Implement a queue with 2 stacks. Your queue should have an enqueue and a dequeue function and it should be "first in first out" (FIFO).

Optimize for the time cost of m function calls on your queue. These can be any mix of enqueue and dequeue calls.

Assume you already have a stack implementation and it gives $O(1)$ time push and pop.

Gotchas

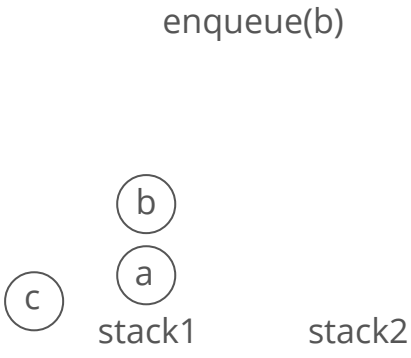
We can get $O(m)$ runtime for m function calls. Crazy, right?

Breakdown

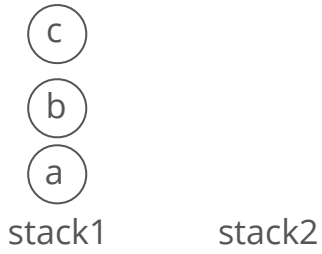
Let's call our stacks `stack1` and `stack2`.

To start, we could just push items onto stack1 as they are enqueued. So if our first 3 function calls are enqueues of `a`, `b`, and `c` (in that order) we push them onto `stack1` as they come in.

But recall that stacks are last in, first out. If our next function call was a `dequeue()` we would need to return `a`, but it would be on the bottom of the stack.



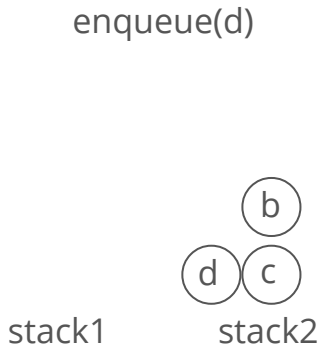
Look at what happens when we pop `c`, `b`, and `a` one-by-one from `stack1` to `stack2`.



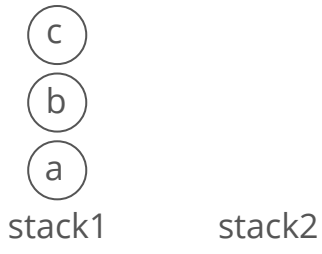
Notice how their order is reversed.

We can pop each item 1-by-1 from `stack1` to `stack2` until we get to `a`.

We could return `a` immediately, but what if our next operation was to enqueue a new item `d`? Where would we put `d`? `d` should get dequeued after `c`, so it makes sense to put them next to each-other . . . but `c` is at the bottom of `stack2`.



Let's try moving the other items back onto `stack1` before returning. This will restore the ordering from before the dequeue, with `a` now gone. So if we enqueue `d` next, it ends up on top of `c`, which seems right.



So we're basically storing everything in `stack1`, using `stack2` only for temporarily "flipping" all of our items during a dequeue to get the bottom (oldest) element.

This is a complete solution. But we can do better.

What's our time complexity for m operations? At any given point we have $O(m)$ items inside our data structure, and if we dequeue we have to move all of them from `stack1` to `stack2` and back again. One dequeue operation thus costs $O(m)$. The number of dequeues is $O(m)$, so our worst-case runtime for these m operations is $O(m^2)$.

Not convinced we can have $O(m)$ dequeues and also have each one deal with $O(m)$ items in the data structure? What if our first $.5m$ operations are enqueues, and the second $.5m$ are alternating enqueues and dequeues. For each of our $.25m$ dequeues, we have $.5m$ items in the data structure.

We can do better than this $O(m^2)$ runtime.

What if we didn't move things back to `stack1` after putting them on `stack2`?

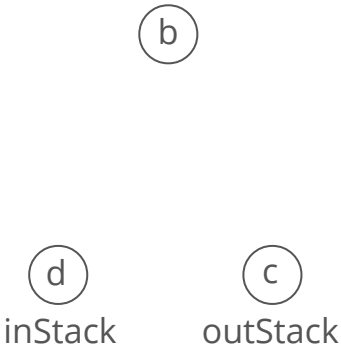
Solution

Let's call our stacks `in_stack` and `out_stack`.

For enqueue, we simply push the enqueued item onto `in_stack`.

For dequeue on an empty `out_stack`, the oldest item is at the bottom of `in_stack`. So we dig to the bottom of `in_stack` by pushing each item one-by-one onto `out_stack` until we reach the bottom item, which we return.

After moving everything from `in_stack` to `out_stack`, the item that was enqueued the $2nd$ longest ago (after the item we just returned) is at the top of `out_stack`, the item enqueued $3rd$ longest ago is just below it, etc. **So to dequeue on a non-empty `out_stack`**, we simply return the top item from `out_stack`.



With that description in mind, let's write some code!

```ruby
class QueueTwoStacks
    def initialize
        @in_stack  = []
        @out_stack = []
    end

    def enqueue(item)
        @in_stack.push(item)
    end

    def dequeue()
        if @out_stack.length == 0
            # Move items from in_stack to out_stack, reversing order
            while @in_stack.length > 0 do
                newest_in_stack_item = @in_stack.pop
                @out_stack.push(newest_in_stack_item)
            end
            # If out_stack is still empty, return nil
            if @out_stack.length == 0
                return nil
            end
        end
        return @out_stack.pop
    end
end
```

Complexity

Each enqueue is clearly $O(1)$ time, and so is each dequeue when `out_stack` has items. Dequeue on an empty `out_stack` is order of the number of items in `in_stack` at that moment, which can vary significantly.

Notice that the more expensive a dequeue on an empty `out_stack` is (that is, the more items we have to move from `in_stack` to `out_stack`), **the more $O(1)$-time dequeues off of a non-empty `out_stack` it wins us in the future**. Once items are moved from `in_stack` to `out_stack` they just sit there, ready to be dequeued in O(1) time. An item never moves "backwards" in our data structure.

We might guess that this "averages out" so that in a set of m enqueues and dequeues the total cost of all dequeues is actually just $O(m)$. To check this rigorously, we can use the accounting method↴ , **counting the time cost *per item* instead of per enqueue or dequeue**.

So let's look at the worst case for a single item, which is the case where it is enqueued and then later dequeued. In this case, the item enters `in_stack` (costing 1 push), then later moves to `out_stack` (costing 1 pop and 1 push), then later comes off `out_stack` to get returned (costing 1 pop).

Each of these 4 pushes and pops is $O(1)$ time. **So our total cost *per item* is $O(1)$**. Our m enqueue and dequeue operations put m or fewer items into the system, giving a total runtime of $O(m)$!

What We Learned

People often struggle with the runtime analysis for this one. The trick is to think of the cost *per item passing through our queue*, rather than the cost per enqueue() and dequeue().

This trick generally comes in handy when you're looking at the time cost of not just one function call, but "m" function calls.
