# 概率论 作业十

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#### 习题七第3题

矩估计:

$$\begin{split} E[X] &= \int_0^\infty x (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} exp(-\frac{1}{2\sigma^2} (\ln x - \mu)^2) \\ &\bar{X} = exp(\mu + \frac{\sigma^2}{2}) \\ E[X^2] &= \int_0^\infty x^2 (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} exp(-\frac{1}{2\sigma^2} (\ln x - \mu)^2) \\ &\frac{1}{n} \sum_{i=1}^n X_i^2 = exp(2\mu + 2\sigma^2) \\ &\sigma^2 = \ln \frac{E[X^2]}{E[X]^2} \\ &\mu = \ln \frac{E[X]^2}{\sqrt{E[X^2]}} \\ &\hat{\sigma}^2 = \ln \left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) - 2\ln \bar{X} \\ &\hat{\mu} = 2\ln \bar{X} - \frac{1}{2} \ln \left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) \end{split}$$

极大似然估计:

$$L(\mu, \sigma) = \prod_{i=1}^{n} p(X_i; \mu, \sigma)$$

$$lnL = \sum_{i=1}^{n} \left(-\frac{1}{2}ln2\pi - ln\sigma - lnX_i - \frac{1}{2\sigma^2}(ln(X) - \mu)^2\right)$$

$$\frac{\partial lnL}{\partial \mu} = \sum_{i=1}^{n} \frac{\mu - lnX_i}{\sigma^2} = 0$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} lnX_i$$

$$\frac{\partial lnL}{\partial \sigma} = \sum_{i=1}^{n} -\frac{1}{\sigma} + \frac{1}{\sigma^3}(lnX - \mu)^2$$

$$\hat{\sigma}^2 = \frac{\sum (lnX_i - \mu)^2}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (lnX_i - \frac{1}{n} \sum_{i=1}^{n} lnX_i)^2$$

#### 习题七第4题

矩估计:

$$egin{aligned} E[X] &= \int_{\mu}^{\infty} x e^{-(x-\mu)/ heta} dx = heta + \mu \ &E[X^2] = 2 heta^2 + \mu^2 + 2 heta \mu \ &\hat{ heta} = \sqrt{(rac{1}{n}\sum_{i=1}^n X_i^2) - (rac{1}{n}\sum_{i=1}^n X_i)^2} = S^* \ &\hat{\mu} = ar{X} - S^* \end{aligned}$$

极大似然估计:

$$L(\mu, \theta) = \prod p(X_i; \mu, \theta)$$
 $lnL(\mu, \theta) = \sum (-ln\theta - (X_i - \mu)/\theta)$ 
 $= -nln\theta + n\mu/\theta - \sum X_i/\theta$ 
 $\frac{\partial lnL}{\partial \mu} = \frac{n}{\theta}$ 
 $\frac{\partial lnL}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$ 

依照定义,  $\exists p(x; \mu, \theta)$ 随 $\mu$ 单调递增,

$$\hat{\mu} = \min_i X_i$$

此时

$$\hat{ heta} = rac{1}{n} \sum_{i=1}^n (X_i - \hat{\mu}) = ar{X} - \min_i X_i$$

#### 习题七第11题

$$egin{aligned} E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] &= \sigma^2 \ \sigma^2 &= E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] \ &= c\sum_{i=1}^{n-1}(E[X_{i+1}-X_i]^2+D[X_{i+1}-X_i]) \ &= c\sum_{i=1}^{n-1}(D[X_{i+1}-X_i]) \ &= c\sum_{i=1}^{n-1}(D[X_{i+1}]+D[X_i]-2cov(X_{i+1},X_i)) \ &= c\sum_{i=1}^{n-1}(2\sigma^2) \ &= 2c(n-1)\sigma^2 \end{aligned}$$

$$c = \frac{1}{2(n-1)}$$

#### 习题七第12题

$$p(x; \sigma^{2}) = \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{(x-\mu)^{2}}{2\sigma^{2}}]$$

$$L(\sigma^{2}) = \prod p(X_{i}; \sigma^{2})$$

$$lnL = \sum_{i=1}^{n} (-ln(\sqrt{2\pi}) - ln\sigma - \frac{(x-\mu)^{2}}{2\sigma^{2}})$$

$$\frac{\partial lnL}{\partial \sigma} = 0$$

$$\hat{\sigma^{2}} = \frac{1}{n} \sum_{i=1}^{n} (X_{i} - \mu)^{2}$$

$$S^{2} = \frac{1}{n-1} \sum (X_{i} - \bar{X})^{2}$$

$$D[\hat{\sigma^{2}}] = \frac{1}{n^{2}} \sum D[(X_{i} - \mu)^{2}]$$

$$= \frac{1}{n^{2}} \sum [E[(X_{i} - \mu)^{4}] - E[(X_{i} - \mu)^{2}]^{2}]$$

$$= \frac{1}{n^{2}} \sum [3\sigma^{4} - \sigma^{4}]$$

$$= \frac{2}{n}\sigma^{4}$$

$$\frac{(n-1)S^{2}}{\sigma^{2}} \sim \chi^{2}(n-1)$$

$$D[S^{2}] = \frac{\sigma^{4}}{(n-1)^{2}} D[\chi^{2}(n-1)]$$

$$= \frac{2}{n-1}\sigma^{4}$$

$$D[\hat{\sigma^{2}}] < D[S^{2}]$$

## 习题七第13题

$$egin{split} rac{ar{X}-\mu}{\sigma/\sqrt{n}} &\sim N(0,1) \ P(-u_{rac{lpha}{2}} < rac{ar{X}-\mu}{\sigma/\sqrt{n}} < u_{rac{lpha}{2}}) = 1-lpha \ L_0 &= 2rac{\sigma}{\sqrt{n}}u_{rac{lpha}{2}} \leq L \ n \geq (rac{2\sigma u_{rac{lpha}{2}}}{L})^2 \end{split}$$

### 习题七 第14题

$$egin{split} rac{ar{X}-\mu}{S/\sqrt{n-1}} &\sim t(n-1) \ P(ar{X}-rac{S}{\sqrt{n-1}}t_{rac{lpha}{2}}(n-1) < \mu < ar{X} + rac{S}{\sqrt{n-1}}t_{rac{lpha}{2}}(n-1)) = 1-lpha \ ar{X} = 6, n = 9, S = 0.5745, lpha = 0.05, t_{rac{lpha}{2}}(n-1) = t_{0.025}(8) = 2.3060 \end{split}$$

置信区间:[5.5584, 6.4416]

# 习题七第18题

$$X \sim P(\lambda)$$
 $E[X] = D[X] = \lambda$ 
 $E[\bar{X}] = E[\frac{1}{n}\sum X] = \lambda$ 
 $D[\bar{X}] = D[\frac{1}{n}\sum X] = \lambda/n$ 
 $\bar{X} \sim N(\lambda, \lambda/n)$ 
 $u = \frac{\bar{X} - \mu}{\sqrt{D[\bar{X}]}} = \frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} \sim N(0, 1)$ 
 $\lambda = E[\frac{1}{n}\sum X_i] = \bar{X}, n \to \infty$ 
 $P(-u_{\alpha/2} < \frac{\bar{X} - \lambda}{\sqrt{\bar{X}/n}} < u_{\alpha/2}) = 1 - \alpha$ 
 $(\bar{X} - u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}}, \bar{X} + u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}})$