Outage Probabilities of Wireless Systems With Imperfect Beamforming

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***Abstract*—This paper investigates the outage probability of a wireless system with conventional beamforming using a uniform linear array beamformer. The focus is to examine the impact of beamforming impairments, such as direction-of-arrival (DOA) estimation errors, signal spatial spreads, antenna array perturbation, and mutual coupling. Fading statistics of Rayleigh, Rician, and Nakagami are used to characterize the desired signal, whereas interferers are assumed to be subject to Rayleigh fading. A simplified beamforming model is used in deriving closed-form outage probability expressions. Numerical results illustrate the increase of outage probabilities in the presence of DOA estimation errors, signal spatial spreads, and antenna perturbation. However, negligible outage performance impact is observed due to mutual coupling.**

***Index Terms*—Array perturbation, beamforming, direction-of-arrival (DOA) estimation errors, mutual coupling, Nakagami fading, outage probability, Rayleigh fading, Rician fading, signal spatial spreads.**

I. Introduction

ANTENNA array techniques are based on utilizing multiple antenna elements to achieve performance and capacity enhancement without the need for additional power or spectrum [1]. There are two categories of antenna array techniques: diversity and beamforming. Diversity and beamforming differ in the spacing requirement among antenna elements. Diversity techniques use a number of antennas that are separated far apart from each other [2]. This approach was first developed for the receiver side, and lately, it has been used at the transmitter side. Transmit antenna diversity and receive antenna diversity can also be implemented at the same time. Space-time coding, which is a popular technique nowadays, is developed based on transmit diversity [3]. The second category of antenna array techniques is beamforming. This technique uses several antenna elements that are placed very close to each other to form an antenna beam [4], [5]. The beam can be steered to focus most signal energy toward a desired direction. At the same time, it reduces interference in other directions. There has been extensive research in beamforming for future wireless systems [6]–[8]. In this paper, beamforming techniques will be investigated.

Beamforming can be implemented through conventional beamforming, minimum-variance distortionless response (MVDR) beamforming, or linear constrained minimum-variance (LCMV) beamforming [4]. The MVDR beamforming is also known as “optimum beamforming.” The LCMV beamforming is developed from MVDR beamforming with additional linear constraints to improve its robustness [4]. In implementing conventional beamforming, the antenna mainlobe is steered toward a desired signal. In MVDR beamforming, an antenna pattern is formed to maximize the output signal-to-interference-plus-noise ratio (SINR) while maintaining a constant gain in the direction of the desired signal [9]. The MVDR beamforming is sensitive to direction-of-arrival (DOA) estimation errors, and its performance decreases significantly when an interferer is inside the mainlobe [4]. The LCMV beamforming can be implemented by placing nulls in the directions of the interferers when multiple interferers are considered [4], [10]–[12]. One limitation of the LCMV beamforming is that the number of antenna elements has to exceed the number of nulls by one [5]. In this paper, we focus on the conventional beamforming in investigating the outage probability performance of wireless systems.

The outage probability is an important performance measure for wireless systems. Performance in terms of the outage probability has been investigated for systems using antenna diversity techniques, and in [13] and [14], several diversity-combining schemes and various fading scenarios are considered. A generalized moment-generation-function (MGF) method was proposed to calculate the outage probability for the diversity systems in [15]. The outage probability of wireless systems with beamforming has also been investigated [10]–[12]. In [10], the interferers are assumed to be Rayleigh faded, and the fading statistics of the desired signal follows a Rayleigh, Rician, or Nakagami distribution. In [11], the desired signal is subject to Rician fading, and the interferers are subject to Nakagami fading. In a recent paper [12], closed-form outage probability expressions are derived for scenarios where both the desired signal and the interferers are subject to Rayleigh, Rician, or Nakagami fading.

All these beamforming related studies [10]–[12] assume that an LCMV beamformer is used. The LCMV beamformer can perfectly cancel the N_I strongest interferers and while all other interferers remain, where N_I is determined by the number of antenna elements. This paper differs from previous research in

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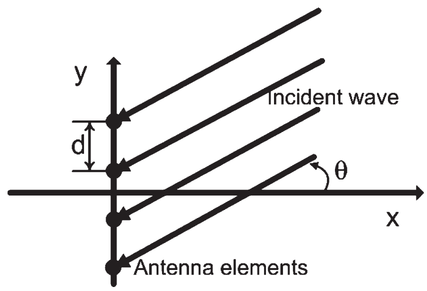


Fig. 1. ULA.

two aspects: First, the beam pattern of a conventional beamformer is used in the derivation of the outage probability expressions. Second, rather than assuming perfect beamforming, impairments in beamforming, such as DOA estimation errors, spatial spreads, array perturbation, and mutual coupling, are considered in outage performance evaluations. In the succeeding sections, closed-form outage probabilities will be derived for scenarios with a Rayleigh, Rician, or Nakagami signal and Rayleigh interferers.

The remainder of this paper is organized as follows: The system model is given in Section II. A simplified beamforming model is also introduced in this section. Outage probability expressions are derived in Section III, considering ideal beamforming. The impact of beamforming impairments is investigated in Section IV. Numerical results are given in Section V, and finally, conclusions are drawn in Section VI.

II. System Model

A. Beamforming

While few antenna elements could be installed at a mobile station, large antenna arrays can be implemented at a base station. At a base station, receive beamforming for each desired signal could be implemented independently without affecting the performance of the other links [16]–[18]. A uniform linear array (ULA) in a two-dimensional environment is considered and shown in Fig. 1. The distance d between the antenna elements is assumed to be 0.5λ, where λ is the carrier wavelength, and θ is an arrival angle of incident waves. In the ULA system, a combining network connects an array of low-gain antenna elements and could generate an ideal antenna pattern [17], [19] of

$$G(\psi,\theta)=\left|\frac{\sin(0.5M\pi\,(\sin\theta-\sin\psi))}{M\sin\left(0.5\pi(\sin\theta-\sin\psi)\right)}\right|^2 \tag{1}$$

where M is the number of antenna elements, and ψ is a scan angle. The beam could be steered to a desired direction by varying ψ. In the remainder of this paper, we will use the antenna pattern specified in (1) to evaluate the outage probability of systems with receive beamforming in reverse-link transmissions.

B. Introduction of Simplified and Accurate Beamforming Model

The complexity considering the exact beam pattern can be high, especially for performance evaluation under beamforming impairments such as DOA estimation errors, due to multiple integrals. A simple Bernoulli model, in which a signal is considered to be within a mainlobe $(G=1)$ or out of the mainlobe $(G=0)$, and the half-power beamwidth is defined as the beamwidth, is introduced in [20]. This model is easy to use, but it neglects the impact of sidelobes and the effect of any specific beam patterns. Spagnolini [21] provides a beamforming model with a triangular pattern to characterize the mainlobe of a beam. In [22], an accurate, yet simple, beamforming model is developed to account the impact of sidelobes and the real beam patterns. In examining interference, this model simplifies the beam pattern to a flat mainlobe and a flat sidelobe. There are two parameters associated with the model: mainlobe width B (normalized by 2π) and sidelobe gain α. The two parameters are determined based on a given beam pattern as

$$\alpha=\frac{\mathrm{E}\left[G^2(\psi,\theta)\right]-\mathrm{E}\left[G(\psi,\theta)\right]}{\mathrm{E}\left[G(\psi,\theta)\right]-1} \tag{2}$$

and

$$B=\frac{\mathrm{E}\left[G^2(\psi,\theta)\right]-\mathrm{E}^2\left[G(\psi,\theta)\right])}{\mathrm{E}\left[G^2(\psi,\theta)\right]+1-2\mathrm{E}\left[G(\psi,\theta)\right]} \tag{3}$$

where $\mathrm{E}[G(\psi,\theta)]$ and $\mathrm{E}[G^2(\psi,\theta)]$ are the antenna gain and square of the antenna gain, respectively, averaged with respect to the uniformly distributed random variables ψ and θ in the region from 0 to 2π. In this simplified beamforming model, there are two beam patterns: One is an exact beam pattern that is used to evaluate the desired signal [Fig. 2(a)], and the other is a simplified beam pattern for the interferers [Fig. 2(b)]. When the beamforming impairments are considered, α and B will be adjusted to reflect the impact of the impairments. We will also show in Sections III-D and V that outage probability evaluations using actual beam patterns agree very well with those based on the simplified beamforming model.

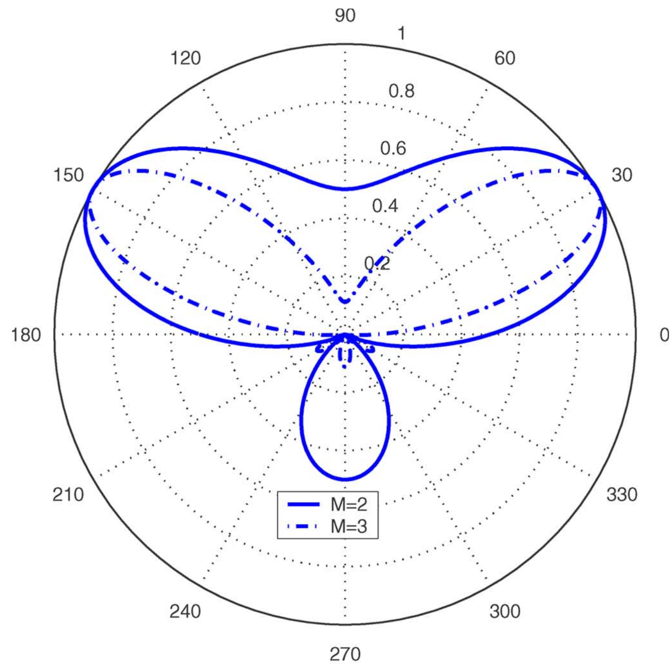
C. Outage Probability

In interference-limited wireless systems, adequate signal-to-interference ratio (SIR) is essential for successful communications [23], [24]. Therefore, the outage probability, which is defined as the probability of failing to achieve a SIR ratio sufficiently to give satisfactory reception, is an appropriate measure to evaluate the performance of the wireless system. Mathematically, the outage probability P_{out} is defined as

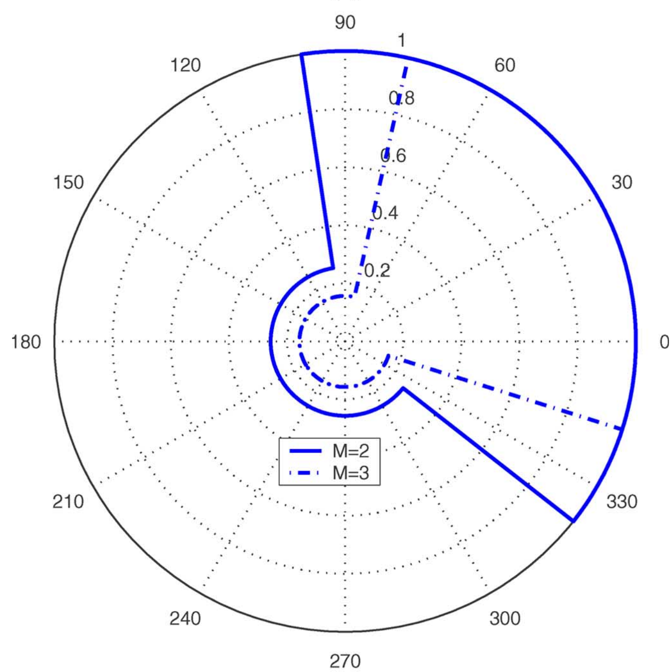
$$P_{\mathrm{out}}=\int_0^{\gamma_{\mathrm{th}}} p_\gamma(\gamma)d\gamma \tag{4}$$

where γ is the instantaneous SIR, and γ_{th} is the required threshold. γ is related to x and y, where x is the power of the desired signal and y is the interference power. In this paper, we consider that the desired signal follows a Rayleigh, Rician, or Nakagami fading, and each interferer follows a Rayleigh distribution.

There are other performance criterions such as bit error rates and symbol error rates in evaluating wireless system



(a)



(b)

Fig. 2. Simplified model for beamforming with an arrival angle of $\theta = 30°$. (a) Signal model. (b) Interference model.

performance. These performance measures and results are related to specific modulation schemes. Therefore, in this paper, we focus on the outage probability evaluation for beamforming systems.

III. Derivation of Outage Probability, Considering Perfect Beamforming

In this section, the outage probability will be derived, considering perfect beamforming. We will start with a simple case with only one interferer. We will next investigate a scenario with multiple interferers, considering actual beam patterns. Finally, the simplified beamforming model will be used to examine the multiple-interferer scenario.

A. Outage Probability Under a Single Interferer

We first analyze a simple case where there is one interferer. The desired signal is subject to Rayleigh fading, and its power x follows an exponential distribution given by

$$p_x(x) = \frac{1}{X}\exp\left(-\frac{x}{X}\right), \qquad x \geqslant 0 \tag{5}$$

where X is the mean of x, $X = \mathrm{E}[x]$. The interferer is also assumed to be Rayleigh faded, and its power y follows an exponential distribution with mean Y given by

$$p_y(y) = \frac{1}{Y}\exp\left(-\frac{y}{Y}\right), \qquad y \geqslant 0. \tag{6}$$

Let $\gamma = x/y$ denote the SIR. The probability density function (pdf) of the SIR can be found as

$$p_\gamma(\gamma) = \int_0^\infty y p_x(\gamma y) p_y(y) dy = \frac{\frac{X}{Y}}{\left(\frac{X}{Y}+\gamma\right)^2}. \tag{7}$$

Considering an interference-limited system, the outage probability is

$$P_{\mathrm{out}} = \int_0^{\gamma_{\mathrm{th}}} p_\gamma(\gamma) d\gamma = 1 - \frac{1}{\frac{\gamma_{\mathrm{th}} Y}{X} + 1}. \tag{8}$$

When beamforming is used, the mean power of the interferer will be $YG(\psi, \theta)$. Therefore, the outage probability becomes

$$\begin{aligned} P_{\mathrm{out}}(\psi,\theta) &= 1 - \frac{1}{\frac{\gamma_{\mathrm{th}} Y G(\psi,\theta)}{X} + 1} \\ &= 1 - \frac{1}{\frac{\gamma_{\mathrm{th}} Y |\sin(0.5M\pi(\sin\theta - \sin\psi))|^2}{X|M\sin(0.5\pi(\sin\theta-\sin\psi))|^2} + 1}. \end{aligned} \tag{9}$$

Notice that the mean power of the desired signal is still X since perfect beamforming is assumed (the mainlobe is steered toward the desired signal), and the normalized antenna gain is 1. Defined a normalized "average" SIR

$$\mathrm{SIR} = \frac{X}{\gamma_{\mathrm{th}} Y} \tag{10}$$

the outage probability becomes

$$\begin{aligned} P_{\mathrm{out}}(\psi,\theta) &= 1 - \frac{1}{\frac{G(\psi,\theta)}{\mathrm{SIR}} + 1} \\ &= 1 - \frac{1}{\frac{|\sin(0.5M\pi(\sin\theta - \sin\psi))|^2}{\mathrm{SIR}|M\sin(0.5\pi(\sin\theta-\sin\psi))|^2} + 1} \end{aligned} \tag{11}$$

where ψ and θ are two random variables uniformly distributed within 0 and 2π. Therefore, the average outage probability using an actual beam pattern is calculated as

$$\begin{aligned} P_{\text{out}} &= \int_0^{2\pi}\int_0^{2\pi} \frac{P_{\text{out}}(\psi,\theta)}{4\pi^2} d\psi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{1-\frac{1}{\frac{|\sin(0.5M\pi(\sin\theta-\sin\psi))|^2}{\text{SIR}|M\sin(0.5\pi(\sin\theta-\sin\psi))|^2}+1}}{4\pi^2} d\psi d\theta. \end{aligned} \tag{12}$$

Equation (12) can be numerically evaluated. We can also use the simplified model (Section II-B) to account the impact of beamforming [22] in calculating the outage probability. According to the simplified model, if an interferer falls into the mainlobe of the beam, using (11), the system outage probability is given by the following by setting $G(\psi,\theta)$ to 1:

$$P_{\text{out}}(\text{mainlobe}) = 1 - \frac{1}{\frac{1}{\text{SIR}}+1} \tag{13}$$

and if the interferer falls in the sidelobe, the system outage probability is

$$P_{\text{out}}(\text{sidelobe}) = 1 - \frac{1}{\frac{\alpha}{\text{SIR}}+1}. \tag{14}$$

Since the probability of an interferer falling in the mainlobe is B, the average outage probability using the simplified model becomes

$$\begin{aligned} P_{\text{out}} &= BP_{\text{out}}(\text{mainlobe}) + (1-B)P_{\text{out}}(\text{sidelobe}) \\ &= B\left(1-\frac{1}{\frac{1}{\text{SIR}}+1}\right)+(1-B)\left(1-\frac{1}{\frac{\alpha}{\text{SIR}}+1}\right). \end{aligned} \tag{15}$$

B. Outage Probability Under Multiple Interferers: Using an Actual Beam Pattern

In Section III-A, we examined a simple single-interferer case using two evaluation approaches: First, we considered an actual beam pattern in calculating the outage probability (12), and then, a simplified model is used in (15). We now investigate cases with multiple interferers. In this subsection, we derive the outage probability using actual beam patterns. Evaluations using the simplified beamforming model will be presented in Section III-C.

If we assume that all the interferers are uniformly distributed geographically, the probability that any two interferers i and j have exactly the same arrival angle is equal to zero. Therefore, interferers i and j are considered to have different mean signal power levels $YG(\psi_i,\theta)$ and $YG(\psi_j,\theta)$. In other words, the mean powers of all the interferers are mutually different. In [23], the outage probability of microcellular mobile systems is derived, and two interference scenarios are considered: One is with the same mean power for all interferers, and the other is with mutually different mean powers. If both desired signal and interferers are subject to Rayleigh fading and their mean powers are mutually different, the pdf of the SIR can be found as (7), [23]

$$p_\gamma(\gamma) = \sum_{i=1}^{N} \frac{b_i}{(\gamma+b_i)^2} \prod_{j=1,j\neq i}^{N} \frac{b_j}{b_j-b_i} \tag{16}$$

in which N is the number of interferers, and

$$b_i = \frac{X}{G(\psi_i,\theta)Y}. \tag{17}$$

Therefore, the conditional outage probability is

$$\begin{aligned} &P_{\text{out}}(\psi_1,\psi_2,\ldots,\psi_N,\theta) \\ &\quad= \int_0^{\gamma_{\text{th}}} \left(\sum_{i=1}^{N} \frac{b_i}{(\gamma+b_i)^2} \prod_{j=1,j\neq i}^{N} \frac{b_j}{b_j-b_i}\right) d\gamma \\ &\quad= \sum_{i=1}^{N}\left(1-\frac{1}{1+\frac{\gamma_{\text{th}}}{b_i}}\right) \prod_{j=1,j\neq i}^{N} \frac{b_j}{b_j-b_i} \\ &\quad= \sum_{i=1}^{N}\left(1-\frac{1}{1+\frac{G(\psi_i,\theta)}{\text{SIR}}}\right) \prod_{j=1,j\neq i}^{N} \left(\frac{1}{1-\frac{G(\psi_j,\theta)}{G(\psi_i,\theta)}}\right). \end{aligned} \tag{18}$$

Considering that all the interferers are uniformly distributed in all directions, we integrate the conditional outage probability (18) and get the average outage probability as

$$\begin{aligned} P_{\text{out}} &= \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{2\pi} \frac{P_{\text{out}}(\psi_1,\psi_2,\ldots,\psi_N,\theta)}{(2\pi)^{N+1}} \\ &\quad\times d\psi_1 d\psi_2\cdots d\psi_N d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{2\pi} \frac{1}{(2\pi)^{N+1}} \\ &\quad\times\left(\sum_{i=1}^{N}\left(1-\frac{1}{1+\frac{G(\psi_i,\theta)}{\text{SIR}}}\right)\prod_{j=1,j\neq i}^{N}\left(\frac{1}{1-\frac{G(\psi_j,\theta)}{G(\psi_i,\theta)}}\right)\right) \\ &\quad\times d\psi_1 d\psi_2\cdots d\psi_N d\theta. \end{aligned} \tag{19}$$

Equation (19) can be numerically calculated, considering actual beam patterns, to evaluate the outage probability of wireless systems with beamforming. However, due to the complexity of the actual beamforming patterns, no closed-form results can be derived from (19), and the multiple integrals remain. This prohibits obtaining numerical results for most systems with a reasonable number of interferers. In the following section, a simplified beamforming model will be used to derive closed-form solutions.

C. Outage Probability Under Multiple Interferers: Using the Simplified Beamforming Model

We now evaluate the outage probability under multiple interferers using the simplified model. N_m interferers are assumed to be within the mainlobe and N_s interferers within the sidelobe. All these interferers are assumed to be Rayleigh faded and have the same mean power, which follows (6). The total power of the N_m interferers y_m follows a chi-square distribution

$$p_{y_m}(y_m) = \frac{y_m^{N_m-1}}{(N_m-1)!Y^{N_m}} \exp\left(-\frac{y_m}{Y}\right) \tag{20}$$

and its MGF is

$$M_{y_m}(s) = \int_0^\infty p_{y_m}(y_m) e^{sy_m} dy_m = \frac{1}{(1-sY)^{N_m}}. \tag{21}$$

Similarly, the total power of N_s interferers y_s follows a chi-square distribution

$$p_{y_s}(y_s) = \frac{y_s^{N_s-1}}{(N_s-1)!(\alpha Y)^{N_s}} \exp\left(-\frac{y_s}{\alpha Y}\right) \tag{22}$$

and its MGF is

$$M_{y_s}(s) = \int_0^\infty p_{y_s}(y_s) e^{sy_s} dy_s = \frac{1}{(1-s\alpha Y)^{N_s}}. \tag{23}$$

The total power of $N_m + N_s$ interferers y is the sum of y_m and y_s. The pdf of y is a convolution of (20) and (22), and its corresponding MGF is

$$M_y(s) = M_{y_m}(s) \cdot M_{y_s}(s) = \frac{1}{(1-sY)^{N_m}(1-s\alpha Y)^{N_s}}. \tag{24}$$

Equation (24) can be rewritten as

$$M_y(s) = \sum_{k=1}^{N_m} \frac{C_k}{(1-sY)^k} + \sum_{l=1}^{N_s} \frac{D_l}{(1-s\alpha Y)^l} \tag{25}$$

where coefficients C_k and D_l are

$$C_k = (1-\alpha)^{-(N_s+k-1)} \alpha^{k-1} \frac{(N_s+k-2)!}{(k-1)!(N_s-1)!} \tag{26}$$

and

$$D_l = \alpha^{N_m} (\alpha-1)^{-(N_m+l-1)} \frac{(N_m+l-2)!}{(l-1)!(N_m-1)!}. \tag{27}$$

From the closed-form MGF of y, we get its pdf as

$$p_y(y) = \sum_{k=1}^{N_m} C_k \frac{y^{k-1}}{(k-1)!Y^k} \exp\left(-\frac{y}{Y}\right) + \sum_{l=1}^{N_s} D_l \frac{y^{l-1}}{(l-1)!(\alpha Y)^l} \exp\left(-\frac{y}{\alpha Y}\right). \tag{28}$$

Inserting (28) into (7), the pdf of SIR can be expressed as follows:

$$\begin{aligned} p_\gamma(\gamma) &= \int_0^\infty y p_x(\gamma y) \sum_{k=1}^{N_m} C_k \frac{y^{k-1}}{(k-1)!Y^k} \exp\left(-\frac{y}{Y}\right) \\ &\quad + y p_x(\gamma y) \sum_{l=1}^{N_s} D_l \frac{y^{l-1}}{(l-1)!(\alpha Y)^l} \exp\left(-\frac{y}{\alpha Y}\right) dy \\ &= \sum_{k=1}^{N_m} C_k \int_0^\infty y p_x(\gamma y) \frac{y^{k-1}}{(k-1)!Y^k} \exp\left(-\frac{y}{Y}\right) dy \\ &\quad + \sum_{l=1}^{N_s} D_l \int_0^\infty y p_x(\gamma y) \frac{y^{l-1}}{(l-1)!(\alpha Y)^l} \exp\left(-\frac{y}{\alpha Y}\right) dy. \end{aligned} \tag{29}$$

Comparing (29) with (7) and (20), we obtain

$$p_\gamma(\gamma) = \sum_{k=1}^{N_m} C_k p_{\gamma|k,Y}(\gamma|k,Y) + \sum_{l=1}^{N_s} D_l p_{\gamma|l,\alpha Y}(\gamma|l,\alpha Y) \tag{30}$$

where

$$p_{\gamma|k,Y}(\gamma|k,Y) = \int_0^\infty y p_x(\gamma y) \frac{y^{k-1}}{(k-1)!Y^k} \exp\left(-\frac{y}{Y}\right) dy \tag{31}$$

and

$$p_{\gamma|l,\alpha Y}(\gamma|l,\alpha Y) = \int_0^\infty y p_x(\gamma y) \frac{y^{l-1}}{(l-1)!(\alpha Y)^l} \exp\left(-\frac{y}{\alpha Y}\right) dy. \tag{32}$$

Notice that $p_{\gamma|k,Y}(\gamma|k,Y)$ and $p_{\gamma|l,\alpha Y}(\gamma|l,\alpha Y)$ are the pdf of the SIR under k Rayleigh-faded interferers, each with mean power Y, and l Rayleigh-faded interferers, each with mean power αY, respectively. Therefore, using the simplified beamforming model, the outage probability under multiple interferers can be derived by integrating the pdf expressed in (30)

$$\begin{aligned} P_{\text{out}}(N_m, N_s) &= \int_0^{\gamma_{\text{th}}} p_\gamma(\gamma) d\gamma \\ &= \sum_{k=1}^{N_m} C_k P_{\text{out}|k,Y} + \sum_{l=1}^{N_s} D_l P_{\text{out}|l,\alpha Y} \end{aligned} \tag{33}$$

where $P_{\text{out}|k,Y}$ and $P_{\text{out}|l,\alpha Y}$ are the outage probabilities under k and l Rayleigh-faded interferers each with mean power Y and αY, respectively. So far, in the derivations in this section, we have assumed Rayleigh-faded interferers. The distribution of the desired signal is not specified. In the following subsection, we derive the exact outage probability expressions where the desired signal is subject to Rayleigh, Rician, or Nakagami fading.

1) Desired Signal: Rayleigh Fading: Following (33) and [24], we are able to get a closed-form outage probability expression when there are N_m interferers within the mainlobe and N_s interferers within the sidelobe, as given by

$$P_{\mathrm{out}}(N_m, N_s) = \sum_{k=1}^{N_m} C_k \left(1 - \frac{1}{\left(1+\frac{1}{\mathrm{SIR}}\right)^k}\right) + \sum_{l=1}^{N_s} D_l \left(1 - \frac{1}{\left(1+\frac{\alpha}{\mathrm{SIR}}\right)^l}\right). \tag{34}$$

The expression can be simplified for special cases. For $N_s = 0$, we have

$$P_{\mathrm{out}}(N_m, 0) = 1 - \frac{1}{\left(1+\frac{1}{\mathrm{SIR}}\right)^{N_m}} \tag{35}$$

and for $N_m = 0$, we have

$$P_{\mathrm{out}}(0, N_s) = 1 - \frac{1}{\left(1+\frac{\alpha}{\mathrm{SIR}}\right)^{N_s}}. \tag{36}$$

2) Desired Signal: Rician Fading: The envelope of the desired signal follows a Rician distribution, and its corresponding power has a noncentral chi-square distribution with two degrees of freedom

$$p_x(x) = \frac{(1+K_d)e^{-K_d}}{X} \times \exp\left(-\frac{(1+K_d)x}{X}\right) \mathrm{I}_0\left(2\sqrt{\frac{K_d(1+K_d)x}{X}}\right) \tag{37}$$

where K_d is a Rician-fading parameter that ranges from 0 to ∞ and $\mathrm{I}_0(\cdot)$ is the modified Bessel function of the first kind and order zero. Following (33) and [24], we get the closed-form outage probability

$$\begin{aligned} P_{\mathrm{out}}&(N_m, N_s) \\ &= \sum_{k=1}^{N_m} C_k \exp\left(-\frac{K_d}{1+\frac{K_d+1}{\mathrm{SIR}}}\right)\left(\frac{1}{1+\frac{\mathrm{SIR}}{K_d+1}}\right) \\ &\quad \times \sum_{n=0}^{k-1}\left(\frac{1}{1+\frac{(K_d+1)}{\mathrm{SIR}}}\right)^n \sum_{j=0}^{n}\binom{n}{n-j}\frac{1}{j!}\left(\frac{K_d}{1+\frac{\mathrm{SIR}}{(K_d+1)}}\right)^n \\ &\quad + \sum_{l=1}^{N_s} D_l \exp\left(-\frac{K_d}{1+\frac{\alpha(K_d+1)}{\mathrm{SIR}}}\right)\left(\frac{1}{1+\frac{\mathrm{SIR}}{\alpha(K_d+1)}}\right) \\ &\quad \times \sum_{n=0}^{l-1}\left(\frac{1}{1+\frac{\alpha(K_d+1)}{\mathrm{SIR}}}\right)^n \\ &\quad \times \sum_{j=0}^{n}\binom{n}{n-j}\frac{1}{j!}\left(\frac{K_d}{1+\frac{\mathrm{SIR}}{\alpha(K_d+1)}}\right)^n. \end{aligned} \tag{38}$$

The expression is simplified for special cases. For $N_s = 0$, we have

$$\begin{aligned} P_{\mathrm{out}}(N_m, 0) &= \exp\left(-\frac{K_d}{1+\frac{K_d+1}{\mathrm{SIR}}}\right)\left(\frac{1}{1+\frac{\mathrm{SIR}}{K_d+1}}\right) \\ &\quad \times \sum_{n=0}^{N_m-1}\left(\frac{1}{1+\frac{(K_d+1)}{\mathrm{SIR}}}\right)^n \sum_{j=0}^{n}\binom{n}{n-j}\frac{1}{j!}\left(\frac{K_d}{1+\frac{\mathrm{SIR}}{(K_d+1)}}\right)^n \end{aligned} \tag{39}$$

and for $N_m = 0$, we have

$$\begin{aligned} P_{\mathrm{out}}(0, N_s) &= \exp\left(-\frac{K_d}{1+\frac{\alpha(K_d+1)}{\mathrm{SIR}}}\right)\left(\frac{1}{1+\frac{\mathrm{SIR}}{\alpha(K_d+1)}}\right) \\ &\quad \times \sum_{n=0}^{N_s-1}\left(\frac{1}{1+\frac{\alpha(K_d+1)}{\mathrm{SIR}}}\right)^n \sum_{j=0}^{n}\binom{n}{n-j}\frac{1}{j!}\left(\frac{K_d}{1+\frac{\mathrm{SIR}}{\alpha(K_d+1)}}\right)^n. \end{aligned} \tag{40}$$

3) Desired Signal: Nakagami Fading: The envelope of the desired signal follows a Nakagami distribution, and its corresponding power has a Gamma distribution

$$p_x(x) = \frac{m_d^{m_d} x^{m_d-1}}{X^{m_d}\Gamma(m_d)} \exp\left(-\frac{m_d}{X}x\right) \tag{41}$$

where $\Gamma(\cdot)$ is the Gamma function, and m_d is a parameter of Nakagami fading. $1/m_d$ is the amount of fading, which is defined as the ratio of variance to the square of the mean of the received energy. Following (33) and [23], we get the closed-form outage probability

$$\begin{aligned} P_{\mathrm{out}}&(N_m, N_s) \\ &= \sum_{k=1}^{N_m} C_k \frac{\Gamma(m_d+k)}{m_d\Gamma(k)\Gamma(m_d)}\left(\frac{m_d}{\mathrm{SIR}}\right)^{-m_d} \\ &\quad \times {}_2\mathrm{F}_1\left(m_d+k, m_d, 1+m_d, -\frac{m_d}{\mathrm{SIR}}\right) \\ &\quad + \sum_{l=1}^{N_s} D_l \frac{\Gamma(m_d+l)}{m_d\Gamma(l)\Gamma(m_d)}\left(\frac{m_d\alpha}{\mathrm{SIR}}\right)^{-m_d} \\ &\quad \times {}_2\mathrm{F}_1\left(m_d+l, m_d, 1+m_d, -\frac{m_d\alpha}{\mathrm{SIR}}\right) \end{aligned} \tag{42}$$

where ${}_2\mathrm{F}_1(.,.,.,.)$ is the Gauss hypergeometric function defined in [25]

$${}_2\mathrm{F}_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!}. \tag{43}$$

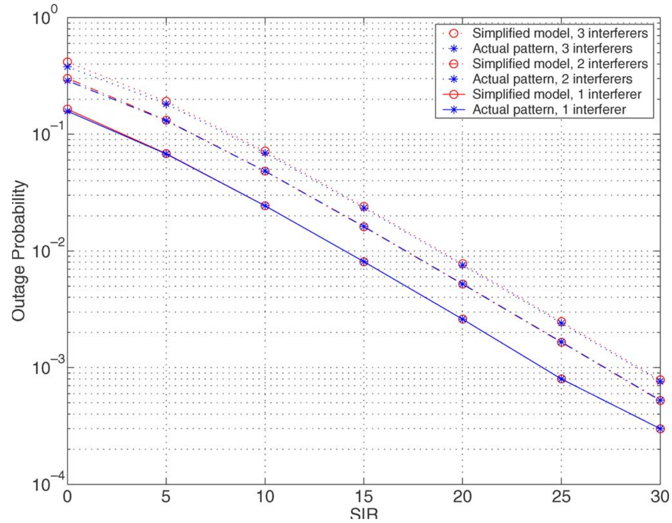


Fig. 3. Accuracy of the simplified beamforming model. The number of antenna elements is $M = 4$.

The outage probability expression is simplified for special cases. For $N_s = 0$, we have

$$P_{\text{out}}(N_m, 0) = \frac{\Gamma(m_d + N_m)}{m_d \Gamma(N_m)\Gamma(m_d)} \left(\frac{m_d}{\text{SIR}}\right)^{-m_d} \times {}_2F_1\left(m_d + N_m, m_d, 1 + m_d, -\frac{m_d}{\text{SIR}}\right) \quad (44)$$

and for $N_m = 0$, we have

$$P_{\text{out}}(0, N_s) = \frac{\Gamma(m_d + N_s)}{m_d \Gamma(N_s)\Gamma(m_d)} \left(\frac{m_d \alpha}{\text{SIR}}\right)^{-m_d} \times {}_2F_1\left(m_d + N_s, m_d, 1 + m_d, -\frac{m_d \alpha}{\text{SIR}}\right). \quad (45)$$

So far, we have considered three different fading scenarios for the desired signal. The conditional outage probabilities for the given N_m and N_s have been derived. Considering that N interferers $N = N_m + N_s$ are uniformly distributed in all directions, the average outage probability using the simplified beamforming model is

$$P_{\text{out}} = \sum_{N_m=0}^{N} \binom{N}{N_m} B^{N_m} (1-B)^{N-N_m} P_{\text{out}}(N_m, N - N_m). \quad (46)$$

The preceding expression gives the outage probability of a wireless system with beamforming, where α and B are determined based on actual beam patterns.

D. Accuracy of the Simplified Model

We use (19) (multiple integrals) and (46) (an expression derived based on a simplified beamforming model) to calculate the outage probabilities of wireless systems with beamforming. The numerical results are shown in Fig. 3. It can be seen that the evaluation results using the simplified model match those using the actual beam pattern very well. Notice that in Fig. 3, only one, two, or three interferers are considered. The results for a larger number of interferers are not obtained due to the computational complexity of multiple integrals in the case of considering actual beam patterns (19). However, it is easy to obtain the results for cases with larger numbers of interferers using the simplified model. This also indicates the need to develop and use the simplified beamforming model. The accuracy of the simplified beamforming model is investigated in the Appendix, which concludes that the model becomes more accurate at higher SIR.

IV. Consideration of Beamforming Impairments

In the previous sections, closed-form outage probability expressions for wireless systems with beamforming have been derived using a simplified beamforming model. These analytical results are based on perfect beamforming conditions. However, a variety of impairments exist in practical beamforming systems [22], [26]–[29]. In [22], the impairments of beamforming, including DOA estimation errors, spatial spreads, array perturbation, and mutual coupling, are analyzed. All the impairments will affect the shape of beam patterns or antenna gains. Notice that in the simplified beamforming model, only α and B need to be modified according to the change of beam patterns. The outage probability expression (46) is still valid. For each beamforming impairment, α and B are uniquely determined based on the shape-changed beam pattern, and their values will be reevaluated. Equation (46) will then be used to calculate the outage probability. In the next four subsections, we will investigate the impacts of each beamforming impairment on the outage probability of wireless systems.

A. DOA Estimation Error

In the implementation of beamforming, DOA, which is used to steer the beams toward the desired signals, is usually obtained through estimation algorithms [4], [17]. Any error in estimating the arrival angles will cause the antenna array to point away from the desired signal and will lead to a reduction of the received power of the desired signal. In this section, the impact of DOA mismatch on the beamforming parameters (α and B) is analyzed. The estimated arrival angles $\hat{\theta}$ for the desired signal and $\hat{\psi}$ for interferers can be characterized as random variables with a uniform distribution or normal distribution [4]. The pdf of $\hat{\theta}$ is expressed as

$$f(\hat{\theta}) = \begin{cases} \frac{1}{2\sqrt{3}\triangle}, -\sqrt{3}\triangle \le (\hat{\theta} - \theta) \le \sqrt{3}\triangle, & \text{uniform} \\ \frac{1}{\sqrt{2\pi}\triangle} \exp\left\{-\frac{(\hat{\theta}-\theta)^2}{2\triangle^2}\right\}, & \text{normal} \end{cases} \quad (47)$$

where θ is the accurate arrival angle, and $\triangle^2$ represents the variance of the estimation errors for uniform or normal distributions. $\hat{\psi}$ has similar pdf expressions as (47). Following (2), (3),

TABLE I
COMPUTATIONAL PARAMETERS α AND B FOR EXAMINING DOA ESTIMATION ERRORS. $\triangle$ IS THE STANDARD DEVIATION. $\triangle_{\max}$ IS DEFINED IN (50)

M		3	4	5	6	7	8	9
ideal BF	α	.157	.113	.089	.073	.062	.055	.049
$\triangle = 0$	B	.265	.207	.172	.147	.129	.115	.104
$\triangle =$	α	.160	.115	.090	.074	.063	.055	.049
$\frac{1}{8}\triangle_{max}$	B	.266	.208	.172	.147	.129	.115	.104
$\triangle =$	α	.170	.121	.094	.077	.066	.057	.051
$\frac{1}{4}\triangle_{max}$	B	.268	.210	.174	.149	.1312	.117	.106
$\triangle =$	α	.188	.130	.101	.082	.070	.060	.054
$\frac{3}{8}\triangle_{max}$	B	.272	.214	.178	.153	.134	.119	.108
$\triangle =$	α	.214	.145	.110	.089	.075	.065	.058
$\frac{1}{2}\triangle_{max}$	B	.285	.224	.186	.160	.140	.125	.113
$\triangle =$	α	.248	.164	.123	.098	.082	.071	.062
$\frac{5}{8}\triangle_{max}$	B	.322	.249	.205	.175	.153	.136	.123
$\triangle =$	α	.230	.165	.124	.099	.082	.070	.062
$\frac{3}{4}\triangle_{max}$	B	.4820	.331	.263	.221	.191	.169	.152

and [22], the equations used to calculate the two parameters α and B are

$$\alpha = \frac{\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}}\left[\frac{G^2\left(\hat{\psi}(\psi),\theta\right)}{G^2\left(\hat{\theta}(\theta),\theta\right)}\right] - \mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G\left(\hat{\psi}(\psi),\theta\right)}{G\left(\hat{\theta}(\theta),\theta\right)}\right]}{\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G\left(\hat{\psi}(\psi),\theta\right)}{G\left(\hat{\theta}(\theta),\theta\right)}\right] - 1} \tag{48}$$

and

$$B = \frac{\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G^2\left(\hat{\psi}(\psi),\theta\right)}{G^2\left(\hat{\theta}(\theta),\theta\right)}\right] - \mathrm{E}^2_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G\left(\hat{\psi}(\psi),\theta\right)}{G\left(\hat{\theta}(\theta),\theta\right)}\right]}{\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G^2\left(\hat{\psi}(\psi),\theta\right)}{G^2\left(\hat{\theta}(\theta),\theta\right)}\right] + 1 - 2\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}(\theta)}\left[\frac{G\left(\hat{\psi}(\psi),\theta\right)}{G\left(\hat{\theta}(\theta),\theta\right)}\right]} \tag{49}$$

where $\mathrm{E}_{\psi,\theta,\hat{\psi},\hat{\theta}}[\cdot]$ denotes the expectation with respect to all the related random variables ψ, θ, $\hat{\psi}$, and $\hat{\theta}$. The two parameters α and B in a variety of DOA estimation error situations are given in Table I. A special case is ideal beamforming ($\Delta = 0$), which is also listed in the table. Standard deviation Δ is normalized by $\Delta_{\max}$, where $\Delta_{\max}$ is the standard deviation of a DOA estimation error that is uniformly distributed from null to null when θ is equal to $0°$ (toward the broadside direction). Using (1), $\Delta_{\max}$ can be found to be

$$\Delta_{\max} = \frac{\arcsin(2/M)}{\sqrt{3}}. \tag{50}$$

B. Spatial Spreads

The obstacles around a transmitter (mobile station), such as buildings, reflect the transmitted waves and result in multiple paths with different arrival angles at a base station (angle spread). A propagation model to characterize the angle spread was reviewed in [30]. Recently, angle spreads have been measured and reported in [26] and [27]. For rural environments, angular spreads between $1°$ and $5°$ have been observed [26]; for urban and hilly terrain environments, considerably larger angular spreads that are as large as $20°$ have been found [27]. The angle spreads not only reduce the received signal power as the DOA estimation becomes random in the interval of arrival angles but also cause DOA estimation uncertainty. Assume that the spatial spread follows a uniform or normal distribution and that the estimated arrival angle follows the same distribution, the expected received signal is averaged, considering both the arrival angle estimations and the spatial spreads. Following (2), (3), and [22], we obtain

TABLE II
COMPUTATIONAL PARAMETERS α AND B FOR EXAMINING SPATIAL SPREADS. δ IS THE STANDARD DEVIATION

M		3	4	5	6	7	8	9
δ	α	.157	.113	.089	.074	.063	.056	.050
$= 1^o$	B	.265	.207	.172	.147	.129	.116	.105
δ	α	.162	.120	.096	.081	.071	.064	.058
$= 3^o$	B	.266	.209	.175	.152	.135	.123	.114
δ	α	.181	.139	.116	.101	.091	.084	.079
$= 6^o$	B	.270	.220	.191	.174	.162	.153	.146

$$\alpha = \frac{\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G^2\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G^2\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right] - \mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right]}{\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right] - 1} \tag{51}$$

and

$$\begin{aligned} B = &\left\{\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G^2\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G^2\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right]\right. \\ &\left. - \mathrm{E}^2_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right]\right\} \\ &\times \left\{\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G^2\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G^2\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right] + 1\right. \\ &\left. - 2\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}\left[\frac{G\left(\hat{\psi}(\psi_s),\theta(\theta_s)\right)}{G\left(\hat{\theta}(\theta),\theta(\theta_s)\right)}\right]\right\}^{-1} \end{aligned} \tag{52}$$

where $\mathrm{E}_{\hat{\psi},\hat{\theta},\theta,\psi_s,\theta_s}[\cdot]$ is the expectation with respect to all the related random variables $\hat{\psi}$, $\hat{\theta}$, θ, ψ_s, and θ_s. ψ_s is the mean value of ψ, and θ_s is the mean value of $\hat{\theta}$ and θ. The values of α and B have been calculated and listed in Table II.

C. Array Perturbation

Even with perfect DOA estimations, array perturbations due to the position errors of antenna elements, which may be caused by wind, can also result in mismatches between exact DOAs and the directions of mainlobes [4]. A method to analyze the effect of array perturbation in beamforming was presented in [22]. An odd number of antenna elements $M = 2C + 1$ is

TABLE III
COMPUTATIONAL PARAMETERS α AND B FOR EXAMINING ARRAY PERTURBATION. σ_p IS THE STANDARD DEVIATION

M		3	5	7	9	11	13	15
σ_p	α	.189	.109	.077	.060	.050	.042	.037
$= .05$	B	.255	.166	.125	.101	.085	.074	.065
σ_p	α	.280	.166	.119	.093	.077	.066	.057
$= .1$	B	.205	.138	.105	.085	.072	.063	.056
σ_p	α	.400	.266	.201	.161	.135	.116	.102
$= .2$	B	.269	.175	.132	.106	.090	.078	.069

assumed, and the position of the bth element is $(x_b, y_b) = (0, b\lambda/2)$ in the absence of perturbation. The position errors due to array perturbation are modeled as random variables with a normal distribution. Let $p_b = x_b/\lambda$, $q_b = y_b/\lambda$, we have the beam pattern (in strength) [4]

$$A = \sum_{b=-C}^{C} \exp\left\{-j2\pi\left[(p_b\cos\theta + q_b\sin\theta) - 0.5b\sin\psi\right]\right\} \tag{53}$$

where p_b and q_b are independent with $\mathrm{E}[p_b] = 0$ and $\mathrm{E}[q_b] = 0.5b$, respectively, and $\mathrm{Var}[p_b] = \mathrm{Var}[q_b] = (\sigma_p/\lambda)^2$, where σ_p^2 is the variance of the position errors normalized by λ^2. Using $\mathbf{p}$ and $\mathbf{q}$ to represent $[p_{-C}\cdots p_0\cdots p_C]$ and $[q_{-C}\cdots q_0\cdots q_C]$, respectively, the antenna gain is

$$\begin{aligned} G(\psi,\theta,\mathbf{p},\mathbf{q}) &= \frac{|AA^*|}{M^2} \\ &= \frac{1}{M} + \frac{1}{M^2} \\ &\quad \times \sum_{a=-C}^{C}\sum_{a\neq b} \exp\left\{-j2\pi\left[((p_a - p_b)\cos\theta + (q_a - q_b)\sin\theta)\right.\right. \\ &\quad \left.\left. - 0.5(a-b)\sin\psi\right]\right\}. \end{aligned} \tag{54}$$

We thus obtain

$$\alpha = \frac{\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G^2(\psi,\theta,\mathbf{p},\mathbf{q})}{G^2(\theta,\theta,\mathbf{p},\mathbf{q})}\right] - \mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G(\psi,\theta,\mathbf{p},\mathbf{q})}{G(\theta,\theta,\mathbf{p},\mathbf{q})}\right]}{\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G(\psi,\theta,\mathbf{p},\mathbf{q})}{G(\theta,\theta,\mathbf{p},\mathbf{q})}\right] - 1} \tag{55}$$

and

$$B = \frac{\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G^2(\psi,\theta,\mathbf{p},\mathbf{q})}{G^2(\theta,\theta,\mathbf{p},\mathbf{q})}\right] - \mathrm{E}^2_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G(\psi,\theta,\mathbf{p},\mathbf{q})}{G(\theta,\theta,\mathbf{p},\mathbf{q})}\right]}{\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G^2(\psi,\theta,\mathbf{p},\mathbf{q})}{G^2(\theta,\theta,\mathbf{p},\mathbf{q})}\right] + 1 - 2\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}\left[\frac{G(\psi,\theta,\mathbf{p},\mathbf{q})}{G(\theta,\theta,\mathbf{p},\mathbf{q})}\right]} \tag{56}$$

where $\mathrm{E}_{\psi,\theta,\mathbf{p},\mathbf{q}}[\cdot]$ is the expectation with respect to all the related random variables ψ, θ, $\mathbf{p}$, and $\mathbf{q}$. The two parameters α and B of the simplified beamforming model with different array perturbation variances are given in Table III.

TABLE IV
COMPUTATIONAL PARAMETERS α AND B EXAMINING FOR MUTUAL COUPLING

M	3	4	5	6	7	8	9
α	.162	.117	.092	.075	.064	.056	.050
B	.262	.205	.169	.144	.126	.113	.102

D. Mutual Coupling

In addition to DOA estimation errors and array perturbation, the existence of mutual coupling between antenna elements also leads to changes in beam patterns [28], [29]. It can affect the estimation of the arrival angles, which results in the disturbance of the weighting vector in beamforming. The corresponding change in the beam pattern could lead to a degradation in system capacity [28]. Considering thin half-wavelength dipoles, mutual coupling (between the ith and jth elements) is characterized by a mutual coupling impedance matrix [29], [31]

$$\mathbf{C}_{i,j} = (Z_T + Z_A)(\mathbf{Z}_{i,j} + Z_T\mathbf{I})^{-1} \tag{57}$$

where Z_A is the antenna impedance, Z_T is the terminating impedance, $\mathbf{I}$ denotes an identity matrix, and $\mathbf{Z}_{i,j}$ is the mutual impedance matrix. Assuming that the arrival angles are estimated correctly, the beam pattern is

$$A = \sum_{m=-C}^{C} \exp\{jm\pi\sin\psi_0\} \sum_{n=-C}^{C} \mathbf{C}_{m,n}\exp\{-j\pi n\sin\theta\} \tag{58}$$

and the normalized beamforming gain is

$$G(\psi,\theta) = \frac{|AA^*|}{M^2}. \tag{59}$$

The two parameters of the simplified model α and B can be found as

$$\alpha = \frac{\mathrm{E}_{\psi,\theta}\left[\frac{G^2(\psi,\theta)}{G^2(\theta,\theta)}\right] - \mathrm{E}_{\psi,\theta}\left[\frac{G(\psi,\theta)}{G(\theta,\theta)}\right]}{\mathrm{E}_{\psi,\theta}\left[\frac{G(\psi,\theta)}{G(\theta,\theta)}\right] - 1} \tag{60}$$

and

$$B = \frac{\mathrm{E}_{\psi,\theta}\left[\frac{G^2(\psi,\theta)}{G^2(\theta,\theta)}\right] - \mathrm{E}^2_{\psi,\theta}\left[\frac{G(\psi,\theta)}{G(\theta,\theta)}\right]}{\mathrm{E}_{\psi,\theta}\left[\frac{G^2(\psi,\theta)}{G^2(\theta,\theta)}\right] + 1 - 2\mathrm{E}_{\psi,\theta}\left[\frac{G(\psi,\theta)}{G(\theta,\theta)}\right]} \tag{61}$$

respectively, where $\mathrm{E}_{\psi,\theta}[\cdot]$ is the expectation with respect to the related random variables ψ and θ, which are modeled as uniform random variables in $[0, 2\pi)$. The two parameters α and B of the simplified beamforming model with mutual coupling are computed and given in Table IV.

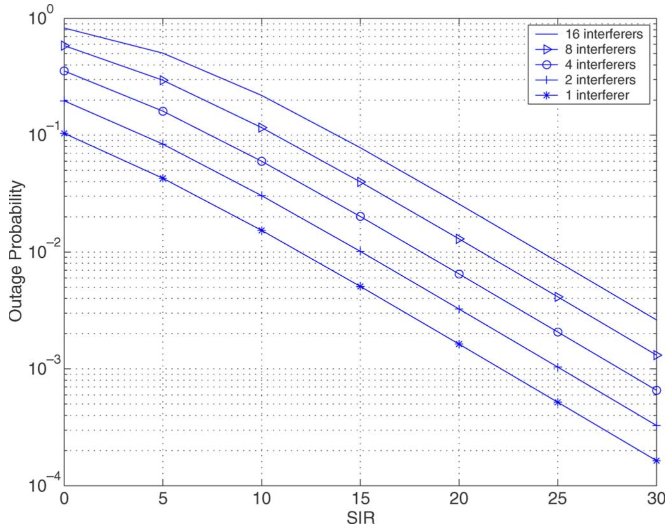


Fig. 4. Outage probability of wireless systems with beamforming under multiple interferers. The number of antenna elements is $M = 8$.

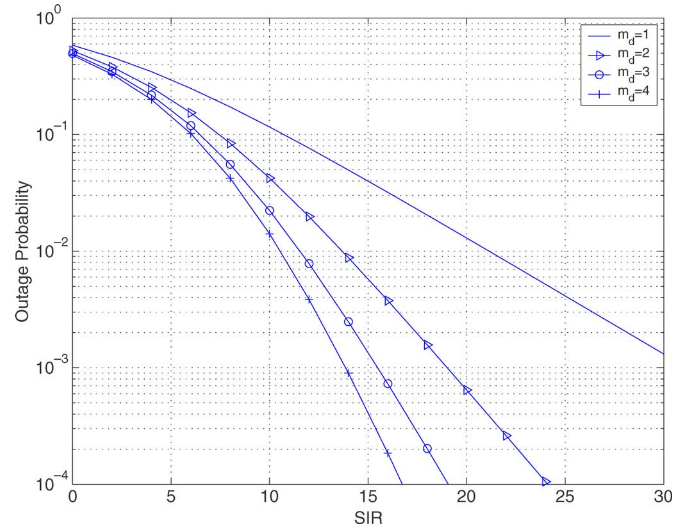


Fig. 6. Outage probability of wireless systems with beamforming. Eight antenna elements and eight interferers are considered. The impact of Nakagami fading is shown (Desired signal: Nakagami fading. Interferers: Rayleigh fading).

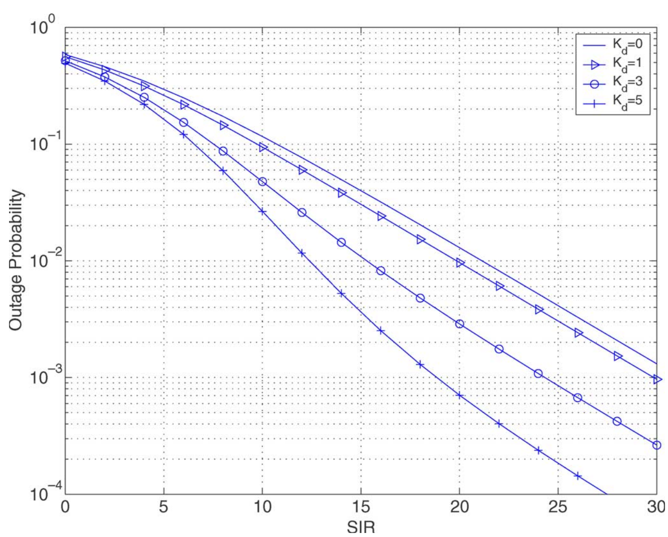


Fig. 5. Outage probability of wireless systems with beamforming. Eight antenna elements and eight interferers are considered. The impact of Rician fading is shown (Desired signal: Rician fading. Interferers: Rayleigh fading).

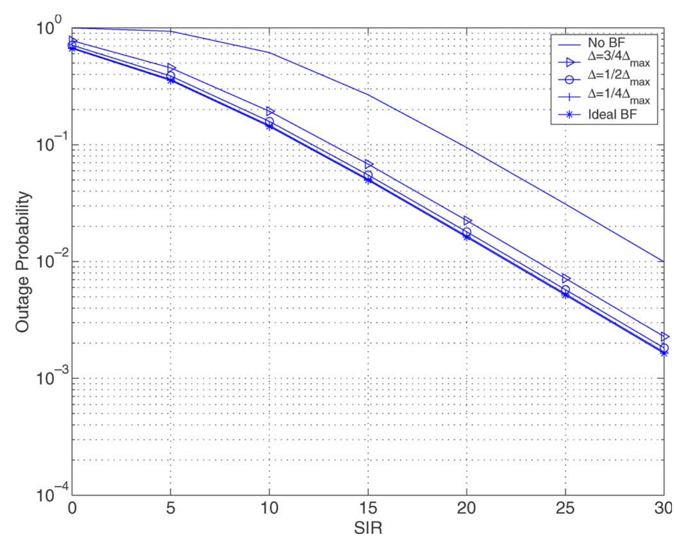


Fig. 7. Outage probability of wireless systems with beamforming with DOA estimation errors. Eight antenna elements and ten interferers are considered. Δ is the standard deviation of the uniformly distributed DOA estimation errors $\Delta_{\max} = 8.4^\circ$.

V. Numerical Results

This section presents numerical evaluations of the outage probability P_{out} versus the normalized SIR, and the results are plotted in Figs. 3–10. The desired signal and interference signals are assumed to follow Rayleigh distributions in most of the evaluations. Scenarios with Rician- or Nakagami-faded signals are examined in Figs. 5 and 6.

Fig. 3 has been discussed in Section III-D. Fig. 4 plots the outage probability with respect to different numbers of interferers. The number of antenna elements is assumed to be 8 in this figure. The simplified model is used for the numerical evaluation. The approach with actual beam patterns is not used due to extremely high calculation complexity in (19) when the number of interferers is more than 3. It is observed in Fig. 4 that the outage probability deteriorates approximately 3 dB while the number of interferers is doubled.

Figs. 5 and 6 show the outage probabilities of wireless communication systems, in which the desired signal follows Rician or Nakagami fading. Eight interferers and a beamformer with eight antenna elements are considered. For two special cases, $K_d = 0$ in Fig. 5 and $m_d = 1$ in Fig. 6, the Rician and Nakagami fadings degenerate into Rayleigh fading and match the Rayleigh fading results plotted in Fig. 4.

Fig. 7 shows the impact of DOA estimation errors. The simplified beamforming model and Table I are used for numerical evaluation. A scenario with eight antenna elements and ten interferers is considered. The DOA estimation errors are assumed to follow a uniform distribution, and Δ is the standard

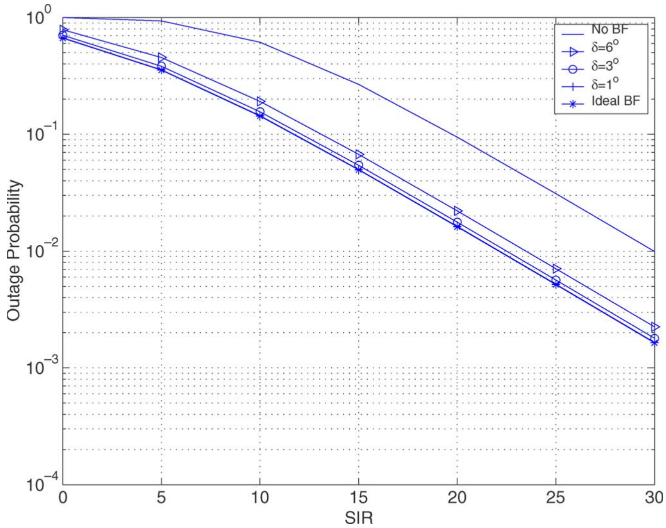


Fig. 8. Outage probability of wireless systems with beamforming with spatial spreads. Eight antenna elements and ten interferers are considered. δ is the standard deviation of the uniformly distributed spatial spreads.

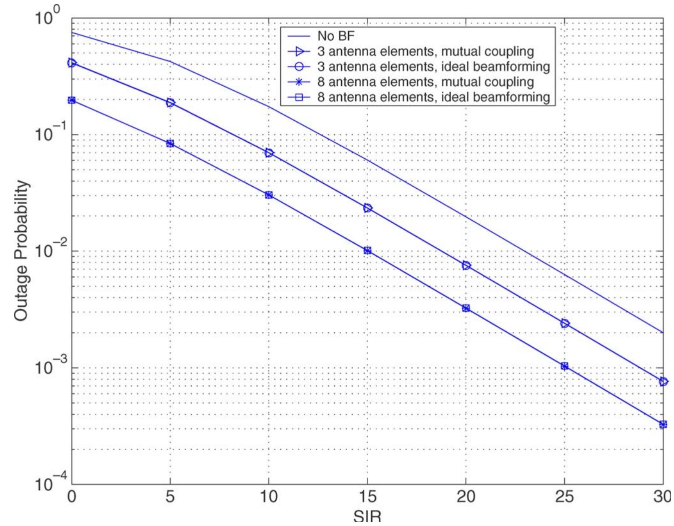


Fig. 10. Outage probability of wireless systems with beamforming with mutual coupling. Ten interferers are considered.

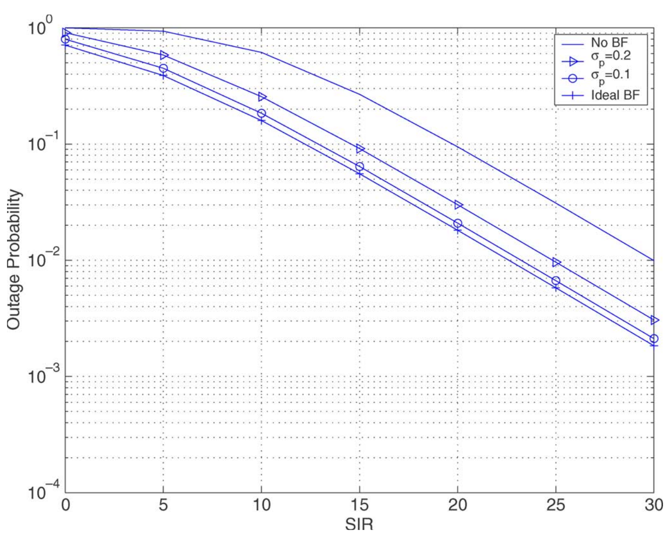


Fig. 9. Outage probability of wireless systems with beamforming with array perturbation. Seven antenna elements and ten interferers are considered. (Array position error: Normal distribution. Standard deviation σ_p).

deviation. As shown in Fig. 7, the outage probability does not change much while the DOA estimation errors are within half of the null-to-null beamwidth ($\Delta \leq 0.5\Delta_{\max}$). For a case with a larger DOA error ($\Delta = 0.75\Delta_{\max}$), a noticeable increase of the outage probability is seen.

Fig. 8 illustrates the spatial spread impact on beamforming. Equation (46) and Table II are used in the numerical calculations. Eight antenna elements and ten interferers are assumed in the evaluation. It is seen that the outage probability deteriorates with increasing spatial spreads, and a SIR loss of approximately 1.3 dB is observed when the standard deviation δ of the spatial spreads is 6°.

The impact of array perturbation is examined in Fig. 9. Seven antenna elements (an odd number of antenna elements) and ten interferers are assumed. Equation (46) and Table III are used in the numerical calculations. The outage probability increases with increasing array perturbation. The σ_p in the figure is the standard deviation of the position errors normalized by λ. With a 1% outage probability, an SIR loss of approximately 2.5 dB is seen when σ_p is equal to 0.2.

Finally, Fig. 10 illustrates the mutual coupling effects in two beamforming scenarios, one with three antenna elements and the other with eight antenna elements, and ten interferers are assumed. Equation (46) and Table IV are used in the numerical calculations, and a negligible performance change is seen due to mutual coupling. This is due to the fact that the distance between adjacent antenna elements is $\lambda/2$, which is large enough to diminish any noticeable mutual coupling.

VI. Conclusion

This paper analyzes the outage probabilities of a wireless system with conventional beamforming. The fading statistics of the desired signal is assumed to be Rayleigh, Rician, or Nakagami, while the interferers are subject to Rayleigh fading. Closed-form expressions are derived based on a simplified beamforming model. Several beamforming impairments, including DOA estimation errors, spatial spreads, antenna array perturbation, and mutual coupling, are examined, and their impacts on the outage probabilities are evaluated. While outage performance degradation is observed due to DOA estimation errors, spatial spreads, and array perturbation, it is noticed that the impact of mutual coupling is negligible.

Appendix

The accuracy of the simplified beamforming model is investigated in this Appendix. As shown in Fig. 3, the simplified beamforming model becomes more accurate at higher SIR. This phenomenon can be explained by comparing the outage probability results, which are based on actual beamforming patterns and the simplified model. Only a single-interferer case is analyzed for illustration. The outage probability with an actual beamforming pattern is given in (12). It can be rewritten using Taylor series as

$$\begin{aligned} P_{\mathrm{out}} &= \int_0^{2\pi}\int_0^{2\pi} \frac{P_{\mathrm{out}}(\psi,\theta)}{4\pi^2} d\psi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{1-\frac{1}{\frac{G(\psi,\theta)}{\mathrm{SIR}}+1}}{4\pi^2} d\psi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{\left(\sum_{i=1}^{\infty}\left(\frac{G(\psi,\theta)}{\mathrm{SIR}}\right)^i (-1)^{i+1}\right)}{4\pi^2} d\psi d\theta \\ &= \sum_{i=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi} \frac{\left(\left(\frac{G(\psi,\theta)}{\mathrm{SIR}}\right)^i (-1)^{i+1}\right)}{4\pi^2} d\psi d\theta \\ &= \sum_{i=1}^{\infty}\left(\frac{E\left[G^i(\psi,\theta)\right]}{\mathrm{SIR}^i}(-1)^{i+1}\right). \end{aligned} \tag{A-1}$$

Similarly, the outage probability based on the simplified beamforming model (15) can be rewritten as

$$\begin{aligned} P_{\mathrm{out}} &= B\left(1-\frac{1}{\frac{1}{\mathrm{SIR}}+1}\right)+(1-B)\left(1-\frac{1}{\frac{\alpha}{\mathrm{SIR}}+1}\right) \\ &= \sum_{i=1}^{\infty}\left(\left(B+(1-B)\alpha^i\right)\frac{(-1)^{i+1}}{\mathrm{SIR}^i}\right). \end{aligned} \tag{A-2}$$

The difference between (A-1) and (A-2) is

$$\begin{aligned} e &= \sum_{i=1}^{\infty}\left(\frac{E\left[G^i(\psi,\theta)\right]}{\mathrm{SIR}^i}(-1)^{i+1}\right) \\ &\quad -\sum_{i=1}^{\infty}\left(\left(B+(1-B)\alpha^i\right)\frac{(-1)^{i+1}}{\mathrm{SIR}^i}\right) \\ &= \sum_{i=3}^{\infty}\left(\left(E\left[G^i(\psi,\theta)\right]-B-(1-B)\alpha^i\right)\frac{(-1)^{i+1}}{\mathrm{SIR}^i}\right) \\ &= O\left(\frac{1}{\mathrm{SIR}^3}\right). \end{aligned} \tag{A-3}$$

In the preceding derivation, the properties of the simplified model

$$E\left[G(\psi,\theta)\right] = B+(1-B)\alpha \tag{A-4}$$

and

$$E\left[G^2(\psi,\theta)\right] = B+(1-B)\alpha^2 \tag{A-5}$$

have been utilized. Equations (A-4) and (A-5) can be derived from (2) and (3). We thus see from (A-3) that the outage probability error e that is introduced by the simplified beamforming model is inversely proportional to the cubic of average SIR. Therefore, the simplified beamforming model becomes more accurate at higher SIR.

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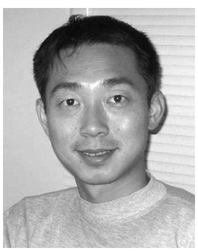

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