Lottery VS Auction: Equality and Efficiency of the Car Plate Control

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Abstract

Air pollution and traffic congestion is getting worse and worse in many areas of China. As a result, some governments have taken measures to control car plates in order to limit the number of cars on road. Up to now, several different mechanisms have already been carried out in big cities, such as lottery in Beijing, auction in Shanghai and hybrid mechanism in Guangzhou. The target of this paper is to compare the efficiency and equality of these mechanism. We introduce social surplus and Gini-coefficient as measures of efficiency and inequality. From calculation I realize that auction and lottery can only achieve either of equality and efficiency, even hybrid mechanism cannot always reach a compromise. Therefore, I show that "auction then lottery" is an improvement of hybrid mechanism and can balance efficiency and equality indeed.

1 Introduction

With the process of China's urbanization, more and more people migrate to cities. The urban population exceeded the rural population for the first time in 2011. In the next 20 years, an estimated number of 350 million rural people would move to cities. Such scale of urban development is both a challenge and an opportunity for urban transportation. Meanwhile, as a result of flourishing economy, people's life quality and life expectancy have improved a lot. More and more families can afford a private car. Over the years, the demand for private cars has been growing rapidly in large cities such as Beijing, Shanghai and Guangzhou. The number of privately owned cars in Shanghai increased from 0.85 million in 2009 to 1.03 million in 2010, then 1.20 million in 2011. Consequently, both air pollution and traffic congestion are getting worse and worse. Beijing, for example, has underwent severe smog and haze, causing great harm to the health of residents. Such bad phenomena prompt governments to take effective measures to control the number of vehicles, especially private cars.

Several major cities have started to limit number of car plates. Every one or two months, only a small number of newly added quotas are permitted to be allocated among a certain large number of potential car buyers. The design of a satisfying mechanism becomes a challenging problem to the policy makers, who have to consider both equality and efficiency when allocating this kind of public resources. Interestingly, the pricing and allocation rules vary from city to city. Shanghai is the first city in China starting the car plate control, which has conducted price mechanism (auction) since 2002. Whereas Beijing and Guiyang started using non-price mechanism (lottery) to allocate new car plates on December 2010 and January 2011 respectively. Due to the disadvantages that lie in these mechanisms (will be discussed in the next chapter), Guangzhou adopted a hybrid mechanism (simultaneous auction and lottery), starting from August 2012, then followed by several cities including Shenzhen, Hangzhou, Tianjin and Shijiazhuang.

The rest of this paper proceeds as follows. In chapter 2, the procedures and issues of different car plate allocation mechanisms will be introduced. In chapter 3, I will model the lottery, auction and hybrid mechanism. Finally, in chapter 4, I will define and compute the equality and efficiency measures for these mechanisms.

2 Policy & Issues

2.1 Beijing

The qualification for participating in the Beijing car plate lottery system requires an applicant to have a Beijing resident certificate and a driving license, and not a holder of a car plate. Non-residents who have social security insurance or working visa for at least five years in Beijing are also eligible. No guarantee fees or expenditures are charged. But every winner received a car plate must purchase a car within 6 months. In the last month of 2017, nearly 2.84 million people applying for the small passenger cars and the winning probability is as low as 0.11%, implying only one in 883 applicants can get the quota. Although a lottery is fair somehow, it may lead to low efficiency and not reveal players' valuations.

The distorted market due to the imbalanced demand and supply of car plates has created externalities. For instance, as the saying goes "When there is a rule, there is a way to get around it.", one can buy a car plate in Hubei province which is next to Beijing for temporary use until his win in the Beijing lottery system. In such case, the car plate control seems to be nullified. In addition, as holding a car plate becomes valuable, this mechanism may attract those who do not necessarily own a car. One interesting phenomenon is "untruthful bidding", in order to increase the winning probability, some families will have two or even more members participating in the lottery system and luckily win more than one quota, which results in a waste standing on other participants' point.

2.2 Shanghai

In 2002, Shanghai issued the Shanghai Metropolitan Transport White Paper. The auction system was then officially adopted, which is a variation of first price sealed-bid auction. All bidders make the initial bid at a preannounced time when other's bids are unknown. After that, there is a 30-minute bid-revision phase during which bidders can submit updated bids twice according to the disclosed lowest winning bid. Any updated bids must be placed within 300 gap around the lowest winning bid at that time.

Bidding prices has risen sharply from about 15,000 RMB in 2002 to 56,000 RMB in

2007. On March 2013, the highest bidding price of a Shanghai car plate reached 91,898 RMB. The car plate then became the world's most expensive sheet-iron.

Shanghai's practice, whist effective and profitable for the government, sparked many suspicions. Many are worried about the "winner's curse", the continuous rising auction price may provoke fierce competition. Eventually, the final bids are more likely to be an over-estimate of the common value of car plates. Moreover, there was criticism that the system is unfair and incline to richer car buyers, since it is costly to win the auction. Still, there are potential buyers who register their cars in other cities or provinces where there is no restriction.

2.3 Guangzhou

To balance the equality and efficiency, it is natural and applicable to come up with the idea of a hybrid mechanism, that is a combination of lottery and auction. Nearly half of the quotas go to auction and the remain is for lottery. Each month, the qualified applicants have to choose between lottery and auction. They are not allowed to participate in both games at the same time. If one loses the lottery or auction, he can submit a new application next month. The average winning price of Guangzhou's auction system is around 30,000 RMB in the last few months of 2017. The number of participants in the auction system is about two to the three times of the number of quotas for auctions while there are a total of 0.6 million car plate applicants competing for around 8000 quotas each month.

3 Model

Suppose the government plans to allocate a total number of k homogeneous car plates which are desired by a total number of n players. Assume that each player $i \in \mathcal{N} = \{1, 2, \cdots, n\}$ is risk-neutral and has unit demand with private value $v_i > 0$. Without loss of generality, let $v_1 \geq v_2 \geq \cdots \geq v_n$. The expected utility function of player $i \in \mathcal{N}$ can be defined as

$$u_i = q_i(v_i - p_i) + (1 - q_i)(0 - p_i'),$$

where q_i denotes player i's probability of winning a car plate, p_i is the winning payment and p_i' is the losing payment. Almost in every mechanism, $p_i' = 0$ except for the deposit of the auction. In Guangzhou's auction system, the deposit is 2,000 RMB and will be returned after the auction (unless a winner of the auction does not pay his bid on time). Whereas Shanghai's auction system not only sets a deposit of 2,000 RMB, but also charges for a commission of 100 RMB which will not be returned.

3.1 Auction

Let us consider auction first. Assume that each bidder i submits a bid $b_i > 0$. Bidders' valuations are drawn independently from a common distribution F and let f denotes the density function. A Nash equilibrium of first-price sealed-bid auction can be derived as follows. For bidder i whose valuation is v_i, he presumes that in equilibrium, other bidders will follow a bidding function B that maps valuation v_j to $B(v_j)$. There are three simple assumptions about B: different valuations produce different bids; higher valuations produce higher bids; bidders can shade their bids down, but never bid above their true valuations. The winning probability with a bid b_i is the probability that all other $n-1$ bidders have valuations v_j such that $B(v_j) < b_i$, which is $q_i = F[B^{-1}(b_i)]^{n-1}$. Once he wins, his payment is $p_i = b_i$. Therefore, bidder i chooses to maximize his expected utility function (payoff) $u_i = (v_i - b_i)F[B^{-1}(b_i)]^{n-1}$. Note that the bidding function B must satisfy $b_i = B(v_i)$. Hence the derivative of $u_i = (v_i - B(v))F(v)^{n-1}$ equals to 0 when $v = v_i$. By applying the Product Rule to differentiate u_i, we get

$$(n-1)F(v_i)^{n-2}f(v_i)\big(v_i - B(v_i)\big) + F(v_i)^{n-1}(-B'(v_i)) = 0,$$

that is,

$$B'(v_i) = (n-1)\frac{f(v_i)\big(v_i - B(v_i)\big)}{F(v_i)}.$$

Finding an explicit expression to the above differential equation is impossible unless the explicit form for the distribution of valuations is given. For the uniform distribution on the interval $[0,1]$ with cumulative distribution function $F(v) = v$ and density function $f(v) = 1$, the differential equation now becomes

$$B'(v_i) = (n-1)\left(1 - \frac{B(v_i)}{v_i}\right)$$

for all v_i between 0 and 1, which is solved by the equation

$$B(v_i) = \frac{n-1}{n}v_i.$$

It means that if each bidder shades his bid down by a factor of $\frac{n-1}{n}$, then this is the optimal behavior given what everyone else bids.

Let me add a comment about Nash equilibrium. The design of bid-revision phase may seem strange, but it can be viewed as an approach to approximate Nash equilibrium. In addition, many potential car buyers have participated in the auction for several times, they will have better estimate of what others will bid.

Actually in the first-price sealed-bid auction, even though he thought of his winning probability as $q_i = F[B^{-1}(b_i)]^{n-1}$, his actual winning probability is

$$q_i = \begin{cases} 1, & i \le k \\ 0, & i > k \end{cases}$$

This is obvious due to our assumption that higher valuations produce higher bids. So the k car plates will eventually go to the first k players, player $i \in \mathcal{K} = \{1,2,\cdots,k\}$ with valuation v_i will pay his bid b_i and receive a payoff of $v_i - b_i$. Losers of the auction pay nothing and receive nothing. The government earns a profit of $b_1 + b_2 + \cdots + b_k$.

3.2 Lottery

The analysis of lottery system is rather straightforward. No matter what valuation a player has, the only strategy for him is to wait (or he can apply "untruthful bidding" as we mentioned before). So the winning probability is same for everyone, that is, $q_i = \frac{k}{n}$. Since it is free to participate in the lottery, player i has a payoff of $\frac{k}{n}v_i$. The government earns no profit.

3.3 Hybrid Mechanism

Hybrid Mechanism is just a combination of auction and lottery. Suppose the quotas for auction and lottery is k_1 and k_2 respectively $(k_1 + k_2 = k)$. Let the set of players who choose auction and lottery as $\mathcal{N}_1$ $(|\mathcal{N}_1| = n_1 > k)$ and $\mathcal{N}_2$ $(|\mathcal{N}_2| = n_2)$ respectively. Then the first k_1 players in $\mathcal{N}_1$ win the auction, denoted as $\mathcal{K}_1$. Whereas in $\mathcal{N}_2$, each player has the same winning probability $\frac{k_2}{n_2}$.

4 Measures

4.1 Efficiency Measure

Define the efficiency measure of a mechanism as the total social welfare $\mathcal{E}$.

For auction, the total social welfare is

$$\mathcal{E}_1 = \sum_{i \in \mathcal{K}} (v_i - b_i) + \sum_{i \in \mathcal{K}} b_i = \sum_{i \in \mathcal{K}} v_i.$$

For lottery, the total social welfare is

$$\mathcal{E}_2 = \frac{k}{n} \sum_{i \in \mathcal{N}} v_i.$$

For hybrid mechanism, the total social welfare is

$$\mathcal{E}_3 = \sum_{i \in \mathcal{K}_1} v_i + \frac{k_2}{n_2} \sum_{i \in \mathcal{N}_2} v_i.$$

We can prove that $\mathcal{E}_1 > \mathcal{E}_2$ and $\mathcal{E}_1 > \mathcal{E}_3$, but the relation between $\mathcal{E}_2$ and $\mathcal{E}_3$ is not clear. We can easily derive $\mathcal{E}_1 > \mathcal{E}_2$ from $\frac{\sum_{i \in \mathcal{K}} v_i}{k} > \frac{\sum_{i \in \mathcal{N}} v_i}{n}$. The proof of $\mathcal{E}_1 > \mathcal{E}_3$ is as follows. First let

$$\mathcal{E}_3' = (v_1 + v_2 + \cdots + v_{k_1}) + \frac{k_2}{n_2}(v_{n_1+1} + \cdots + v_n),$$

then we can prove $\mathcal{E}_3' > \mathcal{E}_3$ with adjustment method. For $j = 1,2,\cdots,k_1$, each time if $j \in \mathcal{N}_2$, then exchange j with the j-th element in $\mathcal{N}_1$. Since the weight of v_j

increases from $\frac{k_2}{n_2}$ to 1, the whole adjustment process must result in an increment from $\mathcal{E}_3$ to $\mathcal{E}_3'$. Next, we can prove $\mathcal{E}_1 \geq \mathcal{E}_3'$ with the fact that

$$\frac{v_{k_1+1} + \cdots + v_k}{k_2} \geq \frac{v_{n_1+1} + \cdots + v_n}{n_2}.$$

To my surprise, $\mathcal{E}_2$ can be larger than $\mathcal{E}_3$. For example, if $n_1 = k_1 = 1$, $n_2 = n - 1$, $k_2 = k - 1$, then

$$\begin{aligned}\mathcal{E}_2 - \mathcal{E}_3 &= \left(\frac{k}{n} - \frac{k-1}{n-1}\right)(v_1 + v_2 + \cdots + v_{n-1}) - \left(1 - \frac{k}{n}\right) v_n \\ &= \left(1 - \frac{k}{n}\right)\left(\frac{v_1 + v_2 + \cdots + v_{n-1}}{n-1} - v_n\right) \geq 0\end{aligned}$$

which reminds us that hybrid mechanism is not necessarily more efficient than lottery. In [1], the authors suggest a variation of hybrid mechanism – “auction then lottery”. The procedure is as follows. Each player submits a bid first, and players with k_1 highest bids get the car plate with probability 1. Then run lottery among other players, each has $\frac{k_2}{n-k_1}$ probability to win the car plate. Now the total social welfare becomes

$$\mathcal{E}_4 = (v_1 + v_2 + \cdots + v_{k_1}) + \frac{k - k_1}{n - k_1}(v_{k_1+1} + \cdots + v_n)$$

We can prove that $\mathcal{E}_4 > \mathcal{E}_2$ through subtraction,

$$\begin{aligned}\mathcal{E}_4 - \mathcal{E}_2 &= \left(1 - \frac{k}{n}\right)(v_1 + v_2 + \cdots + v_{k_1}) - \left(1 - \frac{k}{n}\right)\frac{k_1}{n - k_1}(v_{k_1+1} + \cdots + v_n) \\ &= \left(1 - \frac{k}{n}\right) k_1 \left(\frac{v_1 + v_2 + \cdots + v_{k_1}}{k_1} - \frac{v_{k_1+1} + \cdots + v_n}{n - k_1}\right) \geq 0.\end{aligned}$$

We can also prove that $\mathcal{E}_4 > \mathcal{E}_3$ by comparing $\mathcal{E}_4$ with $\mathcal{E}_3'$ and using the fact that $\mathcal{E}_3' > \mathcal{E}_3$.

$$\begin{aligned}\mathcal{E}_4 - \mathcal{E}_3' &= \frac{k - k_1}{n - k_1}(v_{k_1+1} + \cdots + v_n) - \frac{k_2}{n_2}(v_{n_1+1} + \cdots + v_n) \\ &= k_2\left(\frac{v_{k_1+1} + \cdots + v_n}{n - k_1} - \frac{v_{n_1+1} + \cdots + v_n}{n - n_1}\right) \geq 0.\end{aligned}$$

4.2 Equality Measure

In this paper, we adopt $\eta = 1 - G$ as the equality measure representing the fairness of a mechanism, where G is the Gini-coefficient. Gini-coefficient was originally proposed as a measure of inequality among values of income or wealth. In the case of car plates allocation, since each player has a winning probability, Gini-coefficient could also be used to measure the inequality between winning probabilities.

Formally, Gini-coefficient is defined as

$$G = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|q_i - q_j|}{2n\sum_{i=1}^{n} q_i}$$

where q_i is the winning probability as we defined before.

For auction, the Gini-coefficient is

$$G_1 = \frac{\sum_{i=1}^{k}\sum_{j=k+1}^{n} 2(q_i - q_j)}{2n\sum_{i=1}^{k} q_i} = \frac{2k(n-k)}{2nk} = \frac{n-k}{n},$$

so $\eta_1 = 1 - G_1 = \frac{k}{n}$.

For lottery, the Gini-coefficient is $G_2 = 0$ since all q_i are equal, so $\eta_2 = 1 - G_2 = 1$.

For hybrid mechanism, the Gini-coefficient is

$$G_3 = \frac{\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2} 2|q_i - q_j| + \sum_{i\in\mathcal{K}_1}\sum_{j\in\mathcal{N}_1\setminus\mathcal{K}_1} 2|q_i - q_j|}{2n\left(\sum_{i\in\mathcal{N}_1} q_i + \sum_{j\in\mathcal{N}_2} q_j\right)}$$

$$= \frac{2k_1 n_2\left(1 - \frac{k_2}{n_2}\right) + 2(n_1 - k_1)n_2\frac{k_2}{n_2} + 2k_1(n_1 - k_1)}{2n\left(k_1 + n_2\frac{k_2}{n_2}\right)}$$

$$= \frac{k_1(n-k) + (n_1 - k_1)k_2}{nk}$$

so $\eta_3 = 1 - G_3 = \frac{k_1 k + n_2 k_2 + k_1 k_2}{nk}$.

For "auction then lottery" mechanism, the Gini-coefficient is

$$G_4 = \frac{2k_1(n_1 - k_1)\left(1 - \frac{k_2}{n - k_1}\right)}{2n\left[k_1 + (n - k_1)\frac{k_2}{n - k_1}\right]} = \frac{k_1(n-k)}{nk}$$

so $\eta_4 = 1 - G_4 = \frac{nk_2 + k_1 \mathrm{k}}{nk}$.

Their relation is $\eta_1 < \eta_3 < \eta_4 < \eta_2$. It is trivial that $\eta_4 < 1 = \eta_2$ and

$$k_1 k + n_2 k_2 + k_1 k_2 = k_1(k_1 + k_2) + n_2 k_2 + k_1 k_2 = {k_1}^2 + n_2 k_2 + 2k_1 k_2$$
$$> {k_1}^2 + {k_2}^2 + 2k_1 k_2 = (k_1 + k_2)^2 = k^2$$

gives the first inequality while

$$nk_2 + k_1 k = (n_1 + n_2)k_2 + k_1 k = k_1 k + n_2 k_2 + n_1 k_2 > k_1 k + n_2 k_2 + k_1 k_2$$

gives the second inequality.

4.3 Conclusion

The above mathematical derivation indicates that auction is efficient but unfair and lottery is fair but inefficient. As a combination of auction and lottery, the hybrid mechanism performs better than auction in terms of equality, but does not always perform better than lottery in terms of efficiency. “Auction then lottery” mechanism proves to be fairer and more efficient than hybrid mechanism, can achieve better tradeoff between equality and efficiency.

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