LehrFEM++ Hierarchic Finite Elements

Tobias Rohner

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1 Polynomials

Definition 1. *Given $p \in \mathbb{N}$ and a polynomial $\psi(x) \in \mathcal{P}_p(\mathbb{R})$, we define the scaled polynomial $\psi(x;t)$ as*

$$\psi(x;t) := t^p \psi\left(\frac{x}{t}\right)$$

1.1 Shifted Legendre Polynomials

Definition 2 (Shifted Legendre Polynomials)**.** *The Shifted Legendre Polynomials $P_n(\cdot;t) : [0,t] \to \mathbb{R}$ of degree $n \in \mathbb{N}$ are uniquely defined by the following properties:*

$$\int_0^1 P_m(x;t)P_n(x;t)\,\mathrm{d}x = 0 \text{ if } m \neq n, \quad P_n(t;t) = 1$$

Definition 3 (Integrated Shifted Legendre Polynomials)**.** *We define the Integrated Shifted Legendre Polynomials $L_n(\cdot;t) : [0,t] \to \mathbb{R}$ of degree $n \in \mathbb{N}$ as*

$$L_n(x;t) := \int_0^x P_{n-1}(\xi;t)\,\mathrm{d}\xi$$

We observe that due to the orthogonality condition of the Legendre Polynomials, the Integrated Legendre Polynomials of order $n \geq 2$ are zero at $x \in \{0,t\}$:

$$\begin{aligned}
L_n(0;t) &= \int_0^0 P_{n-1}(\xi;t)\,\mathrm{d}\xi = 0 \\
L_n(t;t) &= \int_0^t P_{n-1}(\xi;t)\,\mathrm{d}\xi = \int_0^t P_{n-1}(\xi;t)P_0(\xi;t)\,\mathrm{d}\xi = 0, \text{ for } n \geq 2
\end{aligned}$$

1.2 Jacobi Polynomials

Definition 4 (Jacobi Polynomials)**.** *The Jacobi Polynomials $\tilde{P}_n^{(\alpha,\beta)} : [-1,1] \to \mathbb{R}$ of degree $n \in \mathbb{N}$ are uniquely defined by the following properties:*

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta \tilde{P}_n^{(\alpha,\beta)}(x)\tilde{P}_m^{(\alpha.\beta)}(x)\,\mathrm{d}x, \quad \tilde{P}_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

Definition 5 (Shifted Legendre Polynomials)**.** *We define the Shifted Legendre Polynomials $P_n^{(\alpha,\beta)} : [0,1] \to \mathbb{R}$ of degree $n \in \mathbb{N}$ as*

$$P_n^{(\alpha,\beta)}(x) := \tilde{P}_n^{(\alpha,\beta)}(2x-1)$$

Definition 6 (Integrated Shifted Legendre Polynomials)**.** *We define the Integrated Shifted Legendre Polynomials $L_n^{(\alpha,\beta)} : [0,1] \to \mathbb{R}$ of degree $n \in \mathbb{N}$ as*

$$L_n^{(\alpha,\beta)}(x) := \int_0^x P_{n-1}^{(\alpha,\beta)}(x)\,\mathrm{d}x$$

For the sake of notational simplicity, we allow ourselves to ignore writing the β parameter if it is equal to zero. We therefore have that

$$P_n^\alpha(x) := P_n^{(\alpha,0)}(x), \quad L_n^\alpha(x) := L_n^{(\alpha,0)}(x)$$

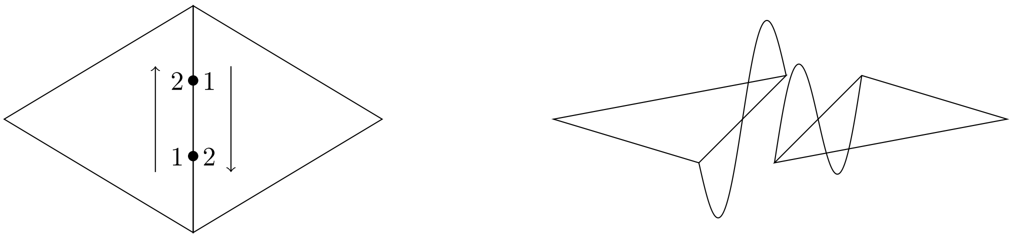

Figure 1: Orientation Problem with Non-Lagrangian Finite Elements

2 Orientation Problems

When using Lagrangian Finite Elements, we rely on the so called "glueing" in order to have a function space containing only continuous functions. This continuity is achieved by having a local numbering of DOFs as illustrated in Figure 1 on the left. A single DOF may have different indices in the neighboring cells. Because the DOFs are symmetrically distributed on the edge, the two local basis functions of the neighboring cells will conveniently be continuous along the shared edge. However, if we do not use Lagrangian FEM, as is the case for Hierarchical FEM where the n-th edge DOF is the basis function coefficient for a polynomial of degree n, we no longer have continuity along the mesh edges (Figure 1 on the right). In order to fix this, we must introduce a global ordering of the edge DOFs, as well as a global orientation of the edges. This is not a big problem though, as LehrFem++ mesh entities provide a function to access the global orientation of their edges. In the case the local orientation is flipped, we simply also flip the order of the local DOFs and the local coordinates at which we evaluate the edge basis functions. In the following sections, we will ignore this ordering problem and one simply has to apply the aforementioned transformations in order to obtain the basis on an arbitrary mesh entity.

3 Basis Functions

We differentiate between three types of basis functions. The vertex basis functions, the edge basis functions and the face bubbles. The vertex basis functions will be nonzero on exactly one vertex and zero on all the others. The edge basis functions will be nonzero on an edge and zero on all other edges and the vertices. The face bubbles are nonzero only on the interior of the mesh entity while being zero on its boundary.

3.1 Segment

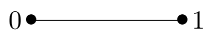

Figure 2: Reference Element for the Segment

Definition 7 (Vertex Basis Functions)**.** *We define the vertex basis functions on the reference segment as*

$$\widehat{b_{\dot{0}}}(x) := 1 - x$$
$$\widehat{b_{\dot{1}}}(x) := x$$

Definition 8 (Edge Basis Functions)**.** *We define the edge basis functions on the reference segment as*

$$\widehat{b_{\bar{n}}}(x) := L_n(x) \text{ for } n \geq 2$$

3.2 Quadrilateral

Each basis function on the quadrilateral is given by the product of two 1D basis functions of the segment. We obtain the following set of basis functions.

Definition 9 (Vertex Basis Functions)**.** *We define the vertex basis functions on the reference quadrilateral as*

$$\widehat{b_{\dot{0}}}(x, y) := (1 - x)(1 - y)$$
$$\widehat{b_{\dot{1}}}(x, y) := x(1 - y)$$
$$\widehat{b_{\dot{2}}}(x, y) := xy$$
$$\widehat{b_{\dot{3}}}(x, y) := (1 - x)y$$

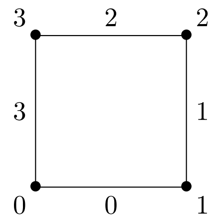

Figure 3: Reference Element for the Quadrilateral

Definition 10 (Edge Basis Functions)**.** *We define the edge basis functions* $\widehat{b^{-}_{e,n}}(x,y)$ *where* $e \in \{0,1,2,3\}$ *is the index of the edge and* $n \geq 2$ *the degree of the basis function as*

$$\widehat{b^{-}_{0,n}}(x,y) := (1-y)L_n(x)$$
$$\widehat{b^{-}_{1,n}}(x,y) := xL_n(y)$$
$$\widehat{b^{-}_{2,n}}(x,y) := yL_n(x)$$
$$\widehat{b^{-}_{3,n}}(x,y) := (1-x)L_n(y)$$

Definition 11 (Face Bubbles)**.** *We define the face bubbles on the reference quadrilateral as*

$$\widehat{b^{\square}_{n,m}}(x,y) := L_n(x)L_m(y)$$

where $n \geq 2$*,* $m \geq 2$*.*

3.3 Triangle

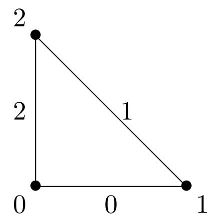

Figure 4: Reference Element for the Triangle

For the ease of notation, we make use of the Barycentric Coordinates given by

$$\lambda_0 = 1 - x - y, \quad \lambda_1 = x, \quad \lambda_2 = y.$$

Definition 12 (Vertex Basis Functions)**.** *We define the vertex basis functions on the reference triangle as*

$$\widehat{b^{\cdot}_{0}}(x,y) := \lambda_0$$
$$\widehat{b^{\cdot}_{1}}(x,y) := \lambda_1$$
$$\widehat{b^{\cdot}_{2}}(x,y) := \lambda_2$$

Definition 13 (Edge Basis Functions)**.** *We define the edge basis functions* $\widehat{b^{-}_{e,n}}(x,y)$ *where* $e \in \{0,1,2\}$ *is the index of the edge and* $n \geq 2$ *the degree of the basis function as*

$$\widehat{b^{-}_{0,n}}(x,y) := L_n(\lambda_1; \lambda_0 + \lambda_1)$$
$$\widehat{b^{-}_{1,n}}(x,y) := L_n(\lambda_2; \lambda_1 + \lambda_2)$$
$$\widehat{b^{-}_{2,n}}(x,y) := L_n(\lambda_0; \lambda_2 + \lambda_0)$$

Note that due to the scaling, the basis functions are only nonzero on the edge they are associated with. Furthermore, they are zero on all vertices.

Definition 14 (Face Bubbles)**.** *We define the face bubbles on the reference triangle as the product of an edge basis function with a blending polynomial.*

$$\widehat{b^{\triangle}_{n,m}}(x,y) := L_m^{2n}(\lambda_2) L_n(\lambda_1; \lambda_0 + \lambda_1)$$

where $n \geq 2$, $m \geq 1$ and $n + m \leq q + 1$ where q is the interior degree of the basis functions.

4 Dual Basis

The dual basis is used to find the basis function coefficients from a set of function evaluations at a predefined set of locations. This must be implemented this way, as a simple matrix inversion to solve for the coefficients is not stable enough for higher polynomial degrees.

We again differentiate between the dual basis associated with the vertices, the edges, and the interior of a cell. It is important to notice that the dual bases must be orthogonalized with respect to the dual bases on the lower dimensional subentities. This is done using Gram-Schmidt and is omitted in the following.

4.1 Segment

Theorem 1 (Vertex Dual Basis)**.** *The vertex dual basis on the reference segment is given by*

$$\begin{aligned} \lambda_0^{\cdot}[f] &:= f(0) \\ \lambda_1^{\cdot}[f] &:= f(1) \end{aligned}$$

Proof. We plug the vertex basis functions into the dual basis to obtain

$$\begin{aligned} \lambda_0^{\cdot}[\widehat{b_0^{\cdot}}] &= \widehat{b_0^{\cdot}}(0) = 1, & \lambda_0^{\cdot}[\widehat{b_1^{\cdot}}] &= \widehat{b_1^{\cdot}}(0) = 0 \\ \lambda_1^{\cdot}[\widehat{b_0^{\cdot}}] &= \widehat{b_0^{\cdot}}(0) = 0, & \lambda_1^{\cdot}[\widehat{b_1^{\cdot}}] &= \widehat{b_1^{\cdot}}(0) = 1 \end{aligned}$$

□

Theorem 2 (Edge Dual Basis)**.** *The edge dual basis on the reference segment is given by*

$$\lambda_n^{-}[f] = (2n-1)\left(P_{n-1}(1)f(1) - P_{n-1}(0)f(0) - \int_0^1 P'_{n-1}(x) f(x)\,\mathrm{d}x\right)$$

Proof. Plugging in the edge basis functions gives

$$\begin{aligned} \frac{1}{2n-1}\lambda_n^{-}[\widehat{b_m^{-}}] &= \frac{1}{2n-1}\lambda_n^{-}[L_m] = P_{n-1}(1)L_m(1) - P_{n-1}(0)L_m(0) - \int_0^1 P'_{n-1}(x)L_m(x)\,\mathrm{d}x \\ &= \int_0^1 P_{n-1}(x)L'_m(x)\,\mathrm{d}x = \int_0^1 P_{n-1}(x)P_{m-1}(x)\,\mathrm{d}x = \frac{1}{2n-1}\delta_{n,m} \end{aligned}$$

□

Note that for the edge dual basis there is no need to orthogonalize with respect to the vertex basis functions because of the orthogonality of the Legendre polynomials.

4.2 Quadrilateral

4.3 Triangle