1 Optimization view of linear systems

Solving a linear system of equations is a special case of linear regression, which at the end of the day is an optimization problem. Solving a system of equations

$$Ax = b$$

can be seen as achieving the minimum value

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 \quad \big(= 0 \text{ if an exact solution exists}\big) \tag{1}$$

As mentioned previously, this is **classical least-squares regression**, where the cost function is the square of the Euclidean norm $\| \cdot \|^2$.

If the system of equations has no solutions (is inconsistent), the optimization problem (1) returns the next best thing, which is equivalent to the smallest value of (1) possible. It will be the case that

$$\min_{x \in \mathbb{R}^n} J(x) = \min_{x \in \mathbb{R}^n} \|Ax - b\|^2 > 0 \leftarrow \text{ if no solution exists} \tag{2}$$

A value of x that achieves the minimum exists? Is it unique?

1.1 Geometric Solution the previous?

Of course, we can use calculus to solve (2), i.e., find the unique optimal value. But we can try the following geometric argument:

1. As you can see in 2-D (maybe in 3-D too), to get from a point $b \in \mathbb{R}^n$ to a hyperplane (in 2-D this is a straight line), you draw a line $\mathcal{L}$ from b to the hyperplane, such that $\mathcal{L}$ **is orthogonal to the hyperplane**.

2 Some assignment hints

In problem 4 we have the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

When does $A^T A$ have an inverse? Let us determine the rank of the column space of A, which for the moment let us denote $\mathbf{C}(A)$. We check if the columns are linearly independent because

$$\mathbf{rank}(A^TA) = \mathbf{rank}(AA^T) = \mathbf{rank}(A) \tag{3}$$

$$A \mapsto \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the column space has rank 2! Thus we cannot guarantee the inverse A^TA exists.