ADVANCED BAYESIAN MODELING

Generalizations and Extensions

General Covariance Structure

Let y have a general covariance structure

$$\mathrm{var}(y \mid \theta, X) \;\; = \;\; \sigma^2 Q_y$$

where Q_y is a *known* $n \times n$ matrix.

The normal-theory linear regression then becomes

$$y \mid \theta, X \;\; \sim \;\; \mathrm{N}(X\beta,\, \sigma^2 Q_y)$$

(For *ordinary* linear regression, $Q_y = I$.)

Assume the standard noninformative prior, and that Q_y is invertible (in addition to the earlier conditions).

Then the posterior exists and has the same form as before, except with

$$\hat{\beta} = \left(X^T Q_y^{-1} X\right)^{-1} X^T Q_y^{-1} y \qquad s^2 = \frac{1}{n-k}(y - X\hat{\beta})^T Q_y^{-1}(y - X\hat{\beta})$$

$$V_\beta = \left(X^T Q_y^{-1} X\right)^{-1}$$

These are *generalized least squares* estimates.

The case of diagonal Q_y is *weighted least squares*.

Remark: Often unrealistic to assume Q_y is fully known – but may still be useful as a component of a larger hierarchical model.

Conjugate Priors

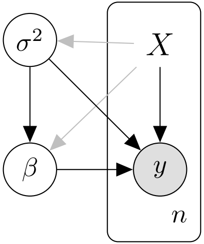
A possible informative prior is

$$\begin{aligned} \beta \mid \sigma^2, X &\sim \mathrm{N}\big(\beta_0,\, \sigma^2 K_0^{-1}\big) \\ \sigma^2 \mid X &\sim \mathrm{Inv}\text{-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

Values β_0, K_0, ν_0, and σ_0^2 may reflect prior information, and may depend on X.

This can be shown to be conjugate.

A particular example is *Zellner's g-prior*: $\quad K_0 = X^T X/g, \quad g$ constant



X is fully conditioned upon, hence treated as constant

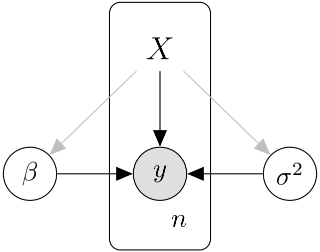
X is on the plate to represent that y_i directly depends on X only through X_i

A *semi*-conjugate prior is

$$\begin{aligned} \beta \mid X &\sim \mathrm{N}\big(\beta_0,\, \Sigma_\beta\big) \\ \sigma^2 \mid X &\sim \mathrm{Inv}\text{-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

Could alternatively replace either the β prior or the σ^2 prior with its improper noninformative prior.

Could put proper priors on only some elements of β (BDA3, Sec. 14.8).


X
β
y
σ²
n