In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

Project 2: Newton Methods

Introduction

Given a set of 2 test scores associated to each student in a current class, we want to predict through a simple model whether a student passes the course or not. We decide to use a linear model to separate who will pass from who will fail.

We use the historical test scores obtained from m past students who passed/failed this course in previous years. We input these historical scores as m data vectors such that $\overline{X}_i = (x_1^{(i)}, x_2^{(i)})$ contains the 2 midterm scores of the ith student $1 \leq i \leq m$. We define a set of associated binary target values $y_i \in \{-1, +1\}$ ($+1$ passed, -1 failed) $1 \leq i \leq m$. The goal then is to find an affine function

$$f(x_1, x_2) = w_1 x_1 + w_2 x_2 + w_3 = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \overline{W} \cdot \overline{X}^T,$$

so that f may be used to predict the outcome of a different (yet unseen) score (z_1, z_2) by

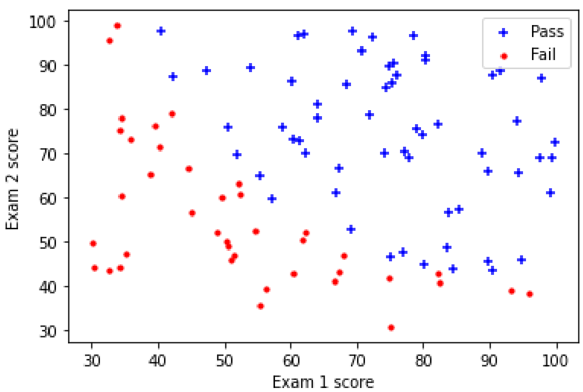
$$f(z_1, z_2) < 0 \rightarrow \text{fail}$$
$$f(z_1, z_2) > 0 \rightarrow \text{pass}$$

Data set and associated Loss Functionals

Suppose you are given the historical data set in the cell/code snippet below

In [16]:
```
# Load and plot the data set
df=pd.read_csv("ex2data1.txt",header=None)
X=df.iloc[:,:-1].values
y=df.iloc[:,-1].values
y=2*y-1 # the original y-values are 0,1 this makes them be -1 or 1.

pos , neg = (y==1).reshape(100,1) , (y==-1).reshape(100,1)
plt.scatter(X[pos[:,0],0],X[pos[:,0],1],c="b",marker="+")
plt.scatter(X[neg[:,0],0],X[neg[:,0],1],c="r", marker="o", s =10)
plt.xlabel("Exam 1 score")
plt.ylabel("Exam 2 score")
plt.legend(["Pass","Fail"],loc=0)
```

Out[16]: <matplotlib.legend.Legend at 0x7f00b789e430>



Introduce the loss functionals

$$J_1(\overline{W}) = \sum_{i=1}^{m} (1 - y_i \overline{W} \cdot \overline{X}_i^T)^2 \quad (\text{linear regression}) \tag{1}$$

$$J_2(\overline{W}) = \sum_{i=1}^{m} \log(1 + e^{-y_i \overline{W} \cdot \overline{X}_i^T}) \quad \text{(logistic regression)} \tag{2}$$

$$J_3(\overline{W}) = \sum_{i=1}^{m} \log(1 + e^{-y_i \overline{W} \cdot \overline{X}_i^T}) + \frac{\lambda}{2} \|\overline{W}\|_2^2 \quad \text{(regularized logistic regression)} \tag{3}$$

1. (40 pts)

Calculate the gradients and the Hessians of J_1, J_2 and J_3, and represent/program each of these as a Python function. Note your function should take as argument variables $\overline{W}$, input data points $\overline{X}_i$, and corresponding y_i values. You can put all the $\overline{X}_i$ into a single X (the ith row of X is $\overline{X}_i$) and all y_i into a vector y, whose ith element is y_i. A possible function prototype outline is

In [17]:
```
# Note students have to append the 1 to X

m , d = X.shape[0], X.shape[1]
X= np.append(np.ones((m,1)),X,axis=1)
y=y.reshape(m,1)
```

In [58]:
```
def CostLinReg(W, X, y): # 10 pts TOTAL correct CostLinReg
    """
    Takes in numpy array W, X and y and returns the corresponding cost function, gradient
    and hessian
    """

    # The W parameter is our initial Guess. X is the data matrix D in [Aggarwal] notation

    m=len(y)
    difference=X@W-y
    cost = difference.T@difference # 2 pts
    grad = X.T@X@W-X.T@y # 4
    hess =X.T@X # 4pts

    return cost , grad, hess
```

In [59]:
```
 def sigmoid(z):
    return 1/ (1 + np.exp(-z))

def CostLogReg(W, X, y):  # 20 PTS TOTAL
    """
    Takes in numpy array W, X and y and returns the corresponding cost function, gradient
    and hessian
    """

    # The W parameter is our initial Guess. X is the data matrix D in [Aggarwal] notation

    predictions = sigmoid(np.multiply(-y,np.dot(X,W))) #<---the matrix P in my Latex notes NOTE: np.multiply .
    error =  -np.log(sigmoid(np.multiply(y,np.dot(X,W)))) #note it is -log(sigmoid), not log(sigmoid)!
    P=np.diagflat(predictions)
    cost = sum(error)

    grad =  -X.T@P@y #10 pts gradient
    hess = np.dot(X.T, predictions*(1-predictions)*X) #<---this was causing the probblem! #10 pts Hessian



    return cost, grad, hess
```

In [87]:
```
def CostLogR2Reg(W, lambd, X, y): #10 pts TOTAL
    """
    Takes in numpy array W, X and y and returns the corresponding cost function, gradient
    and hessian
    """
    d=W.shape[0]
    I=np.eye(d)
    # The W parameter is our initial Guess. X is the data matrix D in [Aggarwal] notation
    cost, grad, hess=CostLogReg(W,X,y)
    cost+=(lambd*W.T@W)[0]
    grad+=2*lambd*W
    hess+=2*lambd*I
```

```
    return cost, grad, hess
```

2. (10 pts)

Are these functionals convex? Explain.

3. (30 pts)

Using the functions you wrote in part 1., implement Newton's method to find the approximate solutions to

$$\nabla J_i(\overline{W}) = 0 \quad \text{(the respective minima)} \tag{1}$$

For a stopping criterion, use

$$\frac{|J_i(\overline{W}_{k+1}) - J_i(\overline{W}_k)|}{J_i(\overline{W}_{k+1})} \leq 10^{-7} \tag{2}$$

You may want to try different initial guesses $\overline{W}_0$, and you may want to try different (relatively small) values for λ (the regularization parameter).

In [75]:

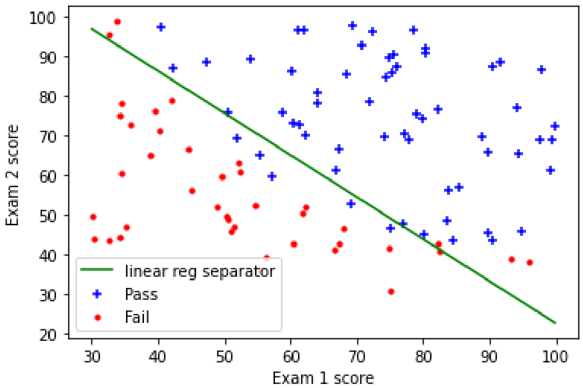
```
 # Your code here
def newton_method(costFunction, X, y, W, tol=1e-8, max_it=100):
    num_it = 0
    cost_prev = 0
    rel_error=1
    cost, grad, hess = costFunction(W,X,y)
    while rel_error>=tol and num_it<max_it: # <---10 pts for putting correct tolerance in a WHILE loop
        # J(x) * delta = fun(x)  <==> delta = J^{-1}fun(x)
        cost_prev = cost
        print("cost is now: ", cost)
        delta = np.linalg.solve(hess, grad) # <---10pts correct delta to add to W_{k} to get W_{k+1}
        W -= delta
        cost, grad, hess = costFunction(W,X,y) #<---reevaluate with new W 5 pts
        rel_error = np.abs(cost-cost_prev)/cost #<---10 pts correct relative error with correct prev value
        num_it+=1
        print("Finished step ", num_it)
    if rel_error<tol:
            print('Newton method converges in %d iteration.' %num_it) #<
    else:
            print('Newton method does not converge in %d iteration.' %max_it)

    return W
```

In [76]:

```
initial_W = np.ones((d+1,1))
cost, grad, hess= CostLinReg(initial_W,X,y)
print("Cost of initial theta is",cost)

W0 = newton_method(CostLinReg,X, y, initial_theta)
print('lin regression W is ', W0)

# plot decision boundary with parameters theta in part c)

plt.scatter(X[pos[:,0],1],X[pos[:,0],2],c="b",marker="+")
plt.scatter(X[neg[:,0],1],X[neg[:,0],2],c="r", marker="o", s =10)
x_value= np.array([np.min(X[:,1]),np.max(X[:,1])])
y_value_lin_reg=-(theta0[0] +theta0[1]*x_value)/W0[2]
plt.plot(x_value,y_value_lin_reg, "g")
plt.xlabel("Exam 1 score")
plt.ylabel("Exam 2 score")
plt.legend(["linear reg separator","Pass","Fail"],loc=0)
```

```
Cost of initial theta is [[1826071.83352504]]
cost is now:  [[37.7975725]]
Finished step  1
Newton method converges in 1 iteration.
lin regression W is  [[-3.59499387]
 [ 0.02968175]
 [ 0.02788434]]
```

Out[76]: <matplotlib.legend.Legend at 0x7f00b6646fd0>

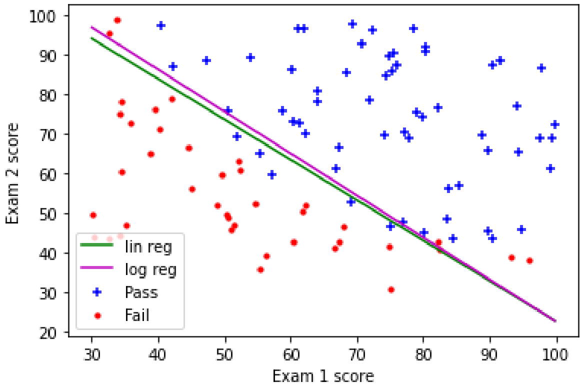


In [80]:

```
# Logistic regression cost function
initial_W = np.zeros((d+1,1))
initial_W[0]=1
initial_W[1]=0
initial_W[2]=0
cost, grad, hess= CostLogReg(initial_W,X,y)
print("Cost of initial theta is",cost)


W_logistic = newton_method(CostLogReg,X, y, initial_W)


print("\n********************************************* ")
print("After Newton method: ")
print("********************************************* ")

new_cost, new_grad, new_hess= CostLogReg(W_logistic,X,y)
print("Cost is", new_cost)

print('W for logistic=', W_logistic)

# plot decision boundary for logistic AND LINEAR REG boundary to compare

plt.scatter(X[pos[:,0],1],X[pos[:,0],2],c="b",marker="+")
plt.scatter(X[neg[:,0],1],X[neg[:,0],2],c="r", marker="o", s =10)
x_value= np.array([np.min(X[:,1]),np.max(X[:,1])])
y_value_log=-(W_logistic[0] +W_logistic[1]*x_value)/W_logistic[2]

plt.plot(x_value,y_value_log, "g")
plt.plot(x_value,y_value_lin_reg, "m")  #<---note I plot linear regression's separating line to compare
plt.xlabel("Exam 1 score")
plt.ylabel("Exam 2 score")
plt.legend(["lin reg", "log reg","Pass","Fail"],loc=0)
```

```
Cost of initial theta is [71.32616875]
cost is now:  [71.32616875]
Finished step  1
cost is now:  [29.86287778]
Finished step  2
cost is now:  [23.38033978]
Finished step  3
cost is now:  [20.92125495]
Finished step  4
cost is now:  [20.3837612]
Finished step  5
cost is now:  [20.3499451]
Finished step  6
cost is now:  [20.34977016]
Finished step  7
Newton method converges in 7 iteration.

********************************************
After Newton method:
********************************************
Cost is [20.34977016]
W for logistic= [[-25.16133355]
 [  0.20623171]
 [  0.2014716 ]]
```

Out[80]: <matplotlib.legend.Legend at 0x7f00b644cee0>



In [99]:

```
# Just for ease, since above functions don't require a lambda parameter and this regularized Tikhonov does.
# we modify its corresponding Newton method. Other options would have been something like if lambda=0 or
# if lambda=FALSE, call Newton in a certain manner and otherwise

def newton_method2(costFunction, X, y, W, lambd, tol=1e-8, max_it=100):
    num_it = 0
    cost_prev = 0
    rel_error=1
    cost, grad, hess = costFunction(W, lambd, X,y)
    while rel_error>=tol and num_it<max_it:
        # J(x) * delta = fun(x)  <==> delta = J^{-1}fun(x)
        cost_prev = cost
        print("cost is now: ", cost)
        delta = np.linalg.solve(hess, grad)
        W -= delta
        cost, grad, hess = costFunction(W,lambd, X,y)
        rel_error = np.abs(cost-cost_prev)/cost
        num_it+=1
        print("Finished step ", num_it)
    if rel_error<tol:
            print('Newton method converges in %d iteration.' %num_it) #<
    else:
            print('Newton method does not converge in %d iteration.' %max_it)

    return W



lambd=0.01

initial_theta = np.zeros((d+1,1))
cost, grad, hess= CostLogR2Reg(initial_theta, lambd,X,y)
print("Cost of initial theta is",cost)


W_r2 = newton_method2(CostLogR2Reg,X, y, initial_theta, lambd)


print("\n******************************************** ")
print("After Newton method: ")
print("******************************************** ")



print('W for Tikhonov logistic=', W_r2)
CostLogR2Reg(W_r2, lambd, X, y)
```

```
Cost of initial theta is [69.31471806]
cost is now:  [69.31471806]
Finished step  1
cost is now:  [33.92631157]
Finished step  2
cost is now:  [26.77844604]
Finished step  3
cost is now:  [24.80156875]
Finished step  4
cost is now:  [24.5929192]
Finished step  5
cost is now:  [24.59031405]
Finished step  6
cost is now:  [24.59031362]
Finished step  7
```

```
Newton method converges in 7 iteration.

********************************************
After Newton method:
********************************************
W for Tikhonov logistic= [[-17.61811209]
 [  0.14597798]
 [  0.14042727]]
```

Out[99]:

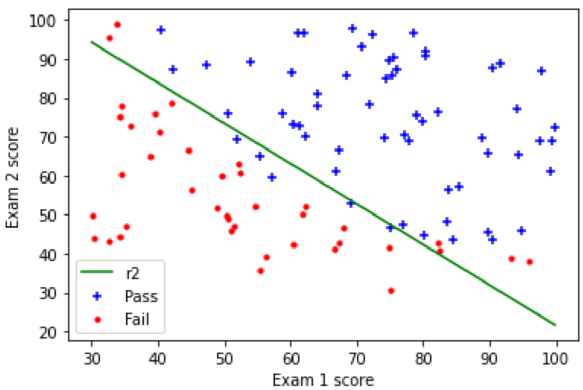
```
(array([24.59031362]),
 array([[-5.38458167e-15],
        [-3.51229896e-13],
        [-3.70549078e-13]]),
 array([[8.65611758e+00, 5.37018734e+02, 5.29685065e+02],
        [5.37018734e+02, 3.59635727e+04, 3.09516772e+04],
        [5.29685065e+02, 3.09516772e+04, 3.49898883e+04]]))
```

In [100…

```
# plot decision boundary with parameters theta in part c)

plt.scatter(X[pos[:,0],1],X[pos[:,0],2],c="b",marker="+")
plt.scatter(X[neg[:,0],1],X[neg[:,0],2],c="r", marker="o", s =10)
x_value= np.array([np.min(X[:,1]),np.max(X[:,1])])
y_value_r2=-(W_r2[0] +W_r2[1]*x_value)/W_r2[2]

plt.plot(x_value,y_value_r2, "g")

plt.xlabel("Exam 1 score")
plt.ylabel("Exam 2 score")
plt.legend(["r2","Pass","Fail"],loc=0)
```

Out[100…

```
<matplotlib.legend.Legend at 0x7f00b60e8640>
```



4. (10 pts)

How many iterations were required for J_1, J_2 and J_3? Can you explain the count obtained for J_1?

ANSWER Q4: To get full marks, you must explain (3pts) 1 iteration (I give 2 pts for 2 iterations), and for the rest of points you must show you relate the gradient of J_1, $\nabla J_1(w) = X^T X w - X^T y$ and the Hessian $\mathrm{Hess}(J_1)(w) = X^T X$ and plug it into the Newton iteration step $w_{k+1} = w_k - (\mathrm{Hess}(J_1))^{-1} \nabla J_1(w_k)$

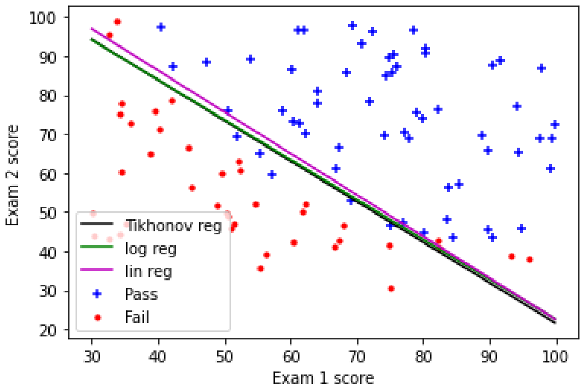
5. (10 pts)

Display the data with the "decision boundaries" obtained from the minimization of J_1, J_2 and J_3.

In [101…

```
# Plot here

# plot decision boundary fo rALL to compare

plt.scatter(X[pos[:,0],1],X[pos[:,0],2],c="b",marker="+")
plt.scatter(X[neg[:,0],1],X[neg[:,0],2],c="r", marker="o", s =10)

plt.plot(x_value,y_value_r2, "k")
plt.plot(x_value,y_value_log, "g")
plt.plot(x_value,y_value_lin_reg, "m")  #<---note I plot linear regression's separating line to compare
plt.xlabel("Exam 1 score")
plt.ylabel("Exam 2 score")
plt.legend(["Tikhonov reg", "log reg", "lin reg" , "Pass","Fail"],loc=0)
```

Out[101… <matplotlib.legend.Legend at 0x7f00b6076b20>



In [ ]: