Assembly Test

Basic function definitions

In[ ]:=
```
a_{i_} := {a_{1,i}, a_{2,i}, a_{3,i}}
a_{i_,j_} := A[[i]][[j]]
s_{i_} := EdgeOrientation[[i + 1]]
Tria := Triangle[{{a_{1,1}, a_{2,1}, a_{3,1}}, {a_{1,2}, a_{2,2}, a_{3,2}}, {a_{1,3}, a_{2,3}, a_{3,3}}}]
```

$$n_{i\_,j\_} := \frac{\text{Cross}[a_i, a_j]}{\text{Sqrt}[\text{Cross}[a_i, a_j].\text{Cross}[a_i, a_j]]}$$

$$\text{nTria} := \frac{\text{Cross}[a_2 - a_1, a_3 - a_1]}{\text{Sqrt}[\text{Cross}[a_2 - a_1, a_3 - a_1].\text{Cross}[a_2 - a_1, a_3 - a_1]]}$$

$$\lambda_0[x\_] := \frac{n_{2,3}.x}{n_{2,3}.a_1}$$

$$\lambda_1[x\_] := \frac{n_{3,1}.x}{n_{3,1}.a_2}$$

$$\lambda_2[x\_] := \frac{n_{1,2}.x}{n_{1,2}.a_3}$$

$$B_0[x\_, n\_] := s_0\,(\lambda_0[x]\ \text{GradGamma}[\lambda_1[x], x, n] - \lambda_1[x]\ \text{GradGamma}[\lambda_0[x], x, n])$$
$$B_1[x\_, n\_] := s_1\,(\lambda_1[x]\ \text{GradGamma}[\lambda_2[x], x, n] - \lambda_2[x]\ \text{GradGamma}[\lambda_1[x], x, n])$$
$$B_2[x\_, n\_] := s_2\,(\lambda_2[x]\ \text{GradGamma}[\lambda_0[x], x, n] - \lambda_0[x]\ \text{GradGamma}[\lambda_2[x], x, n])$$

$$\text{RotB}_{i\_}[x\_, n\_] := \text{RotationMatrix}\left[\frac{\pi}{2}, n\right].B_i[x, n]$$

$$\mu[x\_] := 1$$

Tangential differential Operators

Where the Tangential Nabla operator is defined as the projection of the gradient

$$\nabla_\Gamma = \left(I - \vec{n}\,\vec{n}^T\right)\nabla$$

Which leads to

$$\mathbf{grad}_\Gamma\left(u(\vec{x})\right) = \nabla_\Gamma\, u\,(\vec{x}) = \mathbf{grad}\left(u(\vec{x})\right) - \vec{n}\left(\vec{n}\cdot\mathbf{grad}\left(u(\vec{x})\right)\right)$$

And the Tangential Divergence as

$$\mathrm{div}_\Gamma\left(\vec{u}(\vec{x})\right) = \nabla_\Gamma\cdot\vec{u}\,(\vec{x}) = \nabla\cdot\vec{u}\,(\vec{x}) - \left(\vec{n}\,\vec{n}^T\right)\nabla\cdot\vec{u}\,(\vec{x}) = \nabla\cdot\vec{u}\,(\vec{x}) - \vec{n}^T J_u\,(\vec{x})\,\vec{n}$$

And the following expression for the Tangential Curl

$$\nabla_\Gamma\times\vec{f}\,(\vec{x}) = \left(I - \vec{n}_\Gamma\,\vec{n}_\Gamma^T\right)\nabla\times\vec{f}\,(\vec{x}) = \nabla\times\vec{f}\,(\vec{x}) - \vec{n}\times J_f\,(\vec{x})\,\vec{n}$$

where $J_f(\vec{x})$ is the Jacobian of f at $\vec{x}$

Note that on the surface of the Sphere we get $\vec{n} = \dfrac{\vec{x}}{\|\vec{x}\|}$

But in general $\vec{n}$ just needs to be a normal of some manifold in $\mathbb{R}$ at the point $\vec{x}$

In[ ]:=
```
GradGamma[u_, x_, n_] := Grad[u, x] - n (n.Grad[u, x])
DivGamma[u_, x_, n_] := Div[u, x] - (D[u, {x}].n).n
CurlGamma[u_, x_, n_] := Curl[u, x] - Cross[n, (D[u, {x}].n)]
```

Edge Midpoint Rule for radially scaled function (to 1) multiplied with basis

In[ ]:=
```
EdgeMidpointScalar[{a_, b_, c_}, f_, basis_] :=
 1/3 Area[Triangle[{a, b, c}]] (f[(a + b)/2] × basis[(a + b)/2] +
    f[(c + b)/2] × basis[(c + b)/2] +
    f[(a + c)/2] × basis[(a + c)/2])
```

```
BHack_i_[{xx_, yy_, zz_}] := B_i[{x, y, z}, nTria] /. {x → xx, y → yy, z → zz}
EdgeMidpointVector[{a_, b_, c_}, f_, i_] :=
 1/3 Area[Triangle[{a, b, c}]] (f[(a + b)/2].BHack_i[(a + b)/2] +
    f[(c + b)/2].BHack_i[(c + b)/2] +
    f[(a + c)/2].BHack_i[(a + c)/2])
```

Zero Form

Laplace Matrix

In[ ]:=
```
ElementMatrixZero_ij := Integrate[GradGamma[λ_i[{x, y, z}], {x, y, z}, nTria].
    GradGamma[λ_j[{x, y, z}], {x, y, z}, nTria], {x, y, z} ∈ Tria]
```

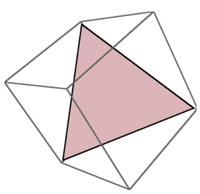
Mass Matrix

In[ ]:=
```
MassMatrixZero_ij := Integrate[λ_i[{x, y, z}] λ_j[{x, y, z}], {x, y, z} ∈ Tria]
```

Test on regular Octagon triangle

In[ ]:=
```
A := {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
Show[Graphics3D[Tria], ImageSize → Tiny]
```

Out[ ]=



Laplace Matrix

In[ ]:=
```
MatrixForm[Table[ElementMatrixZero_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Mass Matrix

In[ ]:=
```
MatrixForm[Table[MassMatrixZero_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4\sqrt{3}} & \frac{1}{8\sqrt{3}} & \frac{1}{8\sqrt{3}} \\ \frac{1}{8\sqrt{3}} & \frac{1}{4\sqrt{3}} & \frac{1}{8\sqrt{3}} \\ \frac{1}{8\sqrt{3}} & \frac{1}{8\sqrt{3}} & \frac{1}{4\sqrt{3}} \end{pmatrix}$$

Load Vector with function $f(\vec{x}) = \dfrac{x_0}{\| \vec{x} \|}$

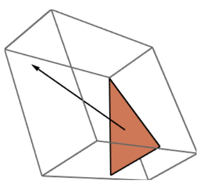
In[ ]:=
```
f[{x_, y_, z_}] := x / Sqrt[{x, y, z}.{x, y, z}]
Table[EdgeMidpointScalar[A, f, λi] , {i , {0, 1, 2}}]
```

Out[ ]= $\left\{\frac{1}{2\sqrt{6}}, \frac{1}{4\sqrt{6}}, \frac{1}{4\sqrt{6}}\right\}$

Test on flat triangle

In[ ]:=
```
A := {{1, 0, 0}, {1/2, 1/2, 0}, {0, 1/2, 1/2}}
Show[Graphics3D[{Tria, Arrow[{(a1 + a2 + a3)/3, (a1 + a2 + a3)/3 + nTria}]}],
 ImageSize → Tiny]
```

Out[ ]=



Laplace Matrix

In[ ]:=
```
MatrixForm[Table[ElementMatrixZeroij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[ElementMatrixZeroij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & 0 \\ -\frac{1}{2\sqrt{5}} & \frac{3}{\sqrt{5}} & -\frac{\sqrt{5}}{2} \\ 0 & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \end{pmatrix}$$

Out[ ]//CForm=

```
List(1/(2.*Sqrt(5)),-0.5*1/Sqrt(5),0,
   -0.5*1/Sqrt(5),3/Sqrt(5),-0.5*Sqrt(5),0,
   -0.5*Sqrt(5),Sqrt(5)/2.)
```

Mass Matrix

In[ ]:=
```
MatrixForm[Table[MassMatrixZeroij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[MassMatrixZeroij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{5}}{48} & \frac{\sqrt{5}}{96} & \frac{\sqrt{5}}{96} \\ \frac{\sqrt{5}}{96} & \frac{\sqrt{5}}{48} & \frac{\sqrt{5}}{96} \\ \frac{\sqrt{5}}{96} & \frac{\sqrt{5}}{96} & \frac{\sqrt{5}}{48} \end{pmatrix}$$

Out[ ]//CForm=

```
List(Sqrt(5)/48.,Sqrt(5)/96.,Sqrt(5)/96.,
   Sqrt(5)/96.,Sqrt(5)/48.,Sqrt(5)/96.,
   Sqrt(5)/96.,Sqrt(5)/96.,Sqrt(5)/48.)
```

Load Vector with function $f(\vec{x}) = \sin(x_0 + x_1)$

In[ ]:=
```
f[{x_, y_, z_}] := Sin[x + y]
Table[EdgeMidpointScalar[{a1, a2, a3}, f, λi] , {i , {0, 1, 2}}]
CForm[Table[EdgeMidpointScalar[{a1, a2, a3}, f, λi] , {i , {0, 1, 2}}]]
```

Out[ ]=
$$\left\{\frac{1}{24}\sqrt{5}\left(\frac{1}{2}\,\mathrm{Sin}\!\left[\frac{3}{4}\right]+\frac{\mathrm{Sin}[1]}{2}\right),\ \frac{1}{24}\sqrt{5}\left(\frac{1}{2}\,\mathrm{Sin}\!\left[\frac{1}{4}\right]+\frac{\mathrm{Sin}[1]}{2}\right),\ \frac{1}{24}\sqrt{5}\left(\frac{1}{2}\,\mathrm{Sin}\!\left[\frac{1}{4}\right]+\frac{1}{2}\,\mathrm{Sin}\!\left[\frac{3}{4}\right]\right)\right\}$$

Out[ ]//CForm=

```
List((Sqrt(5)*(Sin(0.75)/2. + Sin(1)/2.))/
    24.,(Sqrt(5)*
      (Sin(0.25)/2. + Sin(1)/2.))/24.,
   (Sqrt(5)*(Sin(0.25)/2. + Sin(0.75)/2.))/
    24.)
```

One Form

Whitney One Curl Curl Matrix

In[ ]:=
```
WOneCurlCurlMatrixij := Integrate[CurlGamma[Bi[{x, y, z}, nTria], {x, y, z}, nTria].
    CurlGamma[Bj[{x, y, z}, nTria], {x, y, z}, nTria], {x, y, z} ∈ Tria]
```

Whitney One Dot Grad Matrix

In[ ]:=
```
WOneDotMatrixij :=
  Integrate[Bi[{x, y, z}, nTria].GradGamma[λj[{x, y, z}], {x, y, z}, nTria], {x, y, z} ∈ Tria]
```

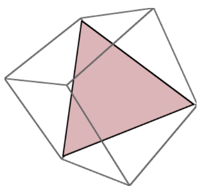
One Form Mass Matrix

In[ ]:=
```
OneFormMassMatrix_ij :=
 Integrate[B_i[{x, y, z} , nTria].B_j[{x, y, z}, nTria], {x, y, z} ∈ Tria]
```

Test on regular Octagon triangle

In[ ]:=
```
A := {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
EdgeOrientation := {1, 1, -1}
Show[Graphics3D[Tria], ImageSize → Tiny]
```

Out[ ]=



Whitney One Curl Curl Matrix

In[ ]:=
```
MatrixForm[Table[WOneCurlCurlMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[WOneCurlCurlMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$

Out[ ]//CForm=

```
List(2/Sqrt(3),2/Sqrt(3),-2/Sqrt(3),2/Sqrt(3),2/Sqrt(3),-2/Sqrt(3),-2/Sqrt(3),-2/Sqrt(3),2/Sqrt(3))
```

Whitney One Dot Grad Matrix

In[ ]:=
```
MatrixForm[Table[WOneDotMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[WOneDotMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} \end{pmatrix}$$

Out[ ]//CForm=

```
List(-0.5*1/Sqrt(3),1/(2.*Sqrt(3)),0,0,-0.5*1/Sqrt(3),1/(2.*Sqrt(3)),-0.5*1/Sqrt(3),0,
  1/(2.*Sqrt(3)))
```

One Form Mass Matrix

In[ ]:=
```
MatrixForm[Table[OneFormMassMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[OneFormMassMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{5}{12\sqrt{3}} & -\frac{1}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} \\ -\frac{1}{12\sqrt{3}} & \frac{5}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} \\ \frac{1}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} & \frac{5}{12\sqrt{3}} \end{pmatrix}$$

Out[ ]//CForm=

```
List(5/(12.*Sqrt(3)),-0.08333333333333333*1/Sqrt(3),1/(12.*Sqrt(3)),-0.08333333333333333*1/S
   5/(12.*Sqrt(3)),1/(12.*Sqrt(3)),1/(12.*Sqrt(3)),1/(12.*Sqrt(3)),5/(12.*Sqrt(3)))
```

Load Vector with function $f(\vec{x}) = \begin{pmatrix} 1 \\ 1 \\ x_0 \end{pmatrix}$

In[ ]:=
```
f[{x_, y_, z_}] := {1, 1, x}
N[Table[EdgeMidpointVector[{a1, a2, a3}, f, i] , {i , {0, 1, 2}}]]
CForm[Table[EdgeMidpointVector[{a1, a2, a3}, f, i], {i , {0, 1, 2}}]]
```

Out[ ]= {-0.0240563, -0.216506, -0.168394}

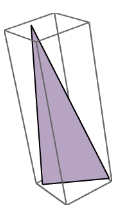
Out[ ]//CForm=

```
List(-0.041666666666666664*1/Sqrt(3),
   -0.125*Sqrt(3),-7/(24.*Sqrt(3)))
```

Test on flat triangle

In[ ]:=
```
A := {{3, 0, 0}, {0.5, 1, 0}, {0, 0.2, 1}}
EdgeOrientation := {-1, 1, -1}
Show[Graphics3D[Tria], ImageSize → Tiny]
```

Out[ ]=



Whitney One Curl Curl Matrix

In[ ]:=
```
MatrixForm[Table[WOneCurlCurlMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[WOneCurlCurlMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.514344 & -0.514344 & 0.514344 \\ -0.514344 & 0.514344 & -0.514344 \\ 0.514344 & -0.514344 & 0.514344 \end{pmatrix}$$

Out[ ]//CForm=
```
List(0.5143444998736394,-0.5143444998736396,0.5143444998736395,-0.5143444998736396,
   0.5143444998736396,-0.5143444998736396,0.5143444998736395,-0.5143444998736396,0.5
```

Whitney One Dot Grad Matrix

In[ ]:=
```
MatrixForm[Table[WOneDotMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[WOneDotMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.126014 & -0.495057 & 0.369042 \\ 0.0411476 & -0.822951 & 0.781804 \\ -0.0848668 & -0.327895 & 0.412761 \end{pmatrix}$$

Out[ ]//CForm=
```
List(0.1260144024690417,-0.49505658112837825,0.3690421786593366,0.04114755998989116,
   -0.8229511997978233,0.7818036398079323,-0.08486684247915052,-0.32789461866944536,
   0.4127614611485959)
```

One Form Mass Matrix

In[ ]:=
```
MatrixForm[Table[OneFormMassMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[OneFormMassMatrixij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Load Vector with function $f(\vec{x}) = \begin{pmatrix} \cos(x_0 - x_2) \\ \sin(x_0 - x_2) \\ x_2 \end{pmatrix}$

In[ ]:=
```
f[{x_, y_, z_}] := {Cos[x - z], Sin[x + y], z}
N[Table[EdgeMidpointVector[{a1, a2, a3}, f, i] , {i , {0, 1, 2}}]]
CForm[Table[EdgeMidpointVector[{a1, a2, a3}, f, i], {i , {0, 1, 2}}]]
```

Out[ ]= {-0.131424, -0.1793, -0.378278}

Out[ ]//CForm=
```
List(-0.13142416550290498,-0.17929962791878098,
   -0.3782779903706242)
```

Two Form

Rot Whitney One Dot Product Matrix

This should be equal to the One Form Mass Matrix

In[∘]:=
```
RotWOneDotMatrix_ij :=
  Integrate[RotB_i[{x, y, z} , nTria].RotB_j[{x, y, z}, nTria], {x, y, z} ∈ Tria]
```

Rot Whitney One Div Matrix

In[∘]:=
```
RotWOneDivMatrix_i_ :=
  Integrate[DivGamma[RotB_i[{x, y, z}, nTria], {x, y, z}, nTria], {x, y, z} ∈ Tria]
```

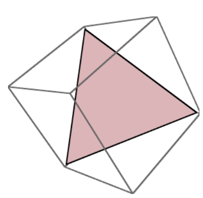
Two Form Mass Matrix

In[∘]:=
```
TwoFormMass := Integrate[μ[{x, y, z}], {x, y, z} ∈ Tria]
```

Test on regular Octagon triangle

In[∘]:=
```
A := {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
EdgeOrientation := {1, 1, -1}
Show[Graphics3D[Tria], ImageSize → Tiny]
```

Out[∘]=



Rot Whitney One Dot Product Matrix

In[ ]:=
```
MatrixForm[Table[RotWOneDotMatrix_{ij}, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[RotWOneDotMatrix_{ij}, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{5}{12\sqrt{3}} & -\frac{1}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} \\ -\frac{1}{12\sqrt{3}} & \frac{5}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} \\ \frac{1}{12\sqrt{3}} & \frac{1}{12\sqrt{3}} & \frac{5}{12\sqrt{3}} \end{pmatrix}$$

Out[ ]//CForm=

```
List(5/(12.*Sqrt(3)),
   -0.08333333333333333*1/Sqrt(3),
   1/(12.*Sqrt(3)),
   -0.08333333333333333*1/Sqrt(3),
   5/(12.*Sqrt(3)),1/(12.*Sqrt(3)),
   1/(12.*Sqrt(3)),1/(12.*Sqrt(3)),
   5/(12.*Sqrt(3)))
```

Rot Whitney One Div Matrix

In[ ]:=
```
MatrixForm[Table[RotWOneDivMatrix_i, {i, {0, 1, 2}}]]
CForm[Table[RotWOneDivMatrix_i, {i, {0, 1, 2}}]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Out[ ]//CForm=

```
List(-1,-1,1)
```

Two Form Mass Matrix

In[ ]:=
```
TwoFormMass
```

Out[ ]= $\frac{\sqrt{3}}{2}$

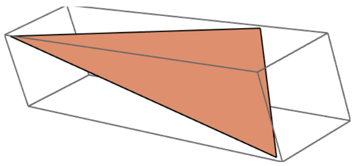
Load Vector with function $f(\vec{x}) = \frac{x_0}{\| \vec{x} \|}$

In[ ]:=
```
f[{x_, y_, z_}] := x / Sqrt[{x, y, z}.{x, y, z}]
EdgeMidpointScalar[A, f, μ]
```

Out[ ]= $\frac{1}{\sqrt{6}}$

Test on flat triangle

In[ ]:=
```
A := {{5, 1, 0}, {0, 1, 0}, {0, 0, 1}}
EdgeOrientation := {1, -1, -1}
Show[Graphics3D[Tria], ImageSize → Small]
```

Out[ ]=


Rot Whitney One Dot Product Matrix

In[ ]:=
```
MatrixForm[Table[RotWOneDotMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]
CForm[Flatten[Table[RotWOneDotMatrix_ij, {i, {0, 1, 2}}, {j, {0, 1, 2}}]]]
```

Rot Whitney One Div Matrix

In[ ]:=
```
MatrixForm[Table[RotWOneDivMatrix_i, {i, {0, 1, 2}}]]
CForm[Table[RotWOneDivMatrix_i, {i, {0, 1, 2}}]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Out[ ]//CForm=
```
List(-1,1,1)
```

Two Form Mass Matrix

In[ ]:=
```
TwoFormMass
```

Out[ ]=

$$\sqrt{\frac{21}{2}}$$

Load Vector with function $f(\vec{x}) = \sin(x_2)$

In[ ]:=
```
f[{x_, y_, z_}] := Sin[y]
EdgeMidpointScalar[{a₁, a₂, a₃}, f, μ]
```

Out[ ]=

$$\sqrt{\frac{14}{3}}\ \mathrm{Sin}\left[\frac{1}{2}\right]$$