Distances of Planets from the Sun and of Satellites from their Primaries in the Satellite Systems of Jupiter, Saturn, and Uranus

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ABSTRACT

It is found that the mean distances of planets from the sun and of satellites from their primaries in the satellite systems of Jupiter, Saturn, and Uranus are in obeisance to a common law of variation with respect to planet (or satellite) ordinal number. The principal variation of distance with respect to ordinal number is exponential and found to be given by $(1.728)^n$. This principal variation is modified by a second variation, individual values of which, for a given system, equal the lengths of radii vectors of an ellipse corresponding to polar angles $(4\pi/13)n$. While each system has its own characteristic distribution ellipse, the various ellipses have certain properties in common. With the exception of a few orbital bodies that subtend unusually small solid angles at the centers of their respective systems, the probable error of computed distances is 0.20 of one per cent, the greatest single error being 0.764 of one per cent.

The purpose of this paper is to demonstrate that a common law of variation exists for the mean distances of planets from the sun and of satellites from their primaries in the satellite systems of Jupiter, Saturn, and Uranus. This common law of variation is with respect to planet (or satellite) ordinal number.[1]

When mean sun-to-planet distances are plotted against planet ordinal numbers, using rectangular semi-logarithmic coördinates, the graph of Figure 1 is obtained. The general trend of the plotted points is given by the straight line of this figure. This line has the equation

$$\log_{10} D_n = \log_{10} D_0 + n \log_{10} \tau \tag{1}$$

where $\log_{10} D_0$ is its intercept with the vertical axis and $\log_{10} \tau$ its slope. Equation (1) corresponds to the simple exponential variation

$$D_n = D_0 \tau^n \tag{2}$$

The straight line, as shown, has values of $\log_{10} D_0$ and of $\log_{10} \tau$ that

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[1] The ordinal number of a planet (or satellite) denotes the order of its remoteness from the central body of the system. In the present paper the maximum values of ordinal numbers used are: Solar system 10, Jupiter's system 10, Saturn's system 10, Uranus' system 5.

were determined by the method of least squares from ten observation equations [of the form of equation (1)], using known (observed) values of $\log_{10} D_n$ for the various planets. The observation equations used are shown in Table I. Their solution yields $\log_{10} D_o = -0.670341$ and $\log_{10} \tau = 0.23325$. The corresponding value of τ is 1.711.

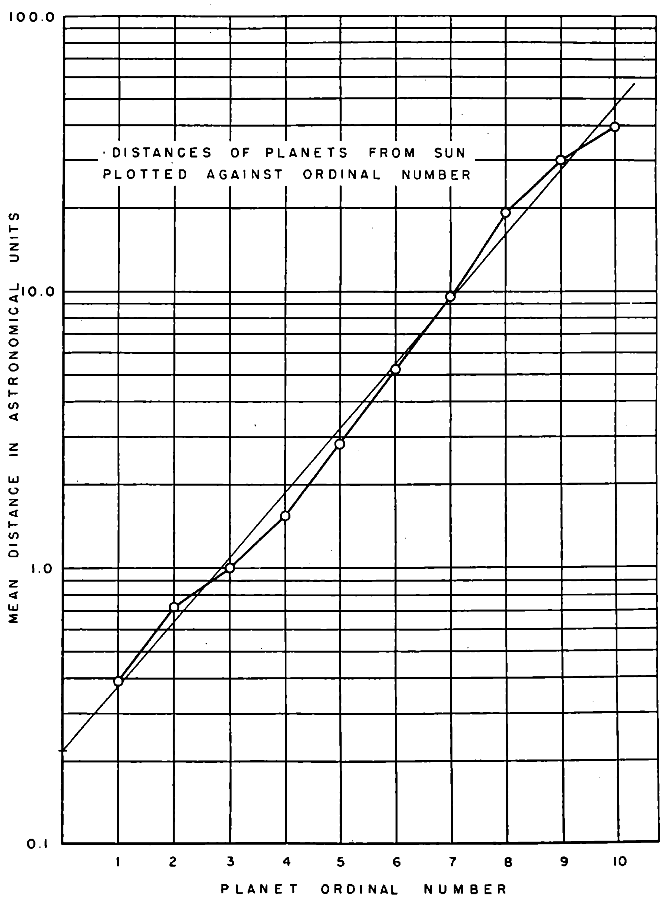

FIGURE 1
SEMI-LOGARITHMIC PLOT OF D_n AGAINST n.

The plotted points of Figure 1 oscillate in a regular manner above and below the straight line of simple exponential variation. This suggests representing values of D_n by an equation of the form

$$D_n = D(\kappa n) \cdot \tau^n \tag{3}$$

in which $D(\kappa n)$ is an oscillatory function of κn, κ being a constant. Since $D(\kappa n)$ is considered oscillatory, it is reasonable to suppose that a polar coördinate plot of D_n/τ^n values against values of κn may prove enlightening. A reasonable (tentative) value of κ appears to be $4\pi/13$.

TABLE I

OBSERVATION EQUATIONS USED FOR DETERMINING THE GENERAL TREND OF PLOTTED POINTS IN FIGURE 1

	$\log_{10} D_0$	+	$n \log_{10} \tau$	=	$\log_{10} D_n$*
1	$\log_{10} D_0$	+	$1 \log_{10} \tau$	=	9.5878221 —10
2	$\log_{10} D_0$	+	$2 \log_{10} \tau$	=	9.8593371 —10
3	$\log_{10} D_0$	+	$3 \log_{10} \tau$	=	0.0000000
4	$\log_{10} D_0$	+	$4 \log_{10} \tau$	=	0.1828961
5	$\log_{10} D_0$	+	$5 \log_{10} \tau$	=	0.4420562
6	$\log_{10} D_0$	+	$6 \log_{10} \tau$	=	0.7162374
7	$\log_{10} D_0$	+	$7 \log_{10} \tau$	=	0.9794957
8	$\log_{10} D_0$	+	$8 \log_{10} \tau$	=	1.2830971
9	$\log_{10} D_0$	+	$9 \log_{10} \tau$	=	1.4781431
10	$\log_{10} D_0$	+	$10 \log_{10} \tau$	=	1.5961288

*Determined from values of D_n taken from "Astronomical Constants" in the *American Ephemeris and Nautical Almanac.*

This follows from observing in Figure 1 that the period of oscillation approximates $n = 6.5$, so that the value of $(4\pi/13)n$ for $n = 6.5$ becomes 2π, which corresponds to one natural cycle in a polar plot. Using known values of D_n and the "most probable" value of τ obtained by the method of least squares, values of D_n/τ^n are plotted as radii vectores against values of $(4\pi/13)n$ as polar angles in Figure 2.

A smooth curve, having the form of a spiral, connects the plotted points in Figure 2. Inspection suggests that an increase in the value

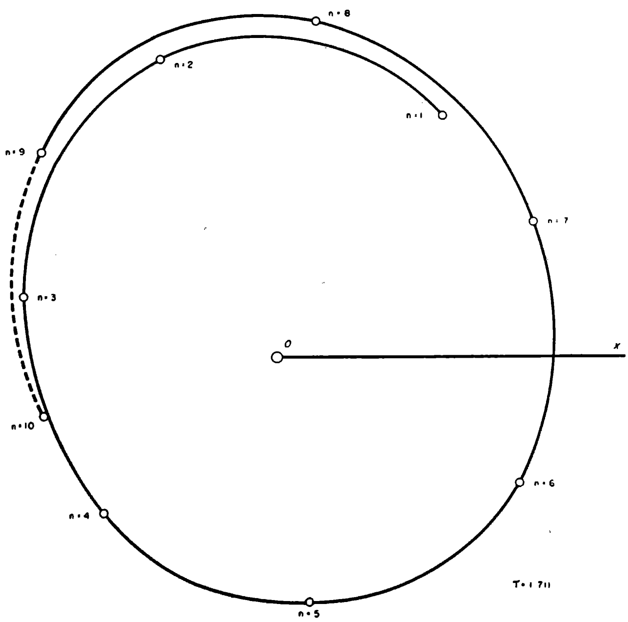

FIGURE 2

VALUES OF D_n/τ^n FOR PLANETS OF THE SOLAR SYSTEM PLOTTED AGAINST $(4\pi/13)n$.

of τ may transform the spiral into a closed figure. This has been found to be true, and for $\tau = 1.728$ the figure approximates a true ellipse as is shown in Figure 3 where values of $D_n/(1.728)^n$ are plotted against $(4\pi/13)n$. The general agreement between the plotted points and corresponding radii vectores of the ellipse (in Figure 3) is good except

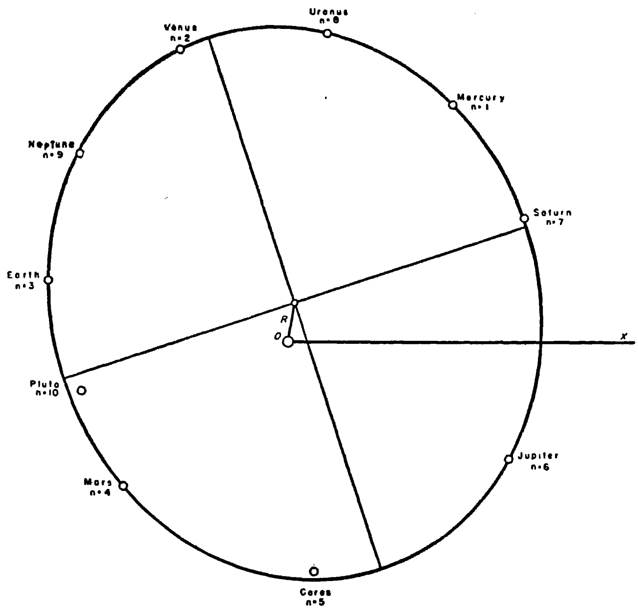

FIGURE 3

VALUES OF $D_n/(1.728)^n$ FOR PLANETS OF THE SOLAR SYSTEM PLOTTED AGAINST $(4\pi/13)n$.

in the case of Ceres and Pluto.[2] It should be remarked that no appreciably better value for κ than $4\pi/13$ has been found.

The foregoing analysis leads to the conclusion that, to a good approximation, *the mean distances of planets from the sun are given by the equation*

$$D_n = D\,[(4\pi/13)\,n] \cdot (1.728)^n \qquad (4)$$

where n *is planet ordinal number and the values of* $D\,[(4\pi/13)n]$ *are the radii vectores of an ellipse, having polar angles* $(4\pi/13)n$. This ellipse will be called the "distribution ellipse for the planets of the solar system." The notation of equation (4) can be simplified by writing ρ_n for the values of $D\,[(4\pi/13)n]$ corresponding to integral values of n; that is, we may regard the ρ_n to be the particular radii vectores of $D\,[(4\pi/13)n]$ that correspond to the various planet ordinal numbers. By this change, equation (4) becomes

$$D_n = \rho_n\,(1.728)^n \qquad (4)'$$

[2] Ceres and Pluto each subtend unusually small solid angles at the system center. This point is discussed near the end of this paper, paragraph 7.

The ellipse of Figure 3, with center displaced a short distance from the pole of the coördinate system, is of the general form shown in Figure 4. The corresponding general equation in terms of the para-

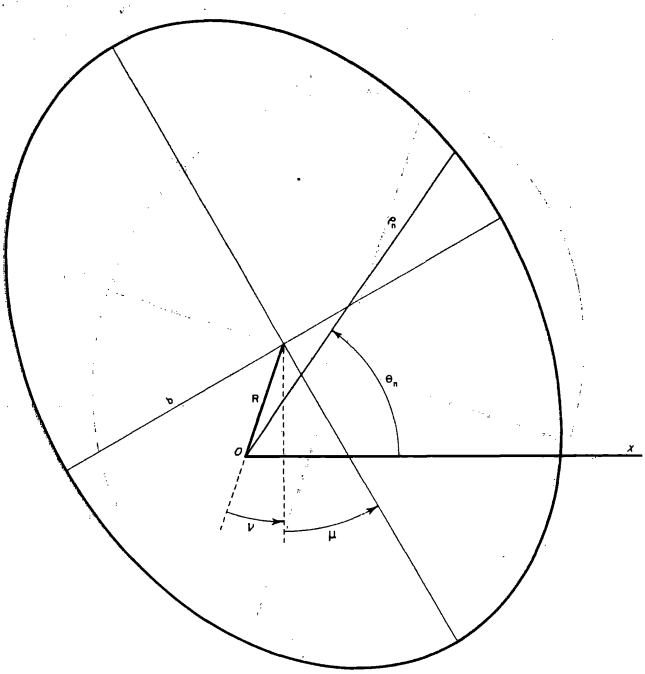

FIGURE 4

TYPICAL FORM OF DISTRIBUTION ELLIPSE

meters of Figure 4 is

$$[1-e^2\sin^2(\theta_n-\mu)]\,\rho_n^2 - R\{(2-e^2)\sin(\theta_n+\nu) - e^2\sin[\theta_n-(2\mu+\nu)]\}\,\rho_n + R^2[1-e^2\cos^2(\mu+\nu)] - b^2 = 0 \qquad (5)$$

where e is the eccentricity of the ellipse, and, as shown in the figure, b is the semi-minor axis, R the displacement of the center of the ellipse from the pole of the coördinate system, ν the angle between the displacement line R and a normal to the initial line of the coördinate system, and μ the angle between the same normal and the major axis of the ellipse.

Values found (by graphical and numerical procedures) for the constants of the solar system distribution ellipse are given in Table II. Values of ρ_n computed from equation (5), using the constants of Table II, appear in Table III.

TABLE II

CONSTANTS OF DISTRIBUTION ELLIPSE FOR THE SOLAR SYSTEM

$e^2 = 0.2352$	$\mu = \theta_1/3$
$b = 0.1900$ A.U.	$\nu = \theta_1/6$
$R/b = 0.1588$	$\theta_1 = 4\pi/13$

Values of D_n computed from equation (4)′, using solar system values of ρ_n from Table III are compared with the known mean distances of the various planets in Table IV.

TABLE III

SOLAR SYSTEM VALUES OF ρ_n COMPUTED FROM EQUATION (5), USING THE DISTRIBUTION ELLIPSE CONSTANTS OF TABLE II

n	ρ_n	n	ρ_n
1	.2238937	6	.1950722
2	.2424635	7	.2066141
3	.1936949	8	.2416496
4	.1710605	9	.2205749
5	.1862469	10	.1762333

TABLE IV

COMPARISON OF OBSERVED VALUES OF D_n WITH THOSE COMPUTED FROM $D_n = \rho_n (1.728)^n$

SOLAR SYSTEM

n	Name of Planet	Observed** D_n	Solid Angle† Subtended at System Center	Computed D_n	Ratio	Percent Difference
1	Mercury	.387099	0.579×10^{-8}	.386888	.99946	− 0.054
2	Venus	.723331	1.032×10^{-8}	.723992	1.00091	+ 0.091
3	Earth	1.000000	0.570×10^{-8}	.999423	0.99942	− 0.058
4	Mars	1.523688	0.0693×10^{-8}	1.525193	1.00099	+ 0.099
5	Ceres*	2.767303	0.00027×10^{-8}	2.869509	1.03693	+ 3.693*
6	Jupiter	5.202803	2.53×10^{-8}	5.193472	0.99821	− 0.179
7	Saturn	9.538843	0.510×10^{-8}	9.505306	0.99648	− 0.352
8	Uranus	19.190978	0.0248×10^{-8}	19.210380	1.00101	+ 0.101
9	Neptune	30.070672	0.0097×10^{-8}	30.300493	1.00764	+ 0.764
10	Pluto*	39.45743	0.00082×10^{-8}	41.833607	1.06022	+ 6.022*

*These bodies subtend unusually small solid angles at the system center. "Percent differences" are correspondingly large.

**Values taken from the *American Ephemeris and Nautical Almanac*.

†In steradian measure.

Equations (4)′ and (5) have permitted us to compute the mean distances of the planets from the sun as a function of planet ordinal number. Good agreement with known distances has been obtained. *These same two equations also hold for the mean distances of satellites from their primaries in the satellite systems of Jupiter, Saturn, and Uranus.* Each system has its own characteristic distribution ellipse so that the values of constants in equation (5) change from system to system.

Values of $D_n/(1.728)^n$ for each of the three satellite systems are given in Table V and are plotted against $(4\pi/13)n$ in Figures 5, 6, and 7. The ellipses of these figures were found by trial with an ellipsograph to be those best fitting the plotted points.[3] The values of constants for use in equation (5) for each satellite system were determined by graphical and numerical methods from the fitted ellipses. These values appear in the summary of constants of Table VI.

In assigning ordinal numbers, it was necessary to assume the existence

[3] Since only four plotted points exist for Uranus, a family of ellipses was possible in this case. The particular ellipse shown in Figure 6 was selected on the assumption that certain properties (discussed later) common to other distribution ellipses were also common to that of Uranus.

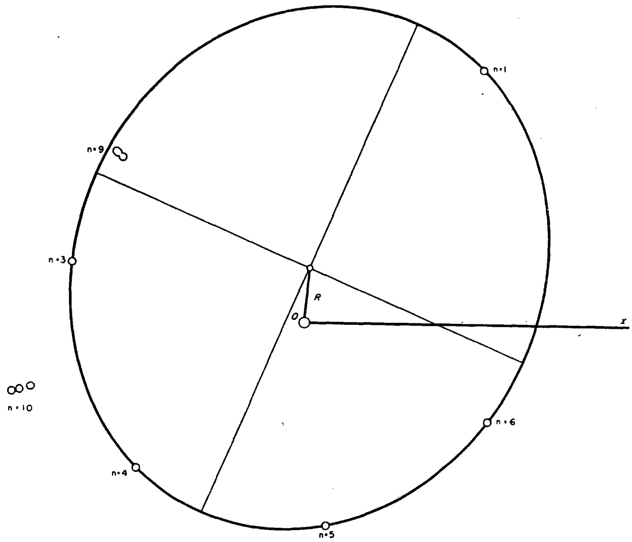

FIGURE 5
VALUES OF $D_n/(1.728)^n$ FOR SATELLITES OF JUPITER
PLOTTED AGAINST $(4\pi/13)n$.

of a few vacancies among known satellites. These were for ordinal numbers 2, 7, 8 in Jupiter's system, number 9 in Saturn's system and number 1 in Uranus' system. Also, in the case of two small satellites of Saturn, it was found necessary to use the ordinal number of an adjacent satellite of superior mass for the exponent of 1.728. The or-

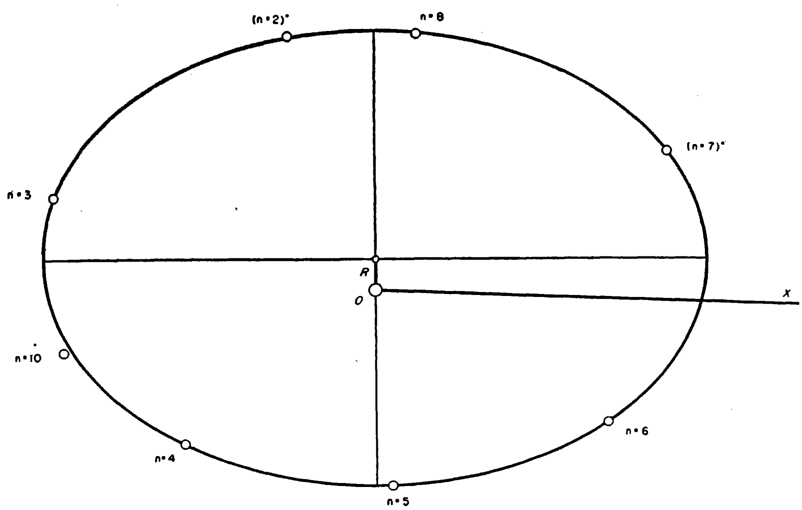

FIGURE 6
VALUES OF $D_n/(1.728)^n$ FOR SATELLITES OF SATURN
PLOTTED AGAINST $(4\pi/13)n$.

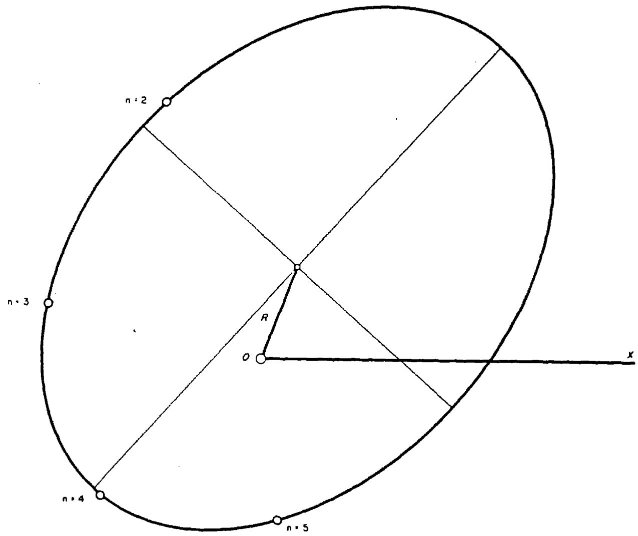

FIGURE 7

VALUES OF $D_n/(1.728)^n$ FOR SATELLITES OF URANUS PLOTTED AGAINST $(4\pi/13)n$.

Note: In the footnote on page 19, for Figure 6 *read* Figure 7; also in Table VI following, for T/2 *read* $\pi/2$.—ED.

dinal number of the large third satellite, Tethys, was used for the exponent in computing D_2 of Enceladus and the ordinal number of the large sixth satellite, Titan, for the exponent in computing D_7 of Hyperion. In all cases, $(4\pi/13)n$ was used for θ_n. The justification for assuming satellite vacancies and for the seemingly arbitrary way of computing D_2 and D_7 of Saturn lies in the results achieved. It should be noted that a unit increase in ordinal number produces a 72.8 per cent

TABLE V

VALUES OF $D_n/(1.728)^n$ FOR THE SATELLITE SYSTEMS OF JUPITER, SATURN, AND URANUS

n	Jupiter	Saturn	Uranus
1	65,162	66,725	.....
2	.....	(28,645)*	39,886
3	50,739	35,467	32,153
4	46,724	26,290	30,529
5	43,110	21,244	23,626
6	43,898	28,501	
7	.....	(34,545)†	
8	.....	27,800	
9	(51,787) (53,083) (53,505)	.....	
10	(59,147) (61,506) (62,865)	33,845	

*$D_2/(1,728)^3$, see text.
†$D_7/(1.728)^6$, see text.

change in $(1.728)^n$. Yet by making the simple changes in ordinal numbering described above, the differences between computed and observed distances immediately became considerably less than one per cent. These results tend to convince one of the correctness of the quantizing-like procedures adopted.

TABLE VI

CONSTANTS OF DISTRIBUTION ELLIPSES FOR THE SATELLITE SYSTEMS OF JUPITER, SATURN, AND URANUS

	Jupiter	Saturn	Uranus
e^2	0.2352	0.5170	0.5170
b	49,210	24,500	30,500
R/b	0.2353	0.1355	0.472
μ	$-5/12\,\theta_1$	$T/2 + 1/24\,\theta_1$	$-3/4\,\theta_1$
ν	$1/12\,\theta_1$	$-1/24\,\theta_1$	$3/8\,\theta_1$
θ_1	$4\pi/13$	$4\pi/13$	$4\pi/13$

TABLE VII

VALUES OF ρ_n COMPUTED FROM EQUATION (5) FOR THE SATELLITE SYSTEMS OF JUPITER, SATURN, AND URANUS

n	(Jupiter) ρ_n	(Saturn) ρ_n	(Uranus) ρ_n
1	65,227	30,374	.....
2	62,091	28,470	39,976
3	50,610	35,377	32,054
4	46,605	26,341	30,601
5	43,110	21,220	23,647
6	44,018	28,430	
7	56,032	34,485	
8	67,217	27,983	
9	55,392	31,722	
10	48,048	32,910	

TABLE VIII

COMPARISON OF OBSERVED AND COMPUTED VALUES OF D_n FOR THE SATELLITE SYSTEMS OF JUPITER, SATURN, AND URANUS

1. SATELLITE SYSTEM OF JUPITER

n	Name of Satellite	Observed* D_n	Solid Angle† Subtended at System Center	Computed D_n	Ratio	Percent Difference
1	Nameless	112,600	60×10^{-8}	112,712	1.00099	+ 0.099
2	....	......	......	......	......	......
3	Io	261,800	6170×10^{-8}	261,137	0.99747	— 0.253
4	Europa	416,600	1740×10^{-8}	415,537	0.99745	— 0.255
5	Ganymede	664,200	1820×10^{-8}	664,196	0.99999	— 0.001
6	Callisto	1,168,700	600×10^{-8}	1,171,913	1.00275	+ 0.275
7	....	......	......	......	......	......
8	....	......	......	......	......	......
9	Nameless‡	7,114,000	0.0013×10^{-8}	7,609,243	Average	Average‡
	Nameless	7,292,000	0.00092×10^{-8}	7,609,243	1.0495	+ 4.95
	Nameless	7,350,000	0.00015×10^{-8}	7,609,243		
10	Nameless‡	14,040,000	0.00004×10^{-8}	11,405,556	Average	Average‡
	Nameless	14,600,000	0.00023×10^{-8}	11,405,556	0.7867	—21.3
	Nameless	14,880,000	0.00013×10^{-8}	11,405,556		

*Distance data from *Sky and Telescope*, Vol. I, No. 6, p. 18. April, 1942.
†In steradian measure.
‡These satellites are all very small. Being also at great distances from the system center, the solid angles subtended by them are unusually small. Per cent differences are correspondingly large.

2. SATELLITE SYSTEM OF SATURN

n	Name of Satellite	Observed* D_n	Solid Angle† Subtended at System Center		Computed D_n	Ratio	Percent Difference
1	Mimas	115,300	950	$\times 10^{-8}$	69,680	‡	‡
2	Enceladus	147,800	900	$\times 10^{-8}$	146,900	0.99391	— 0.609
3	Tethys	183,000	1320	$\times 10^{-8}$	182,538	0.99748	— 0.252
4	Dione	234,400	750	$\times 10^{-8}$	234,860	1.00197	+ 0.197
5	Rhea	327,300	890	$\times 10^{-8}$	326,940	0.99891	— 0.109
6	Titan	758,800	920	$\times 10^{-8}$	756,890	0.99749	— 0.251
7	Hyperion	919,700	8.4	$\times 10^{-8}$	718,110	0.99827	— 0.173
8	Iapetus	2,210,000	20	$\times 10^{-8}$	2,224,500	1.00658	+ 0.658
9	....	......	......		......	......	......
10	Phoebe§	8,034,000	0.027	$\times 10^{-8}$	7,812,060	0.9724	— 2.76§

*Distance data from "Astronomy" by Russell, Dugan, and Stewart, Vol. I, p. iv.

†In steradian measure.

‡Values not computed. Mimas is the only orbital body in all four systems subtending a relatively large central solid angle whose distance-difference exceeds 0.7 of one per cent. See paragraph 8 below for further discussion.

§This satellite subtends an unusually small solid angle at the system center. "Per cent difference" is correspondingly large.

3. SATELLITE SYSTEM OF URANUS

n	Name of Satellite	Observed* D_n	Solid Angle† Subtended at System Center		Computed D_n	Ratio	Percent Difference
1	....	......	......		......	......	......
2	Ariel	119,100	1.53	$\times 10^{-5}$	119,367	1.00224	+ 0.224
3	Umbriel	165,900	0.50	$\times 10^{-5}$	165,394	0.996950	— 0.305
4	Titania	272,200	1.06	$\times 10^{-5}$	272,844	1.00237	+ 0.237
5	Oberon	364,000	0.43	$\times 10^{-5}$	364,334	1.00092	+ 0.092

*Distance data from "Astronomy" by Russell, Dugan, and Stewart, Vol. I, p. iv.

†In steradian measure.

Values of ρ_n for each of the three satellites systems, computed from equation (5) when using the constants of Table VI, appear in Table VII.

Values of D_n for each satellite system *computed* from equation (4)′, using values of ρ_n from Table VII, are compared with corresponding *known* values of D_n in Table VIII. The agreement between known and computed mean distances found here is of the same order as was found in the case of the solar system in Table IV. With few exceptions, differences range from —0.60 to +0.65 of one per cent. These exceptions (in every case and among all systems, except for the first satellite of Saturn) are for orbital bodies that subtend unusually small solid angles at the centers of their corresponding systems. This point is discussed in greater detail in paragraphs 7 and 8 below.

Summarizing, the graphs of Figures 3, 5, 6, *and* 7, *and the numerical comparisons of known and computed values of* D_n *appearing in Tables IV and VIII demonstrate that the mean distances of planets from the sun and of satellites from their primaries in the satellite systems of Jupiter, Saturn, and Uranus are in close agreement with particular values* *of the function,* $D(\kappa n) \cdot \tau^n$, *of equation* (4). *Suitable universal values for* κ *and* τ *have been given as* $4\pi/13$ *and* 1.728, *respectively.*

In conclusion, the following numbered paragraphs relate to certain generalizations. Some of these may prove to be particularly significant, others may be coincidental; all appear to warrant mention.

1. Adding any constant k to each of the ordinal numbers of a system does not alter the *form* of the system's distribution ellipse. Such a change in ordinal numbering does, however, alter the size of the distribution ellipse by the factor $(1.728)^{-k}$ and rotates it through an angle $(4\pi/13)k$ with respect to the initial line of the coördinate system. Implicitly, this property was utilized when the satellites of Uranus were numbered 2-3-4-5 in place of 1-2-3-4, thereby making possible certain generalizations stated in paragraphs 2, 4, 5, and 6 below.

2. Among the three satellite systems the equatorial diameters of the primary bodies are in approximate agreement with the equation

$$d = 2/5\,[\sqrt{ab}/(R/b)] \tag{7}$$

where a and b are the semi-major and semi-minor axes of the corresponding distribution ellipse. This relation was formulated after observing that the distribution ellipses of satellite systems have sizes comparable with the equatorial sections of the corresponding central bodies and that the displacement ratios R/b appear to vary directly with deviations from equality in these dimensions. Figure 8 shows graphically the relative sizes of the various distribution ellipses with respect to the sizes

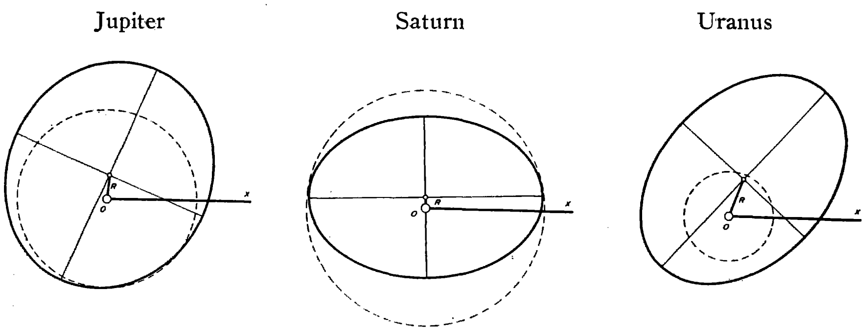

FIGURE 8

DISTRIBUTION ELLIPSES OF THE VARIOUS SATELLITE SYSTEMS COMPARED WITH EQUATORIAL SECTIONS (DOTTED CIRCLES) OF CORRESPONDING CENTRAL BODIES.

TABLE IX

Central Body	$d = 2/5\,[\sqrt{ab}/(R/b)]$	Equatorial Diameter of Central Body
Jupiter	89,450	88,700
Saturn	86,760	75,060
Uranus	31,000	30,880

of corresponding central bodies, the latter being shown by dotted circles. Numerical comparisons of central body diameters with values computed from equation (7) appear in Table IX.

3. The eccentricity of the solar distribution ellipse equals that of Jupiter's distribution ellipse. Similarly, the eccentricities of the distribution ellipses of Saturn and Uranus are equal.

4. In all four systems, the distribution ellipses are displaced *above* the pole of the coördinate system. The angles $90° - \nu$ (between initial line of the coördinate system and the various displacement lines R) range from approximately 69 degrees for Uranus to 92 degrees for Saturn.

5. In each of the four orbital systems considered, the absolute value of the angle between the displacement line R and the major axis of the distribution ellipse is in approximate agreement with the empirical equation

$$|\mu + \nu| = \sin^{-1} 0.178\,(b/R)\,e^2\,(2 - e^2) \tag{8}$$

Agreement in the case of each system appears in Table X.

TABLE X

COMPARISON OF CHARACTERISTIC VALUES OF $|\mu+\nu|$ WITH THOSE GIVEN BY EMPIRICAL EQUATION (8)

System	(Characteristic) $\lvert\mu+\nu\rvert$	(Equation 8) $\lvert\mu+\nu\rvert$
Solar	27° 41′ 32″	27° 43′ 39″
Jupiter	18° 27′ 42″	18° 18′ 01″
Saturn	90° 00′ 00″	90° 24′ 44″
Uranus	20° 46′ 09″	16° 48′ 25″

6. For each of the three satellite system distribution ellipses, the direction of the displacement line R is such as to lie within 4 degrees of a diameter conjugate to a diameter drawn parallel to the initial line of the coördinate system. This is shown graphically in Figure 9.

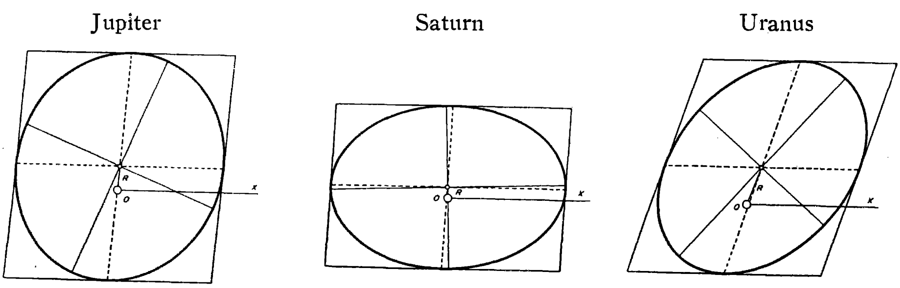

FIGURE 9
CONJUGATE DIAMETERS

7. Relatively large solid angles are subtended at system centers by orbital bodies in 8 out of 10 occupied orbit positions in the solar system, in 5 out of 7 in Jupiter's system, in 8 out of 9 in Saturn's system, and in

4 out of 4 in Uranus' system. If from among these we delete the first satellite of Saturn (whose case is discussed in paragraph 8) and compute the probable error[4] of computed distances of the remaining 24 bodies, we find the probable error to be 0.205 of one per cent, while the greatest single error[4] in computed distance among these bodies is 0.764 of one per cent.

8. Mimas, the first satellite of Saturn, is the only body in all four systems subtending a relatively large solid angle at the center of its system whose computed — observed distance difference exceeds 0.764 of one per cent. The calculated distance for this satellite ($D_1 = 69{,}680$

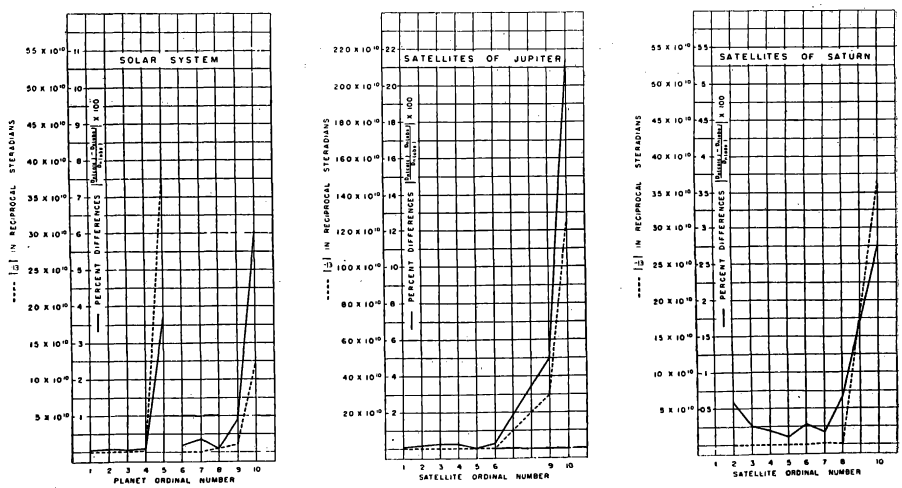

FIGURE 10

GRAPHS SHOWING A DEPENDENCE OF PER CENT DIFFERENCES,

$\left| [D_{n\,(\mathrm{calc.})} - D_{n\,(\mathrm{obs.})}] / D_{n\,(\mathrm{obs.})} \right| \times 100$ ON SOLID ANGLES ω

miles) lies approximately 3000 miles inside the outer edge of the 16,000 mile-wide bright B-ring of Saturn. Thus the ordinal number $n = 1$ appears to be associated with the rings of Saturn as well as with satellite Mimas. It may be significant that this is the only instance among the various systems where the computed value of D_1 is less than the diameter of the central body of the system. Saturn's system is also the only system for which the distribution ellipse lies wholly inside the central body.

[4] The word "error" here applies to the difference between distances computed by equations (4)′ and (5), and the known (observed) distance.