Analysis on the Real Line

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Fall 2023

The Real Numbers

Introduction

- **Lemma**. The equation $x^2 - 2 = 0$ has no solution in $\mathbb{Q}$.

  Proof. By contradiction, assume there exists some $p/q \in \mathbb{Q}$, $p, q \in \mathbb{N}$, $q \neq 0$ such that
  $$\left(\frac{p}{q}\right)^2 - 2 = 0. \quad (*)$$
  Without loss of generality, we can assume that the greatest common divisor between p and q is 1. We rewrite (*) as $p^2 = 2q^2$ which implies that p^2 is even. This means that p is even as well. □

- We say that $\mathbb{N}$ is well-ordered, but not $\mathbb{Q}$ since $\mathbb{Q}$ does not have a least element.

- **Proposition**. There is no natural number such that $0 < n < 1$.

  Proof. Left as an exercise. □

Axioms of the Real Numbers

- Binary operations:
  - $(+)$: $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \qquad x, y \in \mathbb{R},\ x + y \in \mathbb{R}$,
  - $(\cdot)$: $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \qquad x, y \in \mathbb{R},\ x \cdot y \in \mathbb{R}$.
- We have axioms for the real numbers as follows:

  I Algebraic Axioms

  (i) Associativity: For $x, y, z \in \mathbb{R}$, we have
  $$x + (y + z) = (x + y) + z$$
  $$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

(ii) Commutativity: For $x, y \in \mathbb{R}$, we have

$$x + y = y + x$$
$$x \cdot y = y \cdot x$$

(iii) Distributivity: For $x, y, z \in \mathbb{R}$, we have

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

(iv) Identity:

- Addition: There exists some $0 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x + 0 = x$.
- Multiplication: There exists some $1 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \cdot 1 = x$.

(v) Inverses:

- Addition: For all $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that

$$x + y = 0 \iff y = -x.$$

- Multiplication: For all $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that

$$x \cdot y = 1 \iff y = \frac{1}{x} \iff y = x^{-1}.$$

II Ordering

(i) For some $x, y, z \in \mathbb{R}$, we have

$$x \leq y \implies x + z \leq y + z.$$

(ii) For some $x, y \in \mathbb{R}$, we have

$$0 \leq x, 0 \leq y \implies 0 \leq x \cdot y.$$

(iii) For some $x, y, z \in \mathbb{R}$, we have

$$x \leq y, y \leq z \implies x \leq z.$$

(iv) For some $x, y \in \mathbb{R}$, we have

$$x \leq y, y \leq x \implies x = y.$$

(v) For some $x, y \in \mathbb{R}$, we have

$$x \neq y \implies x \leq y \text{ or } y \leq x.$$

III No Hole (not satisfied by $\mathbb{Q}$)

For any non-empty subset X of $\{x > 0\}$, there exists some $a \in \mathbb{R}$ such that

(1) $a \leq x \ \forall x \in X$, and

(2) For all $\varepsilon > 0 (\varepsilon \neq 0, 0 \leq \varepsilon)$, there exists some x_ε such that $x_\varepsilon - a \leq \varepsilon$.

Bounded Intervals

- Given $a < b$, the set X of real numbers is a bounded interval if it is in the form:
  - Open interval with endpoints a, b
  $$\{x \in X \mid a < x < b\} = (a, b).$$
  - Closed intervals
  $$\{x \in X \mid a \leq x \leq b\} = [a, b].$$
  - Half-open intervals
  $$\{x \in X \mid a < x \leq b\} = (a, b].$$

Supremum and Infimum

- **Definition**. Let $X \subseteq \mathbb{R}$ be non-empty. A number $m \in \mathbb{R}$ (not necessarily in X) is a **lower bound** for X if for all $x \in X$, $m \leq x$.

  If a set has a lower bound, it is called **bounded below** (for instance, $\mathbb{Z}$).
- **Remark**. X either has no lower bounds or infinitely many.
- **Definition**. Let $X \subseteq \mathbb{R}$ be non-empty. A number $s \in \mathbb{R}$ (not necessarily in X) is called an **infimum** of the set X if it is
  (1) A lower bound and
  (2) For all other lower bounds r_i of X, $s \geq r_i$.

  The infimum is hence known as the "greatest lower bound." We denote "r is the infimum of X" as
  $$r = \inf(X).$$
- **Definition**. Let $X \subseteq \mathbb{R}$ be non-empty. A number $s \in \mathbb{R}$ (not necessarily in X) is called a **supremum** of the set X if it is
  (1) An upper bound and
  (2) For all other upper bounds r_i of X, $s \leq r_i$.

  The supremum is hence known as the "least upper bound." We denote "r is the supremum of X" as
  $$r = \sup(X).$$
- **Theorem**. Let $X \subseteq \mathbb{R}$ be non-empty. X has an upper bound if and only if X has a finite supremum and the finite supremum is unique.

Proof. If X has a supremum then that supremum is an upper bound for X. Assume that b is an upper bound of X. Let $\overline{X} = \{b+1-y \mid y \in X\}$, then $\overline{X}$ is non empty and $\overline{X} \subset \{x > 0\}$ (there is a 1 so it is surely greater than 0).

So, by Axiom 3, there is a real number a such that

(1) If $z \in \overline{X}$, then $a \leq z$ (i.e., a is a lower bound for $\overline{X}$) and
(2) For every $\varepsilon > 0$, there exists a $z_\varepsilon \in \overline{X}$ such that

$$z_\varepsilon - a \leq \varepsilon.$$

Claim. $b_1 = b + 1 - a$ is the supremum for X.

There are two things to check:

(1) Is b_1 an upper bound for X? Well, if $y \in X$, then $a \leq b + 1 - y$ and we have $y \leq b + 1 - a = b_1$.
(2) Is b_1 the least upper bound?

    Let M be some upper bound for X. Let us show that $b_1 \leq M$. By way of contradiction, assume that $M < b_1 = b + 1 - a$. Let $\varepsilon = b_1 - M > 0$.

    By Axiom 3, there exists some $z \in \overline{X}$ such that $z - a \leq \varepsilon$. We have

$$z = b + 1 - y, y \in X, \qquad \varepsilon = b_1 - M = b + 1 - a - M.$$

    So, supposedly $z - a = b + 1 - y - a \leq b + 1 - a - M$ with $M \leq y$ (this is a contradiction since M is assumed to be an upper bound, it cannot be smaller than y) and therefore $b_1 \leq M$ and $b_1 = \sup(X)$.

□

- **Archimedean Axiom** (satisfied in $\mathbb{R}$). For every $x > 0, y \geq 0$, there exists $n \in \mathbb{N}_*$ such that

$$n \cdot x > y.$$

Proof. By way of contradiction, assume that $n \cdot x \leq y \; \forall n \in \mathbb{N}_*$ (for some $x > 0, y \geq 0$). Then let $S = \{n \cdot x \mid n \in \mathbb{N}_*\} \subset \mathbb{R}$. Clearly S is bounded above by y and S is non-empty. This implies that $\sup(S)$ exists and $n \cdot x \leq \sup(s) \; \forall n \in \mathbb{N}_*$. This all implies that $(m+1) \cdot x \leq \sup(S) \; \forall m \in \mathbb{N}_*$ □

- **Lemma**. Let $X \subset \mathbb{R}$ be non-empty and bounded above. Then for every $\varepsilon > 0$, there exists some x_ε such that

$$\sup(X) - \varepsilon < x_\varepsilon \leq \sup(X).$$

Proof. Since $\sup(X)$ is an upper bound for X, then we know $x \leq \sup(X)\ \forall x \in X$. This implies the second part of the inequality.

To prove the first part of the inequality, assume by way of contradiction that there exists some $\varepsilon_0 > 0$ such that for every $x \in X$, $\sup(x) - \varepsilon_0 \geq x$. However, this is not possible since $\sup(x)$ is an upper bound, i.e., subtract anything from it and it will be less than x. $\square$

- **Lemma**. Let $x, y \in \mathbb{R}$ with $y > x$, then there exists some $r \in \mathbb{Q}$ such that $x < r < y$.

  Proof. Take $y - x > 0$, the Archimidean principle guarantees the existence of $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1 \iff y - x > \frac{1}{n}.$$

  Recall,

$$\begin{aligned} \lfloor x \rfloor \leq x \leq & \rfloor x \lfloor +1 \\ \lfloor nx \rfloor \leq \frac{nx}{n} & < \frac{nx+1}{n}. \end{aligned}$$

  Let $r := {}^{(\lfloor nx \rfloor + 1)}/_{n} \in \mathbb{Q}$. Then we have

$$\begin{aligned} x = \frac{nx}{n} &< \frac{\lfloor nx \rfloor + 1}{n} \\ &\leq \frac{nx+1}{n} \\ &= x + \frac{1}{n} \\ &= x + y - x \\ &< y. \end{aligned}$$

  $\square$

- **Corollary.** Let $p, q \in \mathbb{Q}$ with $p < q$. There exists some $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $p < z < q$.

  Proof. By Lemma,

$$\begin{aligned} \frac{p}{c} < \frac{q}{c} &\iff \frac{p}{c} < \underbrace{r}_{\in \mathbb{Q}} < \frac{q}{c} \\ &\iff p < \underbrace{r \cdot c}_{\in \mathbb{R} \setminus \mathbb{Q}} < q. \end{aligned}$$

  $\square$

Absolute Value

- **Definition**. For some $x \in \mathbb{R}$, the **absolute value** function is defined as follows:
$$|x| := \begin{cases} -x & \text{, if } x < 0 \\ x & \text{, if } x \geq 0. \end{cases}$$
- **Proposition**. Let $x, a \in \mathbb{R}$. Then we have
  (i) $|x| = 0 \iff x = 0$.
  (ii) Let $a > 0$, then
    - $|x| < a \iff -a < x < a$

Sequences

Convergence

- **Definition**. A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is said to be **convergent** if there exists $x \in \mathbb{R}$ such that $\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbb{N}$ such that
$$|x - x_n| \leq \varepsilon\ \forall n \geq N(\varepsilon).$$
Furthermore, we write $\lim_{n\to\infty} x_n = x$.
- **Example**. Show that $\lim_{n\to\infty} 1 + 1/(n+1) = 1$. We want
$$|x - x_n| = \left|1 - \left(1 + \frac{1}{n+1}\right)\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} \leq \varepsilon\ \forall \varepsilon > 0.$$
Let $\varepsilon > 0$, set $N(\varepsilon) = \lfloor 1/\varepsilon \rfloor = N(\varepsilon)$. This holds for
$$\begin{aligned} |x - x_n| = \frac{1}{n+1} &\leq \frac{1}{\lfloor \frac{1}{\varepsilon} \rfloor + 1} \\ &\leq \frac{1}{\frac{1}{\varepsilon}} \leq \varepsilon. \end{aligned}$$
- **Lemma**. A sequence of real numbers has **at most** one limit.

  Proof. Assume $l_1, l_2 \in \mathbb{R}$ are two limits. We will show that $l_1 = l_2$ by showing that for all $\varepsilon > 0$, $|l_1 - l_2| \leq \varepsilon$. Let $\varepsilon > 0$.

  (1) $(x_n)_{n\in\mathbb{N}}$ converges to l_1, i.e., for all $\varepsilon_1 > 0\ \exists N_1(\varepsilon_1)$ such that
  $$|x_n - l_1| \leq \varepsilon_1\ \forall n \geq N_1(\varepsilon_1),$$
  (2) $(x_n)_{n\in\mathbb{N}}$ converges to l_2, i.e., for all $\varepsilon_2 > 0\ \exists N_2(\varepsilon_2)$ such that
  $$|x_n - l_2| \leq \varepsilon_2\ \forall n \geq N_2(\varepsilon_2).$$

Let $\varepsilon > 0$ and set $\varepsilon_1 - {}^{\varepsilon}/_2, \varepsilon_2 = {}^{\varepsilon}/_2$. Then we have

$$\begin{aligned} |l_1 - l_2| &= |l_1 - x_n + x_n - l_2| \\ &\leq |l_1 - x_n| + |l_2 - x_n| \leq \varepsilon \end{aligned}$$

where $|l_1 - x_n| \leq \varepsilon_1 \ \forall n \geq N_1$ and $|l_2 - x_n| \leq \varepsilon_2 \ n \geq N_2 \ n \geq \max\{N_1, N_2\}$. Altogether, this implies that $l_1 = l_2$. $\square$

- **Example of Divergent Sequence**. Take $x_n = n \cdot \sin(n \cdot {}^{\pi}/_2)$. By way of contradiction, assume that $(x_n)_{n\in\mathbb{N}}$ is converging to some $x \in \mathbb{R}$. Say

$$\begin{aligned} x_{p_n} &= x_{4n+1} = (4n+1) \cdot \underbrace{\sin((4n+1) \cdot \frac{\pi}{2})}_{=1} \\ &= 4n+1 \\ x_{q_n} &= x_{4n+5} = (4n+5) \cdot \underbrace{\sin((4n+5) \cdot \frac{\pi}{2})}_{=1} \\ &= 4n+5 \end{aligned}$$

which implies that $|x_{p_n} - x_{x_n}| = 4$. From the contradiction assumption, for $\varepsilon = {}^1/_2$ there exists $N({}^1/_2)$ such that

$$|x_n - x| \leq \frac{1}{2} \ \forall n \geq N(\frac{1}{2}).$$

Take n such that $4n + 1 \geq N({}^1/_2)$. Then we have

$$\begin{aligned} |x_{p_n} - x| &\leq \frac{1}{2} \\ |x_{q_n} - x| &\leq \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 4 &= |x_{p_n} - x_{q_n}| = |x_{p_n} - x + x - x_{q_n}| \\ &\leq |x_{p_n} - x| + |x_{q_n} - x| \\ &\leq \frac{1}{2} + \frac{1}{2} \leq 1 \end{aligned}$$

but this is a contradiction, so $(x_n)_{n\in\mathbb{N}}$ must be divergent.

- **Lemma**. Every convergent sequence is bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence, that is, there exists some $x \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists some $N(\varepsilon) \in \mathbb{N}$ and

$$|x - x_n| \leq \varepsilon \ \forall n \geq N(\varepsilon).$$

In particular, take for $\varepsilon = 1$, there is $N(1)$ such that

$$|x_n - x| \leq 1 \ \forall n \geq N(1).$$

Recall the reverse triangle inequality,

$$\begin{aligned} &|x_n| - |x| \leq |x - x_n| \ \leq 1 \quad n \geq N(1) \\ \implies &|x_n| \leq 1 + |x| \quad n \geq N(1) \end{aligned}$$

Let $M = \max\{|x_0|, |x_1|, \ldots, |x_{N(1)|, 1+|x|}$. Then certainly

$$\begin{aligned} |x_n| &\leq M \qquad n \leq N(1) \\ |x_n| &\leq M \qquad n \geq N(1) \\ \implies |x_n| &\leq M \qquad \forall n \in \mathbb{N}. \end{aligned}$$

□

- **Recall**. A sequence $(x_n)_{n\in\mathbb{N}}$ convergent $\implies \exists M \in \mathbb{N}$ such that $|x_n| \leq M \ \forall n \in \mathbb{N}$.
- **Definition**. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be
  - **Increasing** if $x_m \geq x_n$ when $m \geq n$,
  - **Decreasing** if $x_m \leq x_n$ when $m \geq n$,
  - **Monotone** if $(x_n)_{n\in\mathbb{N}}$ is increasing or decreasing.
- **Lemma**. Any increasing sequence that is bounded above is convergent.

Proof. Let $X = \{x_n \mid n \in \mathbb{N}\}$. Then $x_n \leq M, n \in \mathbb{N} \implies X$ bounded above $\implies x = \sup(X)$.

We want to show that $(x_n)_{n\in\mathbb{N}}$ converges to $x = \sup(X)$. This means (by definition) that for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon \ \forall n \geq N(\varepsilon).$$

Note that $x = \sup(X)$ implies two important things:

(i) $x_n \leq x \ \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$, $\exists x_\varepsilon \in X$ such that $x - x_\varepsilon \leq \varepsilon$ by the characterization of supremum. Note, we can equivalently replace x_ε with $x_{n_\varepsilon} \in \mathbb{N}$ to obtain $\forall \varepsilon > 0$, $\exists x_{n_\varepsilon} \in X$ such that $x - x_{n_\varepsilon} \leq \varepsilon$.

So, let $\varepsilon > 0$, compute

$$\begin{aligned} |x_n - x| &= x - x_n \\ &\leq x - x_{n_\varepsilon} \leq \varepsilon \end{aligned}$$

with $x_{n_\varepsilon} \leq x_n$ and $n \geq n_\varepsilon$ ($(x_n)_{n\in\mathbb{N}}$ increasing) and for all $\varepsilon > 0$, take $N(\varepsilon) = n_\varepsilon$. □

- **Corollary.** A decreasing sequence $(x_n)_{n\in\mathbb{N}}$ that is bounded below is convergent.
- **Remark**. There are two cases:
  (i) $(x_n)_{n\in\mathbb{N}}$ increasing, bounded above $\implies \lim\limits_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$,
  (ii) $(x_n)_{n\in\mathbb{N}}$ decreasing, bounded below $\implies \lim\limits_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$.
- **Example**. Take $x_n = 2^{-n}$ decreasing, bounded below. Then
$$\lim_{n\to\infty} x_n = \inf\{2^{-n} \mid n \in \mathbb{N}\} = 0.$$
- **Corollary**. Let $(x_n)_{n\in\mathbb{N}}$ be a monotone sequence. Then $(x_n)_{n\in\mathbb{N}}$ is convergent if and only if $(x_n)_{n\in\mathbb{N}}$ is bounded.

Algebraic Manipulations of Limits

- **Lemma**. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be convergent sequences. Set $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$. Three things are true:
  (1) The sequence $(x_n + y_n)_{n\in\mathbb{N}}$ is convergent and $lim_{n\to\infty}(x_n + y_n) = x + y$,
  (2) The sequence $(x_n \cdot y_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$,
  (3) Assume $x_n \neq 0, y_n \neq 0$, then the sequence $({}^{y_n}/_{x_n})_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}({}^{y_n}/_{x_n}) = {}^{y}/_{x}$.

Proof. of (2). We want to show that $\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbb{N}$ such that
$$|x_n y_n - xy| \leq \varepsilon\ \forall n \geq N(\varepsilon).$$
Note that
$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + x(y_n - y)| \\ &\leq |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$
by the triangle inequality.

- **Case 1:** $x \neq 0$. Because $(y_n)_{n\in\mathbb{N}}$ is convergent, there exists $N_1 \in \mathbb{N}$ such that $|y_n - y| \leq {}^{\varepsilon}/_{2\cdot|x|}\ \forall n \geq N_1$. Also, $(y_n)_{n\in\mathbb{N}}$ is bounded, i.e., $|y_n| \leq M\ \forall n \in \mathbb{N}\ \exists M \geq 1$.

  Because $(x_n)_{n\in\mathbb{N}}$ is convergent, there exists some $N_2 \in \mathbb{N}$ such that
  $$|x - x_n| \leq \frac{\varepsilon}{2M}\ \forall n \geq N_2.$$
  Therefore, for all $\varepsilon > 0$, we set $N_\varepsilon = \max\{N_1, N_2\}$ and we have that
  $$|x_n y_n - xy| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon\ \forall n \geq N_\varepsilon$$
  which ultimately shows that $\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$.

– **Case 2: $\mathbf{x = 0}$**. Obviously it is true that

$$|x_n y_n - xy| = |x_n y_n| \leq |x_n||y_n|.$$

Then, consider that

* $(y_n)_{n\in\mathbb{N}}$ convergent implies $|y_n| \leq M$, and
* $(x_n)_{n\in\mathbb{N}}$ convergent to 0 implies $\forall\varepsilon > 0\ \exists N_1 \in \mathbb{N}$ such that $|x_n| \leq \varepsilon/2M\ \forall n \geq N_1$. Again, we have

  $$x_n y_n - xy| \leq \frac{\varepsilon}{2M} \cdot M \leq \frac{\varepsilon}{2} \leq \varepsilon\ \forall n \geq N_1$$

  which implies that $\lim_{n\to\infty} x_n y_n = 0$.

□