$1)Let\ X_1, X_2, \cdots, X_n\ be\ a\ random\ sample\ from\ distribution$

$\begin{array}{c|cc} x_i & 0 & 1 \\ \hline P & 1-p & p \end{array}$ $using\ the\ Method\ of\ Moments\ find\ estimator\ of\ p.$

$After\ a\ series\ of\ n = 10\ trials\ this\ observations\ where\ made$

$0,1,1,0,1,1,1,0,0,1\ find\ the\ estimate\ of\ p.$

$$\begin{cases} M_1 = E(X) = \bar{X} = p \\ M_2 = E(X^2) = \frac{1}{n}\sum X_i^2 = p \end{cases}$$

$$M_1 - M_2 \implies 0 = \bar{X} - \frac{1}{n}\sum X_i^2$$

$$\hat{p}_{MM} = \bar{X} = \frac{1}{n}\sum X_i^2$$

$$L(p) = p^{\sum X_i} \cdot (1-p)^{n-\sum X_i}$$

$$\hat{p}_{MLE} = \frac{1}{n}\sum X_i = \bar{X}$$

$$\because \{x_1, \ldots, x_n\} = \{0,1,1,0,1,1,1,0,0,1\}$$

$$\therefore \hat{p}_{MLE} = \frac{6}{10}, \hat{p}_{MM} = \frac{6}{10}$$

$2)Let X_1, X_2, \cdots, X_n\ be\ a\ random\ sample\ from\ distribution$

$\begin{array}{c|ccc} x_i & 0 & 1 & 3 \\ \hline P & 1-p_1-p_2 & p_1 & p_2 \end{array}$ using the Method of Moments find estimators of $p_1\ and\ p_2$

After a series of n $= 10$ trials this observations where made 0, 1, 3, 0, 1, 1, 1, 3, 3, 1 find the estimates of $p_1\ and\ p_2$. Are the obtained estimators unbiased?

$$\begin{cases} M_1 = E(X) = \bar{X} = p_1 + 3p_2 \\ \\ M_2 = E(X^2) = \frac{1}{n}\sum X_i^2 = p_1 + 9p_2 \end{cases} \quad \begin{cases} M_2 - M_1 \implies p_2 = \frac{1}{6}(\frac{1}{n}\sum X_i^2 - \bar{X}) \\ \\ 3M_1 - M_2 \implies p_1 = \frac{1}{2}(3\bar{X} - \frac{1}{n}\sum X_i^2) \end{cases}$$

$$\because \{x_1, \ldots, x_n\} = \{0,1,3,0,1,1,1,3,3,1\}, \bar{X} = \frac{14}{10}, \frac{1}{n}\sum X_i^2 = \frac{32}{10}$$

$$\therefore \hat{p}_1 = \frac{1}{2}, \hat{p}_2 = \frac{3}{10}$$

$$S^2 = \frac{1}{n-1} \cdot \sum (X_i - \bar{X})^2 = \frac{62}{45}$$

$$E(X) = \frac{1}{2} + \frac{9}{10} = \frac{14}{10} = \bar{X}$$

$$D(x) = \frac{1}{2} + \frac{27}{10} = \frac{16}{5} > S^2$$

$$so, \hat{\theta}\ is\ better$$

3)Consider a random sample$X_1, X_2, \cdots, X_n$from the shifted exponential pdf

$$f(x)\begin{cases}\lambda e^{-\lambda(x-\theta)} & x \geq \theta \\ \\ 0 & x < \theta\end{cases}$$

a.Obtain the maximum likelihood estimators of θ and λ.

b. Obtain the method of moment estimators of θ and λ.

c. If n = 10 time observations are made,

resulting in the values 3.11, 0.64, 2.55, 2.20, 5.44, 3.42, 10.39, 8.93, 17.82, and1.30,

calculate the estimates of θ and λ.

$$a)L(\theta,\lambda)=\prod_{i=1}^{n}\lambda e^{-\lambda(x_i-\theta)}=\lambda^n\prod_{i=1}^{n}e^{-\lambda(x_i-\theta)}$$

$$\ln L=n\ln\lambda+(-\lambda)\sum_{i=1}^{n}(x_i-\theta)=n\ln\lambda+\lambda n\theta-\lambda\sum_{i=1}^{n}(x_i)$$

$$\begin{cases}\frac{\partial\ln L}{\partial\theta}=\lambda n>0 & \therefore \theta\uparrow, L(\theta)\uparrow \therefore \theta=X_{min} \implies \hat{\theta}_{MLE}=\{X_1,\ldots,X_n\}_{min} \\ \\ \frac{\partial\ln L}{\partial\lambda}=\frac{n}{\lambda}+n\theta-\sum_{i=1}^{n}x_i=0 & \therefore \lambda=\frac{n}{\sum_{i=1}^{n}x_i-n\theta}=\frac{1}{\bar{x}-\theta} \implies \hat{\lambda}_{MLE}=\frac{1}{\bar{x}-\hat{\theta}_{MLE}}\end{cases}$$

$$b)\mu_1=E(x)=\int_{-\infty}^{+\infty}xf(x)dx=\int_{\theta}^{+\infty}x\lambda e^{-\lambda(x-\theta)}dx=-xe^{-\lambda(x-\theta)}|_{\theta}^{+\infty}-\int_{\theta}^{+\infty}\lambda e^{-\lambda(x-\theta)}dx$$

$$=\lim_{x\to+\infty}xe^{-\lambda(x-\theta)}-(-\theta)+(-\frac{1}{\lambda}e^{-\lambda(x-\theta)}|_{\theta}^{+\infty})=0+\theta+(\lim_{x\to+\infty}e^{-\lambda(x-\theta)}-\frac{1}{\lambda})$$

$$=\theta+\frac{1}{\lambda}$$

$$\mu_2=E(x^2)=\int_{-\infty}^{+\infty}x^2f(x)dx=\int_{\theta}^{+\infty}x^2\lambda e^{-\lambda(x-\theta)}dx$$

$$=-x^2e^{-\lambda(x-\theta)}|_{\theta}^{+\infty}-\int_{\theta}^{+\infty}-e^{-\lambda(x-\theta)}d(x^2)$$

$$=\lim_{x\to+\infty}x^2e^{-\lambda(x-\theta)}-(-\theta^2)+2\int_{\theta}^{+\infty}x\lambda e^{-\lambda(x-\theta)}dx$$

$$=0+\theta^2+2(\theta+\frac{1}{\lambda})=\theta^2+2\theta+\frac{2}{\lambda}$$

$$A_1=\bar{x}=\mu_1 \implies \hat{\theta}_{MM}=\bar{x}-\frac{1}{\lambda}$$

$$A_2=\frac{1}{n}\sum x_i^2=\mu_2 \implies \hat{\lambda}_{MM}=\frac{2}{\frac{1}{n}\sum x_i^2-\theta^2-2\theta}$$

$$c)\hat{\theta}_{MLE}=min(X)=0.64, \hat{\lambda}_{MLE}=\frac{1}{\bar{x}-\hat{\theta}_{MLE}}=\frac{1}{5.58-0.64}\approx 0.2$$

$$\hat{\theta}_{MM}=\infty, \hat{\lambda}_{MM}=0$$