Lecture 9

Plate analysis

Floor and deck slabs

Print version Lecture on *Theory of Elasticity and Plasticity* of

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Contents

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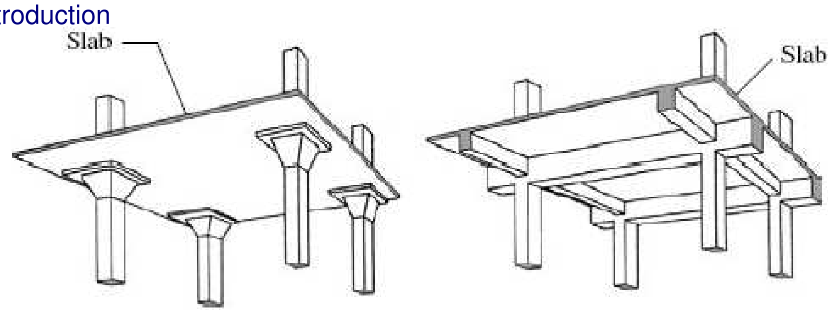
1 Introduction

Introduction



Civil engineering

- The floor slabs in buildings

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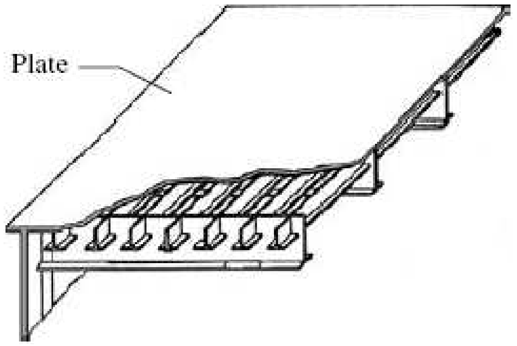
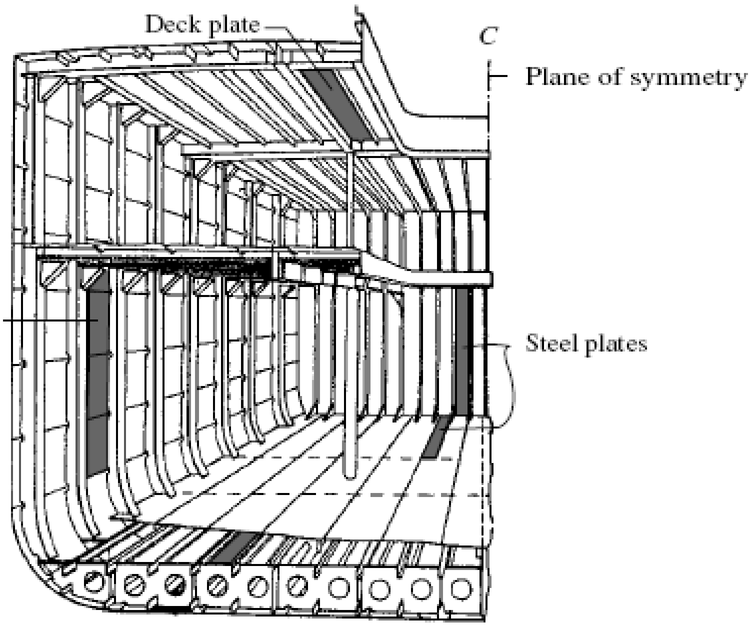
Introduction



Bridge engineering

- The deck slabs of bridges

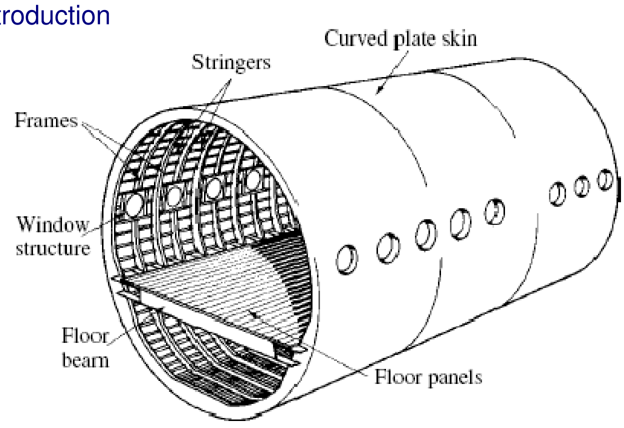
Introduction



Marine engineering

- The ship decks and hull

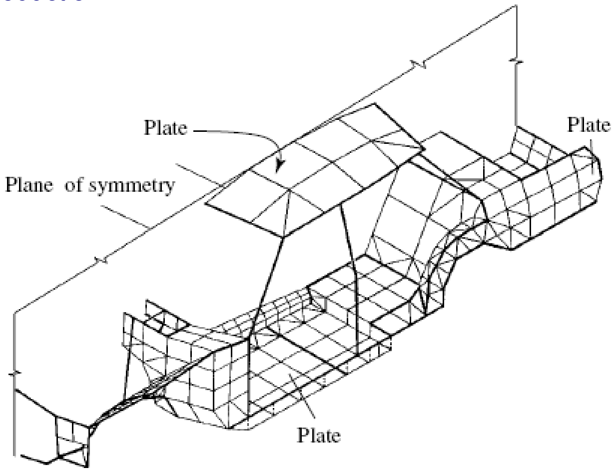
Introduction



Aircraft engineering

- The floor panels and fuselage

Introduction



Automotive engineering

- The car panels

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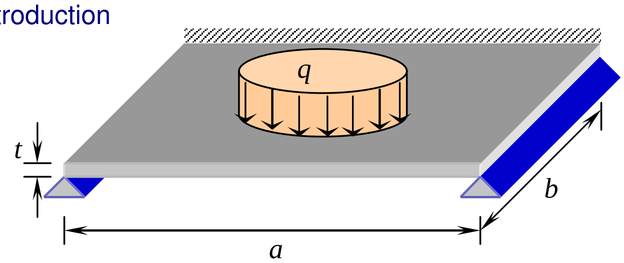
Introduction



Definition

- Plates are plane, 2-D structural components of which one dimension, called thickness t is much smaller than the other dimensions
- The plate loads are mainly transversal to the plane surface
- They are carried by internal bending and twisting moments and shear forces
- The plate edges can be simply supported, fixed or elastically restrained

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2 Assumptions

Assumptions

Classical plate theory

1. The plate is thin- $t \ll a, b$ and $\left(\frac{t}{a}, \frac{t}{b}\right) = \frac{1}{10} - \frac{1}{50}$
2. The in-plane strains are small compared to the unity- $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \ll 1$
3. The transverse normal strain is negligible- $\varepsilon_{zz} \approx 0$
4. The transverse shear stresses are negligible- $\sigma_{xz}, \sigma_{yz} \approx 0$

Note

- Applying the above assumptions we can reduce the 3-D problem to a **2-D plate bending problem**
- This theory is known as a **Kirchhoff-Love plate theory**

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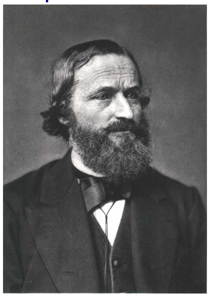
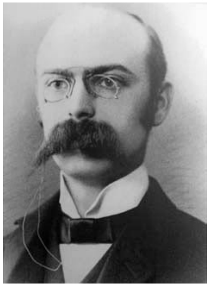
Assumptions



Gustav Kirchhoff (1824-1887)



Augustus Love (1863-1940)

Classical plate theory

- **Kirchhoff-Love plate theory**

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3 Field equations

Field equations

Displacements

- The assumption # 3 implies that $w(x,y,z) = w_0(x,y)$ where w_0 is the transverse displacement of the mid-plane ($z = 0$)
- Applying the assumption of the Kirchhoff hypothesis (*plane section, normal to the mid-surface before deflection remains plane and normal to the deformed surface*) gives

$$u(x,y,z) = z\phi_x$$

- The assumption # 4 gives $\gamma_{xz} = \gamma_{yz} = 0$
- Therefore $\phi_x = -\frac{\partial w}{\partial x}$ and $\phi_y = -\frac{\partial w}{\partial y}$

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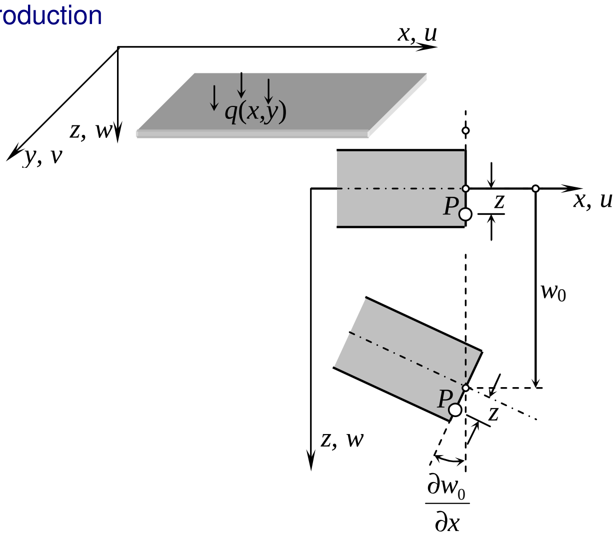
Introduction



Displacements

- Plate kinematics

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Field equations

Plate kinematics

- Displacement field

$$
\begin{aligned}
u(x,y,z) &= -z\frac{\partial w_0}{\partial x} \\
v(x,y,z) &= -z\frac{\partial w_0}{\partial y} \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
$$

- Strain-displacement field

$$
\begin{aligned}
\varepsilon_{xx} &= -z\frac{\partial w^2}{\partial x^2} \\
\varepsilon_{yy} &= -z\frac{\partial w^2}{\partial y^2} \\
\varepsilon_{yy} &= -2z\frac{\partial w^2}{\partial x \partial y}
\end{aligned}
$$

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Field equations

Plate kinematics

- Curvature definition

$$
\begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\frac{\partial w^2}{\partial x^2} \\ -\frac{\partial w^2}{\partial y^2} \\ -2\frac{\partial w^2}{\partial x \partial y} \end{bmatrix}
$$

- Strain-displacement relation becomes

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}
$$

- Or

$$
\varepsilon = z\kappa
$$

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Field equations

Stresses

- Constitutive equations- $\sigma = \mathbf{E}\varepsilon$

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}
$$

- Or

$$
\sigma = z\mathbf{E}\kappa
$$

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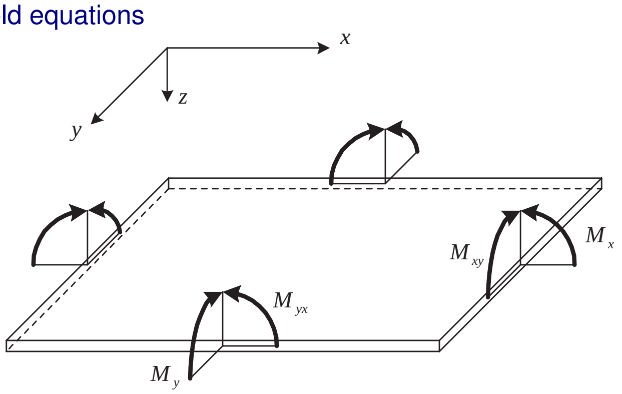
Field equations



Internal forces

- Stress resultants (internal forces)

$$\mathbf{M} = \int_{-t/2}^{t/2} \sigma z \mathrm{d}z = \frac{t^3}{12}\mathbf{E}\kappa$$

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Field equations

Internal forces

- Stress resultants (internal forces)

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

  where $D = \frac{Et^3}{12(1-\nu^2)}$ is called **flexural rigidity** (stiffness) of the plate
- The bending and twisting moments can be expressed in terms of displacements

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{yy} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

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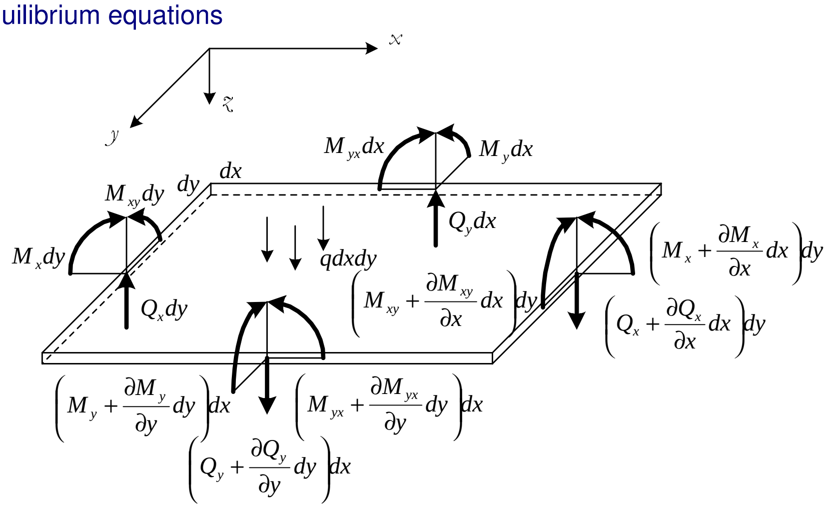
4 Equilibrium equations

Equilibrium equations



Cartesian coordinate system

- Consider the equilibrium of a differential element $\sum Z = 0$

$$q = -\frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y}$$

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Equilibrium equations

Cartesian coordinate system

- The moment equilibrium equations of a differential element lead to

$$\begin{aligned} Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} \\ Q_y &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \\ M_{xy} &= M_{yx} \end{aligned}$$

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Equilibrium equations

Cartesian coordinate system

- Using the above equilibrium relations we may obtain a single equation of the plate equilibrium in terms of the internal forces

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q$$

- Replacement of the moments-displacements relations gives the equilibrium equation in terms of the transversal displacement

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

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Equilibrium equations

Cartesian coordinate system

- In tensor notation is

$$\nabla^4 w(x,y) = \frac{q(x,y)}{D}$$

- The above equilibrium equation is called **Sophie Germain- Lagrange equation**

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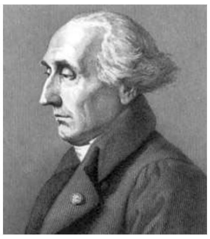
Equilibrium equations

Sophie Germain (1776-1831)



Joseph-Louis Lagrange (1736-1813)

Cartesian coordinate system

- **Sophie Germain- Lagrange equation**

$$\nabla^4 w(x,y) = \frac{q(x,y)}{D}$$

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Equilibrium equations

Shear forces

- The shear forces also can be expressed in terms of the displacements

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

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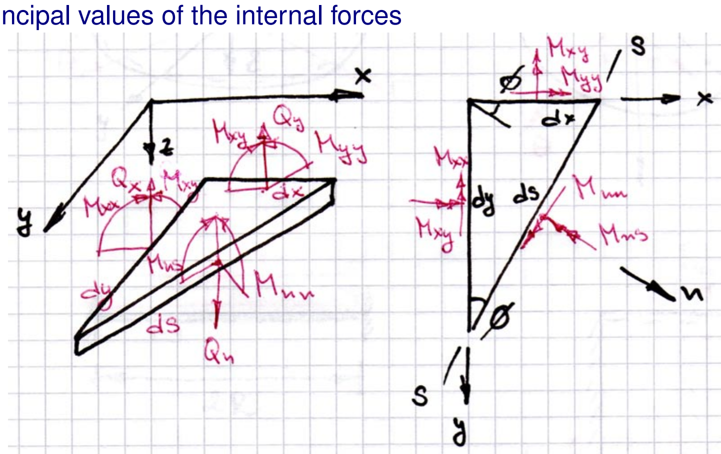
5 Principal values of the internal forces

Principal values of the internal forces



Principal bending moments

- Consider the internal forces acting on plate with arbitrary section cut
- Applying the equilibrium equation $\sum z = 0$ we have

$$Q_n = Q_x \cos\phi + Q_y \sin\phi$$

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Principal values of the internal forces

Principal bending moments

- The moment equations ($M_s = 0$ and $M_n = 0$) gives

$$M_{nn} = M_{xx}\cos^2\phi + 2M_{xy}\sin\phi\cos\phi + M_{yy}\sin^2\phi$$
$$M_{ns} = (M_{yy} - M_{xx})\sin\phi\cos\phi + M_{xy}(\cos^2\phi - \sin^2\phi)$$

- The infinitesimal parts of Q_x, Q_y and q are neglected

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Principal values of the internal forces

Principal bending moments

- The extremum condition $\frac{\partial M_{nn}}{\partial \phi} = 0$ gives the principal direction

$$\tan 2\phi = \frac{2M_{xy}}{M_x - M_y}$$

- The principal moments are

$$M_{1,2} = \frac{1}{2}(M_x + M_y) \pm \frac{1}{2}\left[(M_x + M_y)^2 + 4M_{xy}^2\right]^{1/2}$$

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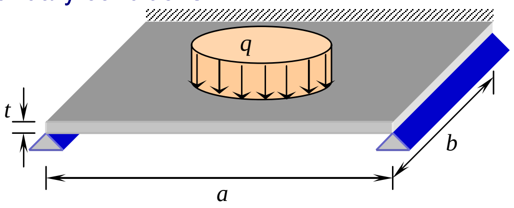
6 Boundary conditions

Boundary conditions



Loading and supports

- An exact solution of the governing plate equations must simultaneously satisfy the differential equations and BCs of any given plate bending problem
- Since the 8-th-order differential equation require **two boundary conditions** at each plate edge

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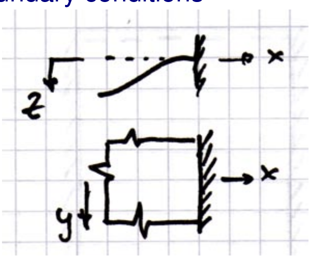
Boundary conditions



Loading and supports

- Essential (displacement) BCs

$$w = 0$$
$$\frac{\partial w}{\partial x} = 0$$

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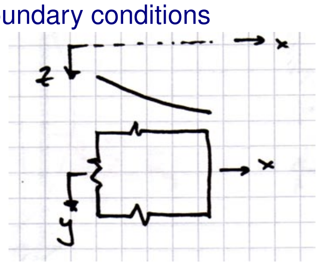
Boundary conditions



Loading and supports

- Natural (force) BCs

$$M_x = 0$$
$$V_x = 0$$

- Modified shear force (Kirchhoff equivalent force)

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$$
$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}$$

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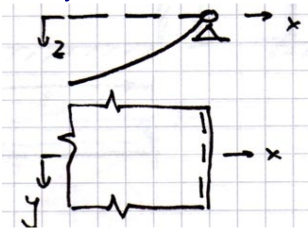
Boundary conditions



Loading and supports

- Mixed BCs

$$M_x = 0$$
$$w = 0$$

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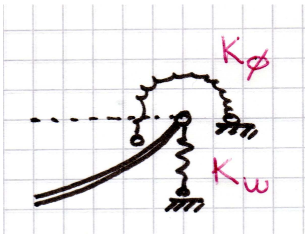
Boundary conditions



Loading and supports

- Elastic restrains

$$w = -\frac{V_x}{k_w}$$
$$\frac{\partial w}{\partial x} = -\frac{M_x}{k_\phi}$$

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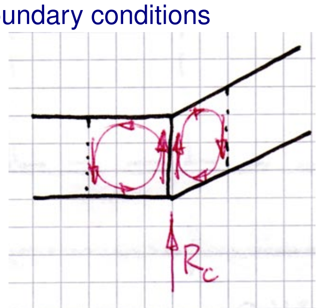
Boundary conditions



Corner forces

- When we have a corner at the plate boundary the twisting moments jump from $+M_{xy}$ to $-M_{xy}$
- The jump of the twisting moment is called **corner force** R_c

$$R_c = 2M_{xy}$$

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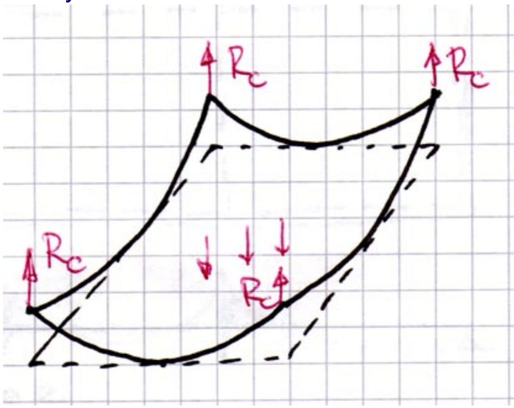
Boundary conditions



Corner forces

- This effect appears at plates with corners with simply supported edges

Question

- What happens when the edges are fixed or free?

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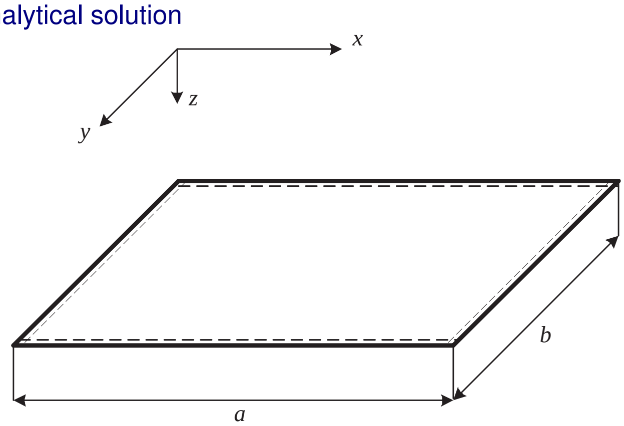
7 Analytical solutions

7.1 Navier's solution

Analytical solution



Navier's solution- double Fourier series

- **Navier's solution**- solution by double trigonometric series
- Rectangular plate
- Boundary conditions

$$\begin{aligned} w &= 0 \quad \text{at} \quad x = 0, x = a \\ w &= 0 \quad \text{at} \quad y = 0, y = b \\ M_x &= 0 \quad \text{at} \quad x = 0, x = a \\ M_y &= 0 \quad \text{at} \quad y = 0, y = b \end{aligned}$$

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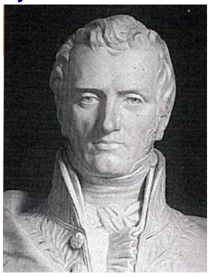
Analytical solution



Navier's solution- double Fourier series

- Claude-Louis Navier (1785-1836)

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Analytical solution

Navier's solution- double Fourier series

- Suppose that the solution is

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

- The above solution satisfies the BCs
- The coefficient of expansion w_{mn} is *unknown*
- The transversal load also can be expanded into double series

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

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Analytical solution

Navier's solution- double Fourier series

- Substitution of the above relations into the equilibrium equation gives

$$w_{mn}\left(\frac{m^4\pi^4}{a^4}+2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}\right)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$
$$=\frac{q_{mn}}{D}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

- Hence

$$w_{mn}=\frac{q_{mn}}{D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$$

- The final solution is

$$w(x,y)=\frac{1}{D\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

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Analytical solutions

Navier's solution- double Fourier series

- The coefficient of the double Fourier series is

$$q_{mn}=\frac{4}{ab}\int_0^b\int_0^a q(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\mathrm{d}x\mathrm{d}y$$

- When the transverse displacements is obtained we may calculate the internal forces

- Internal moments

$$
\begin{aligned}
M_x &= \pi^2 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{m}{a}\right)^2 + \nu\left(\frac{n}{b}\right)^2\right] w_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\\
M_y &= \pi^2 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\nu\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] w_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\\
M_{xy} &= -\pi^2 D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{ab} w_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)
\end{aligned}
$$

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Analytical solution

Navier's solution- double Fourier series

- Shear forces

$$
\begin{aligned}
Q_x &= \frac{\pi^3 D}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{m}{a}\right)^2 + \nu\left(\frac{n}{b}\right)^2 + (1-\nu)\left(\frac{n}{b}\right)^2\right] m w_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\\
Q_y &= \frac{\pi^3 D}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\nu\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + (1-\nu)\left(\frac{m}{a}\right)^2\right] n w_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)
\end{aligned}
$$

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7.2 L evy's solution

Analytical solution

L evy's solution- single Fourier series

- **L evy's solution**- solution by single trigonometric series
- Applicable to rectangular plates, simply supported at two opposite edges
- The solution of the equilibrium equation is given by

$$
w(x,y) = w_h(x,y) + w_p(x)
$$

- The particular solution is

$$
w_p(x) = \sum_{i=1}^{\infty} w_m \sin\frac{m\pi}{a}x
$$

- The homogeneous solution is given by

$$
w_h(x,y) = \sum_{i=1}^{\infty} f_m(y)\sin\frac{m\pi}{a}x
$$

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Analytical solution

L evy's solution- single Fourier series

- The replacement of the above expressions into the equilibrium equation gives a differential equation for y_m

$$
\frac{m^4\pi^4}{a^4} f_m(y) - 2\frac{m^2\pi^2}{a^2} f_m''(y) + f_m''''(y) = 0
$$

- The solution of the above equation is

$$
\begin{aligned}
f_m(y) =& A_m \cosh\frac{m\pi}{a}y + B_m\frac{m\pi}{a}y\sinh\frac{m\pi}{a}y\\
&+ C_m\sinh\frac{m\pi}{a}y + D_m\frac{m\pi}{a}y\cosh\frac{m\pi}{a}y
\end{aligned}
$$

where A_m, B_m, $C_m = 0$ and $D_m = 0$ are constants and can be determined from the BCs.

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Analytical solution

L evy's solution- single Fourier series

- The final solution is

$$w(x,y) = \sum_{i=1}^{\infty} w_m \sin\frac{m\pi}{a}x + \sum_{i=1}^{\infty}\left(A_m \cosh\frac{m\pi}{a}y + B_m\frac{m\pi}{a}y\sinh\frac{m\pi}{a}y\right)\sin\frac{m\pi}{a}x$$

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8 Numerical methods

Numerical methods

Approximate solutions

- The universal analytical solution of the governing plate bending equations for complex domain geometry, BCs and loading is not possible to find
- The engineering practice needs to use approximate solutions to solve the above mentioned problems
- The approximate solution are based on the energy and variational methods of structural mechanics
  - Ritz method
  - Galerkin method
  - Kantorovich method
- Numerical methods
  - Finite differences method
  - Gridwork method
  - Finite elements method
  - Finite strip method

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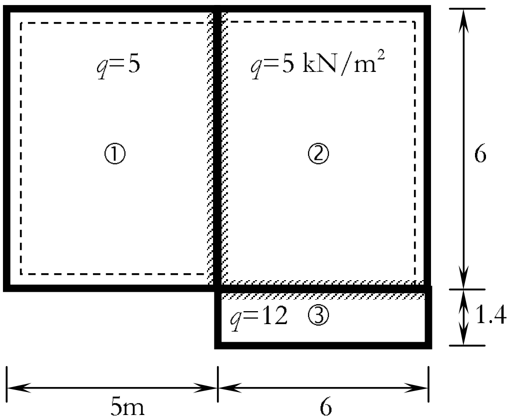
Numerical methods



Example- FE analysis

- Determine the displacements and the moments resultants for the given problem

$$E = 20000000kPa, \qquad \nu = 1/4, \qquad t = 0.1m$$

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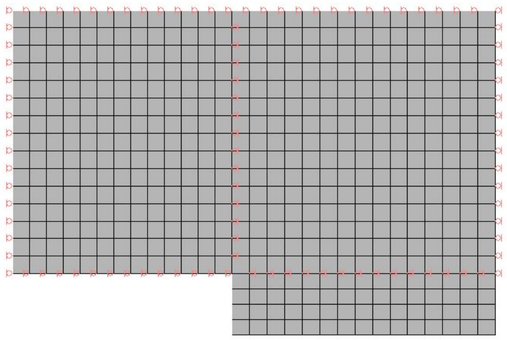
Numerical methods



Example- FE analysis

- Finite element mesh and supported nodes

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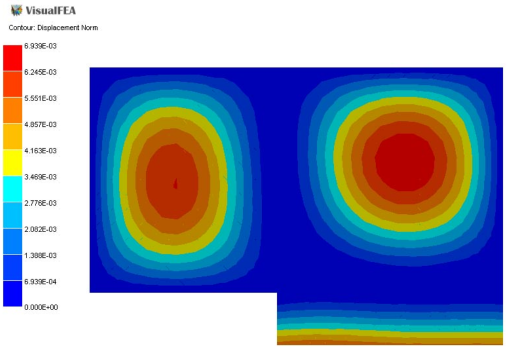
Numerical methods



Example- FE analysis

- Deflections

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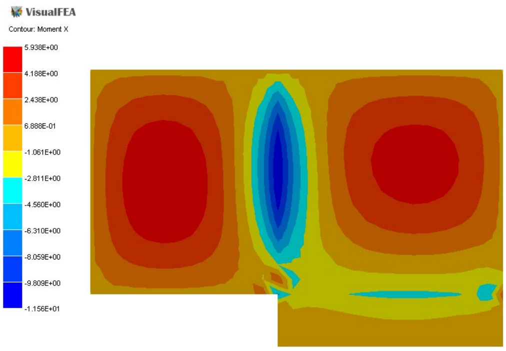
Numerical methods



Example- FE analysis

- Contour plot of the bending moments-M_x

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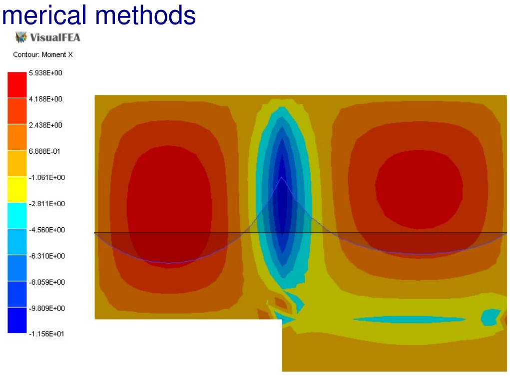
Numerical methods

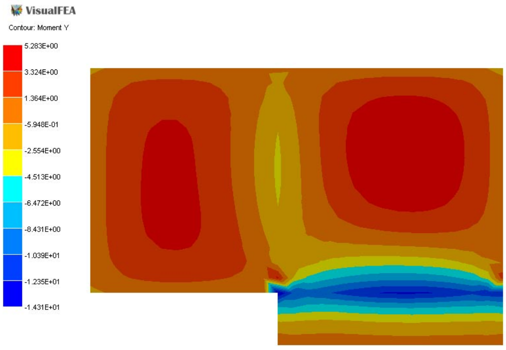


Example- FE analysis

- Section plot of the bending moments-M_x

9.48

Numerical methods

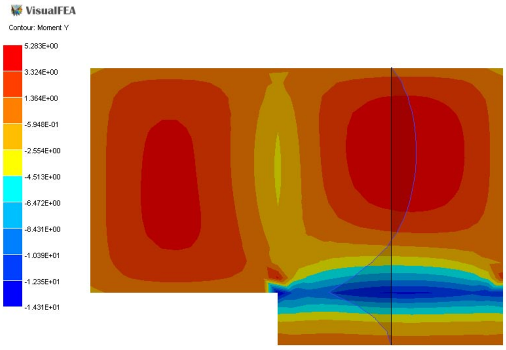


Example- FE analysis

- Contour plot of the bending moments-M_y

9.49

Numerical methods

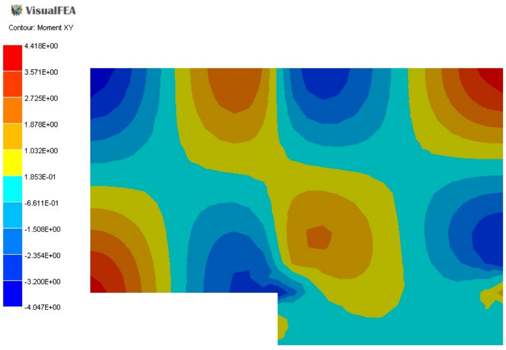


Example- FE analysis

- Section plot of the bending moments-M_y

9.50

Numerical methods

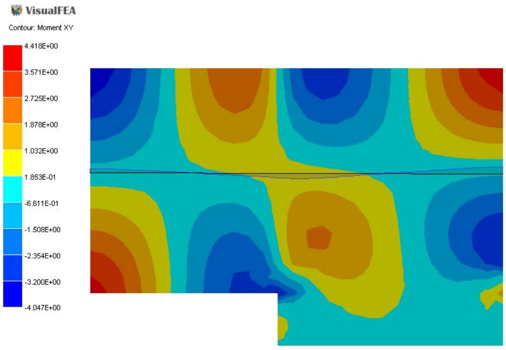


Example- FE analysis

- Contour plot of the bending moments-M_{xy}

9.51

Numerical methods

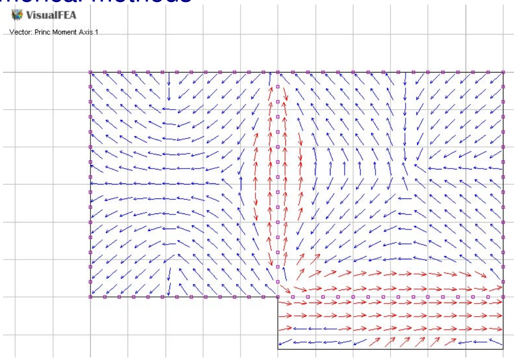


Example- FE analysis

- Section plot of the bending moments-M_{xy}

9.52

Numerical methods

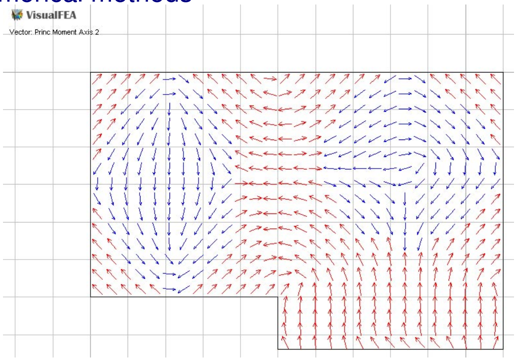


Example- FE analysis

- Vector plot of the principal moments-M_1

9.53

Numerical methods



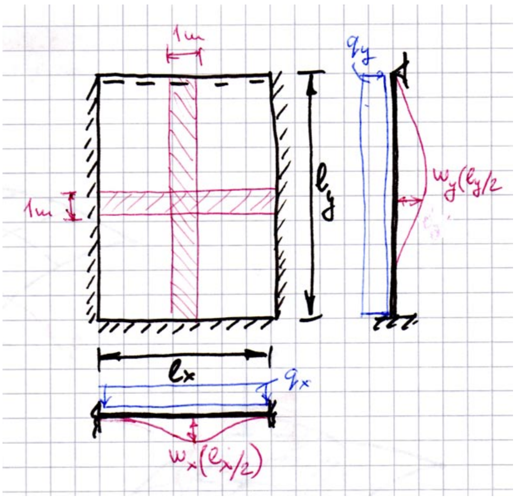
Example- FE analysis

- Vector plot of the principal moments-M_2

9.54

9 Engineering methods

Engineering methods



Elastic web analogy

- The engineering approach known as **Marcus method** (1924)
- The plate is considered as an elastic web consisting of plate strips located at mid-spans of the individual panels
- The application of this method is limited to
  - Uniform load
  - The size difference of the neighboring panels less than 50%
  - The Poisson's ratio is $\nu = \frac{1}{6}$

9.55

Engineering methods

Elastic web analogy

- Mid-span deflections of the webs

$$(w_x)_{\ell_x/2} = \frac{1}{384}\frac{q_x \ell_x^4}{D_x}$$
$$(w_y)_{\ell_y/2} = \frac{2}{384}\frac{q_y \ell_y^4}{D_y}$$

- Because of $D_x = D_y = D$ and

$$(w_x)_{\ell_x/2} = (w_y)_{\ell_y/2}$$
$$q = q_x + q_y$$

9.56

Engineering methods

Elastic web analogy

- We may obtain the *directional loads*

$$q_x = \frac{2\ell_y^4}{\ell_x^4 + 2\ell_y^4} q$$
$$q_y = q - q_x$$

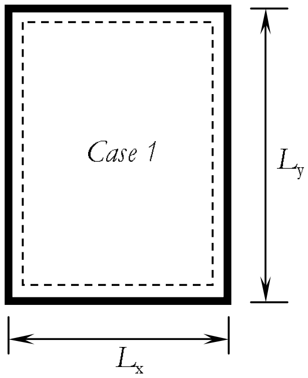
- In general form

$$q_x = \frac{C_y \ell_y^4}{C_x \ell_x^4 + C_y \ell_y^4} q$$

where the factors C_x and C_y depend of the BCs

9.57

Engineering methods



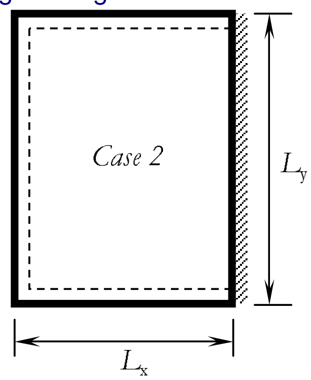
Elastic web analogy

- Case 1

$$C_x = 1$$
$$C_y = 1$$

9.58

Engineering methods



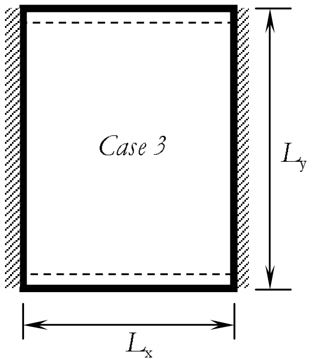
Elastic web analogy

- Case 2

$$C_x = 2$$
$$C_y = 5$$

9.59

Engineering methods



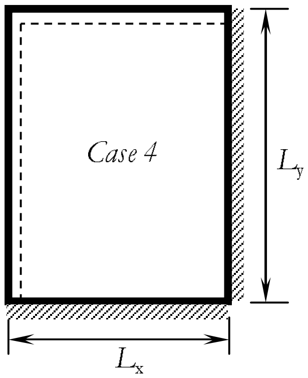
Elastic web analogy

- Case 3

$$C_x = 1$$
$$C_y = 5$$

9.60

Engineering methods



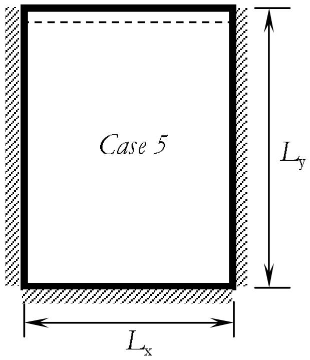
Elastic web analogy

- Case 4

$$C_x = 1$$
$$C_y = 1$$

9.61

Engineering methods



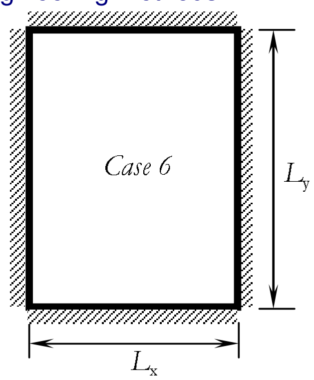
Elastic web analogy

- Case 5

$$C_x = 1$$
$$C_y = 2$$

9.62

Engineering methods



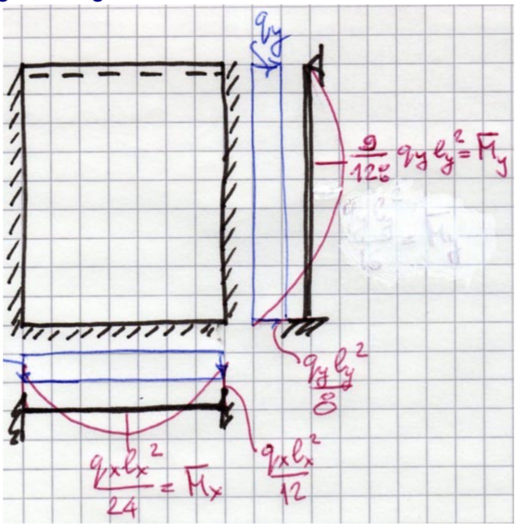
Elastic web analogy

- Case 6

$$C_x = 1$$
$$C_y = 1$$

9.63

Engineering methods



Elastic web analogy

- The fundamental difference between the grillage and the plate is the presence of shear forces between individual strips which produce a torsional resistance an reduce the deflections

9.64

Engineering methods

Elastic web analogy

- The approximated maximum span moments in plate are

$$M_x = \bar{M}_x \left(1 - \frac{5}{6}\frac{\ell_x^2}{\ell_y^2}\frac{\bar{M}_x}{M_x^0}\right)$$
$$M_y = \bar{M}_y \left(1 - \frac{5}{6}\frac{\ell_y^2}{\ell_x^2}\frac{\bar{M}_y}{M_y^0}\right)$$

where $\bar{M}_x$ and $\bar{M}_y$ are maximum span moments in strips and

$$M_x^0 = \frac{q\ell_x^2}{8}, \quad M_y^0 = \frac{q\ell_y^2}{8}$$

9.65

Engineering methods

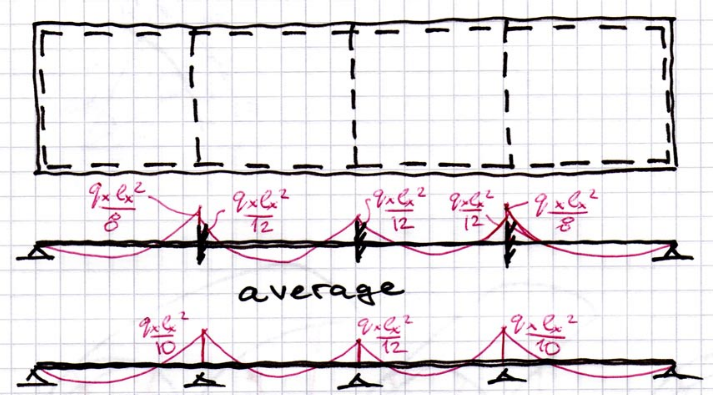
Elastic web analogy

- The edge moments are calculated as a strip supported with the same type of supports as a plate and loaded with directional load q_x or q_y

$$M_x^{supp} = \frac{1}{12} q_x \ell_x^2$$
$$M_y^{supp} = \frac{1}{8} q_y \ell_y^2$$

9.66

Engineering methods

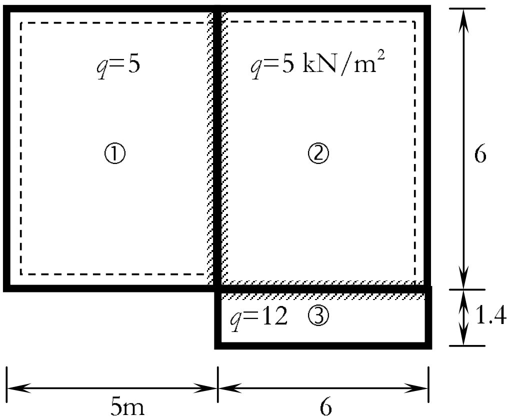


Elastic web analogy

- When the support moments of the neighboring panels does not match the bending moments can be averaged or calculated as support moments in a continuous beam

9.67

Engineering methods

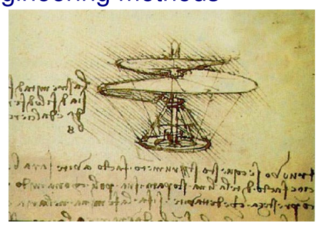


Elastic web analogy-example

- Calculate the bending moments of the slab

9.68

Engineering methods



The End

- Any questions, opinions, discussions?

9.69