Exercise 4

Problem 1: Basic orbits

→The problem: to write a program to model, in 2-D, the orbit of particle around a large stationary mass using a 4th order Runge-Kutta approach.
→To solve the problem I have taken the second order differential equation: $m\ddot{r} = -\frac{mMG}{|r|^2}\hat{r}$ and separated it into 2 first order equations, solving for the x and y components individually: $\dot{x} = v_x$ and $\dot{v}_x = -\frac{MGx}{(x^2+y^2)^{3/2}}$ where the y component is given by swapping x and y.

The position at time t= (n+1)h, where h is the time step, is given by:
$x_{t=(n+1)h} = x_{t=nh} + \frac{h}{6}\left(k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x}\right)$ and same for the velocity, where the k's are the standard Runge-Kutta coefficients which are calculated from the equations above.
The total energy of the orbiting particle is: $E = KE + PE = \frac{mv^2}{2} - \frac{GMm}{r}$, if the energy becomes less than zero the particle is travelling faster than the escape velocity meaning it will not return in orbit; if this happens a message is printed to the screen.
I set the stationary mass, M to 5.92719e24 kg – the earths mass, and the mass m to 1 for simplicity.
The program terminates if the radius of the particle becomes less than that of the earth.
→I was able to simulate stable near perfect circular orbits by starting the particle at a distance **r** from the moon with a velocity = $(GM/r)^{1/2}$ in a direction perpendicular to **r**. A stable elliptical orbit is produced by any initial conditions that produce an orbit that does not hit the earth or reach escape velocity. Energy is conserved to 1 part in 4723726.

Problem 2: Moon shot

→Make a program that models the gravitational effect of the moon and the earth on a particle. Find the correct initial conditions to fire a probe from a low earth orbit around the moon, passing 500km from its surface, and back to earth.
→This program uses the same approach as the previous program, however the $\dot{v}$ equation has another term coming from the force due to the moon. I have chosen my coordinate system such that the earth is centred on the origin with the moon centred on the x-axis at +384400km, this way the y distance from both the moon and the earth is the same as the probe's y position and the x distance from the

moon is the probe's x position minus 384400km. If the probes radial distance from either the moon or earth becomes less than the radius of such then the probe has crashed and the program terminates.

→I found that starting with the probe starting at (-7000, 0)km with initial velocity all in the y direction that the probe could be made to pass either side of the moon before returning to earth as shown in figure 2.1.

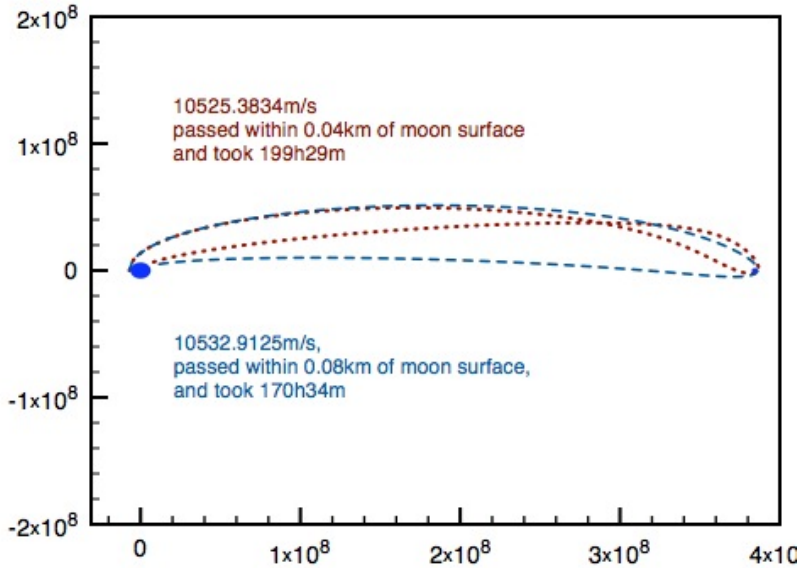

Figure 2.1 – initial starting position (-7000, 0)km initial velocities all in the +y direction. Shows two paths, travelling either side of the moon, both crashing into earth on return.

The probes could be made to go arbitrarily close to the moons surface. However I could not make the probe pass within 500km of the moon's surface without having it crash back into earth on its return. I experimented with starting the probe at points slightly further round its orbit, but once again I found that for the probe to travel within 500km of the moon's surface it needed to have a very low velocity which, after travelling round the moon, led it to crash to earth. The resulting motion with the probe starting at 3 degrees around its orbit is shown in figure 2.2.

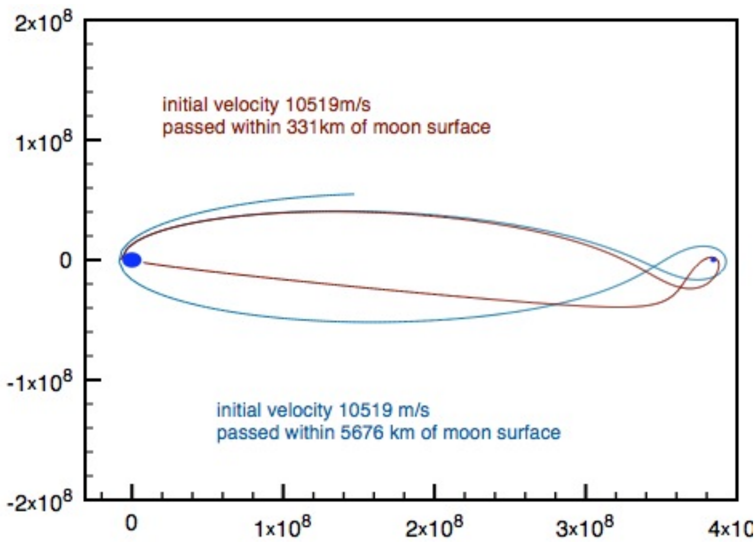

Figure 2.2 – shows the trajectories starting from 3 degrees around the earth orbit, coordinates (-7000cos3, 7000sin3). The initial velocities were both tangent to the orbit.

The blue line shows the velocity that would take the probe closest to the moon without crashing into earth on return.

The red line shows a path that takes the probe within 500km but crashes into earth.

Problem 3: Earth and moon

→Write a program to model 2-body gravitational motion.
→ I used a 4th order Runge-Kutta approach as in the previous programs, solving for each body, but with $\dot{v}_{1x} = -\frac{GM_2}{|r|^3}(x_1 - x_2)$ and $\dot{v}_{2x} = -\frac{GM_2}{|r|^3}(x_2 - x_1)$, where $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, and similarly for the y component. The program stops if r becomes smaller than the radius of M_1 + radius of M_2 as the particles have crashed.
→First simulating two bodies of equal mass: I chose M_1=M_2=1/G, Fig 3.1 shows the motion of the two with initial conditions: r_1 =(-4, 0) v_1 = (0, -0.5) r_2 =(4, 0) v_2 = (0, -1.5).

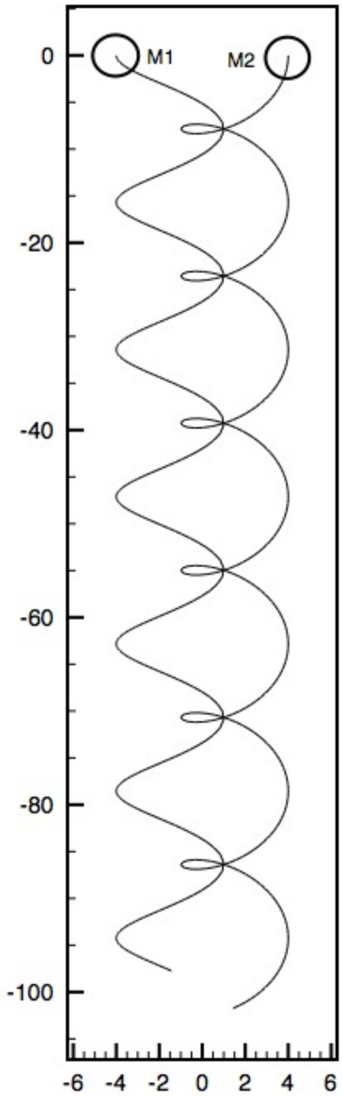

Figure 3.1 – This was using a time step h = 0.001 s for 100 s

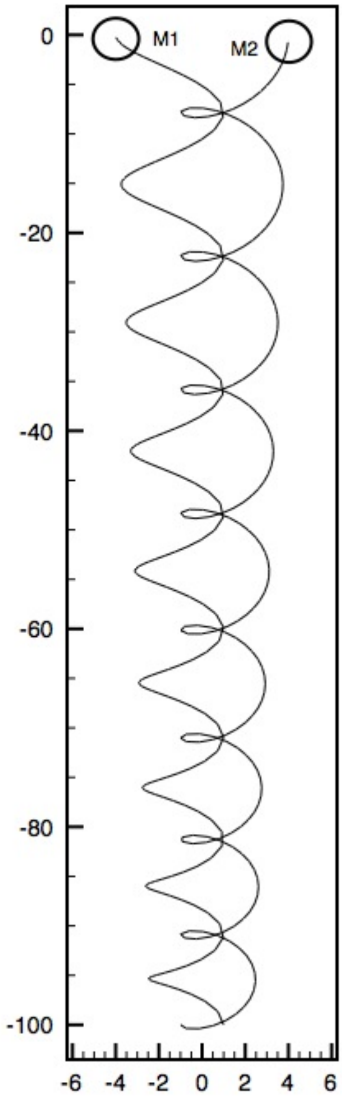

Figure 3.2 – Same initial condition but with h =0.5 s

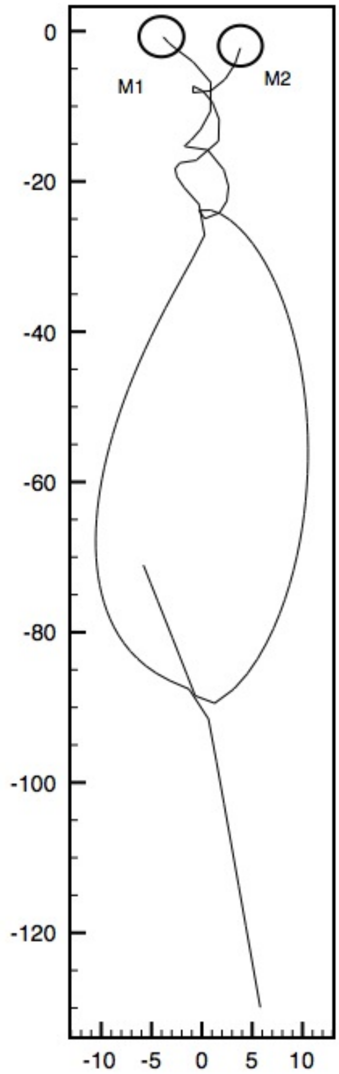

Figure 3.3 - h =1.5 s

Figs 3.2 and 3.3 show the same motion but with larger step sizes. It can be seen in fig 3.2 that energy is not being conserved then in 3.3 that the motion is totally chaotic.

Now where M_1 is the earth and M_2 is the moon: with the origin at the centre of mass, r_e= (-4678.069256, 0) km r_m= (380321.930, 0) km. Giving the moon and earth orbital velocities of 1023 m/s and 12 m/s respectively, their motion – as the centre of mass moves is shown in fig 3.4.

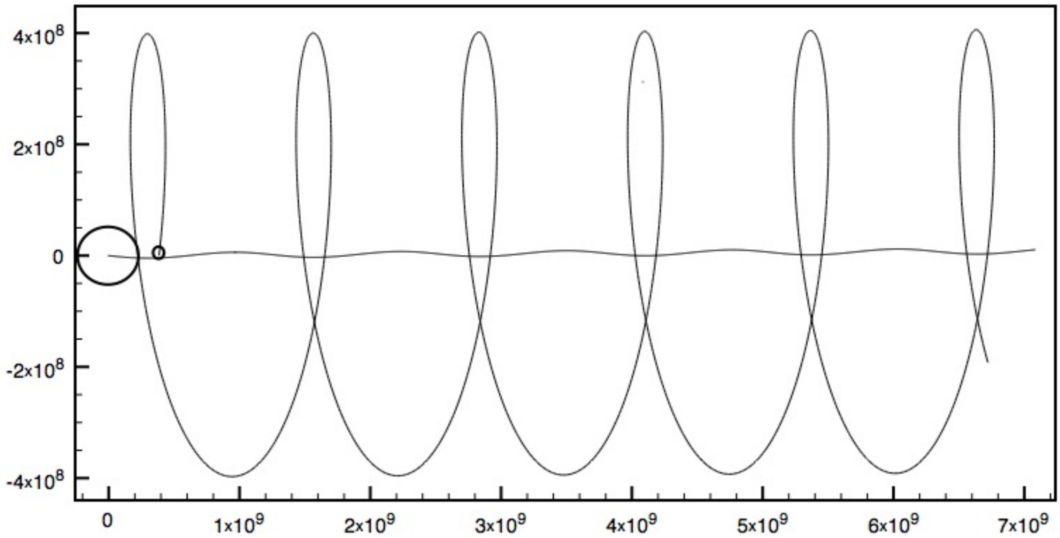

Figure 3.4 – Shows the earth and moon as the orbit their common centre of mass. Planets not to scale.