dy=\frac{x^3}{6}e^{-x},0<x<+\infty$
   $p_y(y)=\int_{-\infty}^{+\infty}p(x,y)dx=\int_{|y|}^{+\infty}\frac{1}{8}(x^2-y^2)e^{-x}dx=\frac{1}{4}(|y|+1)e^{-|y|},-\infty<y<+\infty$
   二者之间并不独立
3. $P_{X|Y=y}(x)=\frac{p(x,y)}{p_y(y)}=\frac{c(x^2-y^2)e^{-x}}{\frac{1}{4}(|y|+1)e^{-|y|}}=\frac{x^2-y^2}{2(|y|+1)}e^{x(x-|y|)},|y|<x<+\infty$
   $P_{Y|X=x}(y)=\frac{p(x,y)}{p_x(x)}=\frac{c(x^2-y^2)e^{-x}}{\frac{x^3}{6}e^{-x}}=\frac{3(x^2-y^2)}{4x^3},|y|<x$

习题三第13题

1. $P(x,y)=P(x)P(y)=\begin{cases}0 & ,others\\ e^{-y} & ,0<x<1\end{cases}$
   $F(z)=P(X+Y\le z)$
   使用卷积公式$p(z)=\int_0^1p_X(x)p_Y(z-x)dx=\int_0^1p_Y(z-x)dx,$
   当$z\ge 1$时，$p(z)=\int_0^1e^{x-z}dx=e^{1-z}-e^{-z},$
   当$0<z<1$时，$p(z)=\int_0^ze^{x-z}dx=1-e^{-z},$

   故$p_z(z)=\begin{cases}1-e^{-z} & ,0<z<1\\ e^{1-z}-e^{-z} & ,z\ge 1\end{cases}$

2. $Z|_{X=x}=F(z|X=x)=P(X+Y\le z|X=x)=P(y\le z-x|X=x)$
   $=F_Y(z-x)=\begin{cases}e^{x-z} & ,0<x<1\\ 0 & ,others\end{cases}$
   $P_{Z|X=x}(z)=(e^{x-z})'=e^{x-z},z>x$

第二题

1. $\because 0<X<+\infty,\therefore 0<Y<1$,Y在[0,1]上严格递增，
   $X=-\frac{1}{\lambda}\ln(1-Y)$
   $P_Y(y)=P_X(-\frac{1}{\lambda}\ln(1-Y))|-\frac{1}{\lambda(1-Y)}|=1$
   因此Y在[0,1]服从均匀分布，Y~U(0,1)

2. 算法如下

```
for(double y from 0 to 1)
    return -ln(1-y)/λ;
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第三题

设n个点分别为x_1到x_n，在x_1处剪开并作为原点，则x_i
在[0,1]上服从均与分布，令事件X_i为当且仅当$x_i x_{i+1} \geq \frac{1}{n}$时等于1，
则$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} P(X_i = 1) = nP(X_i = 1)$,
而$P(X_1 = 1) = \Pi_{i=2}^{n}(1 - F(\frac{1}{n})) = (1 - F(\frac{1}{n}))^{n-1} = (\frac{n-1}{n})^{n-1}$
代入得$E(X) = n(\frac{n-1}{n})^{n-1}$

第四题

1. 令$Z = X_1 + X_2$,应用卷积公式得
   $P_Z(z) = \int_0^{+\infty} P_x(x)P_y(z-x)dx = \int_0^{+\infty} e^{-x} P_y(z-x)dx$
   当$x \leq 0$时，$P_Z(z) = 0$
   当$x > 0$时，$P_Z(z) = \int_0^{z} e^{-z} dx = ze^{-z}$
2. 记$Y_N = X_1 + X_2 + \ldots + X_N$, 对于求解$f_{Y|N}$, 可以使用$N-1$次卷积公式, 当然也可由数学归纳法得

$$
\begin{aligned}
&f_{Y|N=1} = \lambda e^{-\lambda y} \\
&f_{Y|N=2} = \lambda^2 y e^{-\lambda y} \\
&\qquad \cdots \\
&f_{Y|N=n} = \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}
\end{aligned}
$$

由于

$$
\begin{aligned}
f_Y(y) &= P(Y = y) \\
&= \sum_n P(Y = y, N = n) \\
&= \sum_n P(Y = y | N = n) P(N = n) \\
&= \sum_n f_{Y|N=n}(y) P(N = n) \\
&= \sum^{\infty} f_{Y|N=n}(y) P(N = n) \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1} p \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y} \\
&= p\lambda e^{\lambda(1-p)y} e^{-\lambda y} \\
&= p\lambda e^{-p\lambda y}
\end{aligned}
$$

So $Y \sim E(p\lambda)$.