概率论 作业十

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习题七 第3题

矩估计:

$$
\begin{aligned}
E[X] &= \int_0^\infty x(2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}exp(-\frac{1}{2\sigma^2}(lnx-\mu)^2) \\
&= exp(\mu+\frac{\sigma^2}{2}) \\
E[X^2] &= \int_0^\infty x^2(2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}exp(-\frac{1}{2\sigma^2}(lnx-\mu)^2) \\
&= exp(2\mu+2\sigma^2) \\
\sigma^2 &= ln\frac{E[X^2]}{E[X]^2} \\
\mu &= ln\frac{E[X]^2}{\sqrt{E[X^2]}} \\
\hat{\sigma}^2 &= ln\frac{\sum X_i^2}{(\sum X_i)^2} \\
\hat{\mu} &= ln\frac{(\sum X_i)^2}{\sqrt{\sum X_i^2}}
\end{aligned}
$$

极大似然估计:

$$
\begin{aligned}
L(\mu,\sigma) &= \prod_{i=1}^n p(X_i;\mu,\sigma) \\
lnL &= \sum_{i=1}^n(-\frac{1}{2}ln2\pi - ln\sigma - lnX_i - \frac{1}{2\sigma^2}(ln(X)-\mu)^2) \\
\frac{\partial lnL}{\partial\mu} &= \sum_{i=1}^n\frac{\mu - lnX_i}{\sigma^2} = 0 \\
\hat{\mu} &= \frac{\sum lnX_i}{n} \\
\frac{\partial lnL}{\partial\sigma} &= \sum_{i=1}^n -\frac{1}{\sigma}+\frac{1}{\sigma^3}(lnX-\mu)^2 \\
\hat{\sigma}^2 &= \frac{\sum(lnX_i-\mu)^2}{n} \\
&= \frac{1}{n}\sum(lnX_i - \frac{1}{n}\sum lnX_i)^2
\end{aligned}
$$

习题七 第4题

矩估计:

$$
\begin{aligned}
E[X] &= \int_\mu^\infty xe^{-(x-\mu)/\theta}dx = \theta+\mu \\
E[X^2] &= 2\theta^2+\mu^2+2\theta\mu \\
\hat{\theta} &= \sqrt{(\sum X_i^2)-(\sum X_i)^2}
\end{aligned}
$$

$$\hat{\mu} = (\sum X_i) - \sqrt{(\sum X_i^2) - (\sum X_i)^2}$$

极大似然估计:

$$L(\mu, \theta) = \prod p(X_i; \mu, \theta)$$
$$lnL(\mu, \theta) = \sum(-ln\theta - (X_i - \mu)/\theta)$$
$$= -nln\theta + n\mu/\theta - \sum X_i/\theta$$
$$\frac{\partial lnL}{\partial \mu} = \frac{n}{\theta}$$
$$\frac{\partial lnL}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$$
$$\hat{\theta} = \frac{1}{n}\sum X_i$$

依照定义, 且$p(x; \mu, \theta)$随μ单调递增,

$$\hat{\mu} = \min_i X_i$$

习题七 第11题

$$E[c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2] = \sigma^2$$
$$\sigma^2 = E[c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2]$$
$$= c\sum_{i=1}^{n-1}(E[X_{i+1} - X_i]^2 + D[X_{i+1} - X_i])$$
$$= c\sum_{i=1}^{n-1}(D[X_{i+1} - X_i])$$
$$= c\sum_{i=1}^{n-1}(D[X_{i+1}] + D[X_i] - 2cov(X_{i+1}, X_i))$$
$$= c\sum_{i=1}^{n-1}(2\sigma^2)$$
$$= 2c(n-1)\sigma^2$$
$$c = \frac{1}{2(n-1)}$$

习题七 第12题

$$p(x; \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}exp[-\frac{(x-\mu)^2}{2\sigma^2}]$$
$$L(\sigma^2) = \prod p(X_i; \sigma^2)$$
$$lnL = \sum(-ln(\sqrt{2\pi}) - ln\sigma - \frac{(x-\mu)^2}{2\sigma^2})$$
$$\frac{\partial lnL}{\partial \sigma} = 0$$
$$\hat{\sigma} = \frac{1}{n}\sum(X_i - \mu)^2$$
$$S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$
$$D[\hat{\sigma}] = \frac{1}{n^2}\sum D[(X_i - \mu)^2]$$

$$
\begin{gathered}
=\frac{1}{n^2}\sum[E[(X_i-\mu)^4]-E[(X_i-\mu)^2]^2] \\
=\frac{1}{n^2}\sum[3\sigma^4-\sigma^4] \\
=\frac{2}{n}\sigma^4 \\
\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1) \\
D[S^2]=\frac{\sigma^4}{(n-1)^2}D[\chi^2(n-1)] \\
=\frac{2}{n-1}\sigma^4 \\
D[\hat{\sigma}]<D[S^2]
\end{gathered}
$$

习题七 第13题

$$
\begin{gathered}
\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1) \\
P(-u_{\frac{\alpha}{2}}<\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}<u_{\frac{\alpha}{2}})=1-\alpha \\
L_0=2\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}}\le L \\
n\ge(\frac{2\sigma u_{\frac{\alpha}{2}}}{L})^2
\end{gathered}
$$

习题七 第14题

$$
\begin{gathered}
\frac{\bar{X}-\mu}{S/\sqrt{n-1}}\sim t(n-1) \\
P(\bar{X}-\frac{S}{\sqrt{n-1}}t_{\frac{\alpha}{2}}(n-1)<\mu<\bar{X}+\frac{S}{\sqrt{n-1}}t_{\frac{\alpha}{2}}(n-1))=1-\alpha \\
\bar{X}=6,n=9,S=0.542,\alpha=0.05,t_{\frac{\alpha}{2}}(n-1)=t_{0.025}(8)=2.3060
\end{gathered}
$$

置信区间:$[5.583,6.417]$

习题七 第18题

$$
\begin{gathered}
X\sim P(\lambda) \\
E[X]=D[X]=\lambda \\
E[\bar{X}]=E[\frac{1}{n}\sum X]=\lambda \\
D[\bar{X}]=D[\frac{1}{n}\sum X]=\lambda/n \\
\bar{X}\sim N(\lambda,\lambda/n) \\
u=\frac{\bar{X}-\mu}{\sqrt{D[\bar{X}]}}=\frac{\bar{X}-\lambda}{\sqrt{\lambda/n}}\sim N(0,1) \\
\lambda=E[\frac{1}{n}\sum X_i]=\bar{X},n\to\infty \\
P(-u_{\alpha/2}<\frac{\bar{X}-\lambda}{\sqrt{\bar{X}/n}}<u_{\alpha/2})=1-\alpha \\
(\bar{X}-u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}},\bar{X}+u_{\alpha/2}\sqrt{\frac{\bar{X}}{n}})
\end{gathered}
$$