概率论 作业十

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习题七 第3题

矩估计:

$$E[X] = \int_0^\infty x(2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}exp(-\frac{1}{2\sigma^2}(lnx-\mu)^2)$$
$$\bar{X} = exp(\mu + \frac{\sigma^2}{2})$$
$$E[X^2] = \int_0^\infty x^2(2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}exp(-\frac{1}{2\sigma^2}(lnx-\mu)^2)$$
$$\frac{1}{n}\sum_{i=1}^n X_i^2 = exp(2\mu + 2\sigma^2)$$
$$\sigma^2 = ln\frac{E[X^2]}{E[X]^2}$$
$$\mu = ln\frac{E[X]^2}{\sqrt{E[X^2]}}$$
$$\hat{\sigma}^2 = \ln(\frac{1}{n}\sum_{i=1}^n X_i^2) - 2\ln\bar{X}$$
$$\hat{\mu} = 2\ln\bar{X} - \frac{1}{2}\ln(\frac{1}{n}\sum_{i=1}^n X_i^2)$$

极大似然估计:

$$L(\mu,\sigma) = \prod_{i=1}^n p(X_i;\mu,\sigma)$$
$$lnL = \sum_{i=1}^n(-\frac{1}{2}ln2\pi - ln\sigma - lnX_i - \frac{1}{2\sigma^2}(ln(X)-\mu)^2)$$
$$\frac{\partial lnL}{\partial\mu} = \sum_{i=1}^n\frac{\mu - lnX_i}{\sigma^2} = 0$$
$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n lnX_i$$
$$\frac{\partial lnL}{\partial\sigma} = \sum_{i=1}^n -\frac{1}{\sigma} + \frac{1}{\sigma^3}(lnX-\mu)^2$$
$$\hat{\sigma}^2 = \frac{\sum(lnX_i-\mu)^2}{n}$$
$$= \frac{1}{n}\sum_{i=1}^n(lnX_i - \frac{1}{n}\sum_{i=1}^n lnX_i)^2$$

习题七 第4题

矩估计:

$$E[X] = \int_{\mu}^{\infty} xe^{-(x-\mu)/\theta}dx = \theta + \mu$$
$$E[X^2] = 2\theta^2 + \mu^2 + 2\theta\mu$$
$$\hat{\theta} = \sqrt{(\frac{1}{n}\sum_{i=1}^{n} X_i^2) - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2} = S^*$$
$$\hat{\mu} = \bar{X} - S^*$$

极大似然估计:

$$L(\mu, \theta) = \prod p(X_i; \mu, \theta)$$
$$lnL(\mu, \theta) = \sum(-ln\theta - (X_i - \mu)/\theta)$$
$$= -nln\theta + n\mu/\theta - \sum X_i/\theta$$
$$\frac{\partial lnL}{\partial \mu} = \frac{n}{\theta}$$
$$\frac{\partial lnL}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$$

依照定义, 且$p(x; \mu, \theta)$随μ单调递增,

$$\hat{\mu} = \min_i X_i$$

此时

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}) = \bar{X} - \min_i X_i$$

习题七 第11题

$$E[c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2] = \sigma^2$$
$$\sigma^2 = E[c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2]$$
$$= c\sum_{i=1}^{n-1}(E[X_{i+1} - X_i]^2 + D[X_{i+1} - X_i])$$
$$= c\sum_{i=1}^{n-1}(D[X_{i+1} - X_i])$$
$$= c\sum_{i=1}^{n-1}(D[X_{i+1}] + D[X_i] - 2cov(X_{i+1}, X_i))$$
$$= c\sum_{i=1}^{n-1}(2\sigma^2)$$
$$= 2c(n-1)\sigma^2$$

$$c=\frac{1}{2(n-1)}$$

习题七 第12题

$$p(x;\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}exp[-\frac{(x-\mu)^2}{2\sigma^2}]$$
$$L(\sigma^2)=\prod p(X_i;\sigma^2)$$
$$lnL=\sum_{i=1}^{n}(-ln(\sqrt{2\pi})-ln\sigma-\frac{(x-\mu)^2}{2\sigma^2})$$
$$\frac{\partial lnL}{\partial\sigma}=0$$
$$\hat{\sigma^2}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$
$$S^2=\frac{1}{n-1}\sum(X_i-\bar{X})^2$$
$$D[\hat{\sigma^2}]=\frac{1}{n^2}\sum D[(X_i-\mu)^2]$$
$$=\frac{1}{n^2}\sum[E[(X_i-\mu)^4]-E[(X_i-\mu)^2]^2]$$
$$=\frac{1}{n^2}\sum[3\sigma^4-\sigma^4]$$
$$=\frac{2}{n}\sigma^4$$
$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$$
$$D[S^2]=\frac{\sigma^4}{(n-1)^2}D[\chi^2(n-1)]$$
$$=\frac{2}{n-1}\sigma^4$$
$$D[\hat{\sigma^2}]<D[S^2]$$

习题七 第13题

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$
$$P(-u_{\frac{\alpha}{2}}<\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}<u_{\frac{\alpha}{2}})=1-\alpha$$
$$L_0=2\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}}\leq L$$
$$n\geq(\frac{2\sigma u_{\frac{\alpha}{2}}}{L})^2$$

习题七 第14题

$$\frac{\bar{X}-\mu}{S/\sqrt{n-1}}\sim t(n-1)$$
$$P(\bar{X}-\frac{S}{\sqrt{n-1}}t_{\frac{\alpha}{2}}(n-1)<\mu<\bar{X}+\frac{S}{\sqrt{n-1}}t_{\frac{\alpha}{2}}(n-1))=1-\alpha$$
$$\bar{X}=6,n=9,S=0.5745,\alpha=0.05,t_{\frac{\alpha}{2}}(n-1)=t_{0.025}(8)=2.3060$$

置信区间:$[5.5584,6.4416]$

习题七 第18题

$$X \sim P(\lambda)$$
$$E[X] = D[X] = \lambda$$
$$E[\bar{X}] = E[\frac{1}{n}\sum X] = \lambda$$
$$D[\bar{X}] = D[\frac{1}{n}\sum X] = \lambda/n$$
$$\bar{X} \sim N(\lambda, \lambda/n)$$
$$u = \frac{\bar{X} - \mu}{\sqrt{D[\bar{X}]}} = \frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} \sim N(0,1)$$
$$\lambda = E[\frac{1}{n}\sum X_i] = \bar{X}, n \to \infty$$
$$P(-u_{\alpha/2} < \frac{\bar{X} - \lambda}{\sqrt{\bar{X}/n}} < u_{\alpha/2}) = 1 - \alpha$$
$$(\bar{X} - u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}}, \bar{X} + u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}})$$