In [1]:
```
import numpy as np
import numpy.linalg as LA
import scipy.linalg as la
import matplotlib.pyplot as plt
```

Project 1 SVD

The singular value decomposition (SVD) of an $m \times d$ matrix A is a factorization of the form

$$A = V\Sigma U^T \tag{1}$$

where V is an $m \times m$ orthogonal matrix and U is a $d \times d$ orthogonal matrix. Σ is an $m \times d$ diagonal matrix with nonnegative real numbers σ_i on the diagonal.

Alternatively this expresses A as a sum of rank 1 matrices

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{v}_i \mathbf{u}_i^T \tag{2}$$

where σ_i $i = 1, \ldots, r$ are the positive diagonal entries of Σ and $\mathbf{v}_i, \mathbf{u}_i$ are the ith column vectors of V and U, respectively.

Problems

Parts (a)-(c) can be handwritten somewhere else and attached as a separate file (or typed in Latex). If you know how to type Latex within a Jupyter notebook, you are welcome to type it here as long as its done in Latex notation.

a) [5pt]

Show that the columns of U form an orthonormal basis in $\mathbb{R}^d$ of eigenvectors of A^TA. Show that the columns of V form an orthonormal basis in $\mathbb{R}^m$ of eigenvectors of AA^T.

b) [5pt]

Use the fact that $U^TA^TAU = \Sigma^T\Sigma$ to show that

$$U_1^TA^TAU_1 = \Sigma_1^2 \tag{3}$$

where $U_1 = [\mathbf{u}_1 \;\; \mathbf{u}_2 \cdots \mathbf{u}_r]$ and Σ_1 is the $r \times r$ diagonal matrix with all positive σ_i on the diagonal. How does one find the $m \times r$ matrix

$$V_1 = [\mathbf{v}_1 \;\; \mathbf{v}_2 \cdots \mathbf{v}_r]$$

based on A, U_1 and Σ_1?

c) [20pt]

(Two simple examples) With

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 2 & 1/2 \\ 2 & 2 \end{bmatrix}$$

find the eigenvectors and eigenvalues of A_1 and A_2. Find matrices C_1 and C_2 such that

$$A_i = C_i\Sigma_iC_i^{-1} \quad i = 1, 2. \tag{1}$$

Use this decomposition to write A_1 and A_2 as a sum of rank 1 matrices. Find the SVD of A_1 and A_2. Can you explain the differences (and similarities) in what you found above?

d) [30 pts]