An Auction Model of The War of Pursuing Women in *A Beautiful Mind*

An Essay on Game Theory, Supervised by Shiu-Yuen Cheng

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Abstract

In the film *A Beautiful Mind*, John Nash used an hypothetical experiment "The War of Pursuing Women" to illustrate his view on game theory. He offers a better strategy for men who all want to get the most attractive woman with blonde hair. In this paper, we construct an all-pay auction model using equilibrium strategy analysis to compare the two strategies. We also creatively add the refusal factor as a probability into the model.

Keywords A Beautiful Mind; game theory; all-pay auction; equilibrium strategy

1 Introduction

When it comes to game theory, John Nash comes to my mind first. There is a famous American biographical drama film based on the life of John Nash ——— *A Beautiful Mind*, in which an interesting plot enables us to have a glimpse of his idea on game theory.

One day he and his fellow graduate students discuss how to approach a group of women at a bar. One of his fellows, Hansen, quotes Adam Smith's "Individual ambition serves the common good" and "Every man for himself", but Nash claims that "Adam Simth needs revision". Then he did a hypothetical experiment to reason: *We all go for the blonde, we block each other and not a single of us is going to get her. So then we go for her friends, but they will all give us the cold shoulder because nobody likes to be second choice. But what if no one goes to the blonde? We don't get in each other's way and we don't insult the other girls. That's the only way we win.*

In this paper, we model this hypothetical experiment as an all-pay auction where men are bidders, women are prizes, the efforts of pursuing women are bids, all bidders need to pay their bids and the bidder with highest bid win the prize.

2 Model

We model the war of pursing women in the following manner. There are n risk-neutral bidders and a super prize with a series of small prizes to be auctioned. Each bidder i draws his true value v_i independently and uniformly at random from the interval between 0 and 1.

At first, each bidder submits a sealed bid of b_i in the interval $[0, 1]$ to the super prize and the one with the highest bid wins. Then losers turn to small prizes, we assume the choices are diiferent. Each bidder i values his second choice v'_i and bids b'_i, he would lose with a probability of p_i (since nobody likes to be the second choice, the woman may refuse him).

In the following section, we compare two situations shown in the film using our model. One is that each man goes for the blonde at the same time and then the second choice if loses. The other is that all give up the blonde and directly go for the other girls.

3 Comparision

Let's first analyse the first situation which corresponds to each bidder bids for the super prize first and losers turn to their second choice next. Let x_i be the payoff pursuing the second choice. Thus the payoff for bidder i is:

$$u_i = \begin{cases} v_i - b_i, & \text{if } b_i = \max\limits_{1jn} b_j \\ -b_i + x_i, & \text{otherwise} \end{cases}$$

Here we break ties by randomly choosing one with the highest effort(bid) as the winner.

We use a function $s(v) = b$ to represent the bidding strategy that maps a bidder's true value v to a non-negative bid b where $s(\mathring{u})$ follows two assumptions:

(i) $s(\mathring{u})$ is strictly increasing, differentiable function

(ii) $s(v)v$ and $s(0) = 0$

Now consider an equilibrium strategy. From assumption(i) we know that the bidder with the higher value will bid higher. So if bidder i has a value of v_i, the probability that this is higher than the value of other competitors in the interval $[0, 1]$ is v_i^{n-1} since it requires each other bidder to have a value below v_i and values are chosen independently. Therefore, bidder i will win the auction with probability v_i^{n-1}, his expected payoff is:

$$g(v_i) = v_i^{n-1}(v_i - s(v_i)) + (1 - v_i^{n-1})(-s(v_i) + x_i)$$

where the first term corresponds to the payoff when i wins and the second term corresponds to the payoff when i loses.

Suppose there is a deviation from bidding b_i to b where $v = s^{-1}(b)$ is a fake value. According to the definition of equilibrium,

$$v_i^{n-1}(v_i - s(v_i)) + (1 - v_i^{n-1})(-s(v_i) + x_i) v^{n-1}(v_i - s(v)) + (1 - v^{n-1})(-s(v) + x_i)$$

for all v in the interval $[0, 1]$.

We can rewrite the inequality as

$$v_i^n - s(v_i) + (1 - v_i^{n-1})x_i v^{n-1} v_i - s(v) + (1 - v^{n-1})x_i$$

for all v in the interval $[0, 1]$.

If we construct a function as $f(v) = v^{n-1}v_i - s(v) + (1 - v^{n-1})x_i$, we can view the inequality as reaching to the maximum value at $v = v_i$, that is $f'(v_i) = 0$. Note that

$$f'(v) = (n-1)v^{n-2}v_i - s'(v) - (n-1)v^{n-2}x_i = (n-1)v^{n-2}(v_i - x_i) - s'(v),$$

we have

$$s'(v_i) = (n-1)v_i^{n-2}(v_i - x_i),$$

hence

$$s(v_i) = \frac{n-1}{n}v_i^n - v_i^{n-1}x_i.$$

Substitute it into the formula of expected payoff we obtain $g(v_i) = \frac{1}{n}v_i^n + x_i$.

Next we consider x_i, each loser i has a probability of $1 - p_i$ to receive a payoff of $v_i' - b_i'$ and a probablility of p_i to receive a payoff of $-b_i'$. Therefore,

$$x_i = (1 - p_i)(v_i' - b_i') + p_i(-b_i') = (1 - p_i)v_i' - b_i'.$$

Now we can conclude that the expected payoff for each bidder i following the equilibrium strategy in the first situation is

$$g(v_i) = \frac{1}{n}v_i^n + (1 - p_i)v_i' - b_i'.$$

Finally we compare it to the payoff for each bidder i in the second situation which is simply $v_i' - b_i'$ (no risk of being refused). The first situation is worse than the second situation if and only if $\frac{1}{n}v_i^n p_i v_i'$, which is independent of b_i'. The inequality can be interpreted as with the number of bidders n be larger and the probability p_i of being refused by the second choice be increased, the second situation would become better than the first situation.

4 Remark

In the film, Nash argues that men will block each other if all go for the blonde and no one will get her. However, in our model, the blonde would always choose the man who loves her most (pays most efforts). In fact, if the information about each man's valuation is private, no one can win for sure. If we view men putting efforts a bit more and more by observing others, the model can be revised as a "second-prize all-pay auction" just as "the war of attrition" and that's another story.

Also, Nash states that the blonde's friends will give them a cold shoulder because they are unhappy to be a second choice. Here we make an assumption that men will be refused with a probability, actually the probability is hard to know, but it still makes a difference to which strategy to choose.

Last but not least, Nash claims that the only way they win is to go for other girls except the blonde. But we can see in our model that this is not absolute. If pursuers are not that many and the valuation to the blonde is high enough, it is still worth trying to pursue the blonde. We need to note that the left term of the last inequality shrinks quickly as n increases, thus Nash's proposal would siginificantly work when n is larger than some threshold. This also reminds us that we should take risk if the competition for a single prize is not fierce and we value the prize a lot, otherwise we should choose some safe strategy so that avoid to pay a double cost.

References

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