Group All-pay Auction and Application in Team Events

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Abstract

In this paper, I study all-pay auction with several groups of bidders, which is a model of contests with several teams competing for prizes. After formulating the model, I discuss the winning expectation and equilibrium strategy under different forms of group bid and different cases of valuation privacy.

Introduction

Team events are very common in our daily life, from sporing events to large-scale contests seeking innovative solutions and encouraging teamworks (ACM-ICPC, MCM&ICM, KDD-cup etc). Players invest money, time and/or efforts in order to make his group win. They have different valuations (abilities) private to themselves (in some cases known to teammates) and prizes can be one or more, homogeneous or heterogeneous (the values can be same or different).

In the past, all-pay auction is used as a good model for contests where all players pay out but only winners gain utility. So we continue to use all-pay auction here but bidders no longer bid for themselves, they bid for the whole group. In this case, winning rule must be formulated, so I introduce the definition of group bid in the paper and discuss three forms that are common in our daily life.

The paper is organized as follows: In section 2, I present the model of group all-pay auction. I start from the simple case of two teams and players' valuations are private, analyse the winning expectation and the equilibrium strategy under three forms of group bids. Then I let the valuations private to other teams but known to teammates. Finally extend it to multiple teams and multiple prizes. Details of derivations are appeared in appendix. In section 3, I give a few remarks on the results I have derived. In section 4, I propose a plan of experiments that can examine the results.

Model

Assumptions and Notations

Suppose that there are m teams, each with n players. Each player i from team j has a valuation $v_{j,i}$ independently and uniformly distributed between 0 and 1. The distribution infomation is common knowledge among all bidders. The valuation vector of team j and other members in team j are denoted as v_j and $v_{j,-i}$ respectively.

Denote a bid of player i from team j by $b_{j,i}$ and the vectors of bids $(b_{j,1}, b_{j,2}, \cdots, b_{j,n})$ by b_j. Other teams' vectors of bids are denoted as b_{-j}.

Definition 1. *Given a vector of bids of team j's members b_j, team j's **group bid** is a non-negative real number $B(b_j)$. Later we will discuss different forms of group bid $B(\cdot)$.*

Definition 2. *A **strategy** for a player is a function $s(v) = b$ mapping his true value to a non-negative bid b. We make the following simple assumptions about the strategies the bidders are using:*

- *$s(\cdot)$ is a strictly increasing, differentiable function; in particular, if two bidders have different values, they will submit different bids.*
- *$s(v) \leq v$ for all v: bidders never bid above their true values and $s(0) = 0$ since bids are always non-negative.*

For simplicity, we first analyse the case two teams $(m = 2)$ competing for one prize.

Suppose the cost function is a linear function $c(b) = b$ for any bid(effort level) b. Let p_j denote the winning probability of team j, then the contest success function has the following form:

$$p_j(b_j, b_{-j}) = \begin{cases} 1, & B(b_j) > B(b_{-j}) \\ \frac{1}{2}, & B(b_j) = B(b_{-j}) \\ 0, & B(b_j) < B(b_{-j}) \end{cases}$$

The payoff function of player i from team j can be expressed as

$$u_{j,i}(b_j, b_{-j}) = p_j(b_j, b_{-j})v_{j,i} - b_{j,i}$$

Hence, his expected payoff is

$$g(v_{j,i}) = \mathbb{E}[u_{j,i}(b_j, b_{-j})] = \mathbb{E}[B(b_j) \geq B(b_{-j})]v_{j,i} - b_{j,i}$$

Definition 3. *$s(v_{j,i})$ is the **equilibrium strategy** for player i from team j if*

$$u_{j,i}(s(v_{j,i}), s(v_{j,-i}), s(v_{-j})) \geq u_{j,i}(s(v), s(v_{j,-i}), s(v_{-j}))$$

for any possible value v between 0 and 1, which is the condition that player i from team j does not want to deviate from strategy $s(\cdot)$.

In order for $s(\cdot)$ to satisfy above inequality, it must have the property that the expected payoff function

$$g(v) = \mathbb{E}[B(s(v), s(v_{j,-i}))B(s(v_{-j}))]v_{j,i} - s(v)$$

is maximized when $v = v_{j,i}$. Therefore, $v_{j,i}$ should satisfy $g'(v_{j,i}) = 0$ where g' is the first derivative of $g(\cdot)$ with respect to v.

Equilibrium Bidding

Now consider different forms of group bid. (See all derivations in appendix, I just show the result here)

1. $B(b_j) = \max\{b_{j,1}, b_{j,2}, \cdots, b_{j,n}\}$
   The group bid is the highest bid in the team. Given $v_{j,i}$, team j's winning expectation is
   $$\frac{n}{2n-1}{v_{j,i}}^{2n-1} + \frac{n-1}{2n-1}$$
   The equilibrium strategy is
   $$s(v) = \frac{1}{2}v^{2n}.$$
2. $B(b_j) = \min\{b_{j,1}, b_{j,2}, \cdots, b_{j,n}\}$
   The group bid is the lowest bid in the team, which is known as the "Buckets Effect". Given $v_{j,i}$, team j's winning expectation is
   $$\frac{n}{2n-1} - \frac{n}{2n-1}(1 - v_{j,i})^{2n-1}$$
   The equilibrium strategy is
   $$s(v) = \frac{1}{2}(1-v)^{2n} - \frac{n}{2n-1}(1-v)^{2n-1} + \frac{1}{2(2n-1)}.$$
3. $B(b_j) = b_{j,1} + b_{j,2} + \cdots + b_{j,n}$
   The group bid is the sum of all bids in the team, which can best reflect the reality. Unfortunately, given $v_{j,i}$, team j's winning expectation is hard to compute.

Variations

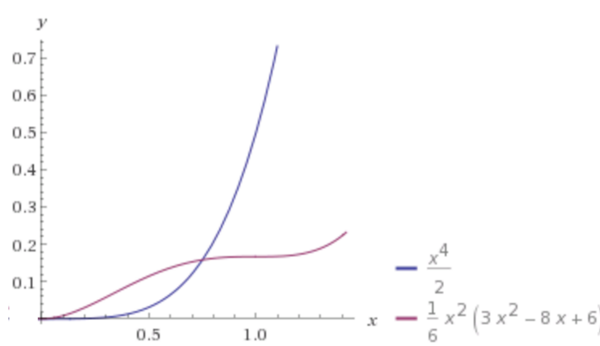
Valuation Known to Teammates In reality, we may know all team members' valuations (abilities) to the contest, then the winning expectation would be different. Suppose the opponents' valuations are still unknown.

1. $B(b_j) = \max\{b_{j,1}, b_{j,2}, \cdots, b_{j,n}\}$
   Given v_j, let $i = \arg\max(v_j)$, team j's winning expectation is ${v_{j,i}}^n$. The equilibrium strategy is player i bids $\frac{n}{n+1}{v_{j,i}}^{n+1}$ and other players bid 0.
2. $B(b_j) = \min\{b_{j,1}, b_{j,2}, \cdots, b_{j,n}\}$
   Given v_j, let $i = \arg\min(v_j)$, team j's winning expectation is $1 - (1 - v_{j,i})^n$. The equilibrium strategy is all players bid $(1 - v_{j,i})^n - \frac{n}{n-1}(1 - v_{j,i})^{n-1} + \frac{1}{n-1}$.
3. $B(b_j) = b_{j,1} + b_{j,2} + \cdots + b_{j,n}$
   Although given v_j, team j's winning expectation is hard to compute, we can suppose that the two teams' sum of valuations are in the same level, that is, there exists integer k such that $\sum_{i=1}^{n} v_{j,i} \in [k, k+1]$ for $j = 1, 2$. Now we can derive the winning expectation and the equilibrium strategy by considering the expectation of
   $$k + 1 \geq \sum_{i=1}^{n} v_{j,i} \geq \sum_{i=1}^{n} v_{-j,i} \geq k$$
   When $n = 2$, player i bids
   $$\frac{(v_{j,1} + v_{j,2})^2 v_{j,i}}{2} - \frac{(v_{j,1} + v_{j,2})^3}{6}$$
   if $v_{j,1} + v_{j,2} \in [0, 1]$ for $j = 1, 2$; player i bids
   $$-\frac{(2 - v_{j,1} - v_{j,2})^2 v_{j,i}}{2} + \frac{(2 - v_{j,1} - v_{j,2})^3}{6}$$
   if $v_{j,1} + v_{j,2} \in [1, 2]$ for $j = 1, 2$.
   When $n > 2$, we can use the same method as appendix.

More than Two Teams Now consider $m > 2$. Note that there can be heterogeneous prizes (multiple prizes with different values). We can suppose that all players value the second prize at a discount of c_2 on the first prize, the third prize at a discount of c_3 on the first prize and so on. Say a contest with first prize 1000 dollars, second prize 500 dollars and third prize 300 dollars, then if a player's valuation on the first prize is v, his valuation on the second and third prize is $0.5v$ and $0.3v$ respectively. Since the calculation would be tediously long, it is unnecessary to go into details here.

Remark

If we plot the equilibrium bid functions of max group bid and min group bid in the private valuation case, we can see that the former one is sensitive to high valuation while the latter one is sensitive to low valuation. We can explain as follows: if the highest bid in the group matters, as one's valuation increases, the possibility to be the key person increases and his bid increases sharply; if the lowest bid in the group matters, player with high valuation makes little difference when producing a high bid but player with low valuation can change a lot. Therefore in the former case, strong players carry the whole team and in the latter case, weak players try to catch up.



However, if valuations are known to teammates, the equilibrium would be quite different. If only the highest bid in the group matters, free riders appear and the one with highest valuation takes the whole responsibility; if only the lowest bid in the group matters, all players pretend as the one with lowest valuation and exert same effort, this reminds me of "the same big bot".

If the group bid is the sum of all members' bids, then one's bid depends on both the sum of all members' valuations and his own valuation, but it follows the rule of "able people should do more work".

Experiments (Planned)

The experiments can be conducted online or offline. If online, each time the system chooses the teammates for players. Players are divided in to several groups with same number of members, they are provided with a picture of an object and must draw their values for the object from a fixed interval (we can rescale it to $[0, 1]$). They are told different rules (max group bid, min group bid or sum group bid) of deciding which group to win and get or lose points as their payoff. We can run the experiments by telling each player the value of his teammates and not telling respectively.

Appendix

For ease of exposition, we use another simple notation system. Denote team 1's valuations as $v_1, v_2, \cdots, v_n$ and bids as $b_1, b_2, \cdots, b_n$, team 2's valuations as $v_1', v_2', \cdots, v_n'$ and bids as $b_1', b_2', \cdots, b_n'$. Without loss of generality, we can only consider the winning expectation of player 1 from team 1.

Valuation private to themselves:

1. $\max\{b_1, b_2, \cdots, b_n\} \geq \max\{b_1', b_2', \cdots, b_n'\}$
   Since $s(\cdot)$ is monotone increasing, higher value means higher bid. So the condition is equivalent to $\max\{v_1, v_2, \cdots, v_n\} \geq \max\{v_1', v_2', \cdots, v_n'\}$.
   If $v_1 = \max\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\int_0^{v_1}\int_0^{v_1}\cdots\int_0^{v_1} dv_2\cdots dv_n dv_1'\cdots dv_n' = v_1^{2n-1}$$
   If $v_1 \neq \max\{v_1, v_2, \cdots, v_n\}$, without loss of generality, assume $v_2 = \max\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\begin{aligned} & \int_{v_1}^1\int_0^{v_2}\cdots\int_0^{v_2} dv_2\cdots dv_n dv_1'\cdots dv_n' \\ = & \int_{v_1}^1 v_2^{2n-2} dv_2 \\ = & \frac{1-v_1^{2n-1}}{2n-1} \end{aligned}$$
   So the overall expectation is
   $$v_1^{2n-1} + \frac{n-1}{2n-1}\left(1 - v_1^{2n-1}\right) = \frac{n}{2n-1}v_1^{2n-1} + \frac{n-1}{2n-1}$$
   So to make
   $$g(v) = \left(\frac{n}{2n-1}v^{2n-1} + \frac{n-1}{2n-1}\right)v_1 - s(v)$$
   maximized when $v = v_1$, substitute $v = v_1$ into
   $$g'(v) = nv^{2n-2}v_1 - s'(v) = 0.$$
   We can obtain that $s'(v_1) = n{v_1}^{2n-1}$, thus
   $$s(v) = \frac{1}{2}v^{2n}.$$

2. $\min\{b_1, b_2, \cdots, b_n\} \geq \min\{b_1', b_2', \cdots, b_n'\}$
   Using the same argument, this is equivalent to $\min\{v_1, v_2, \cdots, v_n\} \geq \min\{v_1', v_2', \cdots, v_n'\}$. For easier calculation, we first consider $\min\{v_1, v_2, \cdots, v_n\} < \min\{v_1', v_2', \cdots, v_n'\}$.
   If $v_1 = \min\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\int_{v_1}^1\int_{v_1}^1\cdots\int_{v_1}^1 dv_2\cdots dv_n dv_1'\cdots dv_n' = (1-v_1)^{2n-1}$$
   If $v_1 \neq \min\{v_1, v_2, \cdots, v_n\}$, without loss of generality, assume $v_2 = \min\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\begin{aligned} & \int_0^{v_1}\int_{v_2}^1\cdots\int_{v_2}^1 dv_2\cdots dv_n dv_1'\cdots dv_n' \\ = & \int_0^{v_1}(1-v_2)^{2n-2}dv_2 \\ = & \frac{1-(1-v_1)^{2n-1}}{2n-1} \end{aligned}$$
   So the overall expectation is
   $$\begin{aligned} & (1-v_1)^{2n-1} + \frac{n-1}{2n-1}\left[1-(1-v_1)^{2n-1}\right] \\ = & \frac{n-1}{2n-1} + \frac{n}{2n-1}(1-v_1)^{2n-1} \end{aligned}$$
   Hence the expectation of $\min\{v_1, v_2, \cdots, v_n\} \geq \min\{v_1', v_2', \cdots, v_n'\}$ should be
   $$\begin{aligned} & 1 - \left[\frac{n-1}{2n-1} + \frac{n}{2n-1}(1-v_1)^{2n-1}\right] \\ = & \frac{n}{2n-1} - \frac{n}{2n-1}(1-v_1)^{2n-1} \end{aligned}$$
   So to make
   $$g(v) = \left[\frac{n}{2n-1} - \frac{n}{2n-1}(1-v_1)^{2n-1}\right]v_1 - s(v)$$
   maximized when $v = v_1$, substitute $v = v_1$ into
   $$g'(v) = n(1-v)^{2n-2}v_1 - s'(v) = 0.$$
   We can obtain that $s'(v_1) = n(1-v_1)^{2n-2}v_1$, thus
   $$s(v) = \frac{1}{2}(1-v)^{2n} - \frac{n}{2n-1}(1-v)^{2n-1} + \frac{1}{2(2n-1)}.$$

3. $b_1 + b_2 + \cdots + b_n \geq b_1' + b_2' + \cdots + b_n'$
   The winning expectation is
   $$\begin{aligned} & \int_0^1\int_0^1\cdots\int_0^1 I[s(v_1) + s(v_2) + \cdots + s(v_n) \\ \geq & s(v_1') + s(v_2') + \cdots + s(v_n')]dv_2\cdots dv_n dv_1'\cdots dv_n' \end{aligned}$$
   where $I[\cdot]$ is the characteristic function, equals to 1 if the inequality is satisfied and 0 otherwise. Since the monotonicity of $s(\cdot)$ can no longer be used and $I[\cdot]$ depends on the inverse function of $s(\cdot)$, solving the winning expectation is somehow involved.

Valuation known to teammates:

1. $\max\{b_1, b_2, \cdots, b_n\} \geq \max\{b'_1, b'_2, \cdots, b'_n\}$
   If $v_1 = \max\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\int_0^{v_1} \cdots \int_0^{v_1} dv'_1 \cdots dv'_n = v_1^n$$
   To make
   $$g(v) = v^n v_1 - s(v)$$
   maximized when $v = v_1$, substitute $v = v_1$ into
   $$g'(v) = nv^{n-1}v_1 - s'(v) = 0.$$
   We can obtain that $s'(v_1) = nv_1^n$, thus
   $$s(v) = \frac{n}{n+1}v^{n+1}$$
   If $v_1 \neq \max\{v_1, v_2, \cdots, v_n\}$, without loss of generality, assume $v_2 = \max\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$\int_0^{v_2} \cdots \int_0^{v_2} dv'_1 \cdots dv'_n = v_2^n$$
   Note that $g(v) = v_2^n v_1 - s(v)$ is maximized when $v = 0$, player 1 has the incentive to pretend his value as 0, that is, to pay no efforts. Actually this is obvious since his bid(effort) makes no difference to the result.
2. $\min\{b_1, b_2, \cdots, b_n\} \geq \min\{b'_1, b'_2, \cdots, b'_n\}$
   If $v_1 = \min\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$1 - \int_{v_1}^{1} \cdots \int_{v_1}^{1} dv'_1 \cdots dv'_n = 1 - (1 - v_1)^n$$
   To make $g(v) = [1-(1-v)^n]v_1 - s(v)$ maximized when $v = v_1$, substitute $v = v_1$ into
   $$g'(v) = n(1-v)^{n-2}v_1 - s'(v) = 0.$$
   We can obtain that $s'(v_1) = n(1-v_1)^{n-2}v_1$, thus
   $$s(v) = (1-v)^n - \frac{n}{n-1}(1-v)^{n-1} + \frac{1}{n-1}$$
   If $v_1 \neq \min\{v_1, v_2, \cdots, v_n\}$, without loss of generality, assume $v_2 = \min\{v_1, v_2, \cdots, v_n\}$, the expectation is
   $$1 - \int_{v_2}^{1} \cdots \int_{v_2}^{1} dv'_1 \cdots dv'_n = 1 - (1 - v_2)^n$$
   Note that this time
   $$g(v) = [1 - (1 - v_2)^n]v_1 - s(v)$$
   is maximized when $v = v_2$, player 1 has the incentive to pretend his value as the lowest one in the team. Actually it's no use for him to produce a high bid.
3. $b_1 + b_2 + \cdots + b_n \geq b'_1 + b'_2 + \cdots + b'_n$
   The winning expectation is
   $$\begin{aligned}&\int_0^1 \int_0^1 \cdots \int_0^1 I[s(v_1) + s(v_2) + \cdots + s(v_n)\\ \geq\ & s(v'_1) + s(v'_2) + \cdots + s(v'_n)]dv'_1 \cdots dv'_n\end{aligned}$$
   where $I[\cdot]$ is the characteristic function, equals to 1 if the inequality is satisfied and 0 otherwise. Although it is still complicated, we can "guess" an equilibrium by adding some constraints and using symmetry.

Claim: If $n = 2$ and $v_1 + v_2 \leq 1$, $v'_1 + v'_2 \leq 1$,
$$s(v_1) = \frac{(v_1+v_2)^2 v_1}{2} - \frac{(v_1+v_2)^3}{6}$$
is an equilibrium strategy for player 1 from team 1.

Proof.
$$s(v_1)+s(v_2) = \frac{(v_1+v_2)^2(v_1+v_2)}{2} - 2\times\frac{(v_1+v_2)^3}{6} = \frac{(v_1+v_2)^3}{6}$$
Similarly,
$$s(v'_1) + s(v'_2) = \frac{(v'_1+v'_2)^3}{6}.$$
So the winning expectation of player 1 (player 2) from team 1 is
$$\begin{aligned}&\int_0^1 \int_0^1 I[s(v_1)+s(v_2) \geq s(v'_1)+s(v'_2) \bigwedge v'_1+v'_2 \leq 1]dv'_1 dv'_2\\ =\ & \int_0^1 \int_0^1 I[\frac{(v_1+v_2)^3}{6} \geq \frac{(v'_1+v'_2)^3}{6} \bigwedge v'_1+v'_2 \leq 1]dv'_1 dv'_2\\ =\ & \int_0^1 \int_0^1 I[1 \geq v_1+v_2 \geq v'_1+v'_2]dv'_1 dv'_2\\ =\ & \frac{(v_1+v_2)^2}{2}\end{aligned}$$
To make
$$g(v) = \frac{(v+v_2)^2}{2}v_1 - s(v)$$
maximized when $v = v_1$, substitute $v = v_1$ into
$$g'(v) = (v+v_2)v_1 - s'(v) = 0.$$
We can obtain that $s'(v_1) = (v_1+v_2)v_1$, thus
$$s(v_1) = \frac{(v_1+v_2)^2 v_1}{2} - \frac{(v_1+v_2)^3}{6}$$
□

We can use a similar argument to show that if $n = 2$ and $v_1 + v_2 \geq 1$, $v'_1 + v'_2 \geq 1$,
$$s(v_1) = -\frac{(2-v_1-v_2)^2 v_1}{2} + \frac{(2-v_1-v_2)^3}{6}$$
is an equilibrium strategy for player 1 from team 1. The proof is omitted here.

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