The exact solution for the Ising model with finite size

According to the equation (39) in [1], we get the expression for the partition function of the Ising model on a finite lattice of size $m \times n$,

$$
\begin{aligned}
Z = \frac{1}{2}(2\sinh 2H)^{mn/2}\Bigg[&\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r}\right) + \prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r}\right) \\
&+\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r-1}\right) + \prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r-1}\right)\Bigg]
\end{aligned}
\tag{1}
$$

with

$$
\cosh\gamma_r = \cosh(2H^*)\cosh(2H) - \sinh(2H^*)\sinh(2H)\cos\frac{r\pi}{n}
\tag{2}
$$

where $H \equiv J/kT$, $\tanh H^* = \mathrm{e}^{-2H}$, and $J > 0$ for ferromagnetic coupling. The calculation program is put in `logz_for_ising.nb`.

Bibliography

[1] Bruria Kaufman. Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis. *Phys. Rev.*, 76(8):1232–1243, oct 1949.