Modeling and Control of Manipulators
Defining Frames

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1 Defining the Frames

When first approaching a manipulator, in order to describe the kinematic chain we need to define a model.
One way is to define a frame attached to each link and to define the transformation matrix of one link with respect to the previous one. In order to uniquely define such frames we can follow the simplified Denavit–Hartenberg convention which states that:

- z axis is in the direction of the link axis;
- x axis pointing towards the next link;
- y axis in order to have a right-handed frame.

Whenever it is not possible to fulfill such convention (i.e. for the x axis) it is suggested to introduce less rotations as possible with respect to the previous link.

2 Transformation Matrices

Once the frames have been defined, in order to describe the robot, one has to compute the transformation matrix of one joint with respect to the previous one.
By defining with the number 0 the base of the manipulator, the following transformation matrices must be defined :

$$ {}^{i-1}_{i}T = \begin{bmatrix} {}^{i-1}_{i}R & {}^{i-1}_{i}\boldsymbol{l}_{<i-1>} \\ 0\,0\,0 & 1 \end{bmatrix} \quad for\ i = 1:n \tag{1} $$

Where

- n is equal to the number of links present in the chain;
- ${}^{i-1}_{i}R$ is the rotation matrix of frame $<i>$ with respect to frame $<i-1>$;

- ${}^{i-1}_{\ i}\boldsymbol{l}_{<i-1>}$ is the translation of frame $< i >$ with respect to frame $< i-1 >$.

This transformation matrix is defined such that

$$ {}^{i-1}_{\ i}T \; {}_{<i>}\boldsymbol{v} = {}_{<i-1>}\boldsymbol{v} \tag{2} $$

Where ${}_{<i>}\boldsymbol{v}$ is a vector expressed in the $< i >$ frame, whereas ${}_{<i-1>}\boldsymbol{v}$ is a vector expressed in the $< i-1 >$ frame.

3 How to Define the Rotation Matrices

In order to define the rotation matrices there are different possible approaches. Let's take into account the two frames depicted in Figure1.

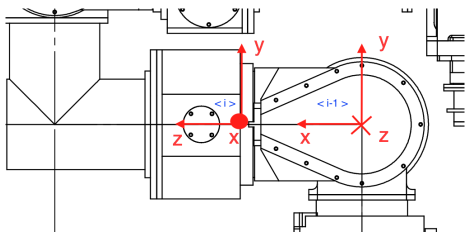

Figure 1: Two frames attached to two links in a manipulator

The first possibility is to detect which rotation occurred in between frame $< i-1 >$ and frame $< i >$ and about which axis and use the following relations:

$$ R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \tag{3} $$

Rotation of θ about x axis

$$ R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{4} $$

Rotation of θ about y axis

$$ R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5} $$

Rotation of θ about z axis

In the example taken into account from frame $< i-1 >$ to frame $< i >$ a rotation of $\frac{\pi}{2}$ about the y axis occurred, hence

$$ {}^{i-1}_{\ i}R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{6} $$

Another way is to remember equation (2) and apply it in case we are dealing with versors.
The $\boldsymbol{i}$ versor in the frame $< i-1 >$ is the versor $\boldsymbol{k}$ in the frame $< i >$.
Therefore

$$ {}^{i-1}_{i}R = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}^{<i>} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = {}^{<i-1>} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow {}^{i-1}_{i}R = \begin{bmatrix} 0 & 0 & 1 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \tag{7} $$

The $\boldsymbol{j}$ versor in the frame $< i-1 >$ is the versor $\boldsymbol{j}$ in the frame $< i >$ (indeed we are dealing with a rotation about the y axis) .
Therefore

$$ {}^{i-1}_{i}R = \begin{bmatrix} 0 & 0 & 1 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}^{<i>} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = {}^{<i-1>} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow {}^{i-1}_{i}R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \tag{8} $$

The $\boldsymbol{k}$ versor in the frame $< i-1 >$ is the versor $-\boldsymbol{i}$ in the frame $< i >$.
Therefore

$$ {}^{i-1}_{i}R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix}^{<i>} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^{<i-1>} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \rightarrow {}^{i-1}_{i}R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{9} $$

To sum up, we are expressing the versors of the frame $< i-1 >$ with respect to the frame $< i >$ and filling the rotation matrix **by rows**.
One can notice that the columns of ${}^{i-1}_{i}R$ are the versors of the frame $< i >$ with respect to the frame $< i-1 >$, so in order to define the rotation matrix one can also decide to express the versors of frame $< i >$ with respect to frame $< i-1 >$ and fill the rotation matrix **by column**

$$ {}^{i-1}_{i}R = \begin{bmatrix} {}^{i-1}_{i}\boldsymbol{i} & {}^{i-1}_{i}\boldsymbol{j} & {}^{i-1}_{i}\boldsymbol{k} \end{bmatrix}. \tag{10} $$

To sum up, knowing that the tranpose of a rotation matrix is equivalent to its inverse, the following relation holds:

$$ {}^{i-1}_{i}R = {}^{i}_{i-1}R^{T} \rightarrow \begin{bmatrix} {}^{i-1}_{i}\boldsymbol{i} & {}^{i-1}_{i}\boldsymbol{j} & {}^{i-1}_{i}\boldsymbol{k} \end{bmatrix} = \begin{bmatrix} {}^{i}_{i-1}\boldsymbol{i}^{T} \\ {}^{i}_{i-1}\boldsymbol{j}^{T} \\ {}^{i}_{i-1}\boldsymbol{k}^{T} \end{bmatrix} \tag{11} $$